3.2 Differentiable Reformulation in the Primal Space

In this section we present a differentiable reformulation of EMFLP in the primal space, and provide some related theorems and lemmas that illustrate the properties of this reformulation, as well as their utility in solving the problem EMFLP. Two examples that support the proposed reformulation are also presented.

Given an instance of EMFLP, let us use the transformation

$$\bar{\mathrm{a}}_{ij} = \{(\bar{x}_i - a_j)^2 + (\bar{y}_i - b_j)^2\}^{1/2} \qquad \forall\, (i, j) \in A_{NE}\text{, and}$$
$$\bar{\mathrm{b}}_{k\ell} = \{(\bar{x}_k - \bar{x}_l)^2 + (\bar{y}_k - \bar{y}_l)^2\}^{1/2}\ \forall\, (k, l) \in A_{NN} \qquad (3.9)$$

and consider the following smooth restatement **REMFLP** of EMFLP.

REMFLP: Minimize $\displaystyle\sum\sum_{(i,j)\in A_{NE}} w_{ij}\, \mathrm{a}_{ij} + \sum\sum_{(k,l)\in A_{NN}} v_{kl}\, \mathrm{b}_{kl}$ (3.10a)

subject to $\mathrm{a}_{ij}^2 \geq (x_i - a_j)^2 + (y_i - b_j)^2 \qquad \forall\, (i, j) \in A_{NE}$ (3.10b)

$\mathrm{b}_{kl}^2 \geq (x_k - x_l)^2 + (y_k - y_l)^2 \qquad \forall\, (k, l) \in A_{NN}$ (3.10c)

$\mathrm{a}_{ij} \geq (x_i - a_j),\ \mathrm{a}_{ij} \geq (a_j - x_i),\ \mathrm{a}_{ij} \geq (y_i - b_j),\ \mathrm{a}_{ij} \geq (b_j - y_i) \quad \forall\, (i, j) \in A_{NE}$ (3.10d)

$\mathrm{b}_{kl} \geq (x_k - x_l),\ \mathrm{b}_{kl} \geq (x_l - x_k),\ \mathrm{b}_{kl} \geq (y_k - y_l),\ \mathrm{b}_{kl} \geq (y_l - y_k) \quad \forall\, (k, l) \in A_{NN}$ (3.10e)

$\mathrm{a}_{ij} \geq 0 \quad \forall\, (i, j) \in A_{NE},\ \mathrm{b}_{kl} \geq 0 \quad \forall\, (k, l) \in A_{NN}.$ (3.10f)

The result below addresses the equivalence of REMFLP and EMFLP problems.

Lemma 2. *REMFLP is an equivalent restatement of EMFLP in that* $(\bar{x}, \bar{y})$ *solves EMFLP if and only if* $(\bar{x}, \bar{y}, \bar{\mathrm{a}}, \bar{\mathrm{b}})$ *solves REMFLP where* $(\bar{\mathrm{a}}, \bar{\mathrm{b}})$ *is given by (3.9) for* $(x,y) \equiv (\bar{x}, \bar{y})$.

Proof. Given any $(\bar{x}, \bar{y})$, it is readily seen that an optimal completion $(\bar{x}, \bar{y}, \bar{\mathrm{a}}, \bar{\mathrm{b}})$ to REMFLP is given by letting $(\bar{\mathrm{a}}, \bar{\mathrm{b}})$ be given by (3.9) with $(x, y) \equiv (\bar{x}, \bar{y})$. Hence, the stated equivalence of EMFLP and REMFLP follows, and this completes the proof.

Although (3.10) is a smooth restatement of EMFLP, note that each of the constraint functions (3.10b) and (3.10c) when written as ≤ 0 inequalities represent nonconvex quadratic functions. However, as the following result shows, the overall feasible region of REMFLP is a convex set. Hence, REMFLP is indeed a smooth, convex programming problem.

Lemma 3. *REMFLP is a convex programming problem.*

Proof. Since the objective function of REMFLP is linear, let us verify that the feasible region is a convex set. Toward this end note that the set of points

$$S_{ij} = \{ (x_i, y_i, \alpha_{ij}): (x_i - a_j)^2 + (y_i - b_j)^2 - \alpha_{ij}^2 \leq 0, \alpha_{ij} \geq 0\} \qquad (3.11a)$$

is equivalently represented by

$$S'_{ij} = \{(x_i, y_i, \alpha_{ij}): \left| \{(x_i - a_j)^2 + (y_i - b_j)^2\}^{1/2} \right| - \alpha_{ij} \leq 0 \}, \quad \forall\ (i, j) \in A_{NE}. \ (3.11b)$$

Since the constraint expression in (3.11b) is a convex function of (x_i, y_i, α_{ij}), S'_{ij} is a convex set, and hence so is S_{ij}, $\forall (i, j) \in A_{NE}$.

Similarly, the set $S_{kl} = \{(x_k, y_k, x_l, y_l, \beta_{kl}): (x_k - a_l)^2 + (y_k - b_l)^2 - \beta_{kl}^2 \leq 0, \beta_{kl} \geq 0\}$ is a convex set $\forall (k, l) \in A_{NN}$. Since the feasible region of REMFLP is composed of the intersection of these sets S_{ij} $\forall (i, j) \in A_{NE}$ and $S_{k\ell}$ $\forall (k, l) \in A_{NN}$ along with the polyhedron defined by (3.10d) and (3.10e), the proof is complete.

Observe that Lemmas 2 and 3 hold true even with the constraints (3.10d) and (3.10e) absent; in fact, these constraints are implied by the other constraints (3.10b), (3.10c) and (3.10f) of REMFLP. As we shall now see, the role of these redundant constraints is to promote the possibility that an optimum to EMFLP turns out to be a KKT solution to REMFLP. This is important because if an optimum to REMFLP is not a KKT point for this problem, then despite its convex, smooth structure, algorithms that are designed to converge to KKT solutions might experience difficulties in approaching such an optimum.

The following result addresses this issue and Example 1 below provides an illustration for the utility of (3.10d) and (3.10e).

Theorem 2. *Let* $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ *be a KKT solution for REMFLP, where* $\bar{a}$ *and* $\bar{b}$ *are given by* (3.9) *for* $(x, y) \equiv (\bar{x}, \bar{y})$. *Then* $(\bar{x}, \bar{y})$ *solves EMFLP. Conversely, let* $(\bar{x}, \bar{y})$ *solve EMFLP, and suppose that the optimality conditions* (3.8) *are satisfied with*

$$|z_{1ij}|+|z_{2ij}|\le 1 \quad \forall (i, j) \in A_{NE} \text{ such that } \bar{a}_{ij} = 0 \text{, and} \tag{3.12a}$$

$$|z_{3k\ell}|+|z_{4k\ell}|\le 1 \quad \forall (k, l) \in A_{NN} \text{ such that } \bar{b}_{k\ell} = 0, \tag{3.12b}$$

where $\bar{a}$ *and* $\bar{b}$ *are given by* (3.9) *for* $(x, y) \equiv (\bar{x}, \bar{y})$. *Then* $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ *is a KKT solution for REMFLP.*

Proof. Let $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ be a KKT solution for REMFLP, where $(\bar{a}, \bar{b})$ is given by (3.9). To show that $(\bar{x}, \bar{y})$ solves EMFLP, by Lemma 2, it is sufficient to show that $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ solves REMFLP. On the contrary, suppose not. Then, by Lemma 3, there must exist a feasible direction at $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ that has a negative directional derivative. However, since this direction is also feasible to the first-order linear programming approximation to the constraint functions at $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ (see Bazaraa *et al*., 1993), it follows that $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ is not a KKT solution, a contradiction. Hence, $(\bar{x}, \bar{y})$ solves EMFLP.

Conversely, let $(\bar{x}, \bar{y})$ solve EMFLP and suppose that (3.12) holds true. Let us show that $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ is then a KKT solution for REMFLP, where $(\bar{a}, \bar{b})$ is defined by (3.9). Toward this end, let us denote nonnegative Lagrange multipliers q_{ij} for each $(i, j) \in A_{NE}$ and $f_{k\ell}$ for each $(k, l) \in A_{NN}$, associated with constraints (3.10b) and (3.10c), respectively, Lagrange multipliers z_{1ij}^{+}, z_{1ij}^{-}, z_{2ij}^{+} and z_{2ij}^{-} associated with the respective inequalities in (3.10d) for each $(i, j) \in A_{NE}$, and similarly, z^{+}_{3ij}, z^{-}_{3ij}, z^{+}_{4ij}, and z^{-}_{4ij} associated with the respective inequalities in (3.10e) for each $(k, l) \in A_{NN}$, and multipliers l_{ij} $\forall (i, j) \in A_{NE}$ and $g_{k\ell}$ $\forall\ (k, \ell) \in A_{NN}$ $\forall (k, l) \in A_{NN}$ associated with the respective sets of inequalities in (3.10f). The KKT conditions for REMFLP at $(\bar{x}, \bar{y}, \bar{a}, \bar{b})$ can then be written as follows, requiring a solution to the following system, where index limits and the vector notation in (3.13h) are obvious and are simplified for ease in presentation, and where the complementary slackness conditions with respect to (3.10b) and (3.10c) are satisfied since these constraints are active at the given solution. (As before, $f_{(i\ell)}$ denotes $f_{i\ell}$ and $i < \ell$, and $f_{\ell i}$ if $\ell < i$.)

$$\sum_j 2(\bar{x}_i - a_j)q_{ij} + \sum_{\ell \neq i} 2(\bar{x}_i - \bar{x}_\ell)f_{(i\ell)} + \sum_j (z^{+}_{1ij} - z^{-}_{1ij}) + \sum_{\ell > i}(z^{+}_{3i\ell} - z^{-}_{3i\ell}) - \sum_{\ell < i}(z^{+}_{3\ell i} - z^{-}_{3\ell i}) = 0$$

$$\forall\ i = 1,.., n \qquad (3.13a)$$

$$\sum_j 2(\bar{y}_i - b_j)q_{ij} + \sum_{\ell \neq i} 2(\bar{y}_i - \bar{y}_\ell)f_{(i\ell)} + \sum_j (z^{+}_{2ij} - z^{-}_{2ij}) + \sum_{\ell > i}(z^{+}_{4i\ell} - z^{-}_{4i\ell}) - \sum_{\ell < i}(z^{+}_{4\ell i} - z^{-}_{4\ell i}) = 0$$

$$\forall\ i = 1,.., n \qquad (3.13b)$$

$$2\bar{a}_{ij}q_{ij} + (z^{+}_{1ij} + z^{-}_{1ij} + z^{+}_{2ij} + z^{-}_{2ij}) + l_{ij} = w_{ij} \qquad \forall (i, j) \in A_{NE} \qquad (3.13c)$$

$$2\bar{b}_{k\ell}f_{k\ell} + (z^{+}_{3k\ell} + z^{-}_{3k\ell} + z^{+}_{4k\ell} + z^{-}_{4k\ell}) + g_{k\ell} = n_{k\ell} \qquad \forall (k, l) \in A_{NN} \qquad (3.13d)$$

$$z^{+}_{1ij}(\bar{a}_{ij} - x_i + a_j) = z^{-}_{1ij}(\bar{a}_{ij} - a_j + x_i) = z^{+}_{2ij}(\bar{a}_{ij} - y_i + b_j) = z^{-}_{2ij}(\bar{a}_{ij} - b_j + y_i) = 0$$

$$\forall (i, j) \in A_{NE} \qquad (3.13e)$$

$$z^{+}_{3k\ell}(\bar{b}_{k\ell} - \bar{x}_k + \bar{x}_\ell) = z^{-}_{3k\ell}(\bar{b}_{k\ell} - \bar{x}_\ell + \bar{x}_k) = z^{+}_{4k\ell}(\bar{b}_{k\ell} - \bar{y}_k + \bar{y}_\ell) = z^{-}_{4k\ell}(\bar{b}_{k\ell} - \bar{y}_\ell + \bar{y}_k) = 0$$

$$\forall (k, l) \in A_{NN} \qquad (3.13f)$$

$$\bar{\alpha}_{ij}\lambda_{ij} = 0 \quad \forall\,(i, j) \in A_{NE},\ \bar{\beta}_{k\ell}\gamma_{k\ell} = 0 \quad \forall\,(k, l) \in A_{NN} \tag{3.13g}$$

$$(\theta, \phi, \zeta_1^{\pm}, \zeta_2^{\pm}, \zeta_3^{\pm}, \zeta_4^{\pm}, \lambda, \gamma) \geq 0\,. \tag{3.13h}$$

Now, select the Lagrange multipliers as follows in order to satisfy (3.13).

(a) For each $(i, j) \in A_{NE}$, if $\bar{\alpha}_{ij} > 0$, let $\lambda_{ij} = 0$ from (3.13g), select $\zeta_{1ij}^{\pm} = \zeta_{2ij}^{\pm} = 0$, and let $\theta_{ij} = w_{ij} / 2\bar{\alpha}_{ij}$ from (3.13c).

(b) For each $(k, l) \in A_{NN}$, if $\bar{\beta}_{k\ell} > 0$, let $\gamma_{k\ell} = 0$ from (3.13g), select $\zeta_{3k\ell}^{\pm} = \zeta_{4k\ell}^{\pm} = 0$, and let $\phi_{k\ell} = \nu_{k\ell} / 2\bar{\beta}_{k\ell}$ from (3.13d).

(c) For each $(i, j) \in A_{NE}$ such that $\bar{\alpha}_{ij} = 0$, noting (3.12a), take $\theta_{ij} = 0$, and select nonnegative $\zeta_{1ij}^{\pm}$ and $\zeta_{2ij}^{\pm}$ such that $\{\zeta_{1ij}^{+} - \zeta_{1ij}^{-} = \zeta_{1ij} w_{ij}, \zeta_{1ij}^{+} + \zeta_{1ij}^{-} = |\zeta_{1ij}|\, w_{ij}\}$, $\{\zeta_{2ij}^{+} - \zeta_{2ij}^{-} = \zeta_{2ij} w_{ij},\ \zeta_{2ij}^{+} + \zeta_{2ij}^{-} = |\xi_{2ij}|\, w_{ij}\}$, and let $\lambda_{ij} = w_{ij}\,[1 - |\zeta_{1ij}| - |\zeta_{2ij}|]$. Note that $\lambda_{ij} \geq 0$ by (3.12a).

(d) For each $(k, l) \in A_{NN}$ such that $\bar{\beta}_{k\ell} = 0$, noting (3.12b), take $\phi_{k\ell} = 0$ and select nonnegative $\zeta_{3kl}^{\pm}$ and $\zeta_{4kl}^{\pm}$ such that $\{\zeta_{3k\ell}^{+} - \zeta_{3k\ell}^{-} = \zeta_{3k\ell}\nu_{k\ell}, \zeta_{3k\ell}^{+} + \zeta_{3k\ell}^{-} = |\zeta_{3k\ell}|\nu_{k\ell}\}$,

$\{\zeta_{4k\ell}^{+} - \zeta_{4k\ell}^{-} = \zeta_{4k\ell}\nu_{k\ell},\ \zeta_{4k\ell}^{+} + \zeta_{4k\ell}^{-} = |\zeta_{4k\ell}|\nu_{k\ell}\}$, and let

$\gamma_{k\ell} = \nu_{k\ell}(1 - |\zeta_{3k\ell}| - |\zeta_{4k\ell}|)$. Note that $\gamma_{kl} \geq 0$ by (3.12b).

Then, examining (3.7) and (3.8), it is readily verified that the prescriptions given by (a) - (d) above satisfy (3.13), and so, $(\bar{x}, \bar{y}, \bar{\alpha}, \bar{\beta})$ is a KKT solution for REMFLP. This completes the proof.

The next example supports Theorem 2 and shows the important role of the redundant constraints (3.10d) and (3.10e) in promoting the satisfaction of KKT conditions of problem REMFLP at the point $(\bar{x}, \bar{y})$ that solves the problem EMFLP.

Example 1

Consider a situation with $m = 2$ existing facilities located at (0, 0) and (1, 0) respectively, and suppose that we wish to locate a single new facility $n = 1$, given $w_{11} = 2$ and $w_{12} = 1$, respectively. Then it is readily verified that $(\bar{x}, \bar{y}) \equiv (0, 0)$ solves EMFLP. We shall consider the problem REMFLP with and without the redundant constraints (3.10d) and (3.10e), in order to demonstrate their usefulness in the equivalent reformulation approach.

Case 1 When the redundant constraints (3.10d) and (3.10e) are considered.

Using Lemma 2, $(\bar{x}, \bar{y}, \bar{\alpha}_{11}, \bar{\alpha}_{12}) = (0, 0, 0, 1)$ solves REMFLP. By Equation (3.7), we have $\bar{\zeta}_x = -1$ and $\bar{\zeta}_y = 0$, and we see that (3.8) holds true by selecting $\zeta_{111} = 1$ and $\zeta_{211} = 0$. Hence, by Theorem 2, since (3.12) holds true, we have that this $(\bar{x}, \bar{y}, \bar{\alpha}_{11}, \bar{\alpha}_{12})$ is a KKT solution for REMFLP. It can be verified that (3.13) holds true by letting $\lambda_{12} = 0$, $\zeta^{\pm}_{112} = \zeta^{\pm}_{212} = 0$, and $\theta_{12} = 1/2$ by selection step (a) in the proof, and by letting $\theta_{11} = 0$, $\zeta^{+}_{111} = 2$, $\zeta^{-}_{111} = 0$, and $\zeta^{\pm}_{211} = \lambda_{11} = 0$ by selection step (3.6) in the proof.

Case 2 When the redundant constraints (3.10d) and (3.10e) are omitted.

If the constraints (3.10d) and (3.10e) are absent, so that $\zeta^{\pm}_{1}$ and $\zeta^{\pm}_{2}$ are vacuous in this example, then $\bar{\alpha}_{12} = 1 > 0$ implies that $\lambda_{12} = 0$ from (3.13g), and so by (3.13c), we get $\theta_{12} = 1/2$. But this yields a value of –1 on the left-hand side of (3.13a), and so, without the constraints (3.10d) and (3.10e), this solution is no longer a KKT point for REMFLP. In fact, without the latter constraints, it can be seen via the proof of Theorem 2 and by Theorem 1 that if $(\bar{x}, \bar{y})$ solves EMFLP, then the corresponding $(\bar{x}, \bar{y}, \bar{\alpha}, \bar{\beta})$, with $(\bar{\alpha}, \bar{\beta})$ given by (3.9), is a KKT solution for REMFLP if and only if $\bar{z}_x = \bar{z}_y = 0$ in (3.7), and by choosing zero subgradients (which is a valid choice) for all nondifferentiable terms in (3.1) at optimality. As a point of interest for this example, without the constraints (3.10d) and (3.10e), the software MINOS 5.2 nonetheless converges to the exact optimum $(x, y) =$ (0,0) for this problem.

The next example illustrates a situation in which an optimum to EMFLP does not correspond to a KKT point for REMFLP.

Example 2

Consider an EMFLP problem with $m = 4$ existing facilities located at the vertices (0, 0), (1, 0), (0, 1) and (1, 1) of a unit square for *j*= 1, 2, 3, 4, respectively, and suppose that we need to locate a single new facility ($n = 1$). Let the interaction weights be $w_{11} = \sqrt{2} + 1$ and $w_{1j} = 1$ for $j = 2, 3, 4$. Consider the solution $(\bar{x}, \bar{y}) = (0, 0)$. By Equation (3.7), we get $\bar{z}_x = \bar{z}_y = -(1 + \sqrt{2})/\sqrt{2}$, and so, (3.8) holds true by selecting $z_{111} = z_{211} = 1/\sqrt{2}$, where $z_{111}^2 + z_{211}^2 = 1$. Hence, $(\bar{x}, \bar{y})$ solves EMFLP and so, $(\bar{x}, \bar{y}, \bar{\alpha}_{11}, \bar{\alpha}_{12}, \bar{\alpha}_{13}, \bar{\alpha}_{14}) = (0, 0, 0, 1, 1, \sqrt{2})$ solves REMFLP. However, this is not a KKT solution for REMFLP since for $(i, j) = (1,1)$, (3.13a) and (3.13b) would require that $z_{111}^+ = z_{211}^+ = (\sqrt{2} + 1)/\sqrt{2}$ (see selection (c) in the proof of Theorem 2), yielding $z_{111}^+ + z_{211}^+ = \sqrt{2} w_{11}$ which then violates (3.13c). As a point of interest, using MINOS 5.2 to solve this instance of REMFLP we nonetheless obtained the exact optimal solution $(x, y) = (0, 0)$ at termination.