Collapsing the Resultant Dipole System

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Abstract.

In this paper we show how to reduce the resultant dipole system developed by Gabor, Hodgkin, and Nelson to a single quadratic equation. This system was developed to determine the resultant dipole of the heart from measurements on the body surface. Reducing the system to a quadratic equation eliminates previous difficulties in numerically solving the system.

Introduction.

In [1] a system of eight nonlinear equations in eight unknowns is derived for the determination of the magnitudes, directions, and locations of two independent dipoles in a two-dimensional conducting region from boundary potential measurements. (The paper [2] also addresses this problem.) We show that this system reduces to a single quadratic equation in one variable, and therefore it is easy to solve. This reduction is significant because the original eight equation formulation is difficult to solve, even by sophisticated numerical techniques [3].

The system in [1] can be written:

$$a + b = \sum M_x \tag{1}$$

$$c + d = \sum M_y \tag{2}$$

$$ta + ub - vc - wd = \sum A \tag{3}$$

$$va + wb + tc + ud = \sum B \tag{4}$$

$$a(t^2 - v^2) - 2ctv + b(u^2 - w^2) - 2duw = \sum C \tag{5}$$

$$c(t^2 - v^2) + 2atv + d(u^2 - w^2) + 2buw = \sum D \tag{6}$$

$$at(t^2 - 3v^2) + cv(v^2 - 3t^2) + bu(u^2 - 3w^2) + dw(w^2 - 3u^2) = \sum E \tag{7}$$

$$ct(t^2 - 3v^2) - av(v^2 - 3t^2) + du(u^2 - 3w^2) - bw(w^2 - 3u^2) = \sum F, \tag{8}$$

where equations (1)–(8) above correspond to equations (1)–(8) of [1], respectively. See Table 1. The notation we use is from [3], where the numerical solution of this system is discussed.

Methods.

We use the approach of "reduction" as discussed in [4], Chapter 7. In this case simple algebra suffices to reduce the system.

Results.

We show that all the solutions of the system consisting of equations (1)–(8) can be obtained from the solutions of the quadratic equation

$$k_1x^2 - k_1k_5x + (k_2k_5 - k_3) = 0, \tag{9}$$

where the constants k_1 through k_5 are given in Table 2. Here x is viewed as a complex unknown number. Thus we will obtain two (complex) solutions to (9) unless $k_1 = 0$.

For each solution x to (9), we generate the four complex numbers

$$y_4 = x, \tag{10}$$
$$y_3 = -y_4 + k_5, \tag{11}$$
$$y_2 = (k_2 - k_1y_3)/(y_4 - y_3), \tag{12}$$
$$y_1 = -y_2 + k_1. \tag{13}$$

(The cases $k_1 = 0$, $y_3 = y_4$, and $k_1k_3 = k_2^2$ are degenerate.)

Thus, we will have two sets of (y_1, y_2, y_3, y_4), unless $k_1 = 0$ or $y_3 = y_4$ or $k_1k_3 = k_2^2$. Then, except for the degenerate cases mentioned, all of the solutions to the system of equations (1)–(8) are obtained by the formulas

$$\begin{aligned} a &= Re(y_1), & t &= Re(y_3),\\ c &= Im(y_1), & v &= Im(y_3),\\ b &= Re(y_2), & u &= Re(y_4),\\ d &= Im(y_2), & w &= Im(y_4), \end{aligned}$$

where "Re" and "Im" denote the real and imaginary parts.

The reduction proceeds in two steps:

STEP 1. Define a new system of four equations (with complex coefficients) in the four unknowns

$$\begin{aligned} y_1 &= a + ci,\\ y_2 &= b + di,\\ y_3 &= t + vi,\\ y_4 &= u + wi, \end{aligned}$$

where i denotes the complex number $\sqrt{-1}$, by adding to equations (1), (3), (5), and (7) i times equations (2), (4), (6), and (8), respectively. The result is the system

$$y_1 + y_2 = k_1, \tag{14}$$
$$y_1y_3 + y_2y_4 = k_2, \tag{15}$$
$$y_1y_3^2 + y_2y_4^2 = k_3, \tag{16}$$
$$y_1y_3^3 + y_2y_4^3 = k_4. \tag{17}$$

STEP 2. Reduce the system consisting of equations (14)–(17) to equation (9) (with $x \equiv y_4$) via successive substitutions as follows: Equation (14) yields (13). Then equations (13) and (15) yield (12). Now equations (12), (13), and (16) yield

$$k_2y_3 + k_2y_4 - k_1y_3y_4 = k_3. \tag{18}$$

Combining (12), (13), (17), and (18) we get (11). (Equation (18) allows us to replace y_3y_4 by a linear relation in y_3 and y_4.) Then, substituting (10) and (11) into (18), we get (9).

Table 1. Correspondence of notation between equations (1)–(8) of this paper and equations (1)–(8) of [1].

This Paper	Reference [1]
a	M_{x1}
b	M_{x2}
c	M_{y1}
d	M_{y2}
t	X_1
u	X_2
v	Y_1
w	Y_2
$\sum M_x$	$k \int V\, dy$
$\sum M_y$	$k \int V\, dx$
$\sum A$	$k \int V(x\, dy - y\, dx)$
$\sum B$	$k \int V(x\, dx + y\, dy)$
$\sum C$	$k \left[\int V(x^2 - y^2)\, dy - 2 \int Vxy\, dx\right]$
$\sum D$	$k \left[\int V(x^2 - y^2)\, dx + 2 \int Vxy\, dy\right]$
$\sum E$	$k \left[\int V(x^3 - 3xy^2)\, dy + \int V(y^3 - 3x^2y)\, dx\right]$
$\sum F$	$k \left[\int V(x^3 - 3xy^2)\, dx - \int V(y^3 - 3x^2y)\, dy\right]$

Discussion.

Frequently, a nonlinear system that is difficult to solve numerically can be reduced algebraically to a form more amenable to solution (as discussed in [4], Chapter 7). The dipole system (1)–(8) has been reduced by this approach to a single quadratic equation. Systems whose physical origin is similar might be successfully reduced via the same procedure.

References.

[1] C. V. Nelson and B. C. Hodgkin, "Determination of magnitudes, directions, and locations of two independent dipoles in a circular conducting region from boundary potential measurements", *IEEE Trans. Biomed. Eng.*, vol. BME-28, pp. 817–823, Dec. 1981.

Table 2. Constants defining the coefficients of equations (9)–(13) from the coefficients of equations (1)–(8). (Here $i = \sqrt{-1}$ is the complex constant.)

Constant	**Formula**
k_1	$\sum M_x + i \sum M_y$
k_2	$\sum A + i \sum B$
k_3	$\sum C + i \sum D$
k_4	$\sum E + i \sum F$
k_5	$(k_1 k_4 - k_2 k_3)/(k_1 k_3 - k_2^2)$

[2] D. Gabor and C. V. Nelson, "Determination of the resultant dipole of the heart from measurements on the body surface", *J. Appl. Phys.*, vol. 25, pp. 413–416, Apr. 1954.

[3] J. E. Dennis, Jr., D. M. Gay, and P. A. Vu, "A new nonlinear equations test problem", preprint, July 1985.

[4] A. P. Morgan, *Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems*, Prentice-Hall, 1987.