The Utility of Mathematical Symbols

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ABSTRACT

Explanations of why mathematics is useful to empirical research focus on mathematics' role as a representation or model. On platonist accounts, the representational relation is one of structural correspondence between features of the real world and the abstract mathematical structures that represent them. Where real numbers are concerned, however, there is good reason to think the world's correspondence with systems of real number symbols, rather than the real numbers themselves, can be utilized for our representational purposes. One way this can be accomplished is through a paraphrase interpretation of real number symbols where the symbols are taken to refer directly to the things in the world real numbers are supposed to represent. A platonist account of structural correspondence between structures of real numbers and the world can be found in the foundations of measurement where a scale of real numbers is applied to quantities of physical properties like length, mass and velocity. This subject will be employed as a demonstration of how abstract real numbers, traditionally construed as modeling features of the world, are superfluous if their symbols are taken to refer directly to those features.

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Introduction

The usefulness of mathematics cannot be fully understood until we have studied the structure of mathematical language. Certain features of mathematical language allow us to learn things about the structure of the world simply by manipulating formal systems of mathematical symbols. One way of explaining this in limited cases is to think of the symbols as referring to aspects of the world and the manipulations as preserving structural relations found among those aspects in terms of relations found among the structures of the symbols. Then systems of mathematical symbols can serve as interactive models and the process begins to look a lot less mysterious. While it is an interesting question whether or not mathematical symbols are actually used, at least implicitly, in this manner it must first be shown whether it is possible to treat them this way.

The idea that the structure of at least some mathematical language (real number symbols) can be used to play a modeling role in applied mathematics has yet to be explored. This may be because the most obvious way of describing the process requires some unusual semantic notions. In particular, the idea is most easily construed as a paraphrase approach. A good way of categorizing theories in the philosophy of mathematics is by differences in the semantic interpretations of mathematical language. Mathematical platonists (realists), including traditional platonists such as Kurt Gödel and structural platonists like Michael Resnik and Stewart Shapiro, take the referents of mathematical symbols to be actual but non-spatiotemporal objects and structures. These entities are referred to as abstract and are highly scrutinized by anti-realists. Anti-realists do not believe that mathematical objects exist. This implies that mathematical objects are not available as denotations for mathematical symbols. The two most prevalent antirealist

views are paraphrase nominalism and fictionalism. The paraphrase nominalist claims that mathematical language should be taken as referring to actual or possible physical objects and structures. This allows her to retain the syntax of mathematics without having to be committed to anything abstract. The fictionalist holds that mathematical language simply makes vacuous statements because there is nothing for its symbols to denote. This does not bar the fictionalist from claiming that mathematical symbols are useful (see Field 1980).

A prevailing assumption in the philosophy of mathematics is that the standard semantic interpretation of mathematical symbols is a platonist one. Mathematics just is about abstract objects. This puts the burden on the paraphrase nominalist to explain how mathematical language can be used in other ways. The most important consideration for the paraphrase approach being presented here is to show how the structure of mathematical language and the structure of its denotations can be brought into correlation. If real number symbols, in particular, are to be taken as referring to actual quantities in the world, there must be rules for how to assign their denotations. If these rules are not followed, the efficacy of the system of real number symbols breaks down. This situation is entirely dependent on a mutual correlation between the structure of systems of real number symbols and the structure of the aspects of the world –quantities of physical properties– that they will be paraphrased as referring to.

In many instances, the symbols we use in mathematics have nothing in common with the objects they are thought to symbolize. In the case of systems of number symbols that is not the case. The way Indo-European numerals are constructed, in particular, makes it possible to define order relations on them as well as purely symbolic operations

like addition, which mirror the relations between their traditional platonist referents in a more disjunctive way. The usefulness of abstract mathematical objects is often taken to result from their ability to serve as reflections or models of the structure of the physical world. But if the structure of our system for symbolizing numbers reflects the structure of real numbers and, if in certain applications, systems of real numbers reflect the structure of the world, then the system for symbolizing numbers must also reflect the structure of the world. Later on, the nature of these structural correspondences will be examined in greater detail.

Besides examining the conditions under which such an interpretation may be conducted, the other question is what are the benefits for our applied mathematical theories? An important distinction between our systems of real number symbols when compared to the real numbers themselves is that the real numbers are abstract and therefore inaccessible while the symbols are always ready at hand. This further implies that what are really doing the representational work in these applications are symbols for real numbers and not the real numbers themselves. While we cannot affix a real number to a physical system with a certain observed temperature state, we can affix a number symbol (Suppes 4). The result is that we can afford ourselves a simpler, less redundant theory if we were to think of the symbol as referring, not to something abstract that in turn represents the temperature state, but rather to the temperature state itself. Further discussion of abstract objects corresponding to these symbols is superfluous. Without their symbols, they cannot be evoked as representations, but once we have established that their symbols can be used as adequate representations of the world, the abstract

objects are no longer needed. So it is that the benefit of this interpretation is the elimination of entities that add unnecessarily to the ontology of our theory.

In order to use this strategy to point to the superfluousness of other abstract mathematical objects, it would have to be true that their symbols reflect the relational nature of what it is in the world that the mathematical objects are used to represent. This is not going to happen because not all mathematical symbols are like that. The implication is only that applications do not give us any good reason to believe in real numbers. Although the possibility will not be explored here, it may subsequently be feasible to paraphrase more complicated applications of mathematics, especially natural laws, using a reinterpretation of real number symbols as a basis. This may be possible insofar as the abstract structures involved in these applications are constructions of real numbers.

The bulk of this paper consists of a presentation of a demonstrative case in which relations between real number symbols strongly reflect the structure of the real world. The case is conjoint measurement, which is normally construed as the application of a model or scale of abstract real numbers to quantities of physical properties such as temperature, mass, conductivity and pressure. The term "quantity" as it is used here has currency in scientific literature, but will seem idiosyncratic in other domains. What is meant by "quantity" is the degree to which something exhibits a physical property e.g., the degree to which something can be hot, massive or conductive. Because such properties as temperature, mass, etc. come in degrees, it is possible to assign numbers to them because numbers vary in size and the size of the number reflects the degree to which something exhibits these properties.

The first section of this paper consists of a discussion of the notion of structural correspondence and its role in explaining the usefulness of mathematics. Explanations from the foundations of measurement are a paradigmatic instance of this notion as it relates to real numbers thus motivating emphasis on the topic. The section further introduces the empirical model, which will be a main topic in the following sections. The second section provides a brief explanation of the foundations of measurement. This consists of an account of the relations of structural correspondence based on shared algebraic properties that facilitate function mappings between quantities of physical properties and their abstract representations. The third section presents a paraphrase approach to real numbers that is stated in terms of simple rules for constructing and assigning physical quantities as the denotation of number symbols. The fourth section examines situations in which number symbols fail to denote because there is nothing to denote. This occurs when quantifiable aspects of the world turn out not to exist. The fifth section examines rival views and possible sources of objection.

Section 1: Background

Standard accounts of the applicability of mathematics to science center on the notion of structural correspondence. The claim is that mathematics acts as a model of certain features of the world by virtue of correspondences between those features of the world and structures of mathematical objects (Maddy 33). When two structures are said to be correspondent, it means that equivalences can be drawn between the ways in which their basic parts are related. For at least some relations obtaining between parts on the one side, there must be relations obtaining between the parts on the other that stand, in a sense, as analogies for the first set of relations. This can be true even though they may be relations of a very different sort.

A no more apt example can be found than Bertrand Russell's illustration of the geographic map. In a map, the directional relations of North, South, East, and West that stand between locations on the spheroid surface of Earth are replaced by analogous relations of above, below, left and right, between points or pictographs on a flat surface (52). In accounts of applied mathematics, similar concepts are at play only the types of relations involved differ more drastically. One begins with essentially physical relations between objects, locations, or quantities of physical properties. Then a structure of mathematical objects is chosen such that the abstract relations that obtain between the mathematical objects are suitable counter-parts of relations that obtain between the features of the physical world they represent.

Stating that a set of physical relations in the world is correspondent to the relations that are found within a particular mathematical representation or model implies the existence of a structure preserving function from whatever features of the world are

involved in the physical relations onto the abstract elements of the model. This follows from the axioms of standard ZF set theory (esp. the axiom of power sets) assuming that physical entities are acceptable elements of sets. Without such functions, it has even been argued that the mathematics is inoperative on the platonist account and does not have a determinate relation to the world (Field 9).

What is meant here by structure preserving is that the function is a pairing up of related elements of its domain with elements of its range that share a counter-part of that relation. Demonstrating that such a function exists between the elements of one structure and those of the other is proof of structural correspondence between the constituents of the domain of the function and those of its range. When no such function is demonstrable, no formal case can be made that two things correspond structurally, although it may be possible to argue this case in far less conclusive terms. Establishing that our mathematical representations are more than just instrumental, but actually correspondent to relations in the world, reduces very neatly to the demonstration of these functions when it possible to do so and thus, for the platonist, special attention should be paid to them.

The demonstration of a structure preserving function from a set having physical content onto a mathematical representation also serves as an explanation of why the mathematical representation is useful. It is useful because it is structurally similar. The structure of the mathematics can demonstrably tell us something about the structure of the world. This is the chief source of interest in structural correspondence in the applications of mathematics. If two structures are correspondent to one another, regardless of whether one is abstract and the other physical, at least some features of the

first structure will be relevant to those of the other in various formally specifiable ways. It is this feature which is supposed to facilitate the use of mathematical structures as models. The relational nature of one structure can tell us things about the relational nature of the other if we reason in terms of a structure of the first sort and then translate its parts and relations into implications for the nature of the structure of the second sort.

Consider, for instance, a mapping of the duration of time which the sun spends in the sky during the days of the year into the real numbers. In order for this mapping to be representational, the numbers have to be assigned in such a way that the greater-than-or-equal-to relations, $\geq$, obtaining between them are analogous to the relations obtaining between the quantities of daylight that occur on each day. The amount of daylight, for instance, that occurs on September 21 at a given latitude stands in a certain physical proportion to the amount of daylight on each of the other days of the year. Thus, the number representing this amount must stand in identical proportion to the numbers representing the amount of daylight on all the other days. By looking at the real numbers, so assigned, it is possible to understand the analogous relations obtaining between the durations of daylight hours which these numbers serve to represent.

A problem facing the structural correspondence account is the question of how to characterize the features of the world to which mathematical entities such as structures of real numbers are applied as representations. This is a relevant problem because it also threatens the possibility of demonstrating correspondence between real number symbols and the world. Most philosophers of mathematics who have endorsed the structural correspondence account do not provide a detailed description of the sort of

correspondence that takes place between the features of the world and their mathematical representations. This may be because it is not clear what is being represented.

A way of characterizing the empirical content of scientific theories is necessary if one expects to show to what exactly our mathematical representations are supposed to correspond. What is meant by empirical content are the sorts of concrete observable entities, such as objects, locations and quantities of physical properties, to which mathematical representations are applied. It is difficult, however, to separate from a scientific theory an empirical component that can be mapped directly onto a mathematical one. One instance where this is possible is within the foundations of measurement. There we find the notion of the "empirical model" of a theory. A model of a theory is any structure which satisfies the axioms of the theory, assuming that the theory is axiomatized at all. It is a realization, within some domain, of the form the axioms describe (Suppes 26). Accordingly, an *empirical* model is a realization of the axioms of a scientific theory having strictly empirical content e.g., observations (58). This empirical content is subsequently mapped directly into a mathematical model of those same axioms. It thus constitutes a set and the relational structure of that set, defined by the axioms, is just the physical relations the individual elements bear to one another.

In this way, the foundations of measurement offer a well-developed account of an application of mathematics in terms of structural correspondence for which there is a clearly specified physical side to the theory, namely an axiomatically described set having strictly empirical content. In the case of measurement theory, the empirical content will be quantities of physical properties regardless of whether they are directly observable or constructed by an interval scale. This type of clear delineation of the

physical side of things to which the mathematics is supposed to correspond, is also important for developing an account of the sort of structural correspondence that can obtain between the world and systems of number symbols. This is because it allows us to look at the relations that exist between the components of the structure that forms the strictly empirical side of the theory, the quantities, and search for a reflection of these relations in the symbols that represent the numbers used to model them.

Because empirical models are sets, they are abstract in nature and this puts them at a degree of removal from the world as we find it. Their purpose, however, is only to facilitate mathematical proof, organization and rigor. In this case, they serve to organize quantities of physical properties and the relations those quantities bear to one another and subsequently allow us to prove that these relations are analogous to relations between real numbers (4). Construing a theory set-theoretically in this way is contentious, but it is also necessary for the platonist if she means to demonstrate the existence of a structure preserving function.

While what we end up looking at are two sets –one containing numbers and one containing physical quantities– that are correspondent to one another, it is the way that the individual elements of the separate sets relate to one another that makes this possible. Relations between particular quantities and relations between particular numbers reflect each other and it is at this level that the correspondence between mathematics and the world is most apparent (e.g., when the relation between a building's width and its length is reflected in the relation that the number 2 bears to the number 1). The set theory is just a way of rigorously organizing these relational features into manageable individual structures which can be compared and related by functions. As idealizations the empirical

models involved do not directly resemble the features of the world that they take as elements, but they are useful. It seems unlikely that mathematics, and thus mathematical symbols, very often correspond to the world as we find it in any straightforward way because theory always stands as an intermediary between the world and mathematics and often theory and mathematics are deeply comingled. A positive note, however, is that empirical models will not be necessary for the paraphrase approach that will ultimately be developed here even if it may not be possible to eliminate them from platonist structural correspondence accounts.

Section 2: The Foundations of Measurement

It is necessary to consider the precise way in which real numbers correspond to physical quantities before considering how we might go about assigning the meaning of real number symbols to those quantities. In this section, some specific topics in measurement theory are outlined as a demonstration.

The key to understanding how a set of quantities of a physical property can be structurally correspondent to a structure of mathematical objects are the algebraic axioms that describe both the properties and their abstract representations in common. A central scientific assumption, as Cohen et al. describe, is that "physical quantities, numerical values, and units may all be manipulated by the ordinary rules of algebra." In other words, there are algebraic properties that apply to all of these things equally, especially physical quantities and numbers (3). These algebraic properties are what are described by the axioms of measurement theory. The algebraic properties they describe are complicated in some instances, although very often they are quite familiar e.g., transitivity, commutativity and associativity. The idea of measurement theory is to define an empirical structure having the quantities of a physical property as its content that satisfies these axioms and then show the kinds of functions that map the empirical structure into correspondent structures of real numbers that also satisfy these axioms.

The listing of algebraic properties, described by a set of axioms, allows us to specify the exact relational nature of an empirical model that maps into a model of real numbers. This begins with the selection of a set of quantities of a given property, A, generally containing all possible quantities of that property. Subsequently, the order in which one quantity of the property is greater than another is established based on physical

measurement procedures such as direct comparison or more indirectly through the assistance of a measurement instrument. The algebraic order properties of the physical property involved are implicit in these operations e.g., properties such as transitivity and reflexivity, and thus end up characterizing the empirical model for that property. In some instances, establishing an ordering may also involve the solving of systems of inequalities but these inequalities between quantities are determined by direct and indirect comparative procedures in the first place (3-6).

An example of a measurement procedure is the use of rods of a standard size to compare the lengths of other objects. If at least as many rods are required to match the length of an object, a, as are required to match the length of an object, b, then $a \gtrsim b$ (here $\gtrsim$ is used to indicate greater-than-or congruent-to/equivalent-to in terms of the quantities of a physical property). These sorts of procedures are often used in practice in order to know what numbers to assign so as to reflect the results of these comparisons, usually in relation to some fixed standard quantity to which the number 1 has been applied e.g., the standard meter (Krantz et al. 4-5). In measurement theory, however, we take these assignments as already established by the existence of functions relating the quantities and their representations. There are more complicated variants of the types of properties and ordering procedures described, but the selection of a set, or sets, containing purely physical quantities, the axiomatization of its order and the assignment of numbers to mirror this order is universal.

Where it is possible to talk about the combination of quantities of a given property in a meaningful way, the next step is to define a concatenation operation ($\circ$), which reflects the additive relations of these quantities. While it is not always possible to

combine two objects or systems to produce a new one that possess the sum of the quantities previously observed in its component parts, various physical procedures can be used to establish that some object c possesses a quantity of a given property that is equivalent to the quantity possessed by additional objects a and b. An example of this is the use of an equal arm balance to compare the mass of one object with that of multiple others. When the balance is level having, for instance, two objects: a and b on one side and a single one: c on the other, it can be assumed that the mass of a and b combined is equivalent to the mass of c (written $a \circ b \sim c$) (89). Algebraic properties such as commutativity and associativity, may end up being implicit in our additive comparisons of the quantities of given property. If they do, they will get adopted as axioms for the empirical model of that property.

Together the ordinal and additive descriptions of a given property determine an algebraic structure: an ordered set with a binary operation (concatenation), symbolized: $\langle A, \gtrsim, \circ \rangle$. This is the full characterization of the empirical model of the physical property. For less familiar properties and applications, empirical models can constitute different kinds of algebraic structures, but the focus here will be on the more familiar cases.

The axioms defining the nature of the order relations and additive properties of the quantities of a property accordingly represent empirically tested truths concerning the results of the observations that come from the comparative procedures of measurement described above, e.g. direct comparison and the solving of systems of linear inequalities (26). As a result, they depend on the type of procedures used and the nature of the property being measured (6). Furthermore, in many cases a given set of axioms does not uniquely characterize an empirical structure and it will be possible to substitute certain

axioms in the place of others. Some individual axioms, however, are necessary and cannot be substituted out, e.g., transitivity as an order property (22).

The axioms governing certain extensive measurements are a very demonstrative case to focus on. An extensive property is a property of a system that depends on the extent or magnitude of the entire system and thus cannot be attributed to a given point. Examples are familiar: weight, length, volume, mass, etc. Extensive properties are contrasted with intensive properties, which are not dependant on the extent of the system i.e., systems of varying extent may feature an intensive property to an equivalent degree. Examples include temperature, pressure and velocity (Cohen et al. 6).

For the extensive properties to which a non-cyclical, as opposed to repeating numerical scale is applied (an example of a repeating numeric scale are hours on a standard clock), the core axioms defined on A are as follows:

1. *Weak order:* $\gtrsim$ *is reflexive* [*If a $\gtrsim$ b, then b $\gtrsim$ a*]*, transitive* [*If a $\gtrsim$ b and b $\gtrsim$ c, then a $\gtrsim$ c*]*, and connected* [*For all a,b, either a $\gtrsim$ b or b $\gtrsim$ a*]*.

2*. Weak associativity: a ∘ (b ∘ c) ∼ (a ∘ b) ∘ c*

3. *Monotonicity: a $\gtrsim$ b if and only if a ∘ c $\gtrsim$ b ∘ c if and only if c ∘ a $\gtrsim$ c ∘ b*

4. *Archimedean: If a $\succ$ b, then for any c,d $\in$ A, there exists a positive integer n such that na ∘ c $\gtrsim$ nb ∘ d, where na is defined* [...] *as 1a = a, (n+1)a = na ∘ a.* (Krantz et al. 73)

Once the axioms have been determined, it is possible to show that there are functions from <A,$\gtrsim$, ∘> into the ordered set of real numbers with addition defined on it: <Re, $\geq$, +>, that preserve all of the algebraic properties described by the axioms and thus that there are mathematical models within <Re, $\geq$, +> of the axioms. The functions (*f:* A→ Re) must satisfy the following rules:

(i) *a $\gtrsim$ b if and only if f(a) $\geq$ f(b)*

(ii) f(a ∘ b) = f(a) + f(b)

The rules restrict the functions between the two types of models to ones that are strictly relation preserving. The reason the plural "models" is used is because multiple subsets of Re can satisfy the axioms in the same way as the sequences {1,2,3,…}, {2,4,6,…} and {4,5,6,…} all satisfy the axioms of Peano arithmetic. Because the existence of these functions is demonstrative of structural correspondence between an empirical model and its mathematical ones,[1] this becomes an important point for explaining the nature and success of our number based representations of physical properties. The practical role of these functions, however, emerges if they are looked at purely as numerical scales: assignments or pairings of real numbers to physical quantities in a relation preserving way that allow us to use particular numbers as stand-ins to help us keep track of the quantities we are talking about (71). This role is essential to providing a detailed explanation of the representational role played by number symbols.

On the traditional platonist account, it is impossible to specify any arbitrary value without evoking numbers. At best, one can apply simple descriptors e.g., the height of the Richardson-Olmstead Complex, the temperature at which water freezes, etc. One can subsequently make comparative statements such as that the height of the Eiffel Tower is greater than the height of the Great Pyramid, but that leaves us at a loss for descriptive power. By applying a scale of real numbers, some determinate number becomes a surrogate for the physical value to which it is applied and, because there are conventional

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[1] More precisely, one maps equivalence classes of magnitudes of attributes into the real numbers. This is because equivalent quantities of length, volume, mass etc. all get mapped onto a given real number. As a result, the function is many-one, but not bijective (16).

syntactic rules for constructing verbal or written symbolizations of real numbers, it becomes possible to specify those real numbers systematically in terms of the symbols that represent them. Subsequently, the symbol or name which applies to a given real number gets applied to the physical value which it has been paired with by the scale. This allows us to apply the infinite symbolic power of a system of number symbols to physical quantities. The objective motivating this research is showing that it is possible to *directly* apply this symbolic power to quantities without using numbers as intermediaries. This amounts to designating number symbols as the names for certain physical quantities and, conversely, directly assigning physical quantities as the denotation of number symbols. If this can be done, there will be no further need to assume that abstract real numbers exist because they will not doing any work for the theory. In the next section, a paraphrase alternative to this account is considered.

A final point involves the axioms just listed. Of the four, the least transparent is the Archimedean. It bars the possibility that the models it defines contain either infinite or infinitesimal elements. These can either be physical quantities or numbers depending on whether the axiom is applied to mathematical of empirical models. Its more common formulation: *If $a \succ b$, then for some positive integer n, an $\succsim b$*, is not used by Krantz et al. in this case because it is not sufficiently powerful for proving certain correspondences between certain empirical and numerical models (73). What is notable about the Archimedean axiom is that it evokes the concept of number in its formulation, as in "*there exists a positive integer n.*" This suggests that numbers are necessary for defining the models involved. This is not a unique instance. The occurrence of numbers in the statement of axioms like this will have to be dealt with in the final section if it is to be

maintained that numbers are not a necessary ontological assumption. If numbers are necessary for characterizing the algebraic properties of physical quantities this will not be possible.

Section 3: A Paraphrase Proposal

An alternative to this picture is suggested if we look at the functions from empirical models into the real numbers as implicitly redefining the denotation of real number symbols. For every element of the model <Re, ≥, +>, it is possible to construct a symbol. In the case of irrational numbers and certain rational numbers, the symbols are always approximations (e.g. "3.14...,") but this does not make a difference in practice because the approximations can be made arbitrarily long depending on the purpose they are needed for. When a model of the quantities of a physical property, e.g. <A,≳, ∘>, is mapped into <Re, ≥, +>, the symbols play the crucial role of denoting which element of <Re, ≥, +> serve as the image of the elements of <A,≳, ∘>. In other words, the symbols tell us which real number serves as a representation of a given quantity in the empirical model. For the most part, individual physical quantities do not have unique designators and it is only by referring to the corresponding real numbers in a given scale (mapping) that we manage to pick out a particular quantity. This suggests that number symbols are implicitly playing a dual role in the theory of measurement. They refer to real numbers and thereby designate the physical quantity that the real number has been assigned on a given scale. The term "implicitly" is meant to evoke the hypothesis that without number symbols, measurement would be inoperative. While the theory of measurement is stated completely in terms of abstract sets, neither empirical nor mathematical models play a role in practice. They are absent from the world if they exist at all. Only number symbols and the elements of empirical models do. Sets and numbers have no active role to play.

If we eliminate the models and the functions, and keep only what is relevant to the practice of measurement, the nature of the paraphrase proposal being presented here

emerges. The idea is simply to get rid of real numbers as the denotation of number symbols and replace them with particular physical quantities. Since this proposal means to do away with real numbers, it will also have to do away with functions between physical quantities and real numbers because these require the existence of real numbers in the first place. This means that we cannot use these functions to fix the denotation of the symbols. The easiest thing to do would be take mappings between empirical models and real numbers and fix the denotation of the symbols normally referring to real numbers to the quantities that correspond to the real numbers. But this is not an option. What we want is to get the same result that would be gained through this process without talking about functions, sets and real numbers. This requires a few extra tools.

Besides quantities and symbols another important component of this proposal is the syntax of the system of real number symbols (numerals), including operations like addition and comparisons of equality and inequality. The syntax of Indo-European number symbols will be considered exclusively due to their prevalence. The syntax of Indo-European numerals (hereafter IEN) and their operators defines what arrangements of instances of certain atomic symbols, e.g., "0," "1," "2," "3," "4," "5," "6," "7," "8," "9," "<" "=" are acceptable and which are not. For example "1 + 7 = 8" is a well formed formula −a formula that is permitted by the syntax− whereas "1 + 6 = 8" and "900 < 7" are not.

Whenever formulas of the above sort are well formed, it implies something about the structure of the IENs that partially comprise them. For instance, when "m < n" is a well formed formula, where "m" and "n" are instances of IEN lacking decimal components (the symbols occurring after the decimal point e.g., "24.623"), it must be

because either "m" is a shorter string of atomic symbols than "n" or that they are strings of the same length constructed from different atomic symbols. In the latter case, "m < n" is a well formed formula only if one of a number of disjunctive situations obtains. Either "n" contains a "9" symbol in the same position that "m" contains an "8," "7," "6," … or "0" symbol, or "n" contains an "8" symbol where "m" contains a "7," "6," "5,"…or "0" symbol and so forth. More complicated compositional rules apply to the IEN appearing in well-formed formulas of the form "m + n = p," but if there were no such rules, it would be impossible for us to do arithmetic by hand or, on a platonist construal, recognize the relations of greater than or less than which the denotations of these symbols bear to one another. They are recognizable by virtue of being reflected in relations between their symbols.

Whenever "m > n" or "m + n = p" are well formed formulas, it is by virtue of the comparative structure of the number symbols involved: "m," "n," and "p." When these are well formed formulas, no matter which particular IEN are involved, certain comparative facts must be true about the structure of these symbols and these structural facts are what ultimately reflect the relational structure of physical quantities. This can be seen if we define some relations on number symbols: $\geqslant$ and $\cdot$. When it is the case that for number symbols "m" and "n" that "m" $\geqslant$ "n" this is by virtue of the comparative structural properties of "m" and "n" that render "m $\geq$ n" a well formed formula. Specifically, because either "m" is a longer string than "n" or they are strings of the same length, but either "m" contains a "9" symbol in the same position that "n" contains an "8," "7," "6," ….etc. Similarly, when it is the case for number symbols "m," "n" and "p" that "m" $\cdot$ "n" = "p" this is by virtue of the structural properties of "m" "n" and "p" that

render "m + n = p" a well formed formula. It can then be shown that $\geqslant$ and $\cdot$ are transitive, monotonic and associative. In this sense, the structure of the system of number symbols reflects the structure of the world, specifically any structure of the world such as extensive properties that also exhibit these algebraic properties. This allows us to reflect relations between things like quantities in terms of relations between the structure of number symbols.

Since this proposal means to do away with functions and real numbers it becomes much more difficult to fix the denotation of IENs. What we want to be the case, however, whether we are interpreting the denotation of number symbols as abstract objects or quantities of physical properties, is that the structural comparison of any two number symbols reflect a comparison of the value they represent. This holds whether that is to be a comparison of abstract number values or physical quantities. If we could not observe differences in the denotation of two number symbols just by comparing the number symbols themselves we would not have a very useful system of number symbols regardless of what we use that system to represent. This amounts to assigning quantities to number symbols in such as way that the relations of $\gtrsim$ and $\circ$, between quantities are reflected in the relations of $\geqslant$ and $\cdot$ between number symbols. Achieving this is relatively simple as it applies to extensive properties. Other types of properties demand further investigation. First what we need is a recursive scheme for constructing number symbols such as the IEN system. Then all we have to do is uphold a modification of *(i)* and *(ii)* recalling that "m" $\geqslant$ "n" whenever "m $\geq$ n" is a well formed formula:

> *(i') If a $\gtrsim$ b then, if "n" is the name of a and "m" is the name of b, "n $\geq$ m" is a well-formed formula according to the ordinary syntax of the IEN.*

(ii') The name of (a ∘ b), p, must be such that if n is the name of a and m is the name of b, "m + n = p" must be a well-formed formula according to the ordinary syntax of the IEN.

Here, *a* and *b*, are physical quantities. In this way, the objective of having comparisons of the structure of symbols reflect comparisons of their denotation will be satisfied insofar as the syntax of the system of number symbols reflects the requisite structural properties.

If these rules have been followed when assigning number symbols as names to quantities of a physical property in a given case, then it will be possible to understand how the quantities they refer to relate in terms of how their symbols relate. Conversely, if we understand how two observed quantities relate, then we will be able to figure out how the symbols assigned to them should be constructed as long as we have the syntax of the number symbols on hand. If we do not follow these rules in making assignments, we will have given number symbols denotations in an entirely useless way.

How to set the assignment of number symbols to particular quantities in the first place is another question. Krantz et al.'s discussion of the practice of measurement, as opposed to the theory, will help illustrate this proposal. One particularly common measurement procedure, called counting-units, begins with the application of the number 1 to a base unit. The kilogram is an instance of this. The number two is subsequently applied to a concatenation of two objects having a mass of one and the number three is assigned to a concatenation of three objects having mass of one. This is a result of requirement *(ii)*. If $f(a) = 1$, then $f(a \circ a) = f(a) + f(a) = 1+1 = 2$. Thus the number assigned to a mass that is twice as large as that to which we assign the number one is the number 2. Effectively, we determine the number that has been assigned by *f* to any new

mass we observe by addition. If object f has a mass that is equivalent to a concatenation of five objects to which the number one is assigned: $a \circ (b \circ (c \circ (d \circ e) \sim f$, as determined by a level arm balance, then we can figure out what number is assigned to it by finding the solution to 1+(1+(1+(1+1) (Krantz et al. 3-4). This is because, in actual practice, pairing an arbitrary quantity with the number 1 and maintaining the truth of *(i)* and *(ii)* rigidly defines a mapping between all quantities of that property and a set of real numbers, but we do not know what number the function assigns to any quantity we may choose at random until we have made these sorts of comparisons and performed the relevant calculations.

The present proposal is similar. We start by arbitrarily designating some quantity of a physical property with the symbol "1" and then determine the names of further quantities by comparison to this base quantity. When we observe quantities that are equivalent to multiple concatenations of the base quantity, say 5 concatenations, we know that the symbol to ascribe to it is that symbol that appears on the other side of the equality sign in the completed version of the formula "1+(1+(1+(1+1) =" according to the syntax of the IEN. Similar considerations apply for other measurement techniques besides counting-units.

Besides maintaining *(i')* and *(ii')* in making assignments, nothing more needs to be done. Wherever we assign number symbols to physical quantities directly in a given situation, it will yield a more parsimonious theory because we need only assume the existence of symbols and the existence of the quantities to which the symbols refer. The real numbers have no role to play, thus they are eliminated as ontological assumptions from the account.

An important point that should be noted about this account is that the denotation of a given number symbol, "32" for instance, will vary depending on which quantity of a physical property has been assigned to be designated by instances of the symbol "1" in a given case. "32" can thus refer to the temperature at which water freezes, the temperature at which water boils and anything in between and beyond as long as *(i')* and *(ii')* are followed in making this assignment. Depending on the domain of discourse it could also refer to 32 meters, 32 kilometers, 32 ohms, etc. None of this matters, however, as long as the domain of discourse is salient and the appropriate rules are followed. Each assignment can be thought of accordingly as establishing a convention, a resolution to use the symbols in a certain definable way in a certain definable situation.

There still remains the issue of the Archimedean axiom at this point. Its evocation of the concept of number in defining the nature of the empirical content of our theory seems much less relevant, but it still needs to be addressed. The particular set of numbers the axiom evokes are the positive integers. If we allow multiplicity to be a physical property of a given system, however, we can interpret these terms just like real number terms as referring to quantities in the world. On its most transparent formulation: $an \gtrsim b$, *n* refers to a certain quantity of iterations of a given procedure. Concatenations of rulers having length *a*, or additions of an object having mass *a* to one side of a level arm balance, for instance. In this sense, the number symbols refer to quantities of measurement procedures, albeit in a hypothetical sense.

Section 4: The Failure of Designators to Denote

This account is presented as if it can be taken for granted that the quantities we intend our number symbols to correspond to really are there to be corresponded with. It often happens, however, that in science a property we think we are quantifying does not exist. Prosper-René Blondlot, for instance, had the bad experience of learning that N-rays did not exist (Pohlmann 22). When number symbol are applied to physical quantities such as N-rays it suggests that they fail to refer in the same way the phrase "Stephen Douglas, 16th President on the United States" fails to refer. This is not at all problematic. The parallel to draw here is with platonist accounts. If structures of real numbers are intended to model physical quantities like N-rays that turn out not to exist, it does not imperil platonism. It implies scientific error and the apparatus of platonism does not fall into question. There is similarly no reason for a failure –even frequent failures— of number symbols to refer to anything to threaten the continued feasibility of using them to refer to physical quantities.

This issue does, however, raise an interesting question. In some instances, it is likely that number symbols, supposedly assigned to some property or quantity that fails to exist, turn out to be useful anyways. This may be because they track something that is influencing our measurement instruments that has a structure like the supposed structure of the property that turns out not to exist.

Section 5: Related Accounts

The nature of the view presented here is difficult to place in relation to other views in the philosophy of mathematics. It is construed as a paraphrase account, but that is not a necessary feature. What is taken as the central question is why is mathematical language useful? This is in contrast to more traditional questions about the truth conditions or ontology of mathematics. Mathematical symbols are paraphrased here as referring to actual quantities in the world, but this does not account for the generic truth of mathematics statements nor is it meant to. In this sense, the view does not fit the mold of traditional paraphrase views such as Philip Kitcher's, which are designed to account for the truth of mathematics.

What is crucial to this account is that the relational structure of the system of real number symbols reflects the relational structure of systems of physical quantities. If this hypothesis is accepted, it opens up multiple ways of describing how mathematical language can be related to physical quantities. One way of doing so is by paraphrasing number symbols as referring to physical quantities as was done here. This amounts to the assignment of particular physical quantities as the denotation of particular number symbols in a relation preserving way. Another option would have been to treat the system of number symbols as a system of indices or quantifiers which are related to one another by virtue of certain consistencies in their structural properties (esp. those used to define $\geqslant$). The notion of number symbols as quantifiers does crop up from time to time, though generally as an expedient for avoiding more lengthy symbolizations.

A further way of relating particular number symbols to particular quantities would have been to operate in fictionalist terms. In this sense, one could utilize abstract objects

qua fictional characters in a story as the denotation of number symbols. The ontology of the theory would be of the same sort as the ontology of *Harry Potter*. Number symbols would then be seen as referring to fictional abstract objects, namely number symbols, which were related by fictional relations preserving functions to actual quantities. By acting *as if* this story were true it would be possible to establish associations between actual number symbols and actual quantities in a way that was sufficient for scientific practice.

The view is not even inimical to platonism, although it was stated at the outset that it is a benefit for our theories if we can restrict the ontology of our scientific theories and thus eliminate abstract objects. If restricting ontology was not something someone had in mind, one could adopt a platonist view of mathematics while still accepting that what is really doing the work underlying our scientific practice is actual number symbols standing in structural correspondence to actual quantities.

The account developed here is simply not meant as a rival to any existing accounts in the philosophy of mathematics. It is only intended to fill an important gap. That gap is the failure to tell us why the most important part of mathematics –physical calculations involving symbols (broadly construed to include computer implementation of calculations)– is useful. Such explanations have nothing to do with ontology or truth. So while the considerations developed here are stated in terms of paraphrase nominalism they could be combined with virtually any anti-realist theory and still satisfy the same explanatory goals and offer the same benefit of restricting the ontology of science to exclude abstract numbers. Simply put, the view is "semantically uncommitted".[2]

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[2] I owe this expression to Michael Trapp.

This does not mean that the particular anti-realist views the central consideration of symbol-quantity correspondence might be combined with do not have their own individual problems. As such it is a good idea to emphasize this issue by talking about the particular paraphrase nominalist construal developed here and comparing it to some related views. In particular, Hartry Field's fictionalism and Philip Kitcher's paraphrase nominalism would be very illustrative, particularly in light of Michael Resnik's criticisms of these positions.

Field's research is motivated by the indispensability argument. The argument stems from Quine principle that we should believe in the entities that our best theories quantify over. Since all scientific theories cannot do without mathematics and thus, given a traditional semantic picture, must quantify over abstract mathematical objects, we should believe that abstract mathematical objects exist. Field therefore seeks to develop scientific theories which do not involve mathematical language and which thus fail to rely on abstract mathematical objects (Field 2).

Field demonstrates the possibility of doing science without mathematics by formulating Newtonian Gravitational Theory in terms of a nominalistic axiomatization of the theory. The axioms he develops refer only to physical quantities, spacetime points and regions, and various relations such as betweenness and congruency. He then proceeds to show that structures of quantities, and structures of spacetime points and regions satisfying these axioms are equivalent to structures of real numbers e.g., fields and ordered sets also satisfying these axioms. This allows him to prove representation and uniqueness theorems that demonstrate that the abstract structures are isomorphic to the nominalistic ones. Accordingly, we can reason in terms of numbers in order to determine

facts about quantities in spacetime as a theoretical expedient, but by no means is it necessary to do so because the same conclusions can be reached through purely nominalistic deductions in terms of spacetime points, quantities, etc., although doing so is exceedingly complicated (61-91).

Michael Resnik identifies a chief problem with Field's view in the lack of nominalist formulations of more complicated theories such as quantum mechanics and general relativity. Since there are many technical barriers to achieving this, it is not clear that Fields strategy generalizes to more complicated theories (Resnik 56). This is not an issue, however, for restricted claims like the paraphrase strategy presented here. Quantum mechanics, for instance, relies on abstract objects such as fields and regions of Hilbert space and the symbol systems for such things are simply not amenable to the paraphrase strategy outlined here for real numbers so this account passes over them. Another issue Resnik points out is raised by probability and statistics. There is nothing, for instance, in the world for our probabilistic use of real numbers to correspond to so this appears to be an issue both for Field and for the current paraphrase account (57).

One option for the paraphrase approach presented here is an appeal to hypothetical reasoning. The process of calculating a distribution, for instance, might work in hypothetical terms e.g., if a sufficiently large number of tests *were* performed and the null hypothesis were true, we would expect to see a certain distribution of actual (ranges of) observed quantities. The apparatus of mathematical syntax along with the relevant background theory allows us to perform the subsequent hypothetical calculation. Without performing a detailed investigation, however, the possibility looms that probability and statistics are equally significant stumbling blocks for this account as they are for Field's.

A further issue is that an appeal to hypothetical reasoning raises the problem of vacuity, as Resnik mentions (66). All hypothetical statements take the form of a conditional statement having a false antecedent. Thus they are true no matter what the consequent. So if our antecedent is the assumption that a certain null hypothesis is true and that a certain number of tests have been performed, classical logic cannot decide between any distribution of the data we might decide to use as the consequent. This is an issue of relevancy, however. There is a fundamental difference between the claims: "If there were a naturally blue rose, it would be one hundred feet tall" and "If I were severely allergic to peanuts I should not shake Jimmy Carter's hand." In the first case, the antecedent has no connection to the consequent, in the latter, there is a clear (though exaggerated) connection between the two based on background information about the constituents of both clauses. Similarly, there will be consequent distributions that are irrelevant to the background information and those that have been calculated in accordance with it. One will be useful, the other will be useless.

Another related account is Philip Kitcher's paraphrase approach. According to Kitcher "mathematics describes the operational activity of an ideal subject" (111). Arithmetic, for instance, is construed as a description of the activities of an ideal agent performing combinations of objects, matching them and segregating them, etc. The ideal agent engaging in these activities does not have to worry about getting tired or running out of time so it can perform infinitely many processes and work with vary large groups of objects. Kitcher's theory is, in this sense, an "idealizing theory," a theory like the ideal gas law which is useful despite not being stated in terms of anything that exists. What it is stated in terms of –ideal gas– is an abstraction from which all of the accidental features

and limitations of actual gases have been removed. The ideal agent, in a similar fashion, is a being from which all of the accidental limitations of actual human agents have been removed. Because none of the features of actual agents that are relevant to the arithmetical operations Kitcher describes are also removed, the theory is useful and, in a hypothetical sense, true (117).

A major problem for Kitcher's account is once again the disciplines of probability statistics. According to Resnik, we will need statistics to relate mathematics qua the theory of ideal agents to actual data (Resnik 66). Kitcher's account is particularly vulnerable to this problem because there is no obvious way to construe probability as a theory of ideal agents. The theory being suggested here might well fare better on the hypothetical framework briefly described above.

Before concluding this section, it would be worthwhile to consider how the present account differs from structural platonist accounts such as those of Michael Resnik and Stuart Shapiro. There are distinct elements of structuralism in the present account, though the elements it contains are distinctly anti-realist. According to Resnik, there are non-mind dependent patterns corresponding to every mathematical structure, although Resnik is neither committed to construing patterns intensionally as universals nor extensionally as sets, so calling him a platonist is not entirely apt (Resnik 202). Any set of positions in these patterns, whatever their metaphysical nature, that are suitably related to one another can act as referents for individual number terms (Resnik 224). Partial physical instantiations of these patterns correspond to them the way a dress corresponds to its template (226).

What is similar about the present account is that it talks about systems of number symbols and the relations between physical quantities as having the same structure. That sounds very similar to saying that they share the same *pattern*. What is meant by the claim that two systems share the same structure, in the present account, is more pragmatic than what Resnik intends.

Underlying this account is the assumption that the world has an objective structure. Its parts, however we happen to take them, are arranged in certain ways independently of whether anyone is there to categorize these arrangements and independently of whether there are, for instance, patterns or structural universals for these arrangements to count as instantiations of. As agents within the world we have certain goals we mean to satisfy so it is instrumental to label certain arrangements that strike us in a certain way or that yield similar perceptual (direct or indirect) experiences. This allows us to build a predictive apparatus as insurance against the future. Structure is fundamental on this account and talk of shared relational nature is a way of categorizing structure in a way that is consistent with the similarities we are met with in getting about in the world. This does not require additional assumption that there are, in addition to individual structures, abstract objects or patterns as Resnik assumes. Because Resnik leaves the precise metaphysical nature of patterns inexact it would be hard to say more about the differences between the two views.

Conclusion

The account is clearly not perfect. Most importantly, there is the question of whether these considerations apply more generally and they certainly do not seem to apply outside of applications of number symbols. It may be that they do not even apply to the more intricate reaches of measurement theory. For instance, it is unlikely that it applies to the measurement of color or measurements in quantum mechanics that involve more complicated mathematical objects such as vectors and regions of Hilbert space. It remains necessary to delve deeper into the subject and examine more complex theories for which it may not be possible to separate mathematics entirely from the empirical elements the theory characterizes.

A further problem is the theoretical power that is lost by eliminating set theoretical structures. Instead of talking about empirical models of axioms, we restrict ourselves to particular observed quantities and the particular symbols that can be ascribed to them in a relation preserving way. The problem with this is that as long as *(i'')* and *(ii'')* have been followed, the relations of quantities will be reflected in certain relations among the symbols that are applied to them, but we will be unable to demonstrate that there is a relation preserving function between all possible values of the quantity and all of the symbols we might attribute to those values in keeping with *(i'')* and *(ii'')*. This amounts to saying that we cannot *prove* in a more powerful deductive system, such as set theory, that the empirical side and the symbolic side found in this proposal are structurally correspondent. It is unclear how much of a loss this constitutes.

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