# Chapter 6: Dynamic Simulation Environment

Previous kinematic and dynamic analysis has demonstrated the singularity-free workspace and the high force-bearing characteristics of the Wrist. This chapter completes the dynamic model of the Carpal Wrist by presenting a closed-form solution and algorithm to the time response (forward dynamic) problem. With this model, simulation of the Wrist in an application environment can be performed to generate various control-system algorithms. An example of such an application would be to create a neural network control device trained via the forward dynamic model. This forward dynamic model will be developed from the equations of motion which have been derived in closed form by direct application of Lagrange's equations to the kinematic model. The dynamic model assumes a relatively massive tool and considers all gravitational, inertial, Coriolis and gyroscopic effects. The forward dynamics presents the equations of motion in a form that is evaluated over time to determine trajectory information based on actuator inputs. The results of these equations is a simulation environment that can represent the Wrist in application for modeling, design, and control purposes.

## 6.1 Introduction

This chapter presents a solution to the forward dynamic analysis for the Carpal Wrist. This time-response or forward dynamic analysis is based on the equations of motion developed using Lagrange's equations and the Wrist kinematics. The results of this analysis will be a procedure that includes the forward dynamic equations and solves for the joint space trajectory given the motor-torque time history. The goal of this forward dynamic model is to provide a simulation environment that can be used for several purposes including developing a high-level Wrist control. For example, the simulation environment can be used to train a neural network control system, based on various operating criteria. Another example demonstrated will be to use the simulation environment as part of an optimization routine solving trajectory synthesis problems.

This chapter evolves the dynamics of the Carpal Wrist into a form that can solve for the joint accelerations explicitly. The starting point are the current equations of motion describing the inverse dynamics (solving required motor inputs explicitly) which were derived using a Lagrangian approach, with the generalized coordinate system chosen as the input joint angles and the kinematics developed in canonical form. The Lagrangian, expressed as a function of the generalized coordinates, depends on the manipulator energy state, i.e., the potential and kinetic energy. The Lagrangian was calculated from position and velocity information, which are available in closed form.. Thus, the Lagrangian may also be expressed in closed form, as shown in Chap. 5. The Lagrangian formulation was used because of its ability to remove internal constraint forces from the equations of motion and because it allowed the choice of generalized coordinates, in this case the joint-space coordinate system. In developing the initial equations of motion, tensor subscript notation was used and will continue to be used throughout, with the standard Einstein summation convention assumed (Frederick and Chang, 1965).

### 6.2 Wrist Dynamics

The Wrist dynamics are based on the kinematic model developed in Chap. 3, which provides a mapping between the input space joint angles (joint space) and the output space tool pose (tool space). This mapping is represented in Eq. 6.1:

$$(\alpha, \beta, p) = f_1(\theta_1, \theta_2, \theta_3), \qquad (\theta_1, \theta_2, \theta_3) = f_1(\alpha, \beta, p) \tag{6.1}$$

where:
$\alpha$, $\beta$, and $p$ represent the output coordinates,
$\theta_1$-$\theta_3$ represent the input space coordinates, and the functions,
$f_1$, $f_2$ represent the forward and inverse kinematic mappings, respectively.

#### 6.2.1 Angular Velocity of the Distal Frame

Recall the rotation of the distal plate (and the rigidly attached tool, Fig. 6.1) is caused by the time rate of change of the midplane normal, $\hat{\mathbf{N}}$ or correspondingly the change of the orientation angles α and β (Chap. 5). The vector ω was defined as the angular velocity of the {**D**} frame relative to the {**B**} frame described in {**B**} frame coordinates and given as:

$$\omega_i = \begin{Bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{Bmatrix} + \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\beta} \\ 0 \end{Bmatrix} = \begin{Bmatrix} -s\alpha\dot{\beta} \\ c\alpha\dot{\beta} \\ \dot{\alpha} \end{Bmatrix} \tag{6.2}$$

#### 6.2.2 Translational Velocity of the Tool Center of Mass

Referring to the frame assignment from Chap. 5, Fig. 5.2, and the mounted tool shown in Fig. 6.1, **v** is the velocity of the tool center of mass, **G** with respect to frame {**B**} expressed in frame

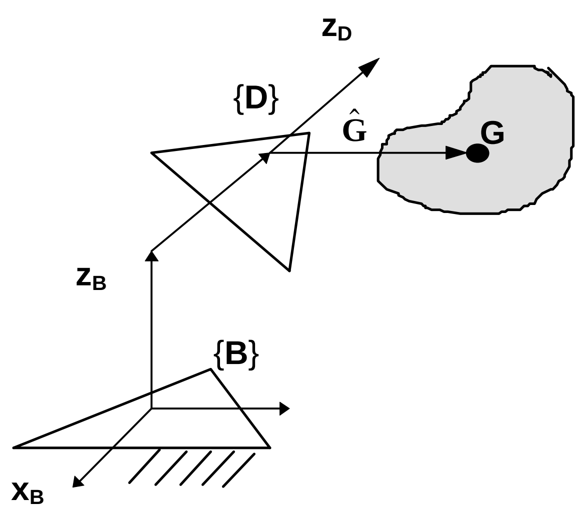


**Figure 6.1: Wrist Mounted Tool**

{**B**} coordinates,

$$v_j = \left(\delta_j + R_{j3}\right) J_{3k} \dot{\theta}_k + \in_{jkl} \omega_k R_{l3} p_d + \in_{jkl} \omega_k R_{lm} {}^D G_m \tag{6.3}$$

where $\varepsilon$ is the permutation tensor and $\delta$ is the Kronecker delta.

## 6.3 Lagrange's Equations for the Wrist-Isolated Problem

Lagrange's equations are again expressed for the parallel structure Carpal Robotic Wrist, as in Chap. 5. In this chapter, the equations will be expanded to solve explicitly for joint space accelerations. Starting with Lagranges equations:

$$\frac{d}{dt}\left( \partial \mathcal{L} \big/ \partial \dot{q}_i \right) + \partial \mathcal{L} \big/ \partial q_i = Q_i \tag{6.4}$$

where $q_i$ represents the generalized coordinates and $Q_i$ the generalized forces, the three input joint parameters are chosen as the generalized coordinates. The generalized forces associated with this choice of coordinates are the input actuator torques, $M_i$. Lagrange's equations become:

$$\frac{d}{dt}\left( \partial \mathcal{L} \big/ \partial \dot{\theta}_i \right) - \partial \mathcal{L} \big/ \partial \theta_i = M_i \tag{6.5}$$

with:

$$\mathcal{L} = T - V \,. \tag{6.6}$$

The kinetic and potential energies are given by:

$$\begin{aligned} T &= \tfrac{1}{2} m v_i v_i + \tfrac{1}{2} \omega_i R_{ij} I_{jk} \omega_k \\ V &= mg\left[ p_d + R_{33} p_d + R_{3j} G_j \right] \end{aligned} \tag{6.7}$$

where:

$I_{jk}$ is the moment of inertia tensor of the tool expressed in the {**D**} frame, and

**G** is the vector locating the center of mass of the tool and distal plate with respect to the center of the distal plate expressed in the {**D**} frame (Fig. 6.1).

### 6.3.1 Derivatives of Kinetic and Potential Energy

The Lagrangian must be differentiated with respect to the inputs to the system, $\theta_i$, the time rate of change of these inputs, $\dot{\theta}_i$, and time. The necessary derivatives of both the potential and kinetic energy are:

$$\begin{aligned} \partial T \big/ \partial \theta_l &= m v_i \, {}^{\partial v_i}\!\big/_{\partial \theta_l} + {}^{\partial \omega_i}\!\big/_{\partial \theta_l} R_{ij} I_{jk} \omega_k + \tfrac{1}{2} \omega_i \, {}^{\partial R_{ij}}\!\big/_{\partial \theta_l} I_{jk} \omega_k \\ \partial T \big/ \partial \dot{\theta}_l &= m v_i \, {}^{\partial v_i}\!\big/_{\partial \dot{\theta}_l} + {}^{\partial \omega_i}\!\big/_{\partial \dot{\theta}_l} R_{ij} I_{jk} \omega_k \\ \frac{d}{dt}\left( \partial T \big/ \partial \dot{\theta}_l \right) &= \frac{\partial^2 T}{\partial \theta_m \partial \dot{\theta}_l} \dot{\theta}_m + \frac{\partial^2 T}{\partial \dot{\theta}_m \partial \dot{\theta}_l} \ddot{\theta}_m \end{aligned} \tag{6.8}$$

$$\frac{d}{dt}\left(\partial T/\partial\dot{\theta}_l\right)=\sum_m m\left[v_i\left(\frac{\partial^2 v_i}{\partial\theta_m\partial\dot{\theta}_l}\dot{\theta}_m+\frac{\partial^2 v_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}\ddot{\theta}_m\right)+\partial v_i/\partial\dot{\theta}_l\left(\partial v_i/\partial\theta_m\,\dot{\theta}_m+\partial v_i/\partial\dot{\theta}_m\,\ddot{\theta}_m\right)\right]+\left[\frac{\partial^2\omega_i}{\partial\theta_m\partial\dot{\theta}_l}\dot{\theta}_m+\frac{\partial^2\omega_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}\ddot{\theta}_m\right]\left(R_{ij}I_{jk}\omega_k\right)$$
$$+\partial\omega_i/\partial\dot{\theta}_l\left(\partial R_{ij}/\partial\theta_m\,\dot{\theta}_m I_{jk}\omega_k+R_{ij}I_{jk}\left(\partial\omega_k/\partial\theta_m\,\dot{\theta}_m+\partial\omega_k/\partial\dot{\theta}_m\,\ddot{\theta}_m\right)\right)\tag{6.9}$$

$$\partial V/\partial\theta_l=mg\left[\partial p/\partial\theta_l\left(1+R_{33}\right)+p\,\partial R_{33}/\partial\theta_l+\partial R_{3j}/\partial\theta_l\,G_j\right].\tag{6.10}$$

Putting these into Lagrange's equation, Eq. 6, gives the equations of motion::

$$M_l=\sum_m m\left[v_i\left(\frac{\partial^2 v_i}{\partial\theta_m\partial\dot{\theta}_l}\dot{\theta}_m+\frac{\partial^2 v_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}\ddot{\theta}_m\right)+\partial v_i/\partial\dot{\theta}_l\left(\partial v_i/\partial\theta_m\,\dot{\theta}_m+\partial v_i/\partial\dot{\theta}_m\,\ddot{\theta}_m\right)\right]+\left[\frac{\partial^2\omega_i}{\partial\theta_m\partial\dot{\theta}_l}\dot{\theta}_m+\frac{\partial^2\omega_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}\ddot{\theta}_m\right]\left(R_{ij}I_{jk}\omega_k\right)$$
$$+\partial\omega_i/\partial\dot{\theta}_l\left(\partial R_{ij}/\partial\theta_m\,\dot{\theta}_m I_{jk}\omega_k+R_{ij}I_{jk}\left(\partial\omega_k/\partial\theta_m\,\dot{\theta}_m+\partial\omega_k/\partial\dot{\theta}_m\,\ddot{\theta}_m\right)\right)$$
$$-mv_i\,\partial v_i/\partial\theta_l+\partial\omega_i/\partial\theta_l\,R_{ij}I_{jk}\omega_k+\tfrac{1}{2}\omega_i\,\partial R_{ij}/\partial\theta_l\,I_{jk}\omega_k$$
$$+mg\left[\partial p/\partial\theta_l\left(1+R_{33}\right)+p\,\partial R_{33}/\partial\theta_l+\partial R_{3j}/\partial\theta_l\,G_j\right]\tag{6.11}$$

Equation 6.11 is expanded to isolate the acceleration components, as demonstrated in the following equations:

$$M_l=\sum_m m\left[\left(v_i\frac{\partial^2 v_i}{\partial\theta_m\partial\dot{\theta}_l}+\partial v_i/\partial\theta_m\,\partial v_i/\partial\dot{\theta}_l\right)\dot{\theta}_m+\left(v_i\frac{\partial^2 v_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}+\partial v_i/\partial\dot{\theta}_l\,\partial v_i/\partial\dot{\theta}_m\right)\ddot{\theta}_m\right]+\left(\frac{\partial^2\omega_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}R_{ij}I_{jk}\omega_k+\partial\omega_i/\partial\dot{\theta}_l\,R_{ij}I_{jk}\,\partial\omega_k/\partial\dot{\theta}_m\right)\ddot{\theta}_m$$
$$\left(\frac{\partial^2\omega_i}{\partial\theta_m\partial\dot{\theta}_l}\dot{\theta}_m R_{ij}I_{jk}\omega_k\right)+\partial\omega_i/\partial\dot{\theta}_l\left(\partial R_{ij}/\partial\theta_m\,\dot{\theta}_m I_{jk}\omega_k+R_{ij}I_{jk}\,\partial\omega_k/\partial\theta_m\,\dot{\theta}_m\right)-mv_i\,\partial v_i/\partial\theta_l+\partial\omega_i/\partial\theta_l\,R_{ij}I_{jk}\omega_k+\tfrac{1}{2}\omega_i\,\partial R_{ij}/\partial\theta_l\,I_{jk}\omega_k$$
$$+mg\left[\partial p/\partial\theta_l\left(1+R_{33}\right)+p\,\partial R_{33}/\partial\theta_l+\partial R_{3j}/\partial\theta_l\,G_j\right]\tag{6.12}$$

$$M_l=\sum_m\left[m\left(v_i\frac{\partial^2 v_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}+\partial v_i/\partial\dot{\theta}_l\,\partial v_i/\partial\dot{\theta}_m\right)+\frac{\partial^2\omega_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}R_{ij}I_{jk}\omega_k+\partial\omega_i/\partial\dot{\theta}_l\,R_{ij}I_{jk}\,\partial\omega_k/\partial\dot{\theta}_m\right]\ddot{\theta}_m$$
$$+m\left(v_i\frac{\partial^2 v_i}{\partial\theta_m\partial\dot{\theta}_l}+\partial v_i/\partial\theta_m\,\partial v_i/\partial\dot{\theta}_l\right)\dot{\theta}_m+\left(\frac{\partial^2\omega_i}{\partial\theta_m\partial\dot{\theta}_l}\dot{\theta}_m R_{ij}I_{jk}\omega_k\right)+\partial\omega_i/\partial\dot{\theta}_l\left(\partial R_{ij}/\partial\theta_m\,\dot{\theta}_m I_{jk}\omega_k+R_{ij}I_{jk}\,\partial\omega_k/\partial\theta_m\,\dot{\theta}_m\right)$$
$$-mv_i\,\partial v_i/\partial\theta_l+\partial\omega_i/\partial\theta_l\,R_{ij}I_{jk}\omega_k+\tfrac{1}{2}\omega_i\,\partial R_{ij}/\partial\theta_l\,I_{jk}\omega_k+mg\left[\partial p/\partial\theta_l\left(1+R_{33}\right)+p\,\partial R_{33}/\partial\theta_l+\partial R_{3j}/\partial\theta_l\,G_j\right]\tag{6.13}$$

$$M_l=\sum_m\{\left[m\left(v_i\frac{\partial^2 v_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}+\partial v_i/\partial\dot{\theta}_l\,\partial v_i/\partial\dot{\theta}_m\right)+\frac{\partial^2\omega_i}{\partial\dot{\theta}_m\partial\dot{\theta}_l}R_{ij}I_{jk}\omega_k+\partial\omega_i/\partial\dot{\theta}_l\,R_{ij}I_{jk}\,\partial\omega_k/\partial\dot{\theta}_m\right]\ddot{\theta}_m$$
$$+\left[\begin{array}{c}m\left(v_i\frac{\partial^2 v_i}{\partial\theta_m\partial\dot{\theta}_l}+\partial v_i/\partial\theta_m\,\partial v_i/\partial\dot{\theta}_l\right)\dot{\theta}_m+\left(\frac{\partial^2\omega_i}{\partial\theta_m\partial\dot{\theta}_l}\dot{\theta}_m R_{ij}I_{jk}\omega_k\right)+\partial\omega_i/\partial\dot{\theta}_l\left(\partial R_{ij}/\partial\theta_m\,\dot{\theta}_m I_{jk}\omega_k+R_{ij}I_{jk}\,\partial\omega_k/\partial\theta_m\,\dot{\theta}_m\right)\\-mv_i\,\partial v_i/\partial\theta_l+\partial\omega_i/\partial\theta_l\,R_{ij}I_{jk}\omega_k+\tfrac{1}{2}\omega_i\,\partial R_{ij}/\partial\theta_l\,I_{jk}\omega_k\end{array}\right]$$
$$+mg\left[\partial p/\partial\theta_l\left(1+R_{33}\right)+p_d\,\partial R_{33}/\partial\theta_l+\partial R_{3j}/\partial\theta_l\,G_j\right]\}\tag{6.14}$$

$$\begin{aligned} M_l = \sum_m \Big\{ & \Big[ m\left( {}^{\partial v_i}\!/\!_{\partial \dot{\theta}_l} \, {}^{\partial v_i}\!/\!_{\partial \dot{\theta}_m} \right) + {}^{\partial \omega_i}\!/\!_{\partial \dot{\theta}_l} \, R_{ij} I_{jk} \, {}^{\partial \omega_k}\!/\!_{\partial \dot{\theta}_m} \Big] \ddot{\theta}_m + m\left( v_i \tfrac{\partial^2 v_i}{\partial \theta_m \partial \dot{\theta}_l} + {}^{\partial v_i}\!/\!_{\partial \theta_m} \, {}^{\partial v_i}\!/\!_{\partial \dot{\theta}_l} \right) \dot{\theta}_m + \left( \tfrac{\partial^2 \omega_i}{\partial \theta_m \partial \dot{\theta}_l} \dot{\theta}_m R_{ij} I_{jk} \omega_k \right) \\ & + {}^{\partial \omega_i}\!/\!_{\partial \dot{\theta}_l} \left( {}^{\partial R_{ij}}\!/\!_{\partial \theta_m} \, \dot{\theta}_m I_{jk} \omega_k + R_{ij} I_{jk} \, {}^{\partial \omega_k}\!/\!_{\partial \theta_m} \, \dot{\theta}_m \right) - m v_i \, {}^{\partial v_i}\!/\!_{\partial \theta_l} + {}^{\partial \omega_i}\!/\!_{\partial \theta_l} \, R_{ij} I_{jk} \omega_k + {}^{1}\!/\!_{2} \, \omega_i \, {}^{\partial R_{ij}}\!/\!_{\partial \theta_l} \, I_{jk} \omega_k \\ & + mg\Big[ {}^{\partial p}\!/\!_{\partial \theta_l} \left(1 + R_{33}\right) + p_d \, {}^{\partial R_{33}}\!/\!_{\partial \theta_l} + {}^{\partial R_{3j}}\!/\!_{\partial \theta_l} \, G_j \Big] \Big\} \end{aligned} \tag{6.15}$$

Equation 6.15 gives the equations of motion in a form that isolates the acceleration of the joint angles, with coefficients that are functions of joint position only. This result is rewritten as:

$$M_l = \sum_m \left[ A(\theta)\ddot{\theta}_m + B\left(\theta, \dot{\theta}\right) + C(\theta) \right] \tag{6.16}$$

where:
$M_l$ is the $l^{th}$ input motor moment,
**A**($\theta$) represents the inertial components of the wrist and tool, a function of *l* and summed over *m*,
$\mathbf{B}\left(\theta, \dot{\theta}\right)$ represents the Coriolis and centrifugal inertia terms, and
**C**($\theta$) represents the gravitational terms, both functions of *l* and summed over *m*.

Equation 6.16 presents the equations of motion in a form which demonstrates the effect of each motor moment, *l*, on the manipulator's motion (inverse dynamics). When the manipulator path is known, the required moment or torque for each motor can be determined. Also, the bending stresses that occur in the Wrist leg members are given by these equations. Note that this is for the bending stress occurring in the plane defined for each leg by the basal and distal revolute axes, generated from the actuators in overcoming static and dynamic inertial loading. Bending stress in the plane of the Wrist leg members is generated from moment reactions at the location of the basal and distal revolute axes. Detailed analysis of these out-of-plane bending loads as well as shear and axial loads in the Wrist members has been presented by Canfield et al., 1995, for the parallel manipulator architecture and by Ganino, 1996, specifically for the Carpal Wrist. Thus, Eq. 6.16 is important both in sizing the actuators, providing voltage control to the motors during operation, and in generating dynamic stress information for mechanical design of the wrist components.

The focus of this chapter is to create a simulation model of the parallel architecture Wrist. The simulation model consists of the equations of motion cast in the form of a solution for the time response problem, i.e., a form that can be solved explicitly for the joint accelerations. These equations, called here the forward dynamic equations, will give the joint accelerations as a function of all input moments, as well as gyroscopic, Coriolis, and gravitational effects. Therefore, the equations given in 6.16 need to be solved for the joint acceleration. The solution proceeds as follows. First, the acceleration components are isolated as:

$$M_l - \sum_m \left[ B_l\left(\theta, \dot{\theta}\right) + C_l(\theta) \right] = \sum_m \left[ A_l(\theta)\ddot{\theta}_m \right]. \tag{6.17}$$

Expanding the independent acceleration terms:

$$M_l - \sum_m \left[ B_{l,m}\left(\theta, \dot{\theta}\right) + C_{l,m}(\theta) \right] = A_{l,m}(\theta)\Big|_{m=1} \ddot{\theta}_1 + A_{l,m}(\theta)\Big|_{m=2} \ddot{\theta}_2 + A_{l,m}(\theta)\Big|_{m=3} \ddot{\theta}_3 . \tag{6.18}$$

Equation 6.18 gives the $l^{\text{th}}$ equation of motion, the input of one motor. To solve uniquely for all the joint accelerations, $\ddot{\theta}_i$, *i=1-3*, the effects of all the inputs are given, *l = 1-3*:

$$\begin{aligned}
M_1 - \sum_m \left[B_{l,m}(\theta,\dot{\theta}) + C_{l,m}(\theta)\right]\Big|_{l=1} &= A_{l,m}(\theta)\Big|_{m=1,l=1}\ddot{\theta}_1 + A_{l,m}(\theta)\Big|_{m=2,l=1}\ddot{\theta}_2 + A_{l,m}(\theta)\Big|_{m=3,l=1}\ddot{\theta}_3 \\
M_2 - \sum_m \left[B_{l,m}(\theta,\dot{\theta}) + C_{l,m}(\theta)\right]\Big|_{l=2} &= A_{l,m}(\theta)\Big|_{m=1,l=2}\ddot{\theta}_1 + A_{l,m}(\theta)\Big|_{m=2,l=2}\ddot{\theta}_2 + A_{l,m}(\theta)\Big|_{m=3,l=2}\ddot{\theta}_3 \cdot \\
M_3 - \sum_m \left[B_{l,m}(\theta,\dot{\theta}) + C_{l,m}(\theta)\right]\Big|_{l=3} &= A_{l,m}(\theta)\Big|_{m=1,l=3}\ddot{\theta}_1 + A_{l,m}(\theta)\Big|_{m=2,l=3}\ddot{\theta}_2 + A_{l,m}(\theta)\Big|_{m=3,l=3}\ddot{\theta}_3
\end{aligned} \tag{6.19}$$

Writing in matrix form,

$$\mathbf{A}\ddot{\theta} = \mathbf{M} \tag{6.20}$$

where:

$$\mathbf{A} = \begin{bmatrix} A_{l,m}(\theta)\Big|_{m=1,l=1} & A_{l,m}(\theta)\Big|_{m=2,l=1} & A_{l,m}(\theta)\Big|_{m=3,l=1} \\ A_{l,m}(\theta)\Big|_{m=1,l=2} & A_{l,m}(\theta)\Big|_{m=2,l=2} & A_{l,m}(\theta)\Big|_{m=3,l=2} \\ A_{l,m}(\theta)\Big|_{m=1,l=3} & A_{l,m}(\theta)\Big|_{m=2,l=3} & A_{l,m}(\theta)\Big|_{m=3,l=3} \end{bmatrix}$$

and,

$$\mathrm{M} = \begin{Bmatrix} M_1 - \sum_m \left[B_{l,m}(\theta,\dot{\theta}) + C_{l,m}(\theta)\right]\Big|_{l=1} \\ M_2 - \sum_m \left[B_{l,m}(\theta,\dot{\theta}) + C_{l,m}(\theta)\right]\Big|_{l=2} \\ M_3 - \sum_m \left[B_{l,m}(\theta,\dot{\theta}) + C_{l,m}(\theta)\right]\Big|_{l=3} \end{Bmatrix}$$

Finally, the joint accelerations are solved:

$$\ddot{\theta} = \mathbf{A}^{-1}\mathbf{M}\,. \tag{6.21}$$

### 6.3.1.1 Expanding the Equations of Motion

Derivatives of the angular and translational velocities are required in the equations of motion. Following the convention applied in taking derivatives of the energy functions, partial derivatives are taken with respect to the generalized coordinates or joint parameters, $\theta_i$ and $\dot{\theta}_i$. For the velocity vector, **v**, the derivatives are given in Eqs. 6.22-6.26.

$$v_i = (\delta_i + R_{i3}) J_{3k}\dot{\theta}_k + \in_{ijk} \left(\omega_j R_{k3} p_d + \omega_j R_{kn}\, {}^D G_n\right) \tag{6.22}$$

$$\partial v_i \big/ \theta_l = (\delta_i + R_{i3})\, {}^{\partial J_{3k}}\!/\!{}_{\partial\theta_l}\, \dot{\theta}_k + {}^{\partial R_{l3}}\!/\!{}_{\partial\theta_l}\, J_{3k}\dot{\theta}_k + \in_{ijk} \begin{pmatrix} \omega_j R_{k3}\, {}^{\partial p}\!/\!{}_{\partial\theta_l} + \omega_j\, {}^{\partial R_{k3}}\!/\!{}_{\partial\theta_l}\, p_d + {}^{\partial\omega_j}\!/\!{}_{\partial\theta_l}\, R_{k3} p_d \\ + \omega_j\, {}^{\partial R_{kn}}\!/\!{}_{\partial\theta_l}\, {}^D G_n + {}^{\partial\omega_j}\!/\!{}_{\partial\theta_l}\, R_{kn}\, {}^D G_n \end{pmatrix} \tag{6.23}$$

$$\partial v_i \big/ \dot{\theta}_l = (\delta_i + R_{i3}) J_{3m} + \in_{ijk} \left({}^{\partial\omega_j}\!/\!{}_{\partial\dot{\theta}_l}\, R_{k3} p_d + {}^{\partial\omega_j}\!/\!{}_{\partial\dot{\theta}_l}\, R_{kn}\, {}^D G_n\right) \tag{6.24}$$

$$\frac{\partial^2 v_i}{\theta_m \dot{\theta}_l} = \left(\delta_i + R_{i3}\right) {}^{\partial J_{3m}}\!/\!_{\partial\theta_m} + {}^{\partial R_{i3}}\!/\!_{\partial\theta_m} J_{3m} + \epsilon_{ijk} \begin{pmatrix} {}^{\partial\omega_j}\!/\!_{\partial\dot{\theta}_l} R_{k3} \, {}^{\partial p}\!/\!_{\partial\theta_m} + {}^{\partial\omega_j}\!/\!_{\partial\dot{\theta}_l} \, {}^{\partial R_{k3}}\!/\!_{\partial\theta_m} p_d + \frac{\partial^2\omega_j}{\partial\theta_m\partial\dot{\theta}_l} R_{k3} p \\ + {}^{\partial\omega_j}\!/\!_{\partial\dot{\theta}_l} \, {}^{\partial R_{kn}}\!/\!_{\partial\theta_m} \, {}^{D}G_n + \frac{\partial^2\omega_j}{\partial\theta_m\partial\dot{\theta}_l} R_{kn} \, {}^{D}G_n \end{pmatrix} \quad (6.25)$$

$$\frac{\partial^2 v_i}{\dot{\theta}_m \dot{\theta}_l} = 0 + \epsilon_{ijk} \left( \frac{\partial^2\omega_j}{\partial\dot{\theta}_m\partial\dot{\theta}_l} \left( R_{k3} p_d + R_{kn} \, {}^{D}G_n \right) \right) = 0 \,. \quad (6.26)$$

Partial derivatives of the angular velocity vector with respect to the generalized coordinates are given in Eqs. 6.27 - 6.31.

$$\omega = \begin{Bmatrix} -s\alpha \cdot \dot{\beta} \\ c\alpha \cdot \dot{\beta} \\ \dot{\alpha} \end{Bmatrix} \quad (6.27)$$

$${}^{\partial\omega}\!/\!_{\partial\theta_l} = \begin{Bmatrix} -s\alpha \cdot {}^{\partial\dot{\beta}}\!/\!_{\partial\theta_l} - c\alpha \, {}^{\partial\alpha}\!/\!_{\partial\theta} \dot{\beta} \\ c\alpha \cdot {}^{\partial\dot{\beta}}\!/\!_{\partial\theta_l} - s\alpha \, {}^{\partial\alpha}\!/\!_{\partial\theta} \dot{\beta} \\ {}^{\partial\dot{\alpha}}\!/\!_{\partial\theta_l} \end{Bmatrix} \quad (6.28)$$

$${}^{\partial\omega}\!/\!_{\partial\dot{\theta}_l} = \begin{Bmatrix} -s\alpha \cdot {}^{\partial\dot{\beta}}\!/\!_{\partial\dot{\theta}_l} \\ c\alpha \cdot {}^{\partial\dot{\beta}}\!/\!_{\partial\dot{\theta}_l} \\ {}^{\partial\dot{\alpha}}\!/\!_{\partial\dot{\theta}_l} \end{Bmatrix} \quad (6.29)$$

$$\frac{\partial^2\omega}{\partial\theta_m\partial\dot{\theta}_l} = \begin{Bmatrix} -s\alpha \cdot \frac{\partial^2\dot{\beta}}{\partial\theta_m\partial\dot{\theta}_l} - c\alpha \, {}^{\partial\alpha}\!/\!_{\partial\theta_m} \, {}^{\partial\dot{\beta}}\!/\!_{\partial\theta_l} \\ c\alpha \cdot \frac{\partial^2\dot{\beta}}{\partial\theta_m\partial\dot{\theta}_l} - s\alpha \, {}^{\partial\alpha}\!/\!_{\partial\theta_m} \, {}^{\partial\dot{\beta}}\!/\!_{\partial\theta_l} \\ \frac{\partial^2\dot{\alpha}}{\partial\theta_m\partial\dot{\theta}_l} \end{Bmatrix} \quad (6.30)$$

$$\frac{\partial^2\omega}{\partial\dot{\theta}_m\partial\dot{\theta}_l} = \begin{Bmatrix} -s\alpha \cdot \frac{\partial^2\dot{\beta}}{\partial\dot{\theta}_m\partial\dot{\theta}_l} \\ c\alpha \cdot \frac{\partial^2\dot{\beta}}{\partial\dot{\theta}_m\partial\dot{\theta}_l} \\ \frac{\partial^2\dot{\alpha}}{\partial\dot{\theta}_m\partial\dot{\theta}_l} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \quad (6.31)$$

Partial Derivatives of the tool space angular coordinate velocities, $\dot{\alpha}$, and $\dot{\beta}$ are also taken with respect to the generalized coordinates, $\theta_i$ and $\dot{\theta}_i$. First, the tool space angular velocities are given as a function of the midplane normal vector, **N**:

$$\dot{\alpha} = \frac{\hat{N}_x \dot{\hat{N}}_y - \hat{N}_y \dot{\hat{N}}_x}{\hat{N}_x^2 + \hat{N}_y^2} \qquad \dot{\beta} = \frac{\hat{N}_x \dot{\hat{N}}_x - \hat{N}_y \dot{\hat{N}}_y}{\sqrt{1 - \left(\hat{N}_x^2 + \hat{N}_y^2\right)} \sqrt{\hat{N}_x^2 + \hat{N}_y^2}} \quad (6.32)$$

The vector **n** is defined to represent the intermediate velocity components:

$$\mathbf{n} = \begin{Bmatrix} \hat{N}_x \\ \hat{N}_y \\ p \end{Bmatrix} \tag{6.33}$$

then:
$$\begin{Bmatrix} \dot{\hat{N}}_x \\ \dot{\hat{N}}_y \\ \dot{p} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{Bmatrix} \quad or \quad \dot{n}_i = J_{ij}\dot{\theta}_j \tag{6.34}$$

and partial derivatives of **n** reduce to elements of the Jacobian as:

$$\partial \dot{n}_i \big/ \partial \dot{\theta}_l = J_{ij} \, \partial \dot{\theta}_j \big/ \partial \dot{\theta}_l = J_{ij}\delta_{jl} = J_{il} \tag{6.35}$$

Now the partial derivatives of the tool space coordinates are expressed in Eqs. 6.36 - 6.41.

$$\partial \dot{\alpha} \big/ \partial \dot{\theta}_l = \frac{\hat{N}_x J_{2l} - \hat{N}_y J_{1l}}{\hat{N}_x^2 + \hat{N}_y^2} \tag{6.36}$$

$$\frac{\partial^2 \dot{\alpha}}{\partial \theta_m \partial \dot{\theta}_l} = \frac{\hat{N}_x \, \partial J_{2l}/\partial \theta_m + \partial \hat{N}_x/\partial \theta_m \, J_{2l} - \hat{N}_y \, \partial J_{1l}/\partial \theta_m - \partial \hat{N}_y/\partial \theta_m \, J_{1l}}{\hat{N}_x^2 + \hat{N}_y^2} - \frac{2\left(\hat{N}_x J_{2l} - \hat{N}_y J_{1l}\right)\left(\hat{N}_x \, \partial \hat{N}_x/\partial \theta_m + \hat{N}_y \, \partial \hat{N}_y/\partial \theta_m\right)}{\left(\hat{N}_x^2 + \hat{N}_y^2\right)^2} \tag{6.37}$$

$$\frac{\partial^2 \dot{\alpha}}{\partial \dot{\theta}_m \partial \dot{\theta}_l} = 0 \tag{6.38}$$

$$\partial \dot{\beta} \big/ \partial \dot{\theta}_l = \frac{\hat{N}_x J_{1l} - \hat{N}_y J_{2l}}{\sqrt{1 - \left(\hat{N}_x^2 + \hat{N}_y^2\right)}\sqrt{\hat{N}_x^2 + \hat{N}_y^2}} \tag{6.39}$$

$$\begin{aligned} \frac{\partial^2 \dot{\beta}}{\partial \theta_m \partial \dot{\theta}_l} = & \frac{\hat{N}_x \, \partial J_{1l}/\partial \theta_m + \partial \hat{N}_x/\partial \theta_m \, J_{1l} - \hat{N}_y \, \partial J_{2l}/\partial \theta_m - \partial \hat{N}_y/\partial \theta_m \, J_{2l}}{\sqrt{1 - \left(\hat{N}_x^2 + \hat{N}_y^2\right)}\sqrt{\hat{N}_x^2 + \hat{N}_y^2}} \\ & - \frac{\left(\hat{N}_x J_{1l} - \hat{N}_y J_{2l}\right)\left(\hat{N}_x \, \partial \hat{N}_x/\partial \theta_m + \hat{N}_y \, \partial \hat{N}_y/\partial \theta_m\right)\left(1 - \left(\hat{N}_x^2 + \hat{N}_y^2\right)\right)}{\left(\left(\hat{N}_x^2 + \hat{N}_y^2\right) - \left(\hat{N}_x^2 + \hat{N}_y^2\right)^2\right)^2} \end{aligned} \tag{6.40}$$

$$\frac{\partial^2 \dot{\beta}}{\partial \dot{\theta}_m \partial \dot{\theta}_l} = 0 \tag{6.41}$$

The rotation matrix that rotates the distal frame into base frame coordinates, ${}_{\mathbf{B}}^{\mathbf{D}}\mathbf{R}$, is a function of the joint space parameters. The partial derivative relative to the joint space parameters is given in Eq. 6.42.

$$\partial R \big/ \partial \theta_i = \begin{bmatrix} -\mathrm{c}\,\alpha\,\mathrm{s}\,\beta\,{}^{\partial\beta}\!/_{\partial\theta_i} - \mathrm{s}\,\alpha\,\mathrm{c}\,\beta\,{}^{\partial\alpha}\!/_{\partial\theta_i} & -\mathrm{c}\,\alpha\,{}^{\partial\alpha}\!/_{\partial\theta_i} & \mathrm{c}\,\alpha\,\mathrm{c}\,\beta\,{}^{\partial\beta}\!/_{\partial\theta_i} - \mathrm{s}\,\alpha\,\mathrm{s}\,\beta\,{}^{\partial\alpha}\!/_{\partial\theta_i} \\ -\mathrm{s}\,\alpha\,\mathrm{s}\,\beta\,{}^{\partial\beta}\!/_{\partial\theta_i} - \mathrm{c}\,\alpha\,\mathrm{c}\,\beta\,{}^{\partial\alpha}\!/_{\partial\theta_i} & -\mathrm{s}\,\alpha\,{}^{\partial\alpha}\!/_{\partial\theta_i} & \mathrm{s}\,\alpha\,\mathrm{c}\,\beta\,{}^{\partial\beta}\!/_{\partial\theta_i} + \mathrm{c}\,\alpha\,\mathrm{s}\,\beta\,{}^{\partial\alpha}\!/_{\partial\theta_i} \\ -\mathrm{c}\,\beta\,{}^{\partial\beta}\!/_{\partial\theta_i} & 0 & -\mathrm{s}\,\beta\,{}^{\partial\beta}\!/_{\partial\theta_i} \end{bmatrix} \tag{6.42}$$

The final elements of the equations of motion are the partial derivatives of the Wrist Jacobian matrix, **J**. These are performed as in the equations above and as demonstrated in Chap. 4.

## 6.4 Solving the Forward Dynamics

The equations of motion cast in the forward dynamic form are given in Eq. 6.21. With position, velocity, and the input motor torques, the resulting joint space accelerations are calculated. Since position and velocity are similarly a function of acceleration, the solution result is instantaneous. In the time response simulation model, a small time interval or step, $\Delta t$, will be introduced. The instantaneous acceleration result will then be assumed valid over the time interval, $\Delta t$. Position and velocity information is calculated at the end of the time interval from the acceleration result and the initial path information. With the updated path position, the process is repeated, calculating a new acceleration, and updating the path information. The result is demonstrated in the following equations:

Let $k$ indicate an initial path position, and $k+1$ a successive path position, separated by the time step, $\Delta t$. Then the parameters over this path interval are defined as:

$t_k$, the time at position $k$,

$t_{k+1}$ the time at position $k+1$

$\Delta t = (t_{k+1} - t_k)$, the time step.

and the position, velocity acceleration parameters:

$\theta_k$, $\theta_{k+1}$ the joint space position vector at path positions $k$ and $k+1$ respectively.

$\dot{\theta}_k$, $\dot{\theta}_{k+1}$ the joint space velocity vector at path positions $k$ and $k+1$ respectively.

$\ddot{\theta}_k$ the joint-space acceleration vector over the path from position $k$ to $k+1$.

$M_k$, $M_{k+1}$ the vector of motor moments at path positions $k$ and $k+1$ respectively.

To begin the time response simulation, given information are the position and velocity, expressed in joint-space coordinates at the initial position, $t_k = 0$,

$$\theta_{k=0} = \theta(0), \dot{\theta}_{k=0} = \dot{\theta}(0) \tag{6.43}$$

Also given are the motor torque time histories, $M_l(t)$, $l=1\text{-}3$. From these functions, torque values are evaluated at the $k^{th}$ path position for motor $l$ as,

$$M_{l,k} = M_l(t_k) \tag{6.44}$$

Starting at position $k$, $\theta_k$ and $\dot{\theta}_k$ as well as the motor torques are used to solve $\ddot{\theta}_k$:

$$\ddot{\theta}_k = \mathbf{A}_k^{-1}\mathbf{M}_k \tag{6.21}$$

where the subscript $k$ denotes the path position. Position and velocity information are updated as:

$$\begin{aligned} \dot{\theta}_{k+1} &= \dot{\theta}_k + \ddot{\theta}_k \cdot \Delta t \\ \dot{\theta}_{k+1} &= \dot{\theta}_k + \ddot{\theta}_k \cdot \Delta t \\ t_{k+1} &= t_k + \Delta t \end{aligned} \qquad (6.45)$$

Finally, the subscript $k$ is updated to trace the path position:

$$\theta_k = \theta_{k+1}, \quad \dot{\theta}_k = \dot{\theta}_{k+1}, \quad t_k = t_{k+1} \qquad (6.46)$$

The results of this technique is a time history of the manipulator path trajectory. This result can be expressed in tool space coordinates using the kinematic relations developed.

In developing the time response model, a mixture of tool-space and joint-space coordinates were used. The functional relationship between these frames, as well as there derivatives are expressed as:

$$\left(\alpha, \dot{\alpha}, \ddot{\alpha}, \beta, \beta, \ddot{\beta}, p, \dot{p}, \ddot{p}\right) = \mathrm{g}\left(\theta_1, \dot{\theta}_1, \ddot{\theta}_1, \theta_2, \dot{\theta}_2, \ddot{\theta}_2, \theta_3, \dot{\theta}_3, \ddot{\theta}_3\right) \qquad (6.47)$$

and have been developed in this and previous work (Chaps. 3, 4, Canfield et al., 1996).

The immediate result of the time response model is the path trajectory time history expressed in joint-space coordinates. Since these are typically desired in tool coordinates, the above kinematic function will be employed to transform the path as desired.

## 6.5 Results and Conclusions

Using Hamilton’s Principle, a forward dynamic analysis was carried out for a parallel-architecture robotic wrist with three inputs actuated relative to a fixed base. From this analysis, a model was developed for full manipulator simulation. Path trajectory time history was developed for an example application. The results of this simulation model could also be demonstrated in creating a high-level controller (Artificial Neural Network).