RÉSUMÉ OF THE DEPARTMENT OF STATISTICS OF VIRGINIA POLYTECHNIC INSTITUTE

by

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TO MARIA JOSEFA

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CHAPTER I

INTRODUCTION

1.1 Statistics

Statistics is a young and vital branch of science. It is difficult to mention a field of science, agriculture, engineering, business, industrial operations, and government work in which statistics is not gaining greater acceptance and use. It is young because most of the presently used statistical techniques have been developed in this century.

The "average person" thinks of statistics as the columns of figures on the business section of newspapers, illustrated with "zig-zag graphs", or as records of births and deaths, etc. This may have been a good interpretation of statistics years ago. It is true that in the beginning statistics was concerned with the collection and compilation of data, for instance census taking. There are many records of census in most of the countries of the world from early times (e.g. in ancient Egypt a census was taken about 3050 years B.C.) But today, besides being descriptive in nature, statistics provides tools for making decisions when conditions of uncertainty prevail. We shall describe <u>statistics as a branch of science which is concerned with the development and application of efficient techniques for the</u>

collection, organization, analysis, and interpretation of information (which can be stated in numerical form) in such a way that the uncertainty of inductive inferences may be evaluated in terms of probability statements.

1.2 Statistics and the scientific method

The scientist may use different ways of obtaining knowledge. Most of these procedures include the following steps:

1. Review of facts, theory and proposals related to the problem raised.
2. Formulation of a logical hypothesis.
3. Objective evaluation of the hypothesis, by means of
   a. Investigations (surveying)
   b. Experiments.
4. Inference, analysis and interpretation of the results of the objective evaluation of the hypothesis.

In general there is no way of deciding when a step ends and the next starts. Their continuous sequence is circular, because the fourth step will open new theories and proposals leading to the first step again.

The scientist cannot observe all the conceivable events related to a given problem. Thus he has to use inductive inference, in other words he has to derive general propositions from the evidence of specific cases under conditions of uncertainty. This process will enable him to draw conclusions about his hypothesis, but he will need to have an idea about the degree of uncertainty of these conclusions. Thus statistics becomes a vital tool of the scientific method.

The application of statistics falls in many aspects of the scientific method. From the initial plan until the collection of data, which calls for appropriate designs of experiments or surveys, and ways of taking observations, and from the tabulations of the data to evaluating the uncertainty of possible inferences to be drawn, which calls for appropriate methods of analysis of the data by means of the theory of probability.

1.3 The statistician's work and career

During the last fifty years, the rapid development of statistical research, especially in England, India and the United States, has produced a large number of fundamental techniques which are being used very profitably in diverse fields of research, to mention a few of them: forecast and improvement of crops, physics including astro-physics and rocket research, production and operations research, medical and biological research, engineering including testing materials and location of factories, etc. Automatic data processing certainly has received its first impetus from statisticians.

In the last three decades, the marked advances in electronic computers has aided considerably the advance of statistics. High speed electronic computers now make it possible to handle extensive numerical analyses.

Therefore there is an ever increasing demand for mathematical and applied statiscians, and for scientists or engineers with statistical training. In almost all major industries and research organizations the statistician is a highly respected and urgently needed specialist. Graduates with advanced statistical training find abundance of opportunities for highly rewarding and well paid work. In 1963 (6) the estimated salary for a graduate with advanced statistical training ranged from $8,000.00 to $12,000 per year.

Today many universities in the United States offer special curricula in statistics, and some have departments or institutes, which offer programs leading to advanced degrees (M.A., M.S., Ph.D.) in statistics, which are engaged in consulting and research work, and which also offer supporting courses for graduate students of other departments. More and more universities are adopting undergraduate instruction in statistical methods.

In many of the so called land-grant universities, with the expansion of the agricultural and engineering experimentation came the need for adequate statistical services. As a consequence they had to organize departments or institutes of statistics. This was the case also at Virginia Polytechnic Institute, where its Department of Statistics has been providing extensive consulting and computing services to the Virginia Agricultural Experiment Station, Virginia Engineering Experiment Station, federal and state agencies, and to the University as a whole.

The objective of this thesis is to give an outline of the organization, importance and objectives of the Department of Statistics of Virginia Polytechnic Institute. Special emphasis is put on the consulting and computing service that the Department of Statistics through its Statistical Laboratory has been providing since its initial organization until September 1966. As the computational work of the Department has considerable aid from the University High-Speed Computer Center, we shall mention briefly its organization and functions.

CHAPTER II
ORGANIZATION AND FUNCTIONS OF THE DEPARTMENT OF STATISTICS AT VPI

2.1 History

The Virginia Polytechnic Institute is one of the so-called land-grant universities organized under the provisions of the Morrill Act passed by the National Congress and approved on July 2, 1862. Virginia - and every other state - was apportioned 30,000 acres of public land (without mineral deposits) for each senator and representative in Congress according to representation based on the 1860 census, (32).

In March, 1872, Governor Gilbert C. Walker signed the bill establishing the Virginia Agricultural and Mechanical College at Blacksburg. Following some of the words of the Morrill Act, the purpose of the new college was stated as: "The curriculum of the Virginia Agricultural and Mechanical College shall embrace such branches of learning as relate to agriculture and mechanic arts, without excluding other scientific and classical studies, and including military tactics". In 1896, the name of the college was changed to make it the Virginia Agricultural and Mechanical College and Polytechnic Institute, as a consequence of the beginning of its great growth. In 1944, the "Agricultural and Mechanical" was dropped and the legal name became the Virginia

Polytechnic Institute (VPI).

Professors in the early days of land-grant colleges soon learned that teaching in the traditional way from textbooks was not enough. Because they had to have more information, they conducted scientific experiments.

The Agricultural Experiment Station was established at VPI in 1887, only 15 years after the university opened its doors, under the federal Hatch Act. And in 1921, the Engineering Experiment Station was established at VPI.

During the 1920's, there was a considerable expansion of the work of the Agricultural Experiment Station, and with it came a need for adequate statistical and computing services, and the training of professional statisticians.

During the early 1930's, two separate installations with tabulating card processing equipment arrived on campus, one to serve the primary need of the Agricultural Experiment Station. It consisted of basic unit record installation with card punches, verifiers, sorters, reproducers, and tabulators. The other was primarily used by the business offices of VPI.

In 1946, undergraduate and graduate offerings in statistics were first announced in the curriculum of the Department: Agricultural Economics, Rural Sociology, and Statistics.

In 1948, a statistical laboratory was organized as a part of the Virginia Agricultural Experiment Station. Dr. Boyd Harshbarger was invited to organize the laboratory with the help of one assistant.

In 1949, the Department of Statistics was established in the School of Applied Science and Business Administration, and was authorized to offer a curriculum leading to the M.S. degree in statistics. In 1952 it added a curriculum leading to the Ph.D. degree. More recently it has offered selected courses for undergraduates in various departments, and since 1957, has a curriculum leading to the B.S. degree with a major in statistics.

In 1957 the Department of Statistics substantially expanded its present program with the aid of a grant from the National Institute of Health which provides for assistance to students as well as the staff.

In 1963, the Department of Statistics, in cooperation with the Department of Civil Engineering-Sanitary Engineering, initiated a training program in Environmental Engineering statistics.

Since 1963, the Department of Statistics belongs to the College of Arts and Sciences.

2.2 Purpose and organization

2.2.1 Purpose

The purposes of the Department of Statistics are:

1) To provide educational programs leading to careers in statistics, and to provide applied courses for research workers and students majoring in other fields.
2) To provide consulting and computing services in applied statistics.
3) To provide statistical research toward the development and extension of basic theory as well as the application of existing statistical techniques to applied problems in various fields.

2.2.2 The Department of Statistics in the organizational structure of VPI

The situation of the Department of Statistics within the present organizational structure of the Virginia Polytechnic Institute is shown by means of the chart No. 1.(29)

The Department includes its faculty and the Statistical Laboratory.

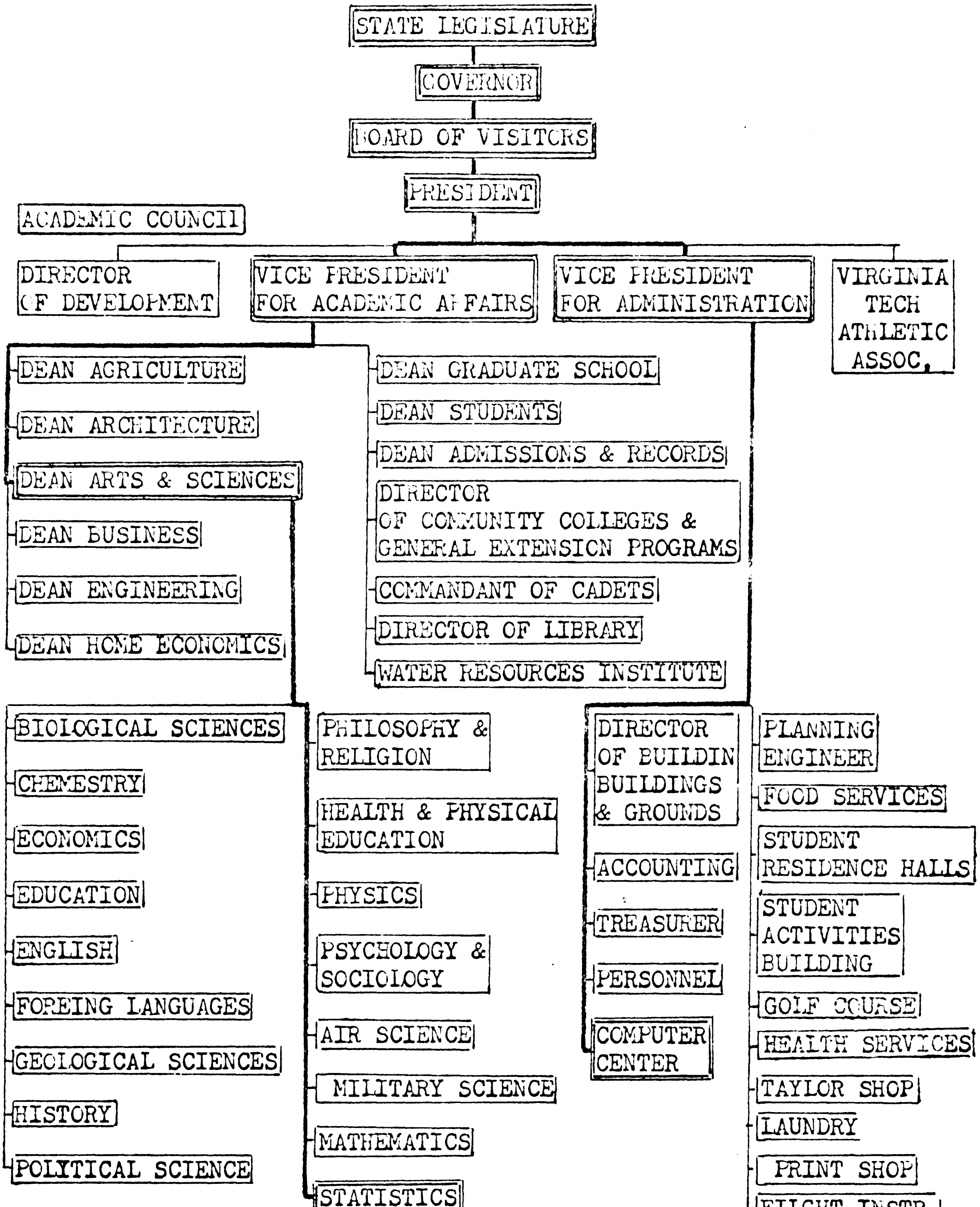

Chart 1. The Department of Statistics in the organizational Structure of VPI.

2.2.3 The faculty

The faculty of the Department consists of mathematical and applied statisticians who participate in teaching, research in statistical theory and methodology, and consulting service in applied statistics.

Below we list the present faculty members with a short biography and major field of interest.

	Major Field of Interest
Head of the Department of Statistics	
Boyd Harshbarger, B.A., M.S., M.A., Ph.D., D.Sc.	Design of Experiments
Visiting Professor	
Jerome Li, B.S., Ph.D.	Design of Experiments
Professors of Statistics	
Brian W. Connolly, B.A., M.A.	Stochastic Processes and Operations Research
Clyde Y. Kramer, B.S., M.S., Ph.D.	Design and Analysis of Experiments
Associate Professors of Statistics	
James P. Barrett, B.S., M.S., Ph.D.	Forestry and Sampling
Whitfield Cobb, A.B., A.M., Ph.D.	Teaching Applied Statistics
Klaus Hinkelmann, B.A., Ph.D.	Statistical Genetics

	Major Field of Interest
Richard G. Krutchkoff, A.B., A.M., Ph.D.	Use of Priori Information in Statistics
Raymod H. Myers, B.S., M.S., Ph.D.	Application to Engineering problems-Design and Analysis, Response Surface methods.
Whitney L. Johnson, B.S., M.S.	Digital Computing and Biostatistical Applications
<u>Assistant Professors of Statistics</u>	
D. R. Jensen, B.S., M.S., Ph.D.	Multivariate Inference
James Pickands, III, B.A., Ph.D.	Stochastic Processes
<u>Instructor of Statistics</u>	
Waldemar E. Heinzelman, B.S.	High speed computing

2.2.4 The Statistical Laboratory

The Statistical Laboratory consists of four trained computers, who provide clerk assistance for the faculty of the Department, the Virginia Agricultural Experiment Station, the Virginia Engineering Experiment Station, other Departments of VPI, Virginia Truck Station and other state and federal agencies. It is supervised by one statistician of the faculty. At the present time Dr. Clyde Kramer is in charge of the Statistical Laboratory.

2.3 Facilities

The Department of Statistics and the Statistical Laboratory occupy the fourth floor of Hutchinson Hall and Smith Hall, where they have offices for administration, staff, secretaries and graduate assistants, classrooms and the Statistical Laboratory.

The Statistical Laboratory is equipped with 4 full automatic desk calculators and one Mathatron electronic calculator, for a direct computing serve; 10 full automatic desk calculators for the use of staff and graduate assistants; and 19 non-full automatic desk calculators for student use.

The Department of Statistics also occupies two rooms of the second floor of the old Elementary School Building. This space is devoted to study desks for graduate students, and it is provided with two full automatic desk calculators.

2.4 Teaching

2.4.1 Courses

At first the Department of Statistics was authorized to offer a curriculum leading to the M.S. degree in statistics, and courses for graduate students majoring in other fields. In 1952 it added a curriculum leading to the Ph.D.

degree in statistics. More recently it has offered selected courses for undergraduates in various departments, and since 1957, has a curriculum leading to a B.S. degree with a major in statistics.

The following courses are offered by the Department of Statistics (27,31)

<u>Courses for Undergraduates</u>

201* Introductory statistics

202 Statistical Laboratory

301 Forestry statistics

302* Computer programming

310 Elementary statistics

313,323 Biological statistics

<u>Courses for graduates and undergraduates</u>

401 Educational statistics

402* Sample Survey methods

403* Experimental designs

404 Elementary econometrics

408* Digital computer techniques

410,420* Statistical methods

415,425,435 Statistics for engineers

419,429,439* Theoretical statistics

<u>Courses for graduates only</u>

5010 Linear Programming

5011 Advanced Econometrics

5012 Computing Techniques in Research

5013 Statistical Methods in Epidemiology

503** Statistical Inference

504** Theory of Linear Hypothesis

505** Probability

507 The Statistics of Biological Assay

508 Sample Survey Theory

516,526** Applied Statistics

517,527 Statistical Theory of Signal Detection

518,528,538 Graduate Seminar

536** Design and Analysis of Experiments

599** Research and Thesis

600 Directed Study

6010 Queuing Theory

601 Methods of Multivariate Analysis

602 Theory of Multivariate Analysis

603 Theory of Sequential Methods

604 Advanced Statistical Inference

605 Analysis of Variance

606 Nonparametric Statistics

607 The Theory and Application of Stochastic Processes

608 Advanced Analysis

609 Order Statistics

610 Response Surfaces

611,621 Statistical Genetics

612 Advanced Probability

799 Research and Dissertation

Courses indicated with one asterisk * are included in the requirements for a B.S. program majoring in statistics. Courses indicated with two asterisks ** are those from which selections are usually made for a M.S. program in statistics.

References (27), (30), (31) give more fully descriptions of the courses indicated above.

Reference (27) gives information about the undergraduate cirriculum, which provides the necessary preparation for students who after graduation expect to work on the master's or doctor's degree in statistics.

Students who expect to specialize in graduate work in statistics are advised to study as much mathematics as possible during their undergraduate work, but some experience in an area of application (physical sciences, agriculture, engineering, economics, biology, or psychology) is also highly desirable. As a rule, graduate students in statistics will have either a full minor in mathematics or a split minor in mathematics and some field of application.(30)

The requirements that a Master of Science and a Doctor of Philosophy program of study must meet are indicated below. (31)

Study Work		Credit hours required	
		M.S.	Ph.D.
Research and Thesis		9-18	
Research and Dissertation(599 &799)			50-70
Courses numbered 500 or higher(excluding courses in "Directed study", numbered 600)	minimum	18	50
Courses numbered 400 and above(including a maximum of 6 hours of Direct Study "600")	maximum	18	
(including a maximum of 9 hours of "Direct Study" "600")	maximum		30
Total Minimum Credit hours required		45	135

2.5 Consulting and Computing Service

2.5.1 Organizations for which consulting and computing service is provided

The Department of Statistics, through the Statistical Laboratory, provides both consulting and computing services to the following research organizations (26,28,29,30)

(1) The Virginia Agricultural Experiment Station of VPI, which includes the following Departments:

Agricultural Economics

Agricultural Engineering

Agronomy

Animal Science

Biochemistry and Nutrition

Biology

Clothing, Textiles, and Related Arts

Dairy Science

Entomology

Forestry and Wildlife

Horticulture

Human Nutrition and Foods

Plant Pathology and Physiology

Poultry Science

Veterinary Science

and the following Research Stations located in several places of the state of Virginia:

Beef Cattle Research Station, Front Royal

Eastern Virginia Research Station, Warsaw

Northern Virginia Pasture Research Station, Middleburg

Piedmont Research Laboratory, Charlottesville

Piedmont Research Station, Orange

Shenandoah Valley Research Station, Steeles Tavern

Southside Virginia Research Station, Charlotte

Southwest Virginia Research Station, Glade Spring

Tidewater Research Station, Holland

Tobacco Disease Research Station, Chatham

Virginia State College Research Station, Petersburg

Winchester Research Laboratory, Winchester Station

(2) The Virginia Engineering Experiment Station of VPI which includes the following Departments

Civil-Sanitary Engineering

Chemical Engineering

Electrical Engineering

Engineering Mechanics

Industrial Engineering

Materials Engineering Science

Wood Construction

Metals and Ceramic Engineering

Mechanical Engineering

(3) Other Departments of the Virginia Polytechnic Institute (included in the Engineering Experiment Station)

Chemistry

Physics and Nuclear Science

Vocational Education

Psychology and Sociology

Economics

Political Science

Business

(4) The Virginia Truck Experiment Station of Norfolk, although not an integral part of VPI is closely affiliated. The director there is a member of the VPI resident faculty.

(5) And other state and federal agencies, like

The National Institutes of Health

U.S. Army Research Office (Durham)

State Highway Department

State Industrial Division

U.S. Department of Agriculture

2.5.2 Consultation

As we said, the use of statistics as a tool in the scientific method starts from the very beginning of the planning of the experiments, thus much of the consulting service given by the faculty of the Department consists of:

1) Discussing the objectives of the research worker's experiments
2) Assisting in setting up a suitable design of the experiment which will furnish answers to the research worker's questions
3) Setting up appropriate plans for surveys
4) Helping in finding the appropriate statistical techniques for the analysis of the data
5) Discussing the results of the statistical analysis with the research worker
6) Assisting the research worker in writing his report and preparing technical papers.

2.5.3 Computing service of the Statistical Laboratory

The Statistical Laboratory provides clerical assistance for the staff of the Department of Statistics, the Virginia Agricultural Experimental Station, the Virginia Engineering Experiment Station, and all other organizations mentioned in 2.5.1. This assistance is provided in order to process data from research experiments and surveys, which require the use of full automatic desk calculators and involve time-consuming computations procedures for the research workers. The computation consist of analysis (most of them analysis of variance) of non-large quantities of data with non-complicated procedures, that can be done easily with the use of desk caculators rather than with elaborate special high-speed computer programs.

The Statistician in charge of the Statistical Laboratory and the Director of the Computer Center help the research worker in deciding where to send this data for computing service.

Requests for computation service are made directly to the Statistician in charge of the Statistical Laboratory, who reserves the right to approve of the methods of analysis. Most of the request are made by personal interview especially if the research worker is from VPI.

Once an analysis is approved, the Statistician in charge of the Statistical Laboratory outlines the necessary computations to be done and hands it to one of the computers. The result of the computations are written in duplicate on special sheets. The original revised copy is sent directly to the research worker and the copy is kept in the files of the laboratory.

The Statistical Laboratory does not keep records of the date when each statistical analysis was finished. This thesis includes a relation of the analyses computed in 1965 and the first seven months of 1966, approximately. Most of the analysis were analysis of variance. In Table 2.5.1 we are listing the different types of analysis of variance computed and their amount expressed in percentage over 3,327 analysis of variances accounted.

The percentage or proportion of each analysis expressed in Table 2.5.1 does not give a complete picture of the amount of work needed for each type, because some of them although few in number demanded more and complicated computations. In order to see this, we are indicating in the next chapter the corresponding model and scheme of analysis associated with each of these analyses of variance.

About 2 % of the total number of analyses of variance computed included missing value techniques. In general, for the case of several missing values, the general method described in (10) was applied.

As a general practice, after computing the analysis of variance according to the procedures for each design, treatment comparisons, when necessary, were carried out by using "Duncan's Multiple Range Test" procedure (8).

Also we found records of computations of the following statistical analyses:

Estimation and test of coefficients of correlations

Estimation of means, standard deviations

t test for paired data

Simple linear regression analysis

Combining ability analysis.

TABLE 2.5.1

	Analysis of variance	Percentage of analyses
1	One-way Classification (Completely Randomized Design)	4.40
2	One-way Classification with unequal numbers of observations per treatment	1.62
3	Two-way Classification with one Observation per cell	0.90
4	Two-way Classification with n observations per cell(axb Factorial in Completely Randomized Design)	2.83
5	Two-way Classification with n observations per cell and equal sampling (s samples/observation)	2.86
6	Two-way Classification with unequal numbers of observations per cell:	
	a) Case of proportional frequencies	2.56

(Table 2.5.1 continued)

	Analysis of variance	Percentage of analyses
	b) "Fitting Constants" method	0.63
	c) "Weighted Squares of means" method	0.27
7	Three-way Classification with one observation per cell	0.36
8	Three-way Classification with n observations per cell(axbxc Factorial in Completely Randomized Design)	9.77
9	Three-way Classification with n observations per cell and equal sampling(s samples/observation)	0.48
10	Multi-way Classification with one observation per cell(highest Interaction negligible)	0.12
11	Multi-way Classification with r observations per cell	0.03
12	Multi-way Classification with equal subsampling	0.18
13	Two-stage Nested Classification with equal samples	0.78
14	Two-stage Nested Classification with unequal samples	4.04
15	Three-stage Nested Classification with equal subsamples	0.03
16	Randomized Complete Block Design	
17	Randomized Complete Block Design with sampling	3.19
18	Randomized Complete Block Design with subsampling	0.12
19	Group of Randomized Complete Block Designs(each with a different level of a fixed factor)	7.51

(Table 2.5.1 continued)

	Analysis of variance	Percentage of analyses
20	Group of Randomized Complete Block Designs with sampling	0.12
21	Two-way Classified group of Randomized Complete Block Designs	0.30
22	axb Factorial in Randomized Complete Block Design	9.38
23	axb Factorial + Additional treatments in Randomized Complete Block Design	1.08
24	axb Factorial in Randomized Block Design with sampling	0.39
25	axb Factorial in Randomized Block Design with subsampling	0.24
26	Group of axb Factorials in Randomized Block Designs	0.30
27	axbxc Factorial in Randomized Block Design	0.78
28	axbxc Factorial + additional treatments in Randomized Block Design	0.18
29	axbxc Factorial in Randomized Block Design with sampling	0.69
30	Multi-Factorial in Randomized Block Design	0.15
31	Latin Square Design	0.45
32	Group of Latin Square Designs	0.15
33	Split-Plot Design	2.29
34	Split-Plot Design considering interactions between Repetitions and each of the two Factors	1.84
35	Split-Plot Design with sampling	0.12

(Table 2.5.1 continued)

	Analysis of variance	Percentage of analyses
36	Split-Plot Design with sampling considering interactions between Repetitions and each of the two Factors	0.75
37	Group of Split-Plot Designs	0.60
38	Split-Plot design with axb Factorial on the whole plots	0.96
39	Split-Plot design with axb Factorial on the Whole plots, considering interactions between Repetitions and each component of the axbxc Factorial	0.21
40	Split-Plot design with axb Factorial on the whole plots with sampling	0.81
41	Split-plot Design with axb Factorial on the whole plots with sampling. Interactions between Repetitions and each component of the axbxc Factorial	0.21
42	Split-Plot design with bxc Factorial on the sub-plots	0.21
43	Split-split-plot design	1.39
44	Split-split-plot design, considering interactions between Repetitions and each of the components of the axbxc Factorial	0.18
45	Split-split-plot design with sampling	0.36
46	Split-split-plot design. Interactions between Repetitions and each of the components of the axbxc Factorial with sampling	0.12
47	Split-split-plot design with cxd Factorial on the sub-subplots	0.06
48	Lattice designs	0.63

2.6 Research

The Department and Laboratory of Statistics are engaged in fundamental and applied research for the purpose of promoting use of efficient statistical techniques in diverse fields of research and advancing statistics by developing new procedures by theoretical investigation (28).

Among recent research conducted by the Department were projects concerning (28)

- The reliability program of the Redstone missile, (the first missile to send an American into space)
- Allocation of cancer patients to different treatments under comparison
- Relationships between the number and types of accidents and the number of types of physical conditions clinically diagnosed
- Statistical analysis of nutritions clinically diagnosed
- Statistical analysis of nutrition studies of preadolescent children
- Analysis of household food expenditures, collection of data on supplies, demands and shipments of livestock within the southern region and comparison of these data with those of other regions
- Simulation of nuclear reactors
- Development and tabulation of statistical functions
- Watershed drainage investigations
- Analyses of economic production functions
- Statistical techniques for the analysis of agricultural experiments

- Environmental Engineering Statistics, etc.

CHAPTER III

COMPUTATIONAL PROCEDURES FOR ANALYZING CERTAIN TYPES OF EXPERIMENTS

3.1 Mathematical Models

We shall now briefly indicate some of the basic concepts and notations used in the following tables in which we outline the analysis of the experiments considered in this thesis.

Models. Since early times, scientists have been using models to describe, demonstrate and predict events in the universe. Using mathematical models is one way of finding the relations between measurements which depend on several kinds of effects operating simultaneously.

Mathematical Models. Following Graybill's (11) definition, a mathematical model is an equation involving random variables, mathematical variables and parameters.

Linear Model. When the mathematical model is linear in the parameters and random variables then we have the so called linear model. This is the kind of model we shall be concerned with.

With regard to the analyses of variance, covariance and regression computed by the Statistical Laboratory, it was assumed that the experimental observations are random variables, which can be expressed in terms of a linear model of the following general form:

$$Y_j = \sum_{i=1}^{p} \theta_i X_{ij} + \epsilon_j \tag{1}$$

where Y_j = j-th observation $(j=1,2,\cdots,n)$

θ_i = parameter (unknown quantity) $(i=1,2,\cdots,p)$

X_{ij} = mathematical variable associated with the observation Y_j and the parameter θ_i (in classification models as used in this thesis the X_{ij} take on only the value of 1 or 0)

ϵ_j = error, associated with the j-th observation

and it was assumed that the ϵ_j are random variables, distributed independently with mean zero and common variance σ_ϵ^2. For test of significance they are assumed to follow the normal distribution.

For purposes of drawing inferences from the analysis of variance one has to distinguish between three types of linear models:

Fixed effect models are those models for which the parameters in (1) are assumed to represent fixed effects or unknown constants.

Random effect models are those models for which the parameters in (1), except the general mean μ, are assumed to represent random effects or random variables.

Mixed effect models are those models for which the parameters in (1) can be divided into two sets one of which contains parameters representing fixed effects and the other contains parameters representing random effects.

3.2 Analysis of variance schemes

In the following tables we indicate for special types of linear models the analysis of variance scheme including sums of squares, degrees of freedom, and expected mean squares corresponding to different factors or sources of variation considered in the model.

Much has been published on the techniques of analysis of variance and for many of the indicated experimental designs part or all of the analysis of variance scheme can be found in many well known books, like those by Anderson & Bancroft(1), Beyer(2), Bennett & Franklin(3), Brownlee(4), Cochran & Cox(5), Davies(6), Federer(9), Graybill(11), Goulden(12), Hald(13), Hicks(14), Huitson(15), Johnson & Leone(16), Kempthorne(17), Li(18), Ostle(20), Rao(21,22), Scheffe(23), Snedecor(24), Steel & Torrie(25), Wine(34),etc. But as Wilk and Kempthorne(33) mention, we will find sometimes that they do not agree with respect to certain rules and results concerning the expectations of mean squares and the choice of error terms, largely because explicit and objective methods for obtaining the appropriate model are not generally available.

For the analyses accounted here and the underlying models we shall state whether each parameter was assumed to be fixed or random. These assumptions determine the corresponding expected mean squares, and consequently the research worker can obtain estimates and tests of hypotheses regarding fixed effects or estimates and tests of hypotheses of variance components regarding random effects.

As we shall see, most of the models are of the mixed type, where fixed effects are related to "treatment effects" of those sources of variation of direct interest to the research worker, and random effects are related to sources of variation like repetitions, sampling, subsampling, etc. It should be noted here that the choice of model, i.e., the choice of assumptions concerning the parameters in the model, depends on the actual experimental situation.

The sums of squares are not affected by the assumptions whether the parameters are fixed or random. The same computing formulae can be used in both cases. However, changing the assumptions changes the expected mean squares, and it may well be that the experimental situation calls for assumptions other than those we have used for each analysis.

<u>General notation</u>:

$Y_{ij\ldots t}$ = individual observation or measurement where $ij\ldots t$ represent m subscripts, each of them indicates a particular level of each of the m factors, under which the observation was taken.

μ = general mean (population mean).

$\epsilon_{ij\ldots t}$ = experimental error, deviation of the actual value of the observation $Y_{ij\ldots t}$ from the true value, due to measurement error and/or other sources of variation not considered in the model (always considered as <u>random variable</u>).

Greek letters (other than μ and ϵ) are used to denote parameters that express the effect of factors under study, i.e.

$\alpha_i, \beta_j, \gamma_k$, etc. denote parameters that express the effect of i-th, j-th, k-th, etc., levels of factors A, B, C, etc., respectively.

$(\alpha\beta)_{ij}$ denotes parameters that express the effect of interaction between the i-th level of factor A and j-th level of factor B.

$(\alpha\beta\gamma)_{hij}$ denotes the effect of interaction between h-th, i-th, and j-th levels of factors A, B, and C respectively.

etc.

Source = Source of Variation

d.f. = degrees of freedom

S.S. = sum of squares

M.S. = mean square

E(M.S.) = expected mean square

To denote the number of observations for particular factor levels or level combinations we use the following general notation:

$$n_{ij\ldots t} = 1$$

$$n_{i\ldots} = \sum_{j,\ldots,t} n_{ij\ldots t}$$ = number of observations for the i-th level of factor A.

$$n_{ij\ldots} = \sum_{k,\ldots,t} n_{ijk\ldots t}$$ = number of observations for the i-th level of factor A and the j-th level of factor B.

etc.

$$N = \sum_{i,j,\ldots,t} n_{ij\ldots t}$$ = total number of observations.

In most cases, however, we shall use a simpler notation as indicated for every model.

Sum of Squares Notation. In defining the sum of squares (S.S.) we shall use the following notation. If an individual observation is denoted by $Y_{ijk\ldots t}$, where ijk...t represent m subscripts, then any particular mean taken over a subset of q subscripts ($q \leq m$) is denoted by a lower case y omitting these q subscripts, but retaining the m-q subscripts. For example, if an individual observation is denoted by Y_{ij}, then

$$y_i = \sum_j Y_{ij}/n_{i.}$$

$$y = \sum_i \sum_j Y_{ij}/N$$

if the individual observation is denoted by Y_{hijk}, then

$$y = \sum_h \sum_i \sum_j \sum_k Y_{hijk}/N$$

$$y_j = \sum_h \sum_i \sum_k Y_{hijk}/n_{..j.}$$

$$y_{ik} = \sum_h \sum_j Y_{hijk}/n_{.i.k}$$

etc.

The greek letter Σ will denote the summation over the ranges of <u>all</u> subscripts defining an observation Y in the model.

<u>Expected Mean Squares Notation</u>. In the expression for the expected mean squares (E(M.S.)), Σ will denote the summation over the ranges of the subscripts defining the respective factors or source of variation. Further,

σ_ϵ^2 = experimental error variance

σ_β^2 = variance of the random variables $\beta_j (j=1,2,\cdots,b)$

$\sigma_{\alpha\beta}^2$ = variance of the random variables $(\alpha\beta)_{ij}$ $(i=1,2,\cdots,a;\ j=1,2,\cdots,b)$

etc.

TABLE 3.2.1

One-way classification

Model:

$$Y_{ij} = \mu+\alpha_i+\epsilon_{ij}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,n)$$

where α_i = fixed effect

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Among treatments	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+n\Sigma\alpha_i^2/(a-1)$
Within treatments	a(n-1)	$\Sigma(Y_{ij}-y_i)^2$	σ_ϵ^2
Total	na-1	$\Sigma(Y_{ij}-y)^2$	

TABLE 3.2.2

One-way classification with unequal observations per treatment

Model:

$$Y_{ij} = \mu+\alpha_i+\epsilon_{ij}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,n_i)$$

where α_i = fixed effect

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Among treatments	a-1	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+\Sigma n_i\alpha_i^2/(a-1)$
Between treatments	N-a	$\Sigma(Y_{ij}-y_i)^2$	σ_ϵ^2
Total	N-1	$\Sigma(Y_{ij}-y)^2$	

TABLE 3.2.3

Two-way classification with one observation per cell

negligible interaction

Model:

$$Y_{ij} = \mu+\alpha_i+\beta_j+\epsilon_{ij}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,b)$$

$$\alpha_i,\ \beta_j = \text{fixed effects}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+b\Sigma\alpha_i^2/(a-1)$
B	(b-1)	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+a\Sigma\beta_j^2/(b-1)$
Error	(a-1)·(b-1)	$\Sigma(Y_{ij}-y_i-y_j+y)^2$	σ_ϵ^2
Total	ab-1	$\Sigma(Y_{ij}-y)^2$	

TABLE 3.2.4

Two-way classification with n observations per cell

Model: $$Y_{ijk} = \mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\epsilon_{ijk}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,b;\ k=1,2,\cdots,n)$$

$$\alpha_i,\ \beta_j,\ (\alpha\beta)_{ij} = \text{fixed effects}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+bn\Sigma\alpha_i^2/(a-1)$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+an\Sigma\beta_j^2/(b-1)$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+n\Sigma(\alpha\beta)_{ij}^2/(a-1)(b-1)$
Error	$ab(n-1)$	$\Sigma(Y_{ijk}-y_{ij})^2$	σ_ϵ^2
Total	$abn-1$	$\Sigma(Y_{ijk}-y)^2$	

TABLE 3.2.5

Two-way classification with n observations per cell and equal sampling

Model:

$$Y_{ijkl} = \mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\epsilon_{ijk}+\varphi_{ijkl}$$

$(i=1,2,3,\cdots,a;j=1,2,\cdots,b;k=1,2,\cdots,n;l=1,2,\cdots,s)$

α_i, β_j, $(\alpha\beta)_{ij}$ = fixed effects

ϵ_{ijk}, φ_{ijkl} = random effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+bns\frac{\Sigma\alpha_i^2}{(a-1)}$
B	(b-1)	$\Sigma(y_j-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+ans\frac{\Sigma\beta_j^2}{(b-1)}$
AB	(a-1)·(b-1)	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+ns\frac{\Sigma(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
Rep. within cell	ab(n-1)	$\Sigma(y_{ijk}-y_{ij})^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2$
Sample within rep.	abn(s-1)	$\Sigma(Y_{ijkl}-y_{ijk})^2$	σ_φ^2
Total	abns-1	$\Sigma(Y_{ijkl}-y)^2$	

TABLE 3.2.6

Two-way classification with unequal numbers of observations per cell

Model:

$$Y_{ijk} = \mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\epsilon_{ijk}$$

$$(i=1,2,\ldots a;\ j=1,2,\ldots,b;\ k=1,2,\ldots,n_{ij})$$

$$\alpha_i,\beta_j,(\alpha\beta)_{ij} = \text{fixed effects}$$

a) Method of proportionate subclass numbers*

(where it is assumed $n_{ij} = n_{i.}n_{.j}/n_{..}$)

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	$(a-1)$	$\Sigma n_{ij}(y_i-y)^2$	$\sigma_\epsilon^2+\frac{\Sigma n_{i.}\alpha_i^2}{(a-1)}$
B	$(b-1)$	$\Sigma n_{ij}(y_j-y)^2$	$\sigma_\epsilon^2+\frac{\Sigma n_{.j}\beta_j^2}{(b-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma n_{ij}(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+\frac{\Sigma n_{ij}(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
Error	$N-ab$	$\Sigma(Y_{ijk}-y_{ij})^2$	σ_ϵ^2
Total	$N-1$	$\Sigma(Y_{ijk}-y)^2$	

* reference (19) indicates a χ^2 test for testing this proportionality.

(Table 3.2.6 continued.)

b) Method of fitting constants

Analysis of variance: 1

Source	d.f.	S.S.	E(M.S.)
B (unadjusted)	(b-1)	$\Sigma(y_j - y)^2$	
A (adjusted)	(a-1)	Q*	$\sigma^2 + \sum_i n_{i.}\alpha_i^2 - \sum_j \frac{(\sum_i n_{ij}\alpha_i)^2}{n_{.j}}$ **
AB	(a-1) $\cdot$(b-1)*	I	$\sigma_\epsilon^2 + f((\alpha\beta)_{ij}\text{'s})$
Between cells	(ab-1)*	$\Sigma(y_{ij} - y)^2$	
Error	(N-ab)*	$\Sigma(Y_{ijk} - y_{ij})^2$	σ_ϵ^2
Total	N-1	$\Sigma(Y_{ijk} - y)^2$	

Analysis of variance: 2

Source	d.f.	S.S.	E(M.S.)
A (unadjusted)	(a-1)	$\Sigma(y_j - y)^2$	
B (adjusted)	(b-1)	R*	$\sigma_\epsilon^2 + \sum_j n_{.j}\beta_{.j} - \sum_i \frac{(\sum_j n_{ij}\beta_j)^2}{n_{i.}}$ **
AB	(a-1) $\cdot$(b-1)*	I	$\sigma_\epsilon^2 + f((\alpha\beta)_{ij}\text{'s})$

*Unless one or more cells are empty.

**When Interaction AB is found to be negligible.

(Table 3.2.6 continued.)

(Analysis of variance:2 continued.)

Source	d.f.	S.S.	E(M.S.)
Between cells	(ab-1)*	$\Sigma(y_{ij}-y)^2$	
Error	(N-ab)*	$\Sigma(Y_{ijk}-y_{ij})^2$	σ_ϵ^2
Total	N-1	$\Sigma(Y_{ijk}-y)^2$	

where:

$$Q = \sum_i \hat{\alpha}_i g_i$$

$\hat{\alpha}_i$ is solution of $C\underline{\hat{\alpha}} = \underline{G}$

$$\underline{\hat{\alpha}}' = (\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_a)$$

$$\underline{G}' = (g_1, g_2, \ldots, g_a)$$

$$g_i = Y_{i..} - \sum_j (n_{ij} Y_{.j.}/n_{.j})$$

$$C = [c_{ii'}]$$

$$c_{ii'} = \delta_{ii'} n_i - \sum_t (n_{it} n_{i't}/n_{.t}), \quad (t=1,2,\ldots,b)$$

$$I = S.S._{(\text{Between cells})} - S.S._{A(\text{adj.})} - S.S._{B(\text{unadj.})}$$

$$R = S.S._{(\text{Between cells})} - S.S._{A(\text{unadj.})} - I$$

*Unless one or more cells are empty.

(Table 3.2.6 continued.)

c) Method of Weighted Squares of means

Analysis of variance:

Source	d.f.	S.S.
A	(a-1)	$b^2 \sum_i w_i(\sum_j y_{ij.}/b)^2 - \frac{(\sum_i w_i(\sum_j y_{ij.}/b))^2}{\sum_i w_i}$
B	(b-1)	$a^2 \sum_j v_j(\sum_i y_{ij.}/a)^2 - \frac{(\sum_j v_j(\sum_i y_{ij.}/a))^2}{\sum_j v_j}$
AxB	(a-1) ·(b-1)	$\Sigma y^2_{ij.} - \sum_i Y_{i..}\hat{\alpha}_i - \sum_j (Y^2_{.j.}/n_{.j})$
Between cells	ab-1	$\Sigma(y_{ij}-y)^2$
Error	N-ab	$\Sigma(Y_{ijk}-y_{ij})^2$
Total	N-1	$\Sigma(Y_{ijk}-y)^2$

where:

$w_i = 1/\sum_i(1/n_{ij})$

$v_j = 1/\sum_j(1/n_{ij})$

$\hat{\alpha}_i$ = a solution of $C\hat{\alpha} = Q$

(C, $\hat{\alpha}$ and Q are the same as in the Fitted constant method.)

TABLE 3.2.7

Three-way classification with one observation per cell

Model:
$$Y_{ijk} = \mu+\alpha_i+\beta_j+\gamma_k+(\alpha\beta)_{ij}+(\alpha\gamma)_{ik}+(\beta\gamma)_{jk}+\epsilon_{ijk}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,b;\ k=1,2,\cdots,c)$$

α_i, β_j, γ_k, $(\alpha\beta)_{ij}$, $(\alpha\gamma)_{ik}$, $(\beta\gamma)_{jk}$ = fixed effects

(Table 3.2.7 continued.)

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+bc\frac{\Sigma\alpha_i^2}{(a-1)}$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+ac\frac{\Sigma\beta_j^2}{(b-1)}$
C	$(c-1)$	$\Sigma(y_k-y)^2$	$\sigma_\epsilon^2+ab\frac{\Sigma\gamma_k^2}{(c-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+c\frac{\Sigma(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma_\epsilon^2+b\frac{\Sigma(\alpha\gamma)_{ik}^2}{(a-1)(c-1)}$
BC	$(b-1)\cdot(c-1)$	$\Sigma(y_{jk}-y_j-y_k+y)^2$	$\sigma_\epsilon^2+a\frac{\Sigma(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$
Residual	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(Y_{ijk}-y_{ij}-y_{ik}-y_{jk}+y_i+y_j+y_k-y)^2$	σ_ϵ^2
Total	$abc-1$	$\Sigma(Y_{ijk}-y)^2$	

TABLE 3.2.8

Three-way classification with n observations per cell

Model: $Y_{hijk} = \mu+\alpha_h+\beta_i+\gamma_j+(\alpha\beta)_{hi}+(\alpha\gamma)_{hj}+(\beta\gamma)_{ij}+(\alpha\beta\gamma)_{hij}+\epsilon_{hijk}$

$(h=1,2,\cdots,a;\ i=1,2,\cdots,b;\ j=1,2,\cdots,c;\ k=1,2,\cdots,n)$

α_h, β_i, γ_j, $(\alpha\beta)_{hi}$, $(\alpha\gamma)_{hj}$, $(\alpha\beta\gamma)_{hij}$ = fixed effects

(Table 3.2.8 continued.)

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	(a-1)	$\Sigma(y_h - y)^2$	$\sigma_\epsilon^2 + nbc\frac{\Sigma\alpha_n^2}{(a-1)}$
B	(b-1)	$\Sigma(y_i - y)^2$	$\sigma_\epsilon^2 + nac\frac{\Sigma\beta_i^2}{(b-1)}$
C	(c-1)	$\Sigma(y_j - y)^2$	$\sigma_\epsilon^2 + nab\frac{\Sigma\gamma_j^2}{(c-1)}$
AB	(a-1) ·(b-1)	$\Sigma(y_{hi} - y_h - y_i + y)^2$	$\sigma_\epsilon^2 + nc\frac{\Sigma(\alpha\beta)_{hi}^2}{(a-1)(b-1)}$
AC	(a-1) ·(c-1)	$\Sigma(y_{hj} - y_h - y_j + y)^2$	$\sigma_\epsilon^2 + nb\frac{\Sigma(\alpha\gamma)_{hj}^2}{(a-1)(c-1)}$
BC	(b-1) ·(c-1)	$\Sigma(y_{ij} - y_i - y_j + y)^2$	$\sigma_\epsilon^2 + na\frac{\Sigma(\beta\gamma)_{ij}^2}{(b-1)(c-1)}$
ABC	(a-1) ·(b-1) ·(c-1)	$\Sigma(y_{hij} - y_{hi} - y_{hj} - y_{ij} + y_h + y_i + y_j - y)^2$	$\sigma_\epsilon^2 + n\frac{\Sigma(\alpha\beta\gamma)_{hij}^2}{(a-1)(b-1)(c-1)}$
Error	abc(n-1)	$\Sigma(Y_{hijk} - y_{hij})^2$	σ_ϵ^2
Total	abcn-1	$\Sigma(Y_{hijk} - y)^2$	

TABLE 3.2.9

Three-way Classification with n observations

per cell and equal samples

Model:

$$Y_{hijkl} = \mu+\alpha_h+\beta_i+\gamma_j+(\alpha\beta)_{hi}+(\alpha\gamma)_{hj}+(\beta\gamma)_{ij}+(\alpha\beta\gamma)_{hij}+\epsilon_{hijk}+\varphi_{hijkl}$$

$(h=1,2,\cdots,a;i=1,2,\cdots,b;j=1,2,\cdots,c;k=1,2,\cdots,n;l=1,2,\cdots,s)$

α_h, β_i, γ_j, $(\alpha\beta)_{hi}$, $(\alpha\gamma)_{hj}$, $(\beta\gamma)_{ij}$, $(\alpha\beta\gamma)_{hij}$ = fixed effects

ϵ_{hijk}, φ_{hijkl} = random effects

(Table 3.2.9 continued.)

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	$(a-1)$	$\Sigma(y_h-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{nsbc}{(a-1)}\Sigma\alpha_h^2$
B	$(b-1)$	$\Sigma(y_i-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{nsac}{(b-1)}\Sigma\beta_i^2$
C	$(c-1)$	$\Sigma(y_j-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{nsab}{(c-1)}\Sigma\gamma_j^2$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{nsc}{(a-1)(b-1)}\Sigma(\alpha\beta)_{hi}^2$
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{hj}-y_h-y_j+y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{nsb}{(a-1)(c-1)}\Sigma(\alpha\gamma)_{hj}^2$
BC	$(b-1)\cdot(c-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{nsa}{(b-1)(c-1)}\Sigma(\beta\gamma)_{ij}^2$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{hij}-y_{hi}-y_{hj}-y_{ij}+y_h+y_i+y_j-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{ns}{(a-1)(b-1)(c-1)}\cdot\Sigma(\alpha\beta\gamma)_{hij}^2$
Reps. within cells	$abc\cdot(n-1)$	$\Sigma(y_{hijk}-y_{hij})^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2$
Sample within reps.	$abcn\cdot(s-1)$	$\Sigma(Y_{hijkl}-y_{hijk})^2$	σ_φ^2
Total	$abcns-1$	$\Sigma(Y_{hijkl}-y)^2$	

TABLE 3.2.10

Multiple-way classification with one observation per cell

(Highest interaction negligible)

Model:

$$Y_{gh\cdots mnp} = \mu+\alpha_g+\beta_h+\cdots+\varphi_p+(\alpha\beta)_{gh}+\cdots+(\lambda\varphi)_{np}+(\alpha\beta\gamma)_{ghi}+\cdots$$

$$+(\delta\lambda\varphi)_{mnp}+\cdots+(\beta\gamma\cdots\lambda\varphi)_{hi\cdots np}+\epsilon_{gh\cdots np}$$

$(g=1,2,\cdots,a;h=1,2,\cdots,b;i=1,2,\cdots,c;\ldots;m=1,2,\cdots,v;n=1,2,\cdots,w;p=1,2,\cdots,z)$

$ghi\cdots mnp$ are a set of t subscripts,

(t=number of factors)

$\alpha_g,\beta_h,\cdots,\varphi_p,(\alpha\beta)_{gh},\cdots,(\lambda\varphi)_{np},(\alpha\beta\gamma)_{ghi},\cdots,(\beta\gamma\cdots\lambda\varphi)_{hi\cdots np}$ = fixed effects.

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	$(a-1)$	$\Sigma(y_g-y)^2$	$\sigma_\epsilon^2+\frac{b\cdots wz\Sigma\alpha_g^2}{(a-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Z	$(z-1)$	$\Sigma(y_p-y)^2$	$\sigma_\epsilon^2+\frac{a\cdots w\Sigma\varphi_p^2}{(z-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{gh}-y_g-y_h+y)^2$	$\sigma_\epsilon^2+\frac{c\cdots wz\Sigma(\alpha\beta)_{gh}^2}{(a-1)(b-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
WZ	$(w-1)\cdot(z-1)$	$\Sigma(y_{np}-y_n-y_p+y)^2$	$\sigma_\epsilon^2+\frac{a\cdots uv\Sigma(\lambda\varphi)_{np}^2}{(w-1)(z-1)}$

(Table 3.2.10 continued.)

Source	d.f.	S.S.	E(M.S.)
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{ghi}-y_{gh}-y_{gi}-y_{hi}+y_g+y_h+y_i-y)^2$	$\sigma_\epsilon^2+\frac{d\cdots wz\Sigma(\alpha\beta\gamma)^2_{ghi}}{(a-1)(b-1)(c-1)}$
⋮	⋮	⋮	⋮
VWZ	$(v-1)\cdot(w-1)\cdot(z-1)$	$\Sigma(y_{mnp}-y_{mn}-y_{mp}-y_{np}+y_m+y_n+y_p+y)^2$	$\sigma_\epsilon^2+\frac{a\cdots u\Sigma(\delta\lambda\varphi)^2_{mnp}}{(v-1)(w-1)(z-1)}$
⋮	⋮	⋮	⋮
BC···VWZ	$(b-1)\cdot(c-1)\cdots\cdot(w-1)\cdot(z-1)$	$\Sigma(y_{hi\ldots np}-y_{hi..n}-\cdots-y_{i\ldots np}+y_{hi\ldots m}+\cdots+y_{j\ldots np}-\cdots+(-1)^{t-1}y)^2$	$\sigma_\epsilon^2+\frac{a\Sigma(\beta\cdots\varphi)^2_{h\ldots p}}{(b-1)\cdots(z-1)}$
Error	$(a-1)\cdot(b-1)\cdots\cdot(w-1)\cdot(z-1)$	$\Sigma(Y_{gh\ldots np}-y_{gh\ldots n}-\cdots-y_{h\ldots np}+y_{gh\ldots n}+\cdots+y_{i\ldots np}-\cdots+(-1)^t y)^2$	σ_ϵ^2
Total	$ab\cdots z-1$	$\Sigma(Y_{gh\ldots np}-y)^2$	

TABLE 3.2.11

Multiple-way classification with r samples per cell

Model:

$$Y_{gh\ldots mnpq} = \mu+\alpha_g+\beta_h+\ldots+\varphi_p+(\alpha\beta)_{gh}+\ldots+(\lambda\varphi)_{np}+(\alpha\beta\gamma)_{ghi}+\ldots$$

$$+(\delta\lambda\varphi)_{mnp}+\ldots+(\alpha\beta\gamma\ldots\delta\lambda\varphi)_{gh\ldots np}+\epsilon_{gh\ldots npq}$$

$(g=1,2,\ldots,a;h=1,2,\ldots,b;i=1,2,\ldots,b;\ldots;m=1,2,\ldots,v;n=1,2,$
$\ldots,w;p=1,2,\ldots,z;q=1,2,\ldots,r)$

$ghi\ldots mnp$ are a set of t subscripts (t=number of factors)

$\alpha_g,\beta_h,\ldots,\varphi_p,(\alpha\beta)_{gh},\ldots,(\lambda\varphi)_{np},(\alpha\beta\gamma)_{ghi},\ldots,(\alpha\beta\gamma\ldots\delta\lambda\varphi)_{gh..np}$
= fixed effects.

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	(a-1)	$\Sigma(y_g-y)^2$	$\sigma_\epsilon^2+\frac{rb\ldots wz\Sigma\alpha_g^2}{(a-1)}$
.	.	.	.
.	.	.	.
.	.	.	.
Z	(z-1)	$\Sigma(y_p-y)^2$	$\sigma_\epsilon^2+\frac{ra\ldots w\Sigma\varphi_p^2}{(z-1)}$
AB	(a-1) ·(b-1)	$\Sigma(y_{gh}-y_g-y_h+y)^2$	$\sigma_\epsilon^2+\frac{rc\ldots wz\Sigma(\alpha\beta)_{gh}^2}{(a-1)(b-1)}$
.	.	.	.
.	.	.	.
.	.	.	.

(Table 3.2.11 continued.)

Source	d.f.	S.S.	E(M.S.)
WZ	$(w-1)\cdot(z-1)$	$\Sigma(y_{np}-y_n-y_p+y)^2$	$\sigma_\epsilon^2+\frac{ra\ldots uv\Sigma(\lambda\varphi)^2_{np}}{(w-1)(z-1)}$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{ghi}-y_{gh}-y_{gi}-y_{hi}+y_g+y_h+y_i-y)^2$	$\sigma_\epsilon^2+\frac{rd\ldots wz\Sigma(\alpha\beta\gamma)^2_{ghi}}{(a-1)(b-1)(c-1)}$
⋮	⋮	⋮	⋮
VWZ	$(v-1)\cdot(w-1)\cdot(z-1)$	$\Sigma(y_{mnp}-y_{mn}-y_{mp}-y_{np}+y_m+y_n+y_p-y)^2$	$\sigma_\epsilon^2+\frac{ra\ldots u\Sigma(\delta\lambda\varphi)^2_{mnp}}{(v-1)(w-1)(z-1)}$
⋮	⋮	⋮	⋮
ABC...VWZ	$(a-1)\cdot(b-1)\cdots\cdot(w-1)\cdot(z-1)$	$\Sigma(y_{gh\ldots np}-y_{gh\ldots n}-\ldots-y_{h\ldots np}+y_{gh\ldots m}+\ldots+y_{i\ldots np}-\ldots+(-1)^t y)^2$	$\sigma_\epsilon^2+\frac{r\Sigma(\alpha\ldots\varphi)^2_{gh\ldots p}}{(a-1)\ldots(z-1)}$
Error	$abc\ldots wz\cdot(r-1)$	$\Sigma(Y_{gh\ldots npq}-y_{gh\ldots np})^2$	σ_ϵ^2
Total	$abc..wzr-1$	$\Sigma(Y_{gh\ldots npq}-y)^2$	

TABLE 3.2.12

Multiple-way classification with equal subsampling

Model:

$$Y_{gh\ldots mnpqk} = \mu+\alpha_g+\ldots+\varphi_p+(\alpha\beta)_{gh}+\ldots+(\lambda\varphi)_{np}+(\alpha\beta\gamma)_{ghi}+\ldots$$
$$+(\delta\lambda\varphi)_{mnp}+\ldots+(\alpha\beta\gamma\ldots\delta\lambda\varphi)_{gh\ldots np}+\epsilon_{gh\ldots pq}$$
$$+\theta_{gh\ldots pqk}$$

$(g=1,2,\ldots,a;h=1,2,\ldots,b;i=1,2,\ldots,c;\ldots;m=1,2,\ldots,v;n=1,$
$2,\ldots,w;p=1,2,\ldots,z;q=1,2,\ldots,r;k=1,2,\ldots,s)$

$\alpha_g,\beta_h,\ldots,\varphi_p,(\alpha\beta)_{gh},\ldots,(\lambda\varphi)_{np},(\alpha\beta\gamma)_{ghi},\ldots,(\alpha\beta\gamma\ldots\delta\lambda\varphi)_{gh\ldots np}$

= fixed effects

$\epsilon_{gh\ldots pq},\theta_{gh\ldots pqk}$ = random effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	$(a-1)$	$\Sigma(y_g-y)^2$	$\sigma^2_\theta+s\sigma^2_\epsilon+\frac{rsb\ldots wz\Sigma\alpha^2_g}{(a-1)}$
.	.	.	.
.	.	.	.
.	.	.	.
Z	$(z-1)$	$\Sigma(y_p-y)^2$	$\sigma^2_\theta+s\sigma^2_\epsilon+\frac{rsa\ldots w\Sigma\varphi^2_p}{(z-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{gh}-y_g-y_h+y)^2$	$\sigma^2_\theta+s\sigma^2_\epsilon+\frac{rsc..wz\Sigma(\alpha\beta)^2_{gh}}{(a-1)(b-1)}$
.	.	.	.
.	.	.	.
.	.	.	.

(Table 3.2.12 continued.)

Source	d.f.	S.S.	E(M.S.)
WZ	$(w-1)\cdot(z-1)$	$\Sigma(y_{np}-y_n-y_p+y)^2$	$\sigma_\theta^2+s\sigma_\epsilon^2+\frac{rsa\ldots uv\Sigma(\lambda\varphi)^2_{np}}{(w-1)(z-1)}$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{ghi}-y_{gh}-y_{gi}-y_{hi}+y_g+y_h+y_i-y)^2$	$\sigma_\theta^2+s\sigma_\epsilon^2+\frac{rsd\ldots wz\Sigma(\alpha\beta\gamma)^2_{ghi}}{(a-1)(b-1)(c-1)}$
⋮	⋮	⋮	⋮
VWZ	$(v-1)\cdot(w-1)\cdot(z-1)$	$\Sigma(y_{mnp}-y_{mn}-y_{mp}-y_{np}+y_m+y_n+y_p-y)^2$	$\sigma_\theta^2+s\sigma_\epsilon^2+\frac{rsa\ldots u\Sigma(\delta\lambda\varphi)^2_{mnp}}{(v-1)(w-1)(z-1)}$
⋮	⋮	⋮	⋮
ABC...VWZ	$(a-1)\cdot(b-1)\cdots\cdot(w-1)\cdot(z-1)$	$\Sigma(y_{gh\ldots np}-y_{gh\ldots n}-\ldots-y_{h\ldots np}+y_{gh\ldots n}+\ldots+y_{i\ldots np}-\ldots+(-1)^t y)^2$	$\sigma_\theta^2+s\sigma_\epsilon^2+\frac{rs\Sigma(\alpha\beta\ldots\varphi)^2_{gh\ldots p}}{(a-1)\ldots(z-1)}$
Experimental Error	$ab\ldots wz\cdot(r-1)$	$\Sigma(y_{gh\ldots npq}-y_{gh\ldots np})^2$	$\sigma_\theta^2+s\sigma_\epsilon^2$
Sampling Error	$ab\ldots wzr\cdot(s-1)$	$\Sigma(Y_{gh\ldots pqk}-y_{gh\ldots pq})^2$	σ_θ^2
Total	$abc\ldots wzrs-1$	$\Sigma(Y_{gh\ldots pqk}-y)^2$	

TABLE 3.2.13

Two-stage nested classification with equal sampling

Model: $$Y_{hij} = \mu+\alpha_h+\beta_{hi}+\gamma_{hij}$$

$$(h=1,2,\cdots,a;\ i=1,2,\cdots,b;\ j=1,2,\cdots,c)$$

α_h = fixed effect

β_{hi}, γ_{hij} = random effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	a-1	$\Sigma(y_h-y)^2$	$\sigma^2_\gamma+c\sigma^2_\beta+\frac{bc}{a-1}\Sigma\alpha^2_i$
B within A	a(b-1)	$\Sigma(y_{hi}-y_h)^2$	$\sigma^2_\gamma+c\sigma^2_\beta$
C within B	ab(c-1)	$\Sigma(Y_{hij}-y_{hi})^2$	σ^2_γ
Total	abc-1	$\Sigma(Y_{hij}-y)^2$	

TABLE 3.2.14

Two-stage nested classification with unequal subsamples

Model:

$$Y_{hij} = \mu+\alpha_h+\beta_{hi}+\gamma_{hij}$$

$$(h=1,2,\cdots,a;\ i=1,2,\cdots,b;\ j=1,2,\cdots,n_{ij})$$

$$\alpha_h = \text{fixed effect}$$

$$\beta_{hi},\gamma_{hij} = \text{random effects}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	(a-1)	$\Sigma(y_h-y)^2$	$\sigma_\gamma^2+k_2\sigma_\beta^2+\Sigma n_{i.}\alpha_i^2/(a-1)$
B within A	a(b-1)	$\Sigma(y_{hi}-y_h)^2$	$\sigma_\gamma^2+k_1\sigma_\beta^2$
C within B	N-ab	$\Sigma(Y_{hij}-y_{hi})^2$	σ_γ^2
Total	N-1	$\Sigma(Y_{hij}-y)^2$	

where

$$k_1 = \left[\sum_{ij} n_{ij} - \sum_i(\sum_j n_{ij}^2/\sum_j n_{ij})\right]/a(b-1)$$

$$k_2 = \left[\sum_i(\sum_j n_{ij}^2/\sum_j n_{ij}) - \sum_{ij} n_{ij}^2/\sum_{ij} n_{ij}\right]/(a-1)$$

TABLE 3.2.15

Three-stage nested classification with equal sub-samples

Model: $$Y_{hijk} = \mu+\alpha_h+\beta_{hi}+\gamma_{hij}+\delta_{hijk}$$

$$(h=1,2,\cdots,a;\ i=1,2,\cdots,b;\ j=1,2,\cdots,c;\ k=1,2,\cdots,n)$$

$$\alpha_h = \text{fixed effect}$$

$$\beta_{h\,i},\ \gamma_{hi\,j},\ \delta_{hij\,k} = \text{random effects}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	a-1	$\Sigma(y_h-y)^2$	$\sigma_\delta^2+n\sigma_\gamma^2+cn\sigma_\beta^2+bcn\frac{\Sigma\alpha_i^2}{(a-1)}$
B within A	a(b-1)	$\Sigma(y_{hi}-y_h)^2$	$\sigma_\delta^2+n\sigma_\gamma^2+cn\sigma_\beta^2$
C within B	ab(c-1)	$\Sigma(y_{hij}-y_{hi})^2$	$\sigma_\delta^2+n\sigma_\gamma^2$
D within C	abc(n-1)	$\Sigma(Y_{hijk}-y_{hij})^2$	σ_δ^2
Total	abcn-1	$\Sigma(Y_{hijk}-y)^2$	

TABLE 3.2.16

Randomized Complete Block Design

Model:

$$Y_{ij} = \mu+\alpha_i+\rho_j+\epsilon_{ij}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,r)$$

α_i = fixed effect

ρ_j = random effect

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Repetitions	(r-1)	$\Sigma(y_j-y)^2$	
Treatments	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+r\frac{\Sigma\alpha_i^2}{a-1}$
Error	(r-1)·(a-1)	$\Sigma(Y_{ij}-y_i-y_j+y)^2$	σ_ϵ^2
Total	ra-1	$\Sigma(Y_{ij}-y)^2$	

TABLE 3.2.17.a

Randomized Complete Block Design with sampling

Model: $$Y_{ijk} = \mu+\alpha_i+\rho_j+(\alpha\rho)_{ij}+\varphi_{ijk}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,r;\ k=1,2,\cdots,s)$$

$$\alpha_i,\ \rho_j,\ (\alpha\rho)_{ij},\ \varphi_{ijk} = \text{random effects}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Repetitions	(r-1)	$\Sigma(y_j-y)^2$	$\sigma_\varphi^2+s\sigma_{\alpha\rho}^2+sa\sigma_\rho^2$
Treatments	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\varphi^2+s\sigma_{\alpha\rho}^2+sr\sigma_\rho^2$
Experiment Error	(r-1) ·(a-1)	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\varphi^2+s\sigma_{\alpha\rho}^2$
Sampling Error	ra ·(s-1)	$\Sigma(Y_{ijk}-y_{ij})^2$	σ_φ^2
Total	ras-1	$\Sigma(Y_{ijk}-y)^2$	

TABLE 3.2.17.b

Randomized Complete Block design with sampling

Model:

$$Y_{ijk} = \mu+\alpha_i+\rho_j+\epsilon_{ij}+\varphi_{ijk}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,r;\ k=1,2,\cdots,s)$$

$$\rho_j,\ \epsilon_{ij},\ \varphi_{ijk} = \text{random effects}$$

$$\alpha_i = \text{fixed effect}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Repetitions	$(r-1)$	$\Sigma(y_j-y)^2$	
Treatments	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+sr\frac{\Sigma\alpha_i^2}{(a-1)}$
Experimental Error	$(r-1)\cdot(a-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2$
Sampling Error	$ra\cdot(s-1)$	$\Sigma(Y_{ijk}-y_{ij})^2$	σ_φ^2
Total	$ras-1$	$\Sigma(Y_{ijk}-y)^2$	

TABLE 3.2.18

Randomized Complete Block Design with subsampling

Model:

$$Y_{ijkl} = \mu+\alpha_i+\rho_j+ \epsilon_{ij}+\varphi_{ijk}+\delta_{ijkl}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,r;\ k=1,2,\cdots,s;\ l=1,2,\cdots,v)$$

$$\rho_j,\ \varphi_{ijk},\ \delta_{ijkl},\ \epsilon_{ij} = \text{random effects}$$

$$\alpha_i = \text{fixed effect}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	(r-1)	$\Sigma(y_j-y)^2$	
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\delta^2+v\sigma_\varphi^2+vs\sigma_\epsilon^2+rvs\Sigma\alpha_i^2/(a-1)$
Experimental Error	(r-1)·(a-1)	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\delta^2+v\sigma_\varphi^2+vs\sigma_\epsilon^2$
Sampling Error	ra·(s-1)	$\Sigma(y_{ijk}-y_{ij})^2$	$\sigma_\delta^2+v\sigma_\varphi^2$
Subsampling Error	ras·(v-1)	$\Sigma(Y_{ijkl}-y_{ijk})^2$	σ_δ^2
Total	rasv-1	$\Sigma(Y_{ijkl}-y)^2$	

TABLE 3.2.19

Group of Randomized Complete Block Designs*

Model: $Y_{ijk} = \mu+\alpha_i+\rho_{ij}+\beta_k+(\alpha\beta)_{ik}+\epsilon_{ijk}$

$(i=1,2,\cdots,a;\ j=1,2,\cdots,r;\ k=1,2,\cdots,b)$

ρ_{ij} = random effect

$\alpha_i, \beta_k, (\alpha\beta)_{ik}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+b\sigma_\rho^2+\frac{br}{(a-1)}\Sigma\alpha_i^2$
Rep. within A	$a(r-1)$	$\Sigma(y_{ij}-y_i)^2$	$\sigma_\epsilon^2+b\sigma_\rho^2$
B	$(b-1)$	$\Sigma(y_k-y)^2$	$\sigma_\epsilon^2+\frac{ar}{(b-1)}\Sigma\beta_k^2$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma_\epsilon^2+\frac{r}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ik}^2$
Error	$a(r-1)\cdot(b-1)$	$\Sigma(Y_{ijk}-y_{ij}-y_{ik}+y_i)^2$	σ_ϵ^2
Total	$abr-1$	$\Sigma(Y_{ijk}-y)^2$	

*Each R C B design has only one level of factor A.

TABLE 3.5.20

Group of Randomized Complete Block design with sampling

Model:

$$Y_{ijkl} = \mu+\alpha_i+\rho_{ij}+\beta_k+(\alpha\beta)_{ik}+\epsilon_{ijk}+\varphi_{ijkl}$$

$(i=1,2,\cdots,a;\ j=1,2,\cdots,r;\ k=1,2,\cdots,b;\ l=1,2,\cdots,s)$

ρ_{ij}, ϵ_{ijk}, φ_{ijkl} = random effects

α_i, β_k, $(\alpha\beta)_{ik}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma^2_\varphi+s\sigma^2_\epsilon+bs\sigma^2_\rho+\frac{brs}{(a-1)}\Sigma\alpha^2_i$
Rep.within A	a(r-1)	$\Sigma(y_{ij}-y_i)^2$	$\sigma^2_\varphi+s\sigma^2_\epsilon+bs\sigma^2_\rho$
B	(b-1)	$\Sigma(y_k-y)^2$	$\sigma^2_\varphi+s\sigma^2_\epsilon+\frac{ars}{(b-1)}\Sigma\beta^2_k$
AB	(a-1) ·(b-1)	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma^2_\varphi+s\sigma^2_\epsilon+\frac{rs}{(a-1)(b-1)}\Sigma(\alpha\beta)^2_{ik}$
Experimental error	a(r-1) ·(b-1)	$\Sigma(y_{ijk}-y_{ij}-y_{ik}+y_i)^2$	$\sigma^2_\varphi+s\sigma^2_\epsilon$
Sampling error	abr ·(s-1)	$\Sigma(Y_{ijkl}-y_{ijk})^2$	σ^2_φ
Total	abrs-1	$\Sigma(Y_{ijkl}-y)^2$	

TABLE 3.2.21

Two-way classified group of Randomized Complete Block designs

Model:

$$Y_{hijk} = \mu+\alpha_h+\beta_i+(\alpha\beta)_{hi}+\rho_{hij}+\gamma_k+(\alpha\gamma)_{hk}+(\beta\gamma)_{ik}+(\alpha\beta\gamma)_{hik}+\epsilon_{hijk}$$

$$(h=1,2,\cdots,a;i=1,2,\cdots,b;j=1,2,\cdots,r;k=1,2,\cdots,c)$$

$\alpha_h,\beta_i,(\alpha\beta)_{hi},\gamma_k,(\alpha\gamma)_{hk},(\beta\gamma)_{ik},(\alpha\beta\gamma)_{hik}$ = fixed effects

ρ_{hij} = random effect

(Table 3.2.21 continued.)

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	(a-1)	$\Sigma(y_h - y)^2$	$\sigma_\epsilon^2 + c\sigma_\rho^2 + \frac{brc\Sigma\alpha_h^2}{(a-1)}$
B	(b-1)	$\Sigma(y_i - y)^2$	$\sigma_\epsilon^2 + c\sigma_\rho^2 + \frac{arc\Sigma\beta_i^2}{(b-1)}$
AB	(a-1) ·(b-1)	$\Sigma(y_{hi} - y_h - y_i + y)^2$	$\sigma_\epsilon^2 + c\sigma_\rho^2 + \frac{rc\Sigma(\alpha\beta)_{hi}^2}{(a-1)(b-1)}$
Rep.within (AB)cells	ab ·(r-1)	$\Sigma(y_{hij} - y_{hi})^2$	$\sigma_\epsilon^2 + c\sigma_\rho^2$
C	(c-1)	$\Sigma(y_k - y)^2$	$\sigma_\epsilon^2 + \frac{abr\Sigma(\gamma)_k^2}{(c-1)}$
AC	(a-1) ·(c-1)	$\Sigma(y_{hk} - y_h - y_k + y)^2$	$\sigma_\epsilon^2 + \frac{br\Sigma(\alpha\gamma)_{hk}^2}{(a-1)(c-1)}$
BC	(b-1) ·(c-1)	$\Sigma(y_{ik} - y_i - y_k + y)^2$	$\sigma_\epsilon^2 + \frac{ar\Sigma(\beta\gamma)_{ik}^2}{(b-1)(c-1)}$
ABC	(a-1) ·(b-1) ·(c-1)	$\Sigma(y_{hik} - y_{hi} - y_{hk} - y_{ik} + y_h + y_i + y_k - y)^2$	$\sigma_\epsilon^2 + \frac{r\Sigma(\alpha\beta\gamma)_{hik}^2}{(a-1)(b-1)(c-1)}$
Error	ab(c-1) ·(r-1)	$\Sigma(Y_{hijk} - y_{hij} - y_{hik} + y_{hi})^2$	σ_ϵ^2
Total	abcr-1	$\Sigma\ (Y_{hijk} - y)^2$	

TABLE 3.5.22

axb factorial in Randomized Complete Block design

Model:

$$Y_{ijk} = \mu+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\rho_k+\epsilon_{ijk}$$

$$(i=1,2,\cdots,a;\ j=1,2,\cdots,b;\ k=1,2,\cdots,r)$$

$$\rho_k = \text{random effect}$$

$$\alpha_i,\ \beta_j,\ (\alpha\beta)_{ij} = \text{fixed effects}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.(Blocks)	(r-1)	$\Sigma(y_k-y)^2$	
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+br\frac{\Sigma\alpha_i^2}{(a-1)}$
B	(b-1)	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+ar\frac{\Sigma\beta_j^2}{(b-1)}$
AB	(a-1)·(b-1)	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+r\frac{\Sigma(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
Error	(ab-1)·(r-1)	$\Sigma(Y_{ijk}-y_{ij}-y_k+y)^2$	σ_ϵ^2
Total	abr-1	$\Sigma(Y_{ijk}-y)^2$	

TABLE 3.2.23

axb Factorial + additional treatments in

Randomized Complete Block design

General Model for any individual observation:

$$Y_{hl} = \mu+\rho_h+T_l+\epsilon_{hl}$$

$(h=1,2,\ldots,r;\ l=1,2,\ldots,t)$

$t = ab+d$

d = number of additional treatments

Model for individuals receiving factorial treatments:

$$Y_{hij} = \mu+\rho_h+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\epsilon_{hij}$$

$(h=1,2,\ldots,r;\ i=1,2,\ldots,a;\ j=1,2,\ldots,b)$

Model for individuals receiving additional treatments:

$$Y_{hg} = \mu+\rho_h+T_g+\epsilon_{hg}$$

$(g=1,2,\ldots,d)$

ρ_h = random effect

$T_l,\alpha_i,\beta_j,(\alpha\beta)_{ij}$ = fixed effects

(Table 3.2.23 continued.)

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	(r-1)	$\Sigma(y_h-y)^2$	
T	(t-1)	$\Sigma(y_l-y)^2$	$\sigma_\epsilon^2+\frac{r}{(t-1)}\Sigma T_l^2$
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+\frac{rb}{(a-1)}\Sigma\alpha_i^2$
B	(b-1)	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+\frac{ra}{(b-1)}\Sigma\beta_j^2$
AB	(a-1) ·(b-1)	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+\frac{r}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$
Residual	d	$\Sigma(y_l-y)^2$ $-\Sigma(y_{ij}-y)^2$	$\sigma_\epsilon^2+f(\varphi)$
Error	(r-1) ·(t-1)	$\Sigma(Y_{hl}-y_h-y_l+y)^2$	σ_ϵ^2
Total	rt-1	$\Sigma(Y_{hl}-y)^2$	

where:

φ includes differences in effects between the axb treatment combinations and the d additional treatments, and differences in effects among additional treatments.

TABLE 3.2.24

axb Factorial in Randomized Complete

Block design with sampling

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\epsilon_{hij}+\varphi_{hijk}$$

$(h=1,2,\cdots,r;\ i=1,2,\cdots,a;\ j=1,2,\cdots,b;\ k=1,2,\cdots,s)$

$\rho_h,\ \epsilon_{hij},\ \varphi_{hijk}$ = random effects

$\alpha_i,\ \beta_j,\ (\alpha\beta)_{ij}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	(r-1)	$\Sigma(y_h-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+ sab\sigma_\rho^2$
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+rsb\frac{\Sigma\alpha_i^2}{(a-1)}$
B	(b-1)	$\Sigma(y_j-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+rsa\frac{\Sigma\beta_j^2}{(b-1)}$
AB	(a-1) ·(b-1)	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+rs\frac{\Sigma(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
Experimental Error	(r-1) ·(ab-1)	$\Sigma(y_{hij}-y_h-y_{ij}+y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2$
Sampling Error	rab ·(s-1)	$\Sigma(Y_{hijk}-y_{hij})^2$	σ_φ^2
Total	rabs-1	$\Sigma(Y_{hijk}-y)^2$	

TABLE 3.2.25

axb Factorial in a Randomized Complete Block Design with subsampling

Model: $Y_{hijkl} = \mu+\rho_h+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\epsilon_{hij}+\varphi_{hijk}+\delta_{hijkl}$

($h=1,2,\cdots,r$; $i=1,2,\cdots,a$; $j=1,2,\cdots,b$; $k=1,2,\cdots,s$; $l=1,2,\cdots,v$)

ρ_h, ϵ_{hij}, φ_{hijk}, δ_{hijkl} = random effects

α_i, β_j, $(\alpha\beta)_{ij}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	$(r-1)$	$\Sigma(y_h-y)^2$	$\sigma_\delta^2+v\sigma_\varphi^2+vs\sigma_\epsilon^2+abvs\sigma_\rho^2$
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\delta^2+v\sigma_\varphi^2+vs\sigma_\epsilon^2+rbvs\frac{\Sigma\alpha_i^2}{(a-1)}$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\delta^2+v\sigma_\varphi^2+vs\sigma_\epsilon^2+ravs\frac{\Sigma\beta_j^2}{(b-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\delta^2+v\sigma_\varphi^2+vs\sigma_\epsilon^2+rvs\frac{\Sigma(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
Experimental Error	$(r-1)\cdot(ab-1)$	$\Sigma(y_{hij}-y_h-y_{ij}+y)^2$	$\sigma_\delta^2+v\sigma_\varphi^2+vs\sigma_\epsilon^2$

(Table 3.2.25 continued.)

Source	d.f.	S.S.	E(M.S.)
Sampling Error	rab $\cdot(s-1)$	$\Sigma(y_{hijk}-y_{hij})^2$	$\sigma_\delta^2+v\sigma_\varphi^2$
Subsampling Error	rabs $\cdot(v-1)$	$\Sigma(Y_{hijkl}-y_{hijk})^2$	σ_δ^2
Total	rabsv-1	$\Sigma(Y_{hijkl}-y)^2$	

TABLE 26

Group of axb factorial Randomized Complete Block designs

Model:

$$Y_{ghij} = \mu+\alpha_g+\rho_{gh}+\beta_i+\gamma_j+(\beta\gamma)_{ij}+(\alpha\beta)_{gi}+(\alpha\gamma)_{gj}+(\alpha\beta\gamma)_{gij}+\epsilon_{ghij}$$

$(g=1,2,\cdots,a;\ h=1,2,\cdots,r;\ i=1,2,\cdots,b;\ j=1,2,\cdots,c)$

ρ_{gh}, ϵ_{ghij} = random effects

$\alpha_g, \beta_i, \gamma_j, (\beta\gamma)_{ij}, (\alpha\beta)_{gi}, (\alpha\gamma)_{gj}, (\alpha\beta\gamma)_{gij}$ = fixed effects

(Table 26 continued.)

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	$(a-1)$	$\Sigma(y_g-y)^2$	$\sigma_\epsilon^2+bc\sigma_\rho^2+\frac{bcr\Sigma\alpha_g^2}{(a-1)}$
Reps.within A	$a(r-1)$	$\Sigma(y_{gh}-y_g)^2$	$\sigma_\epsilon^2+bc\sigma_\rho^2$
B	$(b-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+\frac{acr\Sigma\beta_i^2}{(b-1)}$
C	$(c-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+\frac{abr\Sigma\gamma_j^2}{(c-1)}$
BC	$(b-1)\cdot(c-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+\frac{ar\Sigma(\beta\gamma)_{ij}^2}{(b-1)(c-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{gi}-y_g-y_i+y)^2$	$\sigma_\epsilon^2+\frac{cr\Sigma(\alpha\beta)_{gi}^2}{(a-1)(b-1)}$
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{gj}-y_g-y_j+y)^2$	$\sigma_\epsilon^2+\frac{br\Sigma(\alpha\gamma)_{gj}^2}{(a-1)(c-1)}$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{gij}-y_{gi}-y_{gj}-y_{ij}+y_g+y_i+y_j-y)^2$	$\sigma_\epsilon^2+\frac{r\Sigma(\alpha\beta\gamma)_{gij}^2}{(a-1)(b-1)(c-1)}$
Error	$a(bc-1)\cdot(r-1)$	$\Sigma(Y_{ghij}-y_{gh}-y_{gij}+y_g)^2$	σ_ϵ^2
Total	$abcr-1$	$\Sigma(Y_{ghij}-y)^2$	

TABLE 3.2.27

axbxc factorial in Randomized Complete Block design

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+\beta_j+\gamma_k+(\alpha\beta)_{ij}+(\alpha\gamma)_{ik}+(\beta\gamma)_{jk}+(\alpha\beta\gamma)_{ijk}+\epsilon_{hijk}$$

$$(h=1,2,\cdots,r;i=1,2,\cdots,a;j=1,2,\cdots,b;k=1,2,\cdots,c)$$

ρ_h, ϵ_{hijk} = random effects

α_i, β_j, γ_k, $(\alpha\beta)_{ij}$, $(\alpha\gamma)_{ik}$, $(\beta\gamma)_{jk}$, $(\alpha\beta\gamma)_{ijk}$ = fixed effects

(Table 3.2.27 continued.)

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+rbc\frac{\Sigma\alpha_i^2}{(a-1)}$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+rac\frac{\Sigma\beta_j^2}{(b-1)}$
C	$(c-1)$	$\Sigma(y_k-y)^2$	$\sigma_\epsilon^2+rab\frac{\Sigma\gamma_k^2}{(c-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+rc\frac{\Sigma(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma_\epsilon^2+rb\frac{\Sigma(\alpha\gamma)_{ik}^2}{(a-1)(c-1)}$
BC	$(b-1)\cdot(c-1)$	$\Sigma(y_{jk}-y_j-y_k+y)^2$	$\sigma_\epsilon^2+ra\frac{\Sigma(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{ijk}-y_{ij}-y_{ik}-y_{jk}+y_i+y_j+y_k-y)^2$	$\sigma_\epsilon^2+r\frac{\Sigma(\alpha\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$
Error	$(abc-1)\cdot(r-1)$	$\Sigma(Y_{hijk}-y_{ijk}-y_h+y)^2$	σ_ϵ^2
Total	$abcr-1$	$\Sigma(Y_{hijk}-y)^2$	

TABLE 3.2.28

axbxc Factorial + additional treatments in

Randomized Complete Block design

General model for any individual observation:

$$Y_{hl} = \mu+\rho_h+T_l+\epsilon_{hl}$$

$(h=1,2,\ldots,r;\ l=1,2,\ldots,t)$

$t= abc+d$

$d=$ number of additional treatments

Model for individuals receiving factorial treatments:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+\beta_j+\gamma_k+(\alpha\beta)_{ij}+(\alpha\gamma)_{ik}+(\beta\gamma)_{jk}+(\alpha\beta\gamma)_{ijk}+\epsilon_{hijk}$$

Model for individuals receiving additional treatments:

$$Y_{hg} = \mu+\rho_h+T_g+\epsilon_{hg}\ ,\ (g=1,2,\ldots,d)$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	(r-1)	$\Sigma(y_h-y)^2$	
T	(t-1)	$\Sigma(y_l-y)^2$	$\sigma_\epsilon^2+\frac{r\Sigma T_l^2}{(t-1)}$
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+\frac{rbc\Sigma\alpha_i^2}{(a-1)}$
B	(b-1)	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+\frac{rac\Sigma\beta_j^2}{(b-1)}$

(Table 3.2.28 continued.)

Source	d.f.	S.S.	E(M.S.)
C	(c-1)	$\Sigma(y_k - y)^2$	$\sigma^2_\epsilon + \frac{rab\Sigma\gamma^2_k}{(c-1)}$
AB	(a-1) ·(b-1)	$\Sigma(y_{ij} - y_i - y_j + y)^2$	$\sigma^2_\epsilon + \frac{rc\Sigma(\alpha\beta)^2_{ij}}{(a-1)(b-1)}$
AC	(a-1) ·(c-1)	$\Sigma(y_{ij} - y_i - y_k + y)^2$	$\sigma^2_\epsilon + \frac{rb\Sigma(\alpha\gamma)^2_{ik}}{(a-1)(c-1)}$
BC	(b-1) ·(c-1)	$\Sigma(y_{jk} - y_j - y_k + y)^2$	$\sigma^2_\epsilon + \frac{ra\Sigma(\beta\gamma)^2_{jk}}{(b-1)(c-1)}$
ABC	(a-1) ·(b-1) ·(c-1)	$\Sigma(y_{ijk} - y_{ij} - y_{jk} - y_{ik} + y_i + y_j + y_k - y)^2$	$\sigma^2_\epsilon + \frac{r\Sigma(\alpha\beta\gamma)^2_{ijk}}{(a-1)(b-1)(c-1)}$
Residual	d	$\Sigma(y_l - y)^2 - \Sigma(y_{ij} - y)^2$	$\sigma^2_\epsilon + f(\varphi)$
Error	(r-1) ·(t-1)	$\Sigma(Y_{hl} - y_h - y_l + y)^2$	σ^2_ϵ
Total	rt-1	$\Sigma(Y_h - y)^2$	

where:

φ includes differences in effects between the axbxc treatment combinations and the d additional treatments, and differences in effects among additional treatments.

TABLE 3.2.29

axbxc Factorial in Randomized Complete Block design with sampling

Model:

$$Y_{hijkl} = \mu+\rho_h+\alpha_i+\beta_j+\gamma_k+(\alpha\beta)_{ij}+(\alpha\gamma)_{ik}+(\beta\gamma)_{jk}+(\alpha\beta\gamma)_{ijk}+\epsilon_{hijk}+\varphi_{hijkl}$$

$(h=1,2,\ldots,r;i=1,2,\ldots,a;j=1,2,\ldots,b;k=1,2,\ldots,c;l=1,2,\ldots,s)$

$\rho_h, \epsilon_{hijk}, \varphi_{hijkl}$ = random effects

$\alpha_i, \beta_j, \gamma_k, (\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, (\beta\gamma)_{jk}, (\alpha\beta\gamma)_{ijk}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	(r-1)	$\Sigma(y_h-y)^2$	
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{rbcs\Sigma\alpha_i^2}{(a-1)}$
B	(b-1)	$\Sigma(y_j-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{racs\Sigma\beta_j^2}{(b-1)}$
C	(c-1)	$\Sigma(y_k-y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{rabs\Sigma\gamma_k^2}{(c-1)}$
AB	(a-1) •(b-1)	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{rcs\Sigma(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$
AC	(a-1) •(c-1)	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma_\varphi^2+s\sigma_\epsilon^2+\frac{rbs\Sigma(\alpha\gamma)_{ik}^2}{(a-1)(c-1)}$

(Table 3.2.29 continued.)

Source	d.f.	S.S.	E(M.S.)
BC	$(b-1)\cdot(c-1)$	$\Sigma(y_{jk}-y_{j}-y_{k}+y)^2$	$\sigma_\varphi^2+s\sigma_\varepsilon^2+\frac{ras\Sigma(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{ijk}-y_{ij}-y_{ik}-y_{jk}+y_{i}+y_{j}+y_{k}-y)^2$	$\sigma_\varphi^2+s\sigma_\varepsilon^2+\frac{rs\Sigma(\alpha\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$
Experimental Error	$(abc-1)\cdot(r-1)$	$\Sigma(y_{hijk}-y_{ijk}-y_{h}+y)^2$	$\sigma_\varphi^2+s\sigma_\varepsilon^2$
Sampling Error	$abcr\cdot(s-1)$	$\Sigma(Y_{hijkl}-y_{hijk})^2$	σ_φ^2
Total	$abcrs-1$	$\Sigma(Y_{hijkl}-y)^2$	

TABLE 3.2.30

Multiple factorial in Randomized Complete Block design

Model:

$$Y_{gh\ldots mnp} = \mu+\rho_g+\alpha_h+\beta_i+\ldots+\varphi_p+(\alpha\beta)_{hi}+\ldots+(\lambda\varphi)_{np}+(\alpha\beta\gamma)_{hij}+\ldots$$
$$+(\delta\lambda\varphi)_{mnp}+\ldots+(\beta\gamma\ldots\lambda\varphi)_{ij\ldots np}+\epsilon_{gh\ldots np}$$

$(g=1,2,\ldots,r;h=1,2,\ldots,a;i=1,2,\ldots,b;j=1,2,\ldots,c;\ldots;m=1,$
$2,\ldots,v;n=1,2,\ldots,w;p=1,2,\ldots,z)$

ghi...mnp are a set of t subscripts,(t=number of factors)

ρ_h = random effect

$\alpha_h,\beta_i,\ldots,\varphi_p,(\alpha\beta)_{hi},\ldots,(\lambda\varphi)_{np},(\alpha\beta\gamma)_{hij},\ldots,(\alpha\gamma\ldots\lambda\varphi)_{hi\ldots np}$
= fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	$(r-1)$	$\Sigma(y_g-y)^2$	
A	$(a-1)$	$\Sigma(y_h-y)^2$	$\sigma_\epsilon^2+\frac{rb..wz\Sigma\alpha_h^2}{(a-1)}$
.	.	.	.
.	.	.	.
.	.	.	.
Z	$(z-1)$	$\Sigma(y_p-y)^2$	$\sigma_\epsilon^2+\frac{ra..w\Sigma\varphi_p^2}{(z-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma_\epsilon^2+\frac{rc..wz\Sigma(\alpha\beta)_{hi}^2}{(a-1)(b-1)}$
.	.	.	.
.	.	.	.
.	.	.	.

(Table 3.2.30 continued.)

Source	d.f.	S.S.	E(M.S.)
WZ	$(w-1)\cdot(z-1)$	$\Sigma(y_{np}-y_{n}-y_{p}+y)^2$	$\sigma_\epsilon^2+\frac{ra..uv\Sigma(\lambda\varphi)^2_{np}}{(w-1)(z-1)}$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{hij}-y_{hi}-y_{hj}-y_{ij}+y_{h}+y_{i}+y_{j}-y)^2$	$\sigma_\epsilon^2+\frac{rd..wz\Sigma(\alpha\beta\gamma)^2_{hij}}{(a-1)(b-1)(c-1)}$
⋮	⋮	⋮	⋮
VWZ	$(v-1)\cdot(w-1)\cdot(z-1)$	$\Sigma(y_{mnp}-y_{mn}-y_{np}-y_{mp}+y_{m}+y_{n}+y_{p}-y)^2$	$\sigma_\epsilon^2+\frac{ra..u\Sigma(\delta\lambda\varphi)^2_{mnp}}{(v-1)(w-1)(z-1)}$
⋮	⋮	⋮	⋮
AB...VWZ	$(a-1)\cdot(b-1)\cdots\cdot(w-1)\cdot(z-1)$	$\Sigma(y_{hi...np}-y_{hi...n}-\cdots-y_{i...np}+y_{hi...m}+\cdots+y_{i...np}-\cdots+(-1)^{t-1}y)^2$	$\sigma_\epsilon^2+\frac{ra\Sigma(\alpha..\varphi)^2_{h...p}}{(b-1)...(z-1)}$
Error	$(r-1)\cdot(ab...wz-1)$	$\Sigma(Y_{gh...np}-y_{g}-y_{hi...np}+y)^2$	σ_ϵ^2
Total	$rab...z-1$	$\Sigma(Y_{gh...np}-y)^2$	

TABLE 3.2.31

Latin Square design

Model:

$$Y_{ijk} = \mu+\alpha_i+\beta_j+\gamma_k+\epsilon_{ijk}$$

$(i,j,k = 1,2,\cdots,r)$

α_i, β_j, γ_k = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A (Rows)	$(r-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+r\frac{\Sigma\alpha_i^2}{(r-1)}$
B(Columns)	$(r-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+r\frac{\Sigma\beta_j^2}{(r-1)}$
C(Treatments)	$(r-1)$	$\Sigma(y_k-y)^2$	$\sigma_\epsilon^2+r\frac{\Sigma\gamma_k^2}{(r-1)}$
Error	$(r-1)\cdot(r-2)$	$\Sigma(Y_{ijk}-y_i-y_j-y_k+2y)^2$	σ_ϵ^2
Total	r^2-1	$\Sigma(Y_{ijk}-y)^2$	

TABLE 3.2.32

Group of s Latin Square designs

Model:

$$Y_{hijk} = \mu+\delta_h+\rho_{h(i)}+\gamma_{h(j)}+T_k+(\delta T)_{hk}+\epsilon_{hijk}$$
$$(h=1,2,\ldots,s;\ i,\ j,\ k = 1,2,\ldots,r)$$

$\rho_{h(i)}, \gamma_{h(j)}$ = random effects

$\delta_h, T_k, (\delta T)_{hk}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Squares	$(s-1)$	$\Sigma(y_h-y)^2$	
Rows within squares	$s(r-1)$	$\Sigma(y_{hi}-y_h)^2$	
Columns within squares	$s(r-1)$	$\Sigma(y_{hj}-y_h)^2$	
Treatments	$(r-1)$	$\Sigma(y_k-y)^2$	$\sigma_\epsilon^2+sr\Sigma T_k^2/(r-1)$
Squares x Treatments	$(s-1)\cdot(r-1)$	$\Sigma(y_{hk}-y_h-y_k+y)^2$	$\sigma_\epsilon^2+r\Sigma(\delta T)_{ik}^2/(s-1)(r-1)$
Error	$s(r-1)\cdot(r-2)$	$\Sigma(Y_{hijk}-y_{hi}-y_{hj}-y_{hk}+2y_h)^2$	σ_ϵ^2
Total	sr^2-1	$\Sigma(Y_{hijk}-y)^2$	

TABLE 3.2.33

Split-plot design

Model:

$$Y_{hij} = \mu+\rho_h+\alpha_i+\varphi_{hi}+\beta_j+(\alpha\beta)_{ij}+\epsilon_{hij}$$

$$(h=1,2,\cdots,r;\ i=1,2,\cdots,a;\ j=1,2,\cdots,b)$$

ρ_h = random effect

α_i,β_j = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+b\sigma_\varphi^2+\frac{br}{(a-1)}\Sigma\alpha_i^2$
Error(a)	$(r-1)\cdot(a-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma_\epsilon^2+b\sigma_\varphi^2$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+\frac{ar}{(b-1)}\Sigma\beta_j^2$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+\frac{r}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$
Error (b)	$a(b-1)\cdot(r-1)$	$\Sigma(Y_{hij}-y_{hi}-y_{hj}+y)^2$	σ_ϵ^2
Total	abr-1	$\Sigma(Y_{hij}-y)^2$	

TABLE 3.2.34

Split-plot design, considering interactions between repetitions and each of the two factors

Model:

$$Y_{hij} = \mu+\rho_h+\alpha_i+(\rho\alpha)_{hi}+\beta_j+(\rho\beta)_{hj}+(\alpha\beta)_{ij}+\epsilon_{hij}$$
$$(h=1,2,\ldots,r;\ i=1,2,\ldots,a;\ j=1,2,\ldots,b)$$

$$\rho_h, (\rho\alpha)_{hi}, (\rho\beta)_{hj} = \text{random effects}$$
$$\alpha_i, \beta_j, (\rho\beta)_{hj} = \text{fixed effects}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	(r-1)	$\Sigma(y_h-y)^2$	$\sigma_\epsilon^2+ab\sigma_\rho^2$
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+b\sigma_{\rho\alpha}^2+\frac{rb}{(a-1)}\Sigma\alpha_i^2$
RA	(r-1)·(a-1)	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma_\epsilon^2+b\sigma_{\rho\alpha}^2$
B	(b-1)	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+a\sigma_{\rho\beta}^2+\frac{ra}{(b-1)}\Sigma\beta_j^2$
RB	(r-1)·(b-1)	$\Sigma(y_{hj}-y_h-y_j+y)^2$	$\sigma_\epsilon^2+a\sigma_{\rho\beta}^2$
AB	(a-1)·(b-1)	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+\frac{r}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$
Residual	(r-1)·(a-1)·(b-1)	$\Sigma(Y_{hij}-y_{hi}-y_{hj}-y_{ij}+y_h+y_i+y_j-y)^2$	σ_ϵ^2
Total	rab-1	$\Sigma(Y_{hij}-y)^2$	

TABLE 3.2.35

Split-plot design with sampling

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+\varphi_{hi}+\beta_j+(\alpha\beta)_{ij}+\epsilon_{hij}+\lambda_{hijk}$$

$$(h=1,2,\cdots,r;\ i=1,2,\cdots,a;\ j=1,2,\cdots,b;\ k=1,2,\cdots,s)$$

$$\rho_h,\ \varphi_{hi},\ \epsilon_{hij},\ \lambda_{hijk} = \text{random effects}$$

$$\alpha_i,\ \beta_j,\ (\alpha\beta)_{ij} = \text{fixed effects}$$

(Table 3.2.35 continued.)

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sb\sigma_\varphi^2+\frac{sbr}{(a-1)}\Sigma\alpha_i^2$
Error (a)	$(r-1)\cdot(a-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sb\sigma_\varphi^2$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+\frac{ars}{(b-1)}\Sigma\beta_j^2$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+\frac{sr}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$
Error (b)	$a(b-1)\cdot(r-1)$	$\Sigma(y_{hij}-y_{hi}-y_{ij}+y_i)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2$
Sampling error	$abr\cdot(s-1)$	$\Sigma(Y_{hijk}-y_{hij})^2$	σ_λ^2
Total	$abrs-1$	$\Sigma(Y_{hijk}-y)^2$	

TABLE 3.2.36

Split-plot design with sampling considering interactions between repetitions and each of the two factors

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+(\rho\alpha)_{hi}+\beta_j+(\rho\beta)_{hj}+(\alpha\beta)_{ij}+(\rho\alpha\beta)_{hij}+\epsilon_{hijk}$$

$(h=1,2,\ldots,r;\ i=1,2,\ldots,a;\ j=1,2,\ldots,b;\ k=1,2,\ldots,s)$

$\rho_h, (\rho\alpha)_{hi}, (\rho\beta)_{hj}, (\rho\alpha\beta)_{hij}, \epsilon_{hijk}$ = random effects

$\alpha_i, \beta_j, (\rho\beta)_{hj}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma^2_\epsilon+sb\sigma^2_{\rho\alpha}+\frac{srb}{(a-1)}\Sigma\alpha^2_i$
RA	$(r-1)\cdot(a-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma^2_\epsilon+sb\sigma^2_{\rho\alpha}$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma^2_\epsilon+sa\sigma^2_{\rho\beta}+\frac{sra}{(b-1)}\Sigma\beta^2_j$
RB	$(r-1)\cdot(b-1)$	$\Sigma(y_{hj}-y_h-y_j+y)^2$	$\sigma^2_\epsilon+sa\sigma^2_{\rho\beta}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma^2_\epsilon+\sigma^2_{\rho\alpha\beta}+\frac{sr}{(a-1)(b-1)}\cdot\Sigma(\alpha\beta)^2_{ij}$

(Table 3.2.36 continued.)

Source	d.f.	S.S.	E(M.S.)
RAB	(r-1)·(a-1)·(b-1)	$\Sigma(y_{hij}-y_{hi}-y_{hj}-y_{ij}+y_{h}+y_{i}+y_{j}-y)^2$	$\sigma_{\epsilon}^2+s\sigma_{\rho\alpha\beta}^2$
Sampling Error	rab·(s-1)	$\Sigma(Y_{hijk}-y_{hij})^2$	σ_{ϵ}^2
Total	rabs-1	$\Sigma(Y_{hijk}-y)^2$	

TABLE 3.2.37

Group of split-plot designs

Model:

$$Y_{ijkl} = \mu+\alpha_i+\rho_{i(j)}+\beta_k+(\alpha\beta)_{ik}+\epsilon_{ijk}+\gamma_l+(\alpha\gamma)_{il}+(\beta\gamma)_{kl}+(\alpha\beta\gamma)_{ikl}+\varphi_{ijkl}$$

$(i=1,2,\cdots,a;\ j=1,2,\cdots,r;\ k=1,2,\cdots,b;\ l=1,2,\cdots,c)$

α_i, β_k, $(\alpha\beta)_{ik}$, γ_l, $(\alpha\gamma)_{il}$, $(\beta\gamma)_{kl}$, $(\alpha\beta\gamma)_{ikl}$ = fixed effects

$\rho_{i(j)}$, ϵ_{ijk}, φ_{ijkl} = random effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\varphi^2+c\sigma_\epsilon^2+bc\sigma_\rho^2+\frac{rbc\Sigma\alpha_i^2}{(a-1)}$
Repet. within A	$a(r-1)$	$\Sigma(y_{ij}-y)^2$	$\sigma_\varphi^2+c\sigma_\epsilon^2+bc\sigma_\rho^2$
B	$(b-1)$	$\Sigma(y_k-y)^2$	$\sigma_\varphi^2+c\sigma_\epsilon^2+\frac{rac\Sigma\beta_k^2}{(b-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma_\varphi^2+c\sigma_\epsilon^2+\frac{rc\Sigma(\alpha\beta)_{ik}^2}{(a-1)(b-1)}$
Error Among Plots	$a(r-1)\cdot(b-1)$	$\Sigma(y_{ijk}-y_{ij}-y_{ik}+y_i)^2$	$\sigma_\varphi^2+c\sigma_\epsilon^2$
C	$(c-1)$	$\Sigma(y_l-y)^2$	$\sigma_\varphi^2+\frac{rab\Sigma\gamma_l^2}{(c-1)}$
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{il}-y_i-y_l+y)^2$	$\sigma_\varphi^2+\frac{rb\Sigma(\alpha\gamma)_{il}^2}{(a-1)(c-1)}$

(Table 3.2.37 continued.)

Source	d.f.	S.S.	E(M.S.)
BC	(b-1) $\cdot$(c-1)	$\Sigma(y_{kl}-y_k-y_l+y)^2$	$\sigma_\varphi^2+\frac{ra\Sigma(\beta\gamma)_{kl}^2}{(b-1)(c-1)}$
ABC	(a-1) $\cdot$(b-1) $\cdot$(c-1)	$\Sigma(y_{ikl}-y_{ik}-y_{il}-y_{kl}+y_i+y_k+y_l-y)^2$	$\sigma_\varphi^2+\frac{r\Sigma(\alpha\beta\gamma)_{ikl}^2}{(a-1)(b-1)(c-1)}$
Error Within plots	ab(r-1) $\cdot$(c-1)	$\Sigma(Y_{ijkl}-y_{ijk}-y_{ijl}+y_{ij})^2$	σ_φ^2
Total	abrc-1	$\Sigma(Y_{ijkl}-y)^2$	

TABLE 3.2.38

Split-plot design with

axb factorial on the whole plots

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\varphi_{hij}+\gamma_k+(\alpha\gamma)_{ik}+(\beta\gamma)_{jk}$$
$$+(\alpha\beta\gamma)_{ijk}+\epsilon_{hijk}$$

$(h=1,2,\cdots,r;i=1,2,\cdots,a;j=1,2,\cdots,b;k=1,2,\cdots,c)$

$\rho_h, \epsilon_{hijk}, \varphi_{hij}$ = random effects

$\alpha_i,\ \beta_j,\ (\alpha\beta)_{ij}, \gamma_k,\ (\alpha\gamma)_{ik},\ (\beta\gamma)_{jk},\ (\alpha\beta\gamma)_{ijk}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+c\sigma_\varphi^2+\frac{bcr}{(a-1)}\Sigma\alpha_i^2$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+c\sigma_\varphi^2+\frac{acr}{(b-1)}\Sigma\beta_j^2$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+c\sigma_\varphi^2+\frac{cr}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$
Error(a)	$(ab-1)\cdot(r-1)$	$\Sigma(y_{hij}-y_{ij}-y_h+y)^2$	$\sigma_\epsilon^2+c\sigma_\varphi^2$

(Table 3.2.38 continued.)

Source	d.f.	S.S.	E(M.S.)
C	$(c-1)$	$\Sigma(y_k - y)^2$	$\sigma_\epsilon^2 + \frac{abr}{(c-1)}\Sigma\gamma_k^2$
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{ik} - y_i - y_k + y)^2$	$\sigma_\epsilon^2 + \frac{br}{(a-1)(c-1)}\Sigma(\alpha\gamma)_{ik}^2$
BC	$(b-1)\cdot(c-1)$	$\Sigma(y_{jk} - y_j - y_k + y)^2$	$\sigma_\epsilon^2 + \frac{ar}{(b-1)(c-1)}\Sigma(\beta\gamma)_{jk}^2$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{ijk} - y_{ij} - y_{ik} - y_{jk} + y_i + y_j + y_k - y)^2$	$\sigma_\epsilon^2 + \frac{r}{(a-1)(b-1)(c-1)}\cdot\Sigma(\alpha\beta\gamma)_{ijk}^2$
Error(b)	$ab(r-1)\cdot(c-1)$	$\Sigma(Y_{hijk} - y_{hij} - y_{ijk} + y_{ij})^2$	σ_ϵ^2
Total	$abrc-1$	$\Sigma(Y_{hijk} - y)^2$	

TABLE 3.2.39

Split-plot design with axb factorial on whole plots, Interactions between repetitions and components of the axbxc factorial

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\varphi_{hij}+\gamma_k+(\rho\gamma)_{hk}+(\alpha\gamma)_{ik}+(\beta\gamma)_{jk}$$
$$+(\alpha\beta\gamma)_{ijk}+(\rho\alpha\gamma)_{hik}+(\rho\beta\gamma)_{hjk}+\epsilon_{hijk}$$
$$= \mu+\rho_h+T_{ij}+(\rho T)_{hij}+\gamma_k+(\rho\gamma)_{hk}+(T\gamma)_{ijk}+\epsilon_{hijk}$$

$(h=1,2,\ldots,r;\ i=1,2,\ldots,a;\ j=1,2,\ldots,b;\ k=1,2,\ldots,c)$

where:

$$T_{ij} = \alpha_i+\beta_j+(\alpha\beta)_{ij}$$
$$T_{ij}, \gamma_k, (T\gamma)_{ijk} = \text{fixed effects}$$
$$\rho_h, \varphi_{hij}, (\rho\gamma)_{hk} = \text{random effects}$$

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	$(r-1)$	$\Sigma(y_h-y)^2$	
T	$(ab-1)$	$\Sigma(y_{ij}-y)^2$	$\sigma_\epsilon^2+c\sigma_{\rho T}^2+\frac{rc}{ab-1}\Sigma T_{ij}^2$
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+c\sigma_{\rho T}^2+\frac{rbc}{(a-1)}\Sigma\alpha_i^2$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+c\sigma_{\rho T}^2+\frac{rac}{(b-1)}\Sigma\beta_j^2$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+c\sigma_{\rho T}^2+\frac{rc}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$

(Table 3.2.39 continued.)

Source	d.f.	S.S.	E(M.S.)
RT	(r-1)·(ab-1)	$\Sigma(y_{hij}-y_h-y_{ij}+y)^2$	$\sigma_\epsilon^2+c\sigma_{\rho T}^2$
C	(c-1)	$\Sigma(y_k-y)^2$	$\sigma_\epsilon^2+ab\sigma_{\rho\gamma}^2+\frac{abr}{(c-1)}\Sigma\gamma_k^2$
RC	(r-1)·(c-1)	$\Sigma(y_{hk}-y_h-y_k+y)^2$	$\sigma_\epsilon^2+ab\sigma_{\rho\gamma}^2$
TC	(ab-1)·(c-1)	$\Sigma(y_{ijk}-y_{ij}-y_k+y)^2$	$\sigma_\epsilon^2+\frac{r}{(ab-1)(c-1)}\Sigma(T\gamma)_{ijk}^2$
RTC	(r-1)·(ab-1)·(c-1)	$\Sigma(Y_{hijk}-y_{ijk}-y_{hk}-y_{hij}+y_h+y_{ij}+y_k-y)^2$	σ_ϵ^2
Total	rabc-1	$\Sigma(Y_{hijk}-y)^2$	

TABLE 3.2.40

Split-plot design with axb factorial

on whole plots and sampling

Model:

$$Y_{hijkl} = \mu+\rho_h+\alpha_i+\beta_j+(\alpha\beta)_{ij}+\delta_{hij}+\gamma_k+(\alpha\gamma)_{ik}+(\beta\gamma)_{jk}+(\alpha\beta\gamma)_{ijk}$$

$$+\epsilon_{hijk}+\lambda_{hijkl}$$

$(h=1,2,\ldots,r;\ i=1,2,\ldots,a;\ j=1,2,\ldots,c;\ k=1,2,\ldots,s)$

$\rho_h, \delta_{hij}, \epsilon_{hijk}, \lambda_{hijkl}$ = random effects

$\alpha_i, (\alpha\beta)_{ij}, \gamma_k, (\alpha\gamma)_{ik}, (\beta\gamma)_{jk}, (\alpha\beta\gamma)_{ijk}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sc\sigma_\delta^2+\frac{rbcs}{(a-1)}\Sigma\alpha_i^2$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sc\sigma_\delta^2+\frac{racs}{(b-1)}\Sigma\beta_j^2$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sc\sigma_\delta^2+\frac{rcs}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$
Error(a)	$(ab-1)\cdot(r-1)$	$\Sigma(y_{hij}-y_{ij}-y_h+y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sc\sigma_\delta^2$

(Table 3.2.40 continued.)

Source	d.f.	S.S.	E(M.S.)
C	(c-1)	$\Sigma(y_k - y)^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2 + \frac{rabs}{(c-1)}\Sigma\gamma_k^2$
AC	(a-1) $\cdot$(c-1)	$\Sigma(y_{ik} - y_i - y_k + y)^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2 + \frac{rbs}{(a-1)(c-1)}\Sigma(\alpha\gamma)_{ik}^2$
BC	(b-1) $\cdot$(c-1)	$\Sigma(y_{jk} - y_j - y_k + y)^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2 + \frac{ras}{(b-1)(c-1)}\Sigma(\beta\gamma)_{jk}^2$
ABC	(a-1) $\cdot$(b-1) $\cdot$(c-1)	$\Sigma(y_{ijk} - y_{ij} - y_{ik} - y_{jk} + y_i + y_j + y_k - y)^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2 + \frac{rs}{(a-1)(b-1)(c-1)} \cdot \Sigma(\alpha\beta\gamma)_{ijk}^2$
Error(b)	ab(r-1) $\cdot$(c-1)	$\Sigma(y_{hijk} - y_{hij} - y_{ijk} + y_{ij})^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2$
Sampling Error	abrc $\cdot$(s-1)	$\Sigma(Y_{hijkl} - y_{hijk})^2$	σ_λ^2
Total	abrcs-1	$\Sigma(Y_{hijkl} - y)^2$	

TABLE 3.2.41

Split-plot design with axb factorial on whole plots with sampling, interactions between repetitions and components of the axbxc factorial

Model:

$$Y_{hijkl} = \mu+\rho_h+T_{ij}+(\rho T)_{hij}+\gamma_k+(\rho\gamma)_{hk}+(T\gamma)_{ijk}+(\rho T\gamma)_{hijk}+\lambda_{hijkl}$$

where:

$$T_{ij} = \alpha_i+\beta_j+(\alpha\beta)_{ij}$$

$(h=1,2,\ldots,r;i=1,2,\ldots,a;j=1,2,\ldots,b;k=1,2,\ldots,c;l=1,2,\ldots,s)$

$T_{ij},\gamma_k,(T\gamma)_{ijk}$ = fixed effects

$\rho_h,(\rho T)_{hij},(\rho\gamma)_{hk},(\rho T\gamma)_{hijk}$ = random effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	$(r-1)$	$\Sigma(y_h-y)^2$	
T	$(ab-1)$	$\Sigma(y_{ij}-y)^2$	$\sigma^2_\lambda+cs\sigma^2_{\rho T}+\frac{rcs}{(ab-1)}\Sigma T^2_{ij}$
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma^2_\lambda+cs\sigma^2_{\rho T}+\frac{rcsb}{(a-1)}\Sigma\alpha^2_i$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma^2_\lambda+cs\sigma^2_{\rho T}+\frac{racs}{(b-1)}\Sigma\beta^2_j$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma^2_\lambda+cs\sigma^2_{\rho T}+\frac{rcs}{(a-1)(b-1)}\Sigma(\alpha\beta)^2_{ij}$

(Table 3.2.41 continued.)

Source	d.f.	S.S.	E(M.S.)
RxT	$(r-1)\cdot(ab-1)$	$\Sigma(y_{hij}-y_{h}-y_{ij}+y)^2$	$\sigma_\lambda^2+cs\sigma_{\rho T}^2$
C	$(c-1)$	$\Sigma(y_{k}-y)^2$	$\sigma_\lambda^2+sab\sigma_{\rho\gamma}^2+\frac{rabs}{(c-1)}\Sigma\gamma_k^2$
RC	$(r-1)\cdot(c-1)$	$\Sigma(y_{hk}-y_{h}-y_{k}+y)^2$	$\sigma_\lambda^2+sab\sigma_{\rho\gamma}^2$
TxC	$(ab-1)\cdot(c-1)$	$\Sigma(y_{ijk}-y_{ij}-y_{k}+y)^2$	$\sigma_\lambda^2+s\sigma_{\rho T\gamma}^2+\frac{rs}{(ab-1)(c-1)}\cdot\Sigma(T\gamma)_{ijk}^2$
RxTxC	$(r-1)\cdot(ab-1)\cdot(c-1)$	$\Sigma(y_{hijk}-y_{hij}-y_{hk}-y_{ijk}+y_{h}+y_{ij}+y_{k}-y)^2$	$\sigma_\lambda^2+s\sigma_{\rho T\gamma}^2$
Sampling Error	$rabc\cdot(s-1)$	$\Sigma(Y_{hijkl}-y_{hijk})^2$	σ_λ^2
Total	rabcs-1	$\Sigma(Y_{hijkl}-y)^2$	

TABLE 3.2.42

Split-plot design with bxc Factorial on subplots

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+\delta_{hi}+\beta_j+\gamma_k+(\beta\gamma)_{jk}+(\alpha\beta)_{ij}+(\alpha\gamma)_{ik}+(\alpha\beta\gamma)_{ijk}+\epsilon_{hijk}$$

$(h=1,2,\ldots,r;\ i=1,2,\ldots,a;\ j=1,2,\ldots,b;\ k=1,2,\ldots,c)$

$\rho_h, \delta_{hi}, \epsilon_{hijk}$ = random effects

$\alpha_i, \beta_j, \gamma_k, (\beta\gamma)_{jk}, (\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, (\alpha\beta\gamma)_{ijk}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma^2_\epsilon+bc\sigma^2_\delta+\frac{bcr\Sigma\alpha_i^2}{(a-1)}$
Error(a)	$(r-1)\cdot(a-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma^2_\epsilon+bc\sigma^2_\delta$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma^2_\epsilon+\frac{acr\Sigma\beta_j^2}{(b-1)}$
C	$(c-1)$	$\Sigma(y_k-y)^2$	$\sigma^2_\epsilon+\frac{abr\Sigma\gamma_k^2}{(c-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma^2_\epsilon+\frac{cr\Sigma(\alpha\beta)^2_{ij}}{(a-1)(b-1)}$
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma^2_\epsilon+\frac{br\Sigma(\alpha\gamma)^2_{ik}}{(a-1)(c-1)}$

(Table 3.2.42 continued.)

Source	d.f.	S.S.	E(M.S.)
BC	(b-1) $\cdot$(c-1)	$\Sigma(y_{jk}-y_j-y_k+y)^2$	$\sigma_\epsilon^2+\frac{ar\Sigma(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$
ABC	(a-1) $\cdot$(b-1) $\cdot$(c-1)	$\Sigma(y_{ijk}-y_{ij}-y_{ik}-y_{jk}+y_i+y_j+y_k-y)^2$	$\sigma_\epsilon^2+\frac{r\Sigma(\alpha\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$
Error(b)	a(r-1) $\cdot$(bc-1)	$\Sigma(Y_{hijk}-y_{hi}-y_{ijk}+y_i)^2$	σ_ϵ^2
Total	rabc-1	$\Sigma(Y_{hijk}-y)^2$	

TABLE 3.2.43

Split-split-plot design

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+\delta_{hi}+\beta_j+(\alpha\beta)_{ij}+\varphi_{hij}+\gamma_k+(\alpha\gamma)_{ik}+(\beta\gamma)_{jk}+(\alpha\beta\gamma)_{ijk}+\epsilon_{hijk}$$

$(h=1,2,\ldots,r;\ i=1,2,\ldots,a;\ j=1,2,\ldots,b;\ k=1,2,\ldots,c)$

$\rho_h, \delta_{hi}, \varphi_{hij}, \epsilon_{hijk}$ = random effects

$\alpha_i, \beta_j, \gamma_k$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+c\sigma_\varphi^2+bc\sigma_\delta^2+\frac{bcr}{a-1}\Sigma\alpha_i^2$
Error(a)	$(r-1)\cdot(a-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma_\epsilon^2+c\sigma_\varphi^2+bc\sigma_\delta^2$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+c\sigma_\varphi^2+\frac{acr}{(b-1)}\Sigma\beta_j^2$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+c\sigma_\varphi^2+\frac{cr}{(a-1)(b-1)}\cdot\Sigma(\alpha\beta)_{ij}^2$
Error(b)	$a(r-1)\cdot(b-1)$	$\Sigma(y_{hij}-y_{hi}-y_{ij}+y_i)^2$	$\sigma_\epsilon^2+c\sigma_\varphi^2$
C	$(c-1)$	$\Sigma(y_k-y)^2$	$\sigma_\epsilon^2+\frac{abr}{(c-1)}\Sigma\gamma_k^2$

(Table 3.2.43 continued.)

Source	d.f.	S.S.	E(M.S.)
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma_\epsilon^2+\frac{br}{(a-1)(c-1)}\Sigma(\alpha\gamma)_{ik}^2$
BC	$(b-1)\cdot(c-1)$	$\Sigma(y_{jk}-y_j-y_k+y)^2$	$\sigma_\epsilon^2+\frac{ar}{(b-1)(c-1)}\Sigma(\beta\gamma)_{jk}^2$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{ijk}-y_{ij}-y_{ik}-y_{jk}+y_i+y_j+y_k-y)^2$	$\sigma_\epsilon^2+\frac{r}{(a-1)(b-1)(c-1)}\cdot\Sigma(\alpha\beta\gamma)_{ijk}^2$
Error(c)	$ab(r-1)\cdot(c-1)$	$\Sigma(Y_{hijk}-y_{hij}-y_{ijk}+y_{ij})^2$	σ_ϵ^2
Total	$abrc-1$	$\Sigma(Y_{hijk}-y)^2$	

TABLE 3.2.44

Split-split-plot design-interactions between repetitions and each component of axbxc Factorial

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+(\alpha\rho)_{hi}+\beta_j+(\beta\rho)_{hj}+(\alpha\beta)_{ij}+(\rho\alpha\beta)_{hij}+\gamma_k+(\rho\gamma)_{hk}$$
$$+(\alpha\gamma)_{ik}+(\rho\alpha\gamma)_{hik}+(\beta\gamma)_{jk}+(\rho\beta\gamma)_{hjk}+(\alpha\beta\gamma)_{ijk}+\epsilon_{hijk}$$

$(h=1,2,\cdots,r;\ i=1,2,\cdots,a;\ j=1,2,\cdots,b;\ k=1,2,\cdots,c)$

$\rho_h, (\alpha\rho)_{hi}, (\rho\beta)_{hj}, (\rho\alpha\beta)_{hij}, (\rho\gamma)_{hk}, (\rho\alpha\gamma)_{hik}, (\rho\beta\gamma)_{hjk}, \epsilon_{hijk} =$ random effects.

$\alpha_i, \beta_j, \gamma_k, (\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, (\beta\gamma)_{jk}, (\alpha\beta\gamma)_{ijk}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+bc\sigma_{\rho\alpha}^2+\frac{rbc}{(a-1)}\Sigma\alpha_i^2$
RA	$(r-1)\cdot(a-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma_\epsilon^2+bc\sigma_{\rho\alpha}^2$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+ac\sigma_{\rho\beta}^2+\frac{rac}{(b-1)}\Sigma\beta_j^2$
RB	$(r-1)\cdot(b-1)$	$\Sigma(y_{hj}-y_h-y_j+y)^2$	$\sigma_\epsilon^2+ac\sigma_{\rho\beta}^2$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\epsilon^2+c\sigma_{\rho\alpha\beta}^2+\frac{rc}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$

(Table 3.2.44 continued.)

Source	d.f.	S.S.	E(M.S.)
RAB	$(r-1)\cdot(a-1)\cdot(b-1)$	$\Sigma(y_{hij}-y_{hi}-y_{hj}-y_{ij}+y_h+y_i+y_j-y)^2$	$\sigma_\epsilon^2+c\sigma_{\rho\alpha\beta}^2$
C	$(c-1)$	$\Sigma(y_k-y)^2$	$\sigma_\epsilon^2+ab\sigma_{\rho\gamma}^2+\frac{rab}{(c-1)}\Sigma\gamma_k^2$
RC	$(r-1)\cdot(c-1)$	$\Sigma(y_{hk}-y_h-y_k+y)^2$	$\sigma_\epsilon^2+ab\sigma_{\rho\gamma}^2$
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma_\epsilon^2+b\sigma_{\rho\alpha\gamma}^2+\frac{rb}{(a-1)(b-1)}\Sigma(\alpha\gamma)_{ik}^2$
RAC	$(r-1)\cdot(a-1)\cdot(c-1)$	$\Sigma(y_{hik}-y_{hi}-y_{hk}-y_{ik}+y_h+y_i+y_k-y)^2$	$\sigma_\epsilon^2+b\sigma_{\rho\alpha\gamma}^2$
BC	$(b-1)\cdot(c-1)$	$\Sigma(y_{jk}-y_j-y_k+y)^2$	$\sigma_\epsilon^2+\frac{ar}{(b-1)(c-1)}\Sigma(\beta\gamma)_{jk}^2$
RBC	$(r-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{hjk}-y_{hj}-y_{hk}-y_{jk}+y_h+y_j+y_k-y)^2$	$\sigma_\epsilon^2+a\sigma_{\rho\beta\gamma}^2$
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{ijk}-y_{ij}-y_{ik}-y_{jk}+y_i+y_j+y_k-y)^2$	$\sigma_\epsilon^2+\frac{r}{(a-1)(b-1)(c-1)}\cdot\Sigma(\alpha\beta\gamma)_{ijk}^2$
Residual	$(r-1)\cdot(a-1)\cdot(b-1)\cdot(c-1)$	(o)	σ_ϵ^2
Total	rabc-1	$\Sigma(Y_{hijk}-y)^2$	

$$(o) = \Sigma(Y_{hijk}-y_{hij}-y_{hik}-y_{hjk}-y_{ijk}+y_{hi}+y_{hj}+y_{hk}+y_{ij}+y_{ik}+y_{jk}-y_h-y_i-y_j-y_k+y)^2$$

TABLE 3.2.45

Split-split-plot design with sampling

Model:

$$Y_{hijkl} = \mu+\rho_h+\alpha_i+\delta_{hi}+\beta_j+(\alpha\beta)_{ij}+\varphi_{hij}+\gamma_k+(\alpha\gamma)_{ik}+(\alpha\beta\gamma)_{ijk}$$
$$+\epsilon_{hijk}+\lambda_{hijkl}$$

$(h=1,2,\ldots,r;i=1,2,\ldots,a;j=1,2,\ldots,b;k=1,2,\ldots,c;l=1,2,\ldots,s)$

$\rho_h, \delta_{hi}, \varphi_{hij}, \epsilon_{hijk}, \lambda_{hijkl}$ = random effects

$\alpha_i, \beta_j, (\alpha\beta)_{ij}, \gamma_k, (\alpha\gamma)_{ik}, (\alpha\beta\gamma)_{ijk}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	(r-1)	$\Sigma(y_h-y)^2$	
A	(a-1)	$\Sigma(y_i-y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sc\sigma_\varphi^2+sbc\sigma_\delta^2+\frac{sbcr}{(a-1)}\Sigma\alpha_i^2$
Error(a)	(r-1)·(a-1)	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sc\sigma_\varphi^2+sbc\sigma_\delta^2$
B	(b-1)	$\Sigma(y_j-y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sc\sigma_\varphi^2+\frac{sacr}{(b-1)}\Sigma\beta_j^2$
AB	(a-1)·(b-1)	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sc\sigma_\varphi^2+\frac{scr}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$
Error(b)	a(b-1)·(r-1)	$\Sigma(y_{hij}-y_{hi}-y_{ij}+y_i)^2$	$\sigma_\lambda^2+s\sigma_\epsilon^2+sc\sigma_\varphi^2$

(Table 3.2.45 continued.)

Source	d.f.	S.S.	E(M.S.)
C	(c-1)	$\Sigma(y_k - y)^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2 + \frac{sabr}{(c-1)}\Sigma\gamma_k^2$
AC	(a-1) ·(c-1)	$\Sigma(y_{ik} - y_i - y_k + y)^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2 + \frac{sbr}{(a-1)(c-1)} \cdot \Sigma(\alpha\gamma)_{ik}^2$
BC	(b-1) ·(c-1)	$\Sigma(y_{jk} - y_j - y_k + y)^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2 + \frac{sar}{(b-1)(c-1)} \cdot \Sigma(\beta\gamma)_{jk}^2$
ABC	(a-1) ·(b-1) ·(c-1)	$\Sigma(y_{ijk} - y_{ij} - y_{ik} - y_{jk} + y_i + y_j + y_k - y)^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2 + \frac{sr}{(a-1)(b-1)(c-1)} \cdot \Sigma(\alpha\beta\gamma)_{ijk}^2$
Error(c)	ab(r-1) ·(c-1)	$\Sigma(y_{hijk} - y_{hij} - y_{ijk} + y_{ij})^2$	$\sigma_\lambda^2 + s\sigma_\epsilon^2$
Sampling error	abcr ·(s-1)	$\Sigma(Y_{hijkl} - y_{hijk})^2$	σ_λ^2
Total	abcrs-1	$\Sigma(Y_{hijkl} - y)^2$	

TABLE 3.2.46

Split-split-plot design-interactions between repetitions and each component of axbxc Factorial with sampling

Model:

$$Y_{hijk} = \mu+\rho_h+\alpha_i+(\alpha\rho)_{hi}+\beta_j+(\beta\rho)_{hj}+(\alpha\beta)_{ij}+(\rho\alpha\beta)_{hij}+\gamma_k+(\rho\gamma)_{hk}$$
$$+(\alpha\gamma)_{ik}+(\rho\alpha\gamma)_{hik}+(\beta\gamma)_{jk}+(\rho\beta\gamma)_{hjk}+(\alpha\beta\gamma)_{ijk}+\epsilon_{hijk}$$
$$+\varphi_{hijkl}$$

$(h=1,2,\cdots,r;\ i=1,2,\cdots,a;\ j=1,2,\cdots,b;\ k=1,2,\cdots,s)$

$\rho_h, (\alpha\rho)_{hi}, (\rho\beta)_{hj}, (\rho\alpha\beta)_{hij}, (\rho\gamma)_{hk}, (\rho\alpha\gamma)_{hik}, (\rho\beta\gamma)_{hjk}, \epsilon_{hijk}$,

and φ_{hijkl} = random effects

$\alpha_i, \beta_j, \gamma_k, (\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, (\beta\gamma)_{jk}, (\alpha\beta\gamma)_{ijk}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Reps.	$(r-1)$	$\Sigma(y_h-y)^2$	
A	$(a-1)$	$\Sigma(y_i-y)^2$	$\sigma^2_\varphi+sbc\sigma^2_{\rho\alpha}+\frac{srbc}{(a-1)}\Sigma\alpha^2_i$
RA	$(r-1)\cdot(a-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma^2_\varphi+sbc\sigma^2_{\rho\alpha}$
B	$(b-1)$	$\Sigma(y_j-y)^2$	$\sigma^2_\varphi+sac\sigma^2_{\rho\beta}+\frac{srac}{(b-1)}\Sigma\beta^2_j$
RB	$(r-1)\cdot(b-1)$	$\Sigma(y_{hj}-y_h-y_j+y)^2$	$\sigma^2_\varphi+acs\sigma^2_{\rho\beta}$

(Table 3.2.46 continued.)

Source	d.f.	S.S.	E(M.S.)
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{ij}-y_i-y_j+y)^2$	$\sigma_\varphi^2+cs\sigma_{\rho\alpha\beta}^2 + \frac{rcs}{(a-1)(b-1)}\Sigma(\alpha\beta)_{ij}^2$
RAB	$(r-1)\cdot(a-1)\cdot(b-1)$	$\Sigma(y_{hij}-y_{hi}-y_{hj}-y_{ij}+y_h+y_i+y_j-y)^2$	$\sigma_\varphi^2+cs\sigma_{\rho\alpha\beta}^2$
C	$(c-1)$	$\Sigma(y_k-y)^2$	$\sigma_\varphi^2+sab\sigma_{\rho\gamma}^2 + \frac{srab}{(c-1)}\Sigma\gamma_k^2$
RC	$(r-1)\cdot(c-1)$	$\Sigma(y_{hk}-y_h-y_k+y)^2$	$\sigma_\varphi^2+sab\sigma_{\rho\gamma}^2$
AC	$(a-1)\cdot(c-1)$	$\Sigma(y_{ik}-y_i-y_k+y)^2$	$\sigma_\varphi^2+sb\sigma_{\rho\alpha\gamma}^2 + \frac{rbs}{(a-1)(c-1)}\Sigma(\alpha\gamma)_{ik}^2$
RAC	$(r-1)\cdot(a-1)\cdot(c-1)$	$\Sigma(y_{hik}-y_{hi}-y_{hk}-y_{ik}+y_h+y_i+y_k-y)^2$	$\sigma_\varphi^2+sb\sigma_{\rho\alpha\gamma}^2$
BC	$(b-1)\cdot(c-1)$	$\Sigma(y_{jk}-y_j-y_k+y)^2$	$\sigma_\varphi^2+as\sigma_{\rho\beta\gamma}^2 + \frac{ars}{(b-1)(c-1)}\Sigma(\beta\gamma)_{jk}^2$
RBC	$(r-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{hjk}-y_{hj}-y_{hk}-y_{jk}+y_h+y_j+y_k-y)^2$	$\sigma_\varphi^2+as\sigma_{\rho\beta\gamma}^2$

(Table 3.2.46 continued.)

Source	d.f.	S.S.	E(M.S.)
ABC	$(a-1)\cdot(b-1)\cdot(c-1)$	$\Sigma(y_{ijk}-y_{ij}-y_{ik}-y_{jk}+y_i+y_j+y_k-y)^2$	$\sigma^2_\varphi+s\sigma^2_{\rho\alpha\beta\gamma}+\frac{rs}{(a-1)(b-1)(c-1)}\cdot\Sigma(\alpha\beta\gamma)^2_{ijk}$
RABC	$(r-1)\cdot(a-1)\cdot(b-1)\cdot(c-1)$	(o)	$\sigma^2_\varphi+s\sigma^2_{\rho\alpha\beta\gamma}$
Sampling Error	$rabc\cdot(s-1)$	$\Sigma(Y_{hijkl}-y_{hijk})^2$	σ^2_φ
Total	rabcs-1	$\Sigma(Y_{hijkl}-y)^2$	

$$(o) = \Sigma(y_{hijk}-y_{hij}-y_{hik}-y_{hjk}-y_{ijk}+y_{hi}+y_{hj}+y_{hk}+y_{ij}+y_{ik}$$

$$+y_{jk}-y_h-y_i-y_j-y_k+y)^2$$

TABLE 3.2.47

Split-split-plot design with cxd factorial on the sub-subplots

Model:

$$Y_{ghijk} = \mu+\rho_g+\alpha_h+\theta_{gh}+\beta_i+(\alpha\beta)_{hi}+\varphi_{ghi}+T_{jk}+(\alpha T)_{hjk}+(\beta T)_{ijk}+(\alpha\beta T)_{hijk}+\epsilon_{ghijk}$$

where:

$$T_{jk} = \gamma_i+\delta_j+(\gamma\delta)_{ij}$$

$(g=1,2,\ldots,r;h=1,2,\ldots,a;i=1,2,\ldots,b;j=1,2,\ldots,c;k=1,2,\ldots,d)$

$\rho_g,\theta_{gh},\varphi_{ghi},\epsilon_{ghijk}$ = random effects

$\alpha_h,\beta_i,(\alpha\beta)_{hi},T_{jk},(\alpha T)_{hjk},(\beta T)_{ijk},(\alpha\beta T)_{hijk}$ = fixed effects

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Rep.	$(r-1)$	$\Sigma(y_g-y)^2$	
A	$(a-1)$	$\Sigma(y_h-y)^2$	$\sigma_\epsilon^2+cd\sigma_\varphi^2+bcd\sigma_\theta^2+\frac{rbcd\Sigma\alpha_h^2}{(a-1)}$
Error(a)	$(r-1)\cdot(a-1)$	$\Sigma(y_{gh}-y_g-y_h+y)^2$	$\sigma_\epsilon^2+cd\sigma_\varphi^2+bcd\sigma_\theta^2$
B	$(b-1)$	$\Sigma(y_i-y)^2$	$\sigma_\epsilon^2+cd\sigma_\varphi^2+\frac{racd\Sigma\beta_i^2}{(b-1)}$
AB	$(a-1)\cdot(b-1)$	$\Sigma(y_{hi}-y_h-y_i+y)^2$	$\sigma_\epsilon^2+cd\sigma_\varphi^2+\frac{rcd\Sigma(\alpha\beta)_{hi}^2}{(a-1)(b-1)}$
Error(b)	$a(b-1)\cdot(r-1)$	$\Sigma(y_{ghi}-y_{gh}-y_{hi}+y_h)^2$	$\sigma_\epsilon^2+cd\sigma_\varphi^2$

(Table 3.2.47 continued.)

Source	d.f.	S.S.	E(M.S.)
T	(cd-1)	$\Sigma(y_{jk}-y)^2$	$\sigma_\epsilon^2+\frac{rab\Sigma T_{jk}^2}{(cd-1)}$
C	(c-1)	$\Sigma(y_j-y)^2$	$\sigma_\epsilon^2+\frac{rabd\Sigma\gamma_j^2}{(c-1)}$
D	(d-1)	$\Sigma(y_k-y)^2$	$\sigma_\epsilon^2+\frac{rabc\Sigma\delta_k^2}{(d-1)}$
CD	(c-1) ·(d-1)	$\Sigma(y_{jk}-y_j-y_k+y)^2$	$\sigma_\epsilon^2+\frac{rab\Sigma(\gamma\delta)_{jk}^2}{(c-1)(d-1)}$
AT	(a-1) ·(cd-1)	$\Sigma(y_{hjk}-y_h-y_{jk}+y)^2$	$\sigma_\epsilon^2+\frac{rb\Sigma(\alpha T)_{hjk}^2}{(a-1)(cd-1)}$
BT	(b-1) ·(cd-1)	$\Sigma(y_{ijk}-y_i-y_{jk}+y)^2$	$\sigma_\epsilon^2+\frac{ra\Sigma(\beta T)_{ijk}}{(b-1)(cd-1)}$
ABT	(a-1) ·(b-1) ·(cd-1)	$\Sigma(y_{hijk}-y_{hjk}-y_{hi}-y_{ijk}+y_h+y_i+y_{jk}-y)^2$	$\sigma_\epsilon^2+\frac{r\Sigma(\alpha\beta T)_{hijk}^2}{(a-1)(b-1)(cd-1)}$
Error(c)	ab(r-1) ·(cd-1)	$\Sigma(Y_{ghijk}-y_{ghi}-y_{hijk}+y_{hi})^2$	σ_ϵ^2
Total	rabcd-1	$\Sigma(Y_{ghijk}-y)^2$	

TABLE 3.2.48

Incomplete Block Designs

case of a Simple Lattice

Model:

$$Y_{ijq} = \mu+\rho_i+\beta_{i(j)}+T_q+\epsilon_{ijq}$$

$$(i=1,2;\ j=1,2,\ldots,k;\ q=1,2,\ldots,k^2)$$

$\rho_i, \beta_{i(j)}$ = random effects

T_q = fixed effect

Analysis of variance:

Source	d.f.	S.S.	E(M.S.)
Replications	1	$\Sigma(y_i-y)^2$	
Blocks within rep.(adj.)	$2(k-1)$	$S.S._B$	$\sigma_\epsilon^2+\frac{k}{2}\sigma_\beta^2$
Treatments (unadj.)	k^2-1	$\Sigma(y_q-y)^2$	
Intra-block error	$(k-1)^2$	$S.S._E$	σ_ϵ^2
Total	$2k^2-1$	$\Sigma(Y_{ijq}-y)^2$	

where:

$$S.S._B = \frac{\Sigma\, C_{ij}^2}{2k} - \frac{\Sigma\, R_i^2}{2k^2}$$

C_{ij} = [total (over all replicates) of all treatments in block (ij)-th] $-2B_{ij}$

B_{ij} = total of block ij

R_i = total of replicate i-th.

$S.S._E$ by subtraction

CHAPTER IV

THE COMPUTING CENTER

4.1 Organization

In the chart No. 1 we can see the situation of the computer center within the present organizational structure of VPI(29).

The center consists of a Director (reporting to the Vice-president for administration) with responsibility for an administration of all computing equipment and all personnel associated with this equipment on campus. It also includes one Assistant Director, one staff consultant, two full-time programmers, one system analyst, three operators, one in charge of key-punchers, four key-punchers, three part-time programmers, three to four part-time key-punchers.

4.2 Facilities

The computing center(29) occupies the Building #1365 of VPI. It consists of an IBM 7040 with 32,768 words of magnetic core memory, eight 729 Model II Tape Drives on channel A, and a 1622 Card Read Punch Unit. Associated with this central computer is an IBM 1401 with 8,000 characters of memory, a 1402 Read Punch Unit, and 1403 Printer. The 1401 can be operated in an on-line mode with the 7040. Four of the tape drives are switchable on an individual basis to the 1401. Both the 7040 and the 1401 have a large array of extended performance options available. A complete array of unit record equipment including eight key punchers, two verifiers, two sorters, one producer, one collator, and a tabulator are available to support this operation.

For administrative use in the basement of Burrus Hall, there is a similarly equipped IBM 1401 except that the four tape drives are 1330's. The unit record equipment associated with the 1401 includes three key punchers, one verifier, two sorters, one producer, one collator, one interpreter, and a tabulator.

The Virginia Polytechnic Institute, like some other universities in the South, has easy access to high-speed computing machines of the Oak Ridge National Laboratory and the Langley Research Center.

4.3 Computing service

The computer center offers data processing service for education and research purposes. It offers ample computing tool for staff research and graduate theses.

We shall list some of the classical programs commonly used in their computing service:

Paired Comparisons

*Simple Correlation

*Multiple regression

*Multiway Balanced ANOVA

Completely Unbalanced Nested ANOVA

Paired and Pooled t-test

*One-way ANOVA

Probit analysis

*Simple data description

Factor analysis

Multiple range

*Stepwise regression

*General Least Squares

*Frequency tabulation and Cross tabulation

Guttman scale

Discriminant analysis

*Diallel Analysis

*Lattice

With an asterisk * we indicate those programs that are most frequently used.

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RÉSUMÉ OF THE DEPARTMENT OF STATISTICS OF VIRGINIA POLYTECHNIC INSTITUTE

by

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ABSTRACT

This thesis gives an outline of the organization, importance and objectives of the Department of Statistics of Virginia Polytechnic Institute.

In 1949, the Department of Statistics was established at VPI. Now it offers curriculums leading to a B.S. degree with a major in statistics, and to M.S. and Ph.D. degrees in statistics. It also offers courses for students majoring in other fields.

The Department and its statistical laboratory are engaged in fundamental and applied research toward the development and extension of basic theory as well as the application of existing statistical techniques to applied problems in various fields.

The Department, through its statistical laboratory provides both consulting and computing service to the Virginia Agricultural Experiment Station, Virginia Engineering Experiment Station, other Departments of VPI, Virginia Truck

Experiment Station, and other state and federal research agencies.

Because of the importance of the consulting and computing services, a particular type of analysis of experiments, the analysis of variance, is discussed. To indicate the amount of work needed for such analysis, the computational procedures for 48 different experimental situations and mathematical models are given. The corresponding analysis of variance tables include S.S.,d.f. and E(M.S.).

The Institute's Computer Center gives considerable aid to the computational work of the Department, especially in handling extensive numerical analysis. It includes an IBM 7040/1401 Data processing system and related equipment.