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On Closed-Loop Vibrational Control of Underactuated Mechanical Systems

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Abstract

This paper discusses vibrational stabilization of a class of single-input, two degree-of-freedom mechanical systems. Considering two different control formulations – position-input and force-input – and both open- and closed-loop control, we find that the sets of attainable equilibrium positions for the unactuated coordinate are identical in every case. The subset of positions that are stabilizable, however, depends on the formulation. In general, the set of equilibria that can be stabilized using open-loop force-input is larger than the set that can be stabilized using open-loop position-input. And the use of feedback expands this stabilizable set even further. As examples, this paper presents the dynamic analysis, open- and closed-loop vibrational control, and the mechanics behind the stability of two underactuated systems, the Kapitza pendulum and a one-link horizontal pendulum.

Keywords: Vibrational control, Averaged potential, Potential shaping, Averaging, Underactuated mechanical systems

1 Introduction

Vibrational control is a method for effecting a desired, time-averaged motion of a dynamical system using zero-mean, high-frequency inputs. Vibrational control is most often used as an open-loop control strategy. A simple example of a vibrationally stabilized system is the Stephenson-Kapitza pendulum, a two degree-of-freedom (2-DOF) pendulum that is stabilized in its upright orientation by high-frequency vertical oscillations of its pivot [Stephenson, 1908; Kapitza, 1965]. Using open-loop vibrational control, one may stabilize any number of inverted pendula that are attached to a horizontal plate by fast vertical vibrations of that plate, i.e., using a single actuator and no sensors.

As an open-loop control strategy, vibrational control was first introduced by Meerkov [1977, 1980]. Theorems for vibrational stabilizability, vibrational controllability, and shaping of the transient behavior for linear systems and classes of nonlinear systems are discussed in [Meerkov, 1982; Bellman et al., 1985, 1986a,b]. The robustness of stability of open-loop vibrational control systems is discussed in [Cheng et al., 2018b,a]. Vibrational control has applications in stabilization of underactuated robotic systems [Leroquais and d'Andrea Novel, 1997; Hong et al., 1999; Hong, 2002; Yabuno and Kobayashi, 2020], chemical reactors [Cinar et al., 1987], electromagnetic cantilevered actuators [Nonaka et al., 2006], electrostatic actuators [Berg and Wickramasinghe, 2015], control of principal parametric resonance of beams [Sahoo and Chatterjee, 2021], etc. Though vibrational control was originally developed as an open-loop strategy, closed-loop vibrational control systems have also been developed for zero-placement [Lee et al., 1987b] and pole-placement [Lee et al., 1987a; Kabamba et al., 1998b] of linear systems using periodically varying feedback gains with applications in stabilization and control of decentralized systems [Trave et al., 1985; Runolfsson and Meerkov, 1985]. Closed-loop vibrational control has also been used, for certain classes of systems, to track slowly varying trajectories by modulating the vibrational amplitude with applications to underactuated robotic systems [Bullo, 2002; Bullo and Lewis, 2004; Tahmasian and Woolsey, 2015], electromagnetic actuators [Suzuki and Nonaka, 2011], and biomimetic systems [Tahmasian and Woolsey, 2017]. Robustness of closed-loop vibrational control of linear systems is discussed in [Kabamba et al., 1998a].

Since vibrational control uses zero-mean, time-periodic inputs, averaging is a useful tool for design and dynamic analysis of vibrational control systems. As a perturbation analysis tool, averaging in its current form was developed by Bogoliubov and Mitropolsky [1961]. According to the averaging theorem, if the averaged dynamics of a time-periodic system oscillating at a sufficiently high frequency possesses a hyperbolically stable equilibrium point, then the time-periodic system has a periodic orbit with the same stability characteristics as the equilibrium of the time-averaged dynamics [Guckenheimer and Holmes, 1983; Sanders and Verhulst, 1985]. For a class of mechanical control systems subject to high-frequency, high-amplitude forcing, an analytical expression for the time-averaged dynamics was developed in [Bullo, 2002] and [Bullo and Lewis, 2004, Ch. 9]. In this paper, we use the averaging results in [Bullo, 2002; Bullo and Lewis, 2004] to explore the *equilibrium set* for time-averaged vibrational control systems using position- and force-input with and without feedback. By equilibrium set, we mean the set of configurations that can be an equilibrium for the time-averaged dynamics using the methods suggested in this paper.

Averaging techniques are also used to determine potential-like functions for the time-averaged dynamics. Introduced by Baillieul [1993], the *averaged potential* may be used for stability analysis and open-loop vibrational stabilization of certain classes of mechanical time-periodic systems [Seto and Baillieul, 1994; Weibel et al., 1995, 1997; Baillieul and Weibel, 1998; Weibel and Baillieul, 1998; Baillieul, 1999, 2000]. Considering the position of the actuated coordinate as the input, the averaged potential is determined by averaging the Hamiltonian of the time-periodic system. This paper suggests *averaged potential shaping* via closed-loop vibrational control as a means of expanding the equilibrium set of a class of 2-DOF underactuated mechanical systems. Potential shaping was used for closed-loop vibrational stabilization of a class of underactuated, control-affine mechanical systems in Bullo [2002] and [Bullo and Lewis, 2004, Ch. 12]. In that approach, a proportional feedback controller that uses measurements of the actuated coordinates was combined with an open-loop vibrational input such that the averaged potential energy could be determined from the averaged, controlled Hamiltonian [Nijmeijer and van der Schaft, 1990, Ch. 12].

The equilibrium sets of open-loop vibrational control systems are limited, which in turn limits the possible applications. Of course, closed-loop vibrational control poses a number of practical challenges, aside from the challenge of expanding the equilibrium set. For example, since vibrational control systems use high-frequency inputs, the sensors and actuators must be fast enough to react to the system motion and robust enough to withstand the motion. Assuming such challenges have been overcome, we describe the use of vibrational control for a class of 2-DOF, single-input mechanical systems with feedback from the *unactuated* coordinates.

A bit more generally, we consider the dynamics of open- and closed-loop position- and force-input mechanical systems and compare the equilibrium sets for all of these systems. We show that for the class of systems considered, the equilibrium set of an open- or closed-loop position-input system is not larger than that of the open-loop force-input system. We also show that the equilibrium set of a system cannot be expanded beyond that of the open-loop system. While closed-loop control may be used to control the actuated coordinate for force-input systems, it cannot be used to expand the equilibrium set. We apply these results to two vibrational control systems: the Kapitza pendulum and a one-link horizontal pendulum with a torsional spring at its pivot. Following the stability analyses is a brief discussion of the mechanics underlying the vibrational stabilization of these two systems. The mechanics of stability of the Kapitza pendulum is also discussed in [Butikov, 2001, 2011; Grundy, 2019; Artstein, 2021]. In a subsequent work, the results of this research will be used for stability analysis of insect flight [Tahmasian, 2021].

This paper is organized as follows. In Section 2, we describe the class of systems considered in this paper and present the time-varying and time-averaged equations of motion. In Section 3, the effects of closed-loop vibrational control on the averaged dynamics are discussed as well as the averaged potential energy. The two examples, the Kapitza pendulum and the horizontal pendulum, are discussed in Section 4. Section 5 presents a brief review and discussion of the main results.

2 Equations of Motion and Averaging

This section presents the equations of motion for the class of systems considered in this paper, as well as the averaged dynamics and the averaged potential energy under position- and force-input vibrational control. Consider a 2-DOF underactuated mechanical system described by the Lagrangian

$$L(q_{\mathrm{u}}, \dot{q}_{\mathrm{u}}, \dot{q}_{\mathrm{a}}) = \frac{1}{2} m_1 \dot{q}_{\mathrm{u}}^2 + a(q_{\mathrm{u}}) \dot{q}_{\mathrm{u}} \dot{q}_{\mathrm{a}} + \frac{1}{2} m_2 \dot{q}_{\mathrm{a}}^2 - V(q_{\mathrm{u}}) \tag{1}$$

where q_{u} and q_{a} are the unactuated and actuated coordinates, respectively, m_1 and m_2 are positive parameter values (mass or inertia), $a(q_{\mathrm{u}})$ is an inertial coupling term, and $V(q_{\mathrm{u}})$ is the potential energy. In general, the potential energy can be a linear function of the actuated coordinate q_{a} as well, i.e., the total potential energy V_t of the system can take the form

$$V_t(q_{\mathrm{u}}, q_{\mathrm{a}}) = V(q_{\mathrm{u}}) + b q_{\mathrm{a}} \tag{2}$$

where b is a constant, as is the case for the example in Section 4.1. To simplify the discussion, we consider the Lagrangian with potential energy in the form (1), but the analysis and results also hold for a system with the potential energy (2) if one replaces V with V_t in (1).

2.1 The position-input system

For the underactuated system with Lagrangian (1), the position of the actuated coordinate can be considered as the input. Defining $v = \dot{q}_{\mathrm{a}}$ as the input, the Lagrangian (1) becomes the time-varying Lagrangian

$$L(q_{\mathrm{u}}, \dot{q}_{\mathrm{u}}; v) = \frac{1}{2} m_1 \dot{q}_{\mathrm{u}}^2 + a(q_{\mathrm{u}}) v \dot{q}_{\mathrm{u}} + \frac{1}{2} m_2 v^2 - V(q_{\mathrm{u}}) \tag{3}$$

Consider a small-amplitude, high-frequency displacement of the actuated coordinate

$$q_{\mathrm{a}} = \frac{V_0}{\omega} u(\omega t) \tag{4}$$

so that

$$v = V_0 \beta(\omega t) \tag{5}$$

where ω is the high frequency, relative to the natural dynamics of the system, and $u(t)$ and its derivative $\beta(t) = \dot{u}(t)$ are zero-mean, T-periodic functions. Following [Baillieul, 1993; Weibel et al., 1997] (see Appendix A.1), the averaged potential of the system is determined to be

$$V_A(\bar{q}_{\mathrm{u}}) = \frac{\mu V_0^2}{m_1} a^2(\bar{q}_{\mathrm{u}}) + V(\bar{q}_{\mathrm{u}}) \tag{6}$$

where $\bar{q}_{\mathrm{u}}$ is the coordinate corresponding to q_{u} of the averaged dynamics and

$$\mu = \frac{1}{2T} \int_0^T \beta^2(t) dt \tag{7}$$

Defining the potential-like (or virtual potential) function

$$U(q_{\mathrm{u}}) = a^2(q_{\mathrm{u}}) \tag{8}$$

the averaged potential in (6) can also be written as

$$V_A(\bar{q}_{\mathrm{u}}) = \frac{\mu V_0^2}{m_1} U(\bar{q}_{\mathrm{u}}) + V(\bar{q}_{\mathrm{u}}) \tag{9}$$

The equilibrium values of $\bar{q}_{\mathrm{u}}$ for the averaged dynamics are determined using the equation

$$\frac{dV_A}{d\bar{q}_{\mathrm{u}}} = 0 \quad \Rightarrow \quad \frac{\mu V_0^2}{m_1} U'(\bar{q}_{\mathrm{u}}) + V'(\bar{q}_{\mathrm{u}}) = 0 \tag{10}$$

where $\square'(x) = \frac{d\square}{dx}$. For a certain input waveform $u(t)$ (or $\beta(t)$), for an unactuated average configuration $\bar{q}_{\mathrm{u}}$ to be an equilibrium point of the averaged dynamics, one should choose the amplitude V_0 to be

$$V_0 = \left(-\frac{m_1}{\mu} \frac{V'(\bar{q}_{\mathrm{u}})}{U'(\bar{q}_{\mathrm{u}})} \right)^{\frac{1}{2}} \tag{11}$$

Since $\mu > 0$, from equation (11) it is evident that the sufficient condition for the coordinate $\bar{q}_{\mathrm{u}}$ of the averaged dynamics to be an equilibrium point of the averaged dynamics is

$$\frac{V'(\bar{q}_{\mathrm{u}})}{U'(\bar{q}_{\mathrm{u}})} \leq 0 \tag{12}$$

Additionally, if $V'(\bar{q}_{\mathrm{u}}) \neq 0$, for the input amplitude V_0 to be finite, it is required that $U'(\bar{q}_{\mathrm{u}}) \neq 0$.

Remark 2.1 *: From equation (10) it is evident that there may exist other equilibria whose existence does not depend on applying the high-frequency input. In this paper we call those equilibria the trivial (or non-vibrational) equilibria. A coordinate $\bar{q}_{\mathrm{u}}$ is a trivial equilibrium of the averaged dynamics iff $V'(q_{\mathrm{u}}) = U'(q_{\mathrm{u}}) = 0$. However, high-frequency input may change the stability properties of a trivial equilibrium. Examples of trivial equilibria are the downright and upright orientations of the Kapitza pendulum discussed in Section 4.1. This paper is mostly concerned with the nontrivial (vibrational) equilibria whose existence depends on the high-frequency input, and the trivial equilibria whose stability properties depend on the properties, e.g., frequency and amplitude, of the vibrational input. Unless mentioned otherwise, in this paper, by equilibrium point we mean nontrivial equilibrium point.*

The sufficient condition (12) suggests a subset of the unactuated coordinate space for which the given high-frequency position-input generates an equilibrium of the averaged dynamics. The equilibrium is not necessarily stable; if it is unstable, it may be possible to stabilize it using closed-loop vibrational control, discussed in Section 3. In this paper, the union of the subset of the unactuated coordinate space determined by condition (12) and the set of the trivial equilibria of a system is called the *equilibrium set* of the averaged system.

Note that inequality (12) also suggests the equilibrium set may contain values $\bar{q}_{\mathrm{u}}$ for which the true potential energy $V(\bar{q}_{\mathrm{u}})$ and the potential-like function $U(\bar{q}_{\mathrm{u}})$ in the averaged potential function respond in opposite directions to a displacement from $\bar{q}_{\mathrm{u}}$.

A nontrivial equilibrium point $\bar{q}_{\mathrm{u}} = q_{\mathrm{e}}$ of the averaged dynamics is stable if

$$\left.\frac{d^2 V_A}{d\bar{q}_{\mathrm{u}}^2}\right|_{\bar{q}_{\mathrm{u}}=q_{\mathrm{e}}} > 0 \quad \Rightarrow \quad \frac{\mu V_0^2}{m_1} U''(q_{\mathrm{e}}) + V''(q_{\mathrm{e}}) > 0 \tag{13}$$

Replacing V_0 from (11), the stability condition (13) can be written in the form

$$V''(q_{\mathrm{e}}) - \frac{V'(q_{\mathrm{e}})}{U'(q_{\mathrm{e}})} U''(q_{\mathrm{e}}) > 0 \tag{14}$$

or using (12), in the more succinct form

$$\frac{1}{V'(q_{\mathrm{e}})} \left(\frac{V'(q_{\mathrm{e}})}{U'(q_{\mathrm{e}})}\right)' < 0 \tag{15}$$

The stability of the trivial equilibria can be determined by linearizing the averaged dynamics about that equilibrium or using Lyapunov's stability theorem [Khalil, 1996]. In this paper, a subset of the equilibrium set that can be stabilized using open- or closed-loop vibrational control is called the *stabilizable set*. The union of a subset of the nontrivial equilibrium set satisfying inequality (15) and a subset of the trivial equilibria that are stable or can be vibrationally stabilized, defines the stabilizable set for the open-loop position-input system considered here.

According to the averaging theorem, if equilibrium point q_{e} satisfying the inequality (15) is *asymptotically stable*, then the original system with the Lagrangian (1) possesses an asymptotically stable periodic orbit in an $O\left(\frac{1}{\omega}\right)$ neighborhood of q_{e}, i.e., the unactuated coordinate q_{u} "hovers" in an $O\left(\frac{1}{\omega}\right)$ neighborhood of q_{e} [Guckenheimer and Holmes, 1983; Sanders and Verhulst, 1985].

On the other hand, one may use the averaged dynamics of the original system to determine the equilibrium points and their stability. Using Lagrangian (1) and including a small linear damping term with coefficient $c > 0$ to provide asymptotic stability, the equation of motion of the unactuated coordinate q_{u} is

$$m_1 \ddot{q}_{\mathrm{u}} + c\dot{q}_{\mathrm{u}} + V'(q_{\mathrm{u}}) = -a(q_{\mathrm{u}})\ddot{q}_{\mathrm{a}} \tag{16}$$

Defining the zero-mean, T-periodic acceleration $w(t) = \ddot{u}(t)$ and substituting $\ddot{q}_{\mathrm{a}} = V_0 \omega w(\omega t)$, equation (16) can be written in the first order form

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{c}{m_1} x_2 - \frac{1}{m_1} V'(x_1) - \frac{V_0 a(x_1)}{m_1} \omega w(\omega t)
\end{aligned}
\tag{17}
$$

where $\boldsymbol{x} = (q_{\mathrm{u}}, \dot{q}_{\mathrm{u}})^T$ is the state vector. Following [Bullo and Lewis, 2004, Ch. 9] (see Appendix A.2), the averaged dynamics of the system are determined as

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= -\frac{c}{m_1} \bar{x}_2 - \frac{1}{m_1} V'(\bar{x}_1) - \frac{2\mu V_0^2}{m_1^2} a(\bar{x}_1) a'(\bar{x}_1)
\end{aligned}
\tag{18}
$$

where $\bar{\boldsymbol{x}} = (\bar{q}_{\mathrm{u}}, \dot{\bar{q}}_{\mathrm{u}})^T$ is the state vector of the averaged dynamics and $\mu > 0$ is determined using (7), or alternately, using $w(t)$ [Tahmasian et al., 2018]. Using the virtual potential $U(q_{\mathrm{u}})$, the averaged dynamics (18) can also be rewritten as

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= -\frac{c}{m_1} \bar{x}_2 - \frac{1}{m_1} V'(\bar{x}_1) - \frac{\mu V_0^2}{m_1^2} U'(\bar{x}_1)
\end{aligned}
\tag{19}
$$

The equilibrium points of the averaged dynamics are determined by setting $\dot{\bar{\boldsymbol{x}}} = \mathbf{0}$, which gives equation (10). The stability of an equilibrium point $\bar{\boldsymbol{x}}_{\mathrm{e}}$ can be determined by linearizing the averaged dynamics (19) about that equilibrium and checking that the eigenvalues all have negative real part. The result is the same as using the averaged potential method: if the equilibrium point $\bar{\boldsymbol{x}}_{\mathrm{e}} = (q_{\mathrm{e}}, 0)^T$ satisfies inequality (15), then it is stable.

2.2 The force-input system

Consider the underactuated mechanical system with Lagrangian (1), subject to a small amount of linear damping in the unactuated degree of freedom as described above. Applying the input force F to the actuated coordinate, the equations of motion of the system are

$$
\begin{aligned}
m_1\ddot{q}_\mathrm{u} + a(q_\mathrm{u})\ddot{q}_\mathrm{a} + c\dot{q}_\mathrm{u} + V'(q_\mathrm{u}) &= 0 \\
a(q_\mathrm{u})\ddot{q}_\mathrm{u} + m_2\ddot{q}_\mathrm{a} + a'(q_\mathrm{u})\dot{q}_\mathrm{u}^2 &= F
\end{aligned} \tag{20}
$$

Consider the high-frequency, high-amplitude input force

$$
F = F_0\omega\phi(\omega t) \tag{21}
$$

where $\phi(t)$ is a zero-mean, T-periodic function. Using the state vector $\boldsymbol{y} = (q_\mathrm{u}, q_\mathrm{a}, \dot{q}_\mathrm{u}, \dot{q}_\mathrm{a})^T$, equations of motion (20) can be written as

$$
\begin{aligned}
\dot{y}_1 &= y_3 \\
\dot{y}_2 &= y_4 \\
\dot{y}_3 &= \frac{1}{h(y_1)}\Big(a(y_1)a'(y_1)y_3^2 - m_2\,c\,y_3 - m_2V'(y_1)\Big) - \frac{F_0\,a(y_1)}{h(y_1)}\omega\phi(\omega t) \\
\dot{y}_4 &= \frac{1}{h(y_1)}\Big(-m_1a'(y_1)y_3^2 + c\,a(y_1)y_3 + a(y_1)V'(y_1)\Big) + \frac{m_1\,F_0}{h(y_1)}\omega\phi(\omega t)
\end{aligned} \tag{22}
$$

where $h(q_\mathrm{u}) = m_1m_2 - a^2(q_\mathrm{u})$ is the determinant of the generalized inertia matrix of the system corresponding to the Lagrangian (1). For the mechanical system with the Lagrangian (1), the generalized inertia matrix is positive definite. Thus, for any q_u, the determinant of the symmetric inertia matrix is positive, i.e., $h(q_\mathrm{u}) > 0$.

Following [Bullo and Lewis, 2004, Ch. 9] (see Appendix A.2), the averaged dynamics of system (22) is determined to be

$$
\begin{aligned}
\dot{\bar{y}}_1 &= \bar{y}_3 \\
\dot{\bar{y}}_2 &= \bar{y}_4 \\
\dot{\bar{y}}_3 &= \frac{1}{h(\bar{y}_1)}\Big(a(\bar{y}_1)a'(\bar{y}_1)\bar{y}_3^2 - m_2\,c\,\bar{y}_3 - m_2V'(\bar{y}_1)\Big) - \frac{2\mu\,m_1\,m_2\,F_0^2\,a(\bar{y}_1)\,a'(\bar{y}_1)}{h^3(\bar{y}_1)} \\
\dot{\bar{y}}_4 &= \frac{1}{h(\bar{y}_1)}\Big(-m_1a'(\bar{y}_1)\bar{y}_3^2 + c\,a(\bar{y}_1)\bar{y}_3 + a(\bar{y}_1)V'(\bar{y}_1)\Big) + \frac{2\mu\,m_1\,F_0^2\,a^2(\bar{y}_1)\,a'(\bar{y}_1)}{h^3(\bar{y}_1)}
\end{aligned} \tag{23}
$$

where $\bar{\boldsymbol{y}} = (\bar{q}_\mathrm{u}, \bar{q}_\mathrm{a}, \dot{\bar{q}}_\mathrm{u}, \dot{\bar{q}}_\mathrm{a})^T$ is the state vector of the averaged system and the input parameter $\mu > 0$ is determined as in (7), but using the T-periodic function $\phi(t)$ rather than $\beta(t)$. The equilibrium points of the averaged dynamics are determined by setting $\dot{\bar{\boldsymbol{y}}} = \mathbf{0}$, which gives the following equation in terms of the average unactuated coordinate $\bar{q}_\mathrm{u}$

$$\frac{2\mu\, m_1\, F_0^2\, a(\bar{q}_\mathrm{u})\, a'(\bar{q}_\mathrm{u})}{h^2(\bar{q}_\mathrm{u})} + V'(\bar{q}_\mathrm{u}) = 0 \tag{24}$$

For an unactuated coordinate $\bar{q}_\mathrm{u}$ to be a nontrivial equilibrium point of the averaged dynamics, the force amplitude must be

$$F_0 = h(\bar{q}_\mathrm{u}) \left(-\frac{V'(\bar{q}_\mathrm{u})}{2\mu\, m_1\, a(\bar{q}_\mathrm{u}) a'(\bar{q}_\mathrm{u})} \right)^{\frac{1}{2}} \tag{25}$$

or using the virtual potential $U(q_\mathrm{u})$, the force amplitude is

$$F_0 = h(\bar{q}_\mathrm{u}) \left(-\frac{1}{\mu m_1} \frac{V'(\bar{q}_\mathrm{u})}{U'(\bar{q}_\mathrm{u})} \right)^{\frac{1}{2}} \tag{26}$$

It is evident that the force amplitude F_0 exists if and only if the inequality (12) is satisfied. From equations (10) and (24), it is evident that both the force- and position-input system possess the same trivial equilibria, if any. Therefore the equilibrium sets of the open-loop position-input and force-input systems are identical.

Using the linearization of the averaged dynamics (23) about an equilibrium point and replacing F_0 from (26), it can be shown that the equilibrium point $\bar{q}_\mathrm{u} = q_\mathrm{e}$ is stable if

$$h(q_\mathrm{e}) \left[V''(q_\mathrm{e}) - \left(\frac{a'(q_\mathrm{e})}{a(q_\mathrm{e})} + \frac{a''(q_\mathrm{e})}{a'(q_\mathrm{e})} \right) V'(q_\mathrm{e}) \right] - 4a(q_\mathrm{e}) a'(q_\mathrm{e}) V'(q_\mathrm{e}) > 0 \tag{27}$$

Using $U'(q_\mathrm{u}) = 2a(q_\mathrm{u})a'(q_\mathrm{u})$ and the equilibrium existence condition (12), the stability condition (27) can also be written in the succinct form

$$\frac{1}{V'(q_\mathrm{e})} \left(\frac{V'(q_\mathrm{e})}{U'(q_\mathrm{e})} \right)' < \frac{2}{h(q_\mathrm{e})} \tag{28}$$

The union of the subset of the equilibrium set satisfying inequality (28) and the set of the stable and stabilizable trivial equilibria of the averaged dynamics defines the stabilizable set for the force-input system.

Comparing the stability conditions (15) and (28) for the open-loop position- and force-input systems, respectively, and noting that $h(q_e) > 0$, we see that (15) implies (28). Thus, an equilibrium point $\bar{q}_u = q_e$ that is stable for the force-input system may be unstable for the position-input system, though the converse cannot occur. While the equilibrium sets of both the open-loop position-input and force-input systems are equal, the stabilizable set of the position-input system is a subset of the stabilizable set of the force-input system. The two mechanical systems presented in Section 4 illustrate that this subset can indeed be strict.

3 Closed-Loop Vibrational Control

This section discusses the effects of feedback of the unactuated coordinate on the equilibrium and stabilizable sets of the position- and force-input systems characterized by Lagrangian (1). Consider the position-input system with Lagrangian (3) and suppose that the displacement of the actuated coordinate q_a is the sum of an open-loop oscillatory signal and a feedback term that depends on the unactuated coordinate q_u as follows:

$$q_a = f(q_u) + \frac{V_0}{\omega}u(\omega t) \tag{29}$$

where $f(q_u)$ is a twice differentiable function and $u(t)$ is a zero-mean, T-periodic function. The acceleration of the actuated coordinate is

$$\ddot{q}_a = \ddot{q}_u f'(q_u) + \dot{q}_u^2 f''(q_u) + V_0\omega w(\omega t) \tag{30}$$

where $w(t) = \ddot{u}(t)$ is also zero-mean and T-periodic. Substituting $\ddot{q}_a$ in (30) into equation (16) gives

$$\big(m_1 + a(q_u)f'(q_u)\big)\ddot{q}_u + a(q_u)f''(q_u)\dot{q}_u^2 + V'(q_u) + V_0 a(q_u)\omega w(\omega t) = -c\dot{q}_u \tag{31}$$

Using the state vector $\boldsymbol{x} = (q_u, \dot{q}_u)^T$, equation (31) can be written in the first order form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{a(x_1)f''(x_1)x_2^2 + cx_2 + V'(x_1)}{m_1 + a(x_1)f'(x_1)} - \frac{V_0 a(x_1)}{m_1 + a(x_1)f'(x_1)}\omega w(\omega t) \end{aligned} \tag{32}$$

The averaged dynamics of the system (32) are

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= -\frac{a(\bar{x}_1) f''(\bar{x}_1)\bar{x}_2^2 + c\bar{x}_2 + V'(\bar{x}_1)}{m_1 + a(\bar{x}_1) f'(\bar{x}_1)} - \frac{2\mu V_0^2 a(\bar{x}_1) a'(\bar{x}_1)}{\big(m_1 + a(\bar{x}_1) f'(\bar{x}_1)\big)^3}
\end{aligned} \tag{33}
$$

where μ is a positive parameter determined as in (7) using the function $w(t)$. Defining the state vector $\bar{\boldsymbol{x}} = (\bar{q}_{\rm u}, \dot{\bar{q}}_{\rm u})^T$, the equilibrium points of the averaged dynamics (33) are determined by setting $\dot{\bar{\boldsymbol{x}}} = \mathbf{0}$ which results in $\bar{x}_2 = \dot{q}_{\rm u} = 0$ and

$$
\frac{2\mu V_0^2 a(\bar{q}_{\rm u}) a'(\bar{q}_{\rm u})}{\big(m_1 + a(\bar{q}_{\rm u}) f'(\bar{q}_{\rm u})\big)^2} + V'(\bar{q}_{\rm u}) = 0 \tag{34}
$$

Therefore, for the unactuated coordinate $\bar{q}_{\rm u}$ to be an equilibrium point of the averaged dynamics, one should choose the amplitude V_0 of the oscillatory part of the input to be

$$
V_0 = \Big| m_1 + a(\bar{q}_{\rm u}) f'(\bar{q}_{\rm u}) \Big| \left(-\frac{1}{\mu} \frac{V'(\bar{q}_{\rm u})}{U'(\bar{q}_{\rm u})} \right)^{\frac{1}{2}} \tag{35}
$$

and seek (that is, *design*) the feedback function $f(q_{\rm u})$ to stabilize that equilibrium point.

From equation (35) it is evident that for a certain value of $\bar{q}_{\rm u} = q_{\rm e}$ the amplitude V_0 exists if and only if condition (12) is satisfied. Thus, the closed-loop displacement input (29) does not change the equilibrium set. However, an unstable equilibrium point of the open-loop system may be stabilized by choosing an appropriate feedback function $f(q_{\rm u})$. That is, the closed-loop input (29) may expand the stabilizable set of the open-loop position-input system. This result is illustrated in Section 4.1 for closed-loop control of the Kapitza pendulum system.

Given that the closed-loop position-input system cannot expand the equilibrium set, one might consider using closed-loop force-input control instead. Consider the mechanical system (1) with control force in the form

$$
F = f(q_{\rm a}) + F_0 g(q_{\rm u}) \omega \phi(\omega t) \tag{36}
$$

where $f(q_{\rm a})$ and $g(q_{\rm u})$ are twice differentiable in their arguments. Using the same method as in Section 2.2, it can be shown that for a point $\bar{q}_{\rm u}$ to be an equilibrium point of the averaged system, the function $f(q_{\rm a})$

and the amplitude F_0 must be

$$f(\bar{q}_\text{a}) = \frac{h(\bar{q}_\text{u})g'(\bar{q}_\text{u})V'(\bar{q}_\text{u})}{m_1 g(\bar{q}_\text{u})a'(\bar{q}_\text{u})} \tag{37}$$

and

$$F_0 = \frac{h(\bar{q}_\text{u})}{\left|g(\bar{q}_\text{u})\right|}\left(-\frac{1}{\mu m_1}\frac{V'(\bar{q}_\text{u})}{U'(\bar{q}_\text{u})}\right)^{\frac{1}{2}} \tag{38}$$

From equation (38) it is evident that the amplitude F_0 exists if and only if the unactuated coordinate $\bar{q}_\text{u} = q_\text{e}$ satisfies condition (12). Therefore the force-input system with the closed-loop input (36) also does not expand the equilibrium set of the open-loop position- or force-input systems.

Equation (37) however suggests that the input (36) can be used to control the actuated coordinate q_a. For example, by choosing $f(q_\text{a}) = kq_\text{a}$, the constant k may be selected to make a certain point $(\bar{q}_\text{u}, \bar{q}_\text{a})^T$ a stable equilibrium point of the averaged dynamics. This last remark is also discussed in [Bullo, 2002] and [Bullo and Lewis, 2004, Ch. 12]. In Section 4.2, this observation is used for the control of the actuated and unactuated coordinates of a horizontal pendulum system. Figure 1 summarizes the effects of closed-loop control on stabilizable sets of the position-input and force-input systems.

4 Two Examples of Mechanical Systems

In this section two underactuated mechanical systems are presented and their open- and closed-loop vibrational control are discussed. The systems are the Kapitza pendulum and a horizontal pendulum with a torsional spring at its pivot.

4.1 The Kapitza pendulum

Consider the Kapitza pendulum depicted in Figure 2. The 2-DOF system consists of a simple pendulum with mass m and length l moving in a vertical plane. The pendulum rotates freely about its pivot A, while the pivot moves in the vertical z-direction.

The Lagrangian of the system is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{z}\dot{\theta}\sin\theta + \frac{1}{2}m\dot{z}^2 - mg(z + l\cos\theta) \tag{39}$$

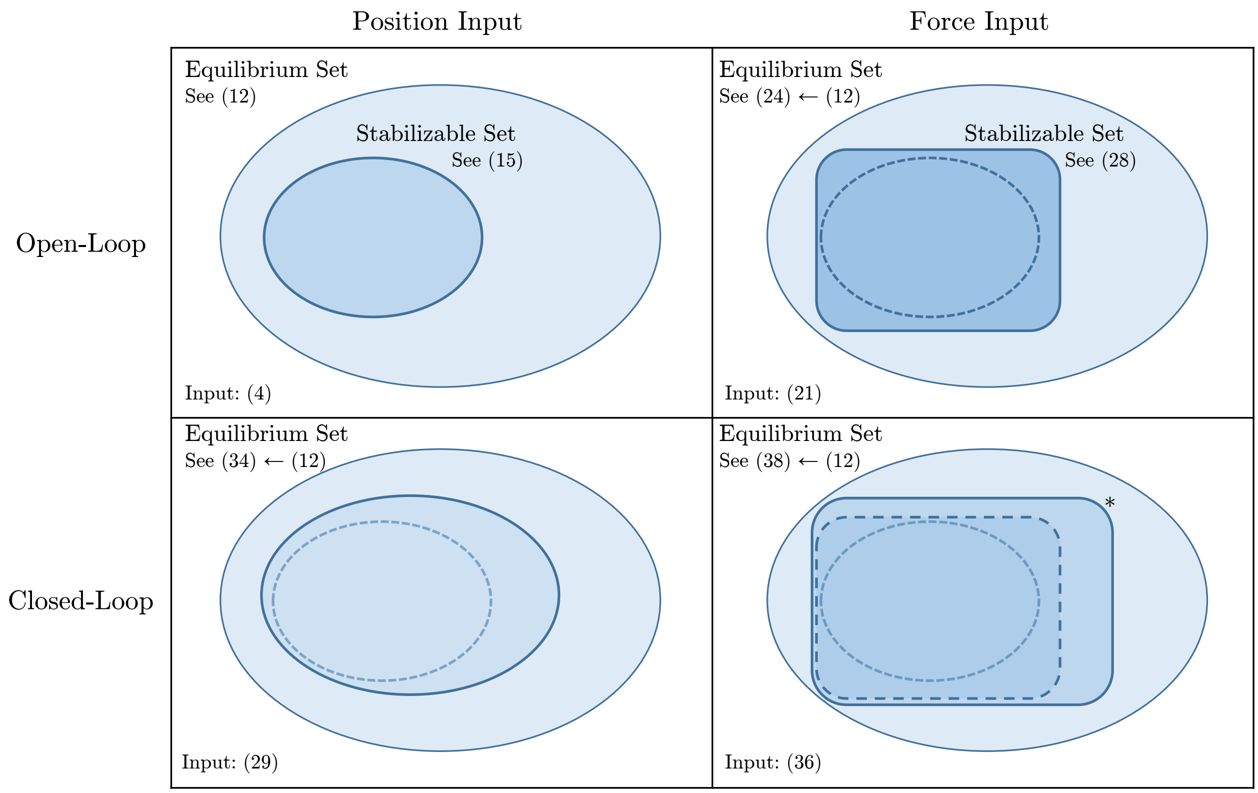

Figure 1: Existence and stability of (time-averaged) equilibria in the uncontrolled coordinate. (* indicates additional control authority over the actuated coordinate.)

where the unactuated coordinate θ is the orientation of the pendulum measured from its upright position, the actuated coordinate z is the position of the pivot, and g is the gravitational acceleration. Comparing with (1), one finds $m_1 = ml^2$, $m_2 = m$, $a(\theta) = -ml\sin\theta$, and $V_t(\theta, z) = mg(z + l\cos\theta)$ which is in the form (2) where $V = mgl\cos\theta$. Using (12), the equilibrium set of the averaged system is determined as the region where $\cos\theta \geq 0$, which is the upper half-plane, plus the downward orientation, i.e., the trivial equilibrium $\theta = \pi$. It must be noted that, regardless of the input amplitude, the trivial equilibrium $\theta = \pi$ is always stable [Thomsen, 2002]. For the open-loop position-input system, the stable nontrivial equilibria in the equilibrium set are determined using condition (15), which for this system simplifies into

$$\frac{\sin^2\theta}{\cos\theta} \leq 0$$

Therefore the only stable equilibria of the open-loop position-input system in its equilibrium set are $\theta = 0$ and the trivial equilibrium $\theta = \pi$; the upright and downright orientations. However, using the closed-loop vibrational control (29), the stabilizable set of the position-input system can be expanded to include the

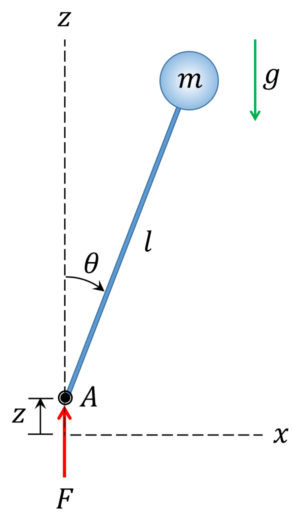

Figure 2: The Stephenson-Kapitza pendulum.

entire equilibrium set, i.e., the upper half-plane.

4.1.1 The closed-loop position-input system

Consider the closed-loop position-input system with the pivot displacement in the form of (29)

$$z = f(\theta) + \frac{V_z}{\omega} u(\omega t) \tag{40}$$

where V_z is the amplitude of the open-loop velocity. Considering a small torsional damping at the pivot with coefficient c_t, the equation of motion of the unactuated coordinate is

$$\big(l - f'(\theta)\sin\theta\big)\ddot{\theta} - \dot{\theta}^2 f''(\theta)\sin\theta + b_t\dot{\theta} - V_z\omega\sin\theta w(\omega t) = 0$$

where $b_t = \frac{c_t}{ml}$ and $w(t) = \ddot{u}(t)$. The averaged dynamics of the system is determined to be

$$\ddot{\bar{\theta}} = \frac{\dot{\bar{\theta}}^2 f''(\bar{\theta})\sin\bar{\theta} - b_t\dot{\bar{\theta}} + g\sin\bar{\theta}}{l - f'(\bar{\theta})\sin\bar{\theta}} - \frac{\mu l V_z^2 \sin 2\bar{\theta}}{\big(l - f'(\bar{\theta})\sin\bar{\theta}\big)^3}$$

where the input parameter μ is determined using $w(t)$.

Using the averaged dynamics it can be shown that for the open-loop system, i.e., $f(\theta) = 0$, the only stable equilibrium point in the equilibrium set is the upright position $\theta = 0$. However, by selecting an appropriate function $f(\theta)$ and using the input (40) with

$$V_z = \left| l - f'(\theta)\sin\theta \right| \sqrt{\frac{g\sec\theta}{2\mu l}}$$

the unactuated coordinate θ can be stabilized in any orientation in the equilibrium set, i.e., $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Figure 3 shows the response of the system using the input

$$z = k\theta - \frac{V_z}{\omega}\cos\omega t$$

where k is a parameter to be selected to stabilize the desired equilibrium. The goal is to stabilize the pendulum at the orientation $\theta = 60°$, on average. The physical parameters of the system are $m = 0.2$ kg, $l = 0.2$ m, $c_t = 0.01$ N.m.s/rad, and $\omega = 300$ rad/s. The selected control parameter is $k = 0.16$, and using the averaged dynamics, the amplitude of the oscillatory input is determined to be $V_z = 0.8605$ m/s. The initial conditions are $\theta_0 = 50°$ and $\dot{\theta}_0 = 0$.

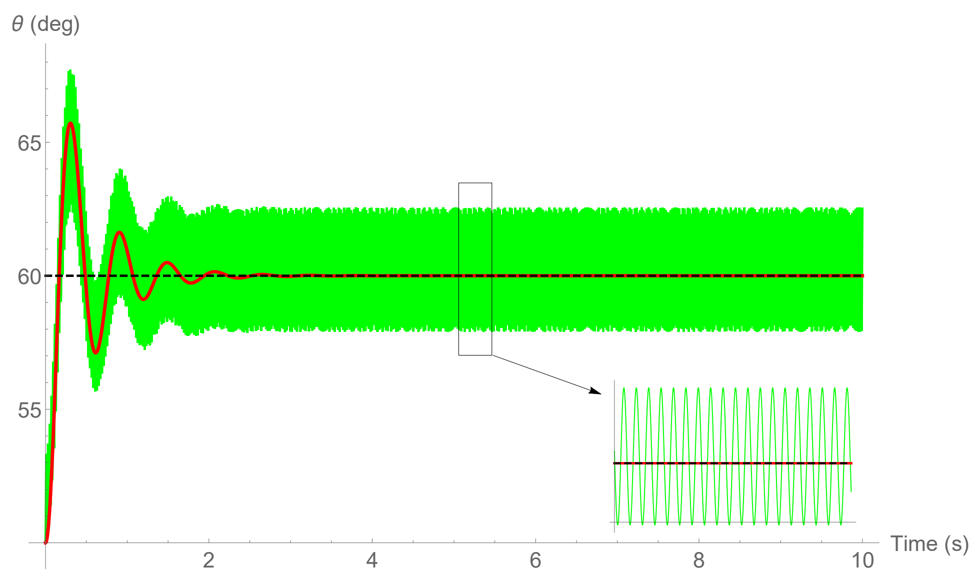

Figure 3: Closed-loop position-input Kapitza pendulum stabilized at the orientation $\theta = 60°$. Solid-green: original system, solid-red: averaged dynamics, dashed-black: desired orientation.

It is noteworthy that one may use the displacement input in the form

$$z = f\big(\theta, \theta_{\mathrm{d}}(t)\big) + \frac{V_z}{\omega}u(\omega t)$$

to follow the slowly time-varying desired trajectory $\theta_\mathrm{d}(t)$ in the equilibrium set of the system on average. In that case, the averaged dynamics is a slowly time-varying system which may be stabilized using the existing theorems on time-varying systems; see [Tsakalis and Ioannou, 1993], for example. Figure 4 shows the response of the system following the slowly time-varying desired trajectory $\theta_\mathrm{d}(t) = 45° + 30° \sin \frac{t}{3}$. The input is

$$z = k_1\big(\theta - \theta_\mathrm{d}(t)\big) + k_2\big(\theta - \theta_\mathrm{d}(t)\big)^2 - \frac{V_z}{\omega} \cos \omega t$$

where $k_1 = 0.16$ and $k_2 = 0.2$ are the control parameters. The physical parameters are the same as before.

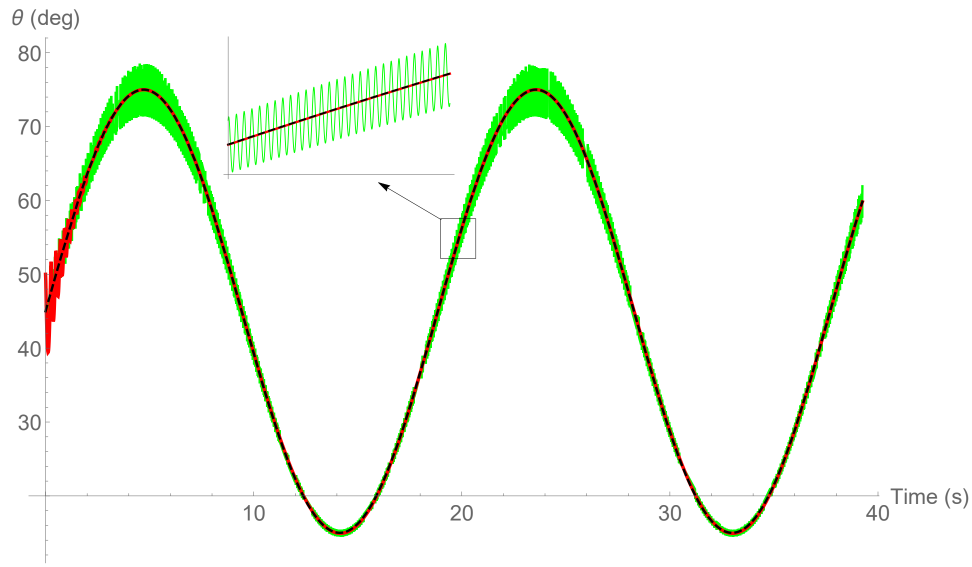

Figure 4: Slowly varying trajectory tracking for the closed-loop Kapitza pendulum under position-input. Solid-green: original system, solid-red: averaged dynamics, dashed-black: desired trajectory.

4.1.2 The force-input system

Alternatively, the force acting on the pivot may be considered as the input. The stabilizable set of the force-input system is determined using the inequality (28). For the Kapitza pendulum, inequality (28) simplifies into $\sin^2\theta \cos\theta > 0$, which is the upper half-plane. Therefore the stabilizable set of the force-input system is equal to the equilibrium set of the system. In other words, for the open-loop force-control system, the pendulum can be stabilized in any orientation in its equilibrium set, i.e., the upper half-plane.

For the force-input system, considering a small linear damping with coefficient c in the actuated coordinate, the equations of motion are

$$\begin{aligned} l\ddot{\theta} - \ddot{z}\sin\theta + b_t\dot{\theta} - g\sin\theta &= 0 \\ m\ddot{z} - ml\ddot{\theta}\sin\theta + c\dot{z} - ml\dot{\theta}^2\cos\theta + mg &= F \end{aligned} \tag{41}$$

Using the open-loop force input

$$F = mg + F_0\omega\phi(\omega t) \tag{42}$$

the averaged dynamics of the system are

$$\begin{aligned} \ddot{\bar{\theta}} &= \dot{\bar{\theta}}^2\tan\theta - \frac{c}{ml}\dot{\bar{z}}\sec\bar{\theta}\tan\bar{\theta} - \frac{b_t}{l}\dot{\bar{\theta}}\sec^2\bar{\theta} + \frac{g}{l}\sec\bar{\theta}\tan\bar{\theta} - \frac{2\mu F_0^2}{m^2l^2}\sec^4\bar{\theta}\tan\bar{\theta} \\ \ddot{\bar{z}} &= l\dot{\bar{\theta}}^2\sec\bar{\theta} - \frac{c}{m}\dot{\bar{z}}\sec^2\bar{\theta} - b_t\dot{\bar{\theta}}\sec\bar{\theta}\tan\bar{\theta} + g\tan^2\bar{\theta} - \frac{2\mu F_0^2}{m^2l}\sec^3\bar{\theta}\tan^2\bar{\theta} \end{aligned} \tag{43}$$

As mentioned, the averaged dynamics and inequality (28) suggest that in the force-input system the pendulum can be stabilized in any orientation in the equilibrium set of the system (upper half-plane), when using the force amplitude

$$F_0 = m\cos\theta\sqrt{\frac{gl\cos\theta}{2\mu}}$$

To control the actuated coordinate z, one may consider the input force in the form

$$F = f(z) + mg + F_0\omega\phi(\omega t) \tag{44}$$

and seek the function $f(z)$ to stabilize the average dynamics at a desired equilibrium point.

In summary, for the Kapitza pendulum, besides the trivial downright stable equilibrium, the open-loop position-input system can be stabilized in its upright position only, i.e., there are at least one and at most two stable equilibria. However, the open-loop force-input and closed-loop position-input systems can be stabilized in any orientation in the equilibrium set of the Kapitza pendulum, i.e., the upper half-plane.

Note that the stabilizable sets for a vibrational mechanical system using open-loop displacement- and force-input may be identical and equal to the equilibrium set of the system. In that case, closed-loop vibrational control does not expand the stabilizable set of the open-loop system. One example is a Kapitza-like pendulum for which the pivot moves horizontally instead of vertically; see Figure 2. Through similar stability analysis as for the Kapitza pendulum, it can be shown that the equilibrium sets and the stabilizable sets using open-loop displacement- and force-input vibrational control are equal and comprise the entire lower half-plane, i.e., $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.

4.2 The horizontal pendulum

The second mechanical system we consider is the 2-DOF horizontal pendulum depicted in Figure 5. The system consists of a simple pendulum with mass m and length l rotating about its pivot A in the horizontal plane: Gravity does not affect the system. The pivot moves along the x-axis in the horizontal plane and a torsional spring with stiffness k_t is connected to the pendulum about its pivot. The unactuated coordinate θ is measured relative to the positive x-axis, as shown. The actuated coordinate x is the position of the pivot. The free orientation of the torsional spring is at $\theta = 0$.

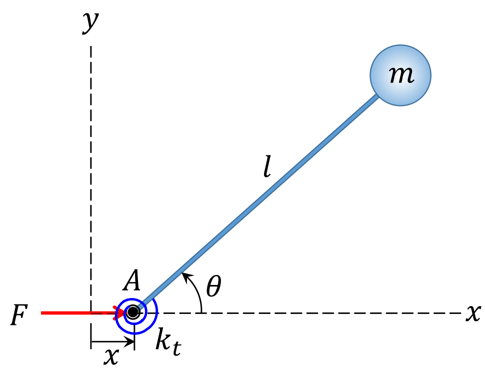

Figure 5: The horizontal pendulum.

The Lagrangian of the system is

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{x}\dot{\theta}\sin\theta + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k_t\theta^2 \tag{45}$$

Comparing with the general system (1), for this system $m_1 = ml^2$, $m_2 = m$, $a(\theta) = -ml\sin\theta$, and $V(\theta) = \frac{1}{2}k_t\theta^2$. Using (12), the equilibrium set of the system is determined to be any orientation satisfying

$$\frac{\theta}{\sin 2\theta} \leq 0 \tag{46}$$

plus the trivial equilibrium $\theta = 0$. It can be shown that, regardless of applying any high-frequency input, the trivial equilibrium $\theta = 0$ is always stable. Therefore, if $\theta > 0$, the equilibrium set of the system is the open second and fourth quadrants, i.e., $\frac{\pi}{2} < \theta < \pi$ and $\frac{3\pi}{2} < \theta < 2\pi$, respectively, and if $\theta < 0$, the equilibrium set is the open third and first quadrants, i.e., $-\frac{\pi}{2} < \theta < -\pi$ and $-\frac{3\pi}{2} < \theta < -2\pi$, respectively. Therefore, considering both clockwise and counterclockwise rotations of the torsional spring, the equilibrium set of the system includes almost any orientation of the pendulum and the system may be stabilized in almost any direction, except two singular orientations $\theta = \pm\frac{\pi}{2}$. The orientations which cannot be stabilized by counterclockwise rotation of the torsional spring, can be stabilized by its clockwise rotation. For example, the pendulum cannot be stabilized in the orientation $\theta = 60°$. However, it can be stabilized at $\theta = -300°$, which is the same orientation $\theta = 60°$ for the pendulum (but not for the spring).

Consider the non-negative orientations, $\theta \geq 0$. For the open-loop position-input system, using (15), the stable orientations θ (i.e., the stabilizable set) within the equilibrium set are determined to be the orientations satisfying

$$1 - \frac{2\theta}{\tan 2\theta} \geq 0 \tag{47}$$

If the pendulum operates in the domain $0 \leq \theta \leq 360°$, then using inequality (47), the stabilizable set of the open-loop position-input system is determined to be $\{128.73° < \theta < 180°\} \bigcup \{312.38° < \theta < 360°\}$ plus the always stable trivial equilibrium $\theta = 0$. However, it is interesting to note that if the pendulum is rotated one full revolution and operates in the domain $360° \leq \theta \leq 720°$, the stabilizable set is slightly smaller than when $0 \leq \theta \leq 360°$. Using inequality (47) it can be shown that if the pendulum operates in the domain $360°n \leq \theta \leq 360°(n+1)$, as $n \to \infty$, the stabilizable set approaches $\{360°n + 135° < \theta < 360°n + 180°\} \bigcup \{360°n + 315° < \theta < 360°(n+1)\}$. Figure 6 presents the stabilizable set of the open-loop position-input system for positive angles $\theta \geq 0$. The solid-dark blue and solid-green regions show the stabilizable set of the pendulum for $n = 0$ and $n \to \infty$, respectively. Similar results can be expressed for the negative orientations $\theta < 0$.

Similar to the Kapitza pendulum, using a closed-loop strategy, the position-input system can be stabilized in any orientation in its equilibrium set, i.e., any desired orientation except the two singular orientations $\theta = \pm\frac{\pi}{2}$. In the following the closed-loop force-input system for control of both actuated and unactuated coordinates is discussed and numerical results are presented.

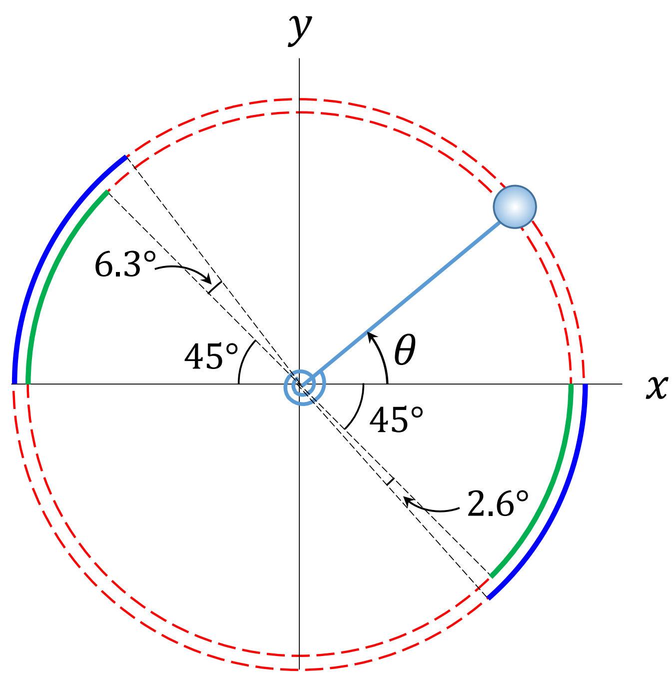

Figure 6: The stabilizable set (orientations) of the open-loop position-input horizontal pendulum for positive orientations, $\theta > 0$. Solid-dark blue: stabilizable set for $0 < \theta < 2\pi$. Solid-green: stabilizable set for $2n\pi < \theta < 2(n+1)\pi$ when $n \to \infty$. Dashed-red: unstabilizable set. (Not to scale.)

Using the Lagrangian (45) and considering small damping in the system, the equations of motion of the system are

$$\begin{aligned} ml^2\ddot{\theta} - ml\ddot{x}\sin\theta + c_t\dot{\theta} + k_t\theta &= 0 \\ m\ddot{x} - ml\ddot{\theta}\sin\theta - ml^2\dot{\theta}^2\cos\theta + c\dot{x} &= F \end{aligned} \tag{48}$$

where c and c_t are small damping coefficients and F is the input force. To control the position x and the angle θ, on average, consider the input force in the form

$$F = f(x) + F_0\omega\phi(\omega t)$$

where $f(x)$ is a twice differentiable function to be selected and $\phi(t)$ is a zero-mean, periodic function. The averaged dynamics of the system are determined to be

$$\begin{aligned}\ddot{\bar{\theta}} &= \dot{\bar{\theta}}^2 \tan\bar{\theta} - \frac{c_t\dot{\bar{\theta}} + k_t\bar{\theta}}{ml^2}\sec^2\bar{\theta} + \frac{f(\bar{x}) - c\dot{\bar{x}}}{ml}\sec\bar{\theta}\tan\bar{\theta} - \frac{2\mu F_0^2}{m^2l^2}\sec^4\bar{\theta}\tan\bar{\theta} \\ \ddot{\bar{x}} &= l\dot{\bar{\theta}}^2\sec\bar{\theta} - \frac{c_t\dot{\bar{\theta}} + k_t\bar{\theta}}{ml}\sec\bar{\theta}\tan\bar{\theta} + \frac{f(\bar{x}) - c\dot{\bar{x}}}{m}\sec^2\bar{\theta} - \frac{2\mu F_0^2}{m^2l}\sec^3\bar{\theta}\tan^2\bar{\theta}\end{aligned} \tag{49}$$

Now one may try to seek a function $f(x)$ to stabilize a desired equilibrium $(\theta_\mathrm{d}, x_\mathrm{d})^T$ for the time-averaged dynamics. Numerical simulation results for stabilizing the desired configuration $\theta_\mathrm{d} = 110^\circ$ and $x_\mathrm{d} = 0.1$ m are shown in Figure 7. The physical parameters of the system are $m = 0.3$ kg, $l = 0.2$ m, $k_t = 0.5$ N.m/rad, $c_t = 0.01$ N.m.s/rad, $c = 0.1$ N.s/m. The input force is considered in the form

$$F = k(x - x_\mathrm{d}) + F_0\omega\cos\omega t \quad \text{(N)}$$

with the frequency $\omega = 500$ rad/s. From the averaged dynamics the value of the open-loop force amplitude is determined to be $F_0 \approx 0.1566$, and any control parameter $k < 0$ stabilizes the desired equilibrium. For the numerical simulations, the control parameter $k = -0.05$ is used. The initial conditions are $\theta_0 = 115^\circ$, $x_0 = 0$, and zero velocities.

For the two vibrational systems presented here – the Kapitza pendulum and the horizontal pendulum – the stabilizable set for the force-input system is equal to the equilibrium set of the system. But this is not the case in general for Lagrangian mechanical systems with Lagrangian (1). One example is a Kapitza-like pendulum whose pivot moves along a line at angle $0 < \phi < \frac{\pi}{2}$ with respect to the vertical axis [Ciezkowski, 2011]. For this system it can be shown that the stabilizable set of the force-input system is a proper subset of the equilibrium set of the system [Tahmasian, 2021].

4.3 The mechanics of stability

This section discusses the mechanics behind the stability of the Kapitza pendulum and the horizontal pendulum and explains why for these two systems the stabilizable set using a vibrational force-input is larger than that obtained using open-loop vibrational position-input.

The stabilizable sets can be explained in terms of the real and virtual moments that act on the systems. The fast oscillation of a simple pendulum's pivot, along a particular line in inertial space, generates a

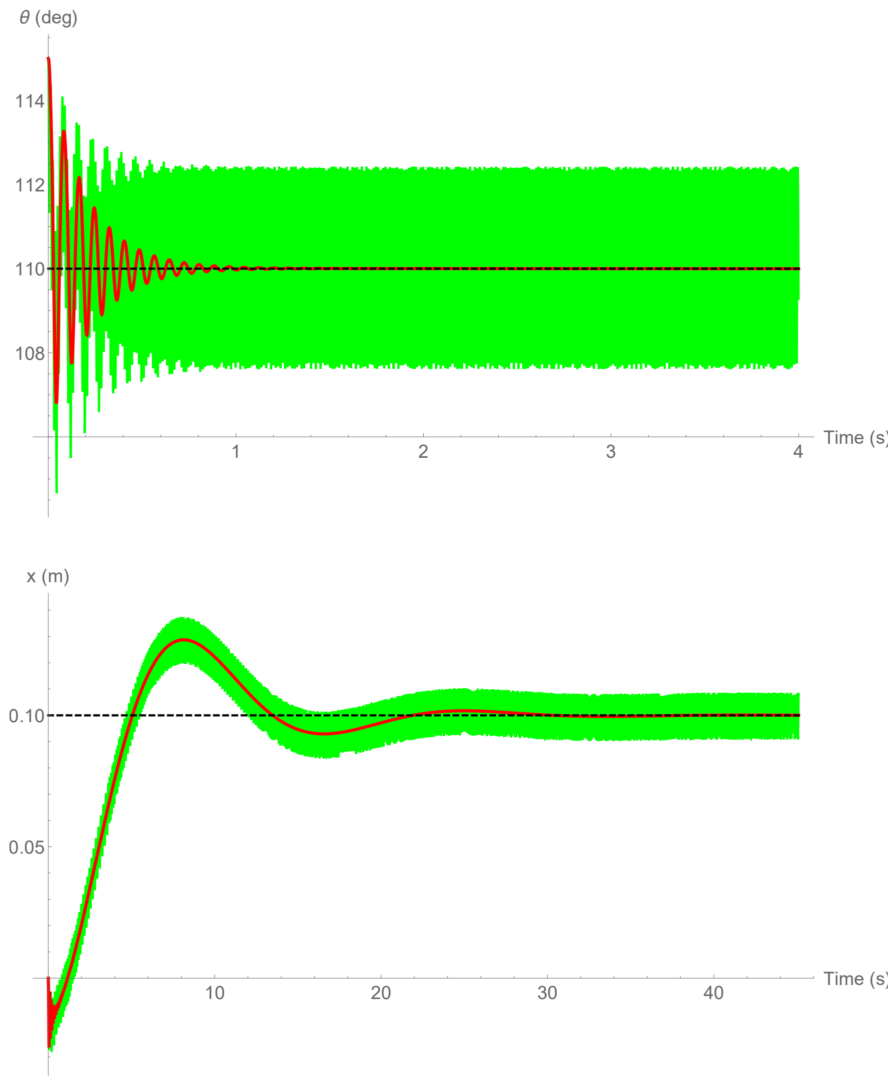

Figure 7: Closed-loop force-input horizontal pendulum stabilized at $\theta = 110^\circ$ and $x = 0.1$ m. Solid-green: original system, solid-red: averaged dynamics, dashed-black: desired coordinate.

virtual moment which, on average, drives the pendulum toward the line along which the pivot oscillates. Consider the simple pendulum with mass m and length l rotating freely about its pivot A in Figure 8. Suppose the pivot experiences a zero-mean, high-frequency acceleration a along a certain axis and that the angle of the pendulum with respect to that axis is θ. Based on Euler's second law of motion, the effect of the pivot acceleration can be considered as a virtual moment $M_\mathrm{v} = mla\sin\theta$, that is, the moment of the inertial force ma about the pivot [Greenwood, 1965]. For a larger angle $0 \le \theta \le \frac{\pi}{2}$, the virtual moment is larger. Due to change of direction of the acceleration in each half-period, the direction of the virtual moment reverses each half-period. Since the pendulum angle θ also changes over each half-period, however, the virtual moments in two successive half-periods are not equal. The *difference* in these two virtual moments over a full period drives the pendulum toward the axis along which the pivot oscillates, i.e., it tends to decrease θ, on average. This phenomenon is summarized in Figure 8.

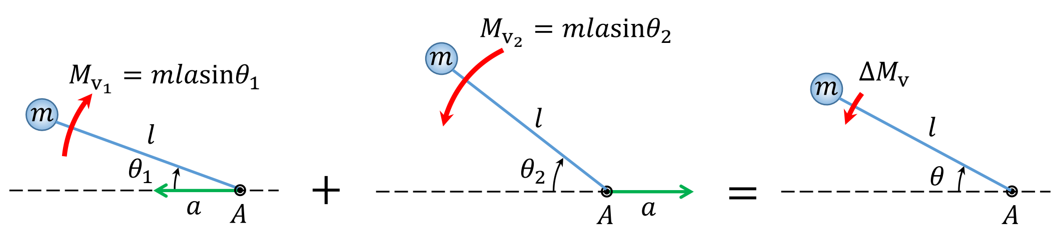

Figure 8: Generation of the net virtual moment ΔM_v during one period of the pivot oscillations.

During one period of fast oscillation, the angle of the pendulum changes by a small value $\Delta\theta$. Therefore, the net moment generated during one cycle is

$$\Delta M_\mathrm{v} = mla\Delta(\sin\theta) \approx mla\cos\theta\Delta\theta$$

However, the change of angle $\Delta\theta$ during one half cycle depends on the magnitude of the moment M_v which, as mentioned, is directly proportional to $\sin\theta$. Therefore the net moment ΔM_v is directly proportional to $\sin\theta\cos\theta$, or to $\sin 2\theta$. In other words, the net virtual moment is maximum when the pendulum orientation is midway between the line of oscillations and the perpendicular to the line of oscillations, i.e, $\theta = \frac{\pi}{4}$. The net virtual moment decreases as the pendulum moves closer to the line of oscillations ($\theta = 0$) or to its perpendicular ($\theta = \frac{\pi}{2}$). The same conclusions follow from the averaged dynamics, where the effect of the pivot's fast vibrations appears as a (virtual) moment that is proportional to $\sin 2\theta$.

Figure 9 depicts the real (red) and virtual (blue) moments acting on the open-loop position-input (a)

Kapitza pendulum and (b) horizontal pendulum, annotated to show the direction of increasing moment. The real moments are due to (a) gravity or (b) the torsional spring while the virtual moment is due to the fast oscillation of the pivots. For each system, the equilibria of the averaged dynamics are in a domain where the moments act in opposite directions to each other. These domains where the real and virtual moments oppose one another are the equilibrium sets, in agreement with the result suggested by inequality (12). The orientations with both of the arrows pointing towards them from both sides are the stable trivial equilibria, i.e., $\theta = \pi$ in the Kapitza pendulum and $\theta = 0$ in the horizontal pendulum. The orientations with the red arrow pointing away from them in both directions and the blue arrow pointing towards them from both directions, e.g., the upward orientation in the Kapitza pendulum, are the unstable trivial equilibria which can be vibrationally stabilized. If within an equilibrium set the moments decrease in magnitude in the direction of their action, then the equilibrium is stable. For the Kapitza pendulum, based on the direction of the moments and the directions in which they increase (or decrease), the only stable equilibria are the upward and downward directions, i.e., $\theta = 0$ and $\theta = \pi$. For the horizontal pendulum, stable equilibria can exist in half of the second and fourth quadrants, i.e., $\frac{3\pi}{4} < \theta < \pi$ and $\frac{7\pi}{4} < \theta < 2\pi$. The unstable equilibria in the equilibrium sets may be stabilized using closed-loop vibrational control, as discussed in Section 3. The domains where the real and virtual moments act in the same direction, e.g., the second and third quadrants of the Kapitza pendulum system, are not stabilizable.

One may wonder, for these two systems, why the stabilizable set obtained using open-loop vibrational force input is larger than that obtained using open-loop vibrational position input. A physical explanation is given below for the Kapitza pendulum. The reasoning is similar for the horizontal pendulum.

As mentioned earlier, the open-loop force-input Kapitza pendulum with harmonic forcing (e.g, $F = F_0 \cos \omega t$) can be stabilized in a non-vertical orientation in the upper-half plane, however the open-loop position-input system with a harmonic input cannot be. The reason is that a harmonic input force at the pivot does not necessarily generate a harmonic acceleration of the pivot. Though the force amplitudes are identical at each half period, the acceleration amplitudes are not. According to the equation of motion of the Kapitza pendulum, for a force F acting at the pivot, the pivot acceleration $a = \ddot{z}$ is a function of $F \sec^2 \theta$ which increases considerably with increasing θ. Therefore the virtual moments over each half period can be considerably different. During a half-period when the angle θ is small, an upward force F generates an acceleration a_{u} and a virtual moment proportional to it which causes the pendulum angle

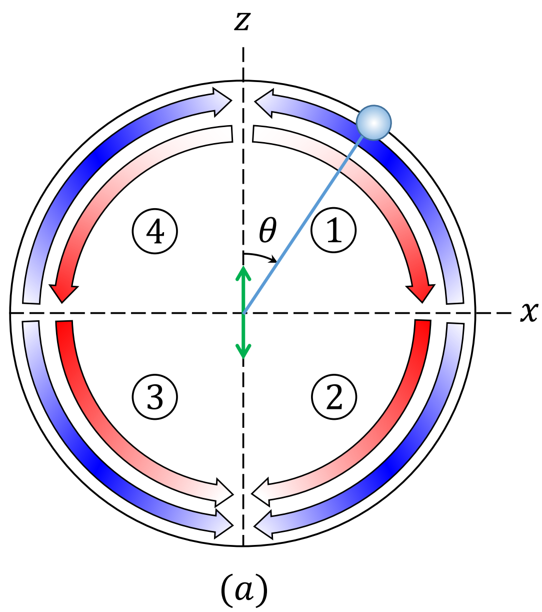

Figure 9: The real and virtual moments acting on a) the Kapitza pendulum, and b) the horizontal pendulum with orientation $-\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}$. The red and blue arrows show the direction of the real and virtual moments, respectively, within each numbered quadrant. Darker shades indicate larger magnitudes. The green arrows show the direction of vibrations.

to increase. In the succeeding half-period, when the angle θ is larger, a downward force F generates a downward pivot acceleration $a_\mathrm{d} > a_\mathrm{u}$ and, as a result, a larger virtual moment that tends to decrease the angle. Therefore, in this case, the net virtual moment (i.e., the difference between the virtual moments during two successive half-periods) is not only the result of different angles in each half-period but also a result of different accelerations during these half-periods. This is in contrast to the position-input system, which exhibits identical accelerations during each half-period. A rough estimate of the net virtual moment during one period is

$$\Delta M_\mathrm{v} = F\Delta(\sec^2\theta) = 2F\sin\theta\sec^3\theta\Delta\theta$$

which is an increasing function in θ. In other words, the color gradient of the blue arrows (virtual moment) in the first and fourth quadrants in Figure 9(a) are changed in such a way that stable equilibria now exist. Therefore, by selecting the right force amplitude, one may generate enough net virtual moment to overcome the moment of gravity and stabilize the pendulum in a non-vertical orientation on average. This phenomenon is shown in Figure 10. In fact, the closed-loop position-input system using input (40) demonstrates the same principle, i.e., different accelerations during each half-period can stabilize the pendulum in a non-vertical orientation.

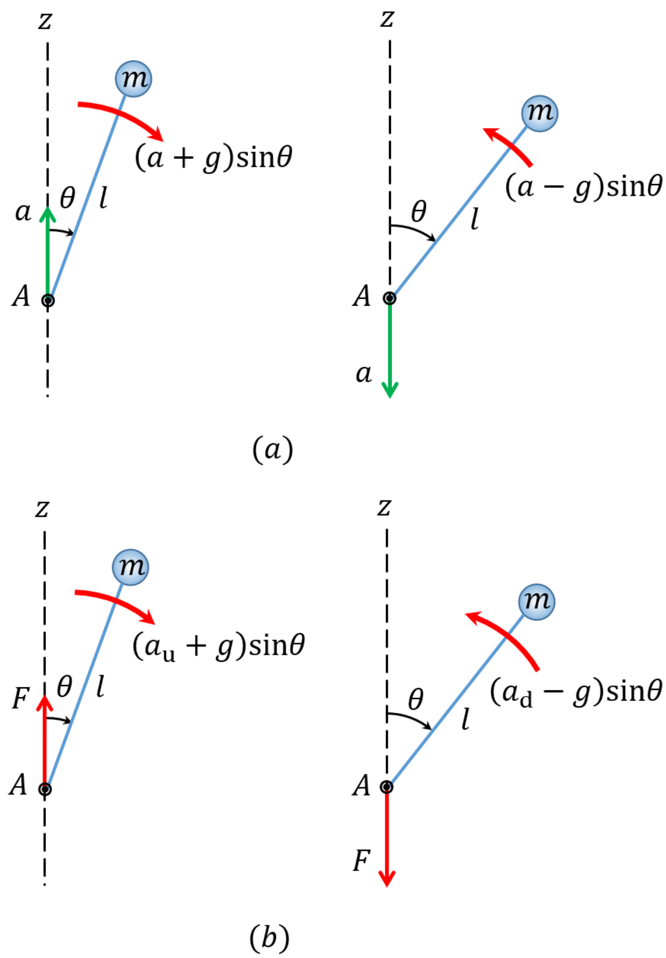

Figure 10: The total moment (virtual plus real moment) per unit mass and length acting on a) the position-input, and b) the force-input Kapitza pendulum. Note that a_{d} and a_{u} are proportional to $F \sec^2 \theta$, and therefore $a_{\mathrm{d}} > a_{\mathrm{u}}$. Also, note the difference in moments during two half-periods in (a).

Figure 11 presents the time history of the pendulum orientation θ and the pivot position z of the open-loop force-input Kapitza pendulum stabilized at $\theta_e = 70°$. The input is (42) with waveform $\phi(t) = \cos t$ and force amplitude $F_0 \approx 0.079$ N, as determined from the averaged dynamics. The physical parameters are the same as those used to generate the results in Figure 3 and $c = 0.05$ N.s/m. The initial conditions are $\theta_0 = 65°$, $z_0 = 0$, and zero initial velocities. The variations of the force F, the pivot acceleration $\ddot{z}$, and the pendulum orientation θ during steady state motion are shown in Figure 12 for comparison.

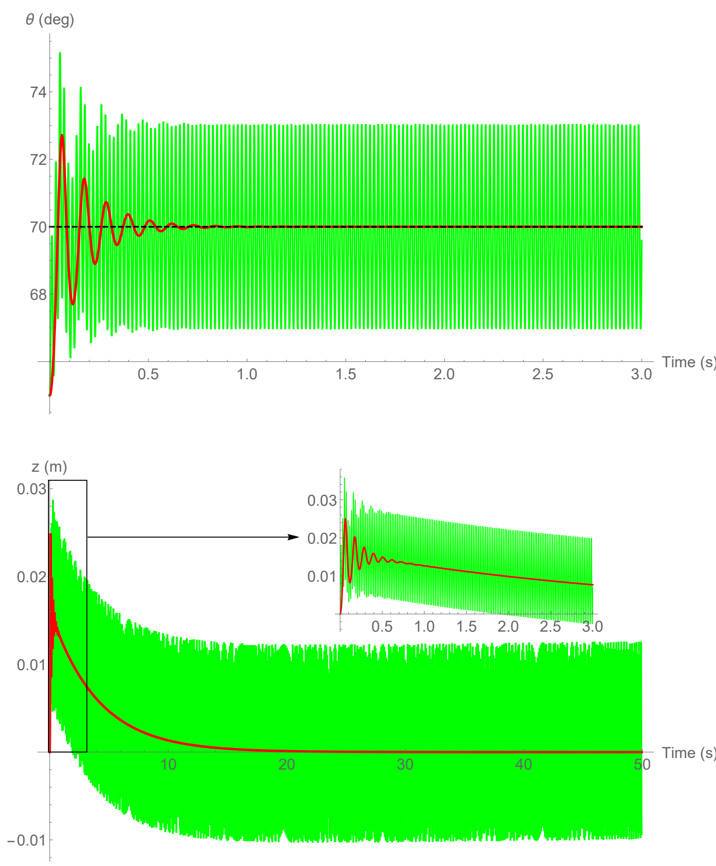

Figure 11: The time history of the coordinates of the force-input Kapitza pendulum stabilized at $\theta = 70°$. Solid-green: actual system, solid-red: averaged dynamics, dashed-black: desired orientation.

5 Conclusions

When considering vibrational control of a single-input, 2-DOF mechanical system, one may be interested in knowing the equilibrium set – the set of unactuated coordinate values which correspond to equilibria

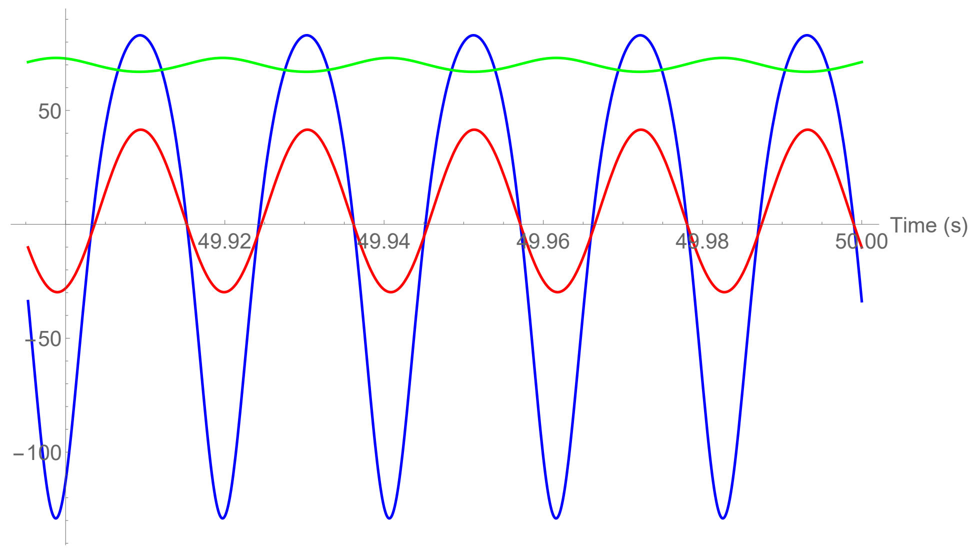

Figure 12: The time history of the force-input Kapitza pendulum stabilized at $\theta = 70°$ during steady-state motion over five periods. Green: orientation angle θ in degrees, red: input force F in N, blue: the pivot acceleration $\ddot{z} \times 10^{-1}$ in m/s^2. Note the non-symmetric pivot acceleration in response to the symmetric force.

of the time-averaged dynamics – and in understanding the stability, or stabilizability, of these equilibria. Closed-loop vibrational control strategies were presented, using either position- or force-input, which can expand the stabilizable set, that is, the subset of coordinate values in the equilibrium set that can be made stable, and depending on the system dynamics, the strategy may expand the stabilizable set to include the entire equilibrium set. The equilibrium set can be determined using the averaged dynamics of the open-loop force-control system. It was shown that the equilibrium sets of the open- and closed-loop position-input systems and of the open-loop force-input system are equal. It was also shown that the stabilizable set for the open-loop position-input system is not larger than that of the open-loop force-control system. Two vibrational control systems, the Kapitza pendulum and a horizontal pendulum with a torsional spring, were discussed, including the mechanics underlying their vibrational stabilization.

A Appendix

A.1 Averaged potential of a class of Lagrangian systems

The following presents an abstract of the concept of averaged potential based on [Baillieul, 1993; Weibel et al., 1997].

Consider an $(n+1)$-DOF, underactuated mechanical system with the n unactuated generalized coordinates $\boldsymbol{q}_{\mathrm{u}} = (q_1, \ldots, q_n)^T$, one actuated coordinate q_{a}, and Lagrangian

$$L(\boldsymbol{q}_{\mathrm{u}}, \dot{\boldsymbol{q}}_{\mathrm{u}}; v) = \frac{1}{2}\dot{\boldsymbol{q}}_{\mathrm{u}}^T \mathbb{M}(\boldsymbol{q}_{\mathrm{u}})\dot{\boldsymbol{q}}_{\mathrm{u}} + v\boldsymbol{A}^T(\boldsymbol{q}_{\mathrm{u}})\dot{\boldsymbol{q}}_{\mathrm{u}} - V_{\mathrm{a}}(\boldsymbol{q}_{\mathrm{u}}; v) \tag{A.1}$$

where $v = v(t) = \dot{q}_{\mathrm{a}}$ is the control input, $\mathbb{M}(\boldsymbol{q}_{\mathrm{u}})$ is the $n \times n$ inertia matrix, $\boldsymbol{A}(\boldsymbol{q}_{\mathrm{u}})$ is an $n \times 1$ inertial coupling vector, and $V_{\mathrm{a}}(\boldsymbol{q}_{\mathrm{u}}; v)$ is called the *augmented potential*.

Defining the momentum vector $\boldsymbol{p} = (p_1, \ldots, p_n)^T$, where $p_i = \frac{\partial L}{\partial \dot{q}_i}$, $i \in \{1, \ldots, n\}$, and using the Legendre transformation $H(\boldsymbol{q}_{\mathrm{u}}, \boldsymbol{p}; v) = \boldsymbol{p}^T\dot{\boldsymbol{q}}_{\mathrm{u}} - L$, the Hamiltonian corresponding to (A.1) is in the form

$$H(\boldsymbol{q}_{\mathrm{u}}, \boldsymbol{p}; v) = \frac{1}{2}\big(\boldsymbol{p} - v\boldsymbol{A}(\boldsymbol{q}_{\mathrm{u}})\big)^T \mathbb{M}^{-1}(\boldsymbol{q}_{\mathrm{u}})\big(\boldsymbol{p} - v\boldsymbol{A}(\boldsymbol{q}_{\mathrm{u}})\big) + V_{\mathrm{a}}(\boldsymbol{q}_{\mathrm{u}}; v) \tag{A.2}$$

Note that since $v = v(t)$, the Hamiltonian H is not conserved. If v is a T-periodic function, then the Hamiltonian (A.2) is also a T-periodic function and its average, called the *averaged Hamiltonian* can be determined as

$$\bar{H}(\bar{\boldsymbol{q}}_{\mathrm{u}}, \bar{\boldsymbol{p}}; \bar{v}) = \frac{1}{T}\int_0^T H\big(\bar{\boldsymbol{q}}_{\mathrm{u}}, \bar{\boldsymbol{p}}; v(t)\big)dt \tag{A.3}$$

where $\bar{\boldsymbol{q}}_{\mathrm{u}}$ is the vector of the unactuated coordinates of the averaged dynamics and $\bar{\boldsymbol{p}}$ is the vector of the average momenta. Replacing H from (A.2) and simplifying, the averaged Hamiltonian $\bar{H}$ is determined in the form

$$\bar{H}(\bar{\boldsymbol{q}}_{\mathrm{u}}, \bar{\boldsymbol{p}}) = \frac{1}{2}\big(\bar{\boldsymbol{p}} - \bar{v}\boldsymbol{A}(\bar{\boldsymbol{q}}_{\mathrm{u}})\big)^T \mathbb{M}^{-1}(\bar{\boldsymbol{q}}_{\mathrm{u}})\big(\bar{\boldsymbol{p}} - \bar{v}\boldsymbol{A}(\bar{\boldsymbol{q}}_{\mathrm{u}})\big) + V_{\mathrm{A}}(\bar{\boldsymbol{q}}_{\mathrm{u}}; \bar{v}) \tag{A.4}$$

where the *averaged potential* V_{A} is

$$V_{\mathrm{A}}(\bar{\boldsymbol{q}}_{\mathrm{u}}; \bar{v}) = \mu\boldsymbol{A}^T(\bar{\boldsymbol{q}}_{\mathrm{u}})\mathbb{M}^{-1}(\bar{\boldsymbol{q}}_{\mathrm{u}})\boldsymbol{A}(\bar{\boldsymbol{q}}_{\mathrm{u}}) + \bar{V}_{\mathrm{a}}(\bar{\boldsymbol{q}}_{\mathrm{u}}) \tag{A.5}$$

and where the input parameter

$$\mu = \frac{1}{2}\left(\overline{v^2} - \bar{v}^2\right) \tag{A.6}$$

and

$$\bar{v} = \frac{1}{T}\int_0^T v(t)dt \tag{A.7}$$

and

$$\bar{v^2} = \frac{1}{T}\int_0^T v^2(t)dt \tag{A.8}$$

Note that if the velocity function $v(t)$ is zero-mean, then $\bar{v} = 0$. However for any function $v(t) \not\equiv 0$, one finds $\bar{v^2} \neq 0$.

For the averaged Hamiltonian $\bar{H}$ one can write

$$\frac{\partial \bar{H}}{\partial t} = 0 \tag{A.9}$$

Therefore the equilibria and stability of the averaged dynamics, and according to the averaging theorem, the existence and stability of the periodic orbits of the original time-periodic system, can be studied using the averaged potential $V_{\mathrm{A}}(\bar{\boldsymbol{q}}_{\mathrm{u}}; \bar{v})$.

A.2 Averaging of mechanical control-affine systems

The following theorem is based on the results developed in [Bullo, 2002; Bullo and Lewis, 2004], and as presented in [Tahmasian et al., 2018].

Consider an n-DOF mechanical control-affine system with m inputs. The dynamics of the system can be written in the general form

$$\ddot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{q}) u_i(t), \qquad \boldsymbol{q}(0) = \boldsymbol{q}_0,\ \dot{\boldsymbol{q}}(0) = \boldsymbol{v}_0 \tag{A.10}$$

where $\boldsymbol{q} = (q_1,\ \ldots,\ q_n)^T$ is the vector of generalized coordinates and $u_i(t)$ are the inputs. Suppose that $\boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\boldsymbol{g}_i(\boldsymbol{q})$ depend polynomially on their arguments, are twice differentiable in $\boldsymbol{q}$, and that the components of $\boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ are homogeneous in $\dot{\boldsymbol{q}}$ of degree two and less. Consider high-frequency, high-amplitude inputs $u_i(t)$, $i \in \{1, \ldots, m\}$, in the following form:

$$u_i(t) = \omega v_i(\omega t) \tag{A.11}$$

where ω is the (high) frequency, and $v_i(t)$ are zero-mean, T-periodic functions.

Using the state vector $\boldsymbol{x} = (\boldsymbol{q}^T,\ \dot{\boldsymbol{q}}^T)^T$ and the inputs defined in (A.11), system (A.10) can be written in the first order form

$$\dot{\boldsymbol{x}} = \boldsymbol{Z}(\boldsymbol{x}) + \sum_{i=1}^{m} \boldsymbol{Y}_i(\boldsymbol{x})\omega v_i(\omega t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 = (\boldsymbol{q}_0^T\ ,\ \boldsymbol{v}_0^T)^T \tag{A.12}$$

where $\boldsymbol{Z}(\boldsymbol{x}) = \big(\dot{\boldsymbol{q}}^T,\ \boldsymbol{f}^T(\boldsymbol{q}, \dot{\boldsymbol{q}})\big)^T$ is the drift vector field and $\boldsymbol{Y}_i(\boldsymbol{x}) = \big(\boldsymbol{0}_{1\times n},\ \boldsymbol{g}_i^T(\boldsymbol{q})\big)^T$ are the input vector fields.

For the inputs (A.11), define scalar parameters κ_i, λ_{ij}, and μ_{ij}, for $i, j \in \{1, \ldots, m\}$, as follows

$$\kappa_i = \frac{1}{T}\int_0^T \int_0^t v_i(\tau)d\tau dt \tag{A.13}$$

$$\lambda_{ij} = \frac{1}{T}\int_0^T \left(\int_0^t v_i(\tau)d\tau\right)\left(\int_0^t v_j(\tau)d\tau\right)dt \tag{A.14}$$

and

$$\mu_{ij} = \frac{1}{2}(\lambda_{ij} - \kappa_i\kappa_j) \tag{A.15}$$

Also, consider the *symmetric product* between two input vector fields $\boldsymbol{Y}_i(\boldsymbol{x})$ and $\boldsymbol{Y}_j(\boldsymbol{x})$ defined as

$$\langle \boldsymbol{Y}_i : \boldsymbol{Y}_j\rangle(\boldsymbol{x}) = \langle \boldsymbol{Y}_j : \boldsymbol{Y}_i\rangle(\boldsymbol{x}) = [\boldsymbol{Y}_j(\boldsymbol{x}), [\boldsymbol{Z}(\boldsymbol{x}), \boldsymbol{Y}_i(\boldsymbol{x})]] \tag{A.16}$$

where $[\cdot,\cdot]$ denotes the Lie bracket of vector fields.

The averaged dynamics of time-periodic system (A.12) then is determined as

$$\dot{\bar{\boldsymbol{x}}} = \boldsymbol{Z}(\bar{\boldsymbol{x}}) - \sum_{i,j=1}^{m} \mu_{ij}\langle \boldsymbol{Y}_i : \boldsymbol{Y}_j\rangle(\bar{\boldsymbol{x}}) \tag{A.17}$$

with the initial condition $\bar{\boldsymbol{x}}(0) = \bar{\boldsymbol{x}}_0 = \boldsymbol{x}_0 + \sum_{i=1}^{m} \kappa_i \boldsymbol{Y}_i(\boldsymbol{x}_0)$, where $\bar{\boldsymbol{x}} = (\bar{\boldsymbol{q}}^T,\ \dot{\bar{\boldsymbol{q}}}^T)^T$ is the state vector of the averaged dynamics [Bullo and Lewis, 2004; Tahmasian et al., 2018].

Declarations

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