BAYES AND MINIMUM VARIANCE UNBIASED ESTIMATORS OF RELIABILITY USING THE TRUNCATED WEIBULL LIFE TESTING MODEL

by

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CHAPTER I

INTRODUCTION

In recent years, statistical theory has become widely used in the study of reliability. Reliability can be defined as the probability of a component (or system) performing its purpose adequately under the design conditions for the period of time intended. Since the reliability function is actually a probability, it has a mathematical representation given by

$$R(t) = Pr\ [T \geq t],$$

where T is the time to failure of the system. Inherent in this definition is the concept of failure, which is that the system fails to perform its intended function under design conditions. A key phrase is "under design conditions." Hence, a system that encounters conditions outside tolerance limits of the design conditions cannot properly be considered a failure when it does not function adequately.

Reliability is not something which has just recently been introduced; however, greater emphasis has been placed on it since World War II as a consequence of technological demands. It has been shown that the exponential distribution can be used to describe failures in electronic equipment; the normal distribution can be used to describe failure experience with mechanical equipment; and the Weibull distribution can be used to describe fatigue life in materials, vacuum tube failures, and ball-bearing failures. Thus, before the reliability of a system can be found we must assume, or be able to develop, an underlying failure distribution which will describe the length of life of the system. It should be obvious that the modes of possible failure will affect the form of the failure distribution. Keeping this in mind, but without further consideration, let $f(x;\underline{\theta})$, where $\underline{\theta}$ is a d-dimensional parameter vector, represent the failure distribution. The probelm is to find suitable estimators of the parameter vector $\underline{\theta}$ and the reliability function $R(t)$ by drawing inferences from a sample of units whose lifetimes are described by $f(x;\underline{\theta})$.

These randomly selected items are then subjected to a life test which simulates actual design conditions until all or a pre-assigned fraction of them have failed. Suppose we decide to choose a sample of size n and terminate the test after $r(r \leq n)$ items have failed as opposed to choosing a sample of size r and waiting for all r units to fail. The test involving r failures out of n has an average waiting time to completion less than the test involving r failures out of r. This fact should be obvious since an extremely long lived item might be included in our sample of size r, but we would not have to wait for this particular item to fail in our sample of size n. Thus, it would be advantageous to terminate the life test after r out of n units have failed. If $X_1, X_2, \ldots, X_r$ represent the times of failure, then $X_1 \leq X_2 \leq \ldots \leq X_r$. This ordered nature of the data is characteristic of life testing.

In this study, we will use the truncated Weibull probability distribution as $f(x;\underline{\theta})$. Thus

$$f(x;\theta,\rho) = \begin{cases} \dfrac{a\,\rho}{\theta}\, x^{\rho-1} \exp\left[-\dfrac{x^{\rho}}{\theta}\right], & (0 < c \leq x,\ 0 < \theta, \rho) \\ 0 & \text{elsewhere} \end{cases} \tag{1.1}$$

where "a" is the normalizing constant, θ is the scale parameter, ρ is the shape parameter, and "c" is the point of truncation. Two types of estimators for the parameter θ and the reliability function $R(t)$ will be found when the parameter ρ is assumed to be known. In Chapter II we shall develop the Bayes estimators and in Chapter III the minimum variance unbiased estimators (MVUE) are developed. A Bayesian analysis implies use of a priori information about θ, which is considered to behave as a random variable, and the use of Bayes' theorem. The general uniform distribution, the exponential distribution, and the inverted gamma distribution are the prior probability density functions examined.

Finally, in Chapter IV the Bayes estimators of θ and the reliability function are compared by Monte Carlo simulation with the corresponding minimum variance unbiased estimators.

Also the Bayes estimators obtained using equation (1.1) as our life testing model will be compared with the Bayesian results of Canavos and Tsokos [5]. Their model was of the form

$$f(x;\theta,\rho,\tau) = \begin{cases} \dfrac{\rho}{\theta}\,(x-\tau)^{\rho-1} \exp\left[-\dfrac{(x-\tau)^{\rho}}{\theta}\right], & (0 < \tau \leq x, 0 < \theta, \rho) \\ 0 & \text{elsewhere} \end{cases} \tag{1.2}$$

where τ is the so called "guaranteed time" parameter; that is, failure will not occur before time τ. In these comparisons, the point c in the present model will be defined as the point of guaranteed time τ as defined in the probability density function (1.2). From a practical point of view, it is desirable to consider the truncated model over the guaranteed time model since, in practice, failures will naturally occur before time τ contrary to theoretical expectations. Furthermore, the system is operating from time zero, not from time τ. Hence, if one must guarantee that a system will not fail before time $\tau = \tau' > 0$, it would seem more practicable to approach the problem as follows: (i) to begin with a distribution having guaranteed time $\tau = 0$; (ii) truncate this distribution at the desired point $c = \tau'$; and (iii) normalize the remaining probability so that the basic properties of a probability density function still hold. The proposed results of this procedure are shown graphically in Figure 1 below. Idealistically, the first two steps would be sufficient since we would have a system that could not fail before time τ' but that actually began operating at zero time. However, the integral from τ' to ∞ would not be equal to one; hence, the third step will be applicable.

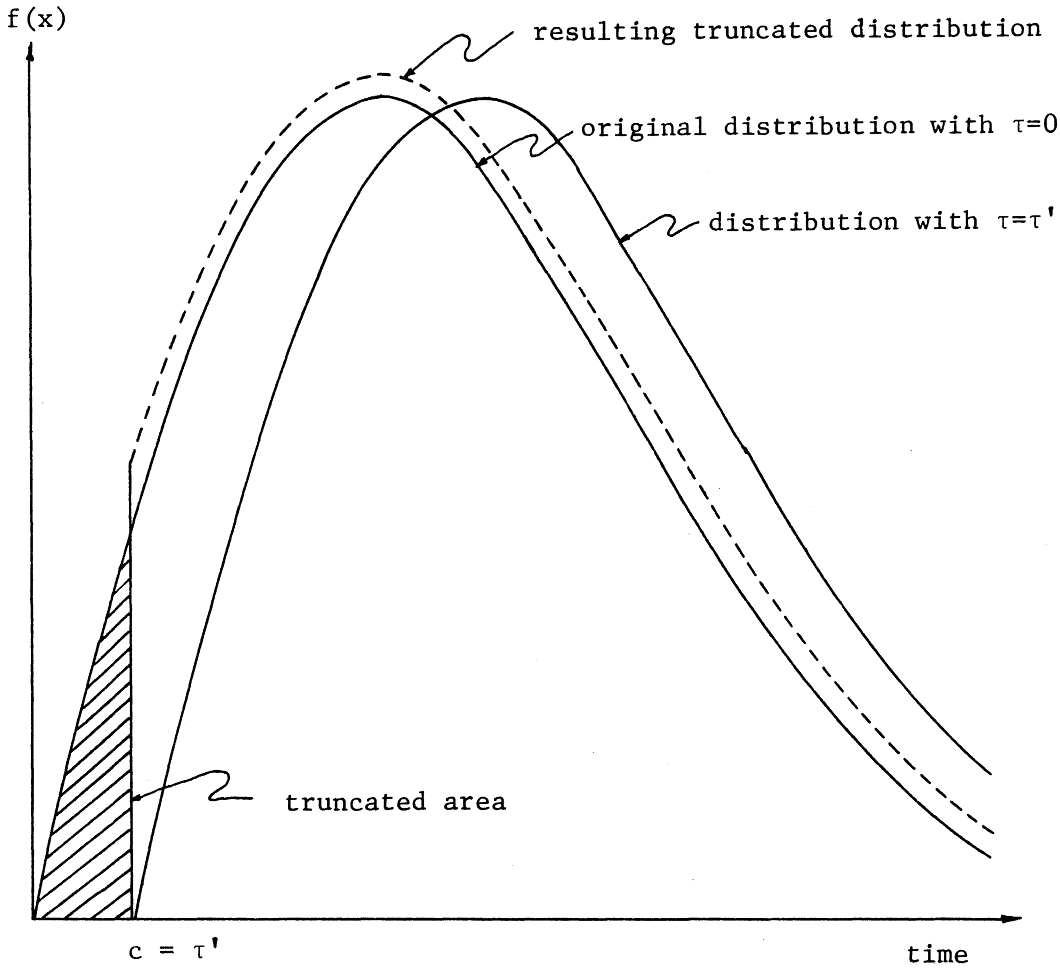

Figure 1. Results of Proposed Procedure in Truncating The Distribution

CHAPTER II

BAYESIAN ANALYSIS OF RELIABILITY

2.1 INTRODUCTION

This chapter is concerned with a Bayesian analysis of the scale parameter θ in the singly truncated Weibull distribution, while the shape parameter ρ is considered a known fixed quantity. Hence, the life testing model is given by the probability density function

$$f(x;\theta,\rho) = \begin{cases} \dfrac{a\ \rho}{\theta}\ x^{\rho-1}\ \exp\left[-\dfrac{x^{\rho}}{\theta}\right], & (0 < c \leq x,\quad 0 < \theta,\ \rho) \\ 0 & \text{elsewhere} \end{cases} \qquad (2.1.1)$$

where a is the normalizing constant (obviously, a function of the point of truncation c). As implied, the scale parameter is not just an unknown fixed parameter but behaves as a random variable.

Suitable prior densities of Θ which we shall examine are:

(i) the general uniform probability density function given by

$$g(\theta;\alpha,\beta) = \begin{cases} \dfrac{(b-1)(\alpha\beta)^{b-1}}{\theta^{b}\ (\beta^{b-1} - \alpha^{b-1})}, & (0 \leq \alpha \leq \theta \leq \beta) \\ 0 & \text{elsewhere} \end{cases} \qquad (2.1.2)$$

which reduces to the uniform desity on $[\alpha,\beta]$ when $b = 0$,

(ii) the exponential probability density function given by

$$g(\theta;\lambda) = \begin{cases} \dfrac{1}{\lambda}\ \exp\left[-\dfrac{\theta}{\lambda}\right], & (0 \leq \theta,\ 0 < \lambda) \\ 0 & \text{elsewhere} \end{cases} \qquad (2.1.3)$$

and

(iii) the inverted gamma probability density function given by

$$g(\theta;\mu,\lambda) = \begin{cases} \dfrac{\left(\frac{\mu}{\theta}\right)^{\lambda+1}}{\mu\,\Gamma(\lambda)} \exp\left[-\dfrac{\mu}{\theta}\right], & (0\leq\theta,\ 0<\mu,\lambda) \\ 0 & \text{elsewhere} \end{cases} \qquad (2.1.4)$$

In addition to the Bayesian estimation of the scale parameter θ, the corresponding reliability function shall be obtained for the above mentioned prior probability density functions.

2.2 PRELIMINARY RESULTS

In order to ensure that the singly truncated Weibull distribution given by equation (2.1.1) is a valid probability density function, we need to obtain the normalization constant. That is,

$$\int_c^\infty f(x;\theta,\rho)dx = a\int_c^\infty \frac{\rho}{\theta}\, x^{\rho-1} \exp\left[-\frac{x^\rho}{\theta}\right] dx = 1 \quad .$$

If we let $y = \frac{x^\rho}{\theta}$, then $dy = \frac{\rho x^{\rho-1}}{\theta}dx$ and upon substitution in the above integral we have

$$a\int_{\frac{c^\rho}{\theta}}^\infty \exp[-y]dy = a \exp\left[-\frac{c^\rho}{\theta}\right] = 1 \quad .$$

Thus, we obtain $\quad a = \exp\left[\dfrac{c^\rho}{\theta}\right] \quad .$

Consequently, we may rewrite the singly truncated Weibull model as

$$f(x;\theta,\rho) = \exp\left[\frac{c^{\rho}}{\theta}\right] \frac{\rho}{\theta} x^{\rho-1} \exp\left[-\frac{x^{\rho}}{\theta}\right]$$

or

$$f(x;\theta,\rho) = \begin{cases} \frac{\rho}{\theta} x^{\rho-1} \exp\left[-\frac{(x^{\rho}-c^{\rho})}{\theta}\right], & (0<c\leq x,\ 0<\theta,\rho) \\ 0 & \text{elsewhere}\ . \end{cases} \qquad (2.2.1)$$

Its reliability at some mission time t, which we will designate as the exact reliability function, is obtained as follows:

$$\begin{aligned} R(t) &= 1 - F(x;\theta,\rho) \\ &= P_r\,[X \geq t] \\ &= \int_t^{\infty} \frac{\rho}{\theta} x^{\rho-1} \exp\left[-\frac{(x^{\rho}-c^{\rho})}{\theta}\right] dx \\ &= \exp\left[-\frac{(t^{\rho}-c^{\rho})}{\theta}\right], \quad (0<c\leq x,\ 0<\theta,\rho)\ . \qquad (2.2.2) \end{aligned}$$

2.3 MAIN RESULTS

The method we shall use when Θ is a random variable is to consider a random sample of n items from a large population whose life-times are characterized by equation (2.2.1). The n items are then subjected to a life test which is terminated after a predetermined $r \leq n$ number of failures have occurred.

Let $\underline{x} = (x_1, x_2, \ldots, x_r)$ denote the observed ordered life-times of the test items. The joint probability of observing r failures at times $x_1, \ldots, x_r$ and (n-r) items which have not failed by time x_r

is given by

$$\begin{aligned} l(\underline{x};\theta,\rho) &= \frac{n!}{(n-r)!} f(x_1;\theta,\rho) \ldots f(x_r;\theta,\rho) \, [1-F(x_r;\theta,\rho)]^{n-r} \\ &= \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left\{ \frac{\rho}{\theta} x_i^{\rho-1} \exp\left[-\frac{(x_i^{\rho}-c^{\rho})}{\theta} \right] \right\} \cdot \left\{ \exp\left[-\frac{(x_r^{\rho}-c^{\rho})}{\theta} \right] \right\}^{n-r} \\ &= \frac{n!}{(n-r)!} \frac{\rho^r}{\theta^r} \prod_{i=1}^{r} x_i^{\rho-1} \cdot \exp\left\{ -\left[\frac{\sum_{i=1}^{r} (x_i^{\rho}-c^{\rho})+(n-r)(x_r^{\rho}-c^{\rho})}{\theta} \right] \right\}. \end{aligned} \quad (2.3.1)$$

Let $S_r = \sum_{i=1}^{r} (x_i^{\rho}-c^{\rho}) + (n-r)\,(x_r^{\rho}-c^{\rho})$ and $T_r = \prod_{i=1}^{r} x_i^{\rho-1}$,

then equation (2.3.1) may be written as

$$l(\underline{x};\theta,\rho) = \frac{n!}{(n-r)!} \frac{\rho^r}{\theta^r} T_r \exp\left[-\frac{S_r}{\theta} \right] . \quad (2.3.2)$$

Now, we shall assume a prior density for Θ, say $g(\theta)$, and apply Bayes' theorem to obtain the posterior density of Θ, according to the following formula;

$$h(\theta;\underline{x}) = \frac{l(\underline{x};\theta,\rho) \cdot g(\theta)}{\int_{\text{all } \theta} l(\underline{x};\theta,\rho) \cdot g(\theta)d\theta} , \quad (\theta \in \text{domain } g(\theta)).$$

2.3.1 General Uniform Prior Probability Density Function

Applying Bayes' theorem when the prior density of Θ is given by the general uniform probability density function, (2.1.2), we obtain the posterior density of Θ to be

$$h(\theta;\underline{x}) = \frac{l(\underline{x};\theta,\rho)\cdot g(\theta;\alpha,\beta)}{\int_\alpha^\beta l(\underline{x};\theta,\rho)\cdot g(\theta;\alpha,\beta)\,d\theta}$$

$$= \frac{\frac{n!}{(n-r)!}\frac{\rho^r}{\theta^r}T_r\exp\left[-\frac{S_r}{\theta}\right]\cdot\frac{(b-1)(\alpha\beta)^{b-1}}{\theta^b(\beta^{b-1}-\alpha^{b-1})}}{\int_\alpha^\beta\frac{n!}{(n-r)!}\frac{\rho^r}{\theta^r}T_r\exp\left[-\frac{S_r}{\theta}\right]\cdot\frac{(b-1)(\alpha\beta)^{b-1}}{\theta^b(\beta^{b-1}-\alpha^{b-1})}\,d\theta}$$

$$= \frac{\frac{1}{\theta^{b+r}}\exp\left[-\frac{S_r}{\theta}\right]}{\int_\alpha^\beta\frac{1}{\theta^{b+r}}\exp\left[-\frac{S_r}{\theta}\right]d\theta},\ (0\le\alpha\le\theta\le\beta). \tag{2.3.1.1}$$

Let $y = \frac{S_r}{\theta}$ or $dy = \frac{-S_r}{\theta^2}\,d\theta$

in the denominator of expression (2.3.1.1). Hence, the integral becomes

$$\frac{-1}{S_r^{\,b+r-1}}\int_{S_r/\alpha}^{S_r/\beta} y^{b+r-2}\exp[-y]\,dy. \tag{2.3.1.2}$$

Using the incomplete gamma function

$$\gamma(n,f) = \int_0^f y^{n-1}\exp[-y]\,dy \tag{2.3.1.3}$$

the integral (2.3.1.2) further reduces to a constant times the difference of two incomplete gamma functions; that is,

$$\frac{-1}{S_r^{b+r-1}} \int_{S_r/\alpha}^{S_r/\beta} y^{(b+r-1)-1} \exp[-y]dy = \frac{\gamma(b+r-1,\frac{S_r}{\alpha})-\gamma(b+r-1,\frac{S_r}{\beta})}{S_r^{b+r-1}} .$$

Thus, the posterior density of Θ for the uniform prior is

$$h(\theta;\underline{x}) = \left\{ \frac{\frac{S_r^{b+r-1}}{\theta^{b+r}} \exp\left[-\frac{S_r}{\theta}\right]}{\gamma(b+r-1, \frac{S_r}{\alpha}) - \gamma(b+r-1, \frac{S_r}{\beta})} \right., \quad (0\leq\alpha\leq\theta\leq\beta). \tag{2.3.1.4}$$

For future notation, we shall denote $\gamma^*(n,f) = \gamma(n,\frac{f}{\alpha}) - \gamma(n,\frac{f}{\beta})$.

The Bayes estimator of θ with respect to squared error loss is given by the posterior mean

$$\hat{\theta}_B = E[\Theta|\underline{x}] = \int_\alpha^\beta \theta h(\theta;\underline{x})d\theta$$

$$= \frac{S_r^{b+r-1}}{\gamma^*(b+r-1,S_r)} \int_\alpha^\beta \frac{1}{\theta^{b+r-1}} \exp\left[-\frac{S_r}{\theta}\right] d\theta .$$

A similar integration approach yields

$$\hat{\theta}_B = S_r \cdot \frac{\gamma^*(b+r-2,S_r)}{\gamma^*(b+r-1,S_r)} . \tag{2.3.1.5}$$

The variance of $\hat{\Theta}_B$ is obtained by evaluating

$$\text{Var}\ [\hat{\Theta}_B|\underline{x}] = E[\Theta^2|\underline{x}] - \{E[\Theta|\underline{x}]\}^2$$

$$= \int_\alpha^\beta \theta^2 h(\theta;\underline{x})\ d\theta - \hat{\theta}_B^2$$

$$= \frac{S_r^{\,b+r-1}}{\gamma^*(b+r-1,S_r)} \int_\alpha^\beta \frac{1}{\theta^{b+r-2}} \exp\left[-\frac{S_r}{\theta}\right] d\theta - \hat{\theta}_B^{\,2} .$$

Therefore,

$$\mathrm{Var}\,[\hat{\Theta}_B|\underline{x}] = S_r^{\,2} \cdot \frac{\gamma^*(b+r-1,S_r)\,\gamma^*(b+r-3,S_r) - [\gamma^*(b+r-2,S_r)]^2}{[\gamma^*(b+r-1,S_r)]^2} . \qquad (2.3.1.6)$$

The Bayes estimator of the exact reliability function, given by equation (2.2.2), with respect to squared error loss is given by

$$\hat{R}(t)_B = E[R(t)|\underline{x}] = \int_\alpha^\beta R(t)\, h(\theta;\underline{x})\, d\,\theta$$

$$= \frac{S_r^{\,b+r-1}}{\gamma^*(b+r-1,S_r)} \int_\alpha^\beta \frac{1}{\theta^{b+r}} \exp\left[-\frac{(S_r+t^\rho-c^\rho)}{\theta}\right] d\,\theta$$

$$= \frac{S_r^{\,b+r-1}}{(S_r+t^\rho-c^\rho)^{b+r-1}} \cdot \frac{\gamma^*(b+r-1,S_r+t^\rho-c^\rho)}{\gamma^*(b+r-1,S_r)}$$

$$= \left[1 + \frac{t^\rho-c^\rho}{S_r}\right]^{1-b-r} \cdot \frac{\gamma^*(b+r-1,S_r+t^\rho-c^\rho)}{\gamma^*(b+r-1,S_r)} . \qquad (2.3.1.7)$$

The variance of the Bayesian estimator $\hat{R}(t)_B$ of reliability is determined as follows:

$$\mathrm{Var}\,[\hat{R}(t)_B|\underline{x}] = E[R^2(t)|\underline{x}] - \{E[R(t)|\underline{x}]\}^2$$

$$= \int_\alpha^\beta R^2(t)\, h(\theta;\underline{x})\, d\theta - \hat{R}^2(t)_B$$

$$= \frac{S_r^{\,b+r-1}}{\gamma^*(b+r-1,S_r)} \int_\alpha^\beta \frac{1}{\theta^{b+r}} \exp\left[-\frac{(S_r+2t^\rho-2c^\rho)}{\theta}\right] d\theta - \hat{R}^2(t)_B$$

$$= \left[1 + \frac{2(t^\rho - c^\rho)}{S_r}\right]^{1-b-r} \cdot \frac{\gamma^*(b+r-1, S_r + 2t^\rho - 2c^\rho)}{\gamma^*(b+r-1, S_r)} - \hat{R}^2(t)_B .$$

Hence,

$$\mathrm{Var}[\hat{R}(t)_B | \underline{x}] = \frac{1}{[\gamma^*(b+r-1, S_r)]^2} \left\{ \left[1 + \frac{2(t^\rho - c^\rho)}{S_r}\right]^{1-b-r} \gamma^*(b+r-1, S_r + 2t^\rho - 2c^\rho) \right.$$

$$\left. \cdot \ \gamma^*(b-r-1, S_r) - \left[1 + \frac{t^\rho - c^\rho}{S_r}\right]^{2(1-b-r)} \left[\gamma^*(b+r-1, S_r + t^\rho - c^\rho)\right]^2 \right\}. \tag{2.3.1.8}$$

The general uniform probability density function restricts the domain of Θ to an interval $[\alpha, \beta]$, but it might be the case that the experimenter lacks sufficient knowledge of the prior density to define α and β . Therefore, he may choose to let $b = 0$ and $(\alpha, \beta) \to (0, \infty)$ in the general uniform probability density, in which case the random variable Θ has a diffuse prior over the non-negative real line. Under these conditions, formulas corresponding to equations (2.3.1.4) - (2.3.1.8) are developed as follows:

$$h(\theta; \underline{x}) = \frac{\dfrac{S_r^{\,b+r-1}}{\theta^{\,b+r}} \exp\left[-\dfrac{S_r}{\theta}\right]}{\gamma^*(b+r-1, S_r)}, \quad (0 \leq \alpha \leq \theta \leq \beta)$$

$$= \frac{\dfrac{S_r^{\,b+r-1}}{\theta^{\,b+r}} \exp\left[-\dfrac{S_r}{\theta}\right]}{\displaystyle\int_{S_r/\beta}^{S_r/\alpha} y^{\,b+r-2} \exp[-y]\, dy} .$$

Assuming $b = 0$ and $(\alpha,\beta) \to (0,\infty)$ then

$$h(\theta;\underline{x}) \to \frac{\dfrac{S_r^{\,r-1}}{\theta^r} \exp\left[-\dfrac{S_r}{\theta}\right]}{\int_0^\infty y^{r-2} \exp[-y]\, dy}$$

$$= \frac{S_r^{\,r-1} \exp\left[-\dfrac{S_r}{\theta}\right]}{\theta^r\, \Gamma(r-1)} \qquad (2.3.1.4')$$

Similarly,

$$\hat{\theta}_B = S_r \cdot \frac{\gamma^*(b+r-2,S_r)}{\gamma^*(b+r-1,S_r)}$$

$$\hat{\theta}_B \to \frac{S_r\ \Gamma(r-2)}{\Gamma(r-1)}$$

$$= \frac{S_r}{r-2}\ , \quad r > 2\ , \qquad (2.3.1.5')$$

and

$$\mathrm{Var}\,[\hat{\theta}_B|\underline{x}] = S_r^{\,2} \cdot \frac{\gamma^*(b+r-1,S_r)\ \gamma^*(b+r-3,S_r) - [\gamma^*(b+r-2,S_r)]^2}{[\gamma^*(b+r-1,S_r)]^2}$$

$$\mathrm{Var}\,[\hat{\theta}_B|\underline{x}] \to S_r^{\,2} \cdot \frac{\Gamma(r-1)\ \Gamma(r-3) - [\Gamma(r-2)]^2}{[\Gamma(r-1)]^2}$$

$$= \frac{S_r^{\,2}}{(r-2)^2(r-3)}\ , \quad r > 3\ . \qquad (2.3.1.6')$$

Also,

$$\hat{R}(t)_B = \left[1 + \frac{t^\rho - c^\rho}{S_r}\right]^{1-b-r} \cdot \frac{\gamma^*(b+r-1,S_r+t^\rho-c^\rho)}{\gamma^*(b+r-1,S_r)}$$

$$\hat{R}(t)_B \to \left[1 + \frac{t^{\rho}-c^{\rho}}{S_r}\right]^{1-r} , \qquad (2.3.1.7')$$

and

$$\mathrm{Var}\,[\hat{R}(t)_B|\underline{x}] = \left[\frac{1}{\gamma^*(b+r-1,S_r)^2}\right]\left\{\left[1 + \frac{2(t^{\rho}-c^{\rho})}{S_r}\right]^{1-b-r}\gamma^*(b+r-1,S_r+2t^{\rho}-2c^{\rho})\right.$$

$$\left.\cdot\ \gamma^*(b+r-1,S_r) - \left[1 + \frac{t^{\rho}-c^{\rho}}{S_r}\right]^{2(1-b-r)}[\gamma^*(b+r-1,S_r+t^{\rho}-c^{\rho})]^2\right\}$$

$$\mathrm{Var}\,[\hat{R}(t)_B|\underline{x}] \to \left[1 + \frac{2(t^{\rho}-c^{\rho})}{S_r}\right]^{1-r} - \left[1 + \frac{t^{\rho}-c^{\rho}}{S_r}\right]^{2(1-r)} . \qquad (2.3.1.8')$$

2.3.2 <u>Exponential Prior Probability Density Function</u>

In this section we shall consider the prior density of θ to be the exponential probability density function, (2.1.3). Applying Bayes' theorem we obtain

$$h(\theta;\underline{x}) = \frac{1(\underline{x};\theta,\rho)\cdot g(\theta;\lambda)}{\int_0^\infty 1(\underline{x};\theta,\rho)\cdot g(\theta;\lambda)d\theta}$$

$$= \frac{\frac{n!}{(n-r)!}\frac{\rho^r}{\theta^r}T_r\exp\left[-\frac{S_r}{\theta}\right]\cdot\frac{1}{\lambda}\exp\left[-\frac{\theta}{\lambda}\right]}{\int_0^\infty\frac{n!}{(n-r)!}\frac{\rho^r}{\theta^r}T_r\exp\left[-\frac{S_r}{\theta}\right]\cdot\frac{1}{\lambda}\exp\left[-\frac{\theta}{\lambda}\right]d\theta}$$

$$= \frac{\frac{1}{\theta^r}\exp\left[-\left(\frac{S_r}{\theta}+\frac{\theta}{\lambda}\right)\right]}{\int_0^\infty\frac{1}{\theta^r}\exp\left[-\left(\frac{S_r}{\theta}+\frac{\theta}{\lambda}\right)\right]d\theta} , \quad (0\leq\theta, 0<\lambda). \qquad (2.3.2.1)$$

If the substitutions $y = \theta$, $f = \frac{2}{\lambda}$, $a^2 = \lambda S_r$ and $v + 1 = r$ are made in the denominator of equation (2.3.2.1), the integral can be recognized as a modified Bessel function $K_v(af)$ of the third kind of order v as given by Erdélyi, et. al. [8]. Using the relation

$$K_v(af) = \tfrac{1}{2} a^v \int_0^\infty \frac{1}{y^{v+1}} \exp\left[-\left(\frac{fy}{2} + \frac{a^2 f}{2y}\right)\right] dy, \qquad (2.3.2.2)$$

the denominator of equation (2.3.2.1) becomes

$$\int_0^\infty \frac{1}{y^{v+1}} \exp\left[-\left(\frac{a^2 f}{2y} + \frac{fy}{2}\right)\right] dy = \frac{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)}{(\lambda S_r)^{\frac{r-1}{2}}} .$$

Hence, on the basis of experimentation the density of Θ is

$$h(\theta;\underline{x}) = \frac{(\lambda S_r)^{\frac{r-1}{2}} \exp\left[-\left(\frac{S_r}{\theta} + \frac{\theta}{\lambda}\right)\right]}{2\theta^r K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} , \quad (0\le\theta, 0<\lambda). \qquad (2.3.2.3)$$

The Bayes estimator of θ is the posterior mean, given by

$$\hat{\theta}_B = E[\theta|\underline{x}] = \int_0^\infty \theta h(\theta;\underline{x})\, d\theta$$

$$= \frac{(\lambda S_r)^{\frac{r-1}{2}}}{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \int_0^\infty \frac{1}{\theta^{r-1}} \exp\left[-\left(\frac{S_r}{\theta} + \frac{\theta}{\lambda}\right)\right] d\theta$$

$$= \sqrt{\lambda S_r} \cdot \frac{K_{r-2}\left(2\sqrt{\frac{S_r}{\lambda}}\right)}{K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} , \qquad (2.3.2.4)$$

and the variance of $\hat{\Theta}_B$ is given by

$$\text{Var}\,[\hat{\Theta}_B|\underline{x}] = E[\Theta^2|\underline{x}] - \{E[\Theta|\underline{x}]\}^2$$

$$= \int_0^\infty \theta^2\, h(\theta;\underline{x})d\theta - \hat{\theta}_B^{\,2}$$

$$= \frac{(\lambda S_r)^{\frac{r-1}{2}}}{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \int_0^\infty \frac{1}{\theta^{r-2}} \exp\left[-\left(\frac{S_r}{\theta} + \frac{\theta}{\lambda}\right)\right] d\theta - \hat{\theta}_B^{\,2}\,.$$

Thus, we have

$$\text{Var}\,[\hat{\Theta}_B|\underline{x}] = \frac{\lambda S_r}{K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \left\{K_{r-3}\left(2\sqrt{\frac{S_r}{\lambda}}\right) - \frac{\left[K_{r-2}\left(2\sqrt{\frac{S_r}{\lambda}}\right)\right]^2}{K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)}\right\}\,. \tag{2.3.2.5}$$

Again, the Bayes estimator of the exact reliability function is given by

$$\hat{R}(t)_B = E[R(t)|\underline{x}] = \int_0^\infty R(t)\; h(\theta;\underline{x})\; d\theta$$

$$= \frac{(\lambda S_r)^{\frac{r-1}{2}}}{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \int_0^\infty \frac{1}{\theta^r} \exp\left[-\left(\frac{S_r + t^\rho - c^\rho}{\theta} + \frac{\theta}{\lambda}\right)\right] d\theta$$

$$= \frac{(\lambda S_r)^{\frac{r-1}{2}}}{K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \cdot \frac{K_{r-1}\left(2\sqrt{\frac{S_r + t^\rho - c^\rho}{\lambda}}\right)}{\left[\lambda(S_r + t^\rho - c^\rho)\right]^{\frac{r-1}{2}}}$$

$$= \left[1 + \frac{t^\rho - c^\rho}{S_r}\right]^{\frac{1-r}{2}} \cdot \frac{K_{r-1}\left(2\sqrt{\frac{S_r + t^\rho - c^\rho}{\lambda}}\right)}{K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \tag{2.3.2.6}$$

Similarly, the variance of $\hat{R}(t)_B$ is obtained by evaluating

$$\text{Var}[\hat{R}(t)_B|\underline{x}] = E[R^2(t)|\underline{x}] - \{E[R(t)|\underline{x}]\}^2$$

$$= \int_0^\infty R^2(t)\, h(\theta;\underline{x})d\theta \; - \; \hat{R}^2(t)_B$$

$$= \frac{(\lambda S_r)^{\frac{r-1}{2}}}{2K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \int_0^\infty \frac{1}{\theta^r} \exp\left[-\left(\frac{S_r+2t^\rho-2c^\rho}{\theta} + \frac{\theta}{\lambda}\right)\right] d\theta - \hat{R}^2(t)_B$$

$$= \left[1 + \frac{2(t^\rho-c^\rho)}{S_r}\right]^{\frac{1-r}{2}} \cdot \frac{K_{r-1}\left(2\sqrt{\frac{S_r+2t^\rho-2c^\rho}{\lambda}}\right)}{K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} - \hat{R}^2(t)_B \,.$$

Therefore,

$$\mathrm{Var}[\hat{R}(t)_B|\underline{x}] = \frac{1}{K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)} \left\{\left[1+ \frac{2(t^\rho-c^\rho)}{S_r}\right]^{\frac{1-r}{2}} K_{r-1}\left(2\sqrt{\frac{S_r+2t^\rho-2c^\rho}{\lambda}}\right) - \left[1 + \frac{t^\rho-c^\rho}{S_r}\right]^{1-r} \cdot \frac{\left[K_{r-1}\left(2\sqrt{\frac{S_r+t^\rho-c^\rho}{\lambda}}\right)\right]^2}{K_{r-1}\left(2\sqrt{\frac{S_r}{\lambda}}\right)}\right\} . \qquad (2.3.2.7)$$

2.3.3 Inverted Gamma Prior Probability Density Function

The last prior density of Θ which we shall consider is the inverted gamma probability density function, (2.1.4). The posterior density of Θ is expressed by

$$h(\theta;\underline{x}) = \frac{l(\underline{x};\theta,\rho) \cdot g(\theta;\mu,\lambda)}{\int_0^\infty l(\underline{x};\theta,\rho) \cdot g(\theta;\mu,\lambda)\, d\theta}$$

$$= \frac{\frac{n!}{(n-r)!} \frac{\rho^r}{\theta^r} T_r \exp\left[-\frac{S_r}{\theta}\right] \cdot \frac{\left(\frac{\mu}{\theta}\right)^{\lambda+1}}{\mu\ \Gamma(\lambda)} \exp\left[-\frac{\mu}{\theta}\right]}{\int_0^\infty \frac{n!}{(n-r)!} \frac{\rho^r}{\theta^r} T_r \exp\left[-\frac{S_r}{\theta}\right] \cdot \frac{\left(\frac{\mu}{\theta}\right)^{\lambda+1}}{\mu\ \Gamma(\lambda)} \exp\left[-\frac{\mu}{\theta}\right] d\theta}$$

$$= \frac{\frac{1}{\theta^{r+\lambda+1}} \exp\left[-\left(\frac{S_r+\mu}{\theta}\right)\right]}{\int_0^\infty \frac{1}{\theta^{r+\lambda+1}} \exp\left[-\left(\frac{S_r+\mu}{\theta}\right)\right] d\theta}, \quad (0\leq\theta, 0<\mu, \lambda). \qquad (2.3.3.1)$$

Using the substitution $y = \frac{S_r+\mu}{\theta}$ or $dy = -\left(\frac{S_r+\mu}{\theta^2}\right) d\theta$ in the denominator of equation (2.3.3.1), the integral becomes

$$\frac{1}{(S_r+\mu)^{r+\lambda}} \int_0^\infty y^{r+\lambda-1} \exp[-y]\, dy = \frac{\Gamma(r+\lambda)}{(S_r+\mu)^{r+\lambda}} .$$

Thus, the posterior density of Θ, when the prior density is the inverted gamma, is

$$h(\theta;\underline{x}) = \frac{(S_r+\mu)^{r+\lambda}}{\theta^{r+\lambda+1}\ \Gamma(r+\lambda)} \exp\left[-\left(\frac{S_r+\mu}{\theta}\right)\right], \quad (0\leq\theta, 0<\mu, \lambda). \qquad (2.3.3.2)$$

Simplifying the above expression we have

$$h(\theta;\underline{x}) = \frac{\left(\frac{S_r+\mu}{\theta}\right)^{r+\lambda+1}}{(S_r+\mu)\ \Gamma(r+\lambda)} \exp\left[-\left(\frac{S_r+\mu}{\theta}\right)\right]$$

which is recognized as another inverted gamma probability density function $g(\theta;\ (S_r+\mu),\ (r+\lambda))$. Hence, we have the property of closure under sampling.

The posterior mean is

$$\hat{\theta}_B = E[\Theta|\underline{x}] = \int_0^\infty \theta h(\theta;\underline{x})\, d\,\theta$$

$$= \frac{(S_r+\mu)^{r+\lambda}}{\Gamma(r+\lambda)} \int_0^\infty \frac{1}{\theta^{r+\lambda}} \exp\left[-\left(\frac{S_r+\mu}{\theta}\right)\right] d\,\theta$$

$$= \frac{S_r+\mu}{r+\lambda-1}\quad,\quad r+\lambda > 1 \qquad (2.3.3.3)$$

while the variance of the Bayes estimator of θ is

$$\mathrm{Var}[\hat{\Theta}_B|\underline{x}] = E[\Theta^2|\underline{x}] - \{E[\Theta|\underline{x}]\}^2$$

$$= \int_0^\infty \theta^2\, h(\theta;\underline{x})\, d\theta - \hat{\theta}_B^{\,2}$$

$$= \frac{(S_r+\mu)^{r+\lambda}}{\Gamma(r+\lambda)} \int_0^\infty \frac{1}{\theta^{r+\lambda-1}} \exp\left[-\left(\frac{S_r+\mu}{\theta}\right)\right] d\theta - \hat{\theta}_B^2 \,.$$

Thus,

$$\mathrm{Var}\ [\hat{\Theta}_B|\underline{x}] = \frac{(S_r+\mu)^2}{(r+\lambda-1)^2(r+\lambda-2)}\quad,\quad r+\lambda>2. \qquad (2.3.3.4)$$

Finally, we shall obtain the Bayes estimate of the exact reliability function, (2.2.2), and its variance. That is,

$$\hat{R}\ (t)_B = E[R(t)|\underline{x}] = \int_0^\infty R(t)\ h(\theta;\underline{x})\ d\,\theta$$

$$= \frac{\left(S_r+\mu\right)^{r+\lambda}}{\Gamma(r+\lambda)} \int_0^\infty \frac{1}{\theta^{r+\lambda+1}} \exp\left[-\left(\frac{S_r+\mu+t^\rho-c^\rho}{\theta}\right)\right] d\,\theta$$

$$= \frac{(S_r+\mu)^{r+\lambda}}{(S_r+\mu+t^\rho-c^\rho)^{r+\lambda}}$$

$$= \frac{1}{\left[1+\frac{t^\rho-c^\rho}{S_r+\mu}\right]^{r+\lambda}}, \qquad (2.3.3.5)$$

and

$$\mathrm{Var}\,[\hat{R}(t)_B|\underline{x}] = E[R^2(t)|\underline{x}] - \{E[R(t)|\underline{x}]\}^2$$

$$= \int_0^\infty R^2(t)\,h(\theta;\underline{x})\,d\theta - \hat{R}^2(t)_B$$

$$= \frac{\left(S_r+\mu\right)^{r+\lambda}}{\Gamma(r+\lambda)} \int_0^\infty \frac{1}{\theta^{r+\lambda+1}} \exp\left[-\left(\frac{S_r+\mu+2t^\rho-2c^\rho}{\theta}\right)\right] d\theta - \hat{R}^2(t)_B$$

$$= \frac{1}{\left[1+\frac{2(t^\rho-c^\rho)}{S_r+\mu}\right]^{r+\lambda}} - \frac{1}{\left[1+\frac{t^\rho-c^\rho}{S_r+\mu}\right]^{2(r+\lambda)}}. \qquad (2.3.3.6)$$

2.4 SUMMARY

In Chapter IV, the Bayesian estimators of θ and the reliability function will be compared through a computer simulation with the corresponding MVUE which will be obtained in the next chapter. The Bayesian estimators will also be compared with those obtained by Canavos and Tsokos [5] as discussed in the introduction. Their results using formula (1.2) as the underlying life testing model are listed below.

The exact reliability is given by

$$R(t) = \exp\left[-\frac{(t-\tau)^{\rho}}{\theta}\right], \qquad (0 < \tau \leq x,\ 0 < \theta,\ \rho). \tag{2.4.1}$$

For the general uniform prior probability density function:

$$\hat{\theta}_B = S_r' \cdot \frac{\gamma^*(b+r-2, S_r')}{\gamma^*(b+r-1, S_r')} \tag{2.4.2}$$

and

$$\hat{R}(t)_B = \left[1 + \frac{(t-\tau)^{\rho}}{S_r'}\right]^{1-b-r} \cdot \frac{\gamma^*(b+r-1, S_r' + (t-\tau)^{\rho})}{\gamma^*(b+r-1, S_r')} \tag{2.4.3}$$

For the exponential prior probability density function:

$$\hat{\theta}_B = \sqrt{\lambda S_r'} \cdot \frac{K_{r-2}\left(2\sqrt{\frac{S_r'}{\lambda}}\right)}{K_{r-1}\left(2\sqrt{\frac{S_r'}{\lambda}}\right)} \tag{2.4.4}$$

and

$$\hat{R}(t)_B = \left[1 + \frac{(t-\tau)^{\rho}}{S_r}\right]^{\frac{1-r}{2}} \cdot \frac{K_{r-1}\left(2\sqrt{\frac{S_r'+(t-\tau)^{\rho}}{\lambda}}\right)}{K_{r-1}\left(2\sqrt{\frac{S_r'}{\lambda}}\right)}. \tag{2.4.5}$$

For the inverted gamma prior probability density function:

$$\hat{\theta}_B = \frac{S_r' + \mu}{r+\lambda-1}, \qquad r + \lambda > 1 \tag{2.4.6}$$

and

$$\hat{R}(t)_B = \frac{1}{\left[1 + \frac{(t-\tau)^\rho}{S_r' + \mu}\right]^{r+\lambda}} . \tag{2.4.7}$$

In all cases, $S_r' = \sum_{i=1}^{r} (x_i-\tau)^\rho + (n-r)(x_r-\tau)^\rho$.

CHAPTER III

MINIMUM VARIANCE UNBIASED ESTIMATOR OF RELIABILITY

3.1 INTRODUCTION

In this chapter we shall be concerned with the derivation of the "best" estimator of reliability for the singly truncated Weibull distribution given by

$$f(x;\theta,\rho) = \begin{cases} \frac{\rho}{\theta}\, x^{\rho-1} \quad \exp\left[-\ \frac{(x^{\rho}-c^{\rho})}{\theta}\right] & , \quad (0<c\leq x,\ 0<\theta,\rho) \\ 0 \qquad \text{elsewhere} & \end{cases} \tag{3.1.1}$$

The estimator is "best" in the sense that it is maximum likelihood, unbiased, minimum variance, efficient, and sufficient. The Rao-Blackwell and Lehmann-Scheffé theorems will be used to find the minimum variance unbiased estimator (MVUE) of reliability. Barton [2] has considered the binomial, the poisson, and the normal probability density functions as the underlying life testing model and derived the MVUE of reliability. Pugh [13] has considered the one parameter exponential probability distribution, while Laurent [11] and Tate [16] have considered the two parameter case. Tate [16] has also considered cases of the gamma and Weibull probability distributions. Basu [3] has derived the MVUE for the gamma, censored one and two parameter exponential, and the censored Weibull distributions.

3.2 PRELIMINARY RESULTS

The method used to derive our MVUE of reliability will be discussed. From a population with probability density function

$f(w;\underline{\theta})$, where $\underline{\theta}$ is a d-dimensional parameter vector, we select a random sample of size n. These n items, say $\underline{w} = (w_1, \ldots, w_n)$, are then placed on a life test which is terminated after a predetermined $r \leq n$ number of failures have occurred. Let $\underline{x} = (x_1, \ldots, x_r)$ be the corresponding order statistics where x_i is the life time of the i^{th} item. Suppose that $\underline{x}$ is divided into two independent components, ξ and $\underline{\eta}$, of sizes one and (r-1), respectively. Let any one of these x_i's be ξ and the remaining (r-1) x_i's comprise $\underline{\eta} = (\eta_1, \ldots, \eta_{r-1})$ and let $\underline{v} = (v_1, \ldots, v_{r-1})$ be its corresponding order statistic. Consider a complete sufficient statistic $\hat{\theta}$, assuming one exists, with probability density $h(\hat{\theta})$ for estimating θ. If $\hat{\theta}^*$ is the MVUE of θ from $\underline{\eta}$, we can find the joint probability density function of ξ and $\hat{\theta}^*$. Then we may obtain the joint density of ξ and $\hat{\theta}$ from which the conditional distribution $g(\xi|\hat{\theta})$ is obtained. Therefore, by the Rao-Blackwell and Lehmann-Schaffé theorems [9], the unique MVUE of the reliability function $R(t)$ is given by

$$R^*(t) = E[I_t(\xi)|\ \hat{\theta}] = \int_t^{\infty} g(\xi|\hat{\theta})\ d\xi ,$$

where $I_t(\xi)$ is an unbiased estimator of R(t) and $I_t(.)$ is the indicator function of the set $[t,\infty]$. It should be noted that if no complete sufficient statistic for θ exists, then the estimator of reliability will not be unique; however, this procedure can be used provided $\hat{\theta}$ is a sufficient statistic.

3.3 MAIN RESULTS

In this section we shall find the MVUE of reliability for the probability distribution given in (2.3.1). Ignoring the permutation coefficient of the joint probability of $x_1, \ldots, x_r$, we obtain the likelihood function given by

$$L(\theta;\underline{x}) = \frac{\rho^r}{\theta^r}\left[\prod_{i=1}^{r} x_i^{\rho-1}\right] \exp\left\{-\left[\frac{\sum_{i=1}^{r}(x_i^\rho-c^\rho)+(n-r)(x_r^\rho-c^\rho)}{\theta}\right]\right\}. \tag{3.3.1}$$

If the shape parameter ρ is assumed to be known, the maximum likelihood estimator $\hat{\theta}_{ML}$ of θ is found by taking the natural logarithm of equation (3.3.1), and then the partial derivative with respect to θ. This yields

$$\ln L(\theta;\underline{x}) = r\ln\rho - r\ln\theta + (\rho-1)\sum_{i=1}^{r}\ln x_i - \frac{1}{\theta}\left[\sum_{i=1}^{r}(x_i^\rho-c^\rho) + (n-r)(x_r^\rho-c^\rho)\right]$$

and

$$\frac{\partial \ln L(\theta;\underline{x})}{\partial\theta} = \frac{-r}{\theta} + \frac{1}{\theta^2}\left[\sum_{i=1}^{r}(x_i^\rho-c^\rho) + (n-r)(x_r^\rho-c^\rho)\right]. \tag{3.3.2}$$

Equating expression (3.3.2) to zero and solving for θ we obtain

$$\hat{\theta} = \hat{\theta}_{ML} = \frac{\sum_{i=1}^{r}(x_i^\rho-c^\rho) + (n-r)(x_r^\rho-c^\rho)}{r}. \tag{3.3.3}$$

Applying the Neyman factorization criterion, $\hat{\theta}$ is shown to be a sufficient statistic as follows:

$$f(\underline{x};\theta,\rho) = \left\{\frac{n!}{(n-r)!}\ \rho^r \prod_{i=1}^{r} x_i^{\rho-1}\right\} \cdot \left\{\frac{1}{\theta^r}\ \exp\left[-\ \frac{r\hat{\theta}}{\theta}\right]\right\}.$$

To obtain the probability distribution of $\hat{\theta}$ we will utilize the following transformation:

$$\begin{aligned} y_1 &= x_1^\rho - c^\rho \\ y_2 &= x_2^\rho - x_1^\rho \\ &\ \ \vdots \\ y_r &= x_r^\rho - x_{r-1}^\rho \end{aligned} \qquad \text{or} \qquad \begin{aligned} y_i &= x_i^\rho - x_{i-1}^\rho \\ x_o &= c \end{aligned} \qquad i=1,\ \ldots,\ r$$

The inverse transformations are given by

$$\begin{aligned} g^{-1}(y_1) &= x_1 = (y_1+c^\rho)^{\frac{1}{\rho}} \\ g^{-1}(y_2) &= x_2 = (y_2+y_1+\ c^\rho)^{\frac{1}{\rho}} \\ &\ \ \vdots \\ g^{-1}(y_r) &= x_r = (y_r+\ \ldots\ +\ y_1+c^\rho)^{\frac{1}{\rho}} \end{aligned} \qquad \text{or} \qquad g^{-1}(y_i) = x_i = \left(\sum_{k=1}^{i} y_k + c^\rho\right)^{\frac{1}{\rho}}$$

The Jacobian of the transformation and its determinate are given by

$$J = \begin{bmatrix} \frac{\partial g^{-1}(y_1)}{\partial y_1} & \frac{\partial g^{-1}(y_1)}{\partial y_2} & \cdots & \frac{\partial g^{-1}(y_1)}{\partial y_r} \\ \frac{\partial g^{-1}(y_2)}{\partial y_1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial g^{-1}(y_r)}{\partial y_1} & \cdots & \cdots & \frac{\partial g^{-1}(y_r)}{\partial y_r} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\rho}(y_1+c^\rho)^{\frac{1}{\rho}-1} & 0 & \cdots & 0 \\ \frac{1}{\rho}(y_2+y_1+c^\rho)^{\frac{1}{\rho}-1} & \frac{1}{\rho}(y_2+y_1+c^\rho)^{\frac{1}{\rho}-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\rho}(y_r+\ldots+y_1+c^\rho)^{\frac{1}{\rho}-1} & \frac{1}{\rho}(y_r+\ldots+y_1+c^\rho)^{\frac{1}{\rho}-1} & \cdots & \frac{1}{\rho}(y_r\ldots+y_1+c^\rho)^{\frac{1}{\rho}-1} \end{bmatrix}$$

and

$$|J| = \frac{1}{\rho^r}(y_1+c^\rho)^{\frac{1-\rho}{\rho}}(y_2+y_1+c^\rho)^{\frac{1-\rho}{\rho}} \cdots (y_r+\ldots+y_1+c^\rho)^{\frac{1-\rho}{\rho}} .$$

Hence, the joint distribution of the y_i's is given by

$$z(y_1, \ldots, y_r;\theta) = f(g^{-1}(y_1), \ldots, g^{-1}(y_r);\theta,\rho) \quad |J|$$

$$= \frac{n!}{(n-r)!}\frac{\rho^r}{\theta^r}(y_1+c^\rho)^{\frac{\rho-1}{\rho}}(y_2+y_1+c^\rho)^{\frac{\rho-1}{\rho}}\ldots(y_r+\ldots+y_1+c^\rho)^{\frac{\rho-1}{\rho}} \quad |J|$$

$$\cdot \exp\left\{-\left[\frac{\sum_{i=1}^{r}\left(\left(\sum_{k=1}^{i} y_k+c^\rho\right)^{\frac{\rho}{\rho}}-c^\rho\right)+(n-r)\left((y_r+\ldots y_1+c^\rho)^{\frac{\rho}{\rho}}-c^\rho\right)}{\theta}\right]\right\}$$

$$= \frac{n!}{(n-r)!}\frac{1}{\theta^r}\exp\left\{-\left[\frac{\sum_{i=1}^{r}\sum_{k=1}^{i} y_k+(n-r)(y_r+\ldots+y_1)}{\theta}\right]\right\}$$

$$= \frac{n!}{(n-r)!}\frac{1}{\theta^r}\exp\left\{-\left[\frac{y_1+(y_1+y_2)+\ldots+(y_1+\ldots+y_r)+(n-r)(y_1+\ldots+y_r)}{\theta}\right]\right\}$$

Therefore,

$$z(y_1,\ldots,y_r;\theta) = \frac{n!}{(n-r)!}\,\frac{1}{\theta^r}\,\exp\left[-\frac{1}{\theta}\sum_{i=1}^{r}(n-i+1)y_i\right]$$

$$= \prod_{i=1}^{r}\left\{\frac{(n-i+1)}{\theta}\,\exp\left[-\frac{(n-i+1)y_i}{\theta}\right]\right\},\quad (y_i \geq 0). \tag{3.3.4}$$

Consequently the y_1's are mutually independent and $(n-i+1)\ y_i$, $(i=1,\ldots,r)$, have common exponential distributions. Furthermore, let $u_i = (n-i+1)y_i$, then

$$\sum_{i=1}^{r} u_i = \sum_{i=1}^{r}(x_i^{\rho}-c^{\rho}) + (n-r)(x_r^{\rho}-c^{\rho}) = r\hat{\theta} \quad \text{or } \hat{\theta} = \bar{u}\ .$$

Hence, we can consider that we have a random sample $(u_1, \ldots, u_r)$ from an exponential distribution. Since $\hat{\theta}$ can be thought of as the mean of a sample of size r from an exponential distribution, $\frac{2r\hat{\theta}}{\theta}$ has a χ^2 distribution with 2r degrees of freedom and $\hat{\theta}$ has the distribution

$$h(\hat{\theta};\theta,r) = \frac{1}{\Gamma(r)}\left(\frac{r}{\theta}\right)^r \hat{\theta}^{r-1}\exp\left[-\frac{r\hat{\theta}}{\theta}\right],\quad (\hat{\theta} > 0). \tag{3.3.5}$$

If we let $\chi^2 = \frac{2r\hat{\theta}}{\theta}$ or $g^{-1}(\chi^2) = \hat{\theta} = \frac{\chi^2\theta}{2r}$

and $\frac{dg^{-1}(\chi^2)}{d\chi^2} = \frac{\theta}{2r}$ then

$$h(\chi^2;2r) = h(g^{-1}(\chi^2);\theta,r)\ \frac{dg^{-1}(\chi^2)}{d\chi^2}$$

$$= \frac{r}{\theta\Gamma(r)} \quad \frac{(\chi^2)^{r-1}}{2^{r-1}} \quad \exp\left[-\frac{\chi^2}{2}\right] \cdot \quad \frac{\theta}{2r}$$

$$= \frac{1}{2^{\frac{2r}{2}}\,\Gamma\left(\frac{2r}{2}\right)} \quad (\chi^2)^{\frac{2r}{2}-1} \quad \exp\left[-\frac{\chi^2}{2}\right] ,$$

which verifies that $\frac{2r\hat{\theta}}{\theta}$ has a chi-squared distribution with 2r degrees of freedom.

The characteristic function of the exponential probability density function is

$$\phi_u(t) = (1 - i\theta t)^{-1} .$$

Thus,

$$\phi_{\hat{\theta}}(t) = \prod_{i=1}^{r} \phi_{u_i}\left(\frac{t}{r}\right) = (1 - \frac{i\theta}{r} t)^{-r}$$

and the distribution of $\hat{\theta}$ is uniquely determined to be a gamma probability density function, given by equation (3.3.5), with parameters $\frac{\theta}{r}$ and r.

Since the gamma density is complete, the maximum likelihood estimator is a complete sufficient statistic. The mean and variance of the gamma distributed random variable are θ and $\frac{\theta^2}{r}$, respectively. Therefore, $\hat{\theta}$ is unbiased, efficient, and minimum variance since the Cramer-Rao lower bound (CRLB) which is derived below, is $\frac{\theta^2}{r}$.

$$\text{CRLB} = \frac{1}{E\left[\frac{\partial \ln f(\underline{x};\theta,\rho)}{\partial\theta}\right]^2}$$

$$= \frac{1}{E\left[-\frac{r}{\theta} + \frac{r\hat{\theta}^2}{\theta^2}\right]^2}$$

$$= \frac{1}{E\left[\frac{r^2}{\theta^2} + \frac{r^2\hat{\theta}^2}{\theta^4} - \frac{2r^2\hat{\theta}}{\theta^3}\right]}$$

$$= \frac{1}{\frac{r^2}{\theta^2} + \frac{r^2}{\theta^4}\left(\frac{\theta^2}{r} + \theta^2\right) - \frac{2r^2\theta}{\theta^3}}$$

$$= \frac{\theta^2}{r} \quad .$$

Our best estimate of θ from $\underline{\eta}$ is given by

$$\hat{\theta}^* = \frac{\sum_{i=1}^{r-1} (v_i^\rho - c^\rho) + (n-r+1)(v_{r-1}^\rho - c^\rho)}{r-1} = \frac{\sum_{i=1}^{r-1} \sigma_i}{r-1} = \bar{\sigma} \tag{3.3.6}$$

where $\sigma_i = (n-i+1)(v_i^\rho - v_{i-1}^\rho)$, $v_o = c$, $i=1,\ldots,r-1$. The probability distribution of $\hat{\theta}^*$ is also a gamma density with parameters $\frac{\theta}{r-1}$ and $r-1$. That is,

$$h^*(\hat{\theta}^*;\theta,r) = \frac{1}{\Gamma(r-1)} \left(\frac{r-1}{\theta}\right)^{r-1} (\hat{\theta}^*)^{r-2} \exp\left[-\frac{(r-1)\hat{\theta}^*}{\theta}\right], (\hat{\theta}^*>0) \tag{3.3.7}$$

while the probability distribution of ξ is given by equation (3.1.1). Hence, the bivariate distribution of ξ and $\hat{\theta}^*$ is

$$q^*(\hat{\theta}^*,\xi;\theta,\rho,r) = f(\xi;\theta,\rho) \cdot h^*(\hat{\theta}^*;\theta,r)$$

$$= \frac{1}{\Gamma(r-1)} \frac{\rho}{\theta} \left(\frac{r-1}{\theta}\right)^{r-1} \xi^{\rho-1} (\hat{\theta}^*)^{r-2} \exp\left\{-\left[\frac{(r-1)\hat{\theta}^*+\xi^\rho-c^\rho}{\theta}\right]\right\},$$

$$(c \leq \xi,\ 0 \leq \hat{\theta}^*,\ 0<\theta,\rho). \tag{3.3.8}$$

Clearly $(r-1)\,\hat{\theta}^* = r\,\hat{\theta} - (n-r)(x_r^\rho - c^\rho) - (\xi^\rho - c^\rho) + (n-r+1)(v_{r-1}^\rho - c^\rho)$

and $$m^{-1}(\hat{\theta}) = \hat{\theta}^* = \frac{r\hat{\theta} - (n-r)(x_r^\rho - c^\rho) - (\xi^\rho - c^\rho) + (n-r+1)(v_{r-1}^\rho - c^\rho)}{r-1} \quad .$$

Also, since $\hat{\theta}^* \geq 0$ then $$\frac{r\hat{\theta} - (n-r)(x_r^\rho - c^\rho) - (\xi^\rho - c^\rho) + (n-r+1)(v_{r-1}^\rho - c^\rho)}{r-1} \geq 0.$$

Solving this expression for ξ we obtain

$$\xi \leq \left[r\hat{\theta} - (n-r)(x_r^\rho - c^\rho) + (n-r+1)(v_{r-1}^\rho - c^\rho) + c^\rho\right]^{\frac{1}{\rho}} \quad .$$

Let this expression be denoted by δ.

The joint probability density function of ξ and $\hat{\theta}$ is given by

$$q(\hat{\theta},\xi;\theta,\rho,r) = q^*(m^{-1}(\hat{\theta}),\xi;\theta,\rho,r)\,\frac{dm^{-1}(\hat{\theta})}{d\hat{\theta}}$$

$$= \frac{r}{\Gamma(r-1)}\,\frac{\rho}{\theta^r}\,\xi^{\rho-1}\left[r\hat{\theta}-(n-r)(x_r^\rho-c^\rho)-(\xi^\rho-c^\rho)+(n-r+1)(v_{r-1}^\rho-c^\rho)\right]^{r-2}$$

$$\cdot \exp\left\{-\frac{\left[r\hat{\theta}-(n-r)(x_r^\rho-c^\rho)+(n-r+1)(v_{r-1}^\rho-c^\rho)\right]}{\theta}\right\}, \quad (c\leq\xi\leq\delta,\ 0\leq\hat{\theta},\ 0<\theta,\rho) \qquad (3.3.9)$$

The conditional distribution of ξ given $\hat{\theta}$, where $q_\xi(\hat{\theta};\theta,\rho,r)$ is the marginal density for $\hat{\theta}$ obtained from $q(\hat{\theta},\xi;\theta,\rho,r)$, is given by

$$g(\xi|\hat{\theta}) = \frac{q(\hat{\theta},\xi;\theta,\rho,r)}{q_\xi(\hat{\theta};\theta,\rho,r)}$$

$$= \frac{\frac{r}{\Gamma(r-1)}\frac{\rho}{\theta^r}\xi^{\rho-1}[r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})-(\xi^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})]^{r-2} \cdot \exp\left\{-\left[\frac{r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})}{\theta}\right]\right\}}{\frac{r}{\Gamma(r)}\frac{1}{\theta^r}[r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})]^{r-1} \cdot \exp\left\{-\left[\frac{r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})}{\theta}\right]\right\}}$$

$$= \frac{(r-1)\rho\xi^{\rho-1}[r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})-(\xi^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})]^{r-2}}{[r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})]^{r-1}}, (c\le\xi\le\delta, 0<\rho).$$

(3.3.10)

Therefore, the minimum variance unbiased estimator of reliability is obtained below.

$$R^*(t) = \int_t^{\delta} g(\xi|\hat{\theta})d\xi .$$

If we let $z(\xi) = r\hat{\theta} - (n-r)(x_r^{\rho}-c^{\rho})-(\xi^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})$ and $dz = -\rho\xi^{\rho-1}d\xi$, then

$$R^*(t) = \frac{r-1}{[r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})]^{r-1}} \int_{z(t)}^{z(\delta)} -z^{r-2}\,dz$$

$$= \frac{[r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})-(t^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})]^{r-1}}{[r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})]^{r-1}}$$

$$= \left[1 - \frac{(t^{\rho}-c^{\rho})}{r\hat{\theta}-(n-r)(x_r^{\rho}-c^{\rho})+(n-r+1)(v_{r-1}^{\rho}-c^{\rho})}\right]^{r-1}, (c\le t\le\delta).$$

(3.3.11)

3.4 SUMMARY

The MVUE of the scale parameter θ in the singly truncated Weibull life testing model and the corresponding reliability function have been obtained. In Chapter IV, the expressions derived for the MVUE of θ and the reliability function will be compared with the results of Chapter II by means of Monte Carlo simulation.

CHAPTER IV

MONTE CARLO SIMULATION

4.1 DESCRIPTION OF PROGRAM

For each prior density of Θ, a separate computer program is used in order to compare the Bayesian estimate and minimum variance unbiased estimate (MVUE) of θ and the reliability function. The number of experiments used in each computation varies as a function of computer time required and cost. When the uniform prior density function is used, we consider 600 repetitions; when the exponential prior density function is used, we consider 2,000 repetitions; and when the inverted gamma prior density function is used, we consider 4,000 repetitions. In all cases, we use a complete sample size 10; that is, $n = r = 10$. For each experiment, 10 random lifetimes from the truncated Weibull life testing model ($c = 1.0$ units) and a single value of θ from its prior distribution are generated. Using these values, the Bayes estimate and the MVUE of θ are calculated. The random lifetimes are also used to compute the Bayes estimate of θ for the Weibull model with guaranteed time $\tau = 1.0$ units. The exact reliability function, its Bayes estimates (both models), and its MVUE are obtained for values of t between τ and 20 units in increment of 0.1 units or until the exact reliability falls below 0.005. Whenever this occurs, a new experiment is begun. Other quantities obtained for each experiment are the squared error loss for all three estimates of θ and of the reliability function; the average squared error loss of the reliability estimates; the accumulated average squared error loss of the reliability estimates for all previous repetitions; and finally, the ratio of the accumulated averages for the Bayes estimate (truncated model) and the MVUE.

At the conclusion of all experiments, the averages are calculated for the generated values of θ, the MVUE of θ, and both Bayes estimates of θ.

In the evaluation of the incomplete gamma function, the trapezoidal rule was used as an approximation to the integral [14].

4.2 LIST OF IMPORTANT VARIABLES USED IN PROGRAMS

The following list of variables are common to all programs. It should be noted that where two names are given for the same function, the first refers to the truncated Weibull model, while the second refers to the guaranteed time Weibull model.

C - truncation point as defined in equation (1.1)

CB, CB1 - Bayes estimates of θ

GAM - shape parameter in both models

NUM - number of experiments simulated

ORDER - subroutine to find the n^{th} order statistic

PCB, PCB1 - accumulated average squared error loss of all experiments for Bayes estimates of reliability

PMV - accumulated average squared error loss of all experiments for MVUE of reliability

R - sample size, number of lifetimes simulated

RANDU - subroutine to generate random numbers

RECB, RECB1 - Bayes estimates of reliability

RELMV - MVUE of reliability

RMS - ratio of PCB to PMV

SCB, SCB1 - averages of Bayes estimates of θ

SEB, SEB1 - accumulated squared error of all experiments loss for Bayes estimates of θ

SEMV - accumulated squared error loss of all experiments for MVUE of θ

STHETA - average of the generated values of θ

SXMV - average of the MVUE of θ

TAU - guaranteed time as defined in equation (1.2)

THETA - generated value of θ from its prior distribution

TREL, TREL1 - exact reliabilities

XMSCB, XMSCB1 - average of the accumulated squared error loss for each experiment for Bayes estimates of reliability

XMSMV - average of the accumulated squared error loss for each experiment for the MVUE of reliability

XMV - MVUE of θ

YFL - random number [0,1]

ZMAX - n^{th} order statistic

The variables below are used in the programs as stated.

ALPHA - lower limit α in the uniform prior distribution

BETA - upper limit β in the uniform prior distribution

INTEGR - subroutine which evaluates the incomplete gamma function in the uniform prior program

BESK - subroutine which evaluates the modified Bessel function $K_v(af)$ of the third kind of order v in the exponential prior program

BNOT - function to generate values of θ according to the inverted gamma prior distribution with μ = 20 and λ = 2

CLM - parameter λ in the exponential prior and the inverted gamma prior distributions

CMU - parameter μ in the inverted gamma prior distribution

ITR2 - subroutine which converges to a value of θ according to BNOT

4.3 RESULTS

The following results were obtained when the Monte Carlo simulation described in section 4.1 was used. Tables I - V are for the general uniform prior probability density function with $b = 0$ and $[\alpha,\beta] = [10,30]$. Tables VI - X are for the exponential prior probability density function with $\lambda = 20$, and Tables XI - XV are for the inverted gamma prior probability density function with $\mu = 20$ and $\lambda = 2$.

TABLE I

RESULTS OF EXPERIMENT 1 FOR UNIFORM
PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.10398578E 02 PMV= 0.65147169E-02
XMV= 0.56361113E 01 PCB= 0.15064445E-02
CB= 0.12413623E 02 PCB1= 0.88498532E-03
CB1= 0.11886759E 02 RMS= 0.23123711E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.90899 0.97084	0.92089 0.97387	0.87958
1.99999	0.80551 0.90831	0.82983 0.91759	0.74567
2.49998	0.69739 0.82565	0.73318 0.84263	0.60987
2.99997	0.59078 0.73166	0.63611 0.75658	0.48114
3.49997	0.49027 0.63345	0.54261 0.66555	0.36575
3.99996	0.39893 0.53661	0.45553 0.57448	0.26741
4.49995	0.31854 0.44532	0.37673 0.48715	0.18755
4.99994	0.24976 0.36236	0.30718 0.40625	0.12573
5.49993	0.19240 0.28934	0.24714 0.33346	0.08020
5.99993	0.14569 0.22686	0.19635 0.26964	0.04838
6.49992	0.10849 0.17476	0.15416 0.21495	0.02740
6.99991	0.07949 0.13233	0.11969 0.16905	0.01442
7.49990	0.05731 0.09855	0.09197 0.13126	0.00696

TABLE II

RESULTS OF EXPERIMENT 150 FOR UNIFORM
PRIOR PROBABILITY DENSITY FUNCTION

THETA=	0.18490112E 02	PMV=	0.15700598E 01
XMV=	0.21121262E 02	PCB=	0.39691079E 00
CB=	0.21641083E 02	PCB1=	0.42325819E 00
CB1=	0.19075577E 02	RMS=	0.25279975E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.94775 0.98349	0.95302 0.98293	0.96701
1.99999	0.88547 0.94735	0.89684 0.94566	0.92662
2.49998	0.81653 0.89785	0.83437 0.89484	0.88042
2.99997	0.74379 0.83886	0.76811 0.83463	0.82979
3.49997	0.66974 0.77354	0.70019 0.76843	0.77597
3.99996	0.59642 0.70464	0.63243 0.69910	0.72011
4.49995	0.52552 0.63448	0.56626 0.62906	0.66324
4.99994	0.45833 0.56502	0.50286 0.56026	0.60629
5.49993	0.39577 0.49786	0.44308 0.49422	0.55007
5.99993	0.33848 0.43420	0.38750 0.43206	0.49530
6.49992	0.28676 0.37494	0.33650 0.37454	0.44258
6.99991	0.24073 0.32066	0.29024 0.32211	0.39242
7.49990	0.20029 0.27167	0.24871 0.27495	0.34518

TABLE III

RESULTS OF EXPERIMENT 300 FOR UNIFORM PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.29648621E 02 PMV= 0.31718025E 01
XMV= 0.13201384E 02 PCB= 0.77483273E 00
CB= 0.16491684E 02 PCB1= 0.85971236E 00
CB1= 0.14488698E 02 RMS= 0.24428779E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.96709 0.98967	0.93770 0.97776	0.94533
1.99999	0.92695 0.96683	0.86466 0.92962	0.87992
2.49998	0.88126 0.93501	0.78538 0.86488	0.80719
2.99997	0.83144 0.89621	0.70362 0.78962	0.73013
3.49997	0.77880 0.85203	0.62241 0.70877	0.65137
3.99996	0.72448 0.80387	0.54413 0.62637	0.57317
4.49995	0.66950 0.75297	0.47052 0.54564	0.49744
4.99994	0.61474 0.70045	0.40274 0.46900	0.42571
5.49993	0.56098 0.64729	0.34148 0.39814	0.35912
5.99993	0.50885 0.59436	0.28700 0.33410	0.29850
6.49992	0.45887 0.54238	0.23924 0.27734	0.24433
6.99991	0.41143 0.49198	0.19791 0.22794	0.19681
7.49990	0.36684 0.44365	0.16257 0.18560	0.15589

TABLE IV

RESULTS OF EXPERIMENT 450 FOR UNIFORM PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.21821091E 02   PMV= 0.46723614E 01
XMV= 0.14495985E 02   PCB= 0.11509037E 01
CB= 0.17425247E 02   PCB1= 0.12843418E 01
CB1= 0.15082579E 02   RMS= 0.24632162E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.95554 0.98599	0.94089 0.97853	0.95537
1.99999	0.90207 0.95521	0.87132 0.93198	0.90140
2.49998	0.84219 0.91274	0.79546 0.86929	0.84059
2.99997	0.77817 0.86166	0.71679 0.79622	0.77513
3.49997	0.71200 0.80447	0.63816 0.71746	0.70699
3.99996	0.64538 0.74332	0.56185 0.63688	0.63791
4.49995	0.57975 0.68011	0.48955 0.55760	0.56941
4.99994	0.51629 0.61647	0.42244 0.48197	0.50279
5.49993	0.45593 0.55378	0.36124 0.41166	0.43911
5.99993	0.39934 0.49317	0.30631 0.34774	0.37924
6.49992	0.34700 0.43551	0.25770 0.29073	0.32380
6.99991	0.29919 0.38146	0.21521 0.24075	0.27321
7.49990	0.25601 0.33146	0.17850 0.19760	0.22772

TABLE V

RESULTS OF EXPERIMENT 600 FOR UNIFORM PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.13670506E 02
XMV= 0.55528851E 01
CB= 0.12387120E 02
CB1= 0.11886042E 02

PMV= 0.62628784E 01
PCB= 0.15467634E 01
PCB1= 0.17153063E 01
RMS= 0.24697322E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.92999 0.97774	0.92077 0.97387	0.88818
1.99999	0.84831 0.92946	0.82956 0.91759	0.76258
2.49998	0.76022 0.86439	0.73278 0.84263	0.63361
2.99997	0.67009 0.78847	0.63560 0.75657	0.50946
3.49997	0.58147 0.70659	0.54201 0.66554	0.39612
3.99996	0.49708 0.62282	0.45487 0.57446	0.29741
4.49995	0.41887 0.54045	0.37604 0.48714	0.21518
4.99994	0.34811 0.46201	0.30649 0.40623	0.14961
5.49993	0.28545 0.38933	0.24647 0.33344	0.09960
5.99993	0.23103 0.32356	0.19572 0.26962	0.06320
6.49992	0.18462 0.26531	0.15358 0.21493	0.03800
6.99991	0.14571 0.21473	0.11917 0.16903	0.02149
7.49990	0.11362 0.17160	0.09151 0.13124	0.01132

STHETA= 0.19930634E 02
SXMV= 0.20112122E 02
SCB= 0.20082001E 02
SCB1= 0.18140717E 02

TABLE VI

RESULTS OF EXPERIMENT 1 FOR EXPONENTIAL PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.62774811E 02 PMV= 0.54331273E-02
XMV= 0.69654388E 02 PCB= 0.19221543E-03
CB= 0.61629608E 02 PCB1= 0.13924311E-02
CB1= 0.56364273E 02 RMS= 0.35378415E-01

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.98432 0.99511	0.98295 0.99417	0.98855
1.99999	0.96481 0.98420	0.96179 0.98119	0.97421
2.49998	0.94204 0.96876	0.93720 0.96288	0.95735
2.99997	0.91651 0.94956	0.90971 0.94022	0.93827
3.49997	0.88863 0.92716	0.87984 0.91395	0.91723
3.99996	0.85880 0.90203	0.84803 0.88467	0.89445
4.49995	0.82737 0.87459	0.81470 0.85295	0.87017
4.99994	0.79469 0.84522	0.78024 0.81929	0.84458
5.49993	0.76108 0.81430	0.74499 0.78416	0.81787
5.99993	0.72682 0.78214	0.70929 0.74799	0.79024
6.49992	0.69218 0.74905	0.67342 0.71117	0.76186
6.99991	0.65741 0.71533	0.63766 0.67404	0.73289
7.49990	0.62274 0.68124	0.60223 0.63694	0.70350

TABLE VII

RESULTS OF EXPERIMENT 500 FOR EXPONENTIAL PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.20615129E 01   PMV= 0.49704695E 01
XMV= 0.30331583E 01   PCB= 0.23415871E 01
CB= 0.36947231E 01   PCB1= 0.73870726E 01
CB1= 0.20835247E 01   RMS= 0.47109973E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.61795 0.86131	0.74397 0.84859	0.78641
1.99999	0.33590 0.61565	0.51819 0.59269	0.57460
2.49998	0.16235 0.38045	0.34256 0.36264	0.38936
2.99997	0.07031 0.20680	0.21736 0.20165	0.24340
3.49997	0.02745 0.09995	0.13363 0.10480	0.13915
3.99996	0.00970 0.04329	0.08023 0.05203	0.07182

TABLE VIII

RESULTS OF EXPERIMENT 1000 FOR EXPONENTIAL PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.17963058E 02	PMV= 0.98944416E 01
XMV= 0.14913827E 02	PCB= 0.48502378E 01
CB= 0.16705353E 02	PCB1= 0.15059362E 02
CB1= 0.13408147E 02	RMS= 0.49019819E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000	1.00000	1.00000
	1.00000	1.00000	
1.50000	0.94626	0.93685	0.95693
	0.98301	0.97498	
1.99999	0.88232	0.86304	0.90475
	0.94585	0.92117	
2.49998	0.81169	0.78322	0.84584
	0.89502	0.84962	
2.99997	0.73736	0.70121	0.78227
	0.83455	0.76762	
3.49997	0.66190	0.62005	0.71592
	0.76774	0.68099	
3.99996	0.58744	0.54210	0.64843
	0.69744	0.59430	
4.49995	0.51569	0.46904	0.58127
	0.62607	0.51103	
4.99994	0.44795	0.40196	0.51569
	0.55564	0.43358	
5.49993	0.38515	0.34146	0.45273
	0.48777	0.36342	
5.99993	0.32789	0.28774	0.39325
	0.42370	0.30128	
6.49992	0.27645	0.24068	0.33787
	0.36430	0.24729	
6.99991	0.23088	0.19997	0.28705
	0.31013	0.20116	
7.49990	0.19106	0.16511	0.24106
	0.26148	0.16232	

TABLE IX

RESULTS OF EXPERIMENT 1500 FOR EXPONENTIAL PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.95379299E 00 PMV= 0.15348947E 02
XMV= 0.56882071E 00 PCB= 0.75833426E 01
CB= 0.70745683E 00 PCB1= 0.21850998E 02
CB1= 0.20020109E 00 RMS= 0.49406272E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.35332 0.72420	0.23379 0.20560	0.26046
1.99999	0.09462 0.35049	0.04917 0.01269	0.03334
2.49998	0.01966 0.12384	0.01027 0.00069	0.00125

TABLE X

RESULTS OF EXPERIMENT 2000 FOR EXPONENTIAL PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.13210494E 02   PMV= 0.20616776E 02
XMV= 0.18323151E 02   PCB= 0.10232390E 02
CB= 0.20108948E 02   PCB1= 0.29812485E 02
CB1= 0.16015976E 02   RMS= 0.49631375E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.92764 0.97697	0.94738 0.97906	0.96066
1.99999	0.84346 0.92710	0.88504 0.93369	0.91281
2.49998	0.75299 0.86001	0.81646 0.87262	0.85853
2.99997	0.66081 0.78197	0.74463 0.80147	0.79961
3.49997	0.57059 0.69810	0.67201 0.72479	0.73770
3.99996	0.48512 0.61264	0.60060 0.64633	0.67425
4.49995	0.40637 0.52900	0.53197 0.56905	0.61057
4.99994	0.33555 0.44976	0.46726 0.49518	0.54779
5.49993	0.27325 0.37675	0.40725 0.42633	0.48690
5.99993	0.21953 0.31109	0.35240 0.36346	0.42870
6.49992	0.17407 0.25333	0.30291 0.30709	0.37383
6.99991	0.13626 0.20353	0.25876 0.25734	0.32277
7.49990	0.10533 0.16138	0.21978 0.21403	0.27588

STHETA= 0.19852600E 02
SXMV= 0.19973343E 02
SCB= 0.19885345E 02
SCB1= 0.16663483E 02

TABLE XI

RESULTS OF EXPERIMENT 1 FOR INVERTED GAMMA PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.58990660E 01	PMV= 0.60961861E-01
XMV= 0.12011658E 02	PCB= 0.33924337E-01
CB= 0.12737870E 02	PCB1= 0.13713542E-01
CB1= 0.97822771E 01	RMS= 0.55648458E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.84517 0.94916	0.91882 0.96631	0.94006
1.99999	0.68301 0.84407	0.82607 0.89495	0.86874
2.49998	0.52977 0.71339	0.72851 0.80242	0.79000
2.99997	0.39544 0.57651	0.63142 0.69981	0.70728
3.49997	0.28465 0.44716	0.53871 0.59565	0.62355
3.99996	0.19792 0.33378	0.45308 0.49611	0.54136
4.49995	0.13311 0.24027	0.37612 0.40525	0.46275
4.99994	0.08669 0.16706	0.30854 0.32532	0.38935
5.49993	0.05473 0.11236	0.25039 0.25711	0.32229
5.99993	0.03352 0.07318	0.20121 0.20038	0.26231
6.49992	0.01994 0.04620	0.16026 0.15423	0.20976
6.99991	0.01152 0.02830	0.12663 0.11740	0.16466
7.49990	0.00647 0.01683	0.09934 0.08850	0.12673

TABLE XII

RESULTS OF EXPERIMENT 1000 FOR INVERTED GAMMA PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.28008835E 02 PMV= 0.10281204E 02
XMV= 0.16734589E 02 PCB= 0.50544624E 01
CB= 0.17031448E 02 PCB1= 0.71642094E 01
CB1= 0.13422120E 02 RMS= 0.49162161E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.96519 0.98907	0.93859 0.97533	0.95627
1.99999	0.92284 0.96493	0.86659 0.92221	0.90333
2.49998	0.87476 0.93134	0.78846 0.85144	0.84361
2.99997	0.82251 0.89048	0.70785 0.77013	0.77924
3.49997	0.76748 0.84408	0.62776 0.68400	0.71212
3.99996	0.71094 0.79366	0.55047 0.59757	0.64395
4.49995	0.65395 0.74057	0.47771 0.51431	0.57622
4.99994	0.59748 0.68600	0.41060 0.43664	0.51019
5.49993	0.54232 0.63102	0.34979 0.36610	0.44692
5.99993	0.48912 0.57653	0.29554 0.30347	0.38726
6.49992	0.43841 0.52330	0.24783 0.24895	0.33185
6.99991	0.39059 0.47197	0.20637 0.20230	0.28112
7.49990	0.34593 0.42304	0.17074 0.16298	0.23533

TABLE XIII

RESULTS OF EXPERIMENT 2000 FOR INVERTED GAMMA PRIOR PROBABILITY DENSITY FUNCTION

THETA=	0.33476608E 02	PMV=	0.19932480E 02
XMV=	0.21887573E 02	PCB=	0.95528460E 01
CB=	0.21715973E 02	PCB1=	0.13805236E 02
CB1=	0.18015991E 02	RMS=	0.47926027E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.97079 0.99085	0.95148 0.98156	0.96836
1.99999	0.93503 0.97057	0.89365 0.94139	0.92957
2.49998	0.89409 0.94222	0.82961 0.88690	0.88512
2.99997	0.84918 0.90751	0.76199 0.82273	0.83630
3.49997	0.80139 0.86777	0.69303 0.75271	0.78429
3.99996	0.75168 0.82419	0.62458 0.68000	0.73017
4.49995	0.70093 0.77780	0.55812 0.60723	0.67489
4.99994	0.64991 0.72956	0.49479 0.53648	0.61936
5.49993	0.59932 0.68030	0.43541 0.46930	0.56434
5.99993	0.54972 0.63079	0.38051 0.40679	0.51052
6.49992	0.50162 0.58168	0.33038 0.34961	0.45850
6.99991	0.45541 0.53355	0.28514 0.29811	0.40876
7.49990	0.41142 0.48686	0.24470 0.25235	0.36170

TABLE XIV

RESULTS OF EXPERIMENT 3000 FOR INVERTED GAMMA PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.10608020E 02　　PMV= 0.29934708E 02
XMV= 0.49759150E 01　　PCB= 0.14144602E 02
CB= 0.63417406E 01　　PCB1= 0.20467728E 02
CB1= 0.48877382E 01　　RMS= 0.47251511E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.91070 0.97140	0.84410 0.93380	0.87690
1.99999	0.80896 0.91004	0.68335 0.80162	0.74043
2.49998	0.70237 0.82877	0.53368 0.64623	0.60258
2.99997	0.59695 0.73619	0.40400 0.49457	0.47254
3.49997	0.49721 0.63918	0.29768 0.36224	0.35666
3.99996	0.40624 0.54325	0.21427 0.25560	0.25857
4.49995	0.32582 0.45249	0.15115 0.17476	0.17955
4.99994	0.25669 0.36970	0.10481 0.11635	0.11897
5.49993	0.19876 0.29651	0.07163 0.07578	0.07484
5.99993	0.15134 0.23360	0.04836 0.04847	0.04442
6.49992	0.11336 0.18088	0.03233 0.03055	0.02467
6.99991	0.08356 0.13773	0.02143 0.01903	0.01268
7.49990	0.06064 0.10316	0.01412 0.01175	0.00595

TABLE XV

RESULTS OF EXPERIMENT 4000 FOR INVERTED GAMMA PRIOR PROBABILITY DENSITY FUNCTION

THETA= 0.63712034E 01   PMV= 0.40233185E 02
XMV= 0.73502607E 01   PCB= 0.18921494E 02
CB= 0.85002365E 01   PCB1= 0.27557159E 02
CB1= 0.65556736E 01   RMS= 0.47029567E 00

TIME	TREL TREL1	RECB RECB1	RELMV
1.00000	1.00000 1.00000	1.00000 1.00000	1.00000
1.50000	0.85578 0.95284	0.88103 0.95018	0.91422
1.99999	0.70258 0.85474	0.75187 0.84768	0.81500
2.49998	0.55530 0.73147	0.62401 0.72107	0.70926
2.99997	0.42358 0.60053	0.50531 0.58930	0.60279
3.49997	0.31243 0.47464	0.40034 0.46527	0.50027
3.99996	0.22316 0.36206	0.31109 0.35648	0.40524
4.49995	0.15456 0.26705	0.23762 0.26606	0.32014
4.99994	0.10392 0.19075	0.17876 0.19412	0.24636
5.49993	0.06788 0.13211	0.13270 0.13886	0.18439
5.99993	0.04311 0.08882	0.09737 0.09767	0.13397
6.49992	0.02665 0.05802	0.07072 0.06772	0.09427
6.99991	0.01604 0.03685	0.05093 0.04638	0.06405
7.49990	0.00940 0.02278	0.03640 0.03145	0.04188

STHETA= 0.19360504E 02
SXMV= 0.19039551E 02
SCB= 0.19126251E 02
SCB1= 0.16034225E 02

4.4 CONCLUSION

Whenever evidence exists that the parameter θ in the underlying life testing model behaves as a random variable, this calls for knowing or deriving the distribution of Θ, say $g(\theta)$ and using it in conjunction with Bayes theorem.

If a priori information about θ is available then, we would expect the squared error loss to be decreased by using a Bayes estimate of reliability as opposed to a minimum variance unbiased estimate. This can be seen in the preceeding tables upon examining the RMS values, which is the ratio of the accumulated average square error loss of the Bayes estimate to that of the minimum variance unbiased estimate. Furthermore, as the number of experiments increase, we notice that for each prior distribution the RMS converges to some number less than one. This is consistent in estimation as we will approach the true state of nature as the number of experiments increase.

For the value of the shape parameter ρ used the Bayes estimates of exact reliability are closer than the corresponding MVUE, as expected. Also, the estimates using the Weibull probability density function with guaranteed time τ are consistently higher than the truncated estimates.

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APPENDIX A

PROGRAM FOR UNIFORM PRIOR PROBABILITY DENSITY FUNCTION

```
      DIMENSION TREL(300),TREL1(300),RECB(300),
     1RECB1(300),RELMV(300),TIM(300),Z(10),ITAB(5)
      DATA       KR,NUM,SSR,STHETA,SXMV,SCB,SSR1,
     1SCB1,SEB,SEMV,PCB,PMV,SEB1,PCB1,KOUNT/
     2  10,600,12*0.,0    /
      DATA  IX,GAM,C /  653, 1.7, 1.    /
      DATA ITAB  / ' I  ',' II ',' III',' IV ',' V   '  /
      KCT= 1
      R=KR
      KCON= KR-1
      KCON1= 1-KR
      BIP=NUM
      TAU=C
      RXM=1./GAM
      CGAM=C**GAM
      ALPHA=10.
      BETA=30.
      DIF= BETA - ALPHA
      DO 1 JN=1,NUM
C     GENERATE UNIFORM RANDOM NUMBER (0,1)
      CALL RANDU(IX,IY,YFL)
      IX=IY
C     CALCULATE VALUE OF THETA FROM UNIFORM PRIOR
      THETA=ALPHA + YFL*DIF
      STHETA = STHETA + THETA
      DO 2 I=1,KR
C     GENERATE UNIFORM RANDOM NUMBER (0,1)
      CALL RANDU(IX,IY,YFL)
      IX=IY
C     COMPUTE VALUES OF LIFETIMES FROM TRUNCATED
C     WEIBULL DISTRIBUTION
    2 Z(I)=(CGAM - THETA * ALOG(1. - YFL) ) ** RXM
      SR=0.
      SR1=0.
      DO 3 I=1,KR
      SR= SR + (Z(I)** GAM - CGAM)
    3 SR1=SR1 +( Z(I) - TAU) ** GAM
      SSR=SSR+SR
      SSR1=SSR1 + SR1
C     CALCULATE MVUE OF THETA
      XMV=SR / R
      SXMV=SXMV + XMV
      KJ=1
      CALL INTEGR(SR ,ALPHA,BETA,SNUM ,SDEN ,KJ,KR)
      CALL INTEGR(SR1,ALPHA,BETA,SNUM2,SDEN2,KJ,KR)
C     CALCULATE BAYES ESTIMATE OF THETA
```

```
      CB =SR  * SNUM  / SDEN
      CB1=SR1 * SNUM2 / SDEN2
      SCB =SCB + CB
      SCB1=SCB1 + CB1
C     COMPUTE SQUARE ERROR LOSS FOR ESTIMATES OF THETA
      SEB =SEB  + (THETA - CB) **2
      SEB1=SEB1 + (THETA - CB1) **2
      SEMV=SEMV + (THETA - XMV) **2
C     GENERATE UNIFORM RANDOM NUMBER (0,1)
      CALL RANDU(IX,IY,YFL)
      IX=IY
      KDEL= 10.*YFL + 1.
C     RANDOMLY DELETE ONE LIFETIME
      Z(KDEL)=0.
      CALL ORDER(Z ,KR,ZMAX )
      TR1=  ZMAX** GAM - CGAM
      SRTR= SR + TR1
      XMSMV=0.
      XMSCB=0.
      XMSCB1=0.
      TIME= TAU
      L=0
    6 L=L+1
      TIM(L)=TIME
      IF(L.EQ.1) GO TO 7
      TRELN= TIME**GAM - CGAM
      TRELN1= (TIME - TAU) ** GAM
C     COMPUTE TRUE RELIABILITY AND MVUE OF RELIABILITY
      TREL(L)= EXP( -TRELN / THETA)
      TREL1(L)= EXP( -TRELN1 / THETA)
      RELMV(L)= (1.- TRELN / SRTR) ** KCON
C     CALCULATE BAYES ESTIMATE OF RELIABILITY
      T1=SR + TRELN
      T11= SR1 + TRELN1
      T2= (1. + TRELN / SR) ** KCON1
      T21= (1. + TRELN1 / SR1) ** KCON1
      KJ=2
      CALL INTEGR(T1 ,ALPHA,BETA,DUMMY,SNUM1 ,KJ,KR)
      CALL INTEGR(T11,ALPHA,BETA,DUMMY,SNUM11,KJ,KR)
      RECB(L)= T2 * SNUM1 / SDEN
      RECB1(L)= T21 * SNUM11 / SDEN2
       GO TO 8
    7 TREL (L)=1.
      TREL1(L)=1.
      RELMV (L)=1.
      RECB (L)=1.
```

```
      RECB1(L)=1.
C     COMPUTE SQUARE ERROR LOSS FOR ESTIMATES
C     OF RELIABILITY
    8 XMSMV= XMSMV + (TREL(L) - RELMV(L))**2
      XMSCB= XMSCB + (TREL(L) - RECB(L))**2
      XMSCB1= XMSCB1 + (TREL1(L) -RECB1(L))**2
      IF(TREL(L).LT. .005 .OR. TIME .GE. 20.) GO TO 9
      TIME = ( 1.0E 60 * TIME + 1.0E 59 ) / 1.0E 60
      GO TO 6
    9 AL=L
      XMSMV= XMSMV / AL
      XMSCB= XMSCB / AL
      XMSCB1= XMSCB1 / AL
      PCB= PCB + XMSCB
      PCB1= PCB1 + XMSCB1
      PMV= PMV + XMSMV
      RMS= PCB / PMV
      IF( JN.EQ.1 .OR. JN.EQ. KOUNT) GO TO 10
      GO TO 1
   10 WRITE(6,110) ITAB(KCT) , JN
  110 FORMAT ('1'///// 33X,'TABLE',A4//17X,' RESULTS ',
     1'OF EXPERIMENT',I4,' FOR UNIFORM'/19X,' PRIOR ',
     2'PROBABILITY DENSITY FUNCTION'//)
      WRITE(6,100) THETA,PMV,XMV,PCB,SEMV,CB,PCB1,
     1SR,SEB,CB1,RMS,SR1,SEB1
  100 FORMAT(14X,'THETA=',E15.8,5X,'PMV=',E15.8/16X,
     1'XMV=',E15.8,5X,'PCB=',E15.8,51X,'SEMV=',
     2E15.8/17X,'CB=',E15.8,4X,'PCB1=',E15.8,29X,'SR=',
     3E15.8,5X,'SEB=',E15.8/16X,'CB1=',E15.8,5X,
     4'RMS=',E15.8,28X,'SR1=',E15.8,4X,'SEB1=',E15.8)
      I67=67
      IF(L .LT. 67 ) I67 = L
      WRITE(6,120) (TIM(I),TREL(I),RECB(I),RELMV(I),
     1TREL1(I),RECB1(I),I=1,I67,5)
  120 FORMAT (/14X,45('-')/
     1          15X,'TIME', 8X,'TREL', 8X,'RECB', 8X,
     2'RELMV'/27X,'TREL1', 7X,'RECB1'/14X,45('-')/(10X,
     3 4F12.5/22X,2F12.5))
      WRITE(6,140)
  140 FORMAT(14X,45('-'))
      KOUNT = KOUNT + 150
      KCT= KCT + 1
    1 CONTINUE
      SSR= SSR / BIP
      SSR1= SSR1 / BIP
      STHETA= STHETA / BIP
```

```
      SXMV= SXMV / BIP
      SCB= SCB / BIP
      SCB1= SCB1 / BIP
      WRITE(6,130) STHETA,SXMV,SCB,SSR,SCB1,SSR1
  130 FORMAT(//26X,'STHETA=',E15.8/28X,'SXMV=',E15.8/
     129X,'SCB=',E15.8,40X,'SSR=',E15.8/28X,'SCB1=',
     2E15.8,39X,'SSR1=',E15.8)
      STOP
      END


      SUBROUTINE ORDER(Z,KR,ZMAX)
C     FINDS LAST ORDER STATISTIC OF THE SAMPLE Z
      DIMENSION Z(10)
      ZMAX=Z(1)
      DO 1 I=2,KR
    1 ZMAX=AMAX1(ZMAX,Z(I))
      RETURN
      END


      SUBROUTINE INTEGR(SR,ALPHA,BETA,SNUM,SDEN,KJ,KR)
C     INTEGRATION OF THE INCOMPLETE GAMMA FUNCTION
C     USING THE TRAPEZOIDAL RULE
      X= SR / BETA
      XN= 45.
      H=     (SR / ALPHA - X)/ XN
      NX=XN -1.
      SNUM=0.
      SDEN=0.
      KR2= KR - 2
      KR3= KR - 3
      DO 1 I=1,NX
      XI=I
      IF (KJ.EQ. 2) GO TO 1
      SNUM=SNUM + 2.*EXP(-(X+ H*XI))*(X+ H*XI) ** KR3
    1 SDEN=SDEN + 2.*EXP(-(X+ H*XI))*(X+ H*XI) ** KR2
      IF (KJ.EQ. 2) GO TO 2
      SNUM=.5*H*(SNUM + EXP(-X)*X ** KR3    +
     1 EXP(-(X+ H*XN))*(X+ H*XN) ** KR3   )
    2 SDEN=.5*H*(SDEN + EXP(-X)*X ** KR2    +
     1 EXP(-(X+ H*XN))*(X+ H*XN) ** KR2   )
      RETURN
      END
```

APPENDIX B

PROGRAM FOR EXPONENTIAL PRIOR PROBABILITY DENSITY FUNCTION

```
      DIMENSION TREL(300),TREL1(300),RECB(300),
     1RECB1(300),RELMV(300),TIM(300),Z(10),ITAB(5)
      DATA        KR,NUM,SSR,STHETA,SXMV,SCB,SSR1,
     1SCB1,SEB,SEMV,PCB,PMV,SEB1,PCB1,KOUNT/
     2 10,2000,12*0.,0    /
      DATA  IX,GAM,C /5437819,  1.7, 1.0    /
      DATA ITAB  / 'VI  ','VII ','VIII','IX  ','X   '  /
      KCT= 1
      R=KR
      KCON= KR-1
      BIP=NUM
      TAU=C
      RXM=1./GAM
      CGAM=C**GAM
      CLM=20.
      CCON= (1.-R)/2.
      N1= KR-1
      N2= KR-2
      DO 1 JN=1,NUM
C     GENERATE UNIFORM RANDOM NUMBER (0,1)
      CALL RANDU(IX,IY,YFL)
      IX=IY
C     CALCULATE VALUE OF THETA FROM EXPONENTIAL PRIOR
      THETA= -CLM* ALOG(1.-YFL)
      STHETA = STHETA + THETA
      DO 2 I=1,KR
C     GENERATE UNIFORM RANDOM NUMBER (0,1)
      CALL RANDU(IX,IY,YFL)
      IX=IY
C     COMPUTE VALUES OF LIFETIMES FROM TRUNCATED
C     WEIBULL DISTRIBUTION
    2 Z(I)=(CGAM - THETA * ALOG(1. - YFL) ) ** RXM
      SR=0.
      SR1=0.
      DO 3 I=1,KR
      SR= SR + (Z(I)** GAM - CGAM)
    3 SR1=SR1 +( Z(I) - TAU) ** GAM
      SSR=SSR+SR
      SSR1=SSR1 + SR1
C     CALCULATE MVUE OF THETA
      XMV=SR / R
      SXMV=SXMV + XMV
      ARG = 2.*SQRT( SR / CLM )
      ARG1= 2.*SQRT( SR1 / CLM )
      CALL BESK(ARG ,N2, BNUM ,IER)
      IF(IER .NE. 0) GO TO 999
```

```
      CALL BESK(ARG ,N1, BDEN ,IER)
      IF(IER .NE. 0) GO TO 999
      CALL BESK(ARG1,N2, BNUM1,IER)
      IF(IER .NE. 0) GO TO 999
      CALL BESK(ARG1,N1, BDEN1,IER)
      IF(IER .NE. 0) GO TO 999
C     CALCULATE BAYES ESTIMATE OF THETA
      CB= SQRT( CLM*SR ) * BNUM / BDEN
      CB1= SQRT( CLM * SR1 ) * BNUM1 / BDEN1
      SCB =SCB + CB
      SCB1=SCB1 + CB1
C     COMPUTE SQUARE ERROR LOSS FOR ESTIMATES OF THETA
      SEB =SEB  + (THETA - CB) **2
      SEB1=SEB1 + (THETA - CB1) **2
      SEMV=SEMV + (THETA - XMV) **2
C     GENERATE UNIFORM RANDOM NUMBER (0,1)
      CALL RANDU(IX,IY,YFL)
      IX=IY
      KDEL= 10.*YFL + 1.
C     RANDOMLY DELETE ONE LIFETIME
      Z(KDEL)=0.
      CALL ORDER(Z ,KR,ZMAX )
      TR1=  ZMAX** GAM - CGAM
      SRTR= SR + TR1
      XMSMV=0.
      XMSCB=0.
      XMSCB1=0.
      TIME= TAU
      L=0
    6 L=L+1
      TIM(L)=TIME
      IF(L.EQ.1) GO TO 7
      TRELN= TIME**GAM - CGAM
      TRELN1= (TIME - TAU) ** GAM
C     COMPUTE TRUE RELIABILITY AND MVUE OF RELIABILITY
      TREL(L)= EXP( -TRELN / THETA)
      TREL1(L)= EXP( -TRELN1 / THETA)
      RELMV(L)= (1.- TRELN / SRTR) ** KCON
C     CALCULATE BAYES ESTIMATE OF RELIABILITY
      T1=SR + TRELN
      T11= SR1 + TRELN1
      ARG2 = 2. * SQRT(T1 / CLM )
      ARG21= 2. * SQRT(T11 / CLM)
      CALL BESK(ARG2 ,N1,BNUM2,IER)
      IF(IER .NE. 0) GO TO 999
      CALL BESK(ARG21,N1,BNUM21,IER)
```

```
      IF(IER .NE. 0) GO TO 999
      T2=(1. + TRELN / SR) ** CCON
      T21= (1. + TRELN1 /SR1) ** CCON
      RECB(L)= T2 * BNUM2 / BDEN
      RECB1(L)= T21 * BNUM21 / BDEN1
       GO TO 8
    7 TREL (L)=1.
      TREL1(L)=1.
      RELMV (L)=1.
      RECB (L)=1.
      RECB1(L)=1.
C     COMPUTE SQUARE ERROR LOSS FOR ESTIMATES
C     OF RELIABILITY
    8 XMSMV= XMSMV + (TREL(L) - RELMV(L))**2
      XMSCB= XMSCB + (TREL(L) - RECB(L))**2
      XMSCB1= XMSCB1 + (TREL1(L) -RECB1(L))**2
      IF(TREL(L).LT. .005 .OR. TIME .GE. 20.) GO TO 9
      TIME = ( 1.0E 60 * TIME + 1.0E 59 ) / 1.0E 60
      GO TO 6
    9 AL=L
      XMSMV= XMSMV / AL
      XMSCB= XMSCB / AL
      XMSCB1= XMSCB1 / AL
      PCB= PCB + XMSCB
      PCB1= PCB1 + XMSCB1
      PMV= PMV + XMSMV
      RMS= PCB / PMV
      IF( JN.EQ.1 .OR. JN.EQ. KOUNT) GO TO 10
      GO TO 1
   10 WRITE(6,110) ITAB(KCT) , JN
  110 FORMAT ('1'///// 32X,'TABLE ',A4//14X,' RESULTS',
     1' OF EXPERIMENT',I5,' FOR EXPONENTIAL'/18X,
     2' PRIOR PROBABILITY DENSITY FUNCTION'//)
      WRITE(6,100) THETA,PMV,XMV,PCB,SEMV,CB,PCB1,
     1SR,SEB,CB1,RMS,SR1,SEB1
  100 FORMAT(14X,'THETA=',E15.8,5X,'PMV=',E15.8/16X,
     1'XMV=',E15.8,5X,'PCB=',E15.8,51X,'SEMV=',
     2E15.8/17X,'CB=',E15.8,4X,'PCB1=',E15.8,29X,'SR=',
     3E15.8,5X,'SEB=',E15.8/16X,'CB1=',E15.8,5X,
     4'RMS=',E15.8,28X,'SR1=',E15.8,4X,'SEB1=',E15.8)
      I67=67
      IF(L .LT. 67 ) I67 = L
      WRITE(6,120) (TIM(I),TREL(I),RECB(I),RELMV(I),
     1TREL1(I),RECB1(I),I=1,I67,5)
  120 FORMAT (/14X,45('-')/
     1          15X,'TIME', 8X,'TREL', 8X,'RECB', 8X,
```

```
     2'RELMV'/27X,'TREL1', 7X,'RECB1'/14X,45('-')/(10X,
     3 4F12.5/22X,2F12.5))
      WRITE(6,150)
  150 FORMAT(14X,45('-'))
      KOUNT = KOUNT + 500
      KCT= KCT + 1
    1 CONTINUE
      SSR= SSR / BIP
      SSR1= SSR1 / BIP
      STHETA= STHETA / BIP
      SXMV= SXMV / BIP
      SCB= SCB / BIP
      SCB1= SCB1 / BIP
      WRITE(6,130) STHETA,SXMV,SCB,SSR,SCB1,SSR1
  130 FORMAT(//26X,'STHETA=',E15.8/28X,'SXMV=',E15.8/
     129X,'SCB=',E15.8,40X,'SSR=',E15.8/28X,'SCB1=',
     2E15.8,39X,'SSR1=',E15.8)
      GO TO 25
  999 WRITE(6,140) IER
  140 FORMAT (/' IER=',I2)
   25 STOP
      END


      SUBROUTINE ORDER(Z,KR,ZMAX)
C     FINDS LAST ORDER STATISTIC OF THE SAMPLE Z
      DIMENSION Z(10)
      ZMAX=Z(1)
      DO 1 I=2,KR
    1 ZMAX=AMAX1(ZMAX,Z(I))
      RETURN
      END
```

APPENDIX C

PROGRAM FOR INVERTED GAMMA PRIOR PROBABILITY DENSITY FUNCTION

```
      DIMENSION TREL(300),TREL1(300),RECB(300),
     1RECB1(300),RELMV(300),TIM(300),Z(10),ITAB(5)
      COMMON YFL,CMU
      EXTERNAL BNOT
      DATA       KR,NUM,SSR,STHETA,SXMV,SCB,SSR1,
     1SCB1,SEB,SEMV,PCB,PMV,SEB1,PCB1,KOUNT/
     2 10,4000,12*0.,0   /
      DATA  IX,GAM,C /365279,  1.7 , 1.0   /
      DATA ITAB  / 'XI  ','XII ','XIII','XIV ','XV  '  /
      KCT= 1
      R=KR
      KCON= KR-1
      BIP=NUM
      TAU=C
      RXM=1./GAM
      CGAM=C**GAM
      AA=.2
      E1=.1E-4
      E2=.1E-8
      ABP=1300.
      DEL=.5
      CMU=20.
      CLM=2.
      KR1= KR + IFIX(CLM)
      M=50
      CCON= R + CLM -1.
      DO 1 JN=1,NUM
C     GENERATE UNIFORM RANDOM NUMBER (0,1)
    5 CALL RANDU(IX,IY,YFL)
      IX=IY
C     COMPUTE VALUE OF THETA FROM INVERTED GAMMA PRIOR
      CALL ITR2(YTEM,AA,ABP,DEL,BNOT,E1,E2,M,ICODE)
      IF(ICODE.NE.0) GO TO 5
      THETA=YTEM
      STHETA = STHETA + THETA
      DO 2 I=1,KR
C     GENERATE UNIFORM RANDOM NUMBER (0,1)
      CALL RANDU(IX,IY,YFL)
      IX=IY
C     COMPUTE VALUES OF LIFETIMES FROM TRUNCATED
C     WEIBULL DISTRIBUTION
    2 Z(I)=(CGAM - THETA * ALOG(1. - YFL) ) ** RXM
      SR=0.
      SR1=0.
      DO 3 I=1,KR
      SR= SR + (Z(I)** GAM - CGAM)
```

```
    3 SR1=SR1 +( Z(I) - TAU) ** GAM
      SSR=SSR+SR
      SSR1=SSR1 + SR1
C     CALCULATE MVUE OF THETA
      XMV=SR / R
      SXMV=SXMV + XMV
      XZ = SR + CMU
      XZ1= SR1 + CMU
C     CALCULATE BAYES ESTIMATE OF THETA
      CB= XZ / CCON
      CB1= XZ1 / CCON
      SCB =SCB + CB
      SCB1=SCB1 + CB1
C     COMPUTE SQUARE ERROR LOSS FOR ESTIMATES OF THETA
      SEB =SEB  + (THETA - CB) **2
      SEB1=SEB1 + (THETA - CB1) **2
      SEMV=SEMV + (THETA - XMV) **2
C     GENERATE UNIFORM RANDOM NUMBER (0,1)
      CALL RANDU(IX,IY,YFL)
      IX=IY
      KDEL= 10.*YFL + 1.
C     RANDOMLY DELETE ONE LIFETIME
      Z(KDEL)=0.
      CALL ORDER(Z ,KR,ZMAX )
      TR1=  ZMAX** GAM - CGAM
      SRTR= SR + TR1
      XMSMV=0.
      XMSCB=0.
      XMSCB1=0.
      TIME= TAU
      L=0
    6 L=L+1
      TIM(L)=TIME
      IF(L.EQ.1) GO TO 7
      TRELN= TIME**GAM - CGAM
      TRELN1= (TIME - TAU) ** GAM
C     COMPUTE TRUE RELIABILITY AND MVUE OF RELIABILITY
      TREL(L)= EXP( -TRELN / THETA)
      TREL1(L)= EXP( -TRELN1 / THETA)
      RELMV(L)= (1.- TRELN / SRTR) ** KCON
C     CALCULATE BAYES ESTIMATE OF RELIABILITY
      RECB(L)= 1. / (1. + TRELN / XZ) **  KR1
      RECB1(L)= 1. /(1. + TRELN1 /XZ1) ** KR1
       GO TO 8
    7 TREL (L)=1.
      TREL1(L)=1.
```

```
      RELMV (L)=1.
      RECB (L)=1.
      RECB1(L)=1.
C     COMPUTE SQUARE ERROR LOSS FOR ESTIMATES
C     OF RELIABILITY
    8 XMSMV= XMSMV + (TREL(L) - RELMV(L))**2
      XMSCB= XMSCB + (TREL(L) - RECB(L))**2
      XMSCB1= XMSCB1 + (TREL1(L) -RECB1(L))**2
      IF(TREL(L).LT. .005 .OR. TIME .GE. 20.) GO TO 9
      TIME = ( 1.0E 60 * TIME + 1.0E 59 ) / 1.0E 60
      GO TO 6
    9 AL=L
      XMSMV= XMSMV / AL
      XMSCB= XMSCB / AL
      XMSCB1= XMSCB1 / AL
      PCB= PCB + XMSCB
      PCB1= PCB1 + XMSCB1
      PMV= PMV + XMSMV
      RMS= PCB / PMV
      IF( JN.EQ.1 .OR. JN.EQ. KOUNT) GO TO 10
      GO TO 1
   10 WRITE(6,110) ITAB(KCT) , JN
  110 FORMAT ('1'///// 33X,'TABLE ',A4//16X,' RESULTS',
     1' OF EXPERIMENT',I5,' FOR INVERTED '/16X,
     2' GAMMA PRIOR PROBABILITY DENSITY FUNCTION'//)
      WRITE(6,100) THETA,PMV,XMV,PCB,SEMV,CB,PCB1,
     1SR,SEB,CB1,RMS,SR1,SEB1
  100 FORMAT(14X,'THETA=',E15.8,5X,'PMV=',E15.8/16X,
     1'XMV=',E15.8,5X,'PCB=',E15.8,51X,'SEMV=',
     2E15.8/17X,'CB=',E15.8,4X,'PCB1=',E15.8,29X,'SR=',
     3E15.8,5X,'SEB=',E15.8/16X,'CB1=',E15.8,5X,
     4'RMS=',E15.8,28X,'SR1=',E15.8,4X,'SEB1=',E15.8)
      I67=67
      IF(L .LT. 67 ) I67 = L
      WRITE(6,120) (TIM(I),TREL(I),RECB(I),RELMV(I),
     1TREL1(I),RECB1(I),I=1,I67,5)
  120 FORMAT (/14X,45('-')/
     1         15X,'TIME', 8X,'TREL', 8X,'RECB', 8X,
     2'RELMV'/27X,'TREL1', 7X,'RECB1'/14X,45('-')/(10X,
     3 4F12.5/22X,2F12.5))
      WRITE(6,140)
  140 FORMAT(14X,45('-'))
      KOUNT = KOUNT + 1000
      KCT= KCT + 1
    1 CONTINUE
      SSR= SSR / BIP
```

```
      SSR1= SSR1 / BIP
      STHETA= STHETA / BIP
      SXMV= SXMV / BIP
      SCB= SCB / BIP
      SCB1= SCB1 / BIP
      WRITE(6,130) STHETA,SXMV,SCB,SSR,SCB1,SSR1
  130 FORMAT(//26X,'STHETA=',E15.8/28X,'SXMV=',E15.8/
     129X,'SCB=',E15.8,40X,'SSR=',E15.8/28X,'SCB1=',
     2E15.8,39X,'SSR1=',E15.8)
      STOP
      END


      SUBROUTINE ITR2 (X,A,B,DELTX,FOFX,
     1                 E1,E2,MAXI,ICODE)
      X=A
      KX=0
      LX=0
      IF (DELTX)111,111,112
  112 IF (B- A ) 113,113,114
  114 I =0
      IF (FOFX(A))1,2,3
    1 XB1=X
      IF(LX.NE.0)GO TO 1001
      X =X+DELTX
      IF(X-B)1000,1000,1004
 1004 X=B
      LX=1
 1000 IF (FOFX(X))1,2,4
    4 XB=X
      X=X-DELTX/(2.**(I+1))
  999 I=I+1
      IF(MAXI.LT.I)GO TO 444
      IF (FOFX(X))6,2,7
    6 L=1
      XX=XB
      GO TO 18
    7 L=2
      XX=XB1
      GO TO 18
    3 XB1=X
      IF(KX.NE.0)GO TO 1001
      X= X+DELTX
      IF(X-B)1002,1002,1003
 1003 X=B
      KX=1
 1002 IF(FOFX(X))5,2,3
```

```
    5 XB=X
      X=X-DELTX/(2.**(I+1))
  998 I=I+1
      IF(MAXI.LT.I)GO TO 444
      IF(FOFX(X))8,2,9
    9 L=3
      XX=XB
      GO TO 18
    8 L =4
      XX=XB1
   18 IF (ABS(X)-E1)36,36,37
   37 IF (ABS((XX-X)/X)-E1)2,2,17
   36 IF (ABS(XX-X)-E2)2,2,17
   17 GO TO (81,4,81,5),L
   81 XB1 =X
      X=X+DELTX/(2.**(I+1))
      GO TO (999,4,998,5),L
  111 ICODE =2
      GO TO 79
  113 ICODE =4
      GO TO 79
 1001 ICODE =3
      GO TO 79
  444 ICODE= 1
      GO TO 79
    2 ICODE =0
   79 CONTINUE
      RETURN
      END


      FUNCTION BNOT(YTEM)
      COMMON YFL,CMU
      BNOT=(1.+CMU/YTEM)*EXP(-CMU/YTEM)-YFL
      RETURN
      END


      SUBROUTINE ORDER(Z,KR,ZMAX)
C     FINDS LAST ORDER STATISTIC OF THE SAMPLE Z
      DIMENSION Z(10)
      ZMAX=Z(1)
      DO 1 I=2,KR
    1 ZMAX=AMAX1(ZMAX,Z(I))
      RETURN
      END
```

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BAYES AND MINIMUM VARIANCE UNBIASED ESTIMATORS OF RELIABILITY USING THE TRUNCATED WEIBULL LIFE TESTING MODEL

Thomas W. Jones

Abstract

In recent years, statistical theory has become widely used in the study of reliability. In this study, we shall consider two types of estimators, namely, the Bayes and minimum variance unbiased estimators for the scale parameter θ in the truncated Weibull life testing model when the shape parameter ρ is assumed known. In addition, we shall estimate the corresponding reliability function of the above estimators. The first type of estimator of θ and the reliability function we will consider is the Bayes estimator using the general uniform, exponential, and inverted gamma distributions as prior probability density functions for the parameter θ. We shall also derive the minimum variance unbiased estimator (MVUE).

For each prior probability density function, the Bayes estimator is compared with the MVUE by Monte Carlo simulation. Also, the Bayes estimators for a given prior density of θ are compared with the results of Canavos and Tsokos [5]. Their model was the Weibull probability density function with guaranteed time τ.