THE F_N METHOD FOR A BARE CRITICAL CYLINDER

by

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c mean number of secondary neutrons produced per collision

I_o modified Bessel function of the first kind of order zero

I_1 modified Bessel function of the first kind of order one

J_o Bessel function of the first kind of order zero

J_1 Bessel function of the first kind of order one

K_o modified Bessel function of the second kind of order zero

K_1 modified Bessel function of the second kind of order one

N number of terms in the F_N expansion

$N(\ )$ psuedo eigenfunction, defined in Eq. (2.11); used for normalization in Chapter III.

P_v Cauchy Principal Value

r radial position in mean free paths

r' position vector in mean free paths

R radial boundary in mean free paths

$R_\nu(\ )$ pseudo eigenfunction, defined in Eq. (2.11)

t radial variable

U union of

β constant, defined in Eq. (4.2)

δ Dirac Delta Function

ε belongs to

$\lambda(\ )$ Case's function, defined in Eq. (2.23)

$\Lambda(\ )$ dispersion function, defined in Eq. (2.19)

$\rho(\ )$ neutron density

$\phi(\ )$ singular eigenfunction of Case, defined in Eq. (2.24)

Φ ( ) psuedo neutron distribution function, defined in Eq. (2.4)

Σ macroscopic neutron cross-section

μ transform variable

ξ collocation variable

ν mean number of neutrons per collision - Chapter IV; eigenvalue elsewhere

Ω( ) function defined in Eq. (3.55)

X( ) function defined in Eq. (2.21)

CHAPTER I

Introduction and Literature Review

A. Background Information

The purpose of this thesis is to develop a more precise and economical means of solving monoenergetic critical cylindrical neutron transport problems. Neutron transport theory has been studied extensively in the last part of this century, but still only a few analytical solutions are known and these are for idealized cases. Many analytical techniques have been employed in an attempt to solve the neutron transport equation explicitly, but the complexity of the equation has made it necessary to employ numerical analysis.

B. Literature Survey

The laws of migration of neutral particles as a function of time, energy, position, and angle were first formulated by Boltzmann. The Boltzmann equation is the basis of all neutron transport theory. The complexity of the Boltzmann equation has made it necessary to deal with it in a reduced form. In 1960, Case[1] solved the plane geometry transport problem in the time-independent, monoenergetic form. Case's method lead to a precise representation of the angular density as a superposition of singular eigenfunctions.

In 1963, Mitsis[2], using Case's methods, obtained transport solutions for monoenergetic plane, spherical, and axially infinite cylindrical critical problems with isotropic scattering. Mitsis replaced

the angular distribution by a density transform and then determined the transform completely using integral equations for the neutron densities. For the spherical and cylindrical cases, Mitsis showed that the density transforms satisfied separated equations whose solutions can be obtained through the use of singular expansion modes. Mitsis applied his boundary conditions yielding a set of singular integral equations for the expansion coefficients. The singular integral equations were reduced to a set of regular Fredholm equations coupled with a criticality condition. Gibbs[3] in 1969 extended Mitsis' work to obtain formal solutions for one-, two-, or three-dimensional bodies of any arbitrary convex geometry. Unfortunately, neither Mitsis nor Gibbs' work could be efficiently applied to numerical computation to produce accurate solutions.

In 1974 Westfall[4] extended the work of Mitsis to obtain transport solutions to the monoenergetic, axially infinite two-region cylinder with a finite outer reflector boundary. Westfall's solution is in the form of two coupled equations for which the expansion coefficients of the criticality equation were computed numerically. Although his computational methods were inefficient, Westfall was able to get up to four-place agreement with other methods which will be presented in Chapter IV.

In 1979, a new numerical technique for solving transport problems was developed by Siewert and Benoist[5]. The technique, known as the F_N method, was derived by using the full-range completeness and orthogonality properties of the singular eigenfunctions. Siewert represented the outgoing angular density $\psi(L,\mu)$ by a power series in the angular

variable μ,

$$\psi(L,\mu) = \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha , \tag{1.1}$$

where a_α are the power coefficients and L is the length of the slab. Siewert got transport solutions for the finite slab and many other applications in plane geometry.

In 1979, Siewert and Grandjean[6] applied this new technique to spherical geometry. They used the F_N method to solve three basic spherical problems: the critical sphere, the albedo problem, and the point source in a finite sphere. Siewert was able to compare his answers to other computational techniques and found that the F_N method was a very concise and economical means of solving spherical and slab-type problems.

C. Objectives

The present work involves extending the F_N method to the bare, axially infinite critical cylinder. The full-range completeness and orthogonality properties of the singular eigenfunctions will be used to derive an expression for the outgoing flux, $\Phi(R,\mu)$, which can be represented by a power series. Unfortunately, the neutron transport equation cannot be solved generally and assumptions must be made. Here the main assumptions will be that the neutron transport equation is in its time-independent, isotropic scattering, monoenergetic form. For fast reactors, the restriction of one neutron speed is not severe.

These assumptions restrict the applicability of the analysis, but they permit ease of computational analysis. The previous computational methods in cylindrical geometry involve complicated coupled equations which unfortunately do not yield easily to efficient computation. On the other hand, we shall see that the F_N method does not readily apply to the cylindrical case due to the appearance of Bessel functions in the criticality equation. The development of the criticality equation will be presented in Chapter III.

Chapter II

Development of General Solution

A. Psuedo Distribution Function

The stationary, monoenergetic, integral transport equation for the neutron density in a homogeneous medium with isotropic scattering as developed by Davison[7] is

$$\rho(r) = \frac{1}{4\pi} \int_{\underline{r}'} \frac{e^{-|r - r'|}}{|r - r'|^2} c(r')\rho(r')\, d^3r' \quad , \tag{2.1}$$

where c, the mean number of secondary neutrons per collision, is a function of r'. If c is a constant, we can write

$$\rho(r) = \frac{c}{4\pi} \int_{r'} \rho(r') \frac{e^{-|r - r'|}}{|r - r'|^2} d^3r' \quad . \tag{2.2}$$

In Fig. 1 the position vector r' is represented by the cylindrical coordinates (t, α, z). Making the transformation to these coordinates Mitsis[2] developed an integral equation for the neutron density of the form

$$\rho(r) = c \int_0^1 \frac{d\mu}{\mu^2} \left[\int_0^r K_o(r/\mu) I_o(t/\mu) t\rho(t)dt \right.$$

$$\left. + \int_r^R K_o(t/\mu) I_o(r/\mu) t\rho(t)dt \right] \quad , \tag{2.3}$$

where t is the radial variable as shown in Fig. 1, and μ is the transform variable.

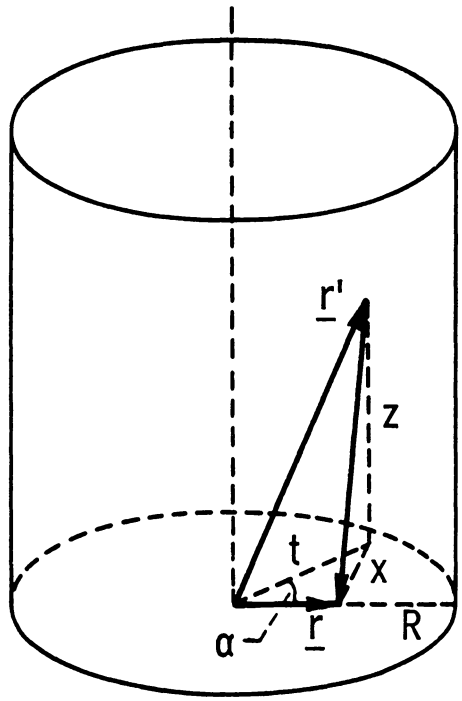

Figure 1. - One region cylindrical geometry.[4]

To simplify this equation, Mitsis introduced a psuedo-distribution function $\Phi(r,\mu)$, such that

$$\rho(r) = \int_0^1 \frac{1}{\mu^2} \Phi(r,\mu) d\mu \ . \tag{2.4}$$

The functions $\rho(r)$ and $\Phi(r,\mu)$ thus form a transform pair, and if $\Phi(r,\mu)$ is known, the neutron density can be calculated by simple integration. To show symmetry for $\Phi(r,\mu)$, Mitsis[2] substituted the Bessel relation

$$I_o(r/\mu)K_o(t/\mu) = \int_0^\infty \frac{J_o(rx)J_o(tx)xdx}{x^2 + (1/\mu)^2} \ ; \ t \geq r \ ; \ \frac{1}{\mu} > 0 \ , \tag{2.5}$$

into Eq. (2.3) and Eq. (2.4) and found the relation

$$\Phi(r,\mu) = c\mu^2 \int_0^R t\rho(t)dt \int_0^\infty \frac{J_o(tx)J_o(rx)}{x^2\mu^2 + 1} dx, \tag{2.6}$$

from which by symmetry

$$\Phi(-r,\mu) = \Phi(r,\mu); \ \Phi(r,-\mu) = \Phi(r,\mu) \ . \tag{2.7}$$

Differentiating $\Phi(r,\mu)$, Mitsis proved that it satisfies the integro-differential transport equation for cylindrical geometry,

$$\frac{\partial^2\Phi(r,\mu)}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi(r,\mu)}{\partial r} - \frac{1}{\mu^2}\Phi(r,\mu)$$

$$= -c \int_0^1 \frac{\Phi(r,\mu')}{(\mu')^2} d\mu' \ . \tag{2.8}$$

The boundary conditions for $\Phi(r,\mu)$ are found from its definition. At the outer boundary it is seen that

$$K_o(R/\mu) \left.\frac{\partial\Phi(r,\mu)}{\partial r}\right|_{r=R} + \frac{K_1(R/\mu)}{\mu}\Phi(R,\mu) = 0. \tag{2.9}$$

The boundary condition at r = 0 is found by substituting r = 0 into Eq. (2.3) and Eq. (2.4) and observing that the integral goes to zero as r approaches zero, and the first term vanishes since $K_o(r)$ diverges as $\ell n(r)$. The substitution of r = 0 yields that

$$\Phi(0,\mu) < \infty \quad . \tag{2.10}$$

B. Singular Eigenfunction Expansion and Solution

To develop the eigenfunctions, Mitsis[2] assumed a solution of Eq. (2.8) of the form

$$\Phi_\nu(r,\mu) = R_\nu(r)N_\nu(\mu) \quad . \tag{2.11}$$

Substituting Eq. (2.11) into Eq. (2.8), one can show that the distribution function is indeed separable. The $R_\nu(r)$ equation is of the form

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r\, d\, R_\nu(r)}{dr}\right) - \frac{1}{\nu^2}R_\nu(r) = 0 \ , \tag{2.12}$$

where $\frac{1}{\nu^2}$ is the separation constant.

Equation (2.12) can be recognized as a form of Bessel's equation with solution

$$R_\nu(r) = \alpha_\nu I_o(r/\nu) + B_\nu K_o(r/\nu) \quad . \tag{2.13}$$

To assure that the solution is finite at the origin, B_ν is required to vanish. The $N_\nu(\mu)$ function must satisfy the equation

$$\left(\frac{1}{\mu^2} - \frac{1}{\nu^2}\right) N_\nu(\mu) = c \int_0^1 \frac{N_\nu(\mu')d\mu'}{(\mu')^2} \; ; \; \mu > 0 \; , \tag{2.14}$$

where $\frac{1}{\nu^2}$ is again the separation constant. If $N_\nu(\mu)$ is normalized by requiring

$$\int_0^1 \frac{1}{\mu^2} N_\nu(\mu) d\mu = 1 \; , \tag{2.15}$$

Eq. (2.14) becomes

$$(\nu^2 - \mu^2) N_\nu(\mu) = c\mu^2\nu^2 \; ; \quad 0 \leq \mu \leq 1 \; . \tag{2.16}$$

If ν does not lie in the interval [-1,1], then Eq. (2.16) becomes

$$N_\nu(\mu) = \frac{c\mu^2\nu^2}{\nu^2 - \mu^2} \; , \tag{2.17}$$

and the normalization condition, Eq. (2.15) requires

$$\int_0^1 \frac{c\nu^2 d\mu}{\nu^2 - \mu^2} = 1 \; , \tag{2.18}$$

or that

$$\Lambda(\nu) \equiv 1 - \frac{c\nu}{2} \int_{-1}^{1} \frac{d\mu}{\nu - \mu} = 0 \; , \tag{2.19}$$

where $\Lambda(\nu)$ is called the dispersion function[8]. The eigenvalues are the zeros of $\Lambda(\nu)$.

Upon examination of $\Lambda(\nu)$, several properties emerge[8]. By symmetry $\Lambda(\nu) = \Lambda(-\nu)$, so if ν_o is a root, $-\nu_o$ is a root also. For $c < 1$, $\Lambda(\nu)$ has two roots, $\pm \nu_o$, $|\nu_o| > 1$, on the real axis; for $c > 1$, $\Lambda(\nu)$ has two roots on the imaginary axis. As c approaches 1, the root approaches infinity. Thus if ν does not lie in the interval $[-1,1]$, $N_\nu(\mu)$ yields the discrete solution

$$N_o(\mu) = \frac{c\nu_o^2}{\nu_o^2 - \mu^2} \quad , \tag{2.20}$$

where ν_o is the positive root of the dispersion function $\Lambda(\nu)$.

For ν in the interval $[-1,1]$, there exist two singularities at $\nu = \pm \mu$. The general solution of Eq. (2.16) is then

$$N_\nu(\mu) = c\, P_\nu \frac{\nu^2 \mu^2}{\nu^2 - \mu^2} + X(\nu)\delta(\mu - \nu) + X(\nu)\delta(\mu + \nu) \quad , \tag{2.21}$$

where Pv denotes the Cauchy principal value. Applying symmetry, it can be observed that $N_\nu(\mu) = N_\nu(-\mu)$. If we apply the normalization condition of Eq. (2.15), we get

$$N_\nu(\mu) = c\, Pv \frac{\nu^2 \mu^2}{\nu^2 - \mu^2} + \nu^2 \lambda(\nu) [\delta(\mu - \nu) + \delta(\mu + \nu)] \; ; \; \mu > 0, \tag{2.22}$$

where

$$\lambda(\nu) = 1 - \frac{c\nu}{2} Pv \int_{-1}^{1} \frac{d\mu}{\nu - \mu} \, . \tag{2.23}$$

Mitsis recognized that his psuedo-eigenfunctions were related to the plane eigenfunctions developed by Case[8] such that

$$N_o(\mu) = \phi_{o+}(\mu) + \phi_{o-}(\mu) \tag{2.24a}$$

and

$$N_\nu(\mu) = \phi_\nu(\mu) + \phi_{-\nu}(\mu) \ . \tag{2.24b}$$

Substituting Eqs. (2.13), (2.20), (2.23) and (2.24) into Eq. (2.11), we obtain the general solution

$$\Phi(r,\mu) = \mu^2 [\phi_{o+}(\mu) + \phi_{o-}(\mu)] \alpha_\nu I_o(r/\nu_o) + \int_0^1 A(\nu) I_o(r/\nu)[\phi_\nu(\mu) + \phi_{-\nu}(\mu)] d\nu \quad ; \tag{2.25}$$

$0 \leq r \leq R$; $0 \leq \mu \leq 1$.

Chapter III

Development of F_N Method for Bare Critical Cylinder

A. Application of Full-Range Orthogonality

Mitsis[2] completed his bare core solutions using the plane geometry eigenfunctions developed by Case. Here we will use the full range orthogonality and normalizations as developed by Case. Case[8] showed that the eigenfunctions were orthogonal in that

$$\int_{-1}^{1} \mu\, \phi_\nu(\mu)\phi_{\nu'}(\mu)d\mu = 0 \ , \ \nu \neq \nu' \ . \tag{3.1}$$

For the case of $\nu = \nu'$, the integral is referred to as the normalization integral,

$$N(\nu)\delta(\nu - \nu') \equiv \int_{-1}^{1} \mu\, \phi_\nu(\mu)\, \phi_{\nu'}(\mu)\, d\mu \ .^{(8)} \tag{3.2}$$

The above eigenfunctions $\phi_\nu(\mu)$ are the continuum eigenfunctions $\phi(\xi,\mu)$, where

$$\phi(\xi,\mu) = \frac{c\xi}{2} \text{Pv} \left(\frac{1}{\xi - \mu}\right) + \lambda(\xi)\delta(\xi - \mu), \ \xi\varepsilon[0,1] \tag{3.3a}$$

whereas for $\xi = \nu_o$

$$\phi(\xi,\mu) = \frac{c\xi}{2} \left(\frac{1}{\xi - \mu}\right)^{(8)} \ . \tag{3.3b}$$

If the general solution for $\Phi(r,\mu)$, Eq. (2.25), is evaluated at $r = R$ and divided by μ, we get

$$\frac{\Phi(R,\mu)}{\mu} = a_o \mu[\phi_{o+}(\mu) + \phi_{o-}(\mu)]I_o(R/\nu_o)$$

$$+ \int_0^1 A(\nu)\mu\,[\phi_\nu(\mu) + \phi_{-\nu}(\mu)]I_o(R/\nu)d\nu \quad . \tag{3.4}$$

If we multiply Eq. (3.4) by $\phi(\xi,\mu)$ and integrate over [-1,1] using the orthogonality and normalization relations, we obtain

$$\int_{-1}^{1} \phi(\xi,\mu)\Phi(R,\mu)\,\frac{d\mu}{\mu} = a_o\, N(\xi)I_o(R/\xi) \quad , \tag{3.5}$$

where again $\xi \in \nu_o\, U[0,1]$.

Taking the partial derivative of $\Phi(r,\mu)/\mu$ with respect to r, and evaluating at r = R, we obtain

$$\frac{1}{\mu}\frac{\partial\Phi(r,\mu)}{\partial r}\bigg|_{r=R} = \frac{a_o}{\nu_o}\,\mu[\phi_{o+}(\mu) + \phi_{o-}(\mu)]I_1(R/\nu_o)$$

$$+ \int_0^1 A(\nu)\,\frac{\mu}{\nu}\,[\phi_\nu(\mu) + \phi_{-\nu}(\mu)]I_1(R/\nu)d\nu \quad . \tag{3.6}$$

If Eq. (3.6) is multiplied by $\phi(\xi,\mu)$ and integrated over [-1,1], we obtain a similar result

$$\int_{-1}^{1} \phi(\xi,\mu)\,\frac{\partial\Phi(R,\mu)}{\partial r}\,\frac{d\mu}{\mu} = \frac{a_o}{\xi}\,N(\xi)I_1\,(R/\xi) \quad . \tag{3.7}$$

Solving Eq. (3.5) and Eq. (3.7) for $a_o N(\xi)$, we find

$$\int_{-1}^{1} \phi(\xi,\mu)\Phi(R,\mu)\,\frac{d\mu}{\mu} - \frac{\xi I_o(R/\xi)}{I_1(R/\xi)} \int_{-1}^{1} \phi(\xi,\mu)\,\frac{\partial\Phi(R,\mu)}{\partial r}\,\frac{d\mu}{\mu} \tag{3.8}$$

B. Development of the Fundamental Equation

Equation (3.8) can be written as

$$I_1(R/\xi) \int_{-1}^{0} \phi(\xi,\mu)\Phi(R,\mu)\,\frac{d\mu}{\mu} + I_1(R/\xi) \int_{0}^{1} \phi(\xi,\mu)\Phi(R,\mu)\,\frac{d\mu}{\mu}$$

$$-\xi I_o(R/\xi) \int_{-1}^{0} \phi(\xi,\mu)\,\frac{\partial\Phi(R,\mu)}{\partial r}\,\frac{d\mu}{\mu}$$

$$-\xi I_o(R/\xi) \int_{0}^{1} \phi(\xi,\mu)\,\frac{\partial\Phi(R,\mu)}{\partial r}\,\frac{d\mu}{\mu} = 0. \tag{3.9}$$

By substituting $-\mu$ for μ in the first and third terms, we write Eq. (3.9) as

$$-I_1(R/\xi) \int_{0}^{1} \phi(-\xi,\mu)\Phi(R,\mu)\,\frac{d\mu}{\mu} + I_1(R/\xi) \int_{0}^{1} \phi(\xi,\mu)\Phi(R,\mu)\,\frac{d\mu}{\mu}$$

$$= \xi I_o(R/\xi) \int_{0}^{1} \phi(-\xi,\mu)\,\frac{\partial\Phi(R,\mu)}{\partial r}\,\frac{d\mu}{\mu}$$

$$- \xi I_o(R/\xi) \int_{o}^{1} \phi(\xi,\mu)\,\frac{\partial\Phi(R,\mu)}{\partial r}\,\frac{d\mu}{\mu}\,. \tag{3.10}$$

From the outer boundary condition, Eq. (2.9), we get

$$\Phi(R,\mu) = -\mu \frac{K_o(R/\mu)}{K_1(R/\mu)} \frac{\partial \Phi(R,\mu)}{\partial r} \quad , \ \mu > 0 \ . \tag{3.11}$$

From the relation found by Mitsis[2] for $\Phi(r,\mu)$, Eq. (2.6), we note that

$$\frac{\partial \Phi(r,\mu)}{\partial r} = - c\mu^2 \int_0^R t\rho(t)dt \int_0^\infty \frac{J_o(tx)J_1(rx)dx}{x^2\mu^2 + 1} \ . \tag{3.12}$$

It can be seen by observation that since $J_1(x)$ is odd,

$$\frac{\partial \Phi(-r,\mu)}{\partial r} = \frac{-\partial \Phi(r,\mu)}{\partial r} \tag{3.13a}$$

and

$$\frac{\partial \Phi(r,-\mu)}{\partial r} = \frac{\partial \Phi(r,\mu)}{\partial r} \ . \tag{3.13b}$$

Therefore substituting Eq. (3.11) and Eq. (3.13) into Eq. (3.10) we find

$$\begin{aligned} &-I_1(R/\xi) \int_0^1 \phi(-\xi,\mu)\Phi(R,\mu) \frac{d\mu}{\mu} + I_1(R/\xi) \int_0^1 \phi(\xi,\mu)\Phi(R,\mu) \frac{d\mu}{\mu} \\ &\quad = \xi I_o(R/\xi) \int_0^1 \phi(-\xi,\mu) \frac{K_1(R/\mu)}{K_o(R/\mu)} \Phi(R,\mu) \frac{d\mu}{\mu^2} \\ &\quad - \xi I_o(R/\xi) \int_0^1 \phi(\xi,\mu) \frac{K_1(R/\mu)}{K_o(R/\mu)} \Phi(R,\mu) \frac{d\mu}{\mu^2} \end{aligned} \tag{3.14}$$

Equation (3.14) is the fundamental equation for our F_N approximation.

For the case $\xi = \nu_o$, the modified Bessel function $I_o(R/\xi)$ and $I_1(R/\xi)$ have imaginary arguments. As discussed later in Chapter IV, the absolute value of the root ν_o is computed. Therefore, we can write the modified Bessel function with order n as

$$I_n(R/\nu_o) = I_n(R/i|\nu_o|) \ , \tag{3.15a}$$

or

$$I_n(R/\nu_o) = I_n(-iR/|\nu_o|) \ . \tag{3.15b}$$

Here we note the Bessel function idenity

$$J_p(ix) = i^p I_p(x) \ , \tag{3.16a}$$

so we have

$$J_p(R/|\nu_o|) = i^p \ I_p(-iR/|\nu_o|) \tag{3.16b}$$

Therefore, the modified Bessel function $I_o(R/\nu_o)$ becomes

$$I_o(R/\nu_o) = i \ J_o(R/|\nu_o|) \ . \tag{3.17a}$$

and $I_1(R/\nu_o)$ becomes

$$I_1(R/\nu_o) = -iJ_1(R/|\nu_o|) \ . \tag{3.17b}$$

C. The F_N Equations

To begin the F_N approximation, we represent the outgoing flux, $\Phi(R,\mu)$, by a power series[5]

$$\Phi(R,\mu) = \sum_{\alpha=2}^{N} a_\alpha \mu^\alpha \ , \tag{3.18}$$

where μ is the angular variable. The summation starts at $\alpha = 2$ to be consistent with Eq. (2.25), the general solution for $\Phi(R,\mu)$.

Substituting the approximation, Eq. (3.18), into the first term of Eq. (3.14) we find

$$\int_0^1 \phi(-\xi,\mu)\Phi(R,\mu)\,\frac{d\mu}{\mu} = \int_0^1 \phi(-\xi,\mu) \sum_{\alpha=0}^{N} a_\alpha \mu^{\alpha+1}\, d\mu \ . \tag{3.19}$$

The summation term can be taken outside the integral yielding

$$\sum_{\alpha=0}^{N} a_\alpha \int_0^1 \phi(-\xi,\mu)\mu^{\alpha+1}\, d\mu \ . \tag{3.20}$$

Following the work of Siewert and Benoist[5], we define $A_\alpha(\xi)$ as

$$A_\alpha(\xi) \equiv \frac{2}{c\xi} \int_0^1 \phi(-\xi,\mu)\mu^{\alpha+1}\, d\mu \ , \tag{3.21a}$$

so Eq. (3.19) can be represented by

$$\int_0^1 \phi(-\xi,\mu)\Phi(R,\mu)\,\frac{d\mu}{\mu} = \sum_{\alpha=0}^{N} a_\alpha \frac{c\xi}{2} A_\alpha(\xi) \ . \tag{3.21b}$$

For the case $\alpha = 0$, $A_\alpha(\xi)$ is

$$A_o(\xi) = \frac{2}{c\xi} \int_0^1 \phi(-\xi,\mu)\mu\, d\mu \ , \tag{3.22a}$$

or by Eq. (3.3),

$$A_o(\xi) = \frac{2}{c\xi} \int_0^1 \frac{c\xi}{2} \frac{\mu}{\xi+\mu} \, d\mu \; ; \tag{3.22b}$$

evaluating the integral we find

$$A_o(\xi) = 1 - \xi \log (1 + 1/\xi) \; . \tag{3.22c}$$

If $\alpha = 1$, $A_\alpha(\xi)$ is

$$A_1(\xi) = \frac{2}{c\xi} \int_0^1 \phi(-\xi,\mu)\mu^2 d\mu \; , \tag{3.23a}$$

or by partial fraction decomposition,

$$A_1(\xi) = \int_0^1 \mu[1 - \xi/(\xi + \mu)] \, d\mu \; ; \tag{3.23b}$$

evaluating the integral we find

$$A_1(\xi) = - \xi A_o(\xi) + 1/2 \; . \tag{3.23c}$$

In general, if $\alpha \neq 0$ $A_\alpha(\xi)$ can be calculated by the recursion relation[5]

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + 1/(\alpha + 1) \; . \tag{3.24}$$

Substituting the power series, Eq. (3.18), into the second term of Eq. (3.14) we find

$$\int_0^1 \phi(\xi,\mu)\Phi(R,\mu) \frac{d\mu}{\mu} = \int_0^1 \phi(\xi,\mu) \sum_{\alpha=0}^{N} a_\alpha \mu^{\alpha+1} \, d\mu \; . \tag{3.25}$$

The summation term can be taken outside the integral yielding

$$\sum_{\alpha=0}^{N} a_\alpha \int_0^1 \phi(\xi,\mu)\mu^{\alpha+1}\, d\mu \quad . \tag{3.26}$$

Define $B_\alpha(\xi)$ such that [5]

$$B_\alpha(\xi) \equiv \frac{2}{c\xi} \int_0^1 \phi(\xi,\mu)\mu^{\alpha+1}\, d\mu \quad , \tag{3.27a}$$

so Eq. (3.22) can be represented by

$$\int_0^1 \phi(\xi,\mu)\Phi(R,\mu)\, \frac{d\mu}{\mu} = \sum_{\alpha=0}^{N} a_\alpha \frac{c\xi}{2} A_\alpha(\xi) \quad . \tag{3.27b}$$

For the case $\alpha = 0$, $B_\alpha(\xi)$ is

$$B_o(\xi) = \frac{2}{c\xi} \int_0^1 [\frac{c\xi}{2} P_v(\frac{1}{\xi-\mu}) + \lambda(\xi)\delta(\xi - \mu)]\mu d\mu, \tag{3.28a}$$

or

$$B_o(\xi) = \frac{2}{c} \lambda(\xi) + P_v \int_0^1 \frac{\mu d\mu}{\xi-\mu} \quad . \tag{3.28b}$$

By using the definition of $\lambda(\nu)$, Eq. (2.23), and evaluating the principal value integral we obtain

$$B_o(\xi) = 2/c - 1 - \xi \log(1 + 1/\xi) \quad . \tag{3.29}$$

For the case $\alpha \neq 0$, $B_\alpha(\xi)$ can be calculated from the recursion relation[5]

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - 1/(\alpha+1) \ , \alpha \geq 1 \ . \tag{3.30}$$

If the power series representation for $\Phi(R,\mu)$, Eq. (3.18), is substituted into the third term of Eq. (3.14) we find

$$\int_0^1 \phi(-\xi,\mu) \frac{K_1(R/\mu)}{K_o(R/\mu)} \Phi(R,\mu) \frac{d\mu}{\mu^2}$$

$$= \int_0^1 \phi(-\xi,\mu) \frac{K_1(R/\mu)}{K_o(R/\mu)} \sum_{\alpha=0}^{N} a_\alpha \mu^{\alpha+2} \frac{d\mu}{\mu^2} \ . \tag{3.31}$$

By bringing the summation term outside the integral and reducing the equation we can write Eq. (3.28) as

$$\sum_{\alpha=0}^{N} a_\alpha \int_0^1 \phi(-\xi,\mu) \frac{K_1(R/\mu)}{K_o(R/\mu)} \mu^\alpha d\mu \ . \tag{3.32}$$

Define $K_{R\alpha}(\mu)$ such that

$$K_{R\alpha}(\mu) \equiv \frac{K_1(R/\mu)}{K_o(R/\mu)} \mu^\alpha, \quad 0 \leq \alpha \leq N \ ; \tag{3.33}$$

where N is the number of power terms. Equation (3.32) can be represented by

$$\sum_{\alpha=0}^{N} a_\alpha \int_0^1 \phi(-\xi,\mu) K_{R\alpha}(\mu) \, d\mu \ . \tag{3.34}$$

Define $C_\alpha(\xi)$ such that

$$C_\alpha(\xi) \equiv \frac{2}{c\xi} \int_0^1 \phi(-\xi,\mu) K_{R\alpha}(\mu)\, d\mu \quad , \tag{3.35a}$$

or by substituting Eq. (3.3) into Eq. (3.35)

$$C_\alpha(\xi) \equiv \int_0^1 K_{R\alpha}(\mu) \frac{d\mu}{\xi+\mu} \, . \tag{3.35b}$$

If $\alpha = 0$, $C_\alpha(\xi)$ is

$$C_o(\xi) = \int_0^1 K_{RO}(\mu) \frac{d\mu}{\xi+\mu} \quad , \tag{3.36}$$

which must be computed numerically.

If $\alpha = 1$, $C_\alpha(\xi)$ becomes

$$C_1(\xi) = \int_0^1 K_{R1}(\mu) \frac{d\mu}{\xi+\mu} \, , \tag{3.37a}$$

or by partial fraction decomposition,

$$C_1(\xi) = \int_0^1 K_{RO}(\mu) \, [1 - \xi/(\xi + \mu)] d\mu \quad ; \tag{3.37b}$$

evaluating the integral we find

$$C_1(\xi) = \int_0^1 K_{RO}(\mu) \, d\mu - \xi C_o(\xi) \, . \tag{3.37c}$$

In general, for $\alpha \geq 1$, $C_\alpha(\xi)$ can be represented by the recursion relation

$$C_\alpha(\xi) = \int_0^1 K_{R(\alpha-1)}(\mu)\, d\mu - \xi C_{\alpha-1}(\xi) \quad . \tag{3.38}$$

Finally, if the power series approximation is substituted into the fourth term of Eq. (3.14) we find

$$\int_0^1 \phi(\xi,\mu) \frac{K_1(R/\mu)}{K_o(R/\mu)} \Phi(R,\mu) \frac{d\mu}{\mu^2}$$

$$= \int_0^1 \phi(\xi,\mu) \frac{K_1(R/\mu)}{K_o(R/\mu)} \sum_{\alpha=0}^{N} a_\alpha \mu^{\alpha+2} \frac{d\mu}{\mu^2} \quad , \tag{3.39a}$$

or by substituting Eq. (3.33) into Eq. (3.39) and reducing, we find Eq. (3.39a) can be represented by

$$\sum_{\alpha=0}^{N} a_\alpha \int_0^1 \phi(\xi,\mu)\, K_{R\alpha}(\mu)\, d\mu \quad . \tag{3.39b}$$

Define $D_\alpha(\xi)$ such that

$$D_\alpha(\xi) \equiv \frac{2}{c\xi} \int_0^1 \phi(\xi,\mu) K_{R\alpha}(\mu) d\mu \quad , \tag{3.40a}$$

so Eq. (3.39) can be represented by

$$\int_0^1 \phi(\xi,\mu)\,\frac{K_1(R/\mu)}{K_o(R/\mu)}\,\Phi(R,\mu)\,\frac{d\mu}{\mu^2}$$

$$= \sum_{\alpha=0}^{N} a_\alpha \frac{c\xi}{2} D_\alpha(\xi) \quad . \tag{3.40b}$$

If Eq. (3.3) is substituted into Eq. (3.40) we find

$$D_\alpha(\xi) = \frac{2}{c\xi}\int_0^1 K_{R\alpha}(\mu)\,[\frac{c\xi}{2} P_v(\frac{1}{\xi-\mu}) + \lambda(\xi)\delta(\xi-\mu)]d\mu \quad . \tag{3.41}$$

By substituting the definition of $\lambda(\nu)$, Eq. (2.23), into Eq. (3.41) and simplifying we find

$$D_\alpha(\xi) = \frac{2}{c\xi}\lambda(\xi)K_{R\xi}(\xi) + Pv\int_0^1 K_{R\alpha}(\mu)\,\frac{d\mu}{\xi-\mu} \quad , \tag{3.42a}$$

where $K_{R\xi}(\xi)$ is defined as

$$K_{R\xi}(\xi) \equiv \frac{K_1(R/\xi)\xi^\alpha}{K_o(R/\xi)} \quad . \tag{3.43b}$$

For the case $\alpha = 0$, $D_\alpha(\xi)$ is

$$D_o(\xi) = \frac{2}{c\xi}\int_0^1 K_{RO}(\mu)\phi(\xi,\mu)d\mu \quad , \tag{3.44a}$$

or by Eq. (3.41)

$$D_o(\xi) = \frac{2}{c\xi}\,\lambda(\xi)K_{RO}(\xi) + P_v \int_0^1 K_{RO}(\mu)\,\frac{d\mu}{\xi-\mu} \quad . \tag{3.44b}$$

By the definition of $\lambda(\nu)$, Eq. (2.23), we can re-write Eq. (3.44) as

$$D_o(\xi) = \frac{2}{c\xi}\left[1 + \frac{c\xi}{2} P_v \int_{-1}^1 \frac{d\mu}{\mu-\xi}\right] K_{RO}(\xi)$$

$$- \int_0^1 \frac{K_{RO}(\mu) - K_{RO}(\xi)}{\mu - \xi} d\mu - K_{RO}(\xi)\, P_v \int_0^1 \frac{d\mu}{\mu-\xi} \,, \tag{3.45a}$$

or

$$D_o(\xi) = \frac{2}{c\xi} K_{RO}(\xi) + K_{RO}(\xi)\, P_v \int_{-1}^1 \frac{d\mu}{\mu-\xi}$$

$$- \int_0^1 \frac{K_{RO}(\mu) - K_{RO}(\mu)}{\mu - \xi}\, d\mu - K_{RO}(\xi)\, P_v \int_0^1 \frac{d\mu}{\mu-\xi} \,. \tag{3.45b}$$

By combining the Cauchy principal value integrals we obtain

$$D_o(\xi) = \frac{2}{c\xi} K_{RO}(\xi) - K_{RO}(\xi) \int_0^1 \frac{d\mu}{\mu+\xi} - \int_0^1 \frac{K_{RO}(\mu) - K_{RO}(\xi)}{\mu - \xi}\, d\mu \,, \tag{3.46a}$$

or, evaluating the integral

$$D_o(\xi) = K_{RO}(\xi)\,[2/c\xi - \log\,(1 + 1/\xi)] - \int_0^1 \frac{K_{RO}(\mu) - K_{RO}(\xi)}{\mu - \xi}\, d\mu \,. \tag{3.46b}$$

If $\alpha = 1$, $D_\alpha(\xi)$ becomes

$$D_1(\xi) = \frac{2}{c\xi} \int_0^1 K_{R1}(\mu)\phi(\xi,\mu)\, d\mu \quad , \tag{3.47a}$$

or by Eq. (3.42)

$$D_1(\xi) = 2/c\xi\ \lambda(\xi)\ K_{R1}(\xi) + P_V \int_0^1 \frac{K_{R1}(\mu)\, d\mu}{\xi - \mu} \ . \tag{3.47b}$$

The above expression for $D_1(\xi)$ reduces to

$$D_1(\xi) = [2/c\xi - \log\ (1 + 1/\xi)]K_{R1}(\xi) - \int_0^1 K_{R0}(\mu)\, d\mu$$

$$- \xi \int_0^1 \frac{K_{R0}(\mu) - K_{R0}(\xi)}{\mu - \xi}\, d\mu \quad . \tag{3.48}$$

The above expression can be observed to be equivalent to

$$D_1(\xi) = \xi\, D_o(\xi) - \int_0^1 K_{R0}(\mu)\, d\mu \quad . \tag{3.49}$$

Upon evaluation of successive terms, the following recursion relation for $D_\alpha(\xi)$ is found

$$D_\alpha(\xi) = \xi^\alpha\ D_{(\alpha-1)}(\xi) - \int_0^1 K_{R(\alpha-1)}(\mu)\, d\mu \quad . \tag{3.50}$$

D. Matrix Form of Equations

Substituting Eqs. (3.21), (3.27), (3.35) and Eq. (3.40) into the fundamental equation, Eq. (3.14), we find

$$-I_1(R/\xi) \sum_{\alpha=0}^{N} a_\alpha[c\xi/2]A_\alpha(\xi)$$

$$+ I_1(R/\xi) \sum_{\alpha=0}^{N} a_\alpha[c\xi/2]\, B_\alpha(\xi)$$

$$= \xi I_o(R/\xi) \sum_{\alpha=0}^{N} a_\alpha[c\xi/2]\, C_\alpha(\xi)$$

$$- \xi I_o(R/\xi) \sum_{\alpha=0}^{N} a_\alpha[c\xi/2]D_\alpha(\xi) \quad . \tag{3.51}$$

By cancelling out the $c\xi/2$ terms and grouping the summation terms, we have

$$\sum_{\alpha=0}^{N} a_\alpha \{- I_1(R/\xi)[A_\alpha(\xi) - B_\alpha(\xi)] -$$

$$\xi I_o(R/\xi)[C_\alpha(\xi) - D_\alpha(\xi)]\} = 0 \quad . \tag{3.52}$$

If we evaluate Eq. (3.52) at $\xi = \nu_o$, we have

$$\sum_{\alpha=0}^{N} a_\alpha \{I_1(R/\nu_o)[B_\alpha(\nu_o) - A_\alpha(\nu_o)] + \nu_o I_o(R/\nu_o)[D_\alpha(\nu_o) - C_\alpha(\nu_o)]\}$$

$$= 0 \quad . \tag{3.53}$$

If we define a_o to be one and separate the $\alpha = 0$ term out of Eq. (3.52), we have

$$\sum_{\alpha=1}^{N} a_\alpha \{ - I_1(R/\xi)[A_\alpha(\xi) - B_\alpha(\xi)] - \xi I_o(R/\xi)[C_\alpha(\xi) - D_\alpha(\xi)]\}$$

$$= I_1(R/\xi)[A_o(\xi) - B_o(\xi)] + \xi I_o(R/\xi)[C_o(\xi) - D_o(\xi)] \ . \qquad (3.54)$$

Define $\Omega_\alpha(\xi)$ such that

$$\Omega_\alpha(\xi) = I_1(R/\xi)[B_\alpha(\xi) - A_\alpha(\xi)] + \xi I_o(R/\xi)[D_\alpha(\xi) - C_\alpha(\xi)] \ , \qquad (3.55a)$$

so Eq. (3.54) becomes

$$\sum_{\alpha=1}^{N} a_\alpha \Omega_\alpha (\xi_\beta) = - \Omega_o (\xi_\beta) \ , \ \beta = 1, 2, \ldots, N; \qquad (3.55b)$$

and Eq. (3.53) becomes

$$\sum_{\alpha=0}^{N} a_\alpha \Omega_\alpha(\nu_o) = 0, \ a_o = 1 \ . \qquad (3.55c)$$

We have now obtained the equations necessary to solve the problem. As an example of the process, we will use the F_2 case. By Eq. (3.55), we have:

$$a_1\,\Omega_1(\xi_1) + a_2\,\Omega_2(\xi_1) = -\,\Omega_o(\xi_1)$$

$$a_1\,\Omega_1(\xi_2) + a_2\,\Omega_2(\xi_2) = -\,\Omega_o(\xi_2)$$

$$a_1\,\Omega_1(\nu_o) + a_2\,\Omega_2(\nu_o) = -\,\Omega_o(\nu_o)\ , \tag{3.56a}$$

or in matrix form:

$$\begin{bmatrix} \Omega_1(\xi_1) & \Omega_2(\xi_1) \\ \Omega_1(\xi_2) & \Omega_2(\xi_2) \end{bmatrix} \begin{bmatrix} a_1 \\ a_1 \end{bmatrix} = - \begin{bmatrix} \Omega_o(\xi_1) \\ \Omega_o(\xi_2) \end{bmatrix} . \tag{3.56b}$$

The equations are solved for the coefficients a_1 and a_2. The coefficients are used in Eq. (3.55c) such that

$$a_1\,\Omega_1(\nu_o) + a_2\,\Omega_2(\nu_o) = -\,\Omega_o(\nu_o)\ . \tag{3.57}$$

If Eq. (3.57) is satisfied, the value of R is the critical radius. If Eq. (3.57) is not satisfied, the value of R must be iterated until it is satisfied.

E. Treatment of Imaginary Eigenvalue

Whenever c > 1, the roots of the dispersion function, Eq. (2.19), are imaginary. However, by Eq. (3.52) we are only interested in the differences $A_\alpha(\xi) - B_\alpha(\xi)$ and $C_\alpha(\xi) - D_\alpha(\xi)$. We can show that these differences are real as follows. If $A_\alpha(\xi) - B_\alpha(\xi)$ is evaluated at $\xi = \nu_o$, we have

$$A_\alpha(\nu_o) - B_\alpha(\nu_o) = \int_0^1 \frac{\mu^{\alpha+1}}{\nu_o+\mu} d\mu - \int_0^1 \frac{\mu^{\alpha+1} d\mu}{\nu_o-\mu} , \tag{3.58a}$$

or by combining integrals,

$$A_\alpha(\nu_o) - B_\alpha(\nu_o) = \int_0^1 \frac{-2\mu^{\alpha+2}}{\nu_o^2 - \mu^2} d\mu \quad . \tag{3.58b}$$

If we substitute iz_o for ν_o, where $z_o = |\nu_o|$, into Eq. (3.58), we have

$$A_\alpha(\nu_o) - B_\alpha(\nu_o) = \int_0^1 \frac{2\mu^{\alpha+2}}{\mu^2+z_o^2} d\mu \quad , \tag{3.59}$$

so the difference is real.

For the case $\alpha = 0$, $A_\alpha(\nu_o) - B_\alpha(\nu_o)$ becomes

$$A_o(\nu_o) - B_o(\nu_o) = \int_0^1 \frac{2\mu^2 d\mu}{\mu^2+z_o^2} \quad , \tag{3.60a}$$

or by partial fraction decomposition,

$$A_o(\nu_o) - B_o(\nu_o) = 2 - 2 z_o^2 \int_0^1 \frac{d\mu}{\mu^2+z_o^2} \quad , \tag{3.60b}$$

or evaluating the integral

$$A_o(\nu_o) - B_o(\nu_o) = 2(1 - 1/c) \ , \tag{3.60c}$$

where we have used Case's[8] relation that $\tan^{-1}\left(\frac{1}{z_o}\right) = 1/cz_o$.

If $\alpha = 1$, $A_\alpha(\nu_o) - B_\alpha(\nu_o)$ becomes

$$A_1(\nu_o) - B_1(\nu_o) = 2\int_0^1 \mu^3 \frac{d\mu}{\mu^2+z_o^2} \ , \tag{3.61a}$$

which upon evaluating becomes

$$A_1(\nu_o) - B_1(\nu_o) = 1 - z_o^2 \log\,(1 + 1/z_o^2) \ . \tag{3.61b}$$

If $\alpha = 2$, $A_\alpha(\nu_o) - B_\alpha(\nu_o)$ becomes

$$A_2(\nu_o) - B_2(\nu_o) = 2\int_0^1 \mu^4 \frac{d\mu}{\mu^2+z_o^2} \ , \tag{3.62a}$$

or by partial fraction decomposition,

$$A_2(\nu_o) - B_2(\nu_o) = 2\int_0^1 \mu^2(1 - z_o^2/(\mu^2 + z_o^2))d\mu \ , \tag{3.62b}$$

or, evaluating the integral,

$$A_2(\nu_o) - B_2(\nu_o) = 2/3 - 2z_o^2(1 - 1/c) \ . \tag{3.62c}$$

If $\alpha = 3$, $A_\alpha(\nu_o) - B_\alpha(\nu_o)$ becomes

$$A_3(\nu_o) - B_3(\nu_o) = 2 \int_0^1 \mu^5 \frac{d\mu}{\mu^2+z_o^2} , \tag{3.63a}$$

which after evaluating the integral becomes

$$A_3(\nu_o) - B_3(\nu_o) = 1/2 - z_o^2 + z_o^4 \log (1 + 1/z_o^2) . \tag{3.63b}$$

In general for $\alpha \geq 2$, $A_\alpha(\nu_o) - B_\alpha(\nu_o)$ satisfies the recursion relation

$$A_\alpha(\nu_o) - B_\alpha(\nu_o) = 2/(\alpha + 1) - z_o^2 [A_{\alpha-2}(\nu_o) - B_{\alpha-2}(\nu_o)] \tag{3.64}$$

When $C_\alpha(\xi) - D_\alpha(\xi)$ is evaluated at $\xi = \nu_o$, we have

$$C_\alpha(\nu_o) - D_\alpha(\nu_o) = \int_o^1 \mu^\alpha \frac{K_1(R/\mu)}{K_o(R/\mu)} \frac{d\mu}{\nu_o+\mu}$$

$$- \int_o^1 \mu^\alpha \frac{K_1(R/\mu)}{K_o(R/\mu)} \frac{d\mu}{\nu_o-\mu} , \tag{3.65a}$$

or by combining integrals and substituting iz_o for ν_o, we have

$$C_\alpha(\nu_o) - D_\alpha(\nu_o) = 2 \int_o^1 \mu^{\alpha+1} \frac{K_1(R/\mu)}{K_o(R/\mu)} \frac{d\mu}{\mu^2+z_o^2} . \tag{3.65b}$$

For the case $\alpha = 0$, $C_\alpha(\nu_o) - D_\alpha(\nu_o)$ is

$$C_o(\nu_o) - D_o(\nu_o) = 2 \int_o^1 \mu \frac{K_1(R/\mu)}{K_o(R/\mu)} \frac{d\mu}{\mu^2+z_o^2} . \tag{3.66}$$

If $\alpha = 1$, $C_\alpha(\nu_o) - D_\alpha(\nu_o)$ becomes

$$C_1(\nu_o) - D_1(\nu_o) = 2 \int_o^1 \mu^2 \frac{K_1(R/\mu)}{K_o(R/\mu)} \frac{d\mu}{\mu^2+z_o^2} , \tag{3.67a}$$

or by partial fraction decomposition

$$C_1(\nu_o) - D_1(\nu_o) = 2 \int_0^1 \frac{K_1(R/\mu)}{K_o(R/\mu)} [1 - z_o^2/(\mu^2 + z_o^2)] \, d\mu . \tag{3.67b}$$

If $\alpha = 2$, $C_\alpha(\nu_o) - D_\alpha(\nu_o)$ becomes

$$C_2(\nu_o) - D_2(\nu_o) = 2 \int_o^1 \mu^3 \frac{K_1(R/\mu)}{K_o(R/\mu)} \frac{d\mu}{\mu^2+z_o^2} , \tag{2.68a}$$

or by partial fraction decomposition,

$$C_2(\nu_o) - D_2(\nu_o) = 2 \int_o^1 \frac{K_1(R/\mu)}{K_o(R/\mu)} [\mu - \mu z_o^2/(\mu^2 + z_o^2)] d\mu . \tag{3.68b}$$

The above expression can be shown to be equivalent to

$$C_2(\nu_o) - D_2(\nu_o) = 2 \int_o^1 K_{R1}(\mu) \, d\mu - z_o^2 \, [D_o(\nu_o) - C_o(\nu_o)] . \tag{3.69}$$

In general for $\alpha \geq 2$, $C_\alpha(\nu_o) - D_\alpha(\nu_o)$ can be expressed by the recursion relation

$$C_\alpha(\nu_o) - D_\alpha(\nu_o) = 2 \int_o^1 K_{R(\alpha-2)}(\mu)\, d\mu$$

$$- z_o^2 \left[D_{(\alpha-2)}(\nu_o) - C_{(\alpha-2)}(\nu_o)\right] . \qquad (3.70)$$

Chapter IV

Numerical Analysis and Results

A. Program Input Data

To implement the iterative process described in Chapter III, the program TRAN FORTRAN was written. The program is designed to iterate on the radius until the incremental radius is less than a specified criterion. The main program has five input variables. The first input variable is c, which is the mean number of secondary neutrons per collision or

$$c \equiv (\nu\Sigma_F + \Sigma_S)/\Sigma_T \ . \tag{4.1}$$

The second input variable is R, the initial guess for the critical radius. To assure proper convergence, the initial value of R must be at least dr below the critical radius, where dr, the incremental radius is the third input variable. For this program, a value of 0.05 for dr was found to be sufficient for proper convergence. The fourth input variable was EPS2, the convergence criterion. A convergence criterion of 10^{-5} was chosen to assure at least five place accuracy. The fifth input variable was P, the number of power terms wanted. As an example, if the value of P was 9, the program would compute an F_9 approximation. The last input variable is IFLAG, which specifies the order that the collocation points are taken by the program. If IFLAG is equal to one, the collocation points are taken in the order which they are input or computed. A value of IFLAG which is unequal to one would cause the program to reverse the collocation order.

B. Numerical Methods

The main program consists of two major parts, one of which is radius-independent. Using the input variable c, the main program sends an initial guess for ν_o to subroutine ROOT. Subroutine ROOT uses Newton's method to refine the estimate and returns to the main program the absolute value of root ν_o which is denoted in the program by z_o. With the value of z_o, the main program can call subroutine INT12 which computes $A_\alpha(\xi)$ and $B_\alpha(\xi)$ from the recursion relationships developed in Chapter III.

Once the radius-independent terms are calculated, the program can begin the iterative cycle. The main program calls subroutine INT34 which computes $C_\alpha(\xi)$ and $D_\alpha(\xi)$. The terms $C_\alpha(\xi)$ and $D_\alpha(\xi)$ are computed using the formulas developed in Chapter III. Once $C_\alpha(\xi)$ and $D_\alpha(\xi)$ are computed, the main program calls subroutine MATREL, which computes $A_\alpha(\xi)-B_\alpha(\xi)$ and $C_\alpha(\xi) - D_\alpha(\xi)$ for the case $\xi = Z_o$. Both subroutines INT34 and MATREL use the nodes and weights computed by the subroutine MAP for integration by Gauss-Legendre quadrature. Once $A_\alpha(\xi)$, $B_\alpha(\xi)$, $C_\alpha(\xi)$ and $D_\alpha(\xi)$ are completely defined the main program can call the routine LEQT2F. Routine LEQT2F is a Internatial Mathematics and Statistics Library routine, hereafter abbreviated by I.M.S.L., which will compute the power series coefficients by solving Eq. (3.56). With the power series coefficients, the main program computes the sum, Eq. (3.55c), and notes the sign. The value of R is incremented and the R-dependent terms are recomputed until a sign change is noted. When a sign change is detected, the incremental radius is halved and the cycle

is repeated until the incremental radius falls below the convergence criterion.

C. Numerical Precision

The objective in seeking numerical results was to obtain critical radii accurate to five significant figures. Therefore, the program was written entirely in double precision. The calculations were done on a IBM-370 computer and the program is designed for an IBM machine. The Bessel functions were all calculated by double precision I.M.S.L. Bessel routines whose maximum allowed argument varies with different machines. The discrete eigenvalues, computed by subroutine ROOT, agreed with the values computed by Case, de Hoffmann, and Placzek[9] up to at least seven significant figures. Figure 2 shows a plot of c vs. $|\nu_o|$.[4]

The converged radius depends on several variables. The order of the power series, N, significantly affects the converged radius. It was found that an F_9 approximation was sufficient to yield four place agreement between different collocation sets. The variation of core radius with F_N approximation is shown in Table I. The values in Table I were computed with forty Gauss-Legendre quadrature points to compute the integrals in Eqs. (3.33), (3.35), (3.46), (3.59) and Eq. (3.65). The zeros of the Chebyshev polynomials, Eq. (4.2), were used as the collocation points. The converged radius also depends on the set of collocation points. The collocation points are the values of ξ in the interval [0,1]. Three different sets of collocation points were used: Gauss-Legendre nodes, equally spaced points, and the zeros of the

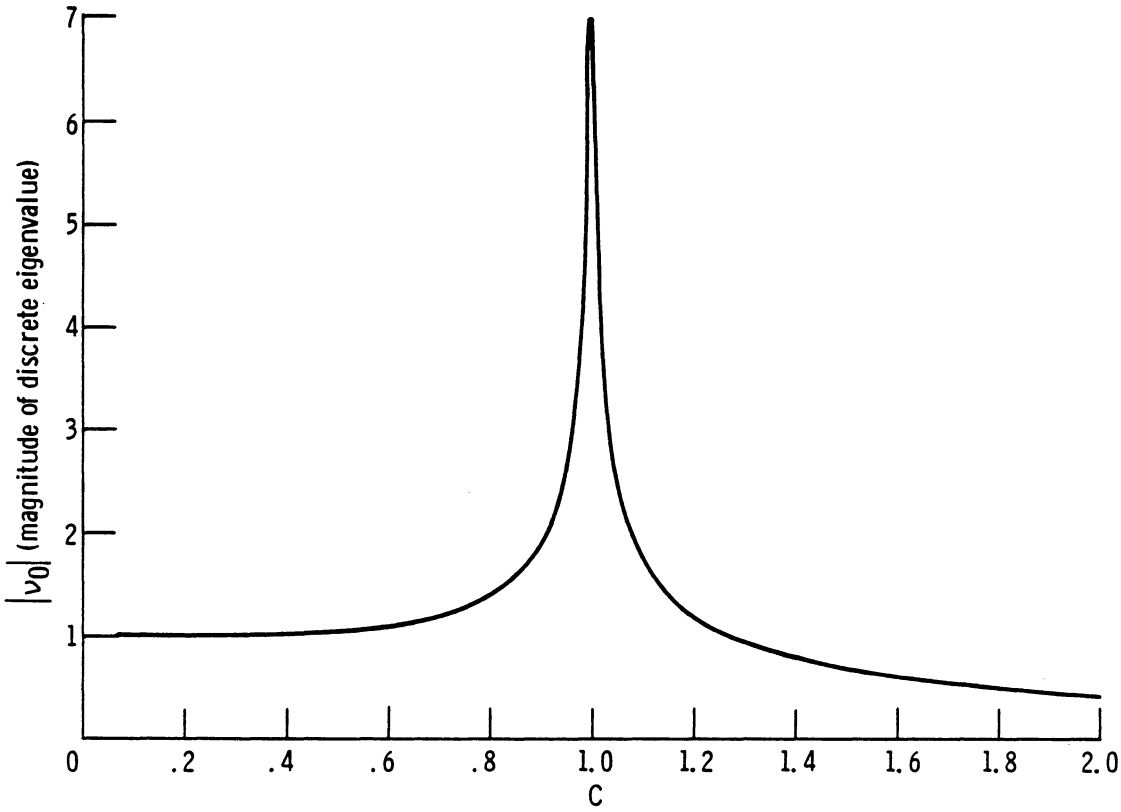

Figure 2. - Variation of the discrete eigenvalue for $0 \leq c \leq 2$. (Mean number of secondaries per collision).

Chebyshev polynomials[10],

$$\xi_\beta = 1/2 + (1/2) \cos \left(\frac{2\beta-1}{2N} \pi\right), \beta = 1, 2, \ldots, N . \quad (4.2)$$

The different collocation sets provided at least four place agreement. The variation of converged radius with the collocation set is shown in Table II, based on forty Gauss-Legendre quadrature points and an F_9 approximation. The converged radius was not very sensitive to the number of Gauss-Legendre points used. The calculations were done with Gauss-Legendre quadratures of twenty, forty, and eighty points. The converged radii showed that twenty points were sufficient for five significant figures.

D. Method Comparison

Critical radii calculated with an F_9 approximation with the zeros of the Chebyshev polynomials as collocation points are compared with other solutions found in the literature in Table III. Westfall[4] extended Mitsis'[2] work to get his values. Carlson and Bell[11] used, for large systems ($R > 1.5$), the extrapolated endpoint method. For small systems, they interpolated between values calcualated with the extrapolated endpoint and the variational method. Hendry used a Fourier expansion to represent the neutron distribution function and integrated using Gauss quadrature. Hembds'[13] IT_n method used a Fourier transformation of the integral equation.

Table I. Variation of Critical Core Radii in Mean Free Paths with F_N Approximation

c	F_3	F_4	F_5	F_6	F_7	F_8	F_9
1.02	9.04368	9.04371	9.04375	9.04376	9.04376	9.04376	9.04376
1.05	5.41195	5.41196	5.41200	5.41201	5.41201	5.41201	5.41201
1.1	3.57787	3.57815	3.57817	3.57820	3.57820	3.57820	3.57820
1.2	2.28646	2.28777	2.28773	2.28776	2.28776	2.28776	2.28776
1.4	1.39318	1.39631	1.39630	1.39630	1.39631	1.39631	1.39631
1.6	1.01505	1.01859	1.01895	1.01890	1.01891	1.01891	1.01891
1.8	0.80072	0.80352	0.80436	0.80430	0.80430	0.80430	0.80430
2.0	0.66169	0.66318	0.66443	0.66445	0.66442	0.66443	0.66443

Table II. Variation of Critical Core Radii (MFP) with Collocation Set

c	Chebyshev	Gauss-Legendre Nodes	Equally Spaced
1.02	9.04376	9.04376	9.04376
1.05	5.41201	5.41201	5.41201
1.1	3.57820	3.57820	3.57820
1.2	2.28776	2.28776	2.28776
1.4	1.39631	1.39631	1.39633
1.6	1.01891	1.01892	1.01892
1.8	0.80430	0.80430	0.80430
2.0	0.66443	0.66442	0.66440

Table III. Comparison of Critical Radii (MFP)

c	Present Solution	Westfall[4]	Carlson-Bell[11]	Hendrey[12]	Hembd[13]
1.02	9.04376	9.043225	9.0433	--	9.04458
1.05	5.41201	5.411288	5.4118	5.414	5.41152
1.1	3.57820	3.577391	3.5783	--	3.57744
1.2	2.28776	2.287209	2.2884	--	2.28727
1.4	1.39631	1.396979	1.3973	--	1.39699
1.6	1.01891	1.020839	1.0209	--	1.02085
1.8	0.80430	0.807427	0.8067	--	0.80743
2.0	0.66443	0.668613	0.6673	0.670	0.66862

Chapter V

Conclusions and Recommendations

A. Conclusions

The F_N approximation developed here for cylindrical geometry is a very efficient means of obtaining approximate critical core radii. This is the first application of the full-range completeness and orthogonality properties of the singular eigenfunctions for cylindrical geometry and the first application of the F_N method to cylindrical geometry. The result of this application was shown in Chapter IV. The F_N method gave four digit agreement with benchmark values for large systems (c = 1.02) and two digit agreement for small systems (c = 2.0). The F_N method was shown to be relatively independent of the collocation set or quadrature order and was found to have a very rapid convergence.

B. Recommendations

Further investigation of the cylindrical transport problem should reveal why the F_N method does not yield at least five place agreement with the other methods presented in Chapter IV. For slab- and spherical-type geometry, the F_N method gave five place agreement with other benchmark values[5],[6]. The appearance of Bessel functions in the general solution for the outgoing flux, Eq. (2.25), hinders the application of the F_N method. However, the Bessel routines used in the program TRAN FORTRAN gave at least twelve place accuracy and were superior to the series approximations which were used by the other methods. Investigation into the discrepency of accuracy would be worthwhile.

Much opportunity exists for the further development of the F_N method for cylindrical geometry. Westfall[4] developed the general solution for the axially infinite two-region cylinder with finite reflector boundary. In 1980, Siewert and Garcia[14] applied the F_N method to a four-region plane geometry transport problem. The F_N method was found to give at least four place agreement with other benchmark values. The application of the F_N method to a multiregion cylinder would be very useful.

In 1981, Siewert and Benoist[15] developed the F_N method for the multigroup transport problem. With the restriction of considering downscattering only, they were able to reduce the multigroup problem to a sequence of one-group problems. Siewert showed that the F_N method gave accurate solutions for the emerging angular fluxes. Siewert and Benoist[15] presented results accurate to five significant figures for a 16-group and 19-group albedo problem. The albedo problem is the problem of obtaining the emerging angular flux everywhere in a source-free half-space if a parallel beam is incident on the slab surface at the origin.[8] With the application of the multigroup F_N method to cylindrical geometry, the restriction to fast reactors will be no longer necessary.

The future for the further advancements in neutron transport theory in cylindrical geometry looks promising. The F_N method has many advantages over other methods and a wide range of applicability. Further study of the cylinder problem may yield a superior power series approximation for the outgoing flux, possibly incorporating Bessel functions. The F_N method has not yet been applied to time-dependent

transport problems and these applications will be worthwhile for practical applications.

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APPENDICES

APPENDIX 1

Listing of Computer Program

TRAN FORTRAN

```
C$JOB
C
C      WRITTEN BY JACK SOUTHERS IN PARTIAL
C      REQUIREMENT FOR MASTER OF SCIENCE.
C      SEPTEMBER 1,1982
C      THIS PROGRAM WILL COMPUTE THE CRITICAL RADIUS
C      FOR A CYLINDRICAL FUEL ROD USING THE FN METHOD
C      ORIGINATED BY C.E.SIEWERT.THE INPUT VARIABLES ARE:
C      C IS THE NUMBER OF SECONDARY NEUTRONS PRODUCED PER COLLISION.
C      R IS THE INITIAL GUESS AT THE CRITICAL RADIUS. THE INITIAL
C       GUESS AT THE CRITICAL RADIUS MUST BE AT LEAST DR BELOW
C       THE ACTUAL RADIUS TO ASSURE PROPER CONVERGENCE.
C      DR IS THE INCREMENTAL RADIUS.
C      EPS2 IS THE CONVERGENCE CRITERIA.
C      P IS THE NUMBER OF POWER TERMS WHICH WILL APPROXIMATE
C       THE FLUX.
C      IFLAG DETERMINES THE ORDER THAT THE COLLOCATION
C       POINTS ARE TAKEN IN.
C
C
C      THE VARIABLES USED IN THE PROGRAM ARE:
C      XI STANDS FOR THE ARRAY OF COLLOCATION POINTS.
C      ZO IS THE ABSOLUTE VALUE OF THE IMAGINERY ROOT OF
C       THE DISPERSION FUNCTION.
C      THE INDEX LL DENOTES THE ORDER.
C      THE INDEX M DENOTES THE COLLOCATION POINTS.
C
C
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 MMBSI0,MMBSI1,I0,I1,MMBSJ0,MMBSJ1,J1,J0
      EXTERNAL MMBSI0,MMBSI1,MMBSJ0,MMBSJ1
      COMMON/BLOCK1/A(160),B(160),N
```

```
      INTEGER P
      DIMENSION C0(11,11),D0(11,11),AB(11,11),RKRP(11)
      DIMENSION XI(20),A0(11,11),B0(11,11),WKAREA(155)
      DIMENSION OMZ02(11,11),CMF(11,1)
      EPS=1.0D-13
      IOPT=2
C     CALL TRAPS(0,0,999999)
      CALL ERRSET(208,256,-1,0,0)
C     DO 2 I=1,N
C   2 WRITE(6,110) A(I),B(I)
C 110 FORMAT(5X,2(D15.8,2X))
      READ(5,*) C,R,DR,EPS2,P,IFLAG
C 112 FORMAT(4D10.5,I2,I2)
      WRITE(6,102)C,R,DR,EPS2,P
      CALL MAP
      WRITE(6,100) C
C
C     THE FOLLOWING SEQUENCE COMPUTES THE ESTIMATE
C     FOR THE EIGENVALUE TO SEND TO SUBROUTINE
C     ROOT.
C
      PP=DABS(1.D0-C)
      Z0=DSQRT(3.D0*PP)*(1.D0-0.4D0*(1.D0-C))
      IF(C.GT.1.D0) GO TO 5
      Z0=1.D0/Z0
    5 CALL ROOT(Z0,C,EPS)
      IF(C.GT.1.D0) Z0=1.D0/Z0
      WRITE(6,104)Z0
C
C     THE FOLLOWING SEQUENCE COMPUTES THE DISCRETE
C     COLLOCATION POINTS.
C
```

```
      XI(1)=Z0
      PI=3.14159265358973D0
      DO 18 J=2,P
      JJ=J-1
      JP=P-1
      XI(J)=0.5D0+0.5D0*DCOS((2.0D0*JJ-1.0D0)*PI/(2.0D0*JP))
   18 CONTINUE
C     DO 18 J=2,P
C     READ(3,*)XI(J)
C18   CONTINUE
C     DX=1.0D0/(P-2)
C     DO 18 J=2,P
C     K=P-J+2
C  18 XI(K)=(J-2)*DX
      IF(IFLAG.NE.1) GO TO 223
      DO 220 J=1,P
  220 WKAREA(J)=XI(J)
      DO 221 J=2,P
      K=P-J+2
  221 XI(J)=WKAREA(K)
  223 CONTINUE
      DO 19 JJ=1,P
   19 WRITE(6,83) XI(JJ)
   83 FORMAT(20X,F10.5)
      CALL INT12 (XI,A0,B0,P,C)
      IT=0
      SUM=0.0D0
      WRITE(6,111)
      ISIGN=1
 6    CONTINUE
      IS=ISIGN
 7    CONTINUE
```

```
      R=R+DR
      IF(DR.LT.EPS2)STOP
      IT=IT+1
      IF(IT.GT.50)GO TO 50
C     WRITE(6,103) IT,R,SUM
      CALL INT34 (R,XI,P,C,CO,DO)
      RM=R/ZO
      J1=MMBSJ1(RM,IER)
      JO=MMBSJO(RM,IER)
      DO 17 LL=1,P
      CALL MATREL(LL,ZO,R,BMA,DMC)
      JJ=1
      F1=-J1/JO*BMA
      F2=ZO*DMC
      CMZO2(LL,JJ)=F1+F2
      DO 17 M=2,P
      X=XI(M)
      IF(X.EQ.0.0DO) GO TO 15
      RM=R/X
      I1=MMBSI1(ICPT,RM,IER)
      IO=MMBSIO(ICPT,RM,IER)
      GO TO 16
   15 IO=1.0DO
      I1=1.0DO
   16 CONTINUE
      F1=I1/IO*(BO(LL,M)-AO(LL,M))
      F2=X*(DO(LL,M)-CO(LL,M))
      CMZO2(LL,M)=F1+F2
 17   CONTINUE
C     DO 19 LL=1,P
C  19 WRITE(6,87) (CMZO2(LL,JJ), JJ=1,P)
   87 FORMAT(5X,6(D13.6,1X))
```

```
      LM=P-1
      DO 20 I=1,LM
      J=I+1
      OMF(I,1)=-OMZ02(1,J)
 20   CONTINUE
C     WRITE(6,87) (OMF(JJ,1),JJ=1,LM)
      DO 21 LL=1,LM
      I=2
      KL=LL+1
      DO 22 M=1,LM
      AB(LL,M)=OMZ02(I,KL)
      I=I+1
 22   CONTINUE
 21   CONTINUE
C     DO 24 II=1,LM
C  24 WRITE(6,87) (AB(II,JJ),JJ=1,LM)
C     DO 23 I=1,LM
C23   WRITE(6,107)(AB(I,J),J=1,LM)
 107  FORMAT(3X,5(D10.3,1X))
      IDGT=8
      IDIM=11
      CALL LEQT2F (AB,1,LM,IDIM,OMF,IDGT,WKAREA,IER)
C     WRITE(6,87) (OMF(JJ,1),JJ=1,LM)
C     DO 99 LL=1,P
C  99 WRITE(6,87) (OMZ02(LL,JJ), JJ=1,P)
      SUM=0.0D0
      DO 29 J=1,LM
      K=J+1
      SUM=SUM+OMF(J,1)*OMZ02(K,1)
 29   CONTINUE
      SUM=SUM+OMZ02(1,1)
      WRITE(6,103) IT,R,SUM
```

```
      ISIGN=SUM/DABS(SUM)
      IF(IT.EQ.1)IS=ISIGN
      IF(ISIGN.EQ.IS)GO TO 6
      R=R-DR
      DR=DR/2.0D0
      GO TO 7
50    WRITE(6,101)IT,EPS2
      STOP
 100  FORMAT(20X,'C=',F7.3)
 104  FORMAT(//,15X,'DISCRETE EIGENVALUE:',F10.6)
102   FORMAT(2X,4F12.7,2X,I4)
101   FORMAT(2X,' DID NOT CONVERGE AFTER',1X,I3,'ITERATIONS TO PRECISION
     2 ',1X,D10.3)
 103  FORMAT(10X,I3,3X,F10.5,5X,D13.6)
105   FORMAT(2X,'THE SUM IS',D23.16)
 111  FORMAT(10X,'IT',8X,'R',12X,'SUM')
      END
```

```
C
C
C      THIS IS THE GAUSS-LEGENDRE QUADRATURE MAPPING
C      SUBROUTINE. THE NODES AND THE WEIGHTS ARE COMPUTED,A(J) AND B(J),
C      FOR THE INTERVAL (0,1).
C
C
       SUBROUTINE MAP
       IMPLICIT REAL*8(A-H,O-Z)
       COMMON/BLOCK1/A(160),B(160),N
       DIMENSION BU(80),BW(80),SINT(20)
       DIMENSION AA(80),BB(80)
       READ(1,*) N,NSINT,END1,END2
C 101  FORMAT(2I5,F20.10,F20.10)
C      L=(N-1)/2+1
       L=N/2
       K=L+1
       CON=2.0D0
       NSINT=NSINT+1
       READ(1,*)(SINT(I),I=1,NSINT)
       NSINT=NSINT-1
       DO 10 I=1,L
       JJ=L+1-I
    10 READ(1,103) BU(JJ),BW(JJ)
   103  FORMAT(2F22.21)
       DO 20 I=1,L
       J=L+1-I
       AA(J)=-BU(I)
    20 BB(J)=BW(I)
       DO 30 I=K,N
       J=I-L
       AA(I)=BU(J)
```

```
 30 BB(I)=BW(J)
 25 CONTINUE
    DO 40 K=1,NSINT
    GRAD=(SINT(K+1)-SINT(K))/CON
    VV=(SINT(K+1)+SINT(K))/2.0D0
    J=N*(K-1)
    DO 40 I=1,N
    A(I+J)=GRAD*AA(I)+VV
    B(I+J)=GRAD*BB(I)
 40 CONTINUE
    N=N*NSINT
    WRITE(6,104) N
    DO 55 I=1,5
    SUM=0.0D0
    DO 50 J=1,N
 50 SUM=SUM+A(J)**I*B(J)
    F=(I+1.0D0)*SUM-1.0D0
 55 WRITE(6,105) I,F
105 FORMAT(15X,I2,3X,D12.4)
104 FORMAT('0','TOTAL QUAD PTS =',I4)
    RETURN
    END
```

```
C
C       THIS SUBROUTINE WILL USE NEWTONS METHOD
C       TO REFINE ESTIMATE ZO FOR ROOT
C       OF THE ONE-SPEED NEUTRON DISPERSION FUNCTION ZL
C       TO PRECISION EPS.THE VALUE RETURNED WILL BE THE
C       ABSOLUTE VALUE OF THE IMAGINERY ROOT ZO.
C
C
        SUBROUTINE ROOT(ZO,C,EPS)
        IMPLICIT REAL*8(A-H,C-Z)
        MAX=50
        IT=0
      1 Z=ZO
        IT=IT+1
        IF(IT.GT.MAX) GO TO 10
        IF(C.GT.1.D0) GO TO 5
        Q=(1.D0+1.D0/Z)/(1.D0-1.D0/Z)
        ZL=1.D0-C*Z*DLOG(Q)/2.D0
        ZLP=(ZL-1.D0)/Z+C*Z/(Z*Z-1.D0)
        GO TO 7
      5 S=Z
        ZL=1.D0-C*DATAN(S)/S
        ZLP=C*DATAN(S)/S/S-C/S/(1.D0+S*S)
      7 ZO=Z-ZL/ZLP
C       WRITE(6,101) IT,ZL,ZLP,ZO
        IF(DABS(Z-ZO).GT.EPS) GO TO 1
C       WRITE(6,102)ZL,ZLP
 102    FORMAT(1X,2D23.16)
        RETURN
     10 WRITE(6,100) EPS,IT
        RETURN
    100 FORMAT(//,20X,'CONVERGENCE OF EIGENVALUE TO PRECISION',
```

```
     21X,D9.2,1X,'NOT ACHIEVED IN',I3,'ITERATIONS')
C 101 FORMAT(10X,I3,3X,3(D10.3,2X))
      END
```

```
C
C      THIS SUBROUTINE WILL FIND AO AND
C      BO  AND PUT THEM IN MATRIX FORM.
C
C
       SUBROUTINE  INT12(XI,AO,BO,P,C)
       IMPLICIT REAL*8(A-H,O-Z)
       COMMON/BLOCK1/A(160),B(160),N
       INTEGER P
       DIMENSION XI(20),AO(11,11),BO(11,11)
       DO 27 M=1,P
       X=XI(M)
       IF(X.GT.0.0D0) GO TO 9
       AO(1,M)=1.0D0
       BO(1,M)=2.0D0/C-1.0D0
       GO TO 27
 9     AO(1,M)=1.0D0-X*DLOG(1.0D0+(1.0D0/X))
       BO(1,M)=(2.0D0/C)-1.0D0-X*DLOG(1.0D0+(1.0D0/X))
 27    CONTINUE
       IF(P.EQ.1)GO TO 39
       DO 31 LL=2,P
       DO 32 M=1,P
       X=XI(M)
       IF(X.GT.0.0D0) GO TO 30
       AO(LL,M)=1.0D0/LL
       BO(LL,M)=-1.0D0/LL
       GO TO 32
   30  KK=LL-1
       AO(LL,M)=-XI(M)*AO(KK,M)+(1.0D0/LL)
       IF(M.EQ.1)GO TO 32
       BO(LL,M)=XI(M)*BO(KK,M)-(1.0D0/LL)
 32    CONTINUE
```

```
31    CONTINUE
39    CONTINUE
      RETURN
      END
```

```
C      THIS SUBROUTINE WILL CALCULATE CO AND DO AND
C      PUT THEM IN MATRIX FORM.
C      THE CO INTEGRAL IS BEING CALCULATED WITH THE NOTATION RSUM.
C      THE DO INTEGRAL IS BEING EVALUATED IN THREE PARTS.
C      KRALPH IS BEING REPRESENTED BY SUM.
C      SSUM IS CALULATING THE SINGULAR INTEGRAL IF EPSILON
C      IS NOT EQUAL TO MU.
C      IF MU IS EQUAL TO EPSILON PRIME IS THE VALUE OF
C      THE INTEGRAL.
C      BECAUSE OF THE SINGULARITY AT EPSILON EQUAL TO ZERO,
C      A SEPARATE LOOP MUST BE ADDED FOR THAT CASE.
C
C
       SUBROUTINE INT34(R,XI,P,C,CO,DO)
       IMPLICIT REAL*8(A-H,O-Z)
       REAL*8 NU,MU,MMBSIO,MMBSI1,MMBSKO,MMBSK1
      2,KO,K1,IO,I1,MMBSJO,MMBSJ1,JO,J1,KOE,K1E
       EXTERNAL MMBSJO,MMBSJ1,MMBSKO,MMBSK1,MMBSIO,MMBSI1
       COMMON/BLOCK1/A(160),B(160),N
       INTEGER P
       DIMENSION DO(11,11),CO(11,11)
       DIMENSION RKRP(11),XI(20)
       DO 29 LL=1,P
       DO 27 M=1,P
       SUM=0.0D0
       SZO=0.0D0
       SSUM=0.0D0
       RSUM=0.0D0
       DO 34 J=1,N
       I=1
       RM=R/A(J)
       TEMP=XI(M)
```

```
      IF(XI(M).LE..001D0)XI(M)=0.01D0
      RN=R/XI(M)
      IF(RN.GT.177.0D0)GO TO 23
      K1E=MMBSK1(I,RN,IER)
      K0E=MMBSK0(I,RN,IER)
      GO TO 33
23    K0E=1.0D0
      K1E=1.0D0
33    IF(RM.GT.177.0D0)GO TO 24
      K0=MMBSK0(I,RM,IER)
      K1=MMBSK1(I,RM,IER)
      GO TO 35
24    K0=1.0D0
      K1=1.0D0
35    XI(M)=TEMP
      KK=LL-1
      SUM=SUM+((K1/K0)*(A(J)**KK))*B(J)
      SZ0=SZ0+K1/K0*A(J)**KK/(XI(1)-A(J))*B(J)
   9  RSUM=RSUM+(K1/K0)*A(J)**KK*1.0D0/(XI(M)+A(J))*B(J)
      IF(LL.GT.1)GO TO 34
      SSUM=SSUM+(((K1/K0)-(K1E/K0E))/(A(J)-XI(M)))*(B(J))
   34 CONTINUE
      IF(M.EQ.1)RKRP(LL)=SUM
      C0(LL,M)=RSUM
      IF(XI(M).NE.0.0D0)GO TO 8
      C0(LL,2)=-C0(LL,M)
      GO TO 27
   8  TERM1=(2.D0/(C*XI(M))-(DLOG(1.D0+(1.D0/XI(M)))))*K1E/K0E
      RFPRM=-(K0E*K0E)-(K0E*K1E)*(XI(M)/R)+(K1E*K1E)
C     RRR=K0E*K0E
C     PRIME=-(RFPRM/RRR)
      D0(LL,1)=SZ0
```

```
      IF(M.GE.2)DO(1,M)=TERM1-SSUM
27    CONTINUE
29    CONTINUE
      IF(P.EQ.1)GO TO 44
      DO 15 LL=2,P
      KK=LL-1
      DO 16 M=2,P
      IF(XI(M).EQ.0.0D0)GO TO 16
      DO(LL,M)=XI(M)*DO(KK,M)-RKRP(KK)
16    CONTINUE
15    CONTINUE
44    CONTINUE
      RETURN
      END
```

```
C      THIS SUBROUTINE SUBTRACTS THE QUANTIES BO AND AO
C      AND DO AND CO FOR THE CASE EPSILON EQUALS THE IMACINERY
C      ROOT ZO.THE DIFFERENCE IN THESE TWO QUANTIES ARE REAL.
C      BMA MEANS BO MINUS AO AND DMC STANDS FOR DO MINUS CO.
C
C
       SUBROUTINE MATREL(IJ,ZO,R,BMA,DMC)
       IMPLICIT REAL*8(A-H,O-Z)
       REAL*8 MMBSKO,MMBSK1,KO,K1,MU
       COMMON/BLOCK1/A(160),B(160),N
       II=1
       XMAX=0.177D 03
       SUM1=0.0D0
       SUM2=0.0D0
       DO 10 I=1,N
       MU=A(I)
       RMU=R/MU
       IF(RMU.GT.XMAX) GO TO 7
       K1=MMBSK1(II,RMU,IER)
       KO=MMBSKO(II,RMU,IER)
       F=K1/KO
       GO TO 8
     7 F=1.0D0
     8 CONTINUE
       SUM1=SUM1+MU**IJ*F*B(I)/(MU**2+ZO**2)
    10 SUM2=SUM2+MU**(IJ+1)*B(I)/(MU**2+ZO**2)
       BMA=-2*SUM2
       DMC=-2*SUM1
       RETURN
       END
```

APPENDIX 2

Example of Output from

TRAN FORTRAN

```
     2.0000000   0.6000000   0.0500000   0.0000100      9
OTOTAL QUAD PTS =  40
                1    -0.3886D-15
                2    -0.3886D-15
                3    -0.4996D-15
                4    -0.4996D-15
                5    -0.4718D-15
                    C=  2.000


              DISCRETE EIGENVALUE:  0.428978
                       0.42898
                       0.88889
                       0.77778
                       0.66667
                       0.55556
                       0.44444
                       0.33333
                       0.22222
                       0.11111
          IT          R                SUM
            1       0.65000       -0.515121D-01
            2       0.70000        0.139968D+00
            3       0.67500        0.396526D-01
            4       0.66250       -0.693673D-02
            5       0.67500        0.396526D-01
            6       0.66875        0.160905D-01
            7       0.66562        0.451208D-02
            8       0.66406       -0.122828D-02
            9       0.66562        0.451208D-02
           10       0.66484        0.163788D-02
           11       0.66445        0.203798D-03
```

12	0.66426	-0.512491D-03
13	0.66445	0.203798D-03
14	0.66436	-0.154409D-03
15	0.66445	0.203798D-03
16	0.66440	0.246789D-04
17	0.66438	-0.648690D-04
18	0.66440	0.246789D-04
19	0.66439	-0.200960D-04
20	0.66440	0.246789D-04

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THE F_N METHOD FOR A BARE CRITICAL CYLINDER

by

Jack Daniel Southers

(Abstract)

The F_N method, originated by C. E. Siewert, is developed for a bare, axially infinite critical cylinder. The full-range completeness and orthogonality properties of the singular eigenfunctions are used to derive an expression for the emerging angular flux, which is represented by a power series. The resulting equations are reduced to matrix form and computer solved.

Examples of the results of this method for different parameters are presented. Comparisons with other models are made. A fourth order approximation was found to be sufficient to achieve up to four digit agreement with benchmark values.