UNITS AND CLASS GROUPS OF IMAGINARY OCTIC FIELDS

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(ABSTRACT)

In this disertation class groups and unit groups of number fields with elementary Galois groups of order 4 and 8 are considered. In chapter 3 we consider bicyclic biquadratic extensions K/k and give a method for determining the structure of the 2-class group of K. In chapters 4 and 5 this method is applied to real and imaginary bicyclic biquadratic extensions of $\mathbf{Q}$. In chapter 6 a method for determining the unit group of an imaginary octic field is given. In the final chapter all imaginary octic fields of class number less than or equal to 16 or prime class number are determined.

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Chapter 1

INTRODUCTION

The algebraic integers in a number field form a ring. If K is an algebraic number field and R is its ring of integers we will say that ideals A and B of R are related if and only if $\alpha A = \beta B$ for some $\alpha, \beta \in R$. Under this equivalence relation the classes form a group known as the class group of K. The order of the class group is the class number of K. This dissertation examines class groups of number fields of degree 4 and 8 having elementary Galois group.

If k is a number field of odd class number and K/k is a bicyclic biquadratic extension, then the odd part of the class group of K is easily shown to be the direct product of the class groups of its subfields. However, the 2-class group of K is more difficult to determine. Lemmermeyer [13] and Kubota [11] give results relating the 2-class group of K to the 2-class groups of its subfields, but neither fully determine the 2-class group. In our recent work [14] we developed a method for determing the 2-class group of K when $k = \mathbf{Q}$. In chapter 2 this method is extended to any field k of odd class number. Two applications of this method are given. In chapter 3, the real bicyclic biquadratic extensions of $\mathbf{Q}$ having cyclic 2-class group are characterized. In chapter 4 it is shown that every abelian group of exponent 2 or 4 occurs as the 2-class group of some imaginary bicyclic biquadratic extension of $\mathbf{Q}$.

In chapters 5 and 6 imaginary octic fields K having elementary Galois group are considered. The class number of K is the product of the class numbers of its quadratic subfields times a unit index divided by 32. In chapter 5 a method is given for computing the unit index. In chapter 6 all octic fields K having class number less than or equal to 16 or prime class number are given. Using the technique of chapter 2 the class group of each field is computed.

Chapter 2

NOTATION

The following notation will be used for the remainder of this dissertation.

k : A number field having odd class number.

K : A bicyclic biquadratic extension of k.

K_1, K_2, K_3 : The subfields of K of degree 2 over k.

H, H_1, H_2, H_3 : The 2-Sylow subgroups of the ideal class groups of K, K_1, K_2 and K_3, respectively.

$\widehat{H}_i$: The group of quadratic character values on H_i.

$\widehat{S}$: The subgroup of $\widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3$ consisting of those character values which are consistent on each pair of H_1, H_2 and H_3.

S : The subgroup of $H_1 \times H_2 \times H_3$ with character group $\widehat{S}$.

θ : The homomorphism $H_1 \times H_2 \times H_3 \to H$ defined by $\theta(C_1, C_2, C_3) = C_1C_2C_3$.

ker : The kernel of θ.

H_0 : The image of θ.

t : The positive integer determined such that 2^t is the product of the ramification indices of all primes, including infinite primes, for the extension K/k.

t_i : The number of primes, including infinite primes, ramified in the extension K_i/k for $i = 1, 2, 3$.

R_a : The rank of $H_1 \times H_2 \times H_3$.

R_2 : The rank of H.

τ : The number of divisors of 2 in k which are totally ramified in K.

(l, q, r) : An element of $H_1 \times H_2 \times H_3$ determined by the ideal classes of prime divisors of l, q and r in K_1, K_2 and K_3, respectively.

ψ : The isomorphism from the multiplicative group $\{\pm 1\}$ to the additive group Z_2.

$\widetilde{A}$: The ideal class determined by the ideal A.

$\left(\frac{a}{b}\right)$: The Kronecker symbol using the convention $\left(\frac{b}{2}\right) = \left(\frac{2}{b}\right)$ for all odd positive integers.

$E_{\_}$: The unit group of the field $\_$.

The following notation applies only when K is an imaginary octic field of type $(2, 2, 2)$.

$k_1, \ldots, k_7$: The quadratic subfields of K with k_1, k_2 and k_3 real.

$h_1, \ldots, h_7$: The class numbers of $k_1, \ldots, k_7$, respectively.

$d_1, \ldots, d_7$: Positive squarefree integers with $k_i = \mathbf{Q}(\sqrt{d_i})$ for $i = 1, 2, 3$, $k_i = \mathbf{Q}(\sqrt{-d_i})$ for $i = 4, \ldots, 7$ and $d_1 < d_2 < d_3$.

$K_0 = \mathbf{Q}(\sqrt{d_1}, \sqrt{d_2})$: The maximal real subfield of K.

ε_i : The fundamental unit of k_i for $i = 1, 2, 3$.

r_i, s_i, a_i : Integers such that $\varepsilon_i = \frac{r_i + s_i\sqrt{d_i}}{2^{a_i}}$ with $a_i = 0$ or 1.

E^* : The subgroup of E_K generated by the units of the proper subfields of K.

W : The roots of unity in K.

W_0 : The roots of unity in $\prod_{i=1}^{7} E_{k_i}$.

Q : The index $[E_K : \prod_{i=1}^{7} E_{k_i}]$.

Q_0 : The index $[E_{K_0} : \prod_{i=1}^{3} E_{k_i}]$.

Q_1 : The index $[E_K : W E_{K_0}]$.

Q_2 : The index $[W : W_0]$.

Δ_i : The absolute value of a nontrivial principal divisor of k_i when $N\varepsilon_i = +1$. If possible take $\Delta_i = 2$. If $N\varepsilon_i = -1$ take $\Delta_i = 1$.

Δ : The semigroup generated by the principal divisors Δ_1, Δ_2 and Δ_3 modulo square factors.

D : The set $\{d_4, d_5, d_6, d_7\}$.

t' : The positive integer determined such that $2^{t'}$ is the product of the ramification indices of all rational primes for the extension $K/\mathbf{Q}$.

t_i' : The number of rational primes which ramify in the extension $k_i/\mathbf{Q}$.

w : The integer determined such that 2^w is the 2-class number of K.

We say that the prime 2 is maximally ramified in K if it ramifies in six quadratic subfields.

Chapter 3

CLASS GROUP STRUCTURE OF BICYCLIC BIQUADRATIC EXTENSIONS

The structure of the odd part of the class group of K is easily shown to be the direct product of the class groups of its subfields. While the structure of H depends on the structures of H_1, H_2 and H_3, the relation is more complicated. In this chapter we describe a method for determining H.

Theorem 1 *The homomorphism θ induces an isomorphism $\frac{S^{2^i}}{S^{2^i} \cap ker} \simeq H^{2^{i+1}}$ for any integer $i \geq 0$.*

Proof Let $(C_1^{2^i}, C_2^{2^i}, C_3^{2^i}) \in S^{2^i}$ with $(C_1, C_2, C_3) \in S$. Since the characters on C_i in $\widehat{H}_i$ are consistent with one another for $i = 1, 2, 3$, there is a prime p of k which satisfies these character values. Now p splits completely in K and has a prime divisor P_0 such that $\mathcal{P}_i = P_0 \cap K_i = P_0 P_i$ where $(p) = P_0 P_1 P_2 P_3$ in K. Note that $(\widetilde{\mathcal{P}}_1^{2^i}, \widetilde{\mathcal{P}}_2^{2^i}, \widetilde{\mathcal{P}}_3^{2^i}) \in S^{2^i}$ with $\widetilde{\mathcal{P}}_i$ and C_i being in the same genus of K_i. Now

$$\theta(\widetilde{\mathcal{P}}_1^{2^i}, \widetilde{\mathcal{P}}_2^{2^i}, \widetilde{\mathcal{P}}_3^{2^i}) = \widetilde{\mathcal{P}}_1^{2^i} \widetilde{\mathcal{P}}_2^{2^i} \widetilde{\mathcal{P}}_3^{2^i} = (\widetilde{\mathcal{P}}_1 \widetilde{\mathcal{P}}_2 \widetilde{\mathcal{P}}_3)^{2^i} = (\widetilde{P}_0^2 \widetilde{p})^{2^i} = \widetilde{P}_0^{2^{i+1}} \in H^{2^{i+1}}.$$

Since $\widetilde{\mathcal{P}}_i C_i^{-1}$ is in the principal genus of K_i, $\widetilde{\mathcal{P}}_i C_i^{-1} = B_i^2$ for some class B_i of K_i. Hence

$$(\widetilde{\mathcal{P}}_1 C_1^{-1}, \widetilde{\mathcal{P}}_2 C_2^{-1}, \widetilde{\mathcal{P}}_3 C_3^{-1}) = (B_1^2, B_2^2, B_3^2),$$

so

$$B_1^2B_2^2B_3^2 = (\widetilde{\mathcal{P}}_1\widetilde{\mathcal{P}}_2\widetilde{\mathcal{P}}_3)(C_1C_2C_3)^{-1} = \widetilde{P}_0^2(C_1C_2C_3)^{-1}.$$

Therefore

$$(B_1B_2B_3)^{2^{i+1}} = \widetilde{P}_0^{2^{i+1}}(C_1^{2^i}C_2^{2^i}C_3^{2^i})^{-1}$$

and $C_1^{2^i}C_2^{2^i}C_3^{2^i} \in H^{2^{i+1}}$.

Conversely, let $C^{2^{i+1}} \in H^{2^{i+1}}$ and $P_0 \in C$ be a prime ideal of degree 1 and index 1 over k. Let $\mathcal{P}_i = P_0 \cap K_i$ for $i = 1,2,3$. Then $\mathcal{P}_1 = P_0P_1, \mathcal{P}_2 = P_0P_2$ and $\mathcal{P}_3 = P_0P_3$ where $P_0 \cap k = (p) = P_0P_1P_2P_3$. Now $(\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \widetilde{\mathcal{P}}_3) \in S$ and $\widetilde{\mathcal{P}}_1\widetilde{\mathcal{P}}_2\widetilde{\mathcal{P}}_3 = \widetilde{P}_0^2 = C^2$. Thus $\widetilde{\mathcal{P}}_1^{2^i}\widetilde{\mathcal{P}}_2^{2^i}\widetilde{\mathcal{P}}_3^{2^i} = \widetilde{P}_0^{2^{i+1}} = C^{2^{i+1}}$. Therefore $\frac{S^{2^i}}{S^{2^i}\cap ker} \simeq H^{2^{i+1}}$.

The characters on H_i must be normalized so that every unit of k belongs to the principal character system. The number of normalizations that occur for the extension K_i/k is η_i, where 2^{η_i} is the number of different unnormalized character values generated by the units of k . Also, the number of normalizations that occur for the extension K/k is η, where 2^{η} different character values are generated by the units of k in the direct product of the unnormalized characters of K_i/k, for $i = 1,2,3$. [8]

Lemma 2 *The order of $\widehat{S}$ is $2^{t-2-\eta}$.*

Proof Each divisor of 2 in k, which ramifies in K, determines either two or one independent characters according as it is totally ramified or not. The other primes of k which ramify in K each determine one character. These t characters must satisfy $\prod_{\chi\in\widehat{H}_i} \chi = +1$, for $i = 1,2,3$, and any two product conditions determine the third. Normalization of characters imposes η more conditions on the characters. Therefore $\widehat{S}$ has $2^{t-2-\eta}$ elements.

Corollary 3 *If $k = \mathbf{Q}$ then*

$$|\widehat{S}| = \begin{cases} 2^{t-2} & \textit{if } K \textit{ is real and no prime congruent to } 3 \textit{ modulo } 4 \textit{ ramifies in } K, \\ 2^{t-3} & \textit{otherwise.} \end{cases}$$

Proof If K is real and no prime congruent to 3 modulo 4 ramifies in K, then $\eta = 0$. Otherwise $\eta = 1$.

Lemma 4 *The order of S is $\frac{h_1 h_2 h_3}{2^{R_a}} |\widehat{S}|$.*

Proof The order of $\widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3$ is 2^{R_a} and the same number of classes of $H_1 \times H_2 \times H_3$ belong to each character value of $\widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3$.

Lemma 5 *The number t is given by $t_1 + t_2 + t_3 = 2t - \tau$. Moreover, $R_a = \sum_{i=1}^{3}(t_i - \eta_i) - 3 = 2t - \tau - 3 - \sum_{i=1}^{3} \eta_i$.*

Proof Each divisor of 2 in k which ramifies in K, ramifies in either two or three intermediate fields. All other primes of k which ramify in K ramify in two intermediate fields. Thus $t_1 + t_2 + t_3 = 2t - \tau$. The rank of H_i is $t_i - \eta_i - 1$, so the expression for R_a follows.

For an extension K/k where k has odd class number, Lemmermeyer [13] shows that $|ker| = \frac{2^{\nu-1}\prod e(p)[\bar{E}_k : E_k^2]}{q(K)}$, where $\nu = 1$ if $K = k(\sqrt{\varepsilon}, \sqrt{\rho})$ for units ε, ρ of k and $\nu = 0$ otherwise; $e(p)$ is the ramification index in K/k of a prime ideal p in k; $\bar{E}_k$ is the group of units in E_k which are norm residues in K/k and $q(K) = [E_K : E_{K_1}E_{K_2}E_{K_3}]$. For $k = \mathbf{Q}$, Lemmermeyer's result reduces to the following Theorem of Kubota [11]:

$$|ker| = \begin{cases} 2^t/q(K) & \text{if K is real and } \eta = 0, \\ 2^{t-1}/q(K) & \text{if K is real and } \eta = 1, \\ 2^{t-2}/q(K) & \text{if K is imaginary.} \end{cases}$$

For the remainder of this chapter let $k = \mathbf{Q}$. In this case we will show that the rank of H is given by the rank of a Z_2-matrix.

Theorem 6 *The rank of H is given by*

$$R_2 = \log_2[H_1 \times H_2 \times H_3 : S \cdot ker] + \begin{cases} t-2 & \textit{if K is real and } \eta = 0, \\ t-3 & \textit{otherwise.} \end{cases}$$

Proof From Kubota [11], $H^2 \subseteq H_0$ and $[H : H_0] = \begin{cases} 2^{t-2} & \text{if K is real and } \eta = 0, \\ 2^{t-3} & \text{otherwise.} \end{cases}$ Thus

$$\begin{aligned} R_2 &= \log_2[H : H^2] \\ &= \log_2[H : H_0] + \log_2[H_0 : H^2] \\ &= \log_2[H_0 : H^2] + \begin{cases} t-2 & \text{if K is real and } \eta = 0, \\ t-3 & \text{otherwise.} \end{cases} \end{aligned}$$

Now $H_0/H^2 \simeq \frac{H_1 \times H_2 \times H_3/ker}{S/S \cap ker}$ and $S/S \cap ker \simeq S \cdot ker/ker$ so $[H_0 : H^2] = [H_1 \times H_2 \times H_3 : S \cdot ker]$.

Corollary 7 *If K is real and $\eta = 0$ then $t-2 \leq R_2 \leq R_a$. Otherwise $t-3 \leq R_2 \leq R_a$.*

Proof It is immediate from Theorem 6 that $R_2 \geq t-2$ if K is real and $\eta = 0$ and $R_2 \geq t-3$ otherwise. Now

$$\begin{aligned}
[H_1 \times H_2 \times H_3 : S \cdot ker] &= \frac{|H_1 \times H_2 \times H_3|}{|S||ker|}|S \cap ker| \\
&\leq \frac{|H_1 \times H_2 \times H_3|}{|S|} = \frac{|H_1 \times H_2 \times H_3|}{\frac{|H_1 \times H_2 \times H_3||\widehat{S}|}{2^{R_a}}} = \frac{2^{R_a}}{|\widehat{S}|} \\
&= \begin{cases} 2^{R_a}/2^{t-2} & \text{if K is real and } \eta = 0, \\ 2^{R_a}/2^{t-3} & \text{otherwise.} \end{cases}
\end{aligned}$$

It now follows from Theorem 6 that $R_2 \leq R_a$.

Theorem 8 *Let m denote the rank of $\widehat{S} \cdot \widehat{ker}$. Then*

$$\begin{aligned}
R_2 &= R_a - m + \begin{cases} t-2 & \textit{if } K \textit{ is real and } \eta = 0, \\ t-3 & \textit{otherwise.} \end{cases} \\
&= -\tau - m - \sum_{i=1}^{3} \eta_i + \begin{cases} 3t-5 & \textit{if } K \textit{ is real and } \eta = 0, \\ 3t-6 & \textit{otherwise.} \end{cases}
\end{aligned}$$

Proof Let $\phi : H_1 \times H_2 \times H_3 \rightarrow \widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3$ be the mapping determined by taking a class C_i of H_i to its character system in $\widehat{H}_i$. Then $\frac{H_1 \times H_2 \times H_3}{ker\phi(S \cdot ker)} \simeq \frac{\widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3}{\phi(S \cdot ker)}$. But $\phi(S \cdot ker) = \phi(S) \cdot \phi(ker) = \widehat{S} \cdot \widehat{ker}$. Moreover, $ker\phi$ is the direct product of the 2-Sylow subgroups of the principal genera of K_1, K_2 and K_3 which is clearly contained in S. Thus $\frac{H_1 \times H_2 \times H_3}{S \cdot ker} \simeq \frac{\widehat{H}_1 \times \widehat{H}_2 \times \widehat{H}_3}{\widehat{S \cdot ker}}$. The result now follows from Lemma 5 and Theorem 6.

In order to determine R_2 we must be able to find a set of generators for ker. If p is a rational prime which ramifies in K then either $(p,p,1), (p,1,p)$ or $(1,p,p)$ is in ker according as p ramifies in K_1 and K_2, K_1 and K_3 or K_2 and K_3. Elements of this form generate ker unless K is real, $\eta = 0$ and $N\varepsilon_i = +1$ for some i. In this case there is an additional generator determined by weak ambiguous classes.

Lemma 9 *Suppose K is real and $\eta = 0$. Then there exist ideals A_i of K_i such that $\widetilde{A}_i$ is an ambiguous class and $A_1A_2A_3 = (\alpha)$ for some $\alpha \in K$ with $N_{K/\mathbf{Q}}(\alpha) < 0$. Futhermore, $\widetilde{A}_i$ is a weak ambiguous class for each i with $N\varepsilon_i = +1$.*

Proof The existence of ideals A_1, A_2 and A_3 such that $\widetilde{A}_i$ is ambiguous and $A_1A_2A_3 = (\alpha)$, for some α with $N_{K/\mathbf{Q}}(\alpha) < 0$, is proven in Lemmas 14 and 15 of [11]. Suppose $N\varepsilon_1 = +1$ and let σ_i be the automorphism of K fixing K_i. Then $A_1^{1-\sigma_2} = \frac{(A_1A_2A_3)^{1+\sigma_1}}{A_1^{1+\sigma_2}A_2^{1+\sigma_1}A_3^{1+\sigma_1}} = \frac{(\alpha)^{1+\sigma_1}}{A_1^{1+\sigma_2}A_2^{1+\sigma_1}A_3^{1+\sigma_1}} = (\rho_1)$ for some $\rho_1 \in K_1$ with $N_{K_1/\mathbf{Q}}(\rho_1) < 0$. Therefore A_1 is not an ambiguous ideal, so $\widetilde{A}_1$ must be a weak ambiguous class.

Now m is the rank of a Z_2-matrix M whose rows correspond to generators of $\widehat{S} \cdot \widehat{ker}$ by means of the isomorphism ψ.

Example Let $K_1 = \mathbf{Q}(\sqrt{lqrs})$, $K_2 = \mathbf{Q}(\sqrt{lq})$ and $K_3 = \mathbf{Q}(\sqrt{rs})$ with $l \equiv q \equiv 3 \pmod 4$ and $r \equiv s \equiv 1 \pmod 4$. The table of consistent characters is:

lq	r	s	lq	r	s
+	+	+	+	+	+
+	−	−	+	−	−.

Here $\widehat{S}$ is generated by $(0,1,1,0,1,1)$ and ker is generated by $\{(l,1,1), (q,1,1), (r,1,r)\}$. Thus

$$M = \begin{pmatrix} 1 & 1 & 1 \\ \psi\left(\frac{l}{r}\right) & \psi\left(\frac{l}{s}\right) & 0 \\ \psi\left(\frac{q}{r}\right) & \psi\left(\frac{q}{s}\right) & 0 \\ \psi(\left(\frac{l}{r}\right)\left(\frac{q}{r}\right)\left(\frac{r}{s}\right)) & \psi\left(\frac{r}{s}\right) & \psi\left(\frac{r}{s}\right) \end{pmatrix}$$

where the first row corresponds to the generator of $\widehat{S}$ and the last three rows correspond to generators of ker. We have deleted one character from each subfield since the product of the characters for a quadratic field is $+1$. The first two columns correspond to characters for K_1, determined by r and s, and the last column to a character for K_3, determined by r. Now $R_a = 3$ and $t = 4$ so $R_2 = 4 - m$.

Chapter 4

REAL BICYCLIC BIQUADRATIC FIELDS OF 2-RANK 1

The real bicyclic biquadratic fields having odd class number have been determined by Hasse [6] using the class number formula. As an application of the techniques developed in chapter 3 we will determine all such fields having 2-class group of rank one.

Theorem 10 *The real bicyclic biquadratic fields whose class groups have 2-rank one are listed below. In each case* $H_1 \times H_2 \times H_3 \simeq Z_{2^a} \times Z_2 \times \cdots \times Z_2$ *for some* $a \geq 1$ *and* $H \simeq Z_{2^{a-1}}, Z_{2^a}$ *or* $Z_{2^{a+1}}$*. In the following table* $[a_1, a_2, a_3]$ *followed by* $[b_1, b_2, \ldots, b_n]$ *indicates that* $\mathbf{Q}(\sqrt{a_1})$*,*$\mathbf{Q}(\sqrt{a_2})$ *and* $\mathbf{Q}(\sqrt{a_3})$ *are the quadratic subfields of* K *and* $b_1, b_2, \ldots, b_n$ *are congruence conditions modulo 4 on the prime divisors of* a_1, a_2 *and* a_3 *listed in alphabetical order. Here* l, q, r *and* s *are distinct primes. The second column gives further conditions that must be satisfied and the third column gives the 2-class group of* K*.*

1. $[lq, l, q]$ $[1 \text{ or } 2, 1]$	$h_1 > 2$	$Z_{2^{a-1}}$
2. $[lq, l, q]$ $[1, 3]$	$\left(\frac{l}{q}\right) = \left(\frac{2}{l}\right) = +1$	$Z_{2^{a-1}}$
3. $[lqr, lq, r]$ $[1 \text{ or } 2, 1, 1 \text{ or } 2]$	$N\varepsilon_1 = N\varepsilon_2 = -1$ *and* $\left(\frac{l}{r}\right) = \left(\frac{q}{r}\right) = -1$	$Z_{2^{a+1}}$
	$N\varepsilon_1 = -1, N\varepsilon_2 = +1$ *and either* $h_2 = 2$ *and* $\left(\frac{l}{r}\right) = -1$ *or* $h_2 = 2$ *and* $\left(\frac{q}{r}\right) = -1$ *or* $h_2 > 2$ *and* $\left(\frac{l}{r}\right) = \left(\frac{q}{r}\right) = -1$	$Z_{2^{a+1}}$

	$N\varepsilon_1 = +1, N\varepsilon_2 = -1,$ $h_1 = 4$ *and* $\left(\frac{l}{r}\right) \neq \left(\frac{q}{r}\right)$	Z_{2^a}
	$N\varepsilon_1 = N\varepsilon_2 = +1, \left(\frac{l}{r}\right) \neq \left(\frac{q}{r}\right)$ *and either* $h_1 = 4$ *and* $h_2 = 2$ *or* $h_1 > 4$ *and* $h_2 = 2$	Z_{2^a}
4. $[lqr, lq, r]$ $[2$ *or* $3, 3, 1$ *or* $2]$	$\left(\frac{l}{r}\right) = \left(\frac{q}{r}\right) = +1$	$Z_{2^{a-1}}$
5. $[lqr, lq, r]$ $[3, 3, 3]$	*If* $\left(\frac{l}{s}\right) = \left(\frac{q}{s}\right)$ *then either* $\left(\frac{l}{r}\right) = +1$ *or* $\left(\frac{q}{r}\right) = +1$	Z_{2^a}
6. $[lqr, lq, r]$ $[1, 3, 3]$	$\left(\frac{l}{r}\right) = -1$ *or* $\left(\frac{l}{s}\right) = -1$	Z_{2^a}
7. $[lqr, lq, r]$ $[1, 1, 3]$	$\left(\frac{l}{r}\right) = \left(\frac{q}{r}\right) = -1$ *or* $\left(\frac{l}{r}\right) = \left(\frac{q}{s}\right) = -1$ *or* $\left(\frac{q}{r}\right) = \left(\frac{l}{s}\right) = -1$	$Z_{2^{a+1}}$ *if* $q(K) = 2$ Z_{2^a} *if* $q(K) = 1$
8. $[lqr, lq, r]$ $[3, 1, 1]$	$\left(\frac{q}{r}\right) = -1$ *and either* $\left(\frac{l}{r}\right) = -1$ *or* $\left(\frac{r}{s}\right) = 1$	$Z_{2^{a+1}}$ *if* $q(K) = 2$ Z_{2^a} *if* $q(K) = 1$
9. $[lqr, lq, r]$ $[1, 3, 2]$	$\left(\frac{l}{r}\right) = -1$	Z_{2^a}
10. $[lqr, lq, r]$ $[2, 1, 3]$	$\left(\frac{l}{q}\right) = -1$ *or* $\left(\frac{q}{r}\right) = -1$	$Z_{2^{a+1}}$ *if* $q(K) = 2$ Z_{2^a} *if* $q(K) = 1$
11. $[lqr, lq, r]$ $[2, 3, 3]$		Z_{2^a}
12. $[lq, lr, qr]$ $[1$ *or* $2, 1$ *or* $2, 1$ *or* $2]$	$N\varepsilon_1 = N\varepsilon_2 = N\varepsilon_3$ *and at least two of* $\left(\frac{l}{q}\right), \left(\frac{l}{r}\right)$ *and* $\left(\frac{q}{r}\right)$ *equal* -1	$Z_{2^{a+1}}$
	$N\varepsilon_1 = +1, N\varepsilon_2 = N\varepsilon_3 = -1$	Z_{2^a}

	with $h_1 = 2$ *and either* $\left(\frac{l}{r}\right) = -1$ *or* $\left(\frac{q}{r}\right) = -1$, *or* $h_1 > 2$ *and* $\left(\frac{l}{r}\right) = \left(\frac{q}{r}\right) = -1$	
	$N\varepsilon_1 = N\varepsilon_2 = +1, N\varepsilon_3 = -1$, $\left(\frac{q}{r}\right) = -1$ *and either* $h_1 = 2$ *and* $h_2 > 2$ *or* $h_1 > 2$ *and* $h_2 = 2$	$Z_{2^{a+1}}$
13. $[lq, lr, qr]$ $[1, 3, 3]$	$\left(\frac{l}{q}\right) = -1$ *or* $\left(\frac{l}{r}\right) = -1$	$Z_{2^{a+1}}$ *if* $q(K) = 2$ Z_{2^a} *if* $q(K) = 1$
14. $[lq, lr, qr]$ $[1, 1, 3]$	$\left(\frac{l}{q}\right) = +1$ *and either* $\left(\frac{l}{r}\right) = \left(\frac{q}{r}\right) = -1$ *or* $\left(\frac{l}{r}\right) = \left(\frac{q}{s}\right) = -1$ *or* $\left(\frac{q}{r}\right) = \left(\frac{l}{s}\right) = -1$	$Z_{2^{a+1}}$ *if* $q(K) = 2$ Z_{2^a} *if* $q(K) = 1$
	$\left(\frac{l}{q}\right) = -1$ *and either* $\left(\frac{l}{r}\right) = -1$ *or* $\left(\frac{q}{r}\right) = -1$	$Z_{2^{a+1}}$ *if* $q(K) = 2$ Z_{2^a} *if* $q(K) = 1$
	$\left(\frac{l}{q}\right) = -1, \left(\frac{l}{r}\right) = \left(\frac{q}{r}\right) = +1$ *and* $\left(\frac{l}{s}\right) \neq \left(\frac{q}{s}\right)$	Z_{2^a} *if* $q(K) = 2$ Z_{2^a} *if* $q(K) = 1$
15. $[lq, lr, qr]$ $[2, 1, 3]$	$\left(\frac{q}{r}\right) = -1$ *or* $\left(\frac{l}{q}\right) = -1$	$Z_{2^{a+1}}$ *if* $q(K) = 4$ Z_{2^a} *if* $q(K) = 2$
16. $[lqrs, lq, rs]$ $[2$ *or* $3, 3, 3, 3]$	$\left(\frac{l}{r}\right) \neq \left(\frac{l}{s}\right)$ *or* $\left(\frac{q}{r}\right) \neq \left(\frac{l}{s}\right)$ *or* $\left(\frac{q}{r}\right) \neq \left(\frac{q}{s}\right)$	Z_{2^a}
17. $[lqrs, lq, rs]$ $[2$ *or*$3, 3, 1, 1$ *or* $2]$	$\left(\frac{l}{r}\right) \neq \left(\frac{q}{r}\right)$ *and* $\left(\frac{l}{s}\right) \neq \left(\frac{q}{s}\right)$ *or* $\left(\frac{l}{r}\right) = \left(\frac{q}{r}\right) = -1$ *and* $\left(\frac{l}{s}\right) \neq \left(\frac{q}{s}\right)$ *or* $\left(\frac{l}{r}\right) \neq \left(\frac{q}{r}\right)$ *and* $\left(\frac{l}{s}\right) = \left(\frac{q}{s}\right) = -1$	$Z_{2^{a+1}}$ *if* $q(K) = 2$ Z_{2^a} *if* $q(K) = 1$
18. $[lqrs, lqr, s]$ $[2$ *or* $3, 3, 1, 1$ *or* $2]$	$\left(\frac{r}{s}\right) = -1$ *and either* $\left(\frac{l}{s}\right) = -1$ *or* $\left(\frac{q}{s}\right) = -1$	$Z_{2^{a+1}}$ *if* $q(K) = 2$ Z_{2^a} *if* $q(K) = 1$
19. $[lqrs, lqr, s]$	$\left(\frac{l}{s}\right) = -1$ *and either*	$Z_{2^{a+1}}$ *if* $q(K) = 2$

$[2,3,3,1]$	$\left(\frac{q}{s}\right)=-1$ *or* $\left(\frac{r}{s}\right)=-1$	Z_{2^a} *if* $q(K)=1$
20. $[lqr, lqs, rs]$ $[1$ *or* $2, 2$ *or* $3, 3, 2$ *or* $3]$	$\left(\frac{l}{r}\right)=-1$ *or* $\left(\frac{l}{s}\right)=-1$	Z_{2^a}
21. $[lqr, lqs, rs]$ $[2$ *or* $3, 3, 1, 1$ *or* $2]$	$\left(\frac{l}{r}\right)=\left(\frac{l}{s}\right)=-1$ *and either* $\left(\frac{q}{r}\right)=+1$ *or* $\left(\frac{q}{s}\right)=+1$	$Z_{2^{a+1}}$ *if* $q(K)=2$ Z_{2^a} *if* $q(K)=1$
	$\left(\frac{l}{r}\right)=-1, \left(\frac{l}{s}\right)=+1$ *and either* $\left(\frac{q}{s}\right)=-1$ *or* $\left(\frac{r}{s}\right)=-1$	$Z_{2^{a+1}}$ *if* $q(K)=2$ Z_{2^a} *if* $q(K)=1$
	$\left(\frac{l}{r}\right)=+1, \left(\frac{l}{s}\right)=-1$ *and either* $\left(\frac{q}{r}\right)=-1$ *or* $\left(\frac{r}{s}\right)=-1$	$Z_{2^{a+1}}$ *if* $q(K)=2$ Z_{2^a} *if* $q(K)=1$
	$\left(\frac{l}{r}\right)=\left(\frac{l}{s}\right)=+1$ *and at least* *two of* $\left(\frac{q}{r}\right), \left(\frac{q}{s}\right)$ *and* $\left(\frac{r}{s}\right)$ *equal* -1	$Z_{2^{a+1}}$ *if* $q(K)=2$ Z_{2^a} *if* $q(K)=1$

Proof Since ker is elementary and $\frac{H_1\times H_2\times H_3}{ker} \simeq H_0$ is a subgroup of H, it follows that if H is cyclic then at most one factor of $H_1 \times H_2 \times H_3$ has order greater than 2. It follows from Corollary 7 that if H is cyclic then $t \le 4$ and $t = 4$ only if $\eta = 1$. The above list follows from a careful analysis of cases. For example, when $d_1 = lqr, d_2 = lqs$ and $d_3 = rs$, with $l \equiv 2$ or $3 \pmod 4$, $q \equiv 3 \pmod 4$, $r \equiv 1 \pmod 4$ and $s \equiv 1$ or $2 \pmod 4$ the table of consistent characters is: $\begin{array}{cc|cc|cc} lq & r & lq & s & r & s \\ \hline + & + & + & + & + & + \\ - & - & - & - & - & - \end{array}$. Here $\widehat{S}$ is generated by $(1,1,1,1,1,1)$ and ker is generated by $(l,l,1),(q,q,1)$ and $(r,1,r)$. Thus

$$M = \begin{pmatrix} 1 & 1 & 1 \\ \psi\left(\frac{l}{r}\right) & \psi\left(\frac{l}{s}\right) & 0 \\ \psi\left(\frac{q}{r}\right) & \psi\left(\frac{q}{s}\right) & 0 \\ \psi(\left(\frac{l}{r}\right)\left(\frac{q}{r}\right)) & 0 & \psi\left(\frac{r}{s}\right) \end{pmatrix}$$

where the first column corresponds to a character for K_1, determined by r, and the last two columns correspond to characters for K_2 and K_3, determined by s. By Theorem 8, $R_2 = 4 - m$. If $\left(\frac{l}{r}\right) = \left(\frac{l}{s}\right) = -1$ then M reduces to

$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ \psi\left(\frac{q}{r}\right) & \psi\left(\frac{q}{s}\right) & 0 \\ 1+\psi\left(\frac{q}{r}\right) & 0 & 0 \end{pmatrix}$ so $R_2 = 1$ if either $\left(\frac{q}{r}\right) = +1$ or $\left(\frac{q}{s}\right) = +1$. If $\left(\frac{l}{r}\right) = \left(\frac{l}{s}\right) = +1$ then M reduces to $\begin{pmatrix} 1 & 1 & 1 \\ \psi\left(\frac{q}{r}\right) & \psi\left(\frac{q}{s}\right) & 0 \\ \psi\left(\frac{q}{r}\right) & 0 & \psi\left(\frac{r}{s}\right) \end{pmatrix}$ so $R_2 = 1$ if at least two of $\left(\frac{q}{r}\right), \left(\frac{q}{s}\right)$ and $\left(\frac{r}{s}\right)$ equal -1. If $\left(\frac{l}{r}\right) = -1$ and $\left(\frac{l}{s}\right) = +1$ then M reduces to $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & \psi\left(\frac{q}{s}\right) & 0 \\ 0 & 0 & \psi\left(\frac{r}{s}\right) \end{pmatrix}$ so $R_2 = 1$ if either $\left(\frac{q}{s}\right) = -1$ or $\left(\frac{r}{s}\right) = -1$. If $\left(\frac{l}{r}\right) = +1$ and $\left(\frac{l}{s}\right) = -1$ then M reduces to $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \psi\left(\frac{q}{r}\right) & 0 & 0 \\ 0 & 0 & \psi\left(\frac{r}{s}\right) \end{pmatrix}$ so $R_2 = 1$ if either $\left(\frac{q}{r}\right) = -1$ or $\left(\frac{r}{s}\right) = -1$. That $H \simeq Z_{2^{a+1}}$ or Z_{2^a} according as $q(K) = 2$ or 1 follows from the class number formula $h = \frac{1}{4}q(K)h_1h_2h_3$. The remaining cases are done similarly.

Chapter 5

GROUPS OCCURING AS CLASS GROUPS OF IMAGINARY BICYCLIC BIQUADRATIC FIELDS

In this section we will show that every abelian group of exponent 2 or 4 occurs as the 2-class group of some imaginary bicyclic biquadratic field. Several technical lemmas preceed the main result.

For any $n \times n$ Z_2-matrix A, let $A(i_1, \ldots, i_k)$ denote the matrix obtained by adding 1 to the $i_j i_j$ entry of A for, $j = 1, \ldots, k$. Define $C_1 = (1)$ and for $n > 1$ define $C_n = (c_{ij})$ to be the $n \times n$ Z_2-matrix given by: $c_{nn} = 1, c_{i\,i+1} = c_{i+1\,i} = 1$ for $i = 1, \ldots, n-1$ and $c_{ij} = 0$ otherwise.

Lemma 11 *The following hold for each n:*

1. $\det C_n = 1$,
2. $\det C_n(1, 2, \ldots, 3k) = 1$,
3. $\det C_n(1, 2, \ldots, 3k+1) = 0$,
4. $\det C_n(1, 2, \ldots, 3k+2) = 1$,

5. $\det C_n(1,2,\ldots,3k,3k+2) = 1.$

Proof Now $\det C_1 = \det C_2 = 1$ and expanding about row 1 of C_n and then about column 1 of the resulting minor we see that $\det C_n = \det C_{n-2}$. Thus $\det C_n = 1$ for each n. It is easily verified that $\det C_1(1) = \det C_2(1) = \det C_3(1) = 0$, $\det C_2(1,2) = \det C_3(1,2) = 1$ and $\det C_3(1,2,3) = 0$. Expanding about row 1 of $C_n(1,\ldots,i)$ and then about column 1 of the resulting 1-2 minor we see that $\det C_n(1) = \det C_{n-1} + \det C_{n-2}$ and $\det C_n(1,\ldots,i) = \det C_{n-1}(1,\ldots,i-1) + \det C_{n-2}(1,\ldots,i-2)$ for $i \geq 2$. Thus (2),(3) and (4) hold. Now $\det C_2(2) = \det C_3(2) = \det C_4(2) = 1$. Expanding about row 1 of $C_n(1,\ldots,3k,3k+2)$ and then about column 1 of the resulting 1-2 minor we see that $\det C_n(2) = \det C_{n-2} = 1$ and $\det C_n(1,\ldots,3k,3k+2) = \det C_{n-1}(1,\ldots,3k-1,3k+1) + \det C_{n-2}(1,\ldots,3k-2,3k)$ for $k \geq 1$. Repeating this for $C_{n-1}(1,\ldots,3k-1,3k+1)$ we see that $\det C_{n-1}(1,\ldots,3k-1,3k+1) = \det C_{n-2}(1,\ldots,3k-2,3k) + \det C_{n-3}(1,\ldots,3k-3,3k-1)$. Therefore $\det C_n(1,\ldots,3k,3k+2) = \det C_{n-3}(1,\ldots,3k-3,3k-1)$ and (5) holds.

For $n \geq 2$ let A_n be the $n \times n$ Z_2-matrix defined by $A_n = C_n(1)$.

Lemma 12 *The following hold for each n :*

1. $\det A_n(1,\ldots,3k) = 0,$

2. $\det A_n(1,\ldots,3k+1) = 1,$

3. $\det A_n(1,\ldots,3k+2) = 1,$

4. $\det A_n(1,\ldots,3k-1,3k+1) = 1.$

Proof Now $\det A_2(1) = \det A_2(1,2) = 1$. For $n \geq 3$, a_{12} and a_{21} are the only nonzero entries in row 1 and column 1 of $A_n(1,\ldots,i)$, respectively. Thus, deleting the first two rows and first two columns of $A_n(1,\ldots,i)$ we see that $\det A_n(1) = \det C_{n-2}$ and $\det A_n(1,\ldots,i) = \det C_{n-2}(1,\ldots,i-2)$ for $i \geq 2$. The result now follows from Lemma 11.

For $n = 3m, m \geq 2$, define $B_n = (b_{ij})$ to be the $n \times n$ Z_2–matrix given by: $b_{12} = a_{12}+1, b_{21} = a_{21}+1, b_{14} = a_{14}+1, b_{41} = a_{41}+1, b_{22} = a_{22}+1, b_{44} = a_{44}+1$ and $b_{ij} = a_{ij}$ otherwise.

Lemma 13 *For each* n, $\det B_n(1,\ldots,n) = 1$.

Proof Note that b_{14} and b_{41} are the only nonzero entries in row 1 and column 1 of $B_n(1,\ldots,n)$, respectively. Thus $B_n(1,\ldots,n)$ can be reduced to the matrix $\begin{pmatrix} B & 0 \\ 0 & C_{n-4}(1,\ldots,n-4) \end{pmatrix}$ where $B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. The result now follows from Lemma 11.

Let $p_1,\ldots,p_s$ be primes with $p_1 \equiv 3 \pmod 4$ and $p_i \equiv 1 \pmod 4, i \neq 1$. For $1 \leq i \leq s-1$ let $y_i = \psi\left(\frac{p_i}{p_s}\right)$.

Lemma 14 *The 2-rank of the class group of* $\mathbf{Q}(\sqrt{-p_1\cdots p_{s-1}}, \sqrt{p_s})$ *is* $(2s-3)-(y_1+\cdots+y_{s-1})$.

Proof Let $K_1 = \mathbf{Q}(\sqrt{-p_1\cdots p_{s-1}})$, $K_2 = \mathbf{Q}(\sqrt{p_s})$ and $K_3 = \mathbf{Q}(\sqrt{-p_1\cdots p_s})$. For $1 \leq i,j \leq s-1, i \neq j$ let $x_{ij} = \psi\left(\frac{p_i}{p_j}\right)$ and $x_{ii} = \sum_{j=1}^{s-1} x_{ij}$. Since *ker* is generated by $\{(p_i,1,p_i)|i =$

$1, \ldots, s-1\}$,

$$M = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & & 0 & 0 & 1 & & 0 & 1 \\ \vdots & & \ddots & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & 1 \\ x_{11} & x_{12} & \cdots & x_{1\,s-2} & x_{11}+y_1 & x_{12} & \cdots & x_{1\,s-2} & x_{1\,s-1} \\ \vdots & & & & & & & & \vdots \\ x_{1\,s-1} & x_{2\,s-1} & \cdots & x_{s-2\,s-1} & x_{1\,s-1} & x_{2\,s-1} & \cdots & x_{s-2\,s-1} & x_{s-1\,s-1}+y_s \end{pmatrix}$$

where the first $s-2$ rows correspond to generators of $\widehat{S}$. Since $\sum_{j=1}^{s-1} x_{ij} = 0$ for each i, M reduces to

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & & 0 & 0 & 1 & & 0 & 1 \\ \vdots & & \ddots & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & y_1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & y_{s-1} \end{pmatrix}.$$

The result now follows from Theorem 8.

For the field $\mathbf{Q}(\sqrt{-p_1 \cdots p_{s-1}}, \sqrt{p_s})$, if $H_1 \times H_2 \times H_3$ is elementary then it follows from Corollary 7 and the class number formula that $s-2 \leq R_2 \leq 2s-4$.

Lemma 15 *For any $s \geq 3, s \neq 4$ and for any l with $s-2 \leq l \leq 2s-4$ there exist primes $p_1, \ldots, p_s$ such that $H_1 \times H_2 \times H_3$ is elementary and the 2-class group of $\mathbf{Q}(\sqrt{-p_1 \cdots p_{s-1}}, \sqrt{p_s})$ has rank l. If $s = 4$ then there exist primes such that the rank is 3 or 4.*

Proof Choose $p_1, \ldots, p_{s-1}$ such that for $i \neq j$, $\psi\left(\frac{p_i}{p_j}\right)$ equals the ij–entry of A_{s-1}. The first row of A_{s-1} is the sum of rows 2 through $s-1$ and these rows are clearly independent, so A_{s-1} has rank $s-2$. Thus H_1 is elementary. Now the character table for K_3 corresponds to $A_{s-1} + \begin{pmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & y_{s-1} \end{pmatrix}$. By Lemma 12, if $1 \leq w \leq s-2$ then p_s can be chosen so that H_3 is elementary and exactly w of $y_1, \ldots, y_{s-1}$ are equal to 1. It also follows from Lemma 12 that if $s \not\equiv 1 \pmod 3$ and p_s is chosen such that $y_1 = \ldots = y_{s-1} = 1$, then H_3 is elementary.

If $s \equiv 1 \pmod 3, s \neq 4$ choose $p_1, \ldots, p_{s-1}$ such that $\psi\left(\frac{p_i}{p_j}\right)$ equals the ij–entry of B_{s-1}. The rows of B_{s-1} are dependent, but after adding row 1 to row 4 and deleting row 2 we are left with $s-2$ independent rows. Thus H_1 is elementary. The character table for H_3 corresponds to $B_{s-1} + \begin{pmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & y_{s-1} \end{pmatrix}$. By Lemma 13, p_s can be chosen so that H_3 is elementary and $y_1 = \ldots = y_{s-1} = 1$. The result now follows from Lemma 14.

With K as in the previous lemma and $s = 4$ the character system of K_1 must be one of the following:

	p_1	p_2	p_3
p_1	$+$	$-$	$-$
p_2	$-$	$+$	$-$
p_3	$-$	$-$	$+$

	p_1	p_2	p_3
p_1	$-$	$+$	$-$
p_2	$+$	$-$	$-$
p_3	$-$	$-$	$+$

	p_1	p_2	p_3
p_1	$-$	$-$	$+$
p_2	$-$	$+$	$-$
p_3	$+$	$-$	$-$

.

A case by case analysis shows that there is no choice of p_4 such that H_3 is elementary and $R_2 = 2$.

Now let $p_1, \dots, p_s$ be primes with $p_1 \equiv \dots \equiv p_{s-2} \equiv 1 \pmod 4$ and $p_{s-1} \equiv p_s \equiv 3 \pmod 4$. Choose $p_1, \dots, p_{s-1}$ so tht $\psi\left(\frac{p_i}{p_j}\right)$ equals he $ij-$entry of A_{s-1}, for $i, j \leq s-1$. For $i = 1, \dots, s-2$, let $y_i = \psi\left(\frac{p_i}{p_s}\right)$.

Lemma 16 *The rank of the 2-class group of* $\mathbf{Q}(\sqrt{-p_1 \dots p_{s-1}}, \sqrt{p_{s-1}p_s})$ *is* $2s - 5 - (y_1 + \dots + y_{s-3})$.

Proof The kernel is generated by $\{(p_i, 1, p_i) | 1 \leq i \leq s-2\}$ and $(p_{s-1}, 1, 1)$ so

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 1+y_1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & y_2 & 1 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & y_3 & \cdots & 0 \\ \vdots & & & & & & & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1+y_{s-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where the first $s-2$ rows correspond to generators of $\widehat{S}$. Now M reduces to

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & y_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & y_2 & 0 & \cdots & 0 \\ \vdots & & & & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & y_{s-2} \end{pmatrix}.$$

The result now follows from Theorem 8.

For the field $\mathbf{Q}(\sqrt{-p_1\cdots p_{s-1}}, \sqrt{p_{s-1}p_s})$, if $H_1 \times H_2 \times H_3$ is elementary then it follows from Corollary 7 and the class number formula that $s-2 \leq R_2 \leq 2s-5$.

Lemma 17 *For any $s \geq 3$ and for any l with $s-2 \leq l \leq 2s-5$ there exist primes $p_1, \ldots, p_s$ such that $H_1 \times H_2 \times H_3$ is elementary and the rank of the 2-class group of $\mathbf{Q}(\sqrt{-p_1\cdots p_{s-1}}, \sqrt{p_{s-1}p_s})$ is l.*

Proof Since A_{s-1} has rank $s-2$, H_1 is elementary. The character table for K_3 corresponds to $A_{s-2} + \begin{pmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & y_{s-2} \end{pmatrix}$. It follows from the proof of Lemma 11 that for $0 \leq w \leq s-3$, if w of $y_1, \ldots, y_{s-3}$ are equal to 1 then y_{s-2} can be chosen such that H_3 is elementary. The result now follows from Lemma 16.

Theorem 18 *Every abelian group of exponent 2 or 4 occurs as the 2-class group of some imaginary bicyclic biquadratic field.*

Proof The result follows immediately from Lemmas 15 and 17 except for the group $Z_4 \times Z_4$. In that case let $K = \mathbf{Q}(\sqrt{-p_1p_2p_3}, \sqrt{p_4})$ with $p_1 \equiv p_2 \equiv p_3 \equiv 3 \pmod 4$ and $p_4 \equiv 1 \pmod 4$. Choose $p_1, \ldots, p_4$ such that $\left(\frac{p_1}{p_3}\right) = +1$ and $\left(\frac{p_i}{p_j}\right) = -1$ for $i = 1,2,3$, $j = 2,3,4$, $(i,j) \neq (1,3)$ and $i < j$. Then $H_1 \times H_2 \times H_3$ is elementary and $H \simeq Z_4 \times Z_4$, since the matrix M has rank 5.

Chapter 6

UNIT GROUPS OF OCTIC FIELDS

For this chapter let K be an imaginary octic field of type $(2,2,2)$. An easy group theoretic argument shows that $Q = Q_0Q_1Q_2$. Now according to Kuroda [12], $Q_0 = 1, 2$ or 4 and $Q_2 = 1$ or 2 according as $\sqrt{-1}, \sqrt{2} \in K$ or not. By Theorem 4.12 of Washington [23], $Q_1 = 1$ or 2 since K is a CM-field. In this section we give conditions for determining Q_1.

Let $\zeta = \frac{1+\iota}{\sqrt{2}}$ be a primitive eighth root of unity.

Lemma 19 *If $e \in E_K - E^*$ then one of the following must hold:*

1. $e^2 = \iota\varepsilon_3$ *with* $N\varepsilon_3 = +1$,
2. $e^2 = -\varepsilon_2\varepsilon_3$ *or* $e^2 = \iota\varepsilon_2\varepsilon_3$ *with* $N\varepsilon_2 = N\varepsilon_3 = +1$,
3. $e^2 = -\varepsilon_1\varepsilon_2\varepsilon_3$ *or* $e^2 = \iota\varepsilon_1\varepsilon_2\varepsilon_3$ *with* $N\varepsilon_1 = N\varepsilon_2 = N\varepsilon_3$,
4. $e^2 = \zeta\varepsilon_2^{1/2}\varepsilon_3^{1/2}$ *or* $e^2 = \zeta\varepsilon_1\varepsilon_2^{1/2}\varepsilon_3^{1/2}$ *with* $N\varepsilon_2 = N\varepsilon_3 = +1$.

Proof Since $[E_K : WE_{K_0}] \leq 2$, $e^2 = \omega\varepsilon$ for some $\omega \in W$ and $\varepsilon \in E_{K_0}$. We may assume that ω is an eighth root of unity since any root of unity of odd order in K is a square. Thus $e^2 = \omega\varepsilon_1^a\varepsilon_2^b\varepsilon_3$ with $a, b \in \{0, 1\}$ or $e^2 = \omega\varepsilon_1^a\varepsilon_2^b\varepsilon_3^{1/2}$ with $a, b \in \{0, \frac{1}{2}, 1, \frac{3}{2}\}$.

Suppose $e^2 = \omega\varepsilon_3$ and note that $\omega \neq \pm 1$ since $e \notin E^*$. If $d_3 = 2$ and $\omega = \zeta$ choose an automorphism σ of K with $\sigma(\iota) = \iota$ and $\sigma(\sqrt{2}) = -\sqrt{2}$. Then $(e^2)^{1+\sigma} = -\iota\varepsilon_3^{1+\sigma} = \iota = \omega^2$ so $e^{1+\sigma} = \pm\omega$ contradicting that ω is not in the fixed field of σ. Thus either $d_3 \neq 2$ or ω is not a primitive eighth root of unity, so there is an automorphism τ of K with $\tau(\omega) = \bar{\omega}$ and $\tau(\sqrt{d_3}) = -\sqrt{d_3}$. Now $(e^2)^{1+\tau} = \varepsilon_3^{1+\tau} = N(\varepsilon_3$. If $N(\varepsilon_3 = -1$ then $\iota = \pm e^{1+\tau}$ is fixed by τ, a contradiction. Thus $N\varepsilon_3 = +1$. If $\omega = \zeta$ choose an automorphism ρ with $\rho(\iota) = \iota$, $\rho(\sqrt{2}) = -\sqrt{2}$ and $\rho(\sqrt{d_3}) = -\sqrt{d_3}$. Then $(e^2)^{1+\rho} = -\iota$ contradicting that ζ is not in the fixed field of ρ. Therefore $\omega = \iota$.

Now suppose $e^2 = \omega\varepsilon_2\varepsilon_3$. If $\omega = \zeta$, choose an automorphism σ with $\sigma(\iota) = \iota$, $\sigma(\sqrt{2}) = -\sqrt{2}$ and $\sigma(\sqrt{d_3}) = \sqrt{d_3}$. Then $(e^2)^{1+\sigma} = \pm\iota\varepsilon_3^2$ contradicting that ζ is not in the fixed field of σ. Thus $\omega \neq \zeta$ so there is an automorphism τ with $\tau(\omega) = \bar{\omega}$, $\tau(\sqrt{d_2}) = \sqrt{d_2}$ and $\tau(\sqrt{d_3}) = -\sqrt{d_3}$. Now $(e^2)^{1+\tau} = \varepsilon_2^2\varepsilon_3^{1+\tau} = \varepsilon_2^2 N\varepsilon_3 = \pm\varepsilon_2^2$. If $(e^2)^{1+\tau} = -\varepsilon_2^2$ then $e^{1+\tau} = \pm\iota\varepsilon_2$ contradicting that ι is not in the fixed field of τ. Thus $N\varepsilon_3 = +1$ and similarly, $N\varepsilon_2 = +1$.

Now supose $e^2 = \omega\varepsilon_1\varepsilon_2\varepsilon_3$. If $\omega = \zeta$ take $d_2 = 2$ and let τ be an automorphism with $\tau(\iota) = \iota$, $\tau(\sqrt{2}) = -\sqrt{2}$ and $\tau(\sqrt{d_1}) = \sqrt{d_1}$. Then $(e^2)^{1+\tau} = -\iota\varepsilon_1^2(\varepsilon_2\varepsilon_3)^{1+\tau} = \iota\varepsilon_1^2\varepsilon_3^{1+\tau} = \pm\iota\varepsilon_1^2$, contradicting that ζ is not in the fixed field of τ. Thus $\omega \neq \zeta$ so there is an automorphism σ with $\sigma(\omega) = \bar{\omega}$, $\sigma(\sqrt{d_1}) = -\sqrt{d_1}$, $\sigma(\sqrt{d_2}) = -\sqrt{d_2}$ and $\sigma(\sqrt{d_3}) = \sqrt{d_3}$. Now $(e^2)^{1+\sigma} = (\varepsilon_1\varepsilon_2)^{1+\sigma}\varepsilon_3^2$ and ι is not in the fixed field of σ so $N\varepsilon_1 = N\varepsilon_2$. Similarly, $N\varepsilon_1 = N\varepsilon_3$ so $N\varepsilon_1 = N\varepsilon_2 = N\varepsilon_3$.

Now suppose $e^2 = \omega\varepsilon_1^a\varepsilon_2^b\varepsilon_3^{1/2}$ with $a, b \in \{0, \frac{1}{2}, 1, \frac{3}{2}\}$. Let σ be an automorphism with $\sigma(\sqrt{d_3}) = \sqrt{d_3}$, $\sigma(\sqrt{d_1}) = -\sqrt{d_1}$, $\sigma(\sqrt{d_2}) = -\sqrt{d_2}$ and $\sigma(\iota) = -\iota$ if $\iota \in K$. Now $e^4 = \omega^2\varepsilon_1^{2a}\varepsilon_2^{2b}\varepsilon_3$ with $\omega^2 = -1$ or ι so $(e^4)^{1+\sigma} = (\varepsilon_1^{1+\sigma})^j(\varepsilon_2^{1+\sigma})^k\varepsilon_3^2$ with $j, k \in \{1, 2\}$. Thus

$(e^4)^{1+\sigma} = \pm\varepsilon_3^2$. If $(e^4)^{1+\sigma} = -\varepsilon_3^2$ then $(e^2)^{1+\sigma} = \pm\iota\varepsilon_3$ contradicting that ι is not in the fixed field of σ. Thus $(e^4)^{1+\sigma} = \varepsilon_3^2$ and $(e^2)^{1+\sigma} = \pm\varepsilon_3$. This implies that $\sqrt{\pm\varepsilon_3}$ is in a biquadratic subfield of K and it follows from [12] that $N\varepsilon_3 = +1$. Now let F_1 and F_2 be the imaginary biquadratic subfields of K containing $\sqrt{d_3}$. Then

$$N_{K/F_1}(e)^2 = N_{K/F_1}(\omega)N_{K/F_1}(\varepsilon_1)^a N_{K/F_1}(\varepsilon_2)^b(\pm\varepsilon_3) = \omega_1\varepsilon_3$$

and

$$N_{K/F_2}(e)^2 = N_{K/F_2}(\omega)N_{K/F_2}(\varepsilon_1)^a N_{K/F_2}(\varepsilon_2)^b(\pm\varepsilon_3) = \omega_2\varepsilon_3$$

where ω_1 and ω_2 are roots of unity in F_1 and F_2,respectively. Since both F_1 and F_2 have unit index 2, $F_1 = k_3(\iota)$ and $F_2 = k_3(\sqrt{-2})$. Thus $K = k_3(\iota, \sqrt{2}) = \mathbf{Q}(\iota, \sqrt{2}, \sqrt{d_3})$. Moreover, we may assume that $k_1 = \mathbf{Q}(\sqrt{2})$ so $N\varepsilon_1 = -1$. Since $e^2 = \varepsilon_1^a\varepsilon_2^b\varepsilon_3^{1/2}$ has no solutions in K, it follows that $\omega = \zeta$. Let τ be the automorphism with $\tau(\iota) = -\iota$, $\tau(\sqrt{2}) = \sqrt{2}$ and $\tau(\sqrt{d_3}) = -\sqrt{d_3}$. Then $(e^2)^{1+\tau} = \varepsilon_1^{2a}(\varepsilon_2^b\varepsilon_3^{1/2})^{1+\tau} = \omega_3\varepsilon_1^{2a}$ for some root of unity ω_3 in the fixed field of τ. If follows from [12] that $a \in Z$. If $b = 0$ then $N_{K/F_1}(e)^2 = (-1)^a\varepsilon_3$ and $N_{K/F_2}(e)^2 = -(-1)^a\varepsilon_3$ so $\sqrt{\varepsilon_3} \in F_1$ or $\sqrt{\varepsilon_3} \in F_2$. This contradicts that F_1 and F_2 are imaginary. If $b = 1$ let ρ be the automorphism with $\rho(\iota) = \iota$, $\rho(\sqrt{2}) = -\sqrt{2}$ and $\rho(\sqrt{d_3}) = -\sqrt{d_3}$. Then $(e^2)^{1+\rho} = -\iota(-1)^a\varepsilon_2^2(\varepsilon_3^{1+\rho})^{1/2} = \pm\iota\varepsilon_2^2$ which is impossible since ζ is not in the fixed field of ρ. Hence $b \in \{\frac{1}{2}, \frac{3}{2}\}$. Since $N_{K/K_0}(e)^2 = \varepsilon_1^{2a}\varepsilon_2^{2b}\varepsilon_3 = \varepsilon_1^{2a}\varepsilon_2^{2b-1}\varepsilon_2\varepsilon_3 = (\varepsilon_1^a\varepsilon_2^{\frac{2b-1}{2}})^2\varepsilon_2\varepsilon_3$ where $2b-1$ is and even integer, it follows that $\sqrt{\varepsilon_1\varepsilon_2} \in K_0$. It follows from [12] that $N\varepsilon_2 = +1$. From symmetry in ε_2 and ε_3 it follows that 2 is a principal divisor of k_2. Thus $\varepsilon_2 = 2\alpha^2$ for some $\alpha \in k_2$. Hence if $e^2 = \omega\varepsilon_1^a\varepsilon_2^{3/2}\varepsilon_3^{1/2}$ then $\left(\frac{e}{\sqrt{2}\alpha}\right)^2 = \omega\varepsilon_1^a\varepsilon_2^{1/2}\varepsilon_3^{1/2}$

so we may take $b = \frac{1}{2}$.

Corollary 20 *If $e^2 = \omega\varepsilon_1^a\varepsilon_2^b\varepsilon_3^{1/2}$ has a solution in K then $\sqrt{-1} \in K$, $k_2 = \mathbf{Q}(\sqrt{m})$ and $k_3 = \mathbf{Q}(\sqrt{2m})$ with $m \equiv 3 \pmod 4$. Moreover, 2 is a principal divisor in both k_2 and k_3.*

Proof As shown in the above proof, $K_0 = \mathbf{Q}(\sqrt{2}, \sqrt{m})$ with $k_2 = \mathbf{Q}(\sqrt{m})$. By symmetry in k_2 and k_3 we may assume that m is odd. Since 2 is a principal divisor in k_2 it follows that $m \equiv 3 \pmod 4$. Also, $\omega = \zeta$ so $\sqrt{-1} \in K$.

In the following lemmas we describe a method for computing units of the form $\sqrt[4]{\iota\varepsilon_2\varepsilon_3}$ and $\sqrt[4]{\iota\varepsilon_1^2\varepsilon_2\varepsilon_3}$. Let $d_1 = 2$, $d_2 = m \equiv 3 \pmod 4$, $K = K_0(\iota)$ and suppose that 2 is a principal divisor in both k_2 and k_3. Write $\varepsilon_2 = r + s\sqrt{m}$ and $\varepsilon_3 = u + v\sqrt{2m}$.

Lemma 21 *There exist integrs a, b, c and d such that $\sqrt{\varepsilon_2} = \frac{a\sqrt{2}+b\sqrt{2m}}{2}$ and $\sqrt{\varepsilon_3} = c\sqrt{2} + d\sqrt{m}$. If $m \equiv 7 \pmod 8$ then $r + 1 = a^2$, $r - 1 = mb^2$,$u + 1 = 4c^2$ and $u - 1 = 2md^2$ and if $m \equiv 3 \pmod 8$ then $r - 1 = a^2$, $r + 1 = mb^2$, $u - 1 = 4c^2$ and $u + 1 = 2md^2$. Moreover, b and d are both odd.*

Proof Since $N\varepsilon_2 = N\varepsilon_3 = +1$ it follows that $\sqrt{\varepsilon_2} = \frac{\sqrt{2(r+1)}+\sqrt{2(r-1)}}{2}$ and $\sqrt{\varepsilon_3} = \frac{\sqrt{2(u+1)}+\sqrt{s(u-1)}}{2}$. Clearly u is odd since $u^2 - 2mr^2 = 1$. Since 2 is a principal divisor of k_2, $2(r \pm 1) = 2a^2$ and $2(r \mp 1) = 2mb^2$ for some $a, b \in Z$. Thus $r \pm 1 = a^2$ and $r \mp 1 = mb^2$. It follows that r is even, for otherwise $r+1 \equiv r-1 \equiv 0 \pmod 4$. Therefore, $\sqrt{\varepsilon_2} = \frac{a\sqrt{2}+b\sqrt{2m}}{2}$ and $\sqrt{\varepsilon_3} = c\sqrt{2} + d\sqrt{m}$ with $\{r + 1, r - 1\} = \{a^2, mb^2\}$ and $\{u + 1, u - 1\} = \{4c^2, 2md^2\}$. Suppose $r + 1 = a^2$ and $u - 1 = 4c^2$. Then $r + u = mb^2 + 2md^2$, $r + u + 2 = a^2 + 2md^2$ and $r + u - 2 = 4c^2 + mb^2$. Thus $a^2 \equiv 2 \pmod m$ and $4c^2 \equiv -2 \pmod m$ so $\left(\frac{a^2}{4c^2}\right) \equiv -1$

(mod m), contradicting that $m \equiv 3 \pmod 4$. The case $r - 1 = a^2$ and $u + 1 = 4c^2$ yields a similar contradiction. Therefore, either $r + 1 = a^2$ and $u + 1 = 4c^2$ or $r - 1 = a^2$ and $u - 1 = 4c^2$. Now r is even and $r \pm 1 = mb^2$ so b must be odd. Also d must be odd, for otherwise $u + 1 \equiv u - 1 \equiv 0 \pmod 4$. Suppose $r + 1 = a^2$ and $m \equiv 3 \pmod 8$. Then $r - 1 = mb^2 \equiv 3 \pmod 8$ so $a^2 = r + 1 \equiv 5 \pmod 8$, which is impossible. Therefore $r - 1 = a^2$ if $m \equiv 3 \pmod 8$ and similarly $r + 1 = a^2$ if $m \equiv 7 \pmod 8$.

Now define $\alpha = mbd + (ac + 1)\sqrt{2}$, $\beta = mbd + (ac - 1)\sqrt{2}$, $\rho_1 = 2mbd + 2a - 4c$, $\rho_2 = 2mbd - 2a + 4c$, $\rho_3 = 2mbd + 2a + 4c$, $\rho_4 = 2mbd - 2a - 4c$, $\gamma_1 = -2mbd + 4ac + 2a + 4c + 4$, $\gamma_2 = -2mbd + 4ac - 2a - 4c + 4$, $\gamma_3 = -2mbd + 4ac + 2a - 4c - 4$ and $\gamma_4 = -2mbd + 4ac - 2a + 4c - 4$ with a, b, c and d as in Lemma 21.

Lemma 22 *If $m \equiv 3 \pmod 8$ then $\sqrt[4]{\varepsilon_2\varepsilon_3} = \frac{1}{4}(1+\iota)(1-\iota+\sqrt{2})(\sqrt{\alpha}+\sqrt{\beta}) = \frac{\sqrt[4]{2}}{2}(\sqrt{\rho_1} + \sqrt{\rho_2} + \sqrt{\rho_3} + \sqrt{\rho_4}$ and $\sqrt[4]{\iota\varepsilon_1^2\varepsilon_2\varepsilon_3} = \frac{1}{4}(1+\iota)(1-\iota+\sqrt{2})(\sqrt{\alpha}+\sqrt{\beta})$. If $m \equiv 7 \pmod 8$ then $\sqrt[4]{\varepsilon_2\varepsilon_3} = \frac{\sqrt[4]{2}}{2}\sqrt{\varepsilon_1}(\sqrt{\alpha\varepsilon_1^{-1}} + \sqrt{\beta\varepsilon_1^{-1}}) = \frac{\sqrt[4]{2}}{2}\sqrt{\varepsilon_1}(\sqrt{\gamma_1}+\sqrt{\gamma_2}+\sqrt{\gamma_3}+\sqrt{\gamma_4})$ and $\sqrt[4]{\iota\varepsilon_2\varepsilon_3} = \frac{1}{4}(1+\iota)(1-\iota+\sqrt{2})(\sqrt{\alpha\varepsilon_1^{-1}}+\sqrt{\beta\varepsilon_1^{-1}})$. Here we take $\sqrt[4]{-2} = \frac{1+\iota}{\sqrt[4]{2}}$ and $\sqrt[4]{\iota}\sqrt{\varepsilon_1} = \frac{\sqrt[4]{2}}{2}(1-\iota+\sqrt{2})$.*

Proof It follows from Lemma 21 that $\sqrt{\varepsilon_2\varepsilon_3} = \frac{1}{2}(2ac + mbd\sqrt{2} + 2bc\sqrt{m} + ad\sqrt{2m})$ and $N_{K_0/k_1}(\sqrt{\varepsilon_2\varepsilon_3}) = +1$. Thus

$$\begin{aligned}
\sqrt[4]{\varepsilon_2\varepsilon_3} &= \frac{1}{2}\left(\sqrt{2ac+2+mbd\sqrt{2}} + \sqrt{2ac-2+mbd\sqrt{2}}\right) \\
&= \frac{\sqrt[4]{2}}{2}\left(\sqrt{mbd+(ac+1)\sqrt{2}} + \sqrt{mbd+(ac-1)\sqrt{2}}\right) \\
&= \frac{\sqrt[4]{2}}{2}(\sqrt{\alpha}+\sqrt{\beta}).
\end{aligned}$$

Note that $(mbd)^2 = 2a^2c^2 + 2 - a^2 - 4c^2$ or $(mbd)^2 = 2a^2c^2 + 2 + a^2 + 4c^2$ according as

$m \equiv 7 \pmod 8$ or $m \equiv 3 \pmod 8$. From this it follows that

$$N(\alpha) = \begin{cases} -(a+2c)^2 & if \quad m \equiv 7 \pmod 8 \\ (a-2c)^2 & if \quad m \equiv 3 \pmod 8 \end{cases}$$

and

$$N(\beta) = \begin{cases} -(a-2c)^2 & if \quad m \equiv 7 \pmod 8 \\ (a+2c)^2 & if \quad m \equiv 3 \pmod 8). \end{cases}$$

Thus $N(\frac{\alpha}{a-2c}) = N(\frac{\beta}{a+2c}) = +1$ if $m \equiv 3 \pmod 8$, so

$$\sqrt{\frac{\alpha}{a-2c}} = \frac{1}{2}\left(\sqrt{\frac{2mbd+2a-4c}{a-2c}} + \sqrt{\frac{2mbd-2a+4c}{a-2c}}\right)$$

and

$$\sqrt{\frac{\beta}{a+2c}} = \frac{1}{2}\left(\sqrt{\frac{2mbd+2a+4c}{a+2c}} + \sqrt{\frac{2mbd-2a-4c}{a+2c}}\right).$$

Therefore

$$\begin{aligned} \sqrt{\alpha} &= \frac{1}{2}(\sqrt{2mbd+2a-4c} + \sqrt{2mbd-2a+4c}) \\ &= \frac{1}{2}(\sqrt{\rho_1} + \sqrt{\rho_2}) \end{aligned}$$

and

$$\begin{aligned} \sqrt{\beta} &= \frac{1}{2}(\sqrt{2mbd+2a+4c} + \sqrt{2mbd-2a-4c}) \\ &= \frac{1}{2}(\sqrt{\rho_3} + \sqrt{\rho_4}). \end{aligned}$$

If $m \equiv 7 \pmod 8$ then $\alpha\varepsilon_1^{-1} = (-mbd+2(ac+1))+(mbd-(ac+1))\sqrt{2}$ and $N(\frac{\alpha\varepsilon_1^{-1}}{a+2c}) = +1$.

Thus

$$\sqrt{\frac{\alpha\varepsilon_1^{-1}}{a+2c}} = \frac{1}{2}\left(\sqrt{\frac{-2mbd+4ac+2a+4c+4}{a+2c}} + \sqrt{\frac{-2mbd+4ac-2a-4c+4}{a+2c}}\right)$$

so $\sqrt{\alpha\varepsilon_1^{-1}} = \frac{1}{2}(\sqrt{\gamma_1} + \sqrt{\gamma_2})$. Similarly $\sqrt{\beta\varepsilon_1^{-1}} = \frac{1}{2}(\sqrt{\gamma_3} + \sqrt{\gamma_4})$. The expressions for $\sqrt[4]{\iota\varepsilon_2\varepsilon_3}$ and $\sqrt[4]{\iota\varepsilon_1^2\varepsilon_2\varepsilon_3}$ are immediate.

Corollary 23 *If $m|(ac+1)$ or $m|(ac-1)$ then either $\sqrt[4]{\iota\varepsilon_2\varepsilon_3} \in K$ or $\sqrt[4]{\iota\varepsilon_1^2\varepsilon_2\varepsilon_3} \in K$ according as $m \equiv 7 \pmod 8$ or $m \equiv 3 \pmod 8$. Conversely, if $\sqrt[4]{\iota\varepsilon_2\varepsilon_3} \in K$ or $\sqrt[4]{\iota\varepsilon_1^2\varepsilon_2\varepsilon_3} \in K$ then $m|(ac+1)$ or $m|(ac-1)$.*

Proof If $m \equiv 7 \pmod 8$ then

$$\begin{aligned} a^2c^2 - 1 &= a^2c^2 - 2c^2 + 2c^2 - 1 \\ &= c^2(a^2 - 2) + 2c^2 - 1 \\ &= c^2(r-1) + \frac{u-1}{2} \\ &= mb^2c^2 + md^2. \end{aligned}$$

Similarly, $a^2c^2 - 1 = mb^2c^2 - md^2$ if $m \equiv 3 \pmod 8$. Now let p be a prime with $p \not| m$, $p|bd$ and $p|(ac+1)$ and note that $p \neq 2$ since bd is odd. Since $a^2c^2 - 1 = mb^2c^2 \pm md^2$ it follows that $p|b$ and $p|d$. Thus $p^2|(ac+1)$. Suppose that $p^i|b$, $p^i|d$ and $p^{2i}|(ac+1)$. Now $\frac{a^2c^2-1}{p^{2i}} = mc^2(\frac{b}{p^i})^2 - m(\frac{d}{p^i})^2$ so if $p|\frac{ac+1}{p^{2i}}$ and $p|\frac{bd}{p^{2i}}$ then $p^2|\frac{ac+1}{p^{2i}}$ and $p^2|\frac{bd}{p^{2i}}$. Thus the highest power of p dividing $\gcd(bd, ac+1)$ must be even. The same argument holds for bd and $ac-1$. Now suppose that $m|(ac+1)$ or $m|(ac-1)$. By Lemma 22 it will suffice to show that $\sqrt{\alpha\varepsilon_1^{-1}}, \sqrt{\beta\varepsilon_1^{-1}} \in K$ or $\sqrt{\alpha}, \sqrt{\beta} \in K$ according as $m \equiv 7 \pmod 8$ or $m \equiv 3 \pmod 8$. The argument above shows that $\alpha = m^i\alpha_1^2\pi_1^{c_1}\cdots\pi_s^{c_s}\varepsilon_1^j$ and $\beta = m^{1-i}\beta_1^2\tau_1^{b_1}\cdots\tau_t^{b_t}\varepsilon_1^k$ where $\alpha_1, \beta_1 \in Z$, $\pi_1, \ldots, \pi_s$ (resp. $\tau_1, \ldots, \tau_t$) are nonconjugate primes in $Z[\sqrt{2}]$ and $i = 0$ or 1 according as $m|(ac+1)$ or $m|(ac-1)$. As in the proof of Lemma 22,

$$N(\alpha) = \begin{cases} -(ac+2)^2 & if \quad m \equiv 7 \pmod 8 \\ (a-2c)^2 & if \quad m \equiv 3 \pmod 8 \end{cases}$$

and

$$N(\beta) = \begin{cases} -(a-2c)^2 & if \quad m \equiv 7 \pmod 8 \\ (a+2c)^2 & if \quad m \equiv 3 \pmod 8). \end{cases}$$

It follows that $c_1, \ldots, c_s, b_1, \ldots, b_t$ are even and j and k are both odd or even according as $m \equiv 7 \pmod 8$ or $m \equiv 3 \pmod 8$. Therefore, $\sqrt{\alpha\epsilon_1^{-1}}, \sqrt{\beta\varepsilon_1^{-1}} \in K$ if $m \equiv 7 \pmod 8$ and $\sqrt{\alpha}, \sqrt{\beta} \in K$ if $m \equiv 3 \pmod 8$. Suppose $m = p_1 \cdots p_x$ with $p_1 \cdots p_y|(ac+1)$ and $p_{y+1} \cdots p_x|(ac-1)$ for some $y < x$. Then as above, $\alpha = p_1 \cdots p_y \alpha_1^2 \pi_1^{c_1} \cdots \pi_s^{c_s} \varepsilon_1^j$ and $\beta = p_{y+1} \cdots p_x \beta_1^2 \tau_1^{b_1} \cdots \tau_t^{b_t} \varepsilon_1^k$ with $c_1, \ldots, c_s, b_1, \ldots, b_t$ even. Now $\sqrt{p_1 \cdots p_y}, \sqrt{p_{y+1} \cdots p_x} \notin K$ so it follows that $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\alpha\varepsilon_1^{-1}}, \sqrt{\beta\varepsilon_1^{-1}} \notin K$ if j and k are odd and $\sqrt{\alpha}, \sqrt{\beta} \notin K$ if j and k are even. Note that if j and k are even and $\sqrt{\alpha\varepsilon_1}, \sqrt{\beta\varepsilon_2} \in K$ then $\sqrt{n\varepsilon_1} \in K$ for some positive integer n. Thus $k_1(\sqrt{n\varepsilon_1}) = K_0$, so $n\varepsilon_1 = mz^2$ for some $z \in k_1$. Thus $-n^2 = m^2 N(z)^2 > 0$, a contradiction.

Corollary 24 *If m is prime then either $\sqrt[4]{\iota\varepsilon_2\varepsilon_3} \in K$ or $\sqrt[4]{\iota\varepsilon_1^2\varepsilon_2\varepsilon_3} \in K$.*

Proof As in Corollary 23, $a^2c^2 - 1 = mb^2c^2 \pm md^2$ so $m|(ac+1)$ or $m|(ac-1)$.

If $N\varepsilon_1 = N\varepsilon_2 = N\varepsilon_3 = -1$ define

$$\begin{aligned} z_1 &= (r_1 + 2^{a_1}\iota)(r_2 + 2^{a_2}\iota)(r_3 + 2^{a_3}\iota), \\ z_2 &= (r_1 + 2^{a_1}\iota)(r_2 + 2^{a_2}\iota)(r_3 - 2^{a_3}\iota), \end{aligned}$$

$$\begin{aligned} z_3 &= (r_1 + 2^{a_1}\iota)(r_2 - 2^{a_2}\iota)(r_3 - 2^{a_3}\iota) \text{ and} \\ z_4 &= (r_1 - 2^{a_1}\iota)(r_2 + 2^{a_2}\iota)(r_3 + 2^{a_3}\iota). \end{aligned}$$

Lemma 25 *If $N\varepsilon_1 = N\varepsilon_2 = N\varepsilon_3 = -1$ then*

$$\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3} = \frac{1}{4}\sqrt{2^{2-a_1-a_2-a_3}}\sum_{j=1}^{4}\sqrt{Re\, z_j + |z_j|}.$$

Proof Now $N_i(\iota\varepsilon_j) = +1$ so

$$\sqrt{\iota\varepsilon_j} = \frac{1}{2}\left(\sqrt{2^{1-a_j}(2^{a_j} + r_j\iota)} + \sqrt{2^{1-a_j}(2^{a_j} - r_j\iota)}\right).$$

Thus

$$\sqrt{\varepsilon_j} = \frac{1}{2}\sqrt{2^{1-a_j}}(\sqrt{r_j + 2^{a_j}\iota} + \sqrt{r_j - 2^{a_j}\iota}).$$

Note that

$$|z_j| = \sqrt{(r_1^2 + 2^{2a_1})(r_2^2 + 2^{2a_1})(r_3^2 + 2^{2a_3})} = s_1 s_2 s_3\sqrt{m_1 m_2 m_3}$$

and

$$Re\sqrt{z_j} = \sqrt{|z_j|}\cos\frac{arg(z_j)}{2} = \sqrt{|z_j|}\sqrt{\frac{1}{2}(\frac{Re\, z_j}{|z_j|} + 1)} = \sqrt{\frac{1}{2}(Re\, z_j + |z_j|)}.$$

Therefore

$$\begin{aligned} \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3} &= \frac{1}{8}\sqrt{2^{3-a_1-a_2-a_3}}\sum_{j=1}^{4}(\sqrt{z_j} + \sqrt{\bar{z}_j}) \\ &= \frac{1}{8}\sqrt{2^{3-a_1-a_2-a_3}}\sum_{j=1}^{4}2Re\sqrt{z_j} \\ &= \frac{1}{4}\sqrt{2^{2-a_1-a_2-a_3}}\sum_{j=1}^{4}\sqrt{Re\, z_j + |z_j|}. \end{aligned}$$

Theorem 26 *If $N\varepsilon_i = +1$ for some i then $[E_K : WE_{K_0}] = 2$ if and only if one of the following holds:*

1. *$\sqrt{-1} \notin K$ and $\Delta \cap D \neq \emptyset$.*

2. *$\sqrt{-1} \in K$, $\sqrt{-2} \notin K$ and $2\Delta \cap D \neq \emptyset$.*

3. *$\sqrt{-1} \in K$, $\sqrt{-2} \in K$,$d_1 = \Delta_2 = \Delta_3 = 2$,$d_2 \equiv 3 \pmod 4$ and either $d_2|(ac+1)$ or $d_2|(ac-1)$.*

If $N\varepsilon_1 = N\varepsilon_2 = N\varepsilon_3 = -1$ then $[E_K : WE_{K_0}] = 2$ if and only if $\sqrt{-1} \notin K$ and $2^{a_1-a_2-a_3}(Re\, z_i + |z_i|) \in D$ for $i = 1, 2, 3, 4$.

Proof Suppose $N\varepsilon_i = +1$ for some i. If $\sqrt{-1} \notin K$ and $\Delta \cap D \neq \emptyset$ then $\sqrt{-\varepsilon_1^a \varepsilon_2^b \varepsilon_3^c} \in E_K - WE_{K_0}$ for some $a, b, c \in \{0, 1\}$ not all zero. If $\sqrt{-1} \in K$, $\sqrt{-2} \notin K$ and $2\Delta \cap D \neq \emptyset$ then $\sqrt{\iota\varepsilon_1^a \varepsilon_2^b \varepsilon_3^c} \in E_K - WE_{K_0}$ for some $a, b, c \in \{0, 1\}$. If $\sqrt{-1}, \sqrt{-2} \in K$, $d_1 = \Delta_2 = \Delta_3 = 2$, $d_3 \equiv 3 \pmod 4$ and either $d_2|(ac+1)$ or $d_2|(ac-1)$ then Corollary 23 shows that either $\sqrt[4]{\iota\varepsilon_2\varepsilon_3} \in E_K - WE_{K_0}$ or $\sqrt[4]{\iota\varepsilon_1^2\varepsilon_2\varepsilon_3} \in E_K - WE_{K_0}$. Thus if (1),(2) or (3) hold then $[E_K : WE_{K_0}] = 2$. Conversely, if $[E_K : WE_{K_0}] = 2$ then either $[E_K : E^*] = 2$ or $[E^* : WE_{K_0}] = 2$. If $[E^* : WE_{K_0}] = 2$, then an imaginary biquadratic subfield of K contains a unit not in WE_{K_0}. Thus either $\sqrt{-1} \notin K$ and $\sqrt{-\varepsilon_j}$ is in a biquadratic subfield for some j or $\sqrt{-1} \in K$, $\sqrt{-2} \notin K$ and $\sqrt{\iota\varepsilon_j}$ is in a biquadratic subfield. Moreover, $N\varepsilon_j = +1$ and $\Delta \cap D \neq \emptyset$. If $[E_K : E^*] = 2$, then by Lemma 18 either $\sqrt{-\varepsilon_1^a\varepsilon_2^b\varepsilon_3}$, $\sqrt{\iota\varepsilon_a^a\varepsilon_2^b\varepsilon_3}$, $\sqrt[4]{\iota\varepsilon_2\varepsilon_3}$ or $\sqrt[4]{\iota\varepsilon_1^2\varepsilon_2\varepsilon_3}$ must be in $E_K - E^*$. If $\sqrt{-\varepsilon_1^a\varepsilon_2^b\varepsilon_3} \in E_K - E^*$ then $\sqrt{-1} \notin K$, for otherwise $\sqrt{-\varepsilon_1^a\varepsilon_2^b\varepsilon_3} = \iota\sqrt{\varepsilon_1^a\varepsilon_2^b\varepsilon_3} \in E^*$. That $\Delta \cap D \neq \emptyset$ follows since $\sqrt{\epsilon_i} = w\sqrt{\Delta_i} + x\sqrt{\frac{d_i}{\Delta_i}}$ for some $w, x \in \mathbf{Q}$. If $\sqrt{\iota\varepsilon_1^a\varepsilon_2^b\varepsilon_3} \in E_K - E^*$ then it follows from the above expression for $\sqrt{\varepsilon_i}$ that $\sqrt{-1} \in K$ and $2\Delta \cap D \neq \emptyset$, However $\sqrt{-2} \notin K$ since $\sqrt{\iota\varepsilon_1^a\varepsilon_2^b\varepsilon_3} \notin E^*$.

If $\sqrt[4]{\iota\varepsilon_2\varepsilon_3} \in E_K - E^*$ or $\sqrt[4]{\iota\varepsilon_1^2\varepsilon_2\varepsilon_3} \in E_K - E^*$ then it follows from Corollary 20 that $\sqrt{-1}, \sqrt{2} \in K$, $d_1 = \Delta_2 = \Delta_3 = 2$ and $d_2 \equiv 3 \pmod 4$. Corollary 23 shows that $d_2 | (ac+1)$ or $d_2 | (ac-1)$. If $N\varepsilon_1 = N\varepsilon_2 = N\varepsilon_3 = -1$ the result follows from Lemmas 19 and 25.

Chapter 7

OCTIC FIELDS OF SMALL CLASS NUMBER

In this chapter we determine all imaginary octic fields of type $(2,2,2)$ having class number less than or equal to 16 or prime class number. For each of these fields we determine the structure of its class group.

For this section K will be an imaginary octic field of type $(2,2,2)$ with imaginary quadratic subfields numbered so that $h_4 \leq h_5 \leq h_6 \leq h_7$. Recall that ker refers to the kernel of the mapping $\theta : H_1 \times H_2 \times H_3 \to H$. The following lemma is used in determining ker.

Lemma 27 *Let $M = L(\sqrt{m})$ be a quadratic extension of L and let A be an ideal of M which is ambiguous for M/L. If A is a principal ideal of M then either $A = (\sqrt{m}\beta)$ for some $\beta \in L$ or there is a unit e of M, with $N_{M/L}(e) = +1$, such that $(1+e) = A(\beta)$ for some $\beta \in L$.*

Proof Let σ be a generator of the Galois group of M/L and let $A = (\alpha)$. Then $A = (\alpha^\sigma)$ so $\alpha = e\alpha^\sigma$ for some unit e of M. Now $e = \alpha^{1-\sigma}$ so $N_{M/L}(e) = e^{1+\sigma} = (\alpha^{1-\sigma})^{1+\sigma} = \alpha^{1-\sigma^2} = 1$. If $e = -1$ then $(\sqrt{m})^\sigma = -\sqrt{m}$ so $\left(\frac{\alpha}{\sqrt{m}}\right)^\sigma = \frac{\alpha}{\sqrt{m}} = \beta \in L$. Suppose $e \neq -1$. Then $(1+e)^\sigma e = e + ee^\sigma = 1 + e$ so $e = \frac{1+e}{(1+e)^\sigma} = \frac{\alpha}{\alpha^\sigma}$ or $\left(\frac{\alpha}{1+e}\right)^\sigma = \frac{\alpha}{1+e}$. Thus $\left(\frac{1+e}{\alpha}\right) = \beta \in L$, so

$(1+e) = (\alpha)(\beta) = A(\beta)$.

Lemma 28 *Let M/L be a quadratic extension and let h_M and h_L be the class numbers of M and L, respectively. Then $h_M \geq \frac{1}{2}h_L$.*

Proof Let F_M and F_L be the Hilbert class fields of M and L, respectively. Then $MF_L \subseteq F_M$ and $h_M = [F_M : M] \geq [MF_L : M] = \frac{1}{2}h_L[MF_L : F_L] \geq \frac{1}{2}h_L$.

Lemma 29 *Let K be an octic field with imaginary quadratic subfields k_i and k_j. Then $h \geq \frac{1}{4}h_i h_j$.*

Proof This follows from Lemma 28 and the class number formula for imaginary biquadratic fields.

Lemma 30 *Let K be an octic field such that $h \leq 16$ or $h = p$ for an odd prime p. If $h_7 \geq 16$ then $h_4, h_5, h_6 \leq 4$.*

Proof By Lemma 29, $h \geq \frac{1}{4}h_i h_7$ for $i = 4, 5, 6$, so if $h \leq 16$ and $h_7 \geq 16$ then $h_6 \leq 4$. If $h = p > 16$ then it follows from Lemma 28 that 4 is the highest power of 2 dividing any h_i. Thus if $h_7 \geq 16$ then h_7 must be divisible by p. Since only one h_i is divisible by p, it follows that $h_i \leq 4$.

Recall that t'_i denotes the number of rational primes which ramify in the extension $k_i/\mathbf{Q}$ and t' denotes the integer such that $2^{t'}$ is the product of the ramification indices of all rational primes for the extension $K/\mathbf{Q}$. Also, w denotes the integer such that 2^w is the 2-class number of K.

Lemma 31 *If 2 is maximally ramified in K then $t' \le \frac{w+17}{4}$. Otherwise $t' \le \frac{w+15}{4}$.*

Proof The 2-class number of k_i is greater than or equal to $2^{t_i'-2}$ or $2^{t_i'-1}$ according as k_i is real or imaginary. Each odd rational prime which ramifies in K ramifies in exactly four quadratic subfields and 2 ramifies in either four or six quadratic subfields. Thus $\sum_{i=1}^{7} t_i' = 4t' - 2$ or $4t'$ according as 2 is maximally ramified or not. According to Wada [22], $h = \frac{1}{32}\mathbf{Q}\prod_{i=1}^{7} h_i$ so we have

$$h \geq 2^{\sum_{i=1}^{7} t_i' - 15} = \begin{cases} 2^{4t'-17} & \text{if 2 is maximally ramified,} \\ 2^{4t'-15} & \text{otherwise.} \end{cases}$$

The result now follows.

Since an imaginary octic field of type $(2,2,2)$ is completely determined by three imaginary quadratic fields, we will consider the following set of fields. Let $\mathcal{F}$ be the set of octic fields determined by choosing three imaginary quadratic fields F_1, F_2 and F_3 with class numbers $h_{F_i} = 2^{f_i} h'_{F_i}$ and h'_{F_i} odd, subject to the following conditions:

1. F_i belongs to the set of fields known to have class number less than 16.

2. $h'_{F_1} h'_{F_2} h'_{F_3} < 8$.

3. $f_1 + f_2 + f_3 \leq 6$.

4. If $h'_{F_1} h'_{F_2} h'_{F_3} > 1$ then $2^{f_i + f_j} h'_{F_1} h'_{F_2} h'_{F_3} \leq 20$.

Theorem 32 *If K is an octic field with $h \leq 16$ or $h = p$ for an odd prime p then $K \in \mathcal{F}$.*

Proof Lemma 29 shows that either $h_7 \geq 16$ and $h_4, h_5, h_6 \leq 4$ or $h_4, h_5, h_6, h_7 \leq 16$. Now all imaginary quadratic fields of class number less than or equal to 4 are known [1, 15, 18, 19]. Moreover, there is at most one imaginary quadratic field of class number less than 16 which is unknown [10]. Thus K is determined by three fields F_1, F_2 and F_3 known to have class number less than 16. Let F_4 be the fourth imaginary quadratic subfield. It follows from that class number formula that $h'_{F_1} h'_{F_2} h'_{F_3} < 16$. However, if $h'_{F_1} h'_{F_2} h'_{F_3} = 9, 11, 13$ or 15 then $h_{F_4} = 1, 2$ or 4 and we may take $h'_{F_1} = 1$ and $h'_{F_2} = 1$ or 3. Thus replacing F_3 with F_4 we have $h'_{F_1} h'_{F_2} h'_{F_3} < 8$. It also follows from the class number formula that F_1, F_2 and F_3 can be chosen so that $f_1 + f_2 + f_3 \leq 6$. Now suppose $h'_{F_1} h'_{F_2} h'_{F_3} > 1$. Since $h'_{F_1} h'_{F_2} h'_{F_3} < 8$ we may assume that $h'_{F_1} = h'_{F_2} = 1$ and $h'_{F_3} = 3, 5$ or 7. Note that if $h = 3, 5$ or 7 then K is generated by three fields F_1, F_2 and F_3 with $h_{F_1}, h_{F_2}, h_{F_3} \leq 4$. Hence we may assume that $h = 6, 9, 10, 12, 14$ or 15. If $h = 9$ or 15 then $h'_{F_3} = 3$ or 5 and $2^{f_i + f_j - 2}$ divides h for any $i, j = 1, 2, 3$. Thus $f_i + f_j \leq 2$ and $2^{f_i + f_j} h'_{F_1} h'_{F_2} h'_{F_3} \leq 20$. If $h = 6, 10$ or 14 then h_{F_4} is a power of 2. We may assume $h_{F_4} \geq 8$, for otherwise K is generated by three fields of class number less than or equal to 4. But $2^{f_i + f_4 - 2}$ divides h, for $i = 1, 2, 3$, so $h_{F_1} = h_{F_2} = 1$ and $h_{F_3} = 3, 5$ or 7. Thus $2^{f_i + f_j} h'_{F_1} h'_{F_2} h'_{F_3} \leq 20$ for any $i, j = 1, 2, 3$. If $h = 12$ then as above we may assume that $h_{F_4} = 2^a$ for some $a \leq 3$. But then $2^{f_i + 1}$ divides h for any $i = 1, 2, 3$. Thus $f_1, f_2, f_3 \leq 1$ and $2^{f_i + f_j} h'_{F_1} h'_{F_2} h'_{F_3} \leq 12$.

Corollary 33 *Let K be and octic field with $h \leq 16$ or $h = p$ for an odd prime p. Then K is determined by imaginary quadratic fields F_1, F_2 and F_3 satisfying one of the following:*

1. $h_{F_1}, h_{F_2}, h_{F_3} \leq 4$,

2. $h_{F_3} = 5, 6$ *or* 10 *and* $h_{F_1} \leq h_{F_2} \leq 2$,

3. $h_{F_3} = 7$ *or* 12 *and* $h_{F_1} = h_{F_2} = 1$,

4. $h_{F_3} = 8$ *and* $h_{F_1} h_{F_2} = 2^i$ *with* $0 \leq i \leq 3$.

Proof If $h \leq 8$ or if h is prime then K is determined by fields with $h_{F_1}, h_{F_2}, h_{F_3} \leq 4$. Thus we may assume that $h = 9, 10, 12, 14, 15$ or 16. By Theorem 32, $h_{F_i} < 16$ for $i = 1, 2, 3$ and $h'_{F_1} h'_{F_2} h'_{F_3} < 8$ so $h_{F_3} = 5, 6, 7, 8, 10$ or 12. If $h_{F_3} = 8$ then $h_{F_1} h_{F_2} = 2^i$ with $0 \leq i \leq 3$ since $f_1 + f_2 + f_3 \leq 6$. If $h_{F_3} = 5, 6, 7, 8$ or 10 then the conditions on h_{F_1} and h_{F_2} follow from the inequality $2^{f_i + f_j} h'_{F_1} h'_{F_2} h'_{F_3} \leq 20$.

Lemma 34 *Let* K *be an octic field with* $h \leq 16$ *or* $h = p$ *for an odd prime* p. *Suppose* K *is determined by imaginary quadratic fields* F_1, F_2 *and* F_3 *satisfying the conditions of Corollary 33 and* $h_{F_1} \leq h_{F_2} \leq h_{F_3}$. *If* $h_{F_2} h_{F_3} > 4$ *then* F_4 *is a known field of class number less than or equal to 16 or* $discF_4 > 4000000$.

Proof It follows from Lemma 29 and the class number formula that if $h_{F_2} h_{F_3} > 4$ then $h_{F_4} \leq 16$. That F_4 is a known field or $discF_4 > 4000000$ follows from Buell [3].

Lemma 35 *If* K *is an octic field with* $h \leq 16$ *or* $h = p$ *for an odd prime* p *then* $t' \leq 5$ *and* $t' = 5$ *only if 2 is maximally ramified and* $h = 8$ *or* 16.

Proof This is immediate from Lemma 31.

Theorem 36 *The imaginary octic fields of type* $(2, 2, 2)$ *having class number less than or equal to 16 or prime class number are listed below. In addition, the class group of each field* *is given. The first two columns give the class number and conductor of K. The next three columns give three imaginary quadratic fields which generate K. The last column gives the class group of K when necessary.*

1	24	-1	-2	-3	
1	40	-1	-2	-5	
1	60	-1	-3	-5	
1	84	-1	-3	-7	
1	88	-1	-2	-11	
1	105	-3	-7	-15	
1	120	-2	-3	-10	
1	132	-1	-3	-11	
1	140	-1	-5	-7	
1	168	-2	-3	-7	
1	228	-1	-3	-19	
1	264	-3	-6	-11	
1	280	-2	-7	-10	
1	364	-1	-7	-13	
1	532	-1	-7	-19	
1	561	-3	-11	-51	
1	627	-3	-11	-19	
2	56	-1	-2	-7	
2	120	-3	-6	-15	
2	156	-1	-3	-13	
2	165	-3	-11	-15	
2	168	-3	-6	-7	
2	204	-1	-3	-17	
2	220	-1	-5	-11	
2	264	-2	-3	-11	
2	273	-3	-7	-39	
2	308	-1	-7	-11	
2	385	-7	-11	-35	
2	408	-3	-6	-51	
2	408	-2	-3	-34	
2	440	-2	-10	-11	
2	456	-2	-3	-19	
2	616	-7	-11	-14	
2	616	-2	-7	-11	
2	748	-1	-11	-17	
2	969	-3	-19	-51	
2	984	-2	-3	-82	
2	1032	-2	-3	-43	
2	1036	-1	-7	-37	
2	1204	-1	-7	-43	
2	1496	-2	-11	-34	
2	1624	-2	-7	-58	
2	1672	-2	-11	-19	
2	3553	-11	-19	-187	
3	104	-1	-2	-13	
3	152	-1	-2	-19	
3	231	-3	-7	-11	
3	232	-1	-2	-29	
3	345	-3	-15	-23	
3	372	-1	-3	-31	
3	456	-3	-6	-19	
3	460	-1	-5	-23	
3	483	-3	-7	-23	
3	516	-1	-3	-43	
3	696	-2	-3	-58	
3	708	-1	-3	-59	
3	728	-2	-7	-26	
3	804	-1	-3	-67	
3	805	-7	-23	-35	
3	920	-2	-10	-23	
3	988	-1	-13	-19	
3	1012	-1	-11	-23	
3	1353	-3	-11	-123	
3	1612	-1	-13	-31	
3	1672	-11	-19	-22	
3	1729	-7	-19	-91	
3	2821	-7	-31	-91	
4	120	-5	-10	-15	(4)
4	120	-3	-5	-6	(2,2)
4	120	-2	-6	-10	(4)
4	120	-2	-5	-6	(4)
4	120	-2	-3	-5	(4)
4	120	-1	-6	-10	(4)
4	120	-1	-5	-6	(4)
4	120	-1	-3	-10	(4)
4	120	-1	-2	-15	(4)
4	136	-1	-2	-17	(2,2)
4	168	-3	-6	-21	(2,2)
4	168	-2	-6	-7	(2,2)
4	168	-1	-6	-7	(2,2)
4	168	-1	-2	-21	(2,2)

4	195	-3	-15	-39	(4)
4	260	-1	-5	-13	(4)
4	264	-2	-6	-11	(2,2)
4	264	-2	-3	-22	(2,2)
4	264	-1	-6	-22	(4)
4	280	-7	-14	-35	(4)
4	280	-5	-10	-35	(2,2)
4	280	-2	-5	-7	(2,2)
4	280	-1	-7	-10	(4)
4	280	-1	-2	-35	(2,2)
4	285	-3	-15	-19	(4)
4	312	-3	-6	-39	(4)
4	312	-2	-3	-13	(2,2)
4	312	-1	-6	-13	(2,2)
4	340	-1	-5	-17	(4)
4	380	-1	-5	-19	(4)
4	399	-3	-7	-19	(4)
4	440	-5	-10	-11	(2,2)
4	444	-1	-3	-37	(4)
4	456	-2	-6	-19	(2,2)
4	492	-1	-3	-41	(4)
4	665	-7	-19	-35	(4)
4	760	-2	-10	-19	(4)
4	760	-2	-5	-19	(2,2)
4	760	-1	-10	-19	(2,2)
4	903	-3	-7	-43	(4)
4	1281	-3	-7	-183	(4)
4	1608	-2	-3	-67	(4)
4	2136	-2	-3	-178	(4)
4	2211	-3	-11	-67	(4)
4	2937	-3	-11	-267	(4)
4	3819	-3	-19	-67	(4)
4	5896	-2	-11	-67	(4)
4	8643	-3	-43	-67	(2,2)
5	296	-1	-2	-37	
5	344	-1	-2	-43	
5	357	-3	-7	-51	
5	572	-1	-11	-13	
5	705	-3	-15	-47	
5	836	-1	-11	-19	
5	1001	-7	-11	-91	

5	1032	-3	-6	-43
5	1064	-2	-7	-19
5	1309	-7	-11	-119
5	1645	-7	-35	-47
5	1880	-2	-10	-47
5	1956	-1	-3	-163
5	2337	-3	-19	-123
5	2552	-2	-11	-58
5	2948	-1	-11	-67
5	3416	-2	-7	-122
5	5092	-1	-19	-67
6	184	-1	-2	-23
6	255	-3	-15	-51
6	276	-1	-3	-23
6	312	-2	-3	-26
6	465	-3	-15	-31
6	520	-2	-10	-26
6	552	-3	-6	-23
6	552	-2	-3	-23
6	609	-3	-7	-87
6	620	-1	-5	-31
6	651	-3	-7	-31
6	741	-3	-19	-39
6	812	-1	-7	-29
6	868	-1	-7	-31
6	1064	-7	-14	-19
6	1068	-1	-3	-89
6	1085	-7	-31	-35
6	1209	-3	-31	-39
6	1240	-2	-10	-31
6	1265	-11	-23	-55
6	1288	-2	-7	-23
6	1771	-7	-11	-23
6	1947	-3	-11	-59
6	1992	-2	-3	-83
6	2193	-3	-43	-51
6	2408	-2	-7	-43
6	3009	-3	-51	-59
6	3052	-1	-7	-109
6	3608	-2	-11	-82
6	3729	-3	-11	-339

6	4123	-7	-19	-31	
6	4233	-3	-51	-83	
6	4521	-3	-11	-411	
6	4564	-1	-7	-163	
6	5289	-3	-43	-123	
6	5336	-2	-23	-58	
6	5379	-3	-11	-163	
6	7372	-1	-19	-97	
6	11033	-11	-59	-187	
7	536	-1	-2	-67	
7	645	-3	-15	-43	
7	860	-1	-5	-43	
7	861	-3	-7	-123	
7	1505	-7	-35	-43	
7	1608	-3	-6	-67	
7	1720	-2	-10	-43	
7	2812	-1	-19	-37	
7	5896	-11	-22	-67	
7	11524	-1	-43	-67	
7	12388	-1	-19	-163	
8	168	-7	-14	-21	(2,4)
8	168	-2	-6	-14	(8)
8	168	-2	-3	-14	(8)
8	168	-1	-6	-14	(8)
8	168	-1	-3	-14	(8)
8	264	-11	-22	-33	(2,4)
8	264	-3	-6	-33	(8)
8	264	-2	-6	-22	(2,4)
8	264	-1	-6	-11	(8)
8	264	-1	-3	-22	(8)
8	264	-1	-2	-33	(2,4)
8	280	-5	-7	-10	(2,4)
8	280	-2	-10	-14	(8)
8	280	-2	-5	-14	(8)
8	280	-1	-10	-14	(8)
8	280	-1	-5	-14	(8)
8	312	-2	-6	-13	(8)
8	312	-1	-2	-39	(8)
8	408	-2	-6	-34	(2,2,2)
8	408	-1	-6	-17	(2,2,2)
8	408	-1	-2	-51	(2,2,2)

8	420	-7	-21	-35	(2,4)
8	420	-5	-15	-35	(2,4)
8	420	-5	-7	-15	(2,4)
8	420	-3	-15	-21	(2,4)
8	420	-3	-5	-21	(2,4)
8	420	-3	-5	-7	(2,4)
8	420	-1	-15	-21	(2,4)
8	420	-1	-7	-15	(2,4)
8	420	-1	-5	-21	(2,4)
8	420	-1	-3	-35	(2,4)
8	440	-2	-5	-22	(8)
8	440	-1	-10	-22	(8)
8	456	-3	-6	-57	(8)
8	456	-1	-6	-19	(8)
8	456	-1	-2	-57	(2,4)
8	520	-2	-5	-13	(2,4)
8	520	-1	-10	-13	(2,4)
8	555	-3	-15	-111	(8)
8	616	-2	-7	-22	(2,2,2)
8	616	-1	-7	-22	(2,4)
8	660	-5	-11	-15	(2,4)
8	660	-3	-15	-33	(2,2,2)
8	660	-3	-5	-11	(2,4)
8	660	-1	-11	-15	(2,2,2)
8	660	-1	-5	-33	(2,4)
8	728	-2	-13	-14	(8)
8	740	-1	-5	-37	(8)
8	780	-3	-5	-13	(2,2,2)
8	780	-1	-13	-15	(2,2,2)
8	840	-15	-30	-35	(2,4)
8	840	-10	-15	-35	(2,2,2)
8	840	-7	-35	-42	(2,4)
8	840	-7	-10	-15	(2,4)
8	840	-6	-7	-30	(2,4)
8	840	-6	-7	-15	(2,4)
8	840	-3	-15	-42	(2,4)
8	840	-3	-10	-35	(2,4)
8	840	-3	-7	-30	(2,4)
8	840	-3	-7	-10	(2,4)
8	840	-3	-6	-35	(2,2,2)
8	840	-2	-15	-35	(2,4)

8	840	-2	-10	-42	(2,4)
8	840	-2	-7	-15	(2,4)
8	840	-2	-3	-35	(2,4)
8	876	-1	-3	-73	(2,2,2)
8	920	-2	-5	-46	(8)
8	920	-1	-10	-46	(8)
8	924	-7	-11	-21	(2,4)
8	1020	-3	-5	-51	(8)
8	1020	-1	-15	-51	(2,4)
8	1032	-1	-6	-43	(8)
8	1064	-2	-14	-19	(8)
8	1092	-7	-21	-91	(2,4)
8	1092	-3	-13	-21	(2,4)
8	1092	-3	-7	-13	(2,4)
8	1092	-1	-13	-21	(2,4)
8	1092	-1	-3	-91	(2,2,2)
8	1140	-5	-15	-19	(2,4)
8	1155	-7	-11	-15	(2,4)
8	1155	-3	-11	-35	(2,4)
8	1164	-1	-3	-97	(2,4)
8	1292	-1	-17	-19	(2,2,2)
8	1320	-11	-15	-22	(2,4)
8	1320	-3	-10	-11	(2,2,2)
8	1320	-2	-11	-15	(2,2,2)
8	1365	-7	-15	-91	(2,2,2)
8	1365	-3	-35	-91	(2,4)
8	1380	-5	-15	-115	(2,2,2)
8	1380	-1	-3	-115	(2,4)
8	1428	-3	-21	-51	(2,4)
8	1428	-1	-7	-51	(2,4)
8	1463	-7	-11	-19	(8)
8	1540	-5	-7	-11	(2,2,2)
8	1540	-1	-11	-35	(2,2,2)
8	1560	-10	-15	-130	(2,4)
8	1560	-2	-3	-130	(2,2,2)
8	1596	-1	-21	-57	(2,2,2)
8	1628	-1	-11	-37	(8)
8	1672	-1	-19	-22	(8)
8	1752	-3	-6	-219	(2,4)
8	1785	-7	-15	-51	(2,4)
8	1820	-5	-13	-35	(2,4)
8	1848	-6	-7	-11	(2,4)
8	1848	-3	-7	-22	(2,4)
8	1924	-1	-13	-37	(8)
8	1995	-7	-15	-19	(2,2,2)
8	1995	-3	-19	-35	(2,2,2)
8	2024	-1	-22	-46	(8)
8	2145	-3	-11	-195	(2,2,2)
8	2184	-6	-7	-78	(2,2,2)
8	2220	-3	-5	-37	(2,2,2)
8	2220	-1	-15	-37	(2,2,2)
8	2236	-1	-13	-43	(8)
8	2244	-1	-33	-51	(2,4)
8	2280	-6	-15	-19	(2,4)
8	2280	-3	-19	-30	(2,4)
8	2380	-5	-35	-85	(2,4)
8	2380	-1	-7	-85	(2,4)
8	2415	-7	-15	-115	(2,4)
8	2415	-3	-35	-115	(2,4)
8	2508	-11	-19	-33	(2,4)
8	2508	-3	-33	-57	(2,4)
8	2508	-1	-19	-33	(2,4)
8	2508	-1	-11	-57	(2,4)
8	2660	-5	-19	-35	(2,4)
8	2860	-5	-11	-13	(2,2,2)
8	3003	-3	-11	-91	(2,4)
8	3080	-11	-22	-35	(2,4)
8	3417	-3	-51	-67	(8)
8	3640	-2	-35	-91	(2,4)
8	3740	-5	-11	-85	(2,2,2)
8	3913	-7	-43	-91	(8)
8	4836	-3	-13	-93	(2,4)
8	5005	-11	-35	-91	(2,4)
8	5016	-6	-19	-22	(2,4)
8	5060	-5	-11	-115	(2,4)
8	5320	-7	-19	-70	(2,2,2)
8	5548	-1	-19	-73	(2,4)
8	5720	-2	-11	-130	(2,2,2)
8	6545	-11	-35	-187	(2,4)
8	6916	-13	-19	-91	(2,4)
8	9291	-3	-19	-163	(8)
8	10659	-19	-51	-187	(2,4)

8	12529	-11	-67	-187	(8)
8	21027	-3	-43	-163	(8)
9	348	-1	-3	-29	(3,3)
9	424	-1	-2	-53	(3,3)
9	472	-1	-2	-59	(3,3)
9	744	-2	-3	-31	(3,3)
9	856	-1	-2	-107	(3,3)
9	996	-1	-3	-83	(3,3)
9	1005	-3	-15	-67	(9)
9	1340	-1	-5	-67	(9)
9	1416	-3	-6	-59	(3,3)
9	1419	-3	-11	-43	(3,3)
9	1668	-1	-3	-139	(3,3)
9	1708	-1	-7	-61	(3,3)
9	1892	-1	-11	-43	(3,3)
9	1976	-2	-19	-26	(3,3)
9	2387	-7	-11	-31	(3,3)
9	2680	-2	-10	-67	(9)
9	2728	-2	-11	-31	(3,3)
9	3224	-2	-26	-31	(3,3)
9	3304	-2	-7	-59	(3,3)
9	3652	-1	-11	-83	(3,3)
9	3892	-1	-7	-139	(3,3)
9	4396	-1	-7	-157	(3,3)
9	4587	-3	-11	-139	(3,3)
9	4588	-1	-31	-37	(3,3)
9	4648	-2	-7	-83	(3,3)
9	4708	-1	-11	-107	(3,3)
9	5192	-11	-22	-59	(3,3)
9	5656	-2	-7	-202	(3,3)
9	6963	-3	-11	-211	(3,3)
9	7172	-1	-11	-163	(9)
9	7657	-19	-31	-247	(3,3)
9	7912	-2	-23	-43	(3,3)
9	8241	-3	-67	-123	(9)
9	9416	-11	-22	-107	(3,3)
9	14003	-11	-19	-67	(9)
10	429	-3	-11	-39	
10	455	-7	-35	-91	
10	476	-1	-7	-17	
10	564	-1	-3	-47	
10	595	-7	-35	-119	
10	615	-3	-15	-123	
10	795	-3	-15	-159	
10	952	-2	-7	-34	
10	987	-3	-7	-47	
10	1128	-2	-3	-47	
10	1659	-3	-7	-79	
10	2072	-2	-7	-74	
10	2136	-3	-6	-267	
10	2163	-3	-7	-103	
10	2261	-7	-19	-119	
10	2408	-7	-14	-43	
10	2409	-3	-11	-219	
10	2451	-3	-19	-43	
10	2585	-11	-47	-55	
10	2632	-2	-7	-47	
10	2667	-3	-7	-127	
10	3212	-1	-11	-73	
10	3423	-3	-7	-163	
10	4323	-3	-11	-131	
10	4539	-3	-51	-267	
10	6104	-2	-7	-218	
10	6232	-2	-19	-82	
10	7189	-7	-79	-91	
10	9417	-3	-43	-219	
10	12556	-1	-43	-73	
10	13737	-3	-19	-723	
10	32763	-3	-67	-163	
11	1869	-3	-7	-267	
11	3484	-1	-13	-67	
11	3912	-3	-6	-163	
11	6364	-1	-37	-43	
11	11481	-3	-43	-267	
11	14344	-11	-22	-163	
11	18361	-7	-43	-427	
11	43684	-1	-67	-163	
12	248	-1	-2	-31	(3,4)
12	312	-3	-6	-13	(3,2,2)
12	312	-2	-6	-26	(3,2,2)
12	312	-1	-3	-26	(3,2,2)
12	435	-3	-15	-87	(3,4)

12	440	-11	-22	-55	(3,4)
12	440	-2	-10	-22	(3,2,2)
12	440	-2	-5	-11	(3,2,2)
12	440	-1	-10	-11	(3,2,2)
12	440	-1	-5	-22	(3,2,2)
12	456	-2	-3	-38	(3,2,2)
12	456	-1	-6	-38	(3,2,2)
12	644	-1	-7	-23	(3,4)
12	680	-2	-10	-34	(3,4)
12	696	-3	-6	-29	(3,2,2)
12	696	-2	-6	-58	(3,4)
12	696	-1	-6	-58	(3,4)
12	696	-1	-3	-58	(3,4)
12	728	-7	-14	-91	(3,4)
12	728	-2	-7	-13	(3,2,2)
12	728	-1	-2	-91	(3,2,2)
12	732	-1	-3	-61	(3,4)
12	744	-3	-6	-93	(3,2,2)
12	744	-3	-6	-31	(3,4)
12	744	-2	-6	-31	(3,2,2)
12	744	-1	-2	-93	(3,2,2)
12	760	-5	-10	-19	(3,2,2)
12	760	-2	-10	-38	(3,2,2)
12	760	-1	-5	-38	(3,2,2)
12	885	-3	-15	-59	(3,4)
12	888	-2	-3	-37	(3,2,2)
12	888	-1	-6	-37	(3,2,2)
12	984	-3	-6	-123	(3,4)
12	1023	-3	-11	-31	(3,4)
12	1032	-2	-6	-43	(3,2,2)
12	1060	-1	-5	-53	(3,4)
12	1064	-2	-7	-38	(3,2,2)
12	1144	-11	-13	-22	(3,2,2)
12	1144	-2	-11	-13	(3,4)
12	1144	-1	-13	-22	(3,4)
12	1180	-1	-5	-59	(3,4)
12	1196	-1	-13	-23	(3,4)
12	1356	-1	-3	-113	(3,4)
12	1416	-2	-6	-59	(3,2,2)
12	1416	-2	-3	-118	(3,2,2)
12	1416	-2	-3	-59	(3,4)
12	1416	-1	-6	-118	(3,2,2)
12	1484	-1	-7	-53	(3,4)
12	1608	-2	-6	-67	(3,2,2)
12	1644	-1	-3	-137	(3,4)
12	1720	-2	-5	-43	(3,4)
12	1720	-1	-10	-43	(3,4)
12	1804	-1	-11	-41	(3,4)
12	1876	-1	-7	-67	(3,4)
12	2065	-7	-35	-59	(3,4)
12	2093	-7	-23	-91	(3,4)
12	2185	-19	-23	-95	(3,4)
12	2332	-1	-11	-53	(3,4)
12	2356	-1	-19	-31	(3,4)
12	2360	-2	-10	-59	(3,4)
12	2552	-11	-22	-29	(3,2,2)
12	2680	-2	-5	-67	(3,4)
12	2680	-1	-10	-67	(3,4)
12	2712	-2	-3	-226	(3,4)
12	2728	-11	-22	-31	(3,4)
12	2739	-3	-11	-83	(3,4)
12	2968	-2	-7	-106	(3,4)
12	3256	-2	-11	-37	(3,2,2)
12	3256	-1	-22	-37	(3,2,2)
12	3336	-2	-3	-139	(3,4)
12	3363	-3	-19	-59	(3,4)
12	3723	-3	-51	-219	(3,4)
12	3752	-2	-7	-67	(3,4)
12	3784	-2	-22	-43	(3,2,2)
12	3784	-2	-11	-43	(3,4)
12	3796	-1	-13	-73	(3,4)
12	4664	-2	-11	-106	(3,4)
12	4712	-2	-19	-31	(3,4)
12	4947	-3	-51	-291	(3,4)
12	5908	-1	-7	-211	(3,4)
12	5992	-2	-7	-107	(3,4)
12	6744	-2	-3	-562	(3,4)
12	6792	-2	-3	-283	(3,4)
12	6923	-7	-23	-43	(3,4)
12	7257	-3	-59	-123	(3,4)
12	7368	-2	-3	-307	(3,4)
12	7756	-1	-7	-277	(3,4)

12	7923	-3	-19	-139	(3,4)
12	8308	-1	-31	-67	(3,4)
12	8344	-2	-7	-298	(3,4)
12	8968	-2	-19	-59	(3,4)
12	9916	-1	-37	-67	(3,4)
12	10209	-3	-83	-123	(3,4)
12	11649	-3	-11	-1059	(3,4)
12	17347	-11	-19	-83	(3,4)
12	20504	-2	-11	-466	(3,4)
13	4408	-2	-19	-58	
13	8313	-3	-51	-163	
13	9976	-2	-43	-58	
13	28036	-1	-43	-163	
13	30481	-11	-163	-187	
14	852	-1	-3	-71	
14	1065	-3	-15	-71	
14	1420	-1	-5	-71	
14	1435	-7	-35	-287	
14	1496	-11	-22	-187	
14	1533	-3	-7	-219	
14	1704	-3	-6	-71	
14	2044	-1	-7	-73	
14	2485	-7	-35	-71	
14	4921	-7	-19	-259	
14	5467	-7	-11	-71	
14	6248	-11	-22	-71	
14	8492	-1	-11	-193	
14	9709	-7	-19	-511	
14	10947	-3	-123	-267	
14	19497	-3	-67	-291	
14	23048	-2	-43	-67	
14	31089	-3	-43	-723	
14	34067	-11	-19	-163	
15	488	-1	-2	-61	
15	636	-1	-3	-53	
15	664	-1	-2	-83	
15	872	-1	-2	-109	
15	940	-1	-5	-47	
15	957	-3	-11	-87	
15	1048	-1	-2	-131	
15	1144	-2	-11	-26	

15	1236	-1	-3	-103
15	1245	-3	-15	-83
15	1272	-2	-3	-106
15	1276	-1	-11	-29
15	1432	-1	-2	-179
15	1545	-3	-15	-103
15	1572	-1	-3	-131
15	1652	-1	-7	-59
15	1660	-1	-5	-83
15	1688	-1	-2	-211
15	1748	-1	-19	-23
15	1992	-3	-6	-83
15	2060	-1	-5	-103
15	2068	-1	-11	-47
15	2121	-3	-7	-303
15	2148	-1	-3	-179
15	2424	-2	-3	-202
15	2445	-3	-15	-163
15	2472	-2	-3	-103
15	2532	-1	-3	-211
15	2596	-1	-11	-59
15	2717	-11	-19	-143
15	2905	-7	-35	-83
15	3260	-1	-5	-163
15	3268	-1	-19	-43
15	3320	-2	-10	-83
15	3404	-1	-23	-37
15	3496	-2	-19	-23
15	3605	-7	-35	-103
15	3619	-7	-11	-47
15	3684	-1	-3	-307
15	3784	-11	-22	-43
15	4120	-2	-10	-103
15	4296	-3	-6	-179
15	4433	-11	-31	-143
15	4548	-1	-3	-379
15	4697	-7	-11	-427
15	4731	-3	-19	-83
15	5064	-3	-6	-211
15	5068	-1	-7	-181
15	5332	-1	-31	-43

15	5405	-23	-47	-115	
15	5705	-7	-35	-163	
15	7089	-3	-51	-139	
15	7304	-11	-22	-83	
15	7611	-3	-43	-59	
15	7689	-3	-11	-699	
15	7924	-1	-7	-283	
15	9273	-3	-11	-843	
15	9331	-7	-31	-43	
15	9339	-3	-11	-283	
15	10564	-1	-19	-139	
15	11528	-11	-22	-131	
15	12328	-2	-23	-67	
15	12331	-11	-19	-59	
15	13237	-7	-31	-427	
15	15752	-11	-22	-179	
15	16683	-3	-67	-83	
15	18568	-11	-22	-211	
15	20009	-11	-107	-187	
15	21508	-1	-19	-283	
15	22161	-3	-83	-267	
15	24497	-11	-131	-187	
15	25993	-11	-139	-187	
16	328	-1	-2	-41	(2,2,4)
16	408	-17	-34	-51	(4,4)
16	408	-3	-6	-17	(4,4)
16	408	-2	-6	-17	(2,8)
16	408	-2	-3	-17	(2,8)
16	408	-1	-6	-34	(2,8)
16	408	-1	-3	-34	(4,4)
16	520	-5	-10	-65	(2,8)
16	520	-2	-10	-13	(2,8)
16	552	-3	-6	-69	(2,8)
16	552	-2	-6	-46	(4,4)
16	552	-2	-3	-46	(4,4)
16	552	-1	-6	-46	(2,8)
16	552	-1	-3	-46	(2,8)
16	552	-1	-2	-69	(2,8)
16	584	-1	-2	-73	(2,8)
16	616	-11	-22	-77	(4,4)
16	616	-2	-14	-22	(2,8)
16	616	-2	-11	-14	(4,4)
16	616	-1	-14	-22	(4,4)
16	616	-1	-11	-14	(2,8)
16	616	-1	-2	-77	(4,4)
16	660	-11	-33	-55	(2,2,4)
16	660	-5	-15	-55	(2,8)
16	660	-3	-5	-33	(2,8)
16	660	-1	-15	-33	(2,8)
16	660	-1	-3	-55	(2,8)
16	663	-3	-39	-51	(2,8)
16	680	-2	-10	-17	(2,8)
16	680	-2	-5	-34	(2,2,4)
16	680	-1	-10	-34	(4,4)
16	744	-2	-3	-62	(2,8)
16	744	-1	-6	-62	(16)
16	760	-5	-10	-95	(16)
16	760	-1	-2	-95	(16)
16	777	-3	-7	-111	(2,8)
16	780	-5	-15	-65	(2,8)
16	780	-5	-13	-15	(2,8)
16	780	-3	-13	-15	(2,8)
16	780	-1	-5	-39	(2,8)
16	780	-1	-3	-65	(2,2,4)
16	820	-1	-5	-41	(2,8)
16	840	-7	-14	-15	(4,4)
16	840	-6	-14	-30	(2,8)
16	840	-6	-14	-15	(2,8)
16	840	-3	-14	-30	(2,8)
16	840	-3	-14	-15	(2,8)
16	884	-1	-13	-17	(4,4)
16	888	-2	-6	-37	(16)
16	915	-3	-15	-183	(2,8)
16	924	-3	-11	-21	(2,8)
16	924	-3	-7	-33	(2,8)
16	924	-1	-21	-33	(2,2,4)
16	984	-1	-6	-82	(2,8)
16	1020	-5	-15	-17	(2,8)
16	1020	-3	-15	-17	(2,8)
16	1045	-11	-19	-55	(2,8)
16	1092	-13	-39	-91	(2,8)
16	1092	-7	-13	-21	(2,8)

16	1092	-3	-21	-39	(2,2,4)
16	1092	-1	-21	-39	(2,8)
16	1092	-1	-7	-39	(2,8)
16	1128	-3	-6	-141	(2,8)
16	1128	-1	-2	-141	(4,4)
16	1140	-3	-15	-57	(2,2,4)
16	1140	-3	-5	-19	(2,8)
16	1140	-1	-15	-19	(2,8)
16	1140	-1	-5	-57	(2,8)
16	1155	-15	-35	-55	(2,8)
16	1155	-3	-7	-55	(2,8)
16	1160	-2	-5	-58	(2,2,4)
16	1240	-5	-10	-155	(4,4)
16	1240	-1	-2	-155	(4,4)
16	1320	-10	-15	-55	(2,8)
16	1320	-10	-11	-15	(2,8)
16	1320	-6	-15	-22	(2,8)
16	1320	-6	-11	-30	(4,4)
16	1320	-3	-22	-30	(2,8)
16	1320	-3	-15	-66	(2,8)
16	1320	-2	-10	-66	(2,2,4)
16	1320	-2	-3	-55	(2,8)
16	1365	-15	-35	-39	(2,8)
16	1365	-7	-35	-39	(2,8)
16	1380	-3	-15	-69	(2,8)
16	1380	-3	-5	-69	(2,8)
16	1380	-1	-15	-69	(2,8)
16	1380	-1	-5	-69	(2,8)
16	1416	-3	-6	-177	(16)
16	1416	-1	-2	-177	(2,8)
16	1428	-7	-17	-21	(2,8)
16	1428	-3	-7	-17	(2,8)
16	1428	-1	-17	-21	(4,4)
16	1496	-11	-17	-22	(4,4)
16	1496	-2	-17	-22	(4,4)
16	1496	-1	-22	-34	(2,8)
16	1540	-7	-35	-77	(2,2,4)
16	1540	-5	-35	-55	(2,8)
16	1540	-5	-11	-35	(2,8)
16	1540	-1	-7	-55	(2,8)
16	1540	-1	-5	-77	(2,8)
16	1560	-3	-10	-39	(4,4)
16	1560	-2	-15	-39	(2,8)
16	1560	-2	-10	-39	(2,8)
16	1596	-7	-19	-21	(2,8)
16	1596	-3	-19	-21	(4,4)
16	1596	-3	-7	-57	(2,8)
16	1608	-1	-6	-67	(2,8)
16	1624	-1	-14	-58	(4,4)
16	1624	-1	-7	-58	(2,2,4)
16	1640	-2	-10	-82	(2,8)
16	1704	-2	-3	-142	(4,4)
16	1716	-11	-13	-33	(2,8)
16	1716	-3	-11	-13	(2,8)
16	1716	-1	-13	-33	(2,8)
16	1740	-5	-15	-145	(2,8)
16	1785	-3	-35	-51	(2,8)
16	1820	-7	-13	-35	(2,8)
16	1820	-5	-7	-65	(2,8)
16	1820	-1	-35	-65	(4,4)
16	1848	-7	-11	-42	(2,8)
16	1848	-6	-14	-22	(2,2,4)
16	1848	-3	-11	-42	(2,8)
16	1848	-3	-11	-14	(2,8)
16	1860	-3	-15	-93	(2,2,4)
16	1860	-3	-5	-93	(2,8)
16	1860	-1	-15	-93	(2,8)
16	1860	-1	-5	-93	(2,8)
16	1880	-2	-5	-94	(16)
16	1932	-3	-21	-69	(2,8)
16	1932	-1	-7	-69	(2,8)
16	2040	-15	-30	-51	(2,2,4)
16	2040	-10	-15	-34	(2,8)
16	2072	-2	-14	-37	(4,4)
16	2072	-2	-7	-37	(2,2,4)
16	2145	-15	-55	-195	(2,8)
16	2145	-11	-15	-39	(2,8)
16	2184	-3	-14	-78	(2,8)
16	2184	-2	-39	-42	(2,8)
16	2244	-11	-33	-187	(2,8)
16	2244	-11	-17	-33	(2,8)
16	2244	-3	-33	-51	(2,8)

16	2244	-3	-17	-33	(2,8)
16	2244	-1	-11	-51	(2,8)
16	2244	-1	-3	-187	(2,8)
16	2280	-10	-15	-19	(2,2,4)
16	2280	-3	-10	-19	(2,2,4)
16	2280	-2	-15	-19	(2,2,4)
16	2316	-1	-3	-193	(2,2,4)
16	2328	-3	-6	-291	(2,8)
16	2380	-7	-17	-35	(2,8)
16	2380	-5	-7	-17	(2,8)
16	2380	-1	-17	-35	(4,4)
16	2392	-1	-13	-46	(4,4)
16	2408	-2	-14	-43	(2,2,4)
16	2460	-3	-5	-123	(2,2,4)
16	2580	-3	-5	-43	(4,4)
16	2580	-1	-15	-43	(2,8)
16	2584	-2	-19	-34	(2,8)
16	2604	-3	-21	-93	(2,2,4)
16	2604	-3	-7	-93	(2,8)
16	2604	-1	-21	-93	(2,8)
16	2604	-1	-7	-93	(2,8)
16	2760	-10	-15	-115	(2,2,4)
16	2760	-6	-15	-46	(2,8)
16	2760	-3	-30	-46	(2,8)
16	2760	-3	-10	-115	(2,8)
16	2760	-2	-15	-115	(2,8)
16	2760	-2	-3	-115	(2,8)
16	2805	-15	-51	-55	(2,8)
16	2820	-3	-5	-141	(2,8)
16	2840	-2	-5	-142	(4,4)
16	2856	-7	-34	-42	(2,8)
16	2856	-7	-14	-51	(2,2,4)
16	2860	-1	-13	-55	(2,8)
16	2860	-1	-11	-65	(2,2,4)
16	2964	-13	-19	-39	(2,8)
16	3003	-7	-11	-39	(2,8)
16	3036	-3	-11	-69	(2,8)
16	3036	-1	-33	-69	(2,8)
16	3045	-7	-15	-203	(2,2,4)
16	3080	-10	-11	-35	(2,2,4)
16	3080	-7	-22	-55	(2,8)

16	3180	-5	-15	-265	(2,8)
16	3192	-7	-19	-42	(2,8)
16	3192	-6	-7	-19	(2,2,4)
16	3220	-5	-35	-115	(2,2,4)
16	3220	-1	-7	-115	(2,8)
16	3224	-2	-13	-62	(16)
16	3315	-3	-51	-195	(2,2,2,2)
16	3432	-11	-22	-39	(2,8)
16	3480	-3	-10	-58	(2,8)
16	3480	-2	-15	-58	(2,8)
16	3612	-3	-21	-43	(2,8)
16	3640	-7	-10	-91	(2,8)
16	3740	-1	-55	-85	(2,8)
16	3784	-1	-22	-43	(16)
16	3795	-3	-11	-115	(2,2,4)
16	3864	-7	-42	-322	(2,2,4)
16	3864	-3	-42	-138	(2,8)
16	3864	-2	-7	-138	(2,8)
16	3864	-2	-3	-322	(2,8)
16	3885	-7	-15	-259	(2,2,4)
16	4004	-11	-13	-77	(2,8)
16	4020	-3	-5	-67	(4,4)
16	4020	-1	-15	-67	(2,8)
16	4161	-3	-19	-219	(2,8)
16	4305	-7	-15	-123	(2,8)
16	4305	-3	-35	-123	(2,8)
16	4488	-11	-22	-51	(2,8)
16	4488	-6	-11	-102	(2,8)
16	4488	-3	-22	-102	(2,8)
16	4488	-2	-51	-66	(2,2,4)
16	4488	-2	-11	-51	(2,2,4)
16	4488	-2	-3	-187	(2,2,4)
16	4515	-7	-15	-43	(2,8)
16	4515	-3	-35	-43	(2,8)
16	4760	-7	-10	-34	(2,8)
16	4760	-2	-34	-35	(4,4)
16	4836	-1	-39	-93	(2,8)
16	4872	-3	-42	-58	(2,8)
16	4935	-7	-15	-235	(2,8)
16	4935	-3	-35	-235	(2,8)
16	4972	-1	-11	-113	(2,8)

16	5005	-7	-55	-91	(2,8)
16	5016	-2	-19	-66	(2,2,4)
16	5016	-2	-11	-114	(2,2,4)
16	5016	-2	-3	-418	(2,2,4)
16	5060	-1	-55	-115	(2,8)
16	5160	-10	-15	-43	(2,8)
16	5160	-3	-10	-43	(2,8)
16	5160	-2	-15	-43	(2,8)
16	5180	-5	-35	-37	(2,8)
16	5187	-3	-19	-91	(2,2,4)
16	5313	-3	-11	-483	(2,8)
16	5320	-14	-19	-35	(2,8)
16	5336	-1	-46	-58	(2,8)
16	5340	-3	-5	-267	(2,2,4)
16	5412	-3	-33	-123	(2,8)
16	5412	-1	-11	-123	(2,8)
16	5640	-3	-10	-235	(2,8)
16	5640	-2	-15	-235	(2,8)
16	5720	-10	-55	-130	(2,8)
16	5740	-5	-35	-205	(2,2,4)
16	5852	-11	-19	-77	(2,8)
16	5896	-1	-22	-67	(2,8)
16	6020	-5	-35	-43	(2,8)
16	6028	-1	-11	-137	(2,8)
16	6045	-3	-155	-195	(2,8)
16	6405	-7	-15	-427	(2,2,4)
16	6440	-7	-10	-115	(2,2,4)
16	6440	-2	-35	-115	(2,8)
16	6545	-7	-55	-187	(2,8)
16	6580	-5	-35	-235	(2,2,4)
16	7035	-3	-35	-67	(2,8)
16	7752	-6	-19	-102	
16	7755	-3	-11	-235	(2,2,4)
16	7788	-3	-33	-177	(2,8)
16	8008	-2	-11	-91	(2,8)
16	8060	-5	-13	-155	(2,8)
16	8463	-3	-91	-403	(2,2,4)
16	8855	-11	-35	-115	(2,8)
16	9944	-2	-11	-226	(2,8)
16	10184	-2	-19	-67	(16)
16	10659	-11	-51	-323	(2,8)
16	10659	-3	-187	-323	(2,8)
16	11284	-7	-13	-217	(2,8)
16	11284	-1	-91	-217	(2,8)
16	11480	-2	-35	-82	(2,2,4)
16	13160	-7	-10	-235	(2,8)
16	13160	-2	-35	-235	(2,8)
16	13468	-13	-37	-91	(2,8)
16	14105	-35	-91	-155	(2,8)
16	15544	-2	-58	-67	(16)
16	15652	-13	-43	-91	(2,8)
16	16120	-2	-130	-155	(2,8)
16	16215	-3	-115	-235	(2,8)
16	19684	-19	-37	-133	(2,8)
16	20049	-3	-123	-163	(16)
16	22204	-7	-13	-427	(2,8)
16	53599	-19	-91	-403	(2,8)
17	8476	-1	-13	-163	
17	14833	-7	-91	-163	

Proof There are 8265 fields satisfying the conditions of Corollary 33, Lemma 34 and Lemma 35. The class number of each field was computed yielding the list below. The structure of the 2-class group of each field was determined using one of the following methods.

If K contains a biquadratic subfield K_i such that a prime divisor of K_i ramifies in K then H contains a subgroup isomorphic to H_i. If H and H_i have the same order then the structure of $H$is determined.

If K contains a biquadratic subfield K_i of odd class number, then the number of ambiguous classes for the extension K/K_i determines the 2-rank of H. This determines the structure of H unless $h = 16$ and $R_2 = 2$.

If the kernel of the mapping θ can be determined then the techniques of section 1 can be used. For example, let $k = \mathbf{Q}(\sqrt{217})$ and $K = k(\sqrt{-35}, \sqrt{65})$. Here $h = 16$ and $|\ker| = 2$. Let $K_1 = k(\sqrt{-35}), K_2 = k(\sqrt{65})$ and $K_3 = k(\sqrt{-91})$. Then $H_1 \simeq Z_4$ and $H_2 \simeq H_3 \simeq Z_2$. The table of consistent characters is:

$P_{\infty_1}P_{\infty_2}$	5	5	$P_{13_1}P_{13_2}$	$P_{\infty_1}P_{\infty_2}$	$P_{13_1}P_{13_2}$
+	+	+	+	+	+
−	−	−	−	−	−

where P_{13_1} and P_{13_2} are divisors of 13 in k and P_{∞_1} and P_{∞_2} are infinite primes of k. Here the characters for each field have been normalized. Since 7 is a principal divisor in $\mathbf{Q}(\sqrt{217})$, $(5,1,1)$ is in the kernel of the mapping $cl(\mathbf{Q}(\sqrt{-35})) \times cl(\mathbf{Q}(\sqrt{217})) \times cl(\mathbf{Q}(\sqrt{-155})) \to H_1$. Since 155 is a principal divisor in $\mathbf{Q}(\sqrt{14105})$ and 31 is a principal divisor in $\mathbf{Q}(\sqrt{217})$, $(1,1,5)$ is in the kernel of the mapping $cl(\mathbf{Q}(\sqrt{217})) \times cl(\mathbf{Q}(\sqrt{65})) \times cl(\mathbf{Q}(\sqrt{14105})) \to H_2$. Hence the divisor of 5 in K_1 is principal as is the divisor of 5 in K_2. Since $\left(\frac{13}{5}\right) = -1$ the divisors of 13 belong to the nonprincipal genus of K_2. Since $N(162 + 11\sqrt{217}) = -13$ the divisors of 13 belong to the nonprincipal genus of K_3. Let C be a class of order 4 in K_1. Then ker $= \{(1,1,1),(1,P_{13_1},P_{13_1})\}$ and $S^2 = \{(1,1,1),(C^2,1,1)\}$. Thus $H^4 \simeq S^2/S^2 \cap \ker \simeq Z_2$ so $H \simeq Z_2 \times Z_8$.

There are seven fields of class number 16 for which the above method cannot be used to determine the structure of H. We now consider each of these fields.

1. $K = \mathbf{Q}(\sqrt{-11},\sqrt{-13},\sqrt{-77})$

   Let $k = \mathbf{Q}(\sqrt{7}), K_1 = k(\sqrt{-11}), K_2 = k(\sqrt{143})$ and $K_3 = k(\sqrt{-13})$. Then $H_1 \simeq Z_4$ and $H_2 \simeq H_3 \simeq Z_2$. In K_1, $(3) = \mathcal{P}_{3_1}\mathcal{P}_{3_2}\mathcal{P}_{3_3}\mathcal{P}_{3_4}$ and $\mathcal{P}_{3_1}$ generates a class of order 4. Also $(13) = \mathcal{P}_{13_1}\mathcal{P}_{13_2}$ and $\mathcal{P}_{13_1}$ generates a class of order 2 since $N(21+5\sqrt{77}) = 13^2\dot{1}4$ and the divisor of 14 becomes principal in K_1. The units of K having relative norm 1 to K_1 are generated by $-1, \varepsilon_2$ and ε_3. Now $(1+\varepsilon_2) = P_{2_1}P_{2_2}P_{13_1}P_{13_2}$ in K where $(2) = (P_{2_1}P_{2_2})^2$ and $(13) = (P_{13_1}P_{13_2})^2$. Note that $P_{2_1}P_{2_2} = (3+\sqrt{7}$ is principal in K_1. Also $(1+\varepsilon_3) = (166)(6931+202\sqrt{1001}) = (166)(\sqrt{7})(\sqrt{-11})$ in K_1. The unit $-1 = \frac{\sqrt{13}}{\sqrt{-13}}$ and $(\sqrt{13}) = P_{13_1}P_{13_2}$ in K. Thus by Lemma 27, if $\mathcal{P}_{13_1}$ becomes principal in K then $(\alpha)\mathcal{P}_{13_1} = (P_{13_1}P_{13_2})^a(\beta)$ for some $\alpha,\beta,\in K_1$. Considering powers of P_{13_2} we see that a must be even. Thus $(\alpha)\mathcal{P}_{13_1} = (\mathcal{P}_{13_1}\mathcal{P}_{13_2})^{a/2}(\beta)$ or $(\alpha)\mathcal{P}_{13_1} = (13)^{a/2}(\beta)$, a contradiction since $\mathcal{P}_{13_1} \not\sim (1)$ in K. Now $S^2 = \{(1,1,1),(\mathcal{P}_{13_1},1,1)\}$ and $(\mathcal{P}_{13_1},1,1) \notin ker$ so $S^2/S^2 \cap ker \simeq Z_2$ and $H \simeq Z_2 \times Z_8$.

2. $K = \mathbf{Q}(\sqrt{-3},\sqrt{-17},\sqrt{-33})$

   Let $k = \mathbf{Q}(\sqrt{-3}), K_1 = k(\sqrt{11}), K_2 = k(\sqrt{51})$ and $K_3 = k(\sqrt{561})$. Then $H_1 \simeq H_3 \simeq$

Z_2 and $H_2 \simeq Z_4$.In K_2, $(3) = (\mathcal{P}_{3_1}\mathcal{P}_{3_2})^2$ and $\mathcal{P}_{3_1}$ generates a class of order 4. Also $(11) = \mathcal{P}_{11_1}\mathcal{P}_{11_2}$ and $\mathcal{P}_{11_1}$ generates a class of order 2 since $N(15+\sqrt{-17}) = 11^2\dot{2}$ and the divisor of 2 becomes principal in K_2. The units of K having relative norm 1 to K_2 are generated by $-1, \varepsilon_1$ and $\sqrt{\varepsilon_3}$. Now $(1-\varepsilon_1) = (3)(3+\sqrt{11}) \sim (3+\sqrt{11}) = P_{2_1}P_{2_2}$ in K, but $P_{2_1}P_{2_2} = (7+\sqrt{51}) \sim 1$ in K_2. Also $(1+\sqrt{-\varepsilon_3}) = \left(\frac{-71+7\sqrt{-17}+5\sqrt{-33}+3\sqrt{561}}{2}\right)(5-3\sqrt{-17}) \sim P_{3_1}P_{11_1}$ where $(3) = (P_{3_1}P_{3_2})^2$ and $(11) = (P_{11_1}P_{11_2})^2$ in K. The unit $-1 = \frac{\sqrt{11}}{-\sqrt{11}}$ and $(\sqrt{11}) = P_{11_1}P_{11_2}$ in K. Thus by Lemma 27, if $\mathcal{P}_{11_1}$ becomes principal in K, then $(\alpha)\mathcal{P}_{11_1} = (P_{3_1}P_{11_1})^a(P_{11_1}P_{11_2})^b(\beta)$ for some $\alpha, \beta \in K_2$. By considering powers of P_{11_1} and P_{11_2} we see that $a+b$ and b must be even, so a is even. Thus $\mathcal{P}_{11_1} \sim P_{3_1}^a P_{11_1}^{a+b} P_{11_2}^b \sim \mathcal{P}_{11_1}^{a/2}\mathcal{P}_{11_1}^{(a+b)/2}\mathcal{P}_{11_2}^b \sim \mathcal{P}_{11_1}^{a+b/2}\mathcal{P}_{11_2}^{b/2}$ so $(1) \sim \mathcal{P}_{11_1}^{a-1}(\mathcal{P}_{11_1}\mathcal{P}_{11_2})^{b/2} \sim \mathcal{P}_{11_1}^{a-1}(11)^{b/2} \sim \mathcal{P}_{11_1}^{a-1}$. But $\mathcal{P}_{11_1}$ has order 2 and $a-1$ is odd, a contradiction. Therefore $\mathcal{P}_{11_1}$ does not become principal in K. Now $S^2 = \{(1,1,1),(1,\mathcal{P}_{11_1},1)\}$ and $(1,\mathcal{P}_{11_1},1) \notin ker$ so $S^2/S^2 \cap ker \simeq Z_2$ and $H \simeq Z_2 \times Z_8$.

3. $K = \mathbf{Q}(\sqrt{-3}, \sqrt{-5}, \sqrt{-141})$

Let $k = \mathbf{Q}(\sqrt{-3}), K_1 = k(\sqrt{15}), K_2 = k(\sqrt{47})$ and $K_3 = k(\sqrt{705})$. Then $H_1 \simeq H_3 \simeq Z_2$ and $H_2 \simeq Z_4$. In K_2, $(5) = \mathcal{P}_{5_1}\mathcal{P}_{5_2}$ and $\mathcal{P}_{5_1}$ generates a class of order 2 since $N(3+\sqrt{-141}) = 5^2\dot{6}$ and the divisor of 6 becomes principal. The units of K having relative norm 1 to K_2 are generated by $-1, \varepsilon_1$ and ε_3. Now $(1+\varepsilon_1) = (5+\sqrt{15}) = P_{2_1}P_{2_2}P_{5_1}P_{5_2}$ in K where $(2) = (P_{2_1}P_{2_2})^2$ and $(5) = (P_{5_1}P_{5_2})^2$. Note that $P_{2_1}P_{2_2} \sim 1$ in K_2. Also $(1+\varepsilon_3) = (58)(4089+154\sqrt{705}) \sim (\sqrt{-3})(\sqrt{47}) \sim 1$ in K_2. The unit $-1 = \frac{\sqrt{-5}}{-\sqrt{-5}}$ and $(\sqrt{-5}) = P_{5_1}P_{5_2}$. Thus by Lemma 27, if $\mathcal{P}_{5_1}$ becomes principal in

K then $(\alpha)\mathcal{P}_{5_1} = (P_{5_1}P_{5_2})^a(\beta)$ for some $\alpha, \beta \in K_2$. By considering powers of P_{5_1} we see that a is even. Thus $(\alpha)\mathcal{P}_{5_1} = (5)^{a/2}(\beta)$, a contradiction since $\mathcal{P}_{5_1} \not\sim 1$ in K_2. Therefore $\mathcal{P}_{5_1}$ does not become principal in K. Now $S^2 = \{(1,1,1),(1,\mathcal{P}_{5_1},1)\}$ and $(1,\mathcal{P}_{5_1},1) \notin ker$ so $S^2/S^2 \cap ker \simeq Z_2$ and $H \simeq Z_2 \times Z_8$.

4. $K = \mathbf{Q}(\sqrt{-3}, \sqrt{-5}, \sqrt{-69})$

Let $k = \mathbf{Q}(\sqrt{23}), K_1 = k(\sqrt{-3}), K_2 = k(\sqrt{15})$ and $K_3 = k(\sqrt{-5})$. Then $H_1 \simeq Z_4$ and $H_2 \simeq H_3 \simeq Z_2$. In K_1, $(5) = \mathcal{P}_{5_1}\mathcal{P}_{5_2}$ and $\mathcal{P}_{5_1}$ generates a class of order 2 since $N(9+\sqrt{-69}) = 5^2\dot{6}$ and the divisor of 6 becomes principal. The units of K having relative norm 1 to K_1 are generated by $-1, \varepsilon_1$ and ε_3. Now $(1+\varepsilon_1) = P_{2_1}P_{2_2}P_{5_1}P_{5_2}$ in K where $(2) = (P_{2_1}P_{2_2})^2$ and $(5) = (P_{5_1}P_{5_2})^2$. Note that $P_{2_1}P_{2_2} \sim 1$ in K_1. Also $(1+\varepsilon_3) = (14)(483+26\sqrt{345}) \sim (\sqrt{-3})(\sqrt{23}) \sim 1$ in K_1. The unit $-1 = \frac{\sqrt{-5}}{-\sqrt{-5}}$ and $(\sqrt{-5}) = P_{5_1}P_{5_2}$ in K. Thus by Lemma 27, if $\mathcal{P}_{5_1}$ becomes principal in K, then $(\alpha)\mathcal{P}_{5_1} = (P_{5_1}P_{5_2})^a(\beta)$ for some $\alpha, \beta \in K_1$. Considering powers of P_{5_2} we see that a is even. Thus $(\alpha)\mathcal{P}_{5_1} = (5)^{a/2}(\beta)$, a contradiction since $\mathcal{P}_{5_1} \not\sim 1$ in K_1. Now $S^2 = \{(1,1,1),(\mathcal{P}_{5_1},1,1)\}$ and $(\mathcal{P}_{5_1},1,1) \notin ker$ so $S^2/S^2 \cap ker \simeq Z_2$ and $H \simeq Z_2 \times Z_8$.

5. $K = \mathbf{Q}(\sqrt{-3}, \sqrt{-5}, \sqrt{-33})$

Let $k = \mathbf{Q}(\sqrt{-3}), K_1 = k(\sqrt{11}), K_2 = k(\sqrt{15})$ and $K_3 = k(\sqrt{165})$. Then $H_1 \simeq H_2 \simeq Z_2$ and $H_3 \simeq Z_4$. In K_3, $(2) = \mathcal{P}_{2_1}\mathcal{P}_{2_2}$ and $\mathcal{P}_{2_1}$ generates a class of order 2 since $N(\frac{5+\sqrt{-55}}{2}) = 4\dot{5}$ and the divisor of 5 becomes principal. The units of K having relative norm 1 to K_3 are generated by $-1, \varepsilon_1$ and ε_2. Now $(1+\varepsilon_2) = P_{2_1}P_{2_2}P_{11_1}P_{11_2}$ in

K where $(2) = (P_{2_1}P_{2_2})^2$ and $(11) = (P_{11_1}P_{11_2})^2$. Note that $P_{11_1}P_{11_2} = \frac{11+\sqrt{165}}{2} \sim 1$ in K_3. Also $(1+\varepsilon_2) = P_{2_1}P_{2_2}P_{5_1}P_{5_2}$ in K where $(5) = (P_{5_1}P_{5_2})^2$. But $P_{5_1}P_{5_2} \sim 1$ in K_3. The unit $-1 = \frac{\sqrt{-5}}{-\sqrt{-5}}$ and $(\sqrt{-5}) = P_{5_1}P_{5_2} \sim 1$ in K_3. Thus by Lemma 27, if $\mathcal{P}_{2_1}$ becomes principal in K then $(\alpha)\mathcal{P}_{2_1} = (P_{2_1}P_{2_2})^a(\beta)$ for some $\alpha, \beta \in K_3$. By considering powers of P_{2_2} we see that a is even. Thus $(\alpha)\mathcal{P}_{2_1} = (2)^{a/2}(\beta)$, a contradiction since $\mathcal{P}_{2_1} \not\sim 1$ in K_3. Therefore $\mathcal{P}_{2_1}$ does not become principal in K. Now $S^2 = \{(1,1,1),(1,1,\mathcal{P}_{2_1})\}$ and $(1,1,\mathcal{P}_{2_1}) \notin ker$ so $S^2/S^2 \cap ker \simeq Z_2$ and $H \simeq Z_2 \times Z_8$.

6. $K = \mathbf{Q}(\sqrt{-5}, \sqrt{-11}, \sqrt{-35})$

Let $k = \mathbf{Q}(\sqrt{7}), K_1 - k(\sqrt{-5}), K_2 = k(\sqrt{55})$ and $K_3 = k(\sqrt{-11})$. Then $H_1 \simeq H_2 \simeq Z_2$ and $H_3 \simeq Z_4$. In K_3, $(3) = \mathcal{P}_{3_1}\mathcal{P}_{3_2}\mathcal{P}_{3_3}\mathcal{P}_{3_4}$. Now $N(2+\sqrt{-77}) = 3^4$ so 3 generates a class of order 4 in $\mathbf{Q}(\sqrt{-77})$. But 3 is principal in k so $P_3 = \mathcal{P}_{3_1}\mathcal{P}_{3_2}$ generates a class of order 2 in K_3. The units of K having relative norm 1 to K_3 are generated by $-1, \varepsilon_1$ and $\sqrt{-\varepsilon_2}$. Now $(1+\epsilon_1) = 3(3+\sqrt{7}) \sim 1$ in K_3. Also $(1+\sqrt{-\varepsilon_2}) = \left(\frac{-7+\sqrt{-5}+\sqrt{-11}-\sqrt{55}}{2}\right)\left(\frac{1-\sqrt{-11}}{2}\right) \sim P_{2_1}P_{2_2}P_{5_1}$ in K where $(2) = (P_{2_1}P_{2_2})^2$ and $(5) = (P_{5_1}P_{5_2})^2$. Note that $P_{2_1}P_{2_2} = (3+\sqrt{7}) \sim (1)$ in K_3. The unit $-1 = \frac{\sqrt{-5}}{-\sqrt{-5}}$ and $(\sqrt{-5}) = P_{5_1}P_{5_2}$. Thus it follows from Lemma 27 that P_3 does not become principal in K. Now $S^2 = \{(1,1,1),(1,1,P_3)\}$ and $(1,1,P_3) \notin ker$ so $S^2/S^2 \cap ker \simeq Z_2$ and $H \simeq Z_2 \times Z_8$.

7. $K = \mathbf{Q}(\sqrt{-1}, \sqrt{-6}, \sqrt{-34})$

Let $k = \mathbf{Q}(\sqrt{-1}), K_1 = k(\sqrt{6}), K_2 = k(\sqrt{34})$ and $K_3 = k(\sqrt{51})$. Then $H_1 \simeq Z_2, H_2 \simeq Z_2 \times Z_4$ and $H_3 \simeq Z_2 \times Z_2$. In K_2, $(5) = \mathcal{P}_{5_1}\mathcal{P}_{5_2}\mathcal{P}_{5_3}\mathcal{P}_{5_4}$. Now K_2 contains the quadratic subfield $\mathbf{Q}(\sqrt{-34})$ which has class group Z_4. Its Hilbert class field contains the subfield $\mathbf{Q}(\sqrt{-2}, \sqrt{17})$ in which the divisors of 5 are inert. Hence they gain degree 4 in the Hilbert class field of $\mathbf{Q}(\sqrt{-34})$. Since 5 splits completely in K_2 and the Hilbert class field of $\mathbf{Q}(\sqrt{-34})$ is contained in the Hilbert class field of K_2, the divisors of 5 in K_2 gain degree at least 4 in the Hilbert class field of K_2. Since $H_2 \simeq Z_2 \times Z_4$, the divisors of 5 in K_2 belong to classes of order 4. We will show that $\mathcal{P}_{5_1}^2$ becomes principal in K. Let $\alpha = \frac{16+3\sqrt{34}+7\sqrt{6}+2\sqrt{51}}{2} = \frac{16+3\sqrt{34}+\sqrt{6}(7+\sqrt{34})}{2}$ and let $P_5 = (5, 2+\sqrt{34})$ be a divisor of 5 in $\mathbf{Q}(\sqrt{34})$. Then $\alpha \equiv 0 \pmod{P_5}$. Now $N(\alpha) = -50$ so $(\alpha) = \mathcal{P}_2 P_5 = \mathcal{P}_2\mathcal{P}_{5_1}\mathcal{P}_{5_2}$ where $\mathcal{P}_2$ is a divisor of 2 in K_2. Also $\mathcal{P}_2^2 \sim (\mathcal{P}_{5_1}\mathcal{P}_{5_2})^2 \sim \mathcal{P}_5^2 \sim (1)$ so $\mathcal{P}_{5_1}^2 \sim \mathcal{P}_{5_2}^2$. Thus $(\mathcal{P}_2\mathcal{P}_{5_1}\mathcal{P}_{5_2})^2 \sim (1)$. Suppose $\mathcal{P}_2\mathcal{P}_{5_1}\mathcal{P}_{5_2} \sim (1)$ in K_2. Since this ideal is ambiguous over $\mathbf{Q}(\sqrt{34})$ there must be a unit e of K_2 of relative norm 1 such that $\mathcal{P}_2\mathcal{P}_{5_1}\mathcal{P}_{5_2} = (\gamma\beta)$ with $\beta \in \mathbf{Q}(\sqrt{34})$ and $\gamma^{1-\sigma} = e$. The group of units of relative norm 1 for $K_2/\mathbf{Q}(\sqrt{34})$ is generated by ι and this gives $\gamma = 1 + \iota$. But $(\gamma)^2 = (2) = (6+\sqrt{34})^2$, so $(\gamma) = (6 + \sqrt{34}$ in K_2. Thus $\mathcal{P}_2\mathcal{P}_{5_1}\mathcal{P}_{5_2} \sim (1)$ in $\mathbf{Q}(\sqrt{34})$, a contradiction,since $\mathcal{P}_2$ is not an ideal of $\mathbf{Q}(\sqrt{34})$. Therefore $\mathcal{P}_2\mathcal{P}_{5_1}\mathcal{P}_{5_2}$ is not principal in K_2. By considering genera of $K_2/\mathbf{Q}(\iota)$ we have the following distribution:

$$\begin{array}{c|ccc} & 2 & 17_1 & 17_2 \\ \hline & + & + & + \\ (1+\iota) & + & - & - \\ (2+\iota) & - & + & - \\ (2-\iota) & - & - & + \end{array}.$$

Thus $\mathcal{P}_2\mathcal{P}_{5_1}\mathcal{P}_{5_2}$ is a nonprincipal class in the principal genus so $\mathcal{P}_2\mathcal{P}_{5_1}\mathcal{P}_{5_2} \sim \mathcal{P}_{5_1}^2$. Thus $\mathcal{P}_{5_1}^2$ becomes principal in K. Now $S^2 = \{(1,1,1),(1,\mathcal{P}_{5_1}^2,1)\}$ and $(1,\mathcal{P}_{5_1}^2,1) \in ker$ so $S^2/S^2 \cap ker \simeq 1$. Thus H has no classes of order 8. From Lemmermeyer [13] it follows that the kernel has order 8 so $H_1 \times H_2 \times H_3/ker$ has order 8. Since $(1,\mathcal{P}_{5_1}^2,1) \in ker$ this factor group has no elements of order 4, so is $Z_2 \times Z_2 \times Z_2$. Hence the rank of H is at least 3. But $S/S \cap ker$ has an element of order 2 so $H \simeq Z_2 \times Z_2 \times Z_4$.

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