

# The Odd Chern Character and Obstruction Theory

by

Constantin Dorin Dumitraşcu

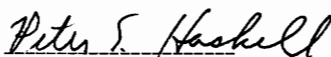
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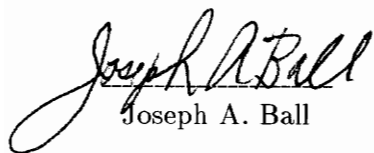
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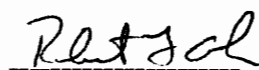
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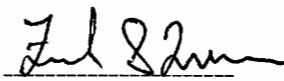
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# The Odd Chern Character and Obstruction Theory

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( ABSTRACT )

Having as starting point a formula described in the paper of Baum & Douglas, [**BmDg**], the odd-degree component of the Chern character is analyzed. Our presentation uses the obstruction theory definition of Chern characteristic classes in order to emphasize the connections with the even-degree component (see Theorem 4.3.1) and leads to a natural justification of the fundamental property of the Chern character, i.e. of being a ring homomorphism. The reader is assumed to have some background in topological  $K$ -theory and algebraic topology.

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Dorin Dumitraşcu  
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# 1 Introduction

This paper has its origins in the desire to understand the odd dimensional case of the isomorphism between analytical and topological  $K$ -homology, as presented in [BmDg]. One essential ingredient for the application of this result to the index theory of elliptic pseudo-differential operators is the Chern character.

The statement of the fundamental theorem for this branch of contemporary mathematics, the Atiyah-Singer index theory, is the following ([AtSi], 2.12):

**1.1 Theorem.** *Let  $P$  be an elliptic operator over a compact manifold  $X$ , and let  $u \in K(TX)$  be the symbol class of  $P$ . Then the index of  $P$  is given by:*

$$\text{index}(P) = (-1)^n \{ch(u) \cdot \mathcal{I}(X)\}[TX] .$$

Here,  $ch(u)$  is the Chern character of the symbol  $u$ ,  $\mathcal{I}(X)$  is the Index class of  $X$  defined as the Todd class of the complexification of the tangent bundle,  $TX$  is oriented as an almost complex manifold with “horizontal part real and vertical part imaginary”, “ $\cdot$ ” denotes the cup product, and  $n = \dim X$ . For the right-hand side of the index formula, the top-dimensional component of  $ch(u) \cdot \mathcal{I}(X)$  is evaluated on the fundamental homology class of  $TX$ .

In the section § 20 , “Odd dimensional  $Spin^c$  manifolds: Toeplitz operators”, of the mentioned paper of Baum&Douglas, it is pointed out that, on a closed odd dimensional  $Spin^c$  manifold  $M$ , the interesting problem is not for the Dirac operator itself (which is

self-adjoint and so has index zero) but rather for Toeplitz operators on the positive space of the Dirac operator. Let  $F$  be a vector bundle over  $M$ , and let  $D : \mathcal{C}^\infty(F) \rightarrow \mathcal{C}^\infty(F)$  be a Dirac operator. Once a hermitian structure for  $F$  and a smooth measure for  $M$  have been chosen, we can view  $D$  as an unbounded self-adjoint operator on the Hilbert space of  $L^2$  sections of  $F$ :  $D : L^2(F) \rightarrow L^2(F)$ . Let  $L_+^2(F)$  be the closed subspace of  $L^2(F)$  spanned by the eigenvectors belonging to non-negative eigenvalues of  $D$ . If  $\beta : M \rightarrow GL(n, \mathbf{C})$  is a continuous map, a bounded Fredholm operator  $T_\beta = \| T_{\beta_{ij}} \|$  can be constructed on  $\mathbf{C}^n \otimes_{\mathbf{C}} L_+^2(F)$ . The following topological formula holds (Theorem (20.3), [BmDg]; also see [Dgls], p.65, for a more general statement):

**1.2 Theorem.**

$$Index(T_\beta) = \{ch(\beta) \cdot Td(TM)\}[M] .$$

Here,  $ch(\beta)$  is the odd-degree Chern character that our paper is about. The explicit formula for it is given below (see  $(\star)$ ).  $Td(TM)$  is the Todd class of the tangent bundle  $TM$  and  $[M]$  is the fundamental homology class of  $M$ .

These two statements tell us immediately that the Chern character, with its even-degree and odd-degree parts, plays an essential role in index formulas. It conveys invariants of the elliptic pseudo-differential operator sitting in the  $K$ -theory groups into cohomological invariants. Using the duality between cohomology and homology, the latter give numerical invariants of the operator.

The literature gives a full treatment of the even-degree part of the Chern character, denoted by  $ch^{even}$  in our paper, which is basically a map

$$ch^{even} : K^0(X) \rightarrow H^{even}(X),$$

$$(\diamond) \quad ch^{even}([\xi]) = \sum_{k=0}^{\infty} ch_k^{even}([\xi]),$$

where

$$ch_k^{even}([\xi]) = \frac{1}{k!} S_k(\mathbf{c}_1(\xi), \mathbf{c}_2(\xi), \dots, \mathbf{c}_n(\xi)), \quad k \geq 0.$$

In the above,  $K^0(X)$  is the first group of topological K-theory of the space  $X$ ,  $H^{even}(X) = \bigoplus_{i=0}^{\infty} H^{2i}(X)$ ,  $\xi$  is an  $n$ -dimensional complex vector bundle over  $X$ ,  $\mathbf{c}_j(\xi)$ 's are the Chern characteristic classes of  $\xi$ , and  $S_k$  are the *power sums*, or *Newton polynomials*. ( $S_k$  expresses the symmetric function  $\sum_{i=1}^n x_i^k$  as unique polynomial of the elementary symmetric function,  $a_r = \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \cdot \dots \cdot x_{i_r}$ . We remark that:  $ch_0^{even}([\xi]) = n = \dim(\xi)$ .) See Subsection 3.3 for more details.

Despite this complete description of the even part, the odd degree part, denoted  $ch^{odd}$  in our paper, is usually treated with only a brief reference to formal constructions in algebraic topology involving suspension ( see [Karb], V.3.26 ), or by *mention* of the resulting formula ( see [BmDg], p.157 ):

$$ch^{odd} : K^{-1}(X) \rightarrow H^{odd}(X),$$

$$(\star) \quad ch^{odd}([f]) = f^* \left( \frac{\omega_1}{0!} - \frac{\omega_3}{1!} + \frac{\omega_5}{3!} - \dots + (-1)^{n-1} \frac{\omega_{2n-1}}{(n-1)!} \right).$$

Here,  $f : X \rightarrow U(n, \mathbf{C})$  is continuous,  $[f]$  is the class of  $f$  in  $K^{-1}(X)$ ,  $\{\omega_1, \omega_3, \dots, \omega_{2n-1}\}$  are the standard generators in the algebra  $H^*(U(n); \mathbf{Z})$  (which is an exterior algebra over these generators; see Subsection 3.2), and  $f^*$  denotes the pull-back map induced by  $f$  at the level of cohomologies.

It is our hope that the treatment given in the present paper will enlighten, among others, the way suspension relates obstruction theory and the even degree Chern character, on one hand, with the formula involving elements of  $H^*(U(n); \mathbf{Z})$ , on the other hand. This will show how the basic constructions in topological  $K$ - theory are reflected in the main property of the Chern character, i.e. of being a ring homomorphism.

The effort to understand the odd Chern character led us to the following diagram (Theorem 4.3.1):

$$\begin{array}{ccccc}
 & K^{-1}(X) & \xrightarrow{gl} & \tilde{K}(S'(X^+)) & \cong & \tilde{K}(S(X^+)) \\
 (\square) \quad ch^{odd} & \downarrow & & & & \downarrow & ch^{even} \\
 & H^{odd}(X) & \xrightarrow{\alpha} & \tilde{H}^{even}(S(X^+)) & & & 
 \end{array}$$

In this diagram, referred to throughout this paper as  $(\square)$ ,  $X^+$  is the disjoint union of  $X$  with a point  $\{\infty\}$ .  $gl$  is the map that can be used to give an equivalent way of defining  $K^{-1}(X)$ : it constructs from an element  $f : X \rightarrow GL(n, \mathbf{C})$  a vector bundle  $\xi_f$  over the reduced suspension of  $X^+$ ,  $S'(X^+)$ . A complete description of  $gl$  is given in Subsection 3.1. The map  $\alpha$  is *essentially* given by the canonical isomorphism from algebraic topology between the reduced cohomologies of the topological space  $X$  and its suspension  $SX$ . A complete characterization is given in Subsection 2.3.

$ch^{odd}$  is given in  $(\star)$ , and  $ch^{even}$  in  $(\diamond)$ .

$\alpha$  being independent of the particular element in  $K^{-1}(X)$  we are starting with, the philosophical problem is to decipher the influence of  $f$  in the formulas for  $ch^{even}$  and  $ch^{odd}$ .

There are, of course, two distinct parts in the above diagram:

$$\begin{array}{ccc}
 f & & f \xrightarrow{gl} [\xi_f] \\
 ch^{odd} \quad \downarrow & \text{and} & \downarrow \quad ch^{even} \\
 f^* \left( \sum_{i=0}^{n-1} (-1)^i \frac{\omega_{2i+1}}{i!} \right) \xrightarrow{\alpha} & & ch^{even}([\xi_f]) \\
 (L.S.) & & (R.S.)
 \end{array}$$

Proving the “commutativity” of  $(\square)$  will be the main step in our presentation.

We understand also that this is not the only possible treatment of the odd Chern character, but we consider the approach worthwhile.

Warning. In the paper we will constantly use results and techniques from algebraic topology and  $K$ -theory. The reader is invited to consult some of the standard texts: [Span], respectively [Atiy]. Nevertheless, the essential references are:

- Hatcher's book ([Htch]) for the concepts used, some of the notation, and the description of the cohomology of  $U(n)$ ;
- Karoubi's book ([Karb]) for the  $K$ -theoretical part of the paper;
- Steenrod's book ([Stnr]) for the obstruction theory.

It is to be mentioned that we tried to make the thesis as self-contained as possible. This is the main reason for the reviews presented in some subsections.

The most important parts of our paper are:

- Subsection 3.2 in which the explicit description of the generators  $\omega_{2k-1}$  of  $H^*(U(n); \mathbf{Z})$  ( $1 \leq k \leq n$ ) is obtained using Thom classes of sphere bundles (see Proposition 3.2.12).
- Subsection 4.2 in which the technical formulas are obtained, using the special structure of the suspension of a  $\Delta$ -complex (see Theorem 4.2.10).

The main result is proved in Subsection 4.3. The property of the Chern character of being a ring homomorphism is discussed in Section 5.

Notations. In our paper  $\mathbf{N}$ ,  $\mathbf{Z}$ ,  $\mathbf{Q}$ ,  $\mathbf{R}$ ,  $\mathbf{C}$  have the usual significance and algebraic properties of the sets of natural, integer, rational, real, and complex numbers, respectively. A reference of the type (2.3.7) sends to the mathematical statement number 7 of Subsection 3 in Section 2. The symbol ■ marks the end of a proof. We used “ $\stackrel{def}{=}$ ” to symbolize a notation we are making between the mathematical expression on the left of the symbol and the one which immediately follows it .

## 2 Homology and Cohomology

### 2.1 General facts regarding homology and cohomology

This subsection summarizes the essential definitions and properties of homology and cohomology. There are also discussed the reduced and unreduced suspensions of a topological space. The purpose is to establish the notations and the background.

**2.1.1 Definition.** A CW-complex is the topological space  $X$  resulting after the following construction:

- (1) Start with a discrete set  $X^0$ .
- (2) Inductively, form the  $k$ -skeleton  $X^k$  from  $X^{k-1}$  by attaching  $k$ -disks  $D_\alpha^k$  via continuous maps  $\phi_\alpha : \partial D_\alpha^k \rightarrow X^{k-1}$ . That is,  $X^k$  is the quotient space of the disjoint union  $X^{k-1} \coprod_\alpha D_\alpha^k$  under the relation  $x \sim \phi_\alpha(x)$ , for  $x \in \partial D_\alpha^k$ . In particular,  $X^k \supset X^{k-1}$ .
- (3) The union  $X = X^0 \cup X^1 \cup \dots$  is given the weak topology: a set  $A \subset X$  is open (closed) iff  $A \cap X^k$  is open (closed) in  $X^k$ , for each  $k$ .

In a CW-complex structure on  $X$ , each attached disk  $D_\alpha^k$  has a natural characteristic map  $\Phi_\alpha : D_\alpha^k \rightarrow X$  extending its attaching map  $\phi_\alpha$ . On the interior of  $D_\alpha^k$ ,  $\Phi_\alpha$  is a homeomorphism onto its image, an open  $k$ -cell  $e_\alpha^k$ , which is an open set in  $X^k$  though not necessarily in  $X$ .

In this paper we shall restrict our attention to a subcategory of the category of CW-complexes, namely the  $\Delta$ -complexes. This extra condition is due to the use of obstruction

theory in the construction of the Chern characteristic classes. Moreover, reasons from  $K$ -theory will force us to consider *compact*  $\Delta$ -complexes. It is to be mentioned that the isomorphism of Baum&Douglas between analytical and topological  $K$ -homology ([**BmDg**], Theorem (18.5)) is given for *finite* simplicial complexes.

For  $k \geq 0$ , let  $\Delta^k = [v_0, v_1, \dots, v_k] = \{(t_0, t_1, \dots, t_k) \in \mathbf{R}^{k+1} \mid \sum_{i=0}^k t_i = 1\}$  denote the standard  $k$ -simplex in  $\mathbf{R}^{k+1}$ . The following definition is in order:

**2.1.2 Definition.** A  $\Delta$ -complex is a  $CW$ -complex  $X$  whose cells  $e_\alpha^k$  are provided with distinguished characteristic maps  $\sigma_\alpha : \Delta^k \rightarrow X$  such that each restriction  $\sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_k]}$  is the distinguished characteristic map  $\sigma_\beta$  of some cell  $e_\beta^{k-1}$ .

It is to be emphasized in this definition the presence of the characteristic maps  $\sigma_\alpha$ , and the condition about the restriction to boundary. They will enable us to use obstruction theory for the category of  $\Delta$ -complexes. Also note that, a  $\Delta$ -complex being a  $CW$ -complex, the interiors of the  $k$ -simplices are pairwise disjoint. Here, and all over the paper, by a  $k$ -simplex we mean, depending on context, either an application  $\sigma_\alpha$  as in Definition 2.1.2, or the image of  $\Delta^k$  in  $X$  under such  $\sigma_\alpha$ .

Implicit in (2.1.2) is the identification of  $[v_0, \dots, \hat{v}_i, \dots, v_k]$  with  $\Delta^{k-1}$  preserving the ordering of the vertices. Note also that the edges of a  $\Delta$ -complex have preferred orientations induced from the natural orientation of  $\Delta^1 = [v_0, v_1]$ , and that the orientations of the edges of each  $k$ -simplex of a  $\Delta$ -complex are related just as they are in  $\Delta^k$ , consistent with the preferred linear ordering of the vertices of  $\Delta^k$ .

It is an easy exercise to show that the second barycentric subdivision of a  $\Delta$ -complex gives rise to a simplicial complex.

## Remarks about the singular (co)chain complexes

Let  $X$  be an arbitrary topological space, and let  $R$  be a ring. (Our main interest will be  $R = \mathbf{Q}$ , the field of rational numbers.) For each  $k \in \mathbf{N}$ , we define  $C_k(X; R) = C_k(X)$  to be the free  $R$ -module having the singular  $k$ -simplices,  $\sigma : \Delta^k \rightarrow X$ , as generators. The *chain complex*  $(C_*(X), \partial)$  gives the singular homology modules of  $X$ ,  $H_*(X; R)$ .

If  $X$  is a  $\Delta$ -complex we consider the  $R$ -modules  $\Delta_k(X)$  generated by the distinguished characteristic maps appearing in Definition 2.1.2, with the same boundary as in the singular case. The resulting homology is isomorphic with the singular one (a non-trivial result!).

The *cochain complex*  $(C^*(X, R), \delta)$  gives the singular cohomology of  $X$  and is obtained by dualizing the chain complex:

$$C^k(X; R) = \text{Hom}_R(C_k(X), R) \quad , \quad \delta = \partial^*.$$

Let  $A \subset X$  be a subspace, and let

$$0 \rightarrow C_k(A) \xrightarrow{i} C_k(X) \xrightarrow{j} C_k(X, A) \rightarrow 0$$

be the short exact sequence which defines the relative homology modules  $H_k(X, A)$ . (Here,  $C_k(X, A) = C_k(X)/C_k(A)$ , and the boundary map is the restriction of  $\partial$  from  $C_*(X)$  to the equivalence classes described above.  $H_*(X, A)$  is the homology of the resulting complex.) By applying  $\text{Hom}_R(-, R)$ , we obtain the short exact sequence:

$$(2.1.3) \quad 0 \leftarrow C^k(A; R) \xleftarrow{i^*} C^k(X; R) \xleftarrow{j^*} C^k(X, A; R) \leftarrow 0.$$

The *relative cohomology modules*  $H^k(X, A; R)$  are defined as  $\ker(\delta)/\text{im}(\delta)$  in the complex

$$\dots \rightarrow C^k(X, A; R) \xrightarrow{\delta} C^{k+1}(X, A; R) \rightarrow \dots,$$

where  $\delta$  is the restriction of the coboundary of the cochain complex  $(C^*(X, R), \delta)$ .

The relative cochains on  $(X, A)$  are absolute cochains on  $X$  which vanish on chains in  $A$ .

The relative cocycles are the relative cochains which vanish on boundaries.

The following result is an easy consequence of the universal coefficient theorem for cohomology:

**2.1.4 Theorem.** *If  $H_{k-1}(X)$  is zero or is a free  $R$ -module, then  $H^k(X, R)$  is canonically isomorphic to the module  $\text{Hom}_R(H_k(X); R)$ . There is a corresponding assertion for pairs  $(X, A)$ .*

#### Remarks about suspensions

Let  $x_0$  be a distinguished point of  $X$ .

Let  $I = [0, 1]$ ,  $\partial I = \{0, 1\}$ . Then  $S^1 \cong I/\partial I$ .

**2.1.5 Definition.** *The reduced suspension of the topological space  $X$ ,  $SX$ , is the smash product of  $X$  with  $S^1$ :*

$$SX = X \wedge S^1 = X \times I / (X \times \{0, 1\} \cup \{x_0\} \times I).$$

Let us next define the cones on  $X$ :

$$C^+X = X \times [\frac{1}{2}, 1] / X \times \{1\} ,$$

$$N \stackrel{\text{not}}{=} X \times \{1\} / X \times \{1\} = \text{the "north" pole of the suspension}$$

and

$$C^-X = X \times [0, \frac{1}{2}] / X \times \{0\} ,$$

$$S \stackrel{\text{not}}{=} X \times \{0\} / X \times \{0\} = \text{the "south" pole of the suspension} .$$

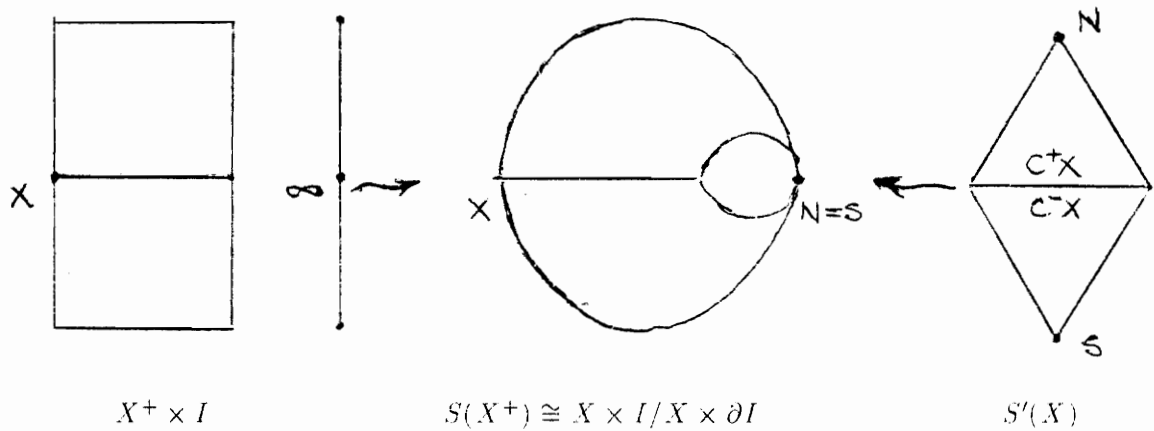
**2.1.6 Definition.** The (unreduced) suspension of  $X$ ,  $S'X$ , is

$$S'X = C^+X \cup C^-X .$$

Let  $X^+ = X \cup \{\infty\}$  be the disjoint union of  $X$  with a point. The next homeomorphism is clear:

$$S(X^+) \cong S'X / (N = S) .$$

The following diagram gives the intuitive feeling about this identification:



**Remark 1.** Even after the identification of  $N$  with  $S$  in forming  $S(X^+)$  from  $S'X$ , we shall continue to refer to the images of  $C^+X$  and  $C^-X$  in  $S(X^+)$  as  $C^+X$  and  $C^-X$  respectively. This identification must be understood without further reference from now on. It is our hope that this is confusion-free.

Our next goal is to give a *canonical*  $\Delta$ -complex structure on  $S(X^+)$ . This will be used extensively in Section 4, and it is very important for our exposition.

All the characteristic maps of  $X$  can be regarded in a natural way, under the inclusion  $X \subseteq S(X^+)$ , as characteristic maps in  $S(X^+)$ . Beside these, from each characteristic map

in  $X$ ,  $\sigma_\alpha : \Delta^k \rightarrow X$ , we shall construct two characteristic maps in  $S(X^+)$ ,

$$(*\sigma_\alpha)^+ \quad \text{and} \quad (*\sigma_\alpha)^- ,$$

in the following way:

$$(*\sigma_\alpha)^+ : \Delta^{k+1} \rightarrow C^+ X$$

$$(2.1.7) \quad (*\sigma_\alpha)^+(t_0, \dots, t_k, t_{k+1}) = \begin{cases} \left( \left( \sigma_\alpha \left( \frac{t_0}{1-t_{k+1}}, \dots, \frac{t_k}{1-t_{k+1}} \right), \frac{t_{k+1}}{2} + \frac{1}{2} \right) \\ N (= S) \end{cases} \quad \text{if } t_{k+1} = 1$$

and

$$(*\sigma_\alpha)^- : \Delta^{k+1} \rightarrow C^- X$$

$$(2.1.8) \quad (*\sigma_\alpha)^-(t_0, \dots, t_k, t_{k+1}) = \begin{cases} \left( \left( \sigma_\alpha \left( \frac{t_0}{1-t_{k+1}}, \dots, \frac{t_k}{1-t_{k+1}} \right), \frac{1}{2} - \frac{t_{k+1}}{2} \right) \\ S (= N) \end{cases} \quad \text{if } t_{k+1} = 1$$

There are several things to be noticed here:

- (1) We used the fact that  $\Delta^{k+1} \subseteq \mathbf{R}^{k+1}$  can be viewed as the join of  $\Delta^k$  with  $(0, \dots, 0, 1) \in \mathbf{R}^{k+1}$ . This is reflected in both formulas (2.1.7) and (2.1.8).
- (2) The continuity of  $(*\sigma_\alpha)^+$  and  $(*\sigma_\alpha)^-$  comes from the definition of quotient topology in  $S(X^+)$ .
- (3) The condition of  $\Delta$ -complex for  $X$  implies that the structure given by (2.1.7) and (2.1.8) is consistent on the “lateral faces” of such  $(k+1)$ -simplices. Consequently, it  $S(X^+)$  becomes a  $\Delta$ -complex.
- (4) The  $(k+1)$ -simplices of  $S(X^+)$  are either  $(k+1)$ -simplices of  $X$ , or simplices of the form  $(*\sigma_\alpha)^+$  or  $(*\sigma_\alpha)^-$  described above. This is the complete characterization of the  $\Delta$ -complex structure for  $S(X^+)$ .

Remark 2. In this paper we work with  $\Delta$ -complexes. Nevertheless, we shall frequently use the terminology from simplicial complexes. We have already pointed out the use of the term  $k$ -simplex ( page 7 ). Another one is the concept of “ $k$ -skeleton”. By this we mean the union of all  $k$ -simplices.

## 2.2 The external cup product

In this section the definition and fundamental properties of the external cup product are given.

Due to the fact that we will use the obstruction theory description of characteristic classes, we have chosen the definition of cup product which involves its characterization at the level of singular chains. This fits very nicely with everything that follows. The main reference for all of this is [Htch].

Let  $X$  be a  $\Delta$ -complex. (By a previous remark,  $X$  could be a  $CW$ -complex, and the whole theory about the external cup product is defined for arbitrary topological spaces. But the use of obstruction theory forces us to restrict ourselves to  $\Delta$ -complexes.)

**2.2.1 Definition.** *At the cochain level, the cup product is a map*

$$C^k(X; R) \times C^l(X; R) \rightarrow C^{k+l}(X; R)$$

$$(\phi, \psi) \rightarrow \phi \cup \psi,$$

where the value of  $\phi \cup \psi$  on a singular simplex  $\sigma : \Delta^{k+l} \rightarrow X$  is defined to be:

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|_{[v_0, \dots, v_k]}) \cdot \psi(\sigma|_{[v_k, \dots, v_{k+l}]})$$

(in the above, “ $\cdot$ ” designates the multiplication in  $R$ .)

Given two topological spaces  $X$  and  $Y$ , let  $pr_1$  and  $pr_2$  be the projections of  $X \times Y$  onto  $X$  and  $Y$ , respectively.

**2.2.2 Definition.** For the product space  $X \times Y$ , the external cup product is the map

$$\begin{aligned} \mu : H^*(X; R) \otimes_R H^*(Y; R) &\rightarrow H^*(X \times Y; R), \\ \mu(a \otimes b) &= pr_1^*(a) \cup pr_2^*(b). \end{aligned}$$

By the standard definition of multiplication in the tensor product of graded rings,  $(a \otimes b)(c \otimes d) = (-1)^{\dim b \cdot \dim c}(ac \otimes bd)$ ,  $\mu$  becomes a ring homomorphism.

There exists also a relative version for the external cup product:

**2.2.3 Definition.** Let  $(X, A)$  and  $(Y, B)$  be CW pairs. Let  $pr_1 : X \times Y \rightarrow X$  and  $pr_2 : X \times Y \rightarrow Y$  be the canonical projections. The relative external cup product

$$(2.2.4) \quad \mu : H^*(X, A; R) \otimes_R H^*(Y, B; R) \rightarrow H^*(X \times Y, A \times Y \cup X \times B; R),$$

is defined by

$$(2.2.5) \quad \mu(a \otimes b) = pr_1^*(a) \cup pr_2^*(b),$$

where  $pr_1^*(a) \in H^*(X \times Y, A \times Y; R)$  and  $pr_2^*(b) \in H^*(X \times Y, X \times B; R)$ .

The following result shows the connection between cup product and product spaces (see [Htch], 3.14.').

**2.2.6 Theorem :** if  $(X, A)$  and  $(Y, B)$  are CW pairs and  $H^k(Y, B; R)$  is a finitely generated free  $R$ -module for all  $k$ , then the external cup product  $\mu$  given by (2.2.4) and (2.2.5) is a ring isomorphism.

This very useful connection between product spaces and the ring structure of cohomology (given by the cup product) has a generalization to certain fiber bundles. This

generalization, known as the Leray-Hirsch theorem, will enable us in Subsection 3.2 to compute the cohomology of  $U(n)$ . For the moment, we use (2.2.6) to describe the suspension isomorphism:

$$(2.2.7) \quad \mu : H^k(X, \{x_0\}; R) \rightarrow H^{k+1}(X \times I, X \times \partial I \cup \{x_0\} \times I; R),$$

which for  $k \geq 1$  gives:

$$H^k(X; R) \cong H^{k+1}(SX).$$

In the statement of (2.2.6), let us consider  $A = \{x_0\}$ , a distinguished point of  $X$ , and  $(Y, B) = (I, \partial I)$ . Applying the theorem we get an isomorphism

$$(2.2.8) \quad \mu : H^*(X, \{x_0\}; R) \otimes_R H^*(I, \partial I; R) \rightarrow H^*(X \times I, X \times \partial I \cup \{x_0\} \times I; R).$$

We have the following result: “ $H^*(I, \partial I; R)$  is an  $R$ -algebra with a unique generator  $e \in H^1(I, \partial I; R)$ , and with relation  $e^2 = 0$ .”

This follows immediately from the cochain complex:

$$0 \rightarrow C^0(I, \partial I) \xrightarrow{\delta} C^1(I, \partial I) \xrightarrow{\delta} 0 \rightarrow \dots$$

$e$  is the class of  $\psi : C_1(I) \rightarrow R$ ,  $\psi(I_i) = i$ ,  $i = 0, 1$ .

Here we use the  $\Delta$ -complex structure of  $I$  with 0-skeleton  $I^0 = \{0, \frac{1}{2}, 1\}$ , and 1-skeleton  $I^1 = \{ I_0 = [0, \frac{1}{2}], I_1 = [\frac{1}{2}, 1] \}$ .

Because  $H^*(I, \partial I; R)$  has a single generator of degree 1, the isomorphism of (2.2.8) can be interpreted to define the isomorphism, also called  $\mu$ , asserted in (2.2.7).

We are ready now for the description of the map  $\alpha$ . This is accomplished in the following subsection.

### 2.3 Description of the map $\alpha$ in $(\square)$

In this subsection the ground ring is  $R = \mathbf{Q}$ , and is omitted in the notation of all cohomology groups. Consequently Theorem 2.1.4 applies. As before, let  $X^+ = X \cup \{\infty\} = X \cup \{x_0\}$ .

Let  $\iota_1$  be the following composition of isomorphisms:

$$H^{odd}(X) \cong H^{odd}(X^+) \cong H^{odd}(X^+, \{x_0\}) .$$

(Both isomorphisms above come from the fact that the dimension of the cohomology groups is strictly greater than zero:  $H^k(X^+) \cong H^k(X) \oplus H^k(\{x_0\})$  .)

We now apply the suspension isomorphism  $\mu$  described in (2.2.7) to the pair  $(X^+, \{x_0\})$ . Given a generator  $\phi$  of  $H^{2k+1}(X)$  ,  $k \geq 0$ , we obtain a generator

$$pr_1^*(\iota_1(\phi)) \cup pr_2^*(e)$$

of

$$H^{2k+2}(X^+ \times I, X^+ \times \partial I \cup \{x_0\} \times I) \xrightarrow{\iota_2} H^{2k+2}(S(X^+)) .$$

Here,  $pr_1^*(\iota_1(\phi)) \in H^{2k+1}(X^+ \times I, \{x_0\} \times I)$  and  $pr_2^*(e) \in H^1(X^+ \times I, X^+ \times \partial I)$ .

$\alpha$  is the map resulting from the compositions  $\iota_2 \circ \mu \circ \iota_1$  :

$$H^{odd}(X) \xrightarrow{\iota_1} H^{odd}(X^+, \{x_0\}) \xrightarrow{\mu} \tilde{H}^{even}(X^+ \times I, X^+ \times \partial I \cup \{x_0\} \times I) \xrightarrow{\iota_2} \tilde{H}^{even}(S(X^+)) ,$$

$$(2.3.1) \quad \alpha(\phi) = \iota_2(pr_1^*(\iota_1(\phi)) \cup pr_2^*(e)) \stackrel{def}{=} pr_1^*(\phi) \cup pr_2^*(e) .$$

### 3 $K$ -theory and the Cohomology of $U(n)$

This section accomplishes the characterization of the maps which form the sides of the diagram ( $\square$ ).

#### 3.1 $K$ -theory facts. Description of the map $gl$ in ( $\square$ )

In this subsection we give the basic definitions and properties of the topological  $K$ -groups  $K(X) = K^0(X)$ ,  $\tilde{K}(X)$ , and  $K^{-1}(X)$ . Using suspensions, some of the connections between these groups are mentioned. Appearing in a natural way in the context of  $K^{-1}$ , the map  $gl$  is described. Our main reference for this subsection is [Karb]. The material can also be found in [Atiy].

Let  $X$  be a *compact* topological space. We denote with  $\theta_n$  the trivial vector bundle of rank  $n$  over  $X$ , i.e.  $\theta_n = X \times \mathbf{C}^n$ .

Let  $\mathcal{C} = \mathcal{E}(X)$  be the additive category of complex vector bundles over  $X$ , where the operation is given by the *Whitney sum* of bundles. For a vector bundle  $E$  we denote with  $[E]$  its isomorphism class in  $\mathcal{E}(X)$ . The set of such isomorphism classes  $\Phi(X)$  can be provided with a monoid structure  $[E] + [F] = [E \oplus F]$ .

**3.1.1 Definition.** The first group of topological  $K$ -theory,  $K(X) = K^0(X)$ , is the

*Grothendieck group (the symmetrization group) of the abelian monoid  $\Phi(X)$ .*

The first  $K$ -theory group is a contravariant functor from the category of compact topological spaces and continuous maps to the category of abelian groups and group homomorphisms. Let a point  $x_0$  be chosen in  $X$ . Then the projection  $p : X \rightarrow \{x_0\}$  and the inclusion  $i : \{x_0\} \rightarrow X$  induce corresponding maps between the  $K$ -groups.

**3.1.2 Definition.** *The reduced  $K$ -theory group of  $X$ ,  $\tilde{K}(X)$ , is the cokernel of the map  $p^* : \mathbf{Z} \cong K(\{x_0\}) \rightarrow K(X)$ .*

The following result holds ([Karb], II.1.21):

**3.1.3 Proposition.** *We have the canonical split exact sequence:*

$$0 \rightarrow \mathbf{Z} \xrightarrow{p^*} K(X) \rightarrow \tilde{K}(X) \rightarrow 0 .$$

For the rest of our discussion we consider  $X$  to be connected. This is justified by the fact that we shall apply the forthcoming results to suspensions and reduced suspensions, which are path connected (and hence connected).

Let  $\Phi_n(X)$  be the set of isomorphism classes of vector bundles of rank  $n$  over  $X$ . Taking the Whitney sum by trivial bundles enables us to define an inductive system of sets:

$$\Phi_0(X) \rightarrow \Phi_1(X) \rightarrow \dots \rightarrow \Phi_n(X) \rightarrow \dots .$$

The direct limit of this system can be regarded as an abelian monoid denoted by  $\Phi'(X)$ .

The following homomorphism of monoids:

$$(3.1.4) \quad d : \Phi'(X) \rightarrow \tilde{K}(X), [E] \mapsto [E] - [\theta_n], \text{ for } [E] \in \Phi_n(X)$$

turns out to be an isomorphism (see [Karb], II.1.31).

By  $[(X, x_0), (Y, y_0)]'$  we denote the homotopy classes of maps between  $X$  and  $Y$  which preserve base points. The following result is important for our treatment ([Karb], I.7.6):

**3.1.5 Theorem.** *If  $X$  is compact we have a bijective map:*

$$[X^+, GL(n, \mathbf{C})]' \longrightarrow \Phi_n(S'(X^+)) .$$

The distinguished point for  $X^+$  is  $x_0 = \infty$ , and for  $GL(n, \mathbf{C})$  is the identity  $e$ . In order to describe this map we need a general construction from fiber bundle theory (see [Stnr], 3.2).

Let  $G$  be a topological group, and  $Z$  a topological space. By a *system of coordinate transformations in  $Z$  with values in  $G$*  is meant an indexed covering  $\{V_j\}_{j \in J}$  of  $Z$  by open sets and a collection of continuous maps

$$g_{ji} : V_i \cap V_j \rightarrow G , \quad \text{for } i, j \in J'$$

such that

$$g_{kj}(x) \cdot g_{ji}(x) = g_{ki}(x) , \quad \text{for } x \in V_i \cap V_j \cap V_k .$$

**3.1.6 Theorem (“clutching of bundles”).** *If  $G$  is a topological transformation group of the fiber  $Y$ , and  $(\{V_j\}, \{g_{ji}\})$  is a system of coordinate transformations in the space  $Z$ , then there exists a bundle  $\mathcal{B}$  with base space  $Z$ , fiber  $Y$ , group  $G$ , and the coordinate transformations  $g_{ji}$ . Any two such bundles are equivalent.*

We are ready now for:

**Sketch of the proof of Theorem 3.1.5.** In this case  $G = GL(n, \mathbf{C})$ . We also mention that Theorem 3.1.6 is true for *locally finite closed covers* of a paracompact space ([Karb], I.3.2).

Let  $f : X \rightarrow GL(n, \mathbf{C})$  be a continuous map. We can view it as a continuous map  $f^+ : X^+ \rightarrow GL(n, \mathbf{C})$ , where  $f^+(\infty) = e$ , and  $f^+(x) = f(x)$ , for  $x \in X$ .

We define a bundle  $\xi_f$  over  $S'(X^+)$  by clutching the trivial bundles  $\xi_1 = C^+(X^+) \times \mathbf{C}^n$  and  $\xi_2 = C^-(X^+) \times \mathbf{C}^n$  with the transition function  $g_{21} = f^+$ . Here we consider a locally finite

closed cover formed by the two cones. The map in the statement of the theorem sends  $[f^+]$  into  $[\xi_f]$ .

The uniqueness assertion via equivalence of bundles of Theorem 3.1.6 combined with the contractibility of  $C^+(X^+)$  and  $C^-(X^+)$  allow us to construct a “transition map”  $f$  once a given fiber bundle over  $S'(X^+)$  is given. This proves the surjectivity of the map. The injectivity follows from the fact that homotopic bundles are isomorphic. ■

**3.1.7 Remark.** *Because  $U(n)$ , the unitary group of order  $n$ , and  $GL(n, \mathbf{C})$  are homotopy equivalent, they can be interchanged in expressions involving homotopy classes. Consequently we have a bijective map:*

$$(3.1.5') \quad [X^+, U(n)]' \longrightarrow \Phi_n(S'(X^+)) ,$$

where  $X$  is a compact topological space.

Consider now the set of pairs  $(E, \beta)$ , where  $E$  is a complex vector bundle over  $X$  and  $\beta$  is an automorphism of  $E$ . Two pairs  $(E, \beta)$  and  $(E', \beta')$  are isomorphic if there is an isomorphism of vector bundles  $h : E \rightarrow E'$ , such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ \beta \downarrow & & \downarrow \beta' \\ E & \xrightarrow{h} & E' \end{array} .$$

The sum  $(\sigma + \tau)$  of two pairs  $\sigma = (E_0, \beta_0)$  and  $\tau = (E_1, \beta_1)$  is defined to be  $(E_0 \oplus E_1, \beta_0 \oplus \beta_1)$ .

A pair  $(E, \beta)$  is elementary if  $\beta$  is homotopic to  $id_E$  within the automorphisms of  $E$ .

**3.1.8 Definition.** *The second topological K-theory group,  $K^{-1}(X)$ , is the quotient set of the above pairs by the equivalence relation:  $\sigma \sim \sigma' \Leftrightarrow \exists \tau$  and  $\tau'$  elementary such that  $(\sigma + \tau)$  and  $(\sigma' + \tau')$  are isomorphic.*

We denote by  $d(E, \beta)$  the class of  $(E, \beta)$  in  $K^{-1}(X)$ .

The following result provides an alternative homotopy-theoretic definition for  $K^{-1}(X)$  ([Karb], II.3.17):

**3.1.9 Theorem.** *Let  $X$  be a compact space and let*

$$[X, GL(\mathbf{C})] \cong \varinjlim_n [X, GL(n, \mathbf{C})]$$

*be the set of homotopy classes of continuous maps from  $X$  to the general linear group  $GL(\mathbf{C})$ , provided with the group structure induced by the product of matrices. Let*

$$u : \varinjlim_n [X, GL(n, \mathbf{C})] \rightarrow K^{-1}(X)$$

*be defined by  $u(\beta_n) = d(\theta_n, \beta_n)$ .*

*Then  $u$  is an isomorphism.*

An immediate corollary of this theorem gives the isomorphism:

$$(3.1.10) \quad u : [X, U] \cong \varinjlim_n [X, U(n)] \rightarrow K^{-1}(X) \quad , \quad u(\beta_n) = d(\theta_n, \beta_n) .$$

This, together with Remark (3.1.7), will enable us from now on to consider in fact maps  $f : X \rightarrow U(n)$  instead of  $f : X \rightarrow GL(n, \mathbf{C})$ .

This subsection ends with the description of the map  $gl$ . It is given by the following compositions:

$$(3.1.11) \quad K^{-1}(X) \xrightarrow{u^{-1} (3.1.10)} \varinjlim_n [X, U(n)] = \varinjlim_n [X^+, U(n)]' \longrightarrow \varinjlim_n \Phi_n (S'(X^+)) \xrightarrow{(3.1.4)} \tilde{K}(S'(X^+)) .$$

The above formalism is nothing but a rigorous mathematical formulation of a simple characterization for  $gl$ : it is the map which constructs classes of vector bundles over  $S'(X^+)$  using representatives of elements in  $K^{-1}(X)$  as transition functions. To complete the description of the top horizontal map in  $(\square)$  we compose  $gl$  with the isomorphism at the level of  $\tilde{K}$  arising from the contraction of  $\{x_0\} \times I$  in  $S'(X^+)$  to a point. This contraction is given by maps

$$r_\tau : S'(X^+) \rightarrow S'(X^+) \quad , \quad \text{for } 0 \leq \tau \leq 1 \quad ,$$

where:

$$r_\tau((x, t)) = \begin{cases} (x, t) & \text{if } x \in X \quad \text{and} \quad t \in (0, 1) \\ (x_0, (1 - \tau/2)) & \text{if } t = 1 \\ (x_0, \tau/2) & \text{if } t = 0 \\ (x_0, (t - 1/2)(1 - \tau) + 1/2) & \text{if } x = x_0 \end{cases}$$

Because the resulting isomorphism  $\tilde{K}(S'(X^+)) \cong \tilde{K}(S(X^+))$  comes from the contraction to a point of a set over which the bundles we are considering are trivialized, the description of  $gl$  remains *essentially* unchanged after applying this isomorphism.

### 3.2 The cohomology of $U(n)$

This is the most important subsection of our paper. The description of the exterior algebra  $H^*(U(n); \mathbf{Z})$  is a consequence of the powerful result called the Leray-Hirsch Theorem (3.2.2). But the need for a more concrete representation of the generators  $\{\omega_1, \omega_3, \dots, \omega_{2n-1}\}$  of  $H^*(U(n); \mathbf{Z})$  led us to consider Thom classes also. The method used is chosen to fit obstruction theory ( see Subsection 4.1 ). Our description is summarized in Proposition 3.2.12 .

Taking into account its definition given in  $(\star)$ , the odd-dimensional Chern character cannot be understood without an aesthetical appreciation of the cohomology of  $U(n)$ . Here is the result we need ([Htch], 6.14):

**3.2.1 Theorem.**  $H^*(U(n); \mathbf{Z}) \cong \Lambda_{\mathbf{Z}}[\omega_1, \omega_3, \dots, \omega_{2n-1}]$  is the exterior algebra on generators  $\omega_i$  of odd dimension  $i$ .

**Proof :** This will be a corollary of a result which generalize Theorem (2.2.6) to certain fiber bundles:

**3.2.2 Theorem (Leray-Hirsch).** Suppose that  $p : E \rightarrow B$  is a fiber bundle with fiber  $F$ , such that  $H^*(F; R)$  is a free  $R$ -module which is finitely generated in each dimension, and suppose also that there exist classes  $x_j \in H^{k_j}(E; R)$  whose restrictions  $i^*(x_j)$  form a basis for  $H^*(F; R)$  in each fiber  $F = p^{-1}(b)$ , where  $i : F \rightarrow E$  is the inclusion. Then the map

$$\phi : H^*(B; R) \otimes H^*(F; R) \rightarrow H^*(E; R)$$

$$\phi\left(\sum_{i,j} \alpha_i \otimes i^*(x_j)\right) = \sum_{i,j} p^*(\alpha_i) \cup x_j$$

is an isomorphism.

We prove Theorem 3.2.1 by induction on  $n$ .

First we notice that  $U(1) \cong S^1$ , and consequently  $H^*(U(1); \mathbf{Z})$  contains two nonzero groups,  $H^0$  and  $H^1$ , both isomorphic with  $\mathbf{Z}$ , and the generator  $x_1$  of  $H^1$  satisfies  $x_1 \cdot x_1 = 0$ . So induction starts.

Suppose next that the result is true for  $U(n-1)$ , and let us denote the generators of  $H^*(U(n-1); \mathbf{Z})$  by  $\{\omega'_1, \omega'_3, \dots, \omega'_{2n-3}\}$ . Using a matricial representation for the unitary groups, we have a natural inclusion  $i : U(n-1) \rightarrow U(n)$ , given by  $A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ . It is known that for a fiber bundle  $p : E \rightarrow B$  with path-connected base space  $B$  the choice of base points  $b_0 \in B$  and  $x_0 \in F = p^{-1}(b_0)$  gives an isomorphism:  $p_* : \pi_j(E, F, x_0) \rightarrow \pi_j(B, b_0)$ , for all  $j$ . We can view  $U(n)$  as a fiber bundle over  $S^{2n-1}$  with fiber  $U(n-1)$ ,  $p_{n,n-1} : U(n) \rightarrow S^{2n-1}$ . (The strange notation for projection,  $p_{n,n-1}$ , will get an explanation later

in this subsection.) By applying the previous remark together with Hurewicz's Theorem, we obtain  $H^j(U(n), U(n-1)) = 0$ , for all  $j \leq 2n-2$ . Next, the long exact sequence of a pair in cohomology gives the isomorphisms  $H^j(U(n)) \xrightarrow{i^*} H^j(U(n-1))$  for  $j \leq 2n-3$ . It follows that there exist  $\omega_{2k-1} \in H^{2k-1}(U(n); \mathbf{Z})$ ,  $1 \leq k \leq n-1$  such that  $i^*(\omega_{2k-1}) = \omega'_{2k-1}$ ,  $1 \leq k \leq n-1$ .

We can now apply the Leray-Hirsch Theorem, with  $\omega_j$ 's as  $x_j$ 's, to obtain

$$H^*(U(n)) \cong H^*(U(n-1)) \otimes_{\mathbf{Z}} H^*(S^{2n-1}) .$$

If  $e_{2n-1}$  is the generator of  $H^{2n-1}(S^{2n-1})$ , then  $\omega_{2n-1} = p_{n,n-1}^*(e_{2n-1})$  is a generator in  $H^*(U(n))$ . In order to accomplish the description of  $H^*(U(n))$  as an exterior algebra on generators  $\{\omega_1, \omega_3, \dots, \omega_{2n-1}\}$  it remains to prove that  $\omega_{2n-1} \cdot \omega_{2n-1} = 0$ . But this is immediate due to its odd dimension and to the graded commutativity of the cup product:

$$\omega_{2n-1} \cdot \omega_{2n-1} = (-1)^{(2n-1)(2n-1)} \omega_{2n-1} \cdot \omega_{2n-1} = -\omega_{2n-1} \cdot \omega_{2n-1} .$$

The proof of Theorem 3.2.1 is complete. ■

The disadvantage of the proof we have just presented lies in the fact that it only asserts the *existence* of the elements  $\omega_{2k-1}$ ,  $1 \leq k \leq n-1$ . An effective construction was done only for  $\omega_{2n-1} = p_{n,n-1}^*(e_{2n-1})$ .  $\omega_{2n-1}$  acts as a sort of “generalized topological index”, its value on a cycle  $c$  in  $U(n)$  being given by the “winding number” computed as  $\epsilon_{2n-1}(p_{n,n-1} \circ c)$ . (The terminology has been chosen according to the case  $n=1$  when  $\omega_1$  does compute the winding number of the projections of various cycles.) We propose to give concrete descriptions of all  $\omega_j$ 's.

We begin with the definition of Stiefel manifolds. These mathematical objects will also be used in Section 4 when obstruction theory is introduced.

**3.2.3 Definition.** *Let  $k$  be such that  $1 \leq k \leq n$ . An orthonormal  $k$ -frame in  $\mathbf{C}^n$  is an ordered set of  $k$ -vectors in  $\mathbf{C}^n$ , of unit length and pairwise orthogonal.*

*The Stiefel manifold  $W_{n,k}$  is the set of all orthonormal  $k$ -frames.*



Now the time has come for making a short review of Thom classes.

Given a bundle  $F \rightarrow E \xrightarrow{p} B$ , with fiber  $F = S^k$  and locally contractible base  $E$ , we denote by  $F_b$  the fiber  $p^{-1}(b)$  over  $b \in B$ . Let  $R$  be a ring. ( Again we mention that we shall be interested mainly in the case  $R = \mathbf{Q}$ . )

**3.2.7 Definition.** *The bundle  $S^k \rightarrow E \xrightarrow{p} B$  is called  $R$ -orientable if the associated bundle of  $R$ -modules  $(H^k(F_b; R))_b$  is the product bundle  $B \times R$ . This is equivalent to the existence of a continuous section  $b \mapsto \alpha_b \in H^k(F_b; R)$ , which is a generator in each fiber. Such a section is called an  $R$ -orientation.*

An  $R$ -orientation always exists if  $B$  is simply connected. Taking into account Theorem 3.2.4 it follows that the bundles described in (3.2.6) are all  $R$ -orientable.

A general construction involving the mapping cylinder of  $p$  shows that the bundle  $p : E \rightarrow B$  with fiber  $F = S^k$  can be extended to a bundle  $p : \bar{E} \rightarrow B$  with fiber  $\bar{F} = D^{k+1}$ , the unit ball in  $\mathbf{R}^{k+1}$ . The corresponding concept of  $R$ -orientability reduces to the bundle of  $R$ -modules  $(H^{k+1}(\bar{F}_b, F_b; R))_b$  of being trivial. The following *Thom isomorphism theorem* can be proved in this framework ([Htch], 6.15):

**3.2.8 Theorem.** *A bundle  $p : \bar{E} \rightarrow B$  with fiber  $D^{k+1}$  is  $R$ -orientable iff there exists a class  $\bar{\varphi} \in H^{k+1}(\bar{E}, E; R)$ , called the Thom class, which restricts to a generator of  $H^{k+1}(\bar{F}_b, F_b; R)$  in each fiber  $\bar{F}_b$ .*

The map

$$\Phi : H^i(B; R) \rightarrow H^{i+k+1}(\bar{E}, E; R), \quad \Phi(\alpha) = p^*(\alpha) \cup \bar{\varphi}$$

is an isomorphism for all  $i \geq 0$ , and  $H^j(\bar{E}, E; R) = 0$  for  $j \leq k$ .

Let us now apply Theorem 3.2.8 to the sphere bundle described in (3.2.6). The long exact sequence of the pair  $(\bar{W}_{n,k}, W_{n,k})$  gives:

(3.2.9)

$$\dots \rightarrow H^{2n-2k+1}(\bar{W}_{n,k}) \rightarrow H^{2n-2k+1}(W_{n,k}) \xrightarrow{\delta} H^{2n-2k+2}(\bar{W}_{n,k}, W_{n,k}) \rightarrow H^{2n-2k+2}(\bar{W}_{n,k}) \rightarrow \dots$$

Now, due to the contractibility of  $\bar{W}_{n,k}$  to the “zero section” of it, we can write (3.2.9) as:

$$(3.2.10) \quad \dots \rightarrow H^{2n-2k+1}(W_{n,k-1}) \rightarrow H^{2n-2k+1}(W_{n,k}) \xrightarrow{\delta} H^{2n-2k+2}(\bar{W}_{n,k}, W_{n,k}) \rightarrow \\ \rightarrow H^{2n-2k+2}(W_{n,k-1}) \rightarrow \dots$$

Theorem 3.2.4 applied to  $W_{n,k-1}$  shows that for  $i \leq 2n - 2k + 2$   $\pi_i(W_{n,k-1}) = 0$ . Next Hurewicz and Theorem 2.1.4 (here we pass to  $R = \mathbf{Q}$  if necessary) give:  $H^{2n-2k+1}(W_{n,k-1}) = H^{2n-2k+2}(W_{n,k-1}) = 0$ . Consequently  $\delta$  is an isomorphism in (3.2.10) and we make the following notation- definition:

$$(3.2.11) \quad \varphi_{n,2n-2k+1} = \delta^{-1}(\bar{\varphi}_{\bar{W}_{n,k}}) , \quad 2 \leq k \leq n .$$

We shall call  $\varphi_{n,2n-2k+1}$  the Thom class of  $W_{n,k}$ , for  $2 \leq k \leq n$ . It has the property that it restricts to a generator of  $H^{2n-2k+1}(F_b)$  for each fiber  $F_b \cong S^{2n-2k+1}$  of  $W_{n,k}$ .

If we insist on having (3.2.11) true for  $1 \leq k \leq n$ , we must look at  $W_{n,1} \cong S^{2n-1}$  as a fiber bundle over a point  $W_{n,0}$ . This fiber bundle will have of course a unique fiber isomorphic with  $S^{2n-1}$ . This means that we complete the right side of diagram (3.2.5) with the following very trivial bundle:

$$\begin{array}{ccc} W_{n,1} & \xrightarrow{q_{n,n}} & W_{n,0} \\ \parallel & & \\ S^{2n-1} & & \end{array} .$$

Now the definition we gave to Thom classes of  $W_{n,k}$  in (3.2.11), for  $2 \leq k \leq n$ , extends very naturally to the case  $k = 1$ :

$$(3.2.11.1) \quad \varphi_{n,2n-1} = e_{2n-1} ,$$

where we recall that  $e_{2n-1}$  is the standard generator of  $H^{2n-1}(S^{2n-1})$ .

We summarize our construction of this section in the following:

**3.2.12 Proposition.** *By considering the maps*

$$p_{n,n-k} : U(n) \rightarrow W_{n,k} , \quad \text{for } 1 \leq k \leq n ,$$

the generators  $\{\omega_1, \omega_3, \dots, \omega_{2n-1}\}$  which appear in the statement of Theorem 3.2.1 can be chosen to be

$$(3.2.13) \quad \omega_{2n-2k+1} = p_{n,n-k}^*(\varphi_{n,2n-2k+1}), \quad \text{for } 1 \leq k \leq n.$$

**Proof.** The case  $k = 1$ , using the convention discussed above in (3.2.11.1), is established in the proof of Theorem 3.2.1. Also in that proof, the use of the Leray-Hirsch Theorem gives us the description of the generators  $\omega_j$ 's: they must restrict to a generator over each fiber of the bundle  $U(n-1) \rightarrow U(n) \rightarrow S^{2n-1}$ . Next, our proof for Proposition 3.2.12 will use induction on  $n$  to show the remaining equalities for  $2 \leq k \leq n$ .

We first check that induction starts. Let  $n = k = 2$ . We have the commutative diagram:

$$\begin{array}{ccccc} U(2) & \xrightarrow{p_{2,0}} & W_{2,2} & \xrightarrow{q_{2,1}} & W_{2,1} \\ \uparrow \iota & & \uparrow \iota_1 & & \\ U(1) & \xrightarrow{p_{1,0}} & S^1 & & \end{array}$$

Here  $\iota$  is the inclusion of  $U(1)$  in  $U(2)$ . Let  $\varphi_{2,1}$  be the Thom class of  $W_{2,2}$ . Then it restricts to a generator of  $S^1$ . Viewing  $\iota_1$  also as an inclusion generated by  $\iota$ , and taking into account that  $p_{2,0}$  and  $p_{1,0}$  are identities, it becomes clear that:

$$(\iota^* \circ p_{2,0}^*)(\varphi_{2,1}) = (p_{1,0}^* \circ \iota_1^*)(\varphi_{2,1}) = \omega'_1.$$

Here  $\omega'_1$  is the generator of  $H^*(U(1))$ , and (\*) is proved in this case.

Suppose that the statement of the proposition is true for  $n = m - 1$ . We shall now show that the restriction to  $U(m-1)$  of  $p_{m,m-k}^*(\varphi_{m,2m-2k+1})$  equals  $p_{m-1,m-k}^*(\varphi_{m-1,2m-2k+1})$ . By the induction hypothesis this implies that the restriction to  $U(m-1)$  of  $p_{m,m-k}^*(\varphi_{m,2m-2k+1})$  is an appropriate generator  $\omega'_{2m-2k+1}$  of  $H^*(U(m-1))$ . Next, by the proof of the Theorem 3.2.1, it will follow that  $p_{m,m-k}^*(\varphi_{m,2m-2k+1}) = \omega_{2m-2k+1}$ .

Let us look to the following commutative diagram:

$$\begin{array}{ccccc}
 U(m) & \xrightarrow{p_{m,m-k}} & W_{m,k} & \rightarrow & W_{m,k-1} \\
 \uparrow \iota & & \uparrow \iota_1 & & \\
 U(m-1) & \xrightarrow{p_{m-1,m-k}} & W_{m-1,k-1} & \rightarrow & W_{m-1,k-2} \\
 & & \uparrow & & \\
 & & S^{2m-2k+1} & & .
 \end{array}$$

For proving its commutativity we consider the embedding of  $U(m-1)$  in  $U(m)$  given by the following  $\begin{pmatrix} 1 & 0 \\ 0 & U(m-1) \end{pmatrix}$ , and the facts that  $W_{m,k} \cong U(m)/U(m-k)$  and  $W_{m-1,k-1} \cong U(m-1)/U(m-k)$ .

In the diagram, we denoted by  $\iota$  the described matricial inclusion of  $U(m-1)$  in  $U(m)$ , and by  $\iota_1$  the inclusion induced by  $\iota$ . Because  $W_{m-1,k-1}$  appears as a subbundle of  $W_{m,k}$ ,  $\iota_1^*(\varphi_{m,2m-2k+1})$  restricts to a generator in each fiber of the bundle  $S^{2m-2k+1} \rightarrow W_{m-1,k-1} \rightarrow W_{m-1,k-2}$ . The uniqueness of Thom class shows that  $\iota_1^*(\varphi_{m,2m-2k+1}) = \varphi_{m-1,2m-2k+1}$ . But then:

$$\begin{aligned}
 \iota^*(p_{m,m-k}^*(\varphi_{m,2m-2k+1})) &= (p_{m-1,m-k})^*(\iota_1^*(\varphi_{m,2m-2k+1})) = \\
 &= (p_{m-1,m-k})^*(\varphi_{m-1,2m-2k+1}) \stackrel{(ind.hyp.)}{=} \omega'_{2m-2k+1} .
 \end{aligned}$$

This accomplishes the induction step and consequently (3.2.13) holds for all  $n$ . ■

We can also generalize the remark about the geometric interpretation of the  $\omega_j$ 's that we made after the proof of Theorem 3.2.1. Taking into account the fact that the Thom classes  $\varphi_{2n-2k+1}$  restrict to generators of the cohomology of the fibers  $S^{2n-2k+1}$ , we can regard all  $\omega_{2n-2k+1}$ 's as acting like “*generalized indices*”: for every cycle  $c$  in  $U(n)$  they measuring the “winding number” or “degree” of  $p_{n-k} \circ c$  over a “*universal sphere*”  $S^{2n-2k+1}$ . Of course, the existence of Thom classes given by (3.2.11) is more subtle than “winding numbers” or “degrees”.

### 3.3 Description of $ch^{even}$

As mentioned in the Introduction, the even dimensional Chern character is given by the formula:

$$(\diamond) \quad ch^{even}([\xi]) = \sum_{k=0}^{\infty} ch_k^{even}([\xi]),$$

where

$$ch_k^{even}([\xi]) = \frac{1}{k!} S_k(\mathbf{c}_1(\xi), \mathbf{c}_2(\xi), \dots, \mathbf{c}_n(\xi)).$$

Here,  $\xi$  is an  $n$ -dimensional complex vector bundle over  $X$ ,  $\mathbf{c}_1(\xi), \mathbf{c}_2(\xi), \dots, \mathbf{c}_n(\xi)$  are the Chern characteristic classes of  $\xi$ , and  $S_k$  are the power sums ( $k \geq 0$ ).

Let  $a_r = \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \cdot \dots \cdot x_{i_r}$  be the unitary or a - functions. We consider  $a_0 =$

1. The r-th power sum,  $S_r(a_1, \dots, a_r)$ , expresses the symmetric function  $\sum_{i=1}^n x_i^r$  as unique polynomial of the unitary functions  $a_1, \dots, a_r$ . The properties of power sums are a purely algebraic subject and form the very beginning of the part of mathematics called ‘‘Symmetric Functions’’. Our reference for this is [DKB] .

In order to find the polynomial  $S_r(a_1, \dots, a_r)$  one has to consider the function:

$$(3.3.1) \quad G(t) = \prod_{i=1}^n (1 - tx_i) = \sum_{r=0}^n a_r (-1)^r t^r .$$

Using the power series representation of the logarithm around zero,  $\ln(1 - x) = -x/1 - x^2/2 - x^3/3 - \dots$ , we obtain:

$$-\ln G(t) = -\sum_{i=1}^n \ln(1 - tx_i) = \sum_{r=1}^{\infty} \frac{S_r}{r} t^r .$$

After taking derivatives, it follows that:

$$-G'(t) = G(t) \cdot \sum_{r=1}^{\infty} S_r \cdot t^{r-1} ,$$

or, using the expression (3.3.1) of  $G(t)$ ,

$$(3.3.2) \quad \sum_{r=1}^n a_r (-1)^{r-1} r t^{r-1} = \left( \sum_{r=0}^n a_r (-1)^r t^r \right) \left( \sum_{r=1}^{\infty} S_r \cdot t^{r-1} \right) .$$

Identifying the coefficients, we obtain:

$$(3.3.3) \quad a_r (-1)^{r-1} r = a_{r-1} (-1)^{r-1} S_1 + a_{r-2} (-1)^{r-2} S_2 + \dots + S_r .$$

Consequently:

$$(3.3.4) \quad S_r = E_r + (-1)^{r-1} r a_r ,$$

where  $E_r = -a_{r-1} (-1)^{r-1} S_1 - \dots - a_1 (-1)^1 S_{r-1}$

Now,  $S_1 = a_1$  , and the uniqueness in the power series representation tells us that (3.3.4) is a formula which express the power sum  $S_r$  as unique polynomials in  $a_r$  's. For example:

$$\begin{aligned} S_1(a_1) &= a_1 , \\ S_2(a_1, a_2) &= a_1^2 - 2a_2 , \\ S_3(a_1, a_2, a_3) &= a_1^3 - 3a_1 a_2 + 3a_3 , \\ S_4(a_1, a_2, a_3, a_4) &= a_1^4 - 4a_1^2 a_2 + 2a_2^2 + 4a_1 a_3 - 4a_4 , \text{ etc.} \end{aligned}$$

Returning now to the topological Chern character, we note that the characteristic Chern classes  $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$  play the role of unitary functions  $a_r$  in the discussion above. This gives the following formula:

$$(3.3.5) \quad \begin{aligned} ch^{even}([\xi]) &= n + \frac{1}{1!} \mathbf{c}_1 + \frac{1}{2!} (\mathbf{c}_1^2 - 2\mathbf{c}_2) + \dots = \\ &= n + \frac{1}{1!} \mathbf{c}_1 + \frac{1}{2!} (E_2 - 2\mathbf{c}_2) + \dots + \frac{1}{n!} \left( E_n + (-1)^{n-1} n \mathbf{c}_n \right) + \frac{1}{(n+1)!} E_{n+1} + \dots , \end{aligned}$$

where  $E_r$  is a sum of various cup products of Chern characteristic classes of total degree  $2r$ , for  $2 \leq r$ .

## 4 Proof of the main result

In this section the main objective of our paper, the commutativity of diagram  $(\square)$  described in the Introduction, is attained (Theorem 4.3.1). The proof is based essentially on Theorem 4.2.10. Once the equalities  $\alpha(f^*(\omega_{2k-1})) = \mathbf{c}_k$  are established for  $k = \overline{1, n}$ , it is only a matter of combinatorics to show the commutativity of  $(\square)$ . These final details are given in Subsection 4.3. Subsection 4.1 gives a short review of obstruction theory.

### 4.1 Remarks about obstruction theory

The main reference here is [Stnr] .

The obstruction-theoretic perspective on characteristic classes is as follows. Let  $\xi$  be an  $n$ -dimensional *complex* vector bundle over the space  $X$ . The  $k$ -th Chern characteristic class of the bundle  $\xi$ ,  $\mathbf{c}_k(\xi)$ , is the *primary obstruction* to finding  $(n - k + 1)$  pointwise independent (equivalently orthogonal) sections of  $\xi$ . ( Sections are always assumed to be continuous. )

This primary obstruction is due to the twisting in the given vector bundle. Hence, it appears that the Chern classes ( like the other types of characteristic classes ) measure to some extent the twisting of the bundle.

In the case of (R.S.) of the diagram  $(\square)$ , the map  $gl$  constructs an  $n$ -dimensional complex vector bundle  $\xi_f$  on  $S(X^+)$ , using  $f : X \rightarrow U(n)$  as transition function. (By (3.1.10) it is

enough to restrict ourselves to maps into  $U(n)$ , instead of  $GL(n, \mathbf{C})$ .)

Let us describe briefly and intuitively the construction of the *obstruction cocycle*  $c_n(\xi_f)$  associated to this vector bundle. The cohomology class of  $c_n(\xi_f)$  will be  $\mathbf{c}_n(\xi_f)$ .

$\xi_f$  is a trivial bundle over  $C^-X$ . Consequently, it is possible to choose a single nonvanishing section (or up to  $n$  pointwise independent sections) defined everywhere on  $C^-X$ . For example, using the chosen trivialization  $\xi_f|_{C^-X} \cong C^-X \times \mathbf{C}^n$ , we can define a section over  $C^-X$  to have a constant value: the vector  $v^1 = (1, 0, \dots, 0) \in \mathbf{C}^n$ . This implicitly gives the value of the section at  $S = N$ . The interesting things for us happen on  $C^+X$ . Viewed from the perspective of the chosen trivialization  $\xi_f|_{C^+X} \cong C^+X \times \mathbf{C}^n$ , the section with constant value  $v^1$  over  $C^-X$  has its values over  $X$  determined by the transition map  $f$ . From the northern perspective, this section, which we shall call  $s^f$  from now on, is defined *over*  $X$  by:  $s^f(x) = f(x)(v^1)$ , for each  $x \in X$ .

We are now in the following situation. We have a non-zero section  $s^f$  defined over the sub- $\Delta$ -complex  $C^-X$  and we want to extend it without vanishing anywhere over the whole  $S(X^+)$ . Because of the nonvanishing condition we want to impose on  $s^f$ , the problem of finding such a section of  $\xi_f$  is equivalent to the problem of finding a continuous section in the *fiber* bundle  $\xi_f^{n,1}$ , obtained from  $\xi_f$  by replacing each fiber  $F_b \cong \mathbf{C}^n$ ,  $b \in S(X^+)$ , with its Stiefel manifold  $(F_b)_{n,1} \cong W_{n,1} \cong S^{2n-1}$ . (Let us observe that the local triviality of the vector bundle  $\xi_f$  transfers to a local triviality for  $\xi_f^{n,1}$ , i.e.  $\xi_f^{n,1}$  is indeed a fiber bundle.) The act of extending a section over the  $q$ -skeleton of the base space to the  $(q+1)$ -skeleton will encounter an obstruction when  $\pi_q(\text{fiber}) \neq 0$ . In our particular case, by Theorem 3.2.4 we know that  $\pi_{2n-1}(W_{n,1}) \cong \mathbf{Z}$  is the first nonvanishing homotopy group of the fiber  $W_{n,1}$ . Consequently  $s^f$  can be extended without any problem over the  $(2n-1)$ -skeleton. In other words, for every  $2n$ -simplex  $\sigma : \Delta^{2n} \rightarrow S(X^+)$  we can extend  $s^f$  over its boundary. We have obtained a map:

$$(4.1.1) \quad s_{\partial\sigma}^f : \partial\Delta^{2n} \cong S^{2n-1} \rightarrow S^{2n-1} .$$

Consequently, the problem of extending  $s^f$  from the boundary of the  $2n$ -cell  $\sigma$  to its interior

is equivalent to extending  $s_{\partial\sigma}^f$  to the interior of  $\Delta^{2n}$ .

The orientation on  $\Delta^{2n}$  induces an orientation on  $\partial\Delta^{2n}$ ; by (4.1.1)  $s_{\partial\sigma}^f$  generates an element in  $\pi_{2n-1}(S^{2n-1})$ , denoted by  $[s_{\partial\sigma}^f]$ . By construction of  $s_f$ , this element is nonzero only if  $\sigma$  is of the form  $(*\sigma_\alpha)^+$ . We next define the function  $c_n(\xi_f)$  by:

$$(4.1.2) \quad c_n(\xi_f)((*\sigma_\alpha)^+) = [s_{\partial(*\sigma_\alpha)^+}^f] \quad , \quad c_n(\xi_f)((*\sigma_\alpha)^-) = 0 \quad ,$$

$$c_n(\xi_f)(\sigma_\beta) = 0 \quad , \quad (\forall) \sigma_\beta : \Delta^{2n} \rightarrow X \quad \text{characteristic map of } X.$$

This function  $c_n(\xi_f)$ , by assigning a number to each  $2n$ -cell in  $S(X^+)$ , is a cochain. The starting point of obstruction theory is a nontrivial theorem ([**Strn**], 32.4) which asserts that  $c_n(\xi_f)$  is in fact a cocycle.

After this heuristic approach and before stating the general theorems, we pause to remark that the problem of constructing *k-pointwise orthonormal sections over  $S(X^+)$*  is not more complicated than the case  $k = 1$  discussed above. This construction reduces to obtaining a continuous section in  $\xi_f^{n,k}$ , where  $\xi_f^{n,k}$  is the *fiber bundle* over  $S(X^+)$  obtained by replacing every fiber  $F_b \cong \mathbf{C}^n$  in  $\xi_f$  with the Stiefel manifold  $(F_b)_{n,k} \cong W_{n,k}$  of  $F_b$ . (Again the local triviality of  $\xi_f$  transfers to a local triviality for  $\xi_f^{n,k}$ .) Theorem 3.2.4 shows that the primary obstruction is due to the nonvanishing of the homotopy group  $\pi_{2n-2k+1}(W_{n,k}) \cong \mathbf{Z}$ . Consequently we have no problems in extending a section  $s^f$  over the  $(2n - 2k + 1)$ -skeleton of  $S(X^+)$ . (The simplest possible choice on  $C^-X$  is used as above to start with.) But the extension over the  $(2n - 2k + 2)$ -skeleton may not be possible. For any simplex  $\sigma : \Delta^{2n-2k+2} \rightarrow S(X^+)$  the section  $s^f$  can be extended over its boundary. The map such obtained:

$$(4.1.3) \quad s_{\partial\sigma}^f : \partial\Delta^{2n-2k+2} \cong S^{2n-2k+1} \rightarrow W_{n,k} \quad ,$$

will generate an element  $[s_{\partial\sigma}^f]$  in  $\pi_{2n-2k+1}(W_{n,k}) \cong \mathbf{Z}$ . This can be interpreted as an assignment of numbers to all  $(2n - 2k + 2)$ -simplices in  $S(X^+)$ . Similarly to (4.1.2), the

obstruction cocycle  $c_{n-k+1}(\xi_f)$  is obtained:

$$(4.1.4) \quad c_{n-k+1}(\xi_f)(\sigma) = [s_{\partial\sigma}^f], \text{ for } \sigma : \Delta^{2n-2k+2} \rightarrow S(X^+) \text{ a characteristic map.}$$

It is to be observed that :

- (1)  $c_{n-k+1}(\xi_f)((*\sigma_\alpha)^-) = 0$ , due to the choice made for  $s^f$  over  $C^-X$ ;
- (2)  $c_{n-k+1}(\xi_f)(\sigma_\beta) = 0$ , ( $\forall$ )  $\sigma_\beta : \Delta^{2n-2k+2} \rightarrow X$  characteristic map of  $X$ , due to the extendability realized by  $f$ .

In §§29-32 of his book, Steenrod describes the problem of extending a section  $f$  of  $\mathcal{B}|_{\text{LUK}^q}$  to the  $(q+1)$ -skeleton  $K^{q+1}$ . Here  $\mathcal{B}$  is a fiber bundle over the *finite* simplicial complex  $K$ , with a fiber  $Y$  which is  $q$ -simple (i.e. arcwise connected, and the computation of homotopy groups independent of the base point), and  $(q-1)$ -connected (i.e.  $\pi_q(Y)$  is the first nonvanishing homotopy group of  $Y$ .)  $L$  is a subcomplex of  $K$ . He uses bundles of coefficients to construct a  $(q+1)$ -cochain of  $K \bmod L$ ,  $\underline{c}(f)$ , with coefficients in the bundle of homotopy groups  $\mathcal{B}(\pi_q)$ . The results of greatest interest for us are the following:

**4.1.5 Theorem :** ([Stnr] , Theorem 32.4) *The obstruction cochain  $c(f)$  is a cocycle.*

and

**4.1.6 Definition.** ([Stnr] , Definition 35.3) *Let  $\mathcal{B}$  be a fiber bundle over  $K$  with fiber  $Y$ , let  $q$  be the least integer such that  $\pi_q(Y) \neq 0$ . Then the cohomology class of the obstruction  $c(f)$ , where  $f$  is any section over  $K^q$ , is called the characteristic cohomology class of  $\mathcal{B}$ . It is the primary obstruction to the construction of a cross section of  $\mathcal{B}$ .*

As pointed out, Steenrod uses the hypothesis that  $K$ , the base space of the fiber bundle, is a finite simplicial complex. But a closer look at his arguments shows that the results will

remain valid for  $\Delta$ -complexes. (We need the interiors of  $k$ -simplices to be pairwise disjoint, the simplices to have an orientation etc. All these conditions are satisfied by a  $\Delta$ -complex.) Let us now apply (4.1.5) and (4.1.6) to our particular situation.

**4.1.7 Definition.**  $c_{n-k+1}(\xi_f)$  defined in (4.1.4) is a cycle, for  $1 \leq k \leq n$ . Its cohomology class, denoted by  $\mathbf{c}_{n-k+1}(\xi_f) = \mathbf{c}_{n-k+1}$ , is the  $(n-k+1)$  - Chern characteristic class of the bundle  $\xi_f$ .

We conclude this section with a result which will be used in the proof of Theorem 4.2.10 :

**4.1.8 Lemma.** For every simplex  $\sigma : \Delta^{2n-2k+2} \rightarrow S(X^+)$ , we have:

$$c_{n-k+1}(\xi_f)(\sigma) = \varphi_{n,2n-2k+1} \left( \sum_{j=0}^{2n-2k+2} (-1)^j s_{[v_0, \dots, \hat{v}_j, \dots, v_{2n-2k+2}]}^f \right).$$

(Here,  $\varphi_{n,2n-2k+1}$  is the Thom class described in (3.2.11), and the sum in parenthesis is  $s_{\partial\sigma}^f$  regarded now as an element in homology.)

**Proof.** Let  $\# = 2n - 2k + 1$ . By definition (4.1.4):

$$c_{n-k+1}(\xi_f)(\sigma) = [s_{\partial\sigma}^f] \in \mathbf{Z} \cong \pi_{\#}(W_{n,k}), \quad \text{where } s_{\partial\sigma}^f : \partial\Delta^{\#+1} \rightarrow W_{n,k}.$$

We have the following isomorphisms:

$$(4.1.9) \quad \pi_{\#}(W_{n,k}) \stackrel{h}{\cong} H_{\#}(W_{n,k}) \cong \text{Hom}_{\mathbf{Z}}(H_{\#}(W_{n,k}); \mathbf{Z}) \stackrel{(2.1.4)}{\cong} H^{\#}(W_{n,k}).$$

In the above  $h$  is the Hurewicz isomorphism,  $h([\alpha]) = \alpha_*([\text{id}])$ , where  $[\text{id}]$  is a standard generator in  $H_{\#}(S^{\#})$ . Using the simplicial structure of  $S^{\#}$ , i.e. its identification with  $\partial\Delta^{\#+1}$ , this standard generator can be taken to be:

$$\sum_{j=0}^{\#+1} (-1)^j \sigma_j, \quad \text{where } \sigma_j : \Delta^{\#} \cong [v_0, \dots, \hat{v}_j, \dots, v_{\#+1}] \xrightarrow{\text{id}} [v_0, \dots, \hat{v}_j, \dots, v_{\#+1}] \subseteq S^{\#}.$$

In other words,  $h$  maps the class of  $\alpha : S^{\#} \cong \partial\Delta^{\#+1} \rightarrow W_{n,k}$ , as element in  $\pi_{\#}$ , into the class of  $\sum_{j=0}^{\#+1} (-1)^j \alpha|_{[v_0, \dots, \hat{v}_j, \dots, v_{\#+1}]}$ , as element in  $H_{\#}(S^{\#})$ .

Finally, we know that  $\varphi_{n,\#}$  is a generator of  $H^\#(W_{n,k})$ . It remains to show that it acts as  $(+1)$  in the third isomorphism of (4.1.9). For this we have to recall the remark in the statement of Theorem 3.2.4 which asserts that the projection of any fiber in the bundle  $W_{n,k} \rightarrow W_{n,k-1}$  can be taken as generator for  $\pi_\#(W_{n,k})$ . This, combined with the description of  $\varphi_{n,\#}$  as element in  $H^\#(W_{n,k})$  which restricts to generators over each fiber, plus a coherent use of orientations in  $S^i$ 's, for all  $i \geq 1$ , shows that

$$c_{n-k+1}(\xi_f)(\sigma) = \varphi_{n,\#} \left( \sum_{j=0}^{\#+1} (-1)^j s_{[v_0, \dots, \hat{v}_j, \dots, v_{\#+1}]}^f \right) .$$

This completes the proof. ■

## 4.2 The equalities $\alpha(f^*(\omega_{2k-1})) = \mathbf{c}_k$ , $k = \overline{1, n}$

In this section the essential result for commutativity of  $(\square)$ , namely Theorem 4.2.10, is proved.

We begin with some remarks about the cycles in  $S(X^+)$  ( see Lemma 4.2.4 ). Given a topological space  $X$ , the definition of its suspension  $SX$  makes one believe that there must exist nice connections between the two spaces. And indeed we have seen some of them at the level of  $K$ -theory in Subsection 3.1. Next, when  $X$  is a  $\Delta$ -complex, the canonical  $\Delta$ -complex structure we described on pages 11-12 for  $S(X^+)$  will help us to characterize its most interesting cycles for obstruction theory.

**4.2.1 Definition.** A  $k$  - s-cycle, or  $k$  - suspension-cycle , in  $S(X^+)$  is a  $k$ -cycle whose summands are only simplices of the form  $(*\sigma_\alpha)^+$  or  $(*\sigma_\alpha)^-$ , where  $\sigma_\alpha : \Delta^{k-1} \rightarrow X$  is a distinguished map of  $X$ , and  $k \geq 1$ .

**4.2.2 Lemma.** *The  $k$  -  $s$ -cycles contain the whole information needed to define, via obstruction theory, the Chern characteristic classes for vector bundles over  $S(X^+)$ .*

**Proof:** Let  $\tilde{c}$  be a  $k$ -cycle in  $S(X^+)$ :

$$(4.2.3) \quad \tilde{c} = \sum_i' n_i \sigma_i' + \sum_j'' n_j \sigma_j''.$$

In the expression for  $\tilde{c}$ ,  $\sum'$  contains simplices of the form  $[v_0, v_1, \dots, v_{k-1}, N]$  or  $[v_0, v_1, \dots, v_{k-1}, S]$ , and  $\sum''$  contains  $k$ -simplices in  $X$ .

Let  $\sigma'' = [v_0, v_1, \dots, v_k]$  be an element appearing in  $\sum''$ . Then

$$(-1)^k \sigma'' = -\partial([v_0, v_1, \dots, v_k, S]) + \sum_{j=0}^k (-1)^j [v_0, \dots, \hat{v}_j, \dots, v_k, S].$$

This shows that, as long as we are working over the field  $\mathbf{Q}$ , i.e.  $H^m(X, \mathbf{Q}) \cong \text{Hom}_{\mathbf{Q}}(H_m(X); \mathbf{Q})$ , we can consider  $\tilde{c}$  given in (4.2.3) only by  $\sum'$ . ■

**4.2.4 Lemma.** *There exists a bijective correspondence between the  $(k-1)$ -cycles in  $X$  and the  $k$  -  $s$ -cycles in  $S(X^+)$ , where  $k \geq 2$ .*

**Proof:** Let  $c$  be a  $(k-1)$ -cycle in  $X$ :  $c = \sum_{i=1}^m n_i \sigma_i$ , where  $n_i \in \mathbf{Q}$ , for all  $i$ . By abuse of notation (see discussion on simplices on the top of page 9), we denote  $\sigma_i$  by  $[v_0^i, v_1^i, \dots, v_{k-1}^i]$ , where actually  $v_j^i = \sigma_i(v_j)$ ,  $j = \overline{0, k-1}$ .

The condition of cycle implies:

$$(4.2.5) \quad 0 = \partial c = \sum_{i=1}^m n_i \partial \sigma_i = \sum_{i=1}^m n_i \left( \sum_{j=0}^{k-1} (-1)^j [v_0^i, \dots, \hat{v}_j^i, \dots, v_{k-1}^i] \right).$$

Define:

$$(4.2.6) \quad Sc = \sum_{i=1}^m n_i ((*\sigma_i)^+ - (*\sigma_i)^-) = \sum_{i=1}^m n_i \left( [v_0^i, \dots, v_{k-1}^i, N] - [v_0^i, \dots, v_{k-1}^i, S] \right).$$

(In the above we used the  $\Delta$ -complex structure of  $S(X^+)$  as explained in Subsection 2.1 . Also the previous remark about the identification of  $N$  and  $S$  applies. We wrote them separate to show that the simplices “with  $N$ ” are in  $C^+X$  and those “with  $S$ ” are in  $C^-X$ .)

$$(4.2.7) \quad \begin{aligned} \partial(Sc) &= \sum_{i=1}^m n_i \left( \sum_{j=0}^{k-1} (-1)^j [v_0^i, \dots, \hat{v}_j^i, \dots, v_{k-1}^i, N] \right) + \sum_{i=1}^m n_i \sigma_i \\ &\quad - \sum_{i=1}^m n_i \left( \sum_{j=0}^{k-1} (-1)^j [v_0^i, \dots, \hat{v}_j^i, \dots, v_{k-1}^i, S] \right) - \sum_{i=1}^m n_i \sigma_i. \end{aligned}$$

Now, (4.2.5) shows that the first and the third sum in (4.2.7) are zero, while the second and the fourth cancel. We obtain that  $Sc$  is a cycle in  $S(X^+)$ . One direction of the correspondence is  $c \mapsto Sc$ .

For the reverse direction, let  $\tilde{c}$  be a  $k$ -s-cycle in  $S(X^+)$ :

$$\tilde{c} = \sum_i n_i \sigma_i .$$

The condition  $\tilde{c}$  a cycle gives:

$$(4.2.8) \quad \begin{aligned} 0 = \partial\tilde{c} &= \sum_{i'} n_{i'} \left( \sum_{j=0}^{k-1} (-1)^j [v_0^{i'}, \dots, \hat{v}_j^{i'}, \dots, v_{k-1}^{i'}, N] + (-1)^k [v_0^{i'}, \dots, v_{k-1}^{i'}] \right) + \\ &\quad + \sum_{i''} n_{i''} \left( \sum_{j=0}^{k-1} (-1)^j [v_0^{i''}, \dots, \hat{v}_j^{i''}, \dots, v_{k-1}^{i''}, S] + (-1)^k [v_0^{i''}, \dots, v_{k-1}^{i''}] \right) . \end{aligned}$$

It follows immediately that the terms in (4.2.8) in which  $N$  appears must cancel and that the terms in which  $S$  appears must cancel. (This observation will be called “cancellation of lateral faces” in the justification of (4.2.13).) Thus we have:

$$(4.2.8.1) \quad 0 = \partial\tilde{c} = (-1)^k \left( \sum_{i'} n_{i'} [v_0^{i'}, \dots, v_{k-1}^{i'}] + \sum_{i''} n_{i''} [v_0^{i''}, \dots, v_{k-1}^{i''}] \right) .$$

Denote the summation over indices  $i'$  in (4.2.8.1) by  $S'$ , and the one over indices  $i''$  by  $S''$ . Assume that these sums have been written so that each simplex appears at most once, and

with nonzero coefficient, in each sum. Then the equality with zero of (4.2.8.1) implies that the simplices of the form  $[v_0^{i'}, \dots, v_{k-1}^{i'}]$  appearing in  $S'$  are exactly the same as the simplices of the form  $[v_0^{i''}, \dots, v_{k-1}^{i''}]$  appearing in  $S''$  and that the coefficient of each such simplex in  $S'$  is the opposite of its coefficient in  $S''$ .

We define the  $(k - 1)$ -chain:

$$(4.2.9) \quad c = \sum_{i'} n_{i'} [v_0^{i'}, \dots, v_{k-1}^{i'}].$$

The claim is that  $c$  is a cycle ( i.e.  $\partial c = 0$  ).

Indeed, viewing  $(*)^+$  as an additive operator, and using the fact that the first sum in  $S'$  must be zero ( again the cocycle condition (4.2.8) for  $\tilde{c}$  ), we obtain:

$$\begin{aligned} 0 &= \sum_{i'} n_{i'} \left( \sum_{j=0}^{k-1} (-1)^j [v_0^{i'}, \dots, \hat{v}_j^{i'}, \dots, v_{k-1}^{i'}, N] \right) = \sum_{i'} n_{i'} \left( *(\partial[v_0^{i'}, \dots, v_{k-1}^{i'}]) \right)^+ = \\ &= \left( * \left( \partial \left( \sum_{i'} n_{i'} [v_0^{i'}, \dots, v_{k-1}^{i'}] \right) \right) \right)^+ = (*(\partial c))^+ . \end{aligned}$$

The equality of the last term with zero implies  $\partial c = 0$ .

Note that for  $\tilde{c}$  as given just before (4.2.8) and  $c$  as given in (4.2.9),  $\tilde{c} = Sc$ . Thus we have established the desired bijective correspondence. ■

**4.2.10 Theorem .** *The following equality holds in  $\tilde{H}^{2n-2k+2}(S(X^+))$ :*

$$\alpha(f^*(\omega_{2n-2k+1})) = \mathbf{c}_{n-k+1} \quad , \quad \text{for } 1 \leq k \leq n \quad .$$

Remark. The conclusion could be formulated in the more natural form:

$$\alpha(f^*(\omega_{2k-1})) = \mathbf{c}_k \quad , \quad \text{for } 1 \leq k \leq n \quad .$$

We preferred the first notation to make easier the references to Subsections 3.2 and 4.1 .

**Proof :** Let us consider  $\tilde{c}$  a  $(2n - 2k + 2)$  - s-cycle in  $S(X^+)$ :

$$\tilde{c} = \sum_i n_i \sigma_i \quad .$$

(See Lemma 4.2.2 .) Here, for each  $i$ ,  $\sigma_i : \Delta^{2n-2k+2} \rightarrow S(X^+)$ . The essential simplices will be those lying in  $C^+X$ , because for those in  $C^-X$  we have  $\mathbf{c}_n(\sigma_i) = 0$ .

Recall that the standard generator of  $H^1(I, \partial I; \mathbf{Z})$  has been denoted by  $e$ .

(L.S.) in diagram ( $\square$ ) gives:

(4.2.11)

$$\alpha(f^*(\omega_{2n-2k+1}))(\tilde{c}) = \alpha(f^*(\omega_{2n-2k+1}) \otimes e)(\tilde{c}) \stackrel{(2.2.6)}{=} \sum_i n_i \{ [pr_1^*(f^*(\omega_{2n-2k+1})) \cup pr_2^*e](\sigma_i) \} =$$

$$= \sum_i n_i \{ pr_1^*(f^*(\omega_{2n-2k+1}))(\sigma|_{[v_0^i, \dots, v_{2n-2k+1}^i]}) \cdot pr_2^*e(\sigma_i|_{[v_{2n-2k+1}^i, v_{2n-2k+2}^i]}) \} =$$

$$= \sum_i n_i \{ \omega_{2n-2k+1}(f \circ \sigma|_{[v_0^i, \dots, v_{2n-2k+1}^i]}) \cdot e(\sigma_i|_{[v_{2n-2k+1}^i, v_{2n-2k+2}^i]}) \} \stackrel{(*)}{=}$$

$$\stackrel{(*)}{=} \sum_{i'} n_{i'} \{ (p_{n,n-k}^*(\varphi_{n,2n-2k+1})) (f \circ \sigma_{i'}|_{[v_0^{i'}, \dots, v_{2n-2k+1}^{i'}]}) \}$$

(4.2.12)

$$= \sum_{i'} n_{i'} \{ \varphi_{n,2n-2k+1}(p_{n,n-k} \circ f \circ \sigma_{i'}|_{[v_0^{i'}, \dots, v_{2n-2k+1}^{i'}]}) \} .$$

Let now explain the significance of (\*) above. We used three facts:

- (i) the first is that  $e(\sigma|_{[v_{2n-2k+1}^i, v_{2n-2k+2}^i]}) = 1$  if  $v_{2n-2k+2}^i = N$ , and it is zero if  $v_{2n-2k+2}^i = S$ . See description of the generator  $e$  on page 14;
- (ii) the second is the description of  $\omega_{2n-2k+1}$  as  $p_{n,n-k}^*(\varphi_{n,2n-2k+1})$ , given in Proposition 3.2.12;
- (iii) the indices  $i'$  are of those simplices for which  $v_{2n-2k+2}^i = N$ , i.e. lying in  $C^+X$ .

For (R.S.) we recall that  $(\tilde{c})$  is an  $s$ -cycle. Consequently:

$$\begin{aligned}
 \mathbf{c}_{n-k+1}(\tilde{c}) &= \sum_i n_i \mathbf{c}_{n-k+1}(\xi_f)(\sigma_i) = \sum_{i'} n_{i'} \mathbf{c}_{n-k+1}(\xi_f)(\sigma_{i'}) \stackrel{(4.1.8)}{=} \\
 (4.2.13) \quad &\stackrel{(4.1.8)}{=} \sum_{i'} n_{i'} \varphi_{n,2n-2k+1} \left( \sum_{j=0}^{2n-2k+2} (-1)^j s^j \underset{[v_0^{i'}, \dots, v_j^{i'}, \dots, v_{2n-2k+2}^{i'}]}{\wedge} \right) = \\
 &= \sum_{i'} n_{i'} \varphi_{n,2n-2k+1} \left( s^j_{[v_0^{i'}, \dots, v_{2n-2k+1}^{i'}]} \right) = \\
 &= \sum_{i'} n_{i'} \varphi_{n,2n-2k+1} \left( p_{n,n-k} \circ f \circ \sigma_{i'}|_{[v_0^{i'}, \dots, v_{2n-2k+1}^{i'}]} \right) .
 \end{aligned}$$

In the first line, the shift from indices  $i$  to indices  $i'$ , that is the shift to simplices in  $C^+X$ , occurs because  $\mathbf{c}_{n-k+1}(\xi_f)$  is zero on simplices in  $C^-X$ . The second equality in line (4.2.13) arises from the ‘‘cancellation of lateral faces’’ when we take the boundary of that part of  $\tilde{c}$  lying in  $C^+X$ . (‘‘Cancellation of lateral faces’’ is discussed following (4.2.8).) For the last equality we used the fact that  $f$  acts as a transition function between the two trivializations of  $\xi_f$ . Consequently, because in that expression it represented an element in  $W_{n,k}$ ,  $p_{n,n-k}$  had to appear.

Comparing the final results in (4.2.12) and (4.2.13) we see that:

$$\alpha(f^*(\omega_{2n-2k+1}))(\tilde{c}) = \mathbf{c}_{n-k+1}(\tilde{c}),$$

and this proves the theorem. ■

Let us remark that we did use the equivalence given by Lemma 4.2.4. The computations (4.2.11) through (4.2.12) would work as well for a single simplex  $\sigma$ . But the evaluation  $\mathbf{c}_{n-k+1}(\sigma)$  does take into consideration the lateral faces of  $\sigma$ , as it was proved in Lemma 4.1.8. Because in process of extending to the whole  $S(X^+)$  the section  $s^f$  (already constructed over  $C^-X$ ) there are no standard procedures and arbitrary choices are possible as long as continuity is preserved, we have no control on what could happen with the values on the lateral faces. The remedy for this was given by restricting our attention to  $s$ -cycles in  $S(X^+)$ . Consequently, the use of  $\mathbf{Q}$ , the field of rational numbers, in the lower line of  $(\square)$  is essential not only for the coefficients in the Chern character formulas to make sense, but also for the technical computations.

### 4.3 Possible motivation for the description $(\star)$ of $ch^{odd}$

After the preliminary material presented in Sections 2, 3, and the two preceding Subsections, we are now able to prove the main result:

**4.3.1 Theorem :** *Let  $X$  be a compact  $\Delta$ -complex. Then the following diagram is commutative:*

$$\begin{array}{ccccc}
 & K^{-1}(X) & \xrightarrow{g^l} & \tilde{K}(S'(X^+)) & \cong & \tilde{K}(S(X^+)) \\
 (\square) & ch^{odd} & & \downarrow & & \downarrow & ch^{even} \\
 & H^{odd}(X) & & \xrightarrow{\alpha} & & \tilde{H}^{even}(S(X^+)) & 
 \end{array}$$

**Proof :** We begin by showing the particular simple form the even-degree Chern character has in the case when the underlying space is a reduced suspension,  $S(X^+)$ . Here we shall use the canonical  $\Delta$ -complex structure of  $S(X^+)$  presented in Subsection 2.1, the definition of cup product (2.2.1), and the characterization of Chern characteristic classes via obstruction theory ( Subsection 4.1 ).

Let  $\mu$  be an element in  $K^{-1}(X)$ . We denote by  $f$  the continuous function  $f : X \rightarrow U(n)$  such that  $\mu = d(\theta_n, f) \stackrel{def}{=} [f]$  (see (3.1.10)).

Let  $\mathbf{c}_k(\xi_f)$  and  $\mathbf{c}_l(\xi_f)$  be two arbitrary characteristic classes of the vector bundle  $\xi_f$  over  $S(X^+)$ , and let  $\sigma : \Delta^{k+l} \rightarrow S(X^+)$  be a  $(k+l)$ -simplex in  $S(X^+)$ . As we pointed out in remark (4) on page 12, only two possibilities can occur: either  $\sigma$  is a  $(k+l)$ -simplex in  $X$ , or  $\sigma(v_{k+l}) = N = S$ . In both cases  $\sigma|_{[v_0, \dots, v_k]}$  lies in  $X$ . By the way an obstruction cocycle is defined,  $\mathbf{c}_k(\sigma|_{[v_0, \dots, v_k]}) = 0$ . We obtain:

$$(\mathbf{c}_k \cdot \mathbf{c}_l)(\sigma) = \mathbf{c}_k(\sigma|_{[v_0, \dots, v_k]}) \cdot \mathbf{c}_l(\sigma|_{[v_k, \dots, v_{k+l}]}) = 0 .$$

The above equality is true for every  $(k+l)$ -simplex  $\sigma$  in  $S(X^+)$ . Consequently  $\mathbf{c}_k \cdot \mathbf{c}_l = 0$ , and all sums of products  $E_r$  in (3.3.5) are zero. Moreover, using the definition of  $\tilde{K}$ , i.e.

formal differences  $[\xi] - [\theta_n]$ , where  $n$  is the dimension of  $\xi$  (see (3.1.4)), and the fact that  $ch^{even}$  is a homomorphism, it is clear that  $ch_0^{even}([\xi]) = 0$ . This discussion on  $ch^{even}$  combined with (3.3.5) shows that:

$$(4.3.2) \quad ch^{even}([\xi_f]) = \frac{1}{0!} \mathbf{c}_1 - \frac{1}{1!} \mathbf{c}_2 + \dots + (-1)^{n-1} \frac{1}{(n-1)!} \mathbf{c}_n = \sum_{i=0}^{n-1} (-1)^i \frac{\mathbf{c}_{i+1}}{i!}.$$

Consequently:

$$\begin{aligned} (\alpha \circ ch^{odd})(\mu) &= \alpha(ch^{odd}([f])) \stackrel{(\star)}{=} \alpha\left(f^*\left(\sum_{i=0}^{n-1} (-1)^i \frac{\omega_{2i+1}}{i!}\right)\right) = \\ &= \sum_{i=0}^{n-1} (-1)^i \frac{1}{i!} \alpha(f^*(\omega_{2i+1})) \stackrel{(4.2.10)}{=} \left(\sum_{i=0}^{n-1} (-1)^i \frac{1}{i!} \mathbf{c}_{i+1}\right) \stackrel{(4.3.2)}{=} ch^{even}([\xi_f]) = (ch^{even} \circ gl)(\mu). \end{aligned}$$

This accomplishes the proof. ■

We tried to suggest in our thesis that the definition  $(\star)$  of the odd-degree Chern character given in the paper of Baum&Douglas ([**BmDg**]) has a correspondence in the even-degree component, or at least can be motivated by  $ch^{even}$  over  $S(X^+)$ . This may not be a surprise taking into account that  $K^{-1}(X) = \tilde{K}(S(X^+))$ . (This is the definition Atiyah gives for  $K^{-1}(X)$ ! See [**Atiy**], page 68.) This possible motivation we presented cannot by any means diminish the beauty and simplicity of formula  $(\star)$ . More arguments about its usefulness can be given by looking at how  $ch^{odd}$  behave with respect to the ring structure of  $K^*(X)$ . This form the subject of the short section that follows.

## 5 Chern character as ring homomorphism

It is clear that simply the commutativity of  $(\square)$ , proved in Theorem 4.3.1, would not have given us the right to speak of the formula  $(\star)$  as the *odd-degree part of the Chern character*. The crucial significance of the Chern character lies in the fact that it is a ring homomorphism. In this section we discuss its behavior with respect to the multiplicative structures at the level of  $K$ -theory and cohomology, respectively.

### 5.1 More $K$ -theory . Multiplicative structures

References for this subsection are [Karb], II.2, II.5, and/or [Atiy], 2.4, 2.6.

Let  $X$  be a compact topological space, and let  $Y$  be a closed subspace of  $X$ . Consider the set of triples  $(E, F, \beta)$ , where  $E$  and  $F$  are complex vector bundles over  $X$  and  $\beta : E|_Y \rightarrow F|_Y$  is an isomorphism. Two triples  $(E, F, \beta)$  and  $(E', F', \beta')$  are called isomorphic if there exist isomorphisms  $f : E \rightarrow E'$  and  $g : F \rightarrow F'$  such that the following diagram commutes:

$$(5.1.1) \quad \begin{array}{ccc} E|_Y & \xrightarrow{\beta} & F|_Y \\ f|_Y \downarrow & & \downarrow g|_Y \\ E'|_Y & \xrightarrow{\beta'} & F'|_Y \end{array} .$$

The sum  $(\sigma + \tau)$  of two triples  $\sigma = (E, F, \beta)$  and  $\tau = (E', F', \beta')$  is defined to be  $(E \oplus E', F \oplus F', \beta \oplus \beta')$ . A triple  $(E, F, \beta)$  is called elementary if  $E = F$  and  $\beta$  is homotopic to  $id_{E|_Y}$  within the automorphisms of  $E|_Y$ .

**5.1.2 Definition.** *The relative  $K$ -theory group of the pair  $(X, Y)$ ,  $K(X, Y)$ , is the quotient set of the above triples by the equivalence relation:  $\sigma \sim \sigma' \Leftrightarrow \exists \tau$  and  $\tau'$  elementary such that  $(\sigma + \tau)$  and  $(\sigma' + \tau')$  are isomorphic.*

We denote by  $d(E, F, \beta)$  the class of  $(E, F, \beta)$  in  $K(X, Y)$ .

The following “excision” property can be proved ([Karb], II.2.35):

**5.1.3 Theorem.** *The projection  $\pi : X \rightarrow X/Y$  induces an isomorphism:*

$$\pi^* : \tilde{K}(X/Y) \cong K(X/Y, Y/Y) \longrightarrow K(X, Y) .$$

In order to show that the Chern character is a ring homomorphism we need an equivalent definition of the map  $gl$  described in (3.1.11). This new definition will enable us to handle general pairs (in the set defining  $K^{-1}(X)$ ) of the form  $(E, \beta)$  instead of just pairs  $(\theta_n, f)$ .

We define:

$$gl : K^{-1}(X) \longrightarrow K^0(X \times I, X \times \partial I)$$

by

$$(5.1.4) \quad gl(d(E, \beta)) = d(E \times I_0 \cup_{\beta} E \times I_1, E \times I, id_{X \times \partial I}) .$$

We recall that  $I_0 = [0, \frac{1}{2}]$ ,  $I_1 = [\frac{1}{2}, 1]$ . In (5.1.4),  $\cup_{\beta}$  denotes the clutching of the bundles  $E \times I_0$  and  $E \times I_1$  over  $X \times \{\frac{1}{2}\}$  using  $\beta$  as transition function. Also,  $id_{X \times \partial I}$  denotes the identity map of the corresponding bundles over  $X \times \partial I$ .

We show that  $gl$  is *well defined* by (5.1.4). The correctness is checked in two steps.

First, let  $(E, \beta)$  and  $(E', \beta')$  be isomorphic pairs. We have to show that:

$$d(E \times I_0 \cup_{\beta} E \times I_1, E \times I, id_{X \times \partial I}) = d(E' \times I_0 \cup_{\beta'} E' \times I_1, E' \times I, id_{X \times \partial I}) .$$

From the definition of isomorphic pairs it follows that there exists an isomorphism  $h : E \rightarrow E'$  such that  $\beta' \circ h = h \circ \beta$ . This implies that the following maps are isomorphisms which

verify (5.1.1):  $f = h \times id_{I_0} \cup h \times id_{I_1}$  and  $g = h \times id_I$ . Consequently the two triples we are interested in are isomorphic and they belong to the same class in  $K^0(X \times I, X \times \partial I)$ .

Next, let us consider an elementary element  $(F, \gamma)$  in the set defining  $K^{-1}(X)$ .  $\gamma$  being homotopic with the identity within the automorphisms of  $F$ , there is no loss of generality in assuming that  $\gamma = id$ . Then it is clear that  $d(E, \beta) = d((E, \beta) + (F, id))$ . We have to show that:

$$d(E \times I_0 \cup_{\beta} E \times I_1, E \times I, id_{X \times \partial I}) =$$

$$d((E \oplus F) \times I_0 \cup_{\beta \oplus id} (E \oplus F) \times I_1, (E \oplus F) \times I, id_{X \times \partial I}) ,$$

in  $K^0(X \times I, X \times \partial I)$ . But this is obvious due to the fact that  $(F \times I, F \times I, id_{X \times \partial I}) = (F \times I_0 \cup_{id} F \times I_1, F \times I, id_{X \times \partial I})$  is elementary and

$$\begin{aligned} & (E \times I_0 \cup_{\beta} E \times I_1, E \times I, id_{X \times \partial I}) + (F \times I_0 \cup_{id} F \times I_1, F \times I, id_{X \times \partial I}) = \\ & = ((E \oplus F) \times I_0 \cup_{\beta \oplus id} (E \oplus F) \times I_1, (E \oplus F) \times I, id_{X \times \partial I}) . \end{aligned}$$

This proves our claim and establishes that  $gl$  is well defined.

It is clear that, if  $E$  is the trivial bundle, the definition (5.1.4) coincides with the map described in (3.1.11). Using excision to identify  $K^0(X \times I, X \times \partial I)$  with  $\tilde{K}(S(X^+))$ , as we did at the end of Subsection 3.1, we have defined in another fashion the top horizontal map in  $(\square)$ . Consequently, the commutativity of  $(\square)$  still holds with this new definition (5.1.4) for  $gl$ .

We next propose to describe the multiplication in  $K$ -theory. The simplest remark is the following. Let  $X$  and  $Y$  be compact spaces, let  $E$  and  $F$  be vector bundles over  $X$  and  $Y$ , respectively, and let  $pr_1 : X \times Y \rightarrow X$  and  $pr_2 : X \times Y \rightarrow Y$  be the corresponding natural projections. We define the “*external tensor product of  $E$  and  $F$* ”,  $E \hat{\otimes} F$ , to be the vector bundle over  $X \times Y$  such that  $(E \hat{\otimes} F)_{(x,y)} = E_x \otimes F_y$ . Let us observe that, when  $X = Y$ , the diagonal map  $\Delta : X \rightarrow X \times X$  gives the product in  $K^0(X)$  by:

$$(5.1.5) \quad ([E], [F]) \mapsto [\Delta^*(E \hat{\otimes} F)] \stackrel{def}{=} [E] \cdot [F] .$$

If  $X$  and  $Y$  have base point, the external tensor product gives a map:

$$(5.1.6) \quad \tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow \tilde{K}(X \wedge Y) \quad .$$

For compact pairs  $(X, A)$  and  $(Y, B)$ , a linear map:

$$(5.1.7) \quad K^{-n}(X, A) \otimes K^{-p}(Y, B) \longrightarrow K^{-n-p}(X \times Y, X \times B \cup A \times Y)$$

is defined using suspensions ( $S^n = S \wedge S \wedge \dots \wedge S$ ,  $S$  denoting the “operator” of taking the reduced suspension):

$$(5.1.8) \quad K^{-n}(X, A) \otimes K^{-p}(Y, B) \cong \tilde{K}(S^n \wedge X/A) \otimes \tilde{K}(S^p \wedge Y/B) \xrightarrow{(5.1.6)}$$

$$\xrightarrow{(5.1.6)} \tilde{K}(S^n \wedge S^p \wedge X/A \wedge Y/B) \cong K^{-n-p}(X/A \wedge Y/B) \cong K^{-n-p}(X \times Y, X \times B \cup A \times Y) \quad .$$

This multiplication is often called cup product in  $K$ -theory, due to its similarity in properties with the external cup product in cohomology that we defined in Subsection 2.2.

We shall give more details for the chain of maps in (5.1.8) in the next subsection.

## 5.2 Chern character as ring homomorphism

We denote the Chern character by “ $ch$ ” :

$$(5.2.1) \quad ch : K^*(X) = K^0(X) \oplus K^{-1}(X) \longrightarrow H^*(X; \mathbf{Q}) \quad ,$$

$$ch = ch^{even} \oplus ch^{odd} \quad .$$

In the subsequent part of the thesis we shall assume as known by the reader that  $ch^{even}$  respects multiplication. This can be easily proved using the following two results:

(1) **The Splitting Principle ([Shan])** : Given a complex vector bundle  $E$  over  $X$ , there exist a space  $Y$  and a mapping  $g : Y \rightarrow X$  such that :

- (i)  $g^*E$  is a sum of complex line bundles;
- (ii)  $g^* : H^*(X; \mathbf{Q}) \rightarrow H^*(Y; \mathbf{Q})$  is a monomorphism.

In other words, the Splitting Principle allows us, when dealing with characteristic classes, to work as if the bundles were direct sums of line bundles.

(2)  $ch^{even}(L_1 \otimes L_2) = ch^{even}(L_1) \cdot ch^{even}(L_2)$ , where  $L_1$  and  $L_2$  are two line bundles.

Let us first show that  $ch^{odd}$  respects the additive structure of  $K^{-1}(X)$ . For this purpose, let  $f : X \rightarrow U(n)$  and  $g : X \rightarrow U(m)$  be continuous functions, and let  $\mu = [\theta_n, f]$  and  $\nu = [\theta_m, g]$  be the corresponding classes in  $K^{-1}(X)$  (see (3.1.10)). Then  $\mu + \nu = [\theta_n \oplus \theta_m, f \oplus g]$ . We have to show that:

$$(5.2.2) \quad ch^{odd}(\mu + \nu) = ch^{odd}(\mu) + ch^{odd}(\nu) .$$

But this is clear, because in the diagram  $(\square)$   $gl$ ,  $ch^{even}$ ,  $\alpha$  are all linear, and  $ch^{odd} = \alpha^{-1} \circ ch^{even} \circ gl$ .

We pause here to remark that, if we denote by  $\{\omega_1, \omega_3, \dots, \omega_{2n+2m-1}\}$  the generators of  $H^*(U(n+m))$ , and by  $\{\omega'_i\}$ ,  $\{\omega''_j\}$  the ones for  $U(n)$ ,  $U(m)$  respectively, the formula (5.2) is equivalent to:

$$(5.2') \quad (f \oplus g)^* \left( \sum_{k=1}^{n+m} \frac{\omega_{2k-1}}{(k-1)!} \right) = f^* \left( \sum_{i=1}^n \frac{\omega'_{2i-1}}{(i-1)!} \right) + g^* \left( \sum_{j=1}^m \frac{\omega''_{2j-1}}{(j-1)!} \right) .$$

And (5.2') is not a priori trivial, because the embedding realized through direct sum,  $\begin{pmatrix} U(n) & 0 \\ 0 & U(m) \end{pmatrix} \subseteq U(n+m)$ , is different from the one we used in Subsection 3.2 when we described the generators  $\omega_i$ 's as pull-backs of Thom classes. Using the geometric significance of  $\omega_i$ 's and the geometric constructions involved in obstruction theory, the reader can easily deduce some beautiful geometric actions on chains (cycles) of the forms appearing in (5.2'). And some things contradicting intuition !

Next we prove that  $ch^{odd}$  preserves the ring structures.

Let  $f, g, \mu, \nu$  be as in the previous two paragraphs. We have to show that:

$$(5.2.3) \quad ch^{even}(\mu \cdot \nu) = ch^{odd}(\mu) \cdot ch^{odd}(\nu) .$$

By definition, we have  $K^{-1}(X) = \tilde{K}(S^1 \wedge X^+) = \tilde{K}(S(X^+))$ . Then the following chain of maps gives the definition of the product between two elements in  $K^{-1}(X)$ :

$$(5.2.4) \quad \begin{aligned} K^{-1}(X) \otimes K^{-1}(X) &\stackrel{(i)}{\cong} \tilde{K}(S^1 \wedge X^+) \otimes \tilde{K}(S^1 \wedge X^+) \stackrel{(5.1.6)}{\xrightarrow{}} \tilde{K}(S^1 \wedge X^+ \wedge S^1 \wedge X^+) \cong \\ &\cong \tilde{K}(S^2 \wedge (X^+ \wedge X^+)) \cong \tilde{K}(S^2 \wedge (X \times X)^+) \stackrel{definition}{=} K^{-2}(X \times X) \stackrel{\Delta^*}{\xrightarrow{}} \\ &\stackrel{\Delta^*}{\xrightarrow{}} K^{-2}(X) \cong K^0(X) \otimes K^{-2}(point) . \end{aligned}$$

The last isomorphism of (5.2.4) is exactly the  $K$ -theory formulation of Bott periodicity. Due to its importance for  $K$ -theory, we shall dedicate the following paragraph to a complete description of the generator  $\beta$  of  $K^{-2}(point) \cong \tilde{K}(S^2)$ .

Let us consider the following two sets:  $P^0 = \{z \mid z \in \mathbf{C}, |z| \leq 1\}$ ,  $P^\infty = \{z \mid z \in \mathbf{C}, |z| \geq 1\} \cup \{\infty\}$ . We look at  $S^2$  as the Riemann sphere, on which the stereographic projection gives a homeomorphism between the southern hemisphere and  $P^0$ , and between the northern hemisphere and  $P^\infty$ . We denote by  $H$  the bundle obtained by clutching trivial bundles over  $P^0$  and over  $P^\infty$  with the map  $z^{-1}$ . (We have to emphasize that this clutching is *from* the southern hemisphere *to* the northern one.) See [Atiy], p.50, [Hsm1],11.2.6, [Karb],III,1.2. We define the following class in  $\tilde{K}(S^2)$ :  $\beta = [H] - [1]$ . With this notation, the complex  $K$ -theory periodicity theorem for reduced  $K$ -groups takes the following form: “The map  $a \mapsto a \otimes \beta$ , from  $\tilde{K}(X)$  to  $\tilde{K}(S^2 \wedge X)$  is an isomorphism, where by  $\otimes$  we denoted the external cup product (or tensor product) in  $K$ -theory.”

We finally note that the suspension isomorphism described in (2.2.7) gives an isomorphism:

$$H^1(I, \partial I) \otimes H^1(I, \partial I) \longrightarrow H^2(I \times I, I \times \partial I \cup \partial I \times I) \cong H^2(S^2) ,$$

$$(5.2.5) \quad e \otimes e \mapsto pr_1^*(e) \cup pr_2^*(e) = e_2 .$$

We are ready now for proving (5.2.3):

$$\begin{aligned}
ch^{even}(\mu \cdot \nu) &\stackrel{(5.2.4)(i)}{=} ch^{even}(gl(\mu) \cdot gl(\nu)) = ch^{even}(gl(\mu)) \cdot ch^{even}(gl(\nu)) \stackrel{(4.3.1)}{=} \\
&= (\alpha \circ ch^{odd})(\mu) \cdot (\alpha \circ ch^{odd})(\nu) = ch^{odd}(\mu) \cdot pr_1^*(e) \cdot ch^{odd}(\nu) \cdot pr_2^*(e) = \\
(5.2.6) \quad &= ch^{odd}(\mu) \cdot ch^{odd}(\nu) \cdot (-1) pr_1^*(e) \cdot pr_2^*(e) = ch^{odd}(\mu) \cdot ch^{odd}(\nu) \cdot (-e_2) .
\end{aligned}$$

Some explanations are in order. In the last equality of (5.2.6) we made an abuse of notation by denoting with  $e_2$  an element which belongs actually to  $H^*(S(S(X^+)))$ . This is obtained after applying twice the suspension isomorphism:

$$\begin{aligned}
&H^*(X^+, x_0) \xrightarrow{\mu(2.2.7)} H^{*+1}(X^+ \times I, X^+ \times \partial I \cup \{x_0\} \times I) \xrightarrow{\mu(2.2.7)} \\
&\xrightarrow{\mu(2.2.7)} H^{*+2}(X^+ \times I \times I, X^+ \times \partial(I \times I) \cup \{x_0\} \times (I \times I)) \cong H^{*+2}(S^2 \wedge X^+) .
\end{aligned}$$

We next propose to explain how the minus sign appears in the result of (5.2.6). The shortest argument can be given by using obstruction theory, as presented in Subsection 4.1. Taking into account a  $\Delta$ -complex structure for  $S^2 \cong S'(S^1)$ , it is not hard to see that  $\mathbf{c}_1(H) = -e_2$ . (This is due to the fact that the transition map in  $H$ , as explained in a previous paragraph, is  $z^{-1}$ , and consequently has index  $(-1)$ .) Another proof can be presented if we consider  $S^2$  as a differentiable manifold. In this case there exists a formula which permits to compute the Euler class  $e(E)$  of a real vector bundle  $E$  in terms of the transition functions of the bundle  $E$ . See [BttT], I.6.38. When having a complex line bundle like  $H$ , we simply make the observation that  $\mathbf{c}_1(H) = e(H_{\mathbf{R}})$ , with  $H_{\mathbf{R}}$  the realification of  $H$ . The same result  $\mathbf{c}_1(H) = -e_2$  is obtained, where now the generator  $e_2$  of  $H^2(S^2)$  must be chosen appropriately (for example,  $d\varphi d\theta$ , for  $\varphi$  measuring the “latitude” from the north pole, and  $\theta$  measuring “longitude”.)

The important fact is that:

$$(5.2.7) \quad ch^{even}(\beta) = -e_2 .$$

Taking into account (5.2.7) and the suspension isomorphisms at the level of  $K$ -theory and cohomology, the computation done in (5.2.6) can be interpreted as being exactly (5.2.3).

We finally show that given  $\xi$ , a vector bundle over  $X$ , and  $f, \mu$  as before, we have:

$$(5.2.8) \quad ch^{odd}(\mu \cdot [\xi]) = ch^{odd}(\mu) \cdot ch^{even}([\xi]) .$$

The proof is similar with the one we used to show (5.2.3), but we are making it in order to give more details for the definition of the product in  $K^*(X)$ . The chain of isomorphism that corresponds to (5.1.8) is:

$$(5.2.9) \quad K^{-1}(X) \otimes K^0(X) \stackrel{(i)}{\cong} \tilde{K}(S^1 \wedge X^+) \otimes \tilde{K}(X^+) \stackrel{(5.1.6)}{\xrightarrow{}} \tilde{K}(S^1 \wedge (X^+ \wedge X^+)) \stackrel{(ii)}{\cong} \\ \stackrel{(ii)}{\cong} \tilde{K}(S^1 \wedge (X \times X)^+) \stackrel{definition}{\cong} K^{-1}(X \times X) \xrightarrow{\Delta^*} K^{-1}(X) .$$

We want to show first that:

$$(5.2.10) \quad gl(\mu \cdot [\xi]) = gl(\mu) \cdot [\xi] .$$

Indeed:

$$\begin{aligned} gl(\mu) \cdot [\xi] &= d(\theta_n \times I_0 \cup_f \theta_n \times I_1, \theta_n \times I, id) \cdot [\xi] = \\ &= \left( pr_1^*(\theta_n \times I_0 \cup_f \theta_n \times I_1) \hat{\otimes} pr_2^*(\xi), pr_1^*(\theta_n \times I) \hat{\otimes} pr_2^*(\xi), pr_1^*(id) \otimes pr_2^*(id_\xi) \right) = \\ &= \left( (\theta_n \hat{\otimes} \xi) \times I_0 \cup_{f \otimes id_\xi} (\theta_n \hat{\otimes} \xi) \times I_1, (\theta_n \hat{\otimes} \xi) \times I, id \otimes id_\xi \right) = \\ &= gl\left(d(\theta_n \hat{\otimes} \xi, f \otimes id_\xi)\right) = gl(\mu \cdot [\xi]) . \end{aligned}$$

In the previous computation, the product space for which the projections  $pr_1$  and  $pr_2$  are defined is  $X \times X \times I$ , and  $\theta_n \hat{\otimes} \xi$  is defined over  $X \times X$ . Because  $ch^{even}$  respects multiplication between elements in  $K^0$ , we have:

$$\begin{aligned} \alpha \circ \left( ch^{odd}(\mu \cdot [\xi]) \right) &\stackrel{(4.3.1)}{=} ch^{even}(gl(\mu \cdot [\xi])) \stackrel{(5.2.10)}{=} ch^{even}(gl(\mu) \cdot [\xi]) = \\ &= ch^{even}(gl(\mu)) \cdot ch^{even}([\xi]) = \left( \alpha \circ ch^{odd} \right) (\mu) \cdot ch^{even}([\xi]) = \\ &= \alpha \left( ch^{odd}(\mu) \cdot ch^{even}([\xi]) \right) . \end{aligned}$$

This proves (5.2.8) ( $\alpha$  is an isomorphism). In the last equality we used the fact that  $ch^{even}([\xi])$  is an even-dimensional cohomology class, and consequently  $pr_2^*(e)$  commutes with it.

The other odd-even possibility, i.e.

$$(5.2.11) \quad ch^{odd}([\xi] \cdot \mu) = ch^{even}([\xi]) \cdot ch^{odd}(\mu) ,$$

is absolutely similar with (5.2.8).

The equations (5.2.3), (5.2.8), (5.2.11), together with the observations we made about  $ch^{even}$  at the beginning of this subsection, show that the Chern character is indeed a ring homomorphism.

We proposed to ourselves at the end of Subsection 4.3 to use explicitly the formula  $(\star)$  in proving the ring homomorphism property of the Chern character. This turned to be a hard task even for checking the linearity . This last section showed us clearly that the most important thing in our description is the commutativity of diagram  $(\square)$  (Theorem 4.3.1). Of course, the simple form of  $(\star)$ , after the description we made in Subsection 3.2 to the generators  $\omega_i$ 's, can be useful in index formulas like (1.2) presented in the Introduction. But for a global characterization of the Chern character it is diagram  $(\square)$  which contains the essential information. Again this must not be considered as a surprise. The Chern character, *as a ring homomorphism*, must reflect the essential constructions of topological  $K$ -theory as a cohomology theory where the suspension plays a crucial role.

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