

GEOMETRIC AND MATERIAL NONLINEAR ANALYSIS
OF THREE-DIMENSIONAL SOIL-STRUCTURE INTERACTION

by

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Dedicated to my parents
and my beloved country,
Vietnam

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Chapter I

INTRODUCTION

I.1. General Remarks

During the recent years, considerable progress has been made in the application of the finite element method to nonlinear problems in solid mechanics. Nonlinear behavior in structural mechanics is assumed to occur due to one or more of the three major factors:

1. Geometric nonlinearity, which arises from the nonlinear terms in the kinematic equations or the expressions characterizing the state variables such as strains,

2. Material nonlinearity, which arises from the nonlinear terms in the constitutive equations, and

3. Boundary nonlinearity which is caused by the interaction between two adjacent bodies with different material properties.

The effect of geometric nonlinearity on the system is the occurrence of large strain and large rotation with respect to the undeformed coordinates, as the external load is applied. On the other hand, the nonlinear behavior of the material causes the changes in its properties during the deformation process. The boundary nonlinearity can exist in crack problems in fracture mechanics, where for instance, a crack is initiated, propagates and closes under the influence of geometrical changes, stress state and surface energy. The boundary nonlinearity also exists in joint

problems in rock mechanics where a gap exists between two adjacent materials with the same property, or in contact problems where two bodies with distinct properties are attached to each other often in the absence of adhesion or cohesion. These contact problems may arise in systems such as concrete structures with reinforcement, soil foundation with different layers, or soil-structure interaction, etc.

There exists a number of analytical and numerical methods to solve nonlinear problems in solid mechanics and geomechanics. Among them, the finite element method is one of the most efficient and versatile numerical methods. The efficiency of the finite element method derives mainly from its capability of solving problems with geometric nonlinearity, material nonlinearity, boundary nonlinearity or any combination of these. Furthermore, in geomechanics, this method is capable of handling other factors such as insitu stresses, stress-paths, physical state of body with arbitrary geometry and loading. This is due to the fact that a body can be represented by a finite element model which is a collection of finite elements. Each element can be treated separately. Its equations can then be established, in which nonlinear kinematical equations and nonlinear constitutive equations are introduced [49, 50]. The process of connecting the finite elements together to form the complete model is purely a topological process and is independent of the physical nature and the nonlinearity of the problem.

The treatment of the boundary nonlinearity is also simple and straightforward by using the finite element method. The study of the contacts or gaps can be carried on independently by modelling them with special finite elements and/or constraints. Their behavior are correlated to other regular elements and assembled to give an assemblage of the system.

Although the application of the finite element method in solving nonlinear problems is well documented in structural mechanics and geomechanics, most of the analyses are performed with the assumption of two-dimensional idealization such as plane strain, plane stress or axisymmetry. For a number of other problems, however, it is necessary to perform an analysis in truly three-dimensional space. For example, in geomechanics, a laterally loaded pile (with arbitrary cross section) embedded in a soft soil body would experience deformation in the three-dimensional space, and an axisymmetric assumption for this problem would be unrealistic. Another example is when a tunnel is being excavated into a rock body with irregular fractured zones and cracks, the analysis of the deformation pattern of the tunnel face must be performed in three-dimensional geometry. The existence of the irregular joints would change the condition of loading on the tunnel if the tunnel is under the application of the rock weight above it. The plane strain assumption, hence, is no longer valid for such a complex situation.

In the current research, the problem of a structure (tillage tool) moving in a soil mass is considered. In this problem, since the dimension of the tool is finite and the soil surrounding the tool would displace in an irregular manner, a three-dimensional analysis should be performed for realistic results.

Although material nonlinearity has long been used in finite element analyses for geomechanical problems, there are very few applications of the geometric nonlinearity in these problems [9,10]. It can be understood that since for a general problem in geomechanics, the failure of a continuum can often occur at a relatively small load, and the consideration of large strain and large displacement may not be necessary. The nonlinear effect for this type of problem is mainly caused by the change in material behavior only. However, for some problems, the change of the geometry of the body can be significant so that the neglect of geometric nonlinearity could lead to unrealistic solutions. For example, in the underground problem discussed above, if the tunnel bottom comprises of a soft material, a large heave of the tunnel floor would be expected when the pressure around the tunnel is very high. Also for the soil-structure interaction problem in the current research, the soil would displace significantly around the tool when the latter is moving horizontally. This type of problem requires a combination of all nonlinear effects with a three-dimensional analysis.

The work presented in this dissertation considers all aspects discussed above in the formulation and implementation for a general three-dimensional finite element procedure. Further discussion about the geometric, material, and boundary nonlinearities will be given in various chapters in this dissertation. The next section in this chapter gives a brief literature review of the recent developments in combined geometric and material nonlinear analyses. The last section presents the aim of the present research.

I.2. Literature Review

Considerable work has been performed on the theory and application of the finite element method in nonlinear problems. In the area of geometric nonlinearity various formulations have been attempted to account for the effect of the geometrical change of the continuum. The basic idea is the inclusion of some additional stiffness, which will be explained later, in the system of the finite element equations to handle this effect.

An earlier attempt was the adoption of such a stiffness called geometric stiffness in the conventional finite element equations [45, 48]. In these analyses, the load is divided into a number of increments, and only an increment of load is applied at a time. The geometry of the body is updated at the end of each load step. The geometric stiffness is then calculated based on the current stress at this load step and the new configuration of the body. No load corrections are accounted in order to satisfy the equilibrium condition at the end of each load step, hence the analysis often yields

unrealistic results. As pointed out by Hibbitt et al [37], a geometric nonlinear analysis using co-moving geometry without load correction is acceptable if only small strain and large rotation is assumed, and if only constant strain elements are used. In the context of the latter, it can be seen that the application of the technique proposed in References [45,48] is limited if a variety of elements is to be used.

A number of workers have attempted to derive the finite element equation for large deformation problems by using the equations of nonlinear continuum mechanics and the principle of virtual work [4, 5,34,37,39,47,52,63,64,67,71,72,77]. Depending on the coordinate definitions used, there are two main approaches on which the finite element equations are based: the Lagrangian approach, and the Eulerian approach.

Lagrangian and Eulerian Approaches

In the Lagrangian approach, the kinematic and static variables are derived and referred to the original or undeformed configuration of the body; and in the Eulerian approach, they are referred to the current configuration of the body. However, the current configuration as used in the Eulerian approach are unknown values, hence by adopting the incremental formulation, the current coordinates are replaced by a coordinate system which is known at the end of the previous load increment. This makes the definition of the Eulerian approach reduce to the mixed Lagrangian-Eulerian approach [39], or, as used in Reference [5], the updated Lagrangian approach.

The formulation using the original geometry, that is Lagrangian or total Lagrangian formulation [5], has been used extensively in structural mechanics [4,34,37,64,77]. In this formulation, with the assumption of large strains and large rotations, the conventional strain-displacement matrix has an extra term due to the change of the geometry at the previous load increment. Another stiffness, called initial stress stiffness matrix and similar to the geometric stiffness matrix given above, is included in the finite element equation. This stiffness takes into account the nonlinear part of the strain and includes the effect of the current stress at the previous load. Also, another type of stiffness matrix, called initial displacement stiffness matrix [37,50], is included to account for the change of load direction under large rotation. The second Piola-Kirchhoff stress, which is a pseudo stress computed at the current configuration but referred to the undeformed configuration [29], is used to define the material nonlinear effect which appears only in the conventional stiffness matrix [5,37,77]. Finally, a load correction vector, which is derived from the equations of nonlinear continuum mechanics, arises automatically in the finite element equation in the Lagrangian formulation.

The updated Lagrangian formulation has also attracted a lot of researches in solid mechanics [5,39,47,52,63,64,71,72,73], especially in geomechanics [1,8,9,10,12,20,55,67,77]. In this formulation, since the geometry of the continuum is updated at each load level and the computation of variables is based on the

current volume, the geometric nonlinear effect can be better controlled as compared to the total Lagrangian formulation. The extra term in the strain-displacement matrix appeared in the total Lagrangian approach does not exist in the updated Lagrangian approach. Similarity exists between these two approaches, except in defining the nonlinear constitutive relationship in the updated Lagrangian approach, the current stress or Cauchy stress is used.

As discussed by Bathe et al. [5], for any formulation based on the principles of continuum mechanics in which all nonlinear effects are included, the final results should be the same. Although this statement holds true for nonlinear elasticity, the solutions for large deformations with inelastic materials can be different for the two formulations [64]. This is partly due to the definition of the constitutive laws being different with various stress states, and partly due to the different numerical schemes adopted to handle the combined nonlinear effects. This can be seen in a simple problem of a cantilever beam on a soft, inelastic foundation under the application of uniform load, as shown in Figure (1.1). At a high load level, the effect of large strain and large rotation could significantly change the geometries of both the beam and the foundation. The stresses in the foundation computed at this load step will largely affect the constitutive law. Hence, the second Piola-Kirchoff stress used in the total Lagrangian formulation and the Cauchy stress in the updated Lagrangian formulation would

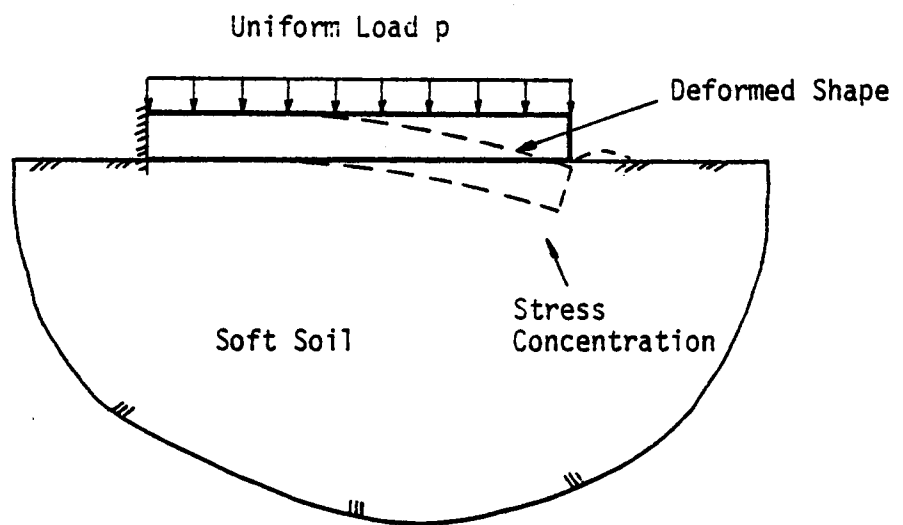


Figure 1.1 Cantilever Beam on Elastic Foundation

lead to different computations of the constitutive matrix. In that respect, the predictions of the deformation of the beam would be different in the two approaches.

The updated Lagrangian approach using co-moving coordinate system has been considered to yield more realistic solutions in terms of generality and computational efficiency [64]. This could be the reason why almost all works in geomechanics relating to inelastic material and large deformation are formulated in the updated Lagrangian approach [9,55]. The relevance of this approach is further illustrated for a problem involving soil deformation [20].

Soil is a highly inelastic material which, in some problems, exhibits large strain and large rotation before it reaches final strength. In laboratory tests for such soil, the material parameters are usually computed based on the current volume of the sample. Hence, they can be used directly in the updated Lagrangian approach.

This research adopts the updated Lagrangian approach based on the above discussions. A contribution is made in coupling this approach with a numerical finite element scheme to solve a general nonlinear problem.

I.3. Aim of the Research

The aims of this research are:

1. to formulate and develop an incremental finite element procedure to handle geometric and material nonlinearities for general three-dimensional problems.

2. to develop a computer program which is capable of solving general nonlinear problems including the contact problem, in particular the problem of soil-structure interaction.
3. to incorporate appropriate constitutive laws defined from parameters obtained from comprehensive laboratory tests for an artificial soil.
4. to include in the analysis the soil-structure interaction effect when a structure such as a tillage tool is moving (horizontally) in a soil body under the application of a draft force. This is done by using special 'interface' finite elements.
5. to compare the numerical results with different elastic-plastic constitutive laws, thus to be able to deduce the best material model in predicting the soil response, and
6. to compare and verify the numerical predictions with the experimental solutions obtained in a specially devised soil-bin test facility.

Chapter II in this dissertation describes the mathematical formulation for a geometric nonlinear problem from the equations of continuum mechanics, and the corresponding finite element equations. Chapter III is devoted to basic theory of plasticity with the assumption of small strain. Two theories of finite deformation in plasticity are then briefly discussed, and difficulties in applying these theories in numerical analysis are pointed out. Various elasto-plastic constitutive models are presented in

Chapter IV, and a new model for the soil in research is subsequently discussed. Chapter V explains the interface effects in soil-structure interaction problems. Chapter VI gives a brief discussion of the convenience of using variable-node finite elements. Numerical techniques used in solving nonlinear problems are presented, and a special consideration for the usage of proper stress in large deformation analysis is given. The stress transfer technique is also mentioned in this chapter.

Chapter VII shows some typical examples in geomechanics where two-dimensional problems are solved by using the three-dimensional code. The predictions of the responses of a three-dimensional soil-moving problem are given and compared with the results from experiments in Chapter XIII. Finally, in Chapter IX, a summary about the work in the current research is given, and some further works are recommended.

Chapter II

GEOMETRIC NONLINEARITY INCLUDING LARGE ROTATION

In the context of geometric nonlinearity, there are two assumptions which have often been made: large displacement with small strain and large displacement with large strain. The differences between these two assumptions are significant if plasticity is also included in the analysis. In this chapter, the finite element formulation for a geometric nonlinear problem with small strain and linear elastic material is discussed first, which is followed by a section devoted to special consideration for large strain and large rotation behaviors. The topic finite strains with plastic behavior is considered in Chapter III.

II.1. Finite Element Formulation

As stated in Chapter I, the finite element formulation in this research is based mainly on the updated Lagrangian approach. Hence this section only discusses the formulation within the framework of this approach. The total Lagrangian approach and its formulation is described elsewhere [5,77].

To start with the formulation of the updated Lagrangian approach, it is necessary to define three typical configurations of a continuous body deforming under the application of an external force. Figure (2.1) shows the configurations of the body at the underformed state (step 0), the deformed state at step n and the deformed state at the subsequent step $n + 1$.

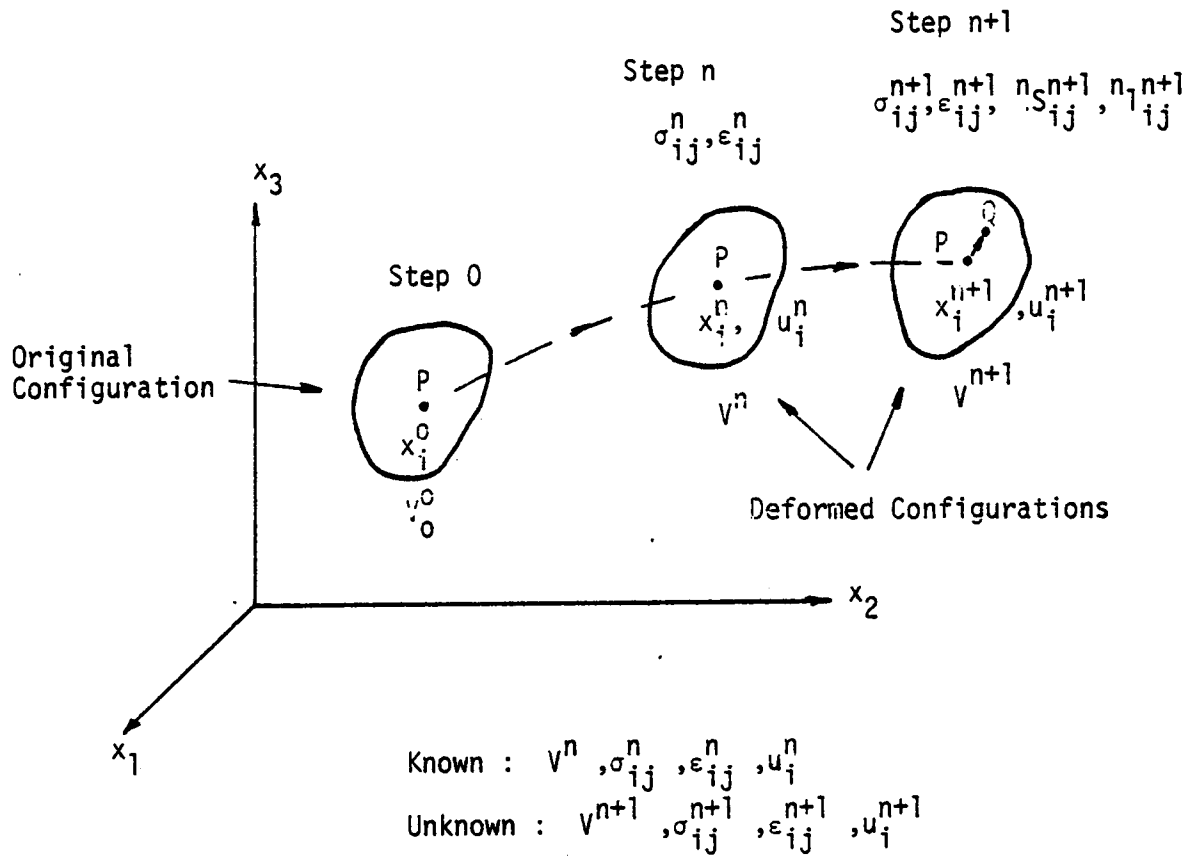


Figure 2.1. Generic Configurations of Body

Suppose that all the static and kinematic variables, σ_{ij}^n , ϵ_{ij}^n , u_{ij}^n , etc. are known at the end of step n , the aim now is to establish the incremental finite element equation to solve for the incremental displacement Δu_i from step n to step $n + 1$. The secondary values such as current stress σ_{ij}^{n+1} at the end of step $n + 1$, will then be computed by using Δu_i . Tensor notation will be used throughout this chapter. However, matrix notation will be shown whenever it is deemed necessary.

II.1.a. Notations and Relationships Among Variables

Due to the nature of the updated Lagrangian approach, the notations for each variable comprise of three scripts. For example, nS_{ij}^{n+1} denotes a value of stress which is a second order tensor with subscripts i and j changing from 1 to 3. The right superscript $n + 1$ denotes the value of stress is computed at the end of incremental load $n + 1$. The left superscript n denotes the value of stress is referred to the configuration of the body at load step n . The left superscript is ignored, however, if the value of the variable is referred to the same configuration of the body as the right superscript, eg. σ_{ij}^{n+1} . The incremental values will be denoted by the symbol Δ (delta), and these incremental values will not have superscripts, i.e. it is implied that these values are calculated from step n to step $n + 1$. Finally the definition of each variable is given as the text goes along.

It is necessary to define some relationships among coordinates, displacements, stresses and strains at the three different

configurations of the body. Imagine an arbitrary point P in the body, Figure (2.1). The coordinates of P are x_i^0 , x_i^n , and x_i^{n+1} at step 0, step n and step $n + 1$, respectively. Similarly, its total displacements are u_i^n and u_i^{n+1} at the end of step n and step $n + 1$, respectively.

The incremental displacement can then be written as

$$\Delta u_i = u_i^{n+1} - u_i^n \quad (2.1)$$

or

$$\Delta u_i = x_i^{n+1} - x_i^n \quad (2.2)$$

Similarly,

$$u_i^n = x_i^n - x_i^0 \quad (2.3)$$

and

$$u_i^{n+1} = x_i^{n+1} - x_i^0 \quad (2.4)$$

At the end of load step $n + 1$, the total strain tensor at point P can be expressed as

$$l_{ij}^{n+1} = \frac{1}{2} \left(\frac{\partial x_k^{n+1}}{\partial x_i^0} \cdot \frac{\partial x_k^{n+1}}{\partial x_j^0} - \delta_{ij} \right) \quad (2.5)$$

or

$$l_{ij}^{n+1} = \frac{1}{2} \left(\frac{\partial u_i^{n+1}}{\partial x_j^0} + \frac{\partial u_j^{n+1}}{\partial x_i^0} + \frac{\partial u_k^{n+1}}{\partial x_i^0} \cdot \frac{\partial u_k^{n+1}}{\partial x_j^0} \right) \quad (2.6)$$

where l_{ij}^{n+1} is called Green-Lagrange strain tensor [24], and is

referred to V^0 , volume at configuration 0, and, δ_{ij} is the Kronecker delta tensor.

Another term of the total strain tensor at point P can be written as

$$\epsilon_{ij}^{n+1} = \frac{1}{2} \left(\delta_{ij} - \frac{\partial x_k^0}{\partial x_i^{n+1}} \cdot \frac{\partial x_k^0}{\partial x_j^{n+1}} \right) \quad (2.7)$$

or

$$\epsilon_{ij}^{n+1} = \frac{1}{2} \left(\frac{\partial u_i^{n+1}}{\partial x_j^{n+1}} + \frac{\partial u_j^{n+1}}{\partial x_i^{n+1}} - \frac{\partial u_k^{n+1}}{\partial x_i^{n+1}} \cdot \frac{\partial u_k^{n+1}}{\partial x_j^{n+1}} \right) \quad (2.8)$$

where ϵ_{ij}^{n+1} is called Almansi strain tensor [24], and is referred to V^n , volume at configuration n.

Similar to Equations (2.6) and (2.8), the incremental strain tensor can be expressed as

$$\Delta l_{ij} = \frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial x_j^n} + \frac{\partial \Delta u_j}{\partial x_i^n} + \frac{\partial \Delta u_k}{\partial x_i^n} \cdot \frac{\partial \Delta u_k}{\partial x_j^n} \right) \quad (2.9)$$

or

$$\Delta l_{ij} = \Delta e_{ij} + \Delta n_{ij} \quad (2.10)$$

where Δl_{ij} is called the incremental Green-Lagrange strain tensor from V^n to V^{n+1} , and is referred to V^n ;

$$\Delta e_{ij} = \frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial x_j^n} + \frac{\partial \Delta u_j}{\partial x_i^n} \right) \quad (2.11)$$

is called the linear or infinitesimal part of Δl_{ij} , and

$$\Delta n_{ij} = \frac{1}{2} \cdot \frac{\partial \Delta u_k}{\partial x_i^n} \cdot \frac{\partial \Delta u_k}{\partial x_j^n} \quad (2.12)$$

is called the nonlinear part of Δl_{ij} .

The incremental strain tensor can also be expressed as

$$\Delta \epsilon_{ij} = \frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial x_j^{n+1}} + \frac{\partial \Delta u_j}{\partial x_i^{n+1}} - \frac{\partial \Delta u_k}{\partial x_i^{n+1}} \cdot \frac{\partial \Delta u_k}{\partial x_j^{n+1}} \right) \quad (2.13)$$

where $\Delta \epsilon_{ij}$ is called the incremental Almansi strain tensor.

An important kinematical property in the theory of finite deformation is the Jacobian matrix. It is essentially a measure of volume change of the continuum at different deformed states; and is defined from V^0 to V^n as

$$[{}^0J^n] = \left[\frac{\partial x_i^0}{\partial x_j^n} \right] \quad (2.14)$$

and from V^n to V^{n+1} as

$$[{}^nJ^{n+1}] = \left[\frac{\partial x_i^n}{\partial x_j^{n+1}} \right] \quad (2.15)$$

where symbol $[\]$ denotes matrix.

In stating the principle of conservation of mass, the determinant of the Jacobian matrix is often used and defined from V^0 to V^n as

$$\det |^0J^n| = \frac{\rho^n}{\rho} \quad (2.16)$$

and from V^n to V^{n+1} as

$$\det |^nJ^{n+1}| = \frac{\rho^{n+1}}{\rho^n} \quad (2.17)$$

To define the kinematical state of point P after deformation, we consider a point Q adjacent to P, Figure (2.1), for the configuration of the continuum at step $n + 1$. Coordinates of Q are $x_i^{n+1} + dx_i^{n+1}$. The difference in incremental displacement between these two points is given as

$$d\Delta u_i = \frac{\partial \Delta u_i}{\partial x_j^{n+1}} dx_j^{n+1} \quad (2.18)$$

$\frac{\partial \Delta u_i}{\partial x_j^{n+1}}$ can be broken as

$$\frac{\partial \Delta u_i}{\partial x_j^{n+1}} = \frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial x_j^{n+1}} + \frac{\partial \Delta u_j}{\partial x_i^{n+1}} \right) + \frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial x_j^{n+1}} - \frac{\partial \Delta u_j}{\partial x_i^{n+1}} \right) \quad (2.19)$$

Now define

$$\Delta v_{ij} = \frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial x_j^{n+1}} + \frac{\partial \Delta u_j}{\partial x_i^{n+1}} \right) \quad (2.20)$$

and

$$\Delta \Omega_{ij} = \frac{1}{2} \left(\frac{\partial \Delta u_i}{\partial x_j^{n+1}} - \frac{\partial \Delta u_j}{\partial x_i^{n+1}} \right) \quad (2.21)$$

where Δv_{ij} is symmetric and is called rate-of-deformation tensor at the end of load step $n + 1$, and $\Delta \Omega_{ij}$ is anti-symmetric and is called spin tensor at the end of load step $n + 1$.

With these definitions, Equation (2.18) becomes

$$d\Delta u_i = (\Delta v_{ij} + \Delta \Omega_{ij}) dx_j^{n+1} \quad (2.22)$$

From Equation (2.22), it is seen that the displacement of a particle is composed of two processes: straining process (deformation) which is represented by Δv_{ij} , and rotational process (rotation) which is represented by $\Delta \Omega_{ij}$.

Stress Transformation

The state of stress of the deformed body at the end of load step $n + 1$ could be expressed by two types of stress tensors: the current (real) stress tensor σ_{ij}^{n+1} which is also called Cauchy stress tensor and is referred to V^{n+1} ; and the pseudo stress tensor ${}^nS_{ij}^{n+1}$ which is also called the second Piola-Kirchoff stress tensor and is referred to V^n . The relationship between these stress tensors are, [5,29,37]

$$\sigma_{ij}^{n+1} = \frac{\rho^{n+1}}{\rho^n} \cdot \frac{\partial x_i^{n+1}}{\partial x_k^n} \cdot \frac{\partial x_j^{n+1}}{\partial x_m^n} {}^nS_{km}^{n+1} \quad (2.23)$$

or, inversely,

$${}^nS_{ij}^{n+1} = \frac{\rho^n}{\rho^{n+1}} \frac{\partial x_i^n}{\partial x_k^{n+1}} \cdot \frac{\partial x_j^n}{\partial x_m^{n+1}} \sigma_{km}^{n+1} \quad (2.24)$$

The process of deforming the body from step n to step $n + 1$ results to an incremental stress tensor which is defined as

$$\Delta S_{ij} = S_{ij}^{n+1} - \sigma_{ij}^n \quad (2.25)$$

where ΔS_{ij} is called the incremental second Piola-Kirchoff stress tensor and is referred to V^n .

ΔS_{ij} is related to the incremental Green-Lagrange strain tensor by

$$\Delta S_{ij} = C_{ijrs}^n \Delta l_{rs} \quad (2.26)$$

where C_{ijrs}^n is the fourth order constitutive tensor and is computed at V^n .

Similarly, the incremental Cauchy stress tensor can be related to the incremental, infinitesimal part of the Almansi strain tensor (or the rate-of-deformation tensor) by

$$\Delta \sigma_{ij} = C_{ijrs}^n \Delta v_{ij} \quad (2.27)$$

With the definitions of relationships of variables as discussed, it is possible now to write the principle of virtual work for a finite deformation problem.

II.1.b. Principle of Virtual Work

The principle of virtual work in the theory of finite deformation is expressed in terms of the configuration V^{n+1} as, [37]

$$\int_{V^{n+1}} \sigma_{ij}^{n+1} \delta v_{ij}^{n+1} dV^{n+1} = \int_{V^{n+1}} f_i^{n+1} \delta u_i^{n+1} dV^{n+1} + \int_{S^{n+1}} p_i^{n+1} \delta u_i^{n+1} dS^{n+1} + F_i^{n+1} \delta u_i^{n+1} \quad (2.28)$$

where

$$v_{ij}^{n+1} = \frac{1}{2} \left(\frac{\partial u_i^{n+1}}{\partial x_j^{n+1}} + \frac{\partial u_j^{n+1}}{\partial x_i^{n+1}} \right) \quad (2.29)$$

is called the infinitesimal strain or rate-of-deformation tensor and is similar to Equation (2.20), F_i^{n+1} is the body force at step $n + 1$, p_i^{n+1} is the total surface traction, F_i^{n+1} is the total concentrated force, and δ denotes variation.

Since all the terms in Equation (2.28) are formulated on the configuration V^{n+1} which is unknown at the time the load is applied, it is necessary to rewrite Equation (2.28) in terms of configuration V^n .

The left hand side of Equation (2.28) becomes [5,37]

$$\int_{V^{n+1}} \sigma_{ij}^{n+1} \delta v_{ij}^{n+1} dV^{n+1} = \int_{V^n} n_{S_{ij}}^{n+1} \delta n_{ij}^{n+1} dV^n \quad (2.30)$$

where n_{ij}^{n+1} is the total Green-Lagrange strain tensor which is computed at the end of the step $n + 1$ and is referred to V^n .

This strain tensor n_{ij}^{n+1} is composed of

$$n_{ij}^{n+1} = n_{ij}^n + \Delta l_{ij} \quad (2.31)$$

Since l_{ij}^n and u_i^n are known values, their variations vanish, and this leads to

$$\delta l_{ij}^{n+1} = \delta \Delta l_{ij} \quad (2.32)$$

and

$$\delta u_i^{n+1} = \delta \Delta u_i \quad (2.33)$$

Equation (2.30) hence becomes

$$\int_{V^{n+1}} \sigma_{ij}^{n+1} \delta v_{ij}^{n+1} dV^{n+1} = \int_{V^n} \sigma_{ij}^{n+1} \delta \Delta l_{ij} dV^{n+1} \quad (2.34)$$

Now define a pseudo body force

$$f_i^{n+1} = \frac{dV^{n+1}}{dV^n} f_i^{n+1} \quad (2.35)$$

and a pseudo surface traction

$$p_i^{n+1} = \frac{dS^{n+1}}{dS^n} p_i^{n+1} \quad (2.36)$$

The integral terms in the right hand side of Equation (2.28) becomes

$$\int_{V^{n+1}} f_i^{n+1} \delta u_i^{n+1} dV^{n+1} = \int_{V^n} f_i^{n+1} \delta u_i^{n+1} dV^n \quad (2.37)$$

and

$$\int_{S^{n+1}} p_i^{n+1} \delta u_i^{n+1} dS^{n+1} = \int_{S^n} p_i^{n+1} \delta u_i^{n+1} dS^n \quad (2.38)$$

Replacing Equations (2.33), (2.34), (2.37) and (2.38) in Equation (2.28) leads to

$$\int_{V^n} S_{ij}^{n+1} \delta \Delta l_{ij} dV^n = \int_{V^n} f_i^{n+1} \delta \Delta u_i dV^n + \int_{S^n} p_i^{n+1} \delta \Delta u_i dS^n + F_i^{n+1} \delta \Delta u_i \quad (2.39)$$

or

$$\int_{V^n} S_{ij}^{n+1} \delta \Delta l_{ij} dV^n = \delta \Delta W \quad (2.40)$$

where

$$\delta \Delta W = \int_{V^n} f_i^{n+1} \delta \Delta u_i dV^n + \int_{S^n} p_i^{n+1} \delta \Delta u_i dS^n + F_i^{n+1} \delta \Delta u_i \quad (2.41)$$

is the virtual work of the externally applied load at load step $n + 1$ and referred to the configuration of the body at step n .

By using Equations (2.10) and (2.25), Equation (2.40) can be rearranged as

$$\int_{V^n} (\sigma_{ij}^n + \Delta S_{ij}) \delta (\Delta e_{ij} + \Delta \eta_{ij}) dV^n = \delta \Delta W \quad (2.42)$$

or

$$\begin{aligned} \int_{V^n} \Delta S_{ij} \delta \Delta e_{ij} dV^n + \int_{V^n} \sigma_{ij}^n \delta \Delta e_{ij} dV^n + \int_{V^n} \Delta S_{ij} \delta \Delta \eta_{ij} dV^n \\ + \int_{V^n} \sigma_{ij}^n \delta \Delta \eta_{ij} dV^n = \delta \Delta W \end{aligned} \quad (2.43)$$

Since ΔS_{ij} is an incremental value and $\Delta \eta_{ij}$ is a nonlinear term of strain tensor, their product is insignificant as compared to other terms in Equation (2.43), hence, as an approximation, the third integral in the left hand side of this equation could be left out. This leads to

$$\int_{V^n} \Delta S_{ij} \delta \Delta e_{ij} dV^n + \int_{V^n} \sigma_{ij}^n \delta \Delta e_{ij} dV^n + \int_{V^n} \sigma_{ij}^n \delta \Delta \eta_{ij} dV^n = \delta \Delta W \quad (2.44)$$

Substituting Equation (2.26) into Equation (2.44) and rearranging terms, we obtain

$$\begin{aligned} & \int_{V^n} C_{ijrs}^n \Delta l_{rs} \delta \Delta e_{ij} dV^n + \int_{V^n} \sigma_{ij}^n \delta \Delta \eta_{ij} dV^n \\ & = \delta \Delta W - \int_{V^n} \sigma_{ij}^n \delta \Delta e_{ij} dV^n \end{aligned} \quad (2.45)$$

Another approximation should be made for the first integral in the left hand side of Equation (2.45). Since Δl_{ij} is composed of a linear term and a nonlinear term, Equation (2.10), its product with $\delta \Delta e_{ij}$ can be approximated by

$$\Delta l_{rs} \delta \Delta e_{ij} \approx \Delta e_{rs} \delta \Delta e_{ij} \quad (2.46)$$

This in turn leads to

$$\int_{V^n} C_{ijrs}^n \Delta e_{rs} \delta \Delta e_{ij} dV^n + \int_{V^n} \sigma_{ij}^n \delta \Delta \eta_{ij} dV^n$$

$$= \delta \Delta W - \int_{V^n} \sigma_{ij}^n \delta \Delta e_{ij} dV^n \quad (2.47)$$

Equation (2.47) is the approximate principle of virtual work formulated at the load step $n + 1$ and referred to the configuration V^n . This equation, after it is properly transformed into finite element equation, gives the incremental displacements from step n to step $n + 1$ when external load is applied at step $n + 1$.

The incremental displacements are then added to the previous total displacements. Another incremental load is applied and the process is repeated until the final load is reached.

II.1.c. Finite Element Equations

Equation (2.47) is valid for any point in the continuum. In the finite element formulation, it is necessary to write this equation in terms of the values at the nodal points of each element. Only brief discussion of how the finite element equation is established will be given here. Additional details can be found in References [6,13,17,30,76].

The values of displacements at any point in the finite element are expressed as

$$\Delta u_i = N_p \Delta q_i^p \quad (2.48)$$

where p is nodal index and ranges from 1 to m , total number of nodes of the element, N_p is the interpolation function at node p , and q_i^p is the incremental displacement of node p in the direction i .

In matrix notation Equation (2.48) can be written as

$$\{\Delta u\} = [N]\{\Delta q\} \quad (2.49)$$

where $\{\Delta u\}$ represents a vector of order 3×1 , $\{\Delta q\}$ represents a vector of order $m \times 1$, and $[N]$ represents a matrix of order $3 \times m$.

Using the interpolation function N_p and its derivatives $\frac{\partial N_p}{\partial x_j^n}$, the gradient of the incremental displacement is

$$\frac{\partial \Delta u_i}{\partial x_j^n} = \frac{\partial N_p}{\partial x_j^n} \cdot \Delta q_i^p \quad (2.50)$$

Replacing Equation (2.50) in Equation (2.11) yields

$$\Delta e_{ij} = \frac{1}{2} \left(\frac{\partial N_p}{\partial x_j^n} \Delta q_i^p + \frac{\partial N_p}{\partial x_i^n} \Delta q_j^p \right) \quad (2.51)$$

or in matrix notation

$$\{e\} = [B_L^n]\{\Delta q\} \quad (2.52)$$

where $\{e\}$ is linear strain vector of order 6×1 , and $[B_L^n]$ is the conventional strain-displacement matrix of order $6 \times 3m$.

Also by replacing Equation (2.48) in Equation (2.12), we obtain

$$\Delta \eta_{ij} = \frac{1}{2} \frac{\partial N_p}{\partial x_j^n} \cdot \frac{\partial N_p}{\partial x_i^n} \Delta q_k^p \Delta q_k^p \quad (2.53)$$

By substituting Equations (2.50) and (2.53) into Equation (2.47), arranging terms, and taking variation with respect to the nodal

incremental displacements, we obtain

$$\begin{aligned} & \left(\int_{V^n} \frac{\partial N_{hi}}{\partial x_k^n} C_{hkrs}^n \frac{\partial N_{rj}}{\partial x_s^n} dV^n + \int_{V^n} \frac{\partial N_{ti}}{\partial x_h^n} \sigma_{hk}^n \frac{\partial N_{tj}}{\partial x_k^n} dV^n \right) \Delta q_j \\ & = Q_i - \int_{V^n} \sigma_{hk}^n \frac{\partial N_{hi}}{\partial x_k^n} dV^n \end{aligned} \quad (2.54)$$

where $\frac{\partial N_{hi}}{\partial x_k^n}$ indicates that the derivatives of the interpolation functions at nodes h and i are taken with respect to coordinate x_n .

Equation (2.54) can also be written as

$$(L_{K_{ij}}^n + NL_{K_{ij}}^n) \Delta q_j = Q_i^{n+1} - Q_i^n \quad (2.55)$$

where

$$L_{K_{ij}}^n = \int_{V^n} \frac{\partial N_{hi}}{\partial x_k^n} C_{hkrs}^n \frac{\partial N_{rj}}{\partial x_s^n} dV^n \quad (2.56)$$

is the conventional or linear stiffness term, (L represents linear),

$$NL_{K_{ij}}^n = \int_{V^n} \frac{\partial N_{ti}}{\partial x_h^n} \sigma_{hk}^n \frac{\partial N_{tj}}{\partial x_k^n} dV^n \quad (2.57)$$

is the geometric or nonlinear stiffness term (NL represents nonlinear), Q_i^{n+1} is the applied load vector at step $n + 1$, and

$$Q_i^n = \int_{V^n} \sigma_{hk}^n \frac{\partial N_{hi}}{\partial x_k^n} dV^n \quad (2.58)$$

is the internal load vector at step n .

In matrix notation, Equation (2.54) can be written as

$$\begin{aligned} & \int_{V^n} [B_L^n]^T [C^n] [B_L^n] dV^n + \int_{V^n} [B_{NL}^n]^T [\sigma^n] [B_{NL}^n] dV^n \{\Delta q\} \\ & = \{Q^{n+1}\} - \int_{V^n} [B_L^n]^T \{\sigma^n\} dV^n \end{aligned} \quad (2.59)$$

or

$$[[K_L^n] + [K_{NL}^n]] \{\Delta q\} = \{Q^{n+1}\} - \{Q_0^n\} \quad (2.60)$$

where

$$[K_L^n] = \int_{V^n} [B_L^n]^T [C^n] [B_L^n] dV^n, \quad (2.61)$$

$$[K_{NL}^n] = \int_{V^n} [B_{NL}^n]^T [\sigma^n] [B_{NL}^n] dV^n, \quad (2.62)$$

and

$$\{Q_0^n\} = \int_{V^n} [B_L^n]^T \{\sigma^n\} dV^n. \quad (2.63)$$

In addition to the matrix $[B_L]$, Equation (2.52), another matrix $[B_{NL}^n]$ appears in the finite element equation, Equation (2.59). This matrix is called the nonlinear strain displacement matrix.

The components of matrices $[B_L^n]$, $[B_{NL}^n]$, $[C^n]$, $[\sigma^n]$, and $\{\sigma^n\}$ are given explicitly in Appendix A.

II.2. Coordinate Independence of Constitutive Law

The first part of this chapter discussed the basic formulation of geometric nonlinearity, and has not considered special cases

where a proper definition of stress must be used in defining the material law. If the material is linear elastic or hyperelastic, the constitutive behavior is independent of the stress rate, and the incremental finite element scheme can be performed without any modification.

At the end of load increment $n + 1$, incremental displacement Δq_i is computed, which in turn is used to compute the incremental second Piola-Kirchhoff stress ΔS_{ij} by using Equations (2.9) and (2.26). Use of ΔS_{ij} and σ_{ij}^n in Equation (2.25) yields ${}^nS_{ij}^{n+1}$, which is then substituted in Equation (2.23) to obtain σ_{ij}^{n+1} . At this step, if the material is linear elastic, the constitutive tensor C_{ijrs} is constant i.e. $C_{ijrs}^{n+1} = C_{ijrs}^n$, and the finite element procedure is repeated for next load increment $n + 2$ where C_{ijrs}^{n+1} and σ_{ij}^{n+1} are used. If the material is hyperelastic, the stress-strain relations are functions only of the strain energy which is defined by using the total Cauchy stress tensor σ_{ij}^{n+1} . Hence C_{ijrs}^{n+1} can be derived from σ_{ij}^{n+1} .

However, for a material whose constitutive relations are of rate-type such as hypoelastic, or dependent on the stress path such as elasto-plastic, C_{ijrs}^{n+1} is often computed from the previous stress and the incremental stress. In this case, the stress-strain relations are defined on a rate-type or incremental basis, and a proper incremental stress should be used.

When the body undergoes rotation, the stress state in the body is unchanged. However, the components of stress vary as the body rotates [29]. The constitutive tensor C_{ijrs}^{n+1} must be

independent of such a rotational change of the geometry. Under this circumstance, to relate the incremental Cauchy stress tensor to the rate-of-deformation tensor by using C_{ijrs}^{n+1} , Equation (2.27), could lead to inconsistency.

It is necessary, therefore, to define another type of incremental stress which is a function of only the deformation of the body. This type of stress is called the Jaumann's rate of stress and is defined as [29].

$$\Delta\sigma_{ij}^J = C_{ijrs}^{n+1} \Delta v_{rs} \quad (2.64)$$

The Jaumann's rate of stress tensor is related to the incremental Cauchy stress tensor, which is a function of both deformation and rotation, as

$$\Delta\sigma_{ij}^J = \Delta\sigma_{ij} - \Delta\Omega_{ik} \sigma_{kj}^{n+1} - \Delta\Omega_{jk} \sigma_{ki}^{n+1} \quad (2.65)$$

where $\Delta\sigma_{ij}$ is the incremental Cauchy stress tensor, and $\Delta\Omega_{ik}$ is defined in Equation (2.21).

Substituting Equation (2.65) into Equation (2.64) yields

$$\Delta\sigma_{ij} = C_{ijrs}^{n+1} \Delta v_{rs} + \Delta\Omega_{ik} \sigma_{kj}^{n+1} + \Delta\Omega_{jk} \sigma_{ki}^{n+1} \quad (2.66)$$

The consideration of the Jaumann's rate of stress is especially important in large deformation analysis with elasto-plastic material, particularly, during (large) plastic deformations. At the end of load step $n + 1$, σ_{ij}^{n+1} is computed and is checked with the yield criterion; the latter is fully described in Chapter III. If σ_{ij}^{n+1}

violates the yield function, iterations must be performed at this load step. C_{ijrs}^{n+1} , which is invariant with respect to coordinate transformation, is computed from σ_{ij}^{n+1} . Equation (2.66) is utilized to compute the incremental Cauchy stress tensor $\Delta\sigma_{ij}$. $\Delta\sigma_{ij}$ is then added to the previous stress to give another value of σ_{ij}^{n+1} which satisfies the invariance requirement and also may provide an improved computation of stress and constitutive matrix. Provided the incremental load from step n to step $n + 1$ is small, the above procedure coupled with the usage of Jaumann's rate of stress, Equation (2.64), is consistent with the constitutive definition whose parameters are independent with respect to coordinate transformation.

The scheme of using Jaumann's rate of stress will be further discussed in Chapter VI.

Chapter III

FINITE DEFORMATION ELASTO-PLASTIC THEORY

Solution from any closed-form analytical or numerical procedures are dependent on the appropriateness of the stress-strain or constitutive laws. The parameters defining the constitutive laws are found from laboratory tests. There is hardly any one constitutive model presently available that can describe the behaviors of all materials.

The behavior of a solid material is often divided into two components: elastic behavior and plastic behavior. A combination of elasticity and plasticity in the behavior of material yields an elasto-plastic material model.

Perhaps the simplest material law for solids is the Hooke's law. This is a linear elastic law which needs two material parameters, namely Young's modulus and Poisson's ratio, to describe the stress-strain relation. Following the Hooke's law, other constitutive models with nonlinear features as hyperelastic, hypoelastic, and elasto-plastic models have been developed. However, due to the analytical or mathematical complexity of these models relating to practical application, not much work has been performed satisfactorily.

The development of the high speed digital computer and the finite element method have attracted renewed interests among many investigators about constitutive modelling. Since it is relatively easy to incorporate complex constitutive models in the finite element procedure, recently considerable attention has been given to mathematical

development of advanced models and their determination from laboratory tests.

One of the most important inherent properties or modes of behavior of many geologic materials is the plastic effects, which occur almost from the start of loading. Although in some metals, linear or nonlinear elasticity has been observed, there is hardly any material that fully recovers its original state after the initially applied load is totally removed. This is especially true for most soils, rocks, and also the construction material such as concrete and composites.

The assumption of ideal elasticity would be unrealistic for these materials. Hence, almost all the recent developments of constitutive laws for soil and other geologic materials are based on the theory of plasticity.

III.1. Theory of Plasticity

There are generally two theories of plasticity [38,69]. The first one is called the deformation theory. In this theory, the constitutive relationship at some deformed state of the body is uniquely defined by the total stress at that state and is independent of the path of stress which occurs during the application of the external load. The second theory is called the flow theory or the incremental theory. This theory differs from the deformation theory by assuming that the final state of the deformed body is governed by the stress path at subsequent loading intervals. In other words, the constitutive relationship at some deformed state is fully dependent

on the stress state of the body at the previous load step.

The deformation theory has been found inconsistent with the laboratory experiments for many geologic materials, and there has not been many constitutive models based on this theory. The flow theory, on the other hand, appears to be more attractive due to its applicability to real situations and its simplicity for use in incremental numerical analyses. Hence, it has been used extensively in research and engineering applications.

In geomechanics, the flow theory of plasticity is essential in those instances where the behaviors of the materials are path dependent and exhibit dilatancy.

For an inherently isotropic material, which follows the flow theory and the associated law of plasticity, there are three criteria needed to describe the plastic behavior of a continuum. These criteria are:

- 1) a yield function which is a combination of stress invariants and material parameters. It specifies the stress state at which the material yields plastically,

- 2) a flow rule that defines the direction of the plastic growth and magnitude of plastic yielding, and

- 3) a hardening (or softening) rule that specifies the rate of growth of the plastic strain, and shows how the yield surface is modified during plastic flow.

Mathematically, the yield or strength criterion is defined by a surface called yield surface or loading surface with the function

$$f = f(\sigma_{ij}, \epsilon_{ij}^p, \kappa) = 0 \quad (3.1)$$

where f is a scalar, σ_{ij} is the total stress tensor, ϵ_{ij}^p is the total plastic strain tensor, and κ is a hardening parameter.

The behavior of a material is defined as

$f < 0$, elastic deformation or unloading occurs,

$f = 0$, plastic deformation or neutral loading occurs,

and $f > 0$ does not exist.

These conditions are depicted in Fig. (3.1).

In order to prescribe the change of the plastic strain when yielding occurs, i.e. $f = 0$, the flow rule is used and is expressed as

$$d\epsilon_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}} \quad (3.2)$$

where $d\epsilon_{ij}^p$ is the incremental plastic strain tensor, and λ is a constant of proportionality to be determined.

Finally, to modify the shape of the yield surface at different steps of loading, the hardening rule is defined by parameter κ which is often a function of incremental plastic strain,

$$\kappa = \kappa(d\epsilon_{ij}^p) \quad (3.3)$$

or a function of plastic potential,

$$\kappa = \kappa\left(\int_0^{\epsilon_{ij}^p} \sigma_{ij} d\epsilon_{ij}^p\right) \quad (3.4)$$

Depending on how κ is defined, there are two types of work-hardening rules: isotropic hardening and kinematic hardening [38,40].

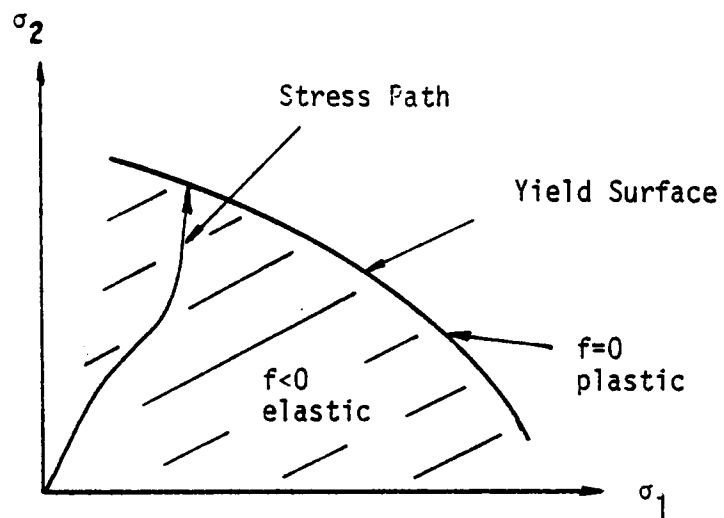


Figure 3.1. Symbolic Representation of Yield Surface in 2-D Stress Space

Isotropic hardening occurs when the yield surface expands uniformly or homogeneously about its axis, i.e. its axis is unchanged, Figure (3.2a). Kinematic hardening occurs when the shape of the yield surface is essentially kept constant but its local axes are translated and rotated to another position in the stress space, Figure (3.2b). The material, in many instances, can be assumed to be subjected to a combination of isotropic and kinematic hardening behaviors.

Explanation of the theory of plasticity with work hardening rules has been given elsewhere [29,38,40,77]. Only salient details of the mathematical formulation are given here.

During loading, the yield surface changes from one state to another such that the condition of consistency, as shown below, is satisfied

$$df = 0 \quad (3.5)$$

or

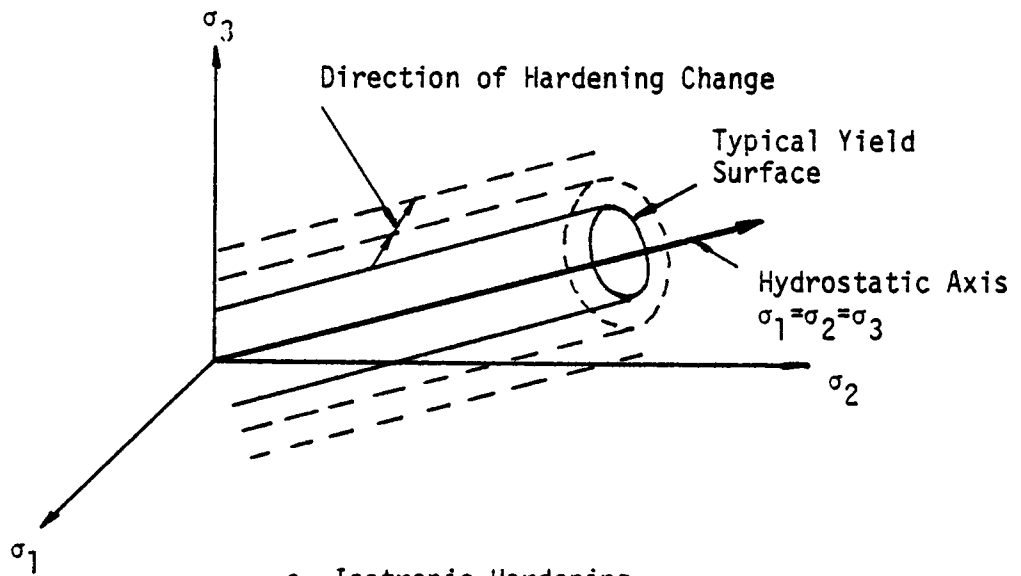
$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \epsilon_{ij}^p} d\epsilon_{ij}^p + \frac{\partial f}{\partial \kappa} d\kappa = 0 \quad (3.6)$$

By assuming the incremental strain to be small, a linear decomposition of elastic and plastic parts is possible, i.e.

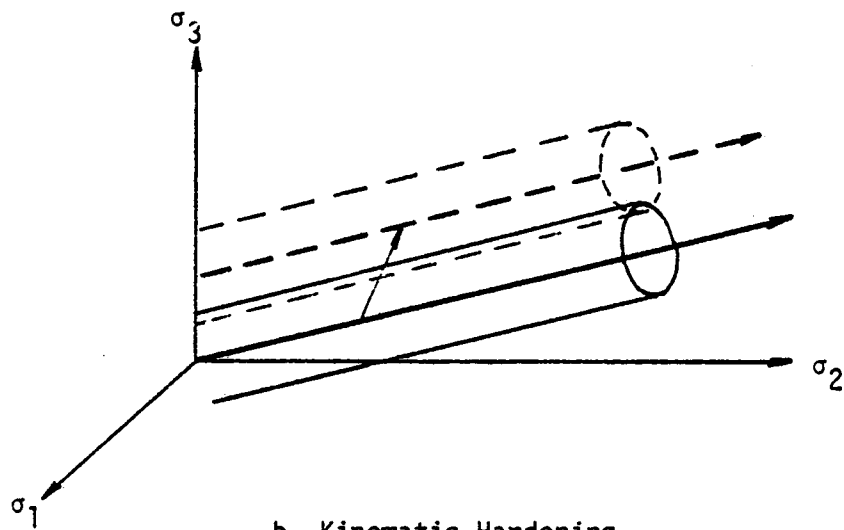
$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p \quad (3.7)$$

Also if the incremental stress is assumed to relate linearly to the elastic strain,

$$d\sigma_{ij} = C_{ijrs}^e d\epsilon_{rs}^e \quad (3.8)$$



a. Isotropic Hardening



b. Kinematic Hardening

Figure 3.2. Conventional Hardening Behaviors of Material

where C_{ijrs}^e is the linear elastic tensor, then substitution of Equations (3.2) and (3.8) in Equation (3.6) leads to

$$\frac{\partial f}{\partial \sigma_{ij}} C_{ijrs}^e d\epsilon_{rs}^e + \frac{\partial f}{\partial \epsilon_{ij}^p} \frac{\partial f}{\partial \sigma_{ij}} \lambda + \frac{\partial f}{\partial \kappa} d\kappa = 0 \quad (3.9)$$

Now by rearranging Equation (3.7) as

$$d\epsilon_{ij}^e = d\epsilon_{ij} - d\epsilon_{ij}^p \quad (3.10)$$

and substituting Equations (3.2) and (3.10) in Equation (3.9), we obtain

$$\frac{\partial f}{\partial \sigma_{ij}} C_{ijrs}^e (d\epsilon_{rs} - \lambda \frac{\partial f}{\partial \sigma_{rs}}) + \frac{\partial f}{\partial \epsilon_{ij}^p} \frac{\partial f}{\partial \sigma_{ij}} \lambda + \frac{\partial f}{\partial \kappa} d\kappa = 0 \quad (3.11)$$

Equation (3.11), after rearranging terms, yields

$$\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}} C_{ijrs}^e d\epsilon_{rs} + \frac{\partial f}{\partial \kappa} d\kappa}{\frac{\partial f}{\partial \sigma_{ij}} C_{ijrs}^e \frac{\partial f}{\partial \sigma_{rs}} - \frac{\partial f}{\partial \epsilon_{ij}^p} \frac{\partial f}{\partial \sigma_{ij}}} \quad (3.12)$$

After substituting Equation (3.12) back in Equation (3.2), the incremental plastic strain is derived as

$$d\epsilon_{ij}^p = \frac{\frac{\partial f}{\partial \sigma_{hk}} C_{hkrs}^e \frac{\partial f}{\partial \sigma_{ij}} d\epsilon_{rs} + \frac{\partial f}{\partial \kappa} d\kappa \frac{\partial f}{\partial \sigma_{ij}}}{\frac{\partial f}{\partial \sigma_{hk}} C_{hkrs}^e \frac{\partial f}{\partial \sigma_{rs}} - \frac{\partial f}{\partial \epsilon_{hk}^p} \frac{\partial f}{\partial \sigma_{hk}}} \quad (3.13)$$

However, the hardening parameter κ , as in Equations (3.3) and (3.4), is a function of the incremental plastic strain $d\epsilon_{ij}^p$. Hence

the appearance of the second term in the numerator of Equation (3.13) can be insignificant, and is ignored.

Equation (3.13) then becomes

$$d\epsilon_{ij}^p = \frac{\frac{\partial f}{\partial \sigma_{hk}} C_{hkrs}^e \frac{\partial f}{\partial \sigma_{ij}} d\epsilon_{rs}}{\frac{\partial f}{\partial \sigma_{hk}} C_{hkrs}^e \frac{\partial f}{\partial \sigma_{rs}} - \frac{\partial f}{\partial \epsilon_{hk}^p} \frac{\partial f}{\partial \sigma_{hk}}} \quad (3.14)$$

Substitution of Equation (3.10) in Equation (3.8),

$$d\sigma_{ij} = C_{ijrs}^e (d\epsilon_{rs} - d\epsilon_{rs}^p) \quad (3.15)$$

and Equation (3.14) in Equation (3.15) leads to

$$d\sigma_{mn} = \left[C_{mnij}^e - \frac{\frac{\partial f}{\partial \sigma_{hk}} C_{hkij}^e C_{mnrs}^e \frac{\partial f}{\partial \sigma_{rs}}}{\frac{\partial f}{\partial \sigma_{hk}} C_{hkrs}^e \frac{\partial f}{\partial \sigma_{rs}} - \frac{\partial f}{\partial \epsilon_{hk}^p} \frac{\partial f}{\partial \sigma_{hk}}} \right] d\epsilon_{ij} \quad (3.16)$$

or

$$d\sigma_{mn} = C_{mnij}^{ep} d\epsilon_{ij} \quad (3.17)$$

where

$$C_{mnij}^{ep} = C_{mnij}^e - \frac{\frac{\partial f}{\partial \sigma_{hk}} C_{hkij}^e C_{mnrs}^e \frac{\partial f}{\partial \sigma_{rs}}}{\frac{\partial f}{\partial \sigma_{hk}} C_{hkrs}^e \frac{\partial f}{\partial \sigma_{rs}} - \frac{\partial f}{\partial \epsilon_{hk}^p} \frac{\partial f}{\partial \sigma_{hk}}} \quad (3.18)$$

is called the elasto-plastic constitutive tensor.

Equation (3.18) is the general equation for any plastic material,

perfectly plastic or work-hardening. It gives a relationship between the incremental stress and incremental strain based on the current stress level and the total plastic strain at some deformed state of the body. Specific material models differ from each other essentially by the yield function f , Equation (3.1), and the work-hardening rule, Equations (3.3) and (3.4). Some elasto-perfectly plastic and work-hardening plastic models used in the current research will be discussed in Chapter IV.

III.2. Finite Deformation in Elasto-Plasticity

The first part of this chapter described the theory of elasto-plasticity with the assumption of small strain, that is the incremental strain is of a small order so that the elastic part and plastic part can be regarded as independent variables. Hence a linear decomposition of the incremental strain is valid, Equation (3.7),

$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p \quad (3.7)$$

However, when the body undergoes large deformation in the plastic range, this assumption may no longer be valid. The elastic recovery is dependent on the change of plastic deformation. Such a dependence is not of a linear nature, but it is rather irregular and following the characteristics of large deformation.

This section presents briefly two theories which deal with finite deformation elasto-plasticity, and discusses the difficulties encountered when these theories are implemented in numerical analysis scheme. An alternative method is proposed which is based on a

numerical approximation technique. This method will be discussed in Chapter VI.

III.2.a. Theory Proposed by Green and Nagdhi

Green and Nagdhi [33] proposed a general theory of an elasto-plastic continuum where the elastic recovery upon unloading is not independent but rather is a function of the plastic deformation. The assumption, however, is that it is possible to linearly decompose the incremental strain into two parts as in Equation (3.7).

$$dl_{ij} = dl_{ij}^e + dl_{ij}^p \quad (3.19)$$

where at this time,

$$dl_{ij} = dl_{ij}(S_{mn}, dl_{mn}^p) \quad (3.20)$$

and

$$dl_{ij}^e = dl_{ij}^e(S_{mn}, dl_{mn}^p) \quad (3.21)$$

Suppose all variables are known at some loading step n , an incremental load is applied at step $n + 1$. Similarly to Chapter II, the aim now is to compute the displacement from step n to step $n + 1$ and the deformed volume V^{n+1} . For ease of discussion, the superscript notations are modified as shown in Figure (3.3), where V^{ep} stands for the final volume, i.e. V^{n+1} , V^p stands for the unrecovered volume upon unloading, and V^0 stands for the initial configuration of the body before the load is applied, i.e. V^n .

The principle of virtual work at load step $n + 1$, expressed in terms of Lagrangian coordinate system at V^0 , is

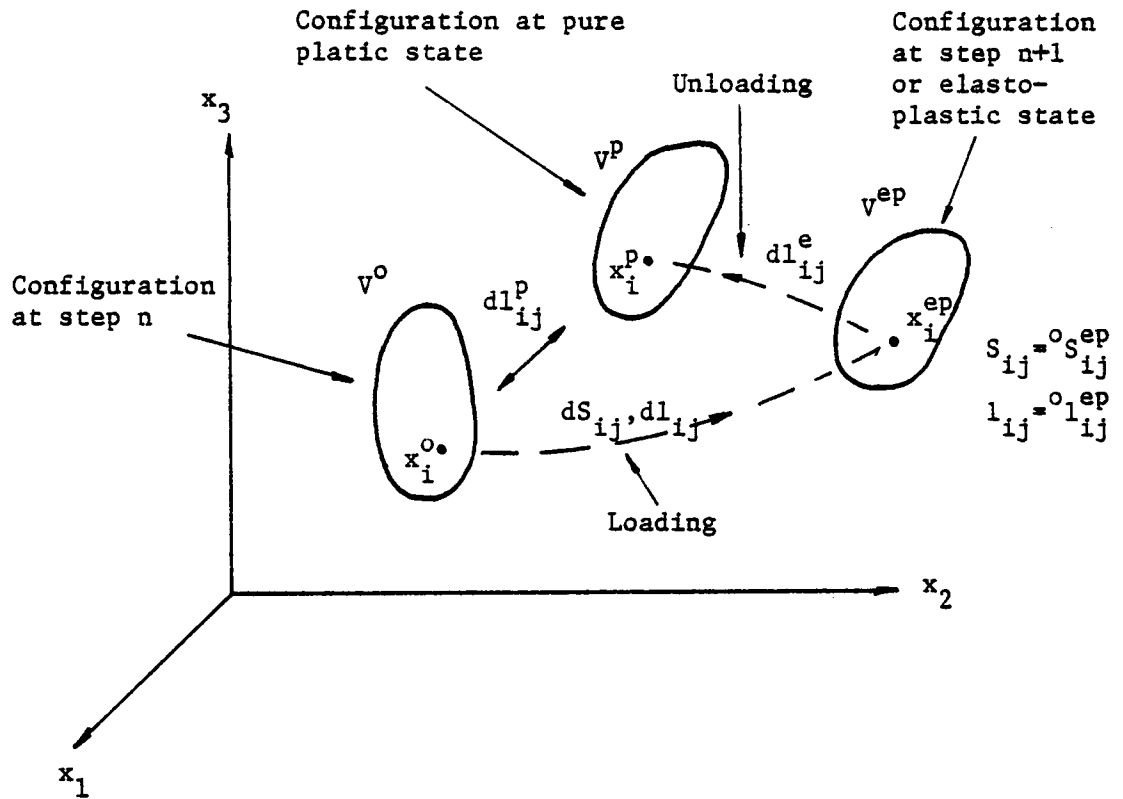


Figure 3.3. Configurations of Deformed Body at Different States and Variables used in the Theory proposed by Green and Nagdhi.

$$\int_{V^0} S_{ij} \delta l_{ij} dV^0 = \delta W^0 \quad (3.22)$$

where S_{ij} is the second Piola-Kirchoff stress tensor, l_{ij} is the Green-Lagrange strain tensor, and W^0 is the total external work at step $n + 1$.

The process of loading from V^0 to V^{ep} causes a plastic flow. The condition of plastic flow is defined by a yield function

$$f(S_{ij}, l_{ij}^p) = \kappa \quad (3.23)$$

where κ is a 'hardening' parameter, and is a function of stresses and plastic strains.

The rate of plastic strain that occurs during the expansion of the yield function is in general a function of the total stress, the incremental stress and the total plastic strain. This is expressed as

$$dl_{ij}^p = g_{ij}(S_{kl}, dS_{kl}, l_{kl}^p) \quad (3.24)$$

Similar to Equation (3.6), it is necessary to identify the manner of the change of yielding condition during plastic flow by assuming

$$df = \frac{\partial f}{\partial S_{ij}} dS_{ij} + \frac{\partial f}{\partial l_{ij}^p} dl_{ij}^p = d\kappa \quad (3.25)$$

in which, for the case of loading, $f \geq \kappa$,

$$dl_{ij}^p = g_{ij} \quad (3.26)$$

and

$$\frac{\partial f}{\partial S_{ij}} dS_{ij} > 0, \quad (3.27)$$

for the case of neutral loading, $f = \kappa$,

$$dl_{ij}^p = 0 \quad (3.28)$$

and

$$\frac{\partial f}{\partial S_{ij}} dS_{ij} = 0, \quad (3.29)$$

and for the case of unloading (elastic recovery), $f < \kappa$,

$$dl_{ij}^p = 0 \quad (3.30)$$

and

$$\frac{\partial f}{\partial S_{ij}} dS_{ij} < 0 \quad (3.31)$$

The total derivative of the hardening parameter κ in general can be assumed as

$$d\kappa = d\kappa(S_{ij}, l_{ij}^p, dS_{ij}, dl_{ij}^p) \quad (3.32)$$

with a condition

$$d\kappa = 0 \quad (3.33)$$

when $dl_{ij}^p = 0$.

Without loss of generality, $d\kappa$ can also be assumed to be a linear

function of dS_{ij} and dI_{ij}^P , such that

$$d\kappa = h_{ij}(S_{kl}, I_{kl}^P) dI_{ij}^P + t_{ij}(S_{kl}, I_{kl}^P) dS_{ij} \quad (3.34)$$

However, since $d\kappa = 0$ when $dI_{ij}^P = 0$, Equation (3.33), the second term in Equation (3.34) is insignificant and is left out, which yields

$$d\kappa = h_{ij}(S_{kl}, I_{kl}^P) dI_{ij}^P \quad (3.35)$$

Substituting Equation (3.35) in Equation (3.25), and factoring out term dI_{ij}^P , leads to

$$dI_{ij}^P = p_{ijkl} dS_{kl} \quad (3.36)$$

with the condition $dI_{ij}^P = 0$ during neutral loading (no plastic potential contribution), or

$$p_{ijkl} dS_{kl} = 0 \quad (3.37)$$

if

$$\frac{\partial f}{\partial S_{ij}} dS_{ij} = 0 \quad (3.38)$$

Equations (3.35) and (3.36) define the constitutive equations for an elasto-plastic material with large deformation.

To obtain the explicit form of the tensor p_{ijkl} , we proceed further by multiplying both sides of Equation (3.38) by a scalar λ and a tensor a_{ij} ,

$$\lambda a_{ij} \frac{\partial f}{\partial S_{k1}} dS_{k1} = 0 \quad (3.39)$$

and subtracting Equation (3.39) to Equation (3.37),

$$(\lambda a_{ij} \frac{\partial f}{\partial S_{k1}} - p_{ijk1}) dS_{k1} = 0 \quad (3.40)$$

Since dS_{k1} is arbitrary, the term in the parentheses must vanish, and we obtain

$$p_{ijk1} = \lambda a_{ij} \frac{\partial f}{\partial S_{k1}} \quad (3.41)$$

Substitution of Equation (3.41) in Equation (3.36) leads to

$$dl_{ij}^p = \lambda a_{ij} \frac{\partial f}{\partial S_{k1}} dS_{k1} \quad (3.42)$$

By using Equation (3.27), we can assume $\lambda > 0$ without loss of generality.

Now rearranging Equations (3.25) and (3.35) as

$$\frac{\partial f}{\partial S_{ij}} dS_{ij} + \frac{\partial f}{\partial l_{ij}^p} dl_{ij}^p = h_{ij} dl_{ij}^p \quad (3.43)$$

and substituting Equation (3.42) in Equation (3.43), yields

$$(1 + \lambda a_{ij} \frac{\partial f}{\partial l_{ij}^p} - \lambda a_{ij} h_{ij}) \frac{\partial f}{\partial S_{k1}} dS_{k1} = 0 \quad (3.44)$$

Since $\frac{\partial f}{\partial S_{k1}}$ and dS_{k1} are arbitrary, the terms in the bracket of Equation (3.44) must vanish, i.e.

$$1 + \lambda a_{ij} \frac{\partial f}{\partial l_{ij}^p} - \lambda a_{ij} h_{ij} = 0 \quad (3.45)$$

or

$$\lambda = \frac{1}{a_{ij} (h_{ij} - \frac{\partial f}{\partial l_{ij}^p})} \quad (3.46)$$

Substituting λ from Equation (3.46) in Equation (3.42), we obtain

$$dl_{rs}^p = \frac{a_{rs} \frac{\partial f}{\partial S_{kl}}}{a_{ij} (h_{ij} - \frac{\partial f}{\partial l_{ij}^p})} dS_{kl} \quad (3.47)$$

Equation (3.47) is an alternative form of Equation (3.36).

It should be noted that although Equation (3.47) only gives the incremental plastic strain in terms of the incremental stress, the total incremental strain can be evaluated by using Equations (3.19) and (3.20).

Thermodynamic Restrictions

The condition of plastic growth, although well described by Equations (3.19), (3.23), (3.25) and (3.47), is restricted by the thermodynamic condition that the plastic work must be nonnegative for irreversible deformation process.

The total strain energy U of the body at any deformed shape is expressible in terms of the total strain at that deformed shape, i.e.

$$U = U(l_{ij}) = U(l_{ij}^e, l_{ij}^p) \quad (3.48)$$

Taking total derivative of Equation (3.48) gives

$$dU = \frac{\partial U}{\partial l_{ij}^e} dl_{ij}^e + \frac{\partial U}{\partial l_{ij}^p} dl_{ij}^p \quad (3.49)$$

The incremental internal work caused by a change of strain can also be expressed as

$$dU^* = \frac{1}{\rho} S_{ij} dl_{ij} \quad (3.50)$$

where ρ is the density of the body at the deformed shape.

Substituting Equation (3.19) in Equation (3.50) gives

$$dU^* = \frac{1}{\rho} S_{ij} (dl_{ij}^e + dl_{ij}^p) \quad (3.51)$$

The thermodynamic restriction states that

$$dU^* - dU \geq 0 \quad (3.52)$$

for all conditions of loading, which, after substituting Equations (3.43) and (3.51) in Equation (3.52), leads to

$$\left(\frac{S_{ij}}{\rho} - \frac{\partial U}{\partial l_{ij}^e} \right) dl_{ij}^e + \left(\frac{S_{ij}}{\rho} - \frac{\partial U}{\partial l_{ij}^p} \right) dl_{ij}^p \geq 0 \quad (3.53)$$

During neutral loading or unloading

$$dl_{ij}^p = 0 \quad (3.54)$$

and the equality in Equation (3.53) holds, hence

$$S_{ij} = \rho \frac{\partial U}{\partial l_{ij}^e} \quad (3.55)$$

Equation (3.55) is a restriction upon the elastic strain at the stress state S_{ij} .

During loading, however, the inequality in Equation (3.52) must be retained. The first term on the left hand side of Equation (3.53) has no effect on the plastic growth, only the second term contributes to the increase of the incremental plastic work. Therefore

$$(S_{ij} - \rho \frac{\partial U}{\partial l_{ij}^p}) dl_{ij}^p \geq 0 \quad (3.56)$$

Substitution of Equation (3.42) in Equation (3.56) leads to

$$\lambda a_{ij} \frac{\partial f}{\partial S_{kl}} dS_{kl} (S_{ij} - \rho \frac{\partial U}{\partial l_{ij}^p}) \geq 0 \quad (3.57)$$

or

$$[\lambda \frac{\partial f}{\partial S_{kl}} dS_{kl}] [a_{ij} (S_{ij} - \rho \frac{\partial U}{\partial l_{ij}^p})] \geq 0 \quad (3.58)$$

Using Equation (3.27) and the condition of $\lambda > 0$ during loading yields

$$\lambda \frac{\partial f}{\partial S_{kl}} dS_{kl} > 0, \quad (3.59)$$

hence

$$a_{ij} (S_{ij} - \rho \frac{\partial U}{\partial l_{ij}^p}) \geq 0 \quad (3.60)$$

This is the restriction on the choice of the tensor a_{ij} for use in Equation (3.42).

Difficulties in Implementation

In applying this theory in numerical analysis, it is found difficult to obtain the incremental plastic strain from the total incremental strain which is known. From Equation (3.20), it can be seen that dl_{ij} is a function of the total stress S_{mn} and the plastic strain dl_{mn}^P . The choice of this function is dependent on the configuration V^P , Figure (3.1), as unloading occurs. However, this configuration V^P is unknown since the numerical scheme is a marching-forward technique. Furthermore, the total stress S_{mn} is an unknown value at the time the load is applied. With this difficulty, the incremental elastic strain cannot be computed in order to satisfy the thermodynamic condition, Equation (3.55).

Another aspect is that although the hardening parameter κ can be defined in advance, which gives a priori value of the tensor h_{ij} in Equation (3.35), the tensor a_{ij} , which is a function of the total stress and the total strain, is difficult to compute to satisfy Equation (3.42). Finally, Equation (3.47) is, in general, non-symmetric and hence, the implementation of this equation could be inconvenient.

III.2.b. Theory Proposed by Lee

Lee [42] and Lee et al [43] proposed another theory for finite deformations with elasto-plasticity by using kinematical equations to describe the relationship between elastic deformation and plastic deformation. The gradients of the coordinates are mobilized to handle the change of the body when plastic deformation occurs. The conven-

tional decomposition of strain into an elastic part and a plastic part, Equation (3.7), is not used in this theory since the strains are no longer linear in displacements [7,42].

Similar to Figure (3.3), Figure (3.4) shows the configurations of the body at V^0 , V^{ep} and V^p , where V^p is the volume of the body if unloading occurs [7].

The deformation gradient of the final configuration, V^{ep} , with respect to the original configuration, V^0 , is defined as

$$g_{ij}^{ep} = \frac{\partial x_i^{ep}}{\partial x_j^0} \quad (3.61)$$

The right hand side of Equation (3.61) can be decomposed as

$$\frac{\partial x_i^{ep}}{\partial x_j^0} = \frac{\partial x_i^{ep}}{\partial x_k^p} \cdot \frac{\partial x_k^p}{\partial x_j^0} \quad (3.62)$$

which leads to

$$g_{ij}^{ep} = g_{ik}^e \cdot g_{kj}^p \quad (3.63)$$

where

$$g_{ik}^e = \frac{\partial x_i^{ep}}{\partial x_k^p} \quad (3.64)$$

is the deformation gradient of V^{ep} with respect to V^p , and

$$g_{kj}^p = \frac{\partial x_k^p}{\partial x_j^0} \quad (3.65)$$

is the deformation gradient of V^p with respect to V^0 .

Substitution of Equation (3.61) and the chain rule

$$\frac{\partial x_k^o}{\partial x_j^{ep}} = \frac{\partial x_k^o}{\partial x_m^p} \frac{\partial x_m^p}{\partial x_j^{ep}} \quad (3.72)$$

in Equation (3.71) yields

$$\frac{\partial du_i}{\partial x_j^{ep}} = dg_{ik}^{ep} \frac{\partial x_k^o}{\partial x_m^p} \frac{\partial x_m^p}{\partial x_j^{ep}} \quad (3.73)$$

But from Equation (3.63)

$$dg_{ik}^{ep} = d(g_{in}^e g_{nk}^p) \quad (3.74)$$

or

$$dg_{ik}^{ep} = g_{in}^e dg_{nk}^p + g_{nk}^p dg_{in}^e \quad (3.75)$$

Hence Equation (3.73) becomes

$$\frac{\partial du_i}{\partial x_j^{ep}} = (g_{in}^e dg_{nk}^p + g_{nk}^p dg_{in}^e) \frac{\partial x_k^o}{\partial x_m^p} \frac{\partial x_m^p}{\partial x_j^{ep}} \quad (3.76)$$

Also from Equation (3.63) we obtain

$$\det(g_{ij}^{ep}) = \det(g_{ik}^e g_{kj}^p) \quad (3.77)$$

or

$$\det(g_{ij}^{ep}) = \det(g_{ik}^e) \det(g_{kj}^p) \quad (3.78)$$

Now substituting Equations (3.76) and (3.78) in Equation (3.67) gives

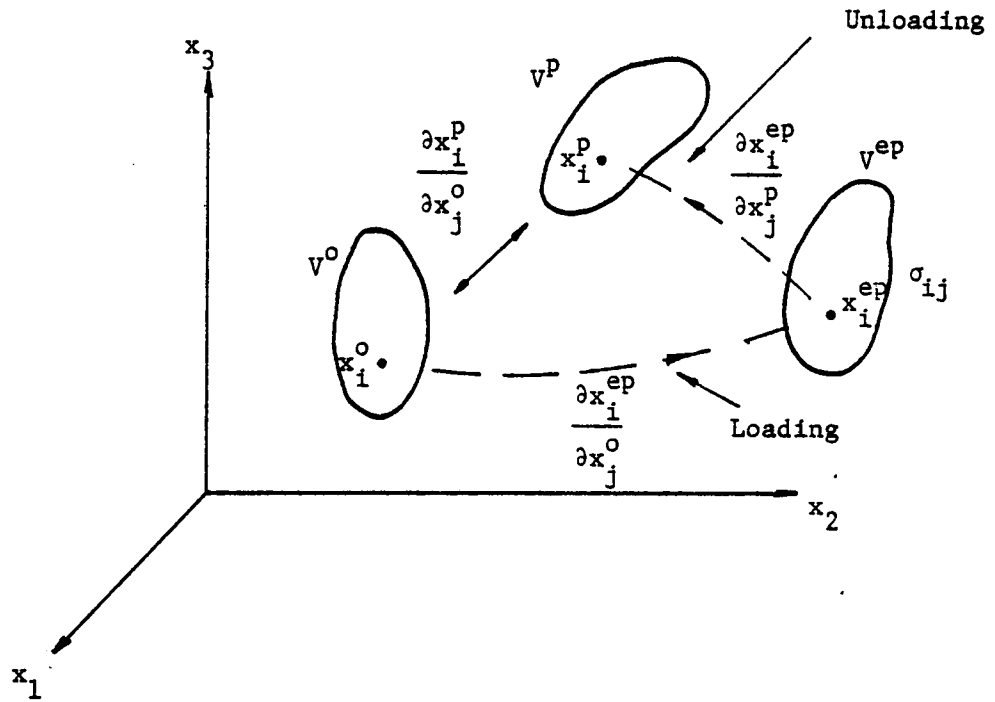


Figure 3.4. Configurations of Deformed Body at Different States and Variables used in the Theory proposed by Lee.

Suppose the incremental displacement from V^0 to V^{ep} is du_i , the incremental work (strain energy) which causes the body to deform from V^0 to V^{ep} is

$$dW = \int_{V^{ep}} \sigma_{ij} \frac{\partial du_i}{\partial x_j^{ep}} dV^{ep} \quad (3.66)$$

where σ_{ij} is the current total stress computed on the configuration V^{ep} .

Since V^{ep} is unknown, it is more convenient to transform Equation (3.66) into V^0 . This yields

$$dW = \int_{V^0} \sigma_{ij} \frac{\partial du_i}{\partial x_j^{ep}} \det(g_{ij}^{ep}) dV^0 \quad (3.67)$$

where $\det$ represents the determinant.

Using the chain rule

$$\frac{\partial du_i}{\partial x_j^{ep}} = \frac{\partial du_i}{\partial x_k^0} \frac{\partial x_k^0}{\partial x_j^{ep}} \quad (3.68)$$

and the condition

$$du_i = dx_i^{ep} \quad (3.69)$$

leads to

$$\frac{\partial du_i}{\partial x_j^{ep}} = \frac{\partial dx_i^{ep}}{\partial x_k^0} \frac{\partial x_k^0}{\partial x_j^{ep}} \quad (3.70)$$

or

$$\frac{\partial du_i}{\partial x_j^{ep}} = d\left(\frac{\partial x_i^{ep}}{\partial x_k^0}\right) \frac{\partial x_k^0}{\partial x_j^{ep}} \quad (3.71)$$

$$dW = \int_{V^0} \sigma_{ij} (g_{in}^e dg_{nk}^p + g_{nk}^p dg_{in}^e) \frac{\partial x_k^0}{\partial x_m^p} \frac{\partial x_m^p}{\partial x_j^{ep}} \det(g_{ik}^e) \det(g_{kj}^p) dV^0 \quad (3.79)$$

or

$$dW = dW^p + dW^e \quad (3.80)$$

where

$$dW^p = \int_{V^0} \sigma_{ij} g_{in}^e dg_{nk}^p \frac{\partial x_k^0}{\partial x_m^p} \frac{\partial x_m^p}{\partial x_j^{ep}} \det(g_{ik}^e) \det(g_{kj}^p) dV^0 \quad (3.81)$$

and

$$dW^e = \int_{V^0} \sigma_{ij} g_{nk}^p dg_{in}^e \frac{\partial x_k^0}{\partial x_m^p} \frac{\partial x_m^p}{\partial x_j^{ep}} \det(g_{ik}^e) \det(g_{kj}^p) dV^0 \quad (3.82)$$

or

$$dW^e = \int_{V^0} \sigma_{ij} dg_{in}^e \frac{\partial x_n^p}{\partial x_j^{ep}} \det(g_{ik}^e) \det(g_{kj}^p) dV^0 \quad (3.83)$$

Assuming further that the plastic volumetric strain is zero during plastic flow, i.e.

$$\det(g_{kj}^p) = 1 \quad (3.84)$$

Equations (3.81) and (3.83) become

$$dW^p = \int_{V^0} \sigma_{ij} dg_{nk}^p g_{in}^e \frac{\partial x_k^0}{\partial x_m^p} \frac{\partial x_m^p}{\partial x_j^{ep}} \det(g_{ik}^e) dV^0 \quad (3.85)$$

and

$$dW^e = \int_{V^0} \sigma_{ij} dg_{in}^e \frac{\partial x_n^p}{\partial x_j^e} \det(g_{ik}^e) dV^0 \quad (3.86)$$

Equation (3.86) has only terms corresponding to elastic recovery, that is from V^{ep} to V^p , and hence is called the incremental elastic work. Equation (3.85), on the other hand, has the coupled terms between elastic and plastic gradients, and is called the incremental plastic work. As discussed in References [42,43], Equations (3.85) and (3.86) are coupled, therefore elastic and plastic deformations are not independent of each other.

The condition of breaking dW into dW^e and dW^p , Equation (3.80), is acceptable only if the plastic configuration V^p is independent of rotational mode, that is the work done is caused by deformation only. In this respect, V^p is not chosen arbitrarily, but must be selected in such a manner that the rotational independent condition is satisfied. A special gradient tensor $\bar{g}_{ij}^e$ was proposed [42] to handle this effect. The gradient tensor $\bar{g}_{ij}^e$ is symmetric and represents only pure deformation in the directions of the principal axes of stress.

$\bar{g}_{ij}^e$ can be obtained by decomposing the tensor g_{ij}^e as

$$g_{ij}^e = v_{ik} r_{kj} \quad (3.87)$$

where v_{ik} is a symmetric tensor and represents pure deformation and r_{kj} is a skew-symmetric tensor and represents rotation.

Substituting Equation (3.87) in Equation (3.63), we obtain

$$g_{ij}^{ep} = v_{im} r_{mk} g_{kj}^p \quad (3.88)$$

or

$$g_{ij}^{ep} = \bar{g}_{im}^e \bar{g}_{mj}^p \quad (3.89)$$

where

$$\bar{g}_{im}^e = v_{im} \quad (3.90)$$

and

$$\bar{g}_{mj}^p = r_{mk} g_{kj}^p \quad (3.91)$$

With this equivalent $\bar{g}_{ij}^e$ and $\bar{g}_{ij}^p$, we can rewrite Equation (3.85) as

$$dW^p = \int_{V^0} \sigma_{ij} d\bar{g}_{nk}^p \bar{g}_{in}^e \frac{\partial \bar{x}_k^0}{\partial \bar{x}_m^p} \frac{\partial \bar{x}_m^p}{\partial \bar{x}_j^{ep}} \det(\bar{g}_{ik}^e) dV^0 \quad (3.92)$$

where $\bar{x}_m^p$ denote the specific configuration which is independent of rotation when unloading occurs, and $\bar{x}_m^p$ are parallel to the directions of the principal stresses at σ_{ij} .

Rearranging indices in the integrand of Equation (3.92) and noting that $\bar{g}_{ij}^e$ is symmetric and $\bar{g}_{ij}^p$ is skew-symmetric, we obtain

$$dW^p = \int_{V^0} \sigma_{ij} d\bar{g}_{ik}^p \frac{\partial \bar{x}_k^0}{\partial \bar{x}_j^p} \det(\bar{g}_{mn}^e) dV^0 \quad (3.93)$$

By using Equation (3.65) for the specific coordinates $\bar{x}_i^p$, Equation (3.93) can be changed to

$$dW^P = \int_{V^0} \sigma_{ij} \frac{d\bar{x}_i^P}{\partial \bar{x}_k^P} \frac{\partial \bar{x}_k^0}{\partial \bar{x}_j^P} \det(\bar{g}_{mn}^e) dV^0 \quad (3.94)$$

or

$$dW^P = \int_{V^0} \sigma_{ij} \frac{\partial d\bar{x}_i^P}{\partial \bar{x}_j^P} \det(\bar{g}_{mn}^e) dV^0 \quad (3.95)$$

From Equation (3.95), it can be seen that the difference between the incremental plastic works using the infinitesimal strain theory and the finite strain theory is the term $\det(\bar{g}_{mn}^e)$.

Lee [42] proposed to change the yield function and other criteria in the conventional theory of elasto-plasticity by multiplying a scalar term $\det(\bar{g}_{mn}^e)$ to the current stress σ_{ij} . The yield function is then redefined as

$$f = f[\sigma_{ij} \det(\bar{g}_{mn}^e)] = c \quad (3.96)$$

where c is a hardening parameter.

This is the main contribution in this theory in handling problems of finite-deformation elasto-plasticity.

Difficulties in Implementation

In applying this theory in numerical analysis, it is necessary to transform variables in proper configurations. This causes some numerical difficulties, which, in turn, could lead to inconsistent formulation. For example, although the stress tensor σ_{ij} computed on V^{eP} can be transformed to the second Piola-Kirchoff stress tensor S_{ij} computed on V^0 , it is not applicable to transform σ_{ij} to configu-

ration V^P since V^P is not known in the computational process. The volume V^P can be computed backward by evaluating the plastic deformation based on the known incremental stress S_{ij} , but this scheme could be unstable and could cause non-unique solutions.

Furthermore, the assumption in Equation (3.84) might be valid for metals with no plastic volumetric change, but for geologic materials such as soils and rocks, this assumption may no longer be valid. Under such circumstances, it is unrealistic to break the plastic work dW into elastic and plastic parts as in Equations (3.81) and (3.82). Hence, Lee's theory is rather restricted for special materials such as metals only.

The technique proposed in this research adopts the theory of elasto-plasticity for infinitesimal deformation but modifies the numerical scheme. The modification can take into account large changes in deformation by updating the geometry of the body for every numerical iteration. This will be discussed further in Chapter VI.

Chapter IV

VARIOUS CONSTITUTIVE MODELS

In this chapter, some of the constitutive models often used in geomechanics will be discussed. The details of the formulation of these models, however, will not be presented. Readers are referred to References such as [16,19,59,61,66,75] for additional details.

IV.1. Elasto-Perfectly Plastic Models

The Von-Mises model has been the source for much of the early work on elasto-plastic constitutive laws because of its simplicity and its relevance in metal plasticity. In geomechanics, this model and the Prandtl-Reuss relations have been used for predicting the behavior of saturated undrained clay [74]. The yield function of this model is expressed as

$$f(\sigma_{ij}, k) = J_{2D}^{1/2} - k = 0 \quad (4.1)$$

where

$$J_{2D} = \frac{1}{6} [(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{xx} - \sigma_{zz})^2] \\ + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 \quad (4.2)$$

is the second invariant of deviator stress, and k is a yield stress.

In the principal stress space, the Von-Mises yield function is a cylinder with the axis being the hydrostatic axis, i.e. $\sigma_1 = \sigma_2 = \sigma_3$, Figure (4.1). The cylindrical shape of this model results from the assumption that $f(\sigma_{ij}) = f(-\sigma_{ij})$; here the

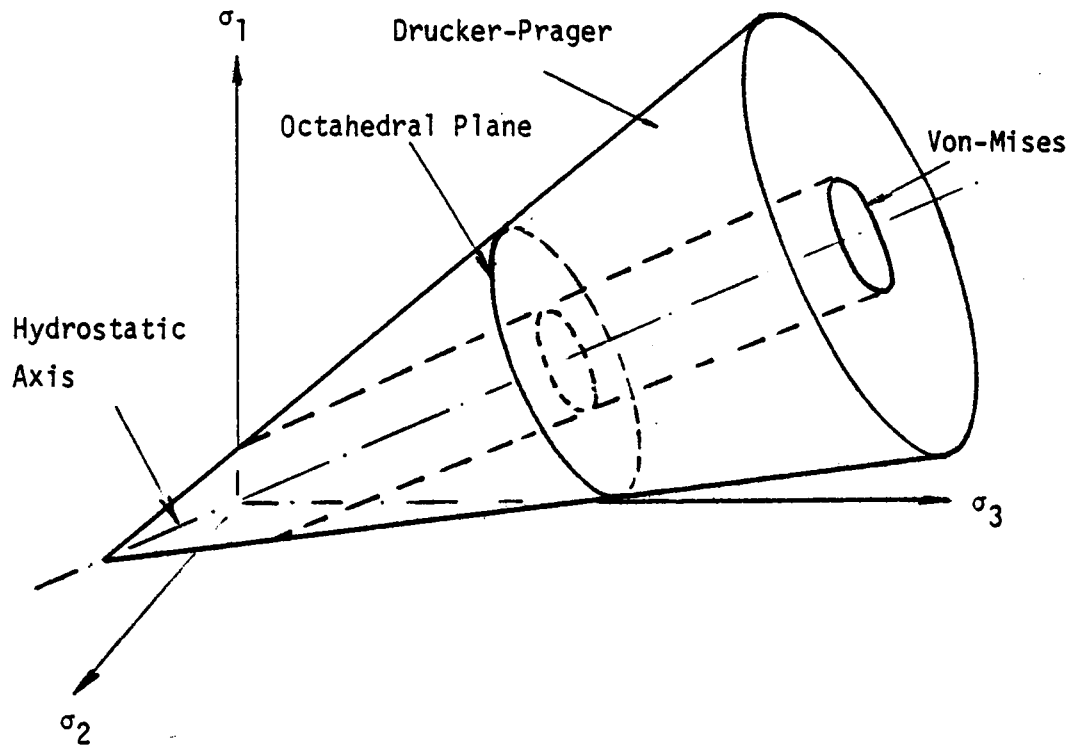


Figure 4.1. Elasto-perfectly Plastic Models

Bauschinger effects are ignored [36]. From Equation (4.1), it can be seen that only shear loading causes plastic deformation. Also by repeating indices i and j in Equation (3.2),

$$d\epsilon_{ij}^p = \lambda \frac{\partial f}{\partial \sigma_{ij}} \quad (4.2)$$

we obtain

$$d\epsilon_{ii}^p = \lambda \frac{\partial f}{\partial \sigma_{ii}} = 0 \quad (4.3)$$

for this model, which shows that there is no plastic volumetric strain and hence is relevant in metal plasticity. The application of the Von-Mises model, however, is rather limited in geomechanics although some good results have been observed in analyses of structures on saturated clay under undrained condition.

The limitation of the Von-Mises model in handling the volumetric changes in soils and rocks under hydrostatic loading, proved in experiments for many geologic materials, has lead to the extended Von-Mises model. It is often terms as Drucker-Prager model.

Drucker et al [25] proposed an elasto-perfectly plastic model which accounts for volumetric change by including the effect of the first invariant of stress in the yield function,

$$f = aJ_1 + J_2^{1/2} - k = 0 \quad (4.4)$$

where $J_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ is the first invariant of stress and a is a coefficient depending on the cohesion and friction of the material.

This function is represented by a right cone, Figure (4.1), with the hydrostatic axis being its axis. The Drucker-Prager elasto-perfectly plastic model is a modification of the Mohr-Coulomb model [75], in which the hydrostatic pressure contributes a significant part in the yielding condition and the plastic deformation.

However, the Drucker-Prager model often predicts too much volumetric change at high pressures which is usually not observed in the laboratory experiments for many types of soil or other geologic materials. This aspect is qualitatively shown in Figure (4.2). In this figure, the plastic strain vector is shown perpendicular to the yield surface following the associated law. The projection of this vector on the J_1 axis is the plastic volumetric strain which denotes the volume change at stress levels σ_{ij} . When the body undergoes further loading, the stress level of a point in the continuum would increase. This increase is again denoted in Figure (4.2) where the arrow marks the direction along the yield surface where the stress point moves. The new plastic strain vector computed from the new state of stress and its projection predicts more volume change than the previous volume change.

The geologic materials often behave as metals under high pressure conditions, i.e. they can become nearly incompressible. In that respect, the Von-Mises model appears to be more attractive than the Drucker-Prager model in predicting the behavior of the material under such loading range. A modification could be made

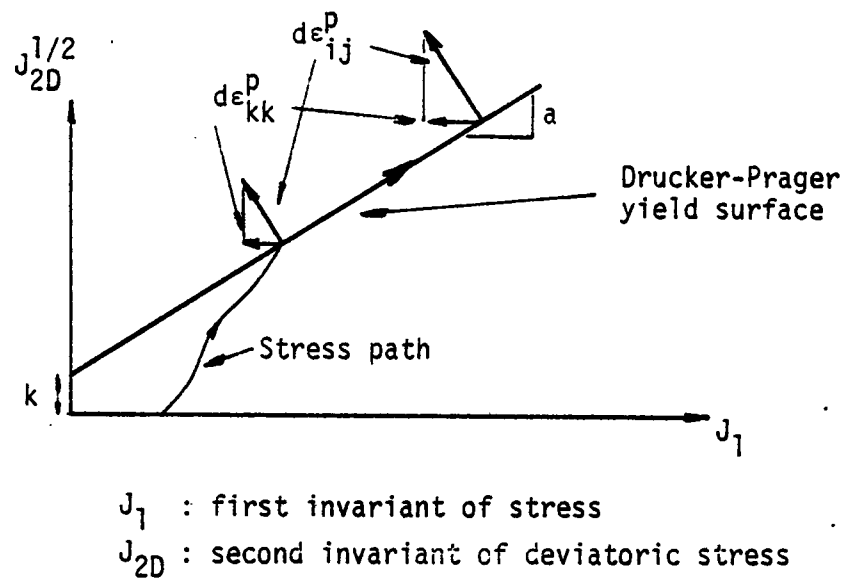


Figure 4.2. Drucker-Prager Model in Predicting Volume Change

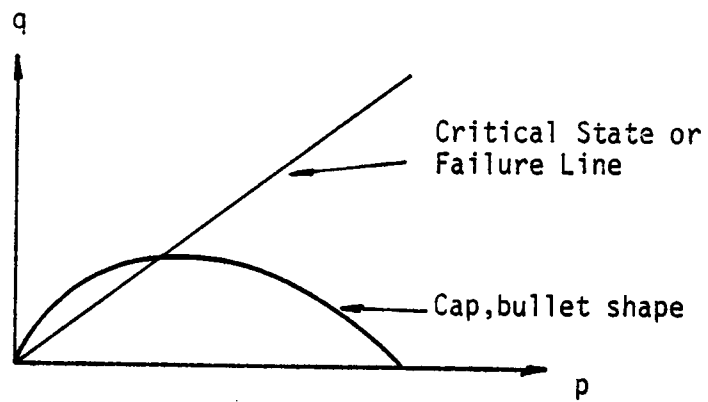
to account for this effect, and it will be shown later in this chapter.

Soils exhibit highly nonlinear stress-strain behavior. As observed in the laboratory, they often show plastic deformations even when subjected to very small loads. However, the elasto-perfectly plastic models assume the material behavior to be elastic until a load level that plastic effect suddenly occurs. Hence in predicting the soil deformation, these models could lead to unrealistic solution.

IV.2. Hardening Plastic Models

To improve on the elastic-perfectly plastic models for geologic media, a number of workers have proposed various models which include the work-hardening effects and can predict realistically the actual behavior of the materials. One class of these work-hardening elastic-plastic models, which will be described next, follows strictly the classical theory of plasticity; and although termed hardening models, they are not restricted to the conventional definitions of isotropic or kinematic hardening rules as discussed in Chapter III.

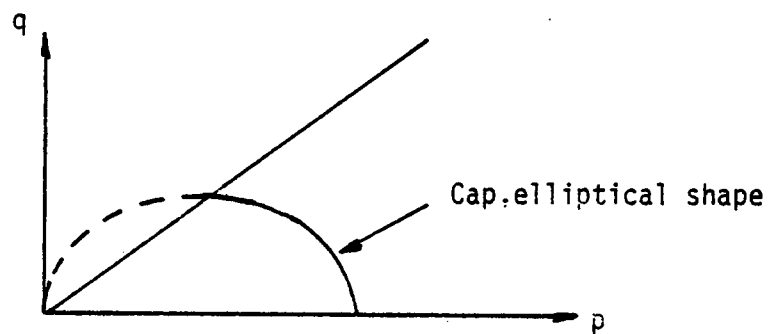
The hardening models of this class originated from the idea of using a yield surface to control the plastic volumetric deformation first suggested by Drucker et al [24]. This yield surface has the shape of a cap, Figure (4.3a), hence these models are often called cap models. The cap model is based on two types of yield surfaces, Figure (4.3a). The first surface is a cap which can expand or



$$p = (\sigma_3 + 2\sigma_1)/3$$

$$q = \sigma_3 - \sigma_1$$

a. Original Cam-Clay Model



b. Modified Cam-Clay Model

Figure 4.3. Critical State Models

contract depending on loading or unloading conditions, respectively. The second surface has the function of a yield surface but is used as an ultimate strength bounding surface or a failure surface. The cap might have any shape depending on the behavior of a given material.

One of the earliest cap models is called the critical state model and was proposed by Roscoe and the Cambridge group [58,61]. The cap of this model has a bullet-like shape, Figure (4.3a). Since this model violates the Drucker's postulates [22,23], it is only used infrequently.

A modification for the critical state model was proposed by the same group [57], in which the new model has an elliptical cap, Figure (4.3b). The intersection of the cap and the critical state line is the apex of the ellipse. The part of the ellipse which is above the critical state line and violates Drucker's postulates is ignored in this modified cap model. Details of the yield functions for the original critical state model and the modified critical state model are given in Appendix B.

Since the developments of the critical state models, many researchers have proposed new constitutive models based on the concept of the cap. Reviews of these models are given in References [19, 56,62]. Probably, the most widely used cap model in geomechanics is the model proposed by DiMaggio et al [21]. This model is defined in a general three-dimensional stress state and could be used for cohesive and cohesionless materials. The

most significant feature of this model is that it can predict better the stress-strain-strength situation at failure during high mean pressure states as compared to the prediction provided by the critical state models. As shown in Figure (4.4), the behavior of the material is initially controlled by a Drucker-Prager failure line coupled with an elliptical cap. Then a transitional failure lines is adopted until it coincides with a Von-Mises line [21,59,60]. The transition from the Drucker-Prager line to Von-Mises line at increasing load has the effect of controlling the volume change discussed previously. The yield function based on the cap model proposed by DiMaggio et al [21] is given again in Appendix B.

An interesting point related to the cap model formulation is that depending on the data obtained in the laboratory for different tests such as uniaxial strain, hydrostatic compression, conventional triaxial, or truly triaxial tests, one can derive different failure surfaces and cap surfaces for various types of soil. This is what happens for the artificial soil in the current research.

IV.3. Constitutive Model for the Artificial Soil

The current research deals with an artificial soil which is composed of sand, clay, and spindle oil, [55,65,66]. This type of soil exhibits significant plastic deformation even when it is subjected to isotropic compression. Few analytical constitutive models of those types discussed above match the real behavior of the artificial soil. Significant amount of volume change has

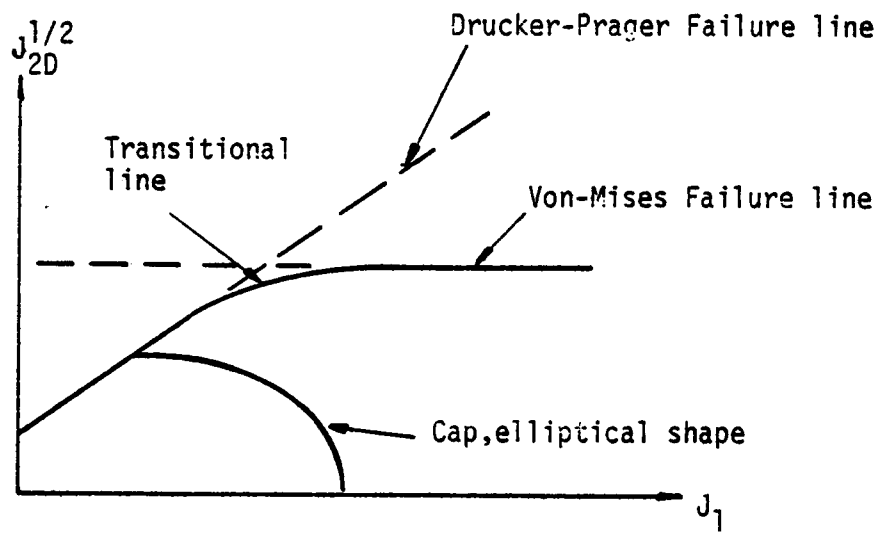


Figure 4.4. Cap Model proposed by DiMaggio et al [21]

been observed in hydrostatic compression tests. In simple shear tests, the soil behavior is highly stress path dependent, and significant differences have been found among the stress-strain curves for various initial pressure conditions in conventional triaxial compression tests. A thorough description of the artificial soil was given by Sture et al [65,66].

A new mathematical model for the artificial soil has been proposed [66], and is found suitable in predicting the behavior of this soil under different loading situations. This model, which is a modification of the cap model proposed by Sandler et al [59,60], comprises of two Drucker-Prager failure surfaces connected to each other by a transitional surface. This is shown in Figure (4.5).

The hardening cap is assumed to be elliptical with constant eccentricity. Since the data from laboratory tests [65,66] show normality of the plastic strain increment with respect to the cap surface, the associated law is assumed throughout the model. The cap function of this model is shown in Appendix B.

Verification

In order to verify the accuracy of the proposed hardening cap model, some numerical finite element analyses have been performed and the results are compared with the experimental data.

It is found that the model predicts well the soil behavior, in the hydrostatic compression tests and in the conventional triaxial compression test. For the conventional triaxial extension

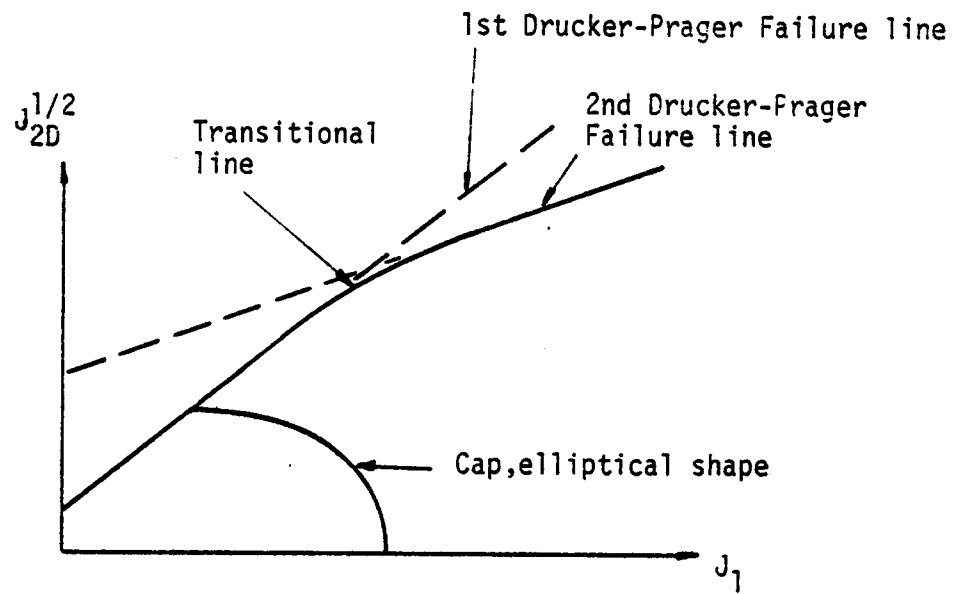


Figure 4.5. Cap Model used in Current Research

test and the simple shear test, however, the comparisons are only fair.

The results of these analyses are shown in Appendix C, where further discussions and comments on this aspect will be given.

CHAPTER V

INTERACTION EFFECTS

In the finite element method, the body under consideration is assumed to be a continuum. This may be valid for the domains that consist of only one type of material and do not possess discontinuities and materials of widely varying stiffness characteristics. For example, in structural mechanics, the finite element analyses of beams, plates or shells can be performed by assuming these structures to be continuous. In geomechanics there are also some problems where such an analysis can be performed, e.g. the problem of a foundation consisting of uniform soil, or the problem of an underground structure with a uniform mass of rock.

However, there are some special problems where the domain under consideration consists of different materials and the deformation of one material could relatively effect the deformation of the other material. This interaction effect may no longer follow the principles of continuum mechanics such as compatibility and equilibrium. The compatible deformation process may not be maintained between these materials, and a special scheme must be considered to handle this effect.

The interaction effects can be visualized for the problem of a concrete structure with steel reinforcement. A point on the reinforcement would deform differently than the adjacent point in concrete if the reinforcement is pulled by an axial force, Figure (5.1). Also, for a problem of axially loaded pile, the side effect between

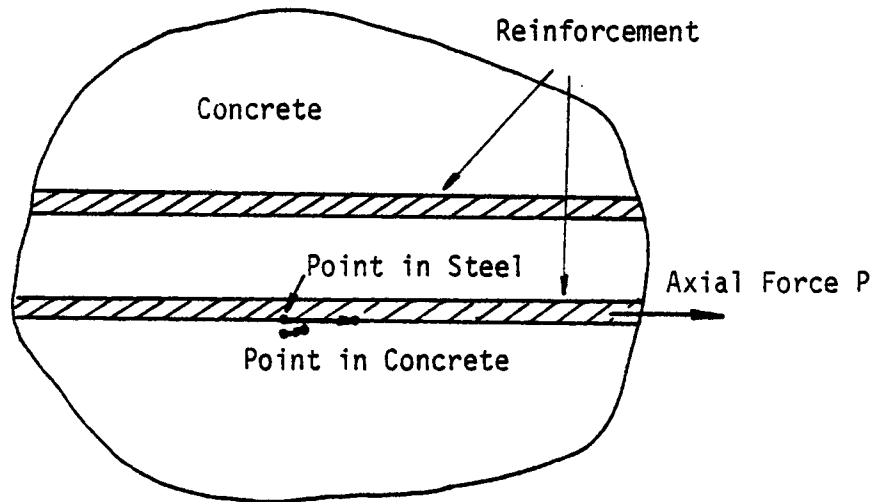


Figure 5.1. Concrete-Reinforcement Contact Problem

the pile and the surrounding soil would be an important factor in the deformation pattern.

For these contact problems, it is necessary to properly model the interaction behavior by adopting some kinds of special finite elements which could take into account the relative movement between two adjacent solid materials. There are two types of interface elements, namely compatible interface element and equilibrium interface element. They are described subsequently.

V.1. Interface Element: Compatible Model

This type of element assumes the displacement of a point on the side of a body is connected to an adjacent point on the side of another body by a relative movement. This is shown in Figure (5.2). The idea was first formalized by Goodman et al [32], where a one-dimensional interface element was used in a two-dimensional space.

From Figure (5.2), the relative displacement is defined as

$$u_i^r = u_i^A - u_i^B \quad (5.1)$$

where u_i^r is the relative displacement, and u_i^A and u_i^B are the displacements of two arbitrary nodes (with the same coordinates) on the boundaries of Body 1 and Body 2, respectively.

u_i^r plays the role of strain in this interface model, and can be expressed in terms of nodal displacements at nodes 1, 2, 3 and 4 by using the interpolation functions. In order to make the formulation consistent, an 'imaginary' stress or interactive force is assumed to relate to the 'strain' u_i^r by a constitutive matrix C_{ij} , given by

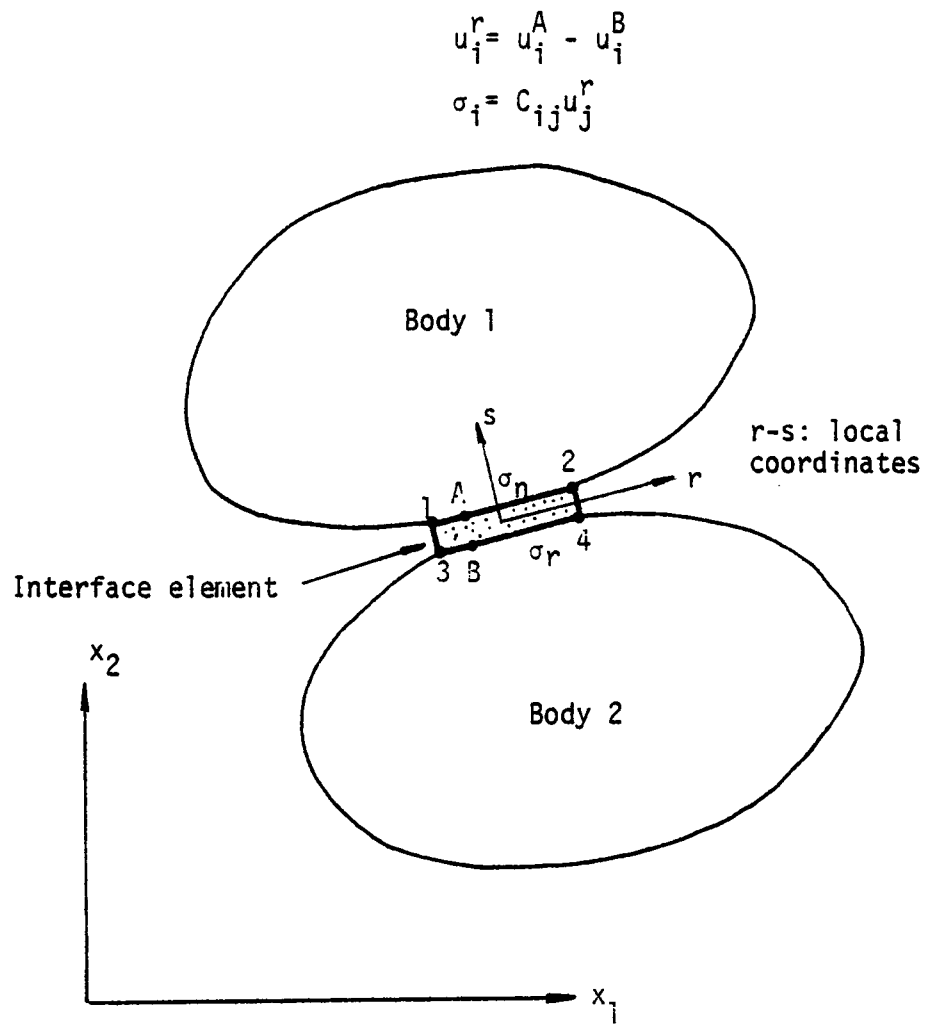


Figure 5.2. Compatible Interface Element

$$\sigma_i = c_{ij} u_j^r \quad (5.2)$$

where

$$c_{ij} = \begin{bmatrix} k_s & 0 \\ 0 & k_n \end{bmatrix} \quad (5.3)$$

where k_s is called the shearing stiffness of the interface element and is calculated based on the direct shear test in the laboratory, and k_n is called the normal stiffness of the interface element and, for soil-structure interaction, is often assumed to have a very high value.

A potential energy expression can be established for such an element, and the finite element equation can be derived for this element. This equation is then assembled to the overall finite element equations by the conventional direct stiffness method. The mathematical formulation of this type of interface element for two-dimensional space is given in Reference [32], and for three-dimensional space is given in Reference [14, 18], hence will not be repeated here.

The Goodman's model [32] has some drawbacks in the prediction of the real behavior of soil-structure interaction. One of its drawbacks is that the model allows the adjacent solid elements to penetrate to each other, which is not physically acceptable. Another shortcoming is that the model could not handle the situation of the existence of a gap, when two close elements are in tension.

Zienkiewicz et al [81] proposed another type of interface element which is a modification of the Goodman's model, and assumed uniform

strain in the normal direction. This element, however, might show ill-conditioning in the finite element equations when gaps occur at the interface.

To further modify the interface element using the compatible concept, Ghaboussi et al [31] proposed an element where the relative displacements are chosen to represent the independent degrees-of-freedom. Thus for each interface element in two-dimensional space, in addition to the four regular nodes, there are two extra nodes with four degrees-of-freedom. These degrees-of-freedom serve as a constraint between the displacements of the two adjacent regular elements. This type of interface element may show better solutions for the contact problems where compression or slip occur; however, for the separation mode of behavior, such as gaps, it still may not provide satisfactory predictions. A slightly modification of this element was proposed in explicit form by Wilson [70], where, instead of increasing the degrees-of-freedom for the interface element, one of the two adjacent elements have some extra degrees-of-freedom to handle the relative movement.

The constitutive behavior of the interface element is often of the Mohr-Coulomb type. This behavior is given in Appendix B.

The application of the interface element, compatible type, can be found in numerous researchers, e.g. by Desai [15], Desai et al [19].

V.2. Interface Element: Equilibrium Model

The compatible interface models as discussed may be valid for some kinds of problems. The interface behavior usually has three modes: non-slip, slip and separation. When the opposing elements separate

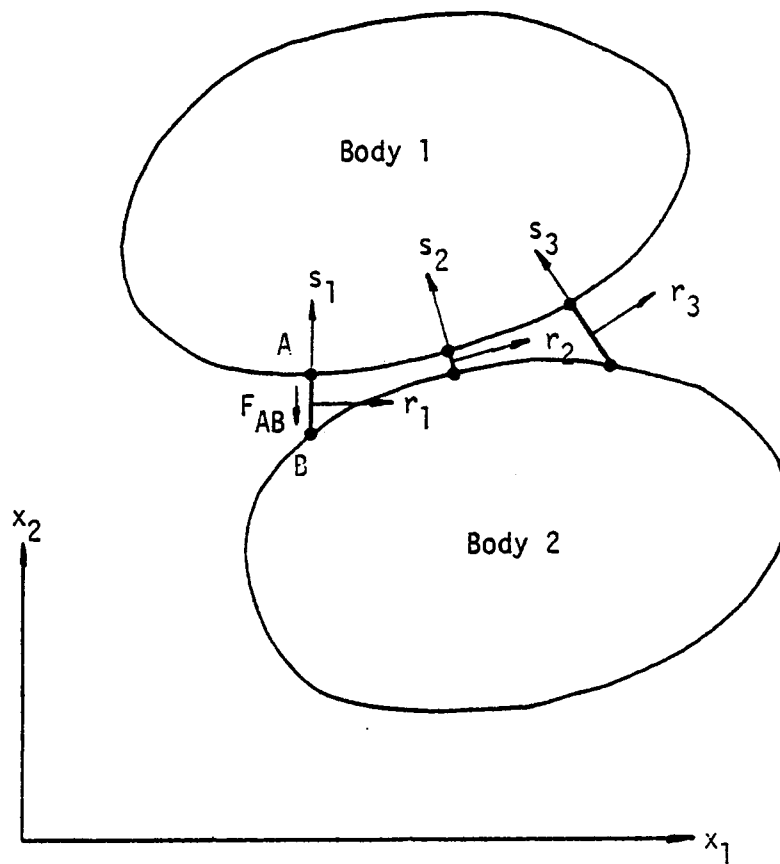


Figure 5.3. Equilibrium Interface Element

from each other, i.e., gaps occur, the compatible models no longer hold. It is necessary to use a model which is based on the equilibrium of the system in order to handle the separation mode. Such a model was proposed by Katona et al [41].

In this model, the nodes at adjacent elements are connected by an 'imaginary' one-dimensional element in a two-dimensional space, Figure (5.3). This element plays the role of a kinematic constraint, such as the Lagrange multiplier constraint [30, 76]. The interaction between two nodes A and B are defined by the force F_{AB} of node A acting on node B. Two independent degrees-of-freedom, which have the dimension of force, are assigned for this interface element. A modified principle of virtual work is written in formulating this equilibrium model [41]. The model shows good approximation to the real interface behavior. However, in order to obtain equilibrium, iterations must be performed within each incremental load step. Such iterations are also necessary in calculating the proper mode of interaction of the system during the application of load, and at the end of the load step. Furthermore, a solution subroutine must be used to solve the finite element equations with a semi-definite symmetric stiffness matrix [53]; which arises from zero diagonal terms in the equations corresponding to the force-type degrees-of-freedom of the interface element. This, in terms of generality, may cause some drawbacks if the model is to be incorporated in a finite element program using displacement approach.

Hermann [35] proposed another type of equilibrium interface element in which he combined the relative displacement and the interactive

force as degrees-of-freedom for the element. Based on the work by Peterson [54] and Fredriksson et al [28] for a general contact problem, Hermann modified his model further [36], and gave a general description for the interface element in any modes; non-slip, slip or separation. This model, however, appears difficult to implement in a finite element program due to the complexity in defining the modes and forces at the beginning of the application of load.

V.3. Application of Interface Elements in Three-Dimensional Space

The application of the compatible and equilibrium models in three-dimensional space appears to be very complicated due to various factors, such as the nature of mesh layout. All of the above models have rarely been used in three-dimensional analyses. Figure (5.4) shows typically a three-dimensional mesh of a tool embedded in a soil foundation. It can be seen that difficulties arise in defining a proper mesh to handle the interface behavior between the tool and the soil. Furthermore, if the tool is under the application of the load P , element 1 could be in non-slip mode, element 2 in slip mode, and element 3 in separation mode. Also there can exist a combination of these modes in these elements. A proper initial mesh setup for this problem is also difficult. For example, the connection of a node on the solid element, say node 6, Figure (5.4) to either node 5 or node 7 of the interface elements is only chosen arbitrarily; which, in some cases, could lead to unrealistic solutions for the deformation pattern of the whole domain.

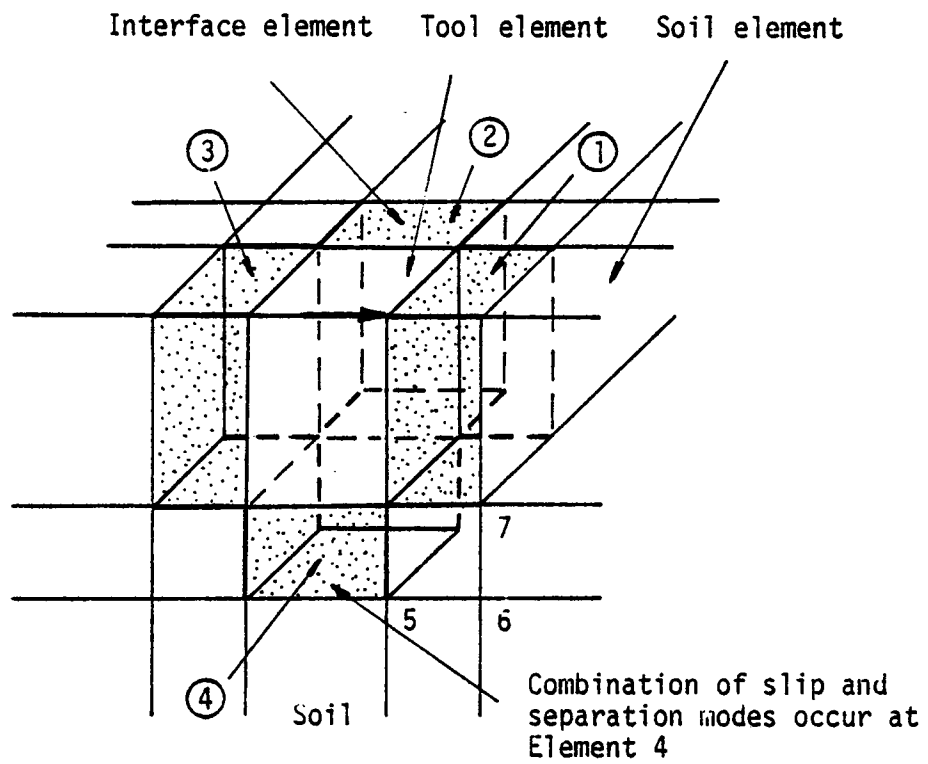
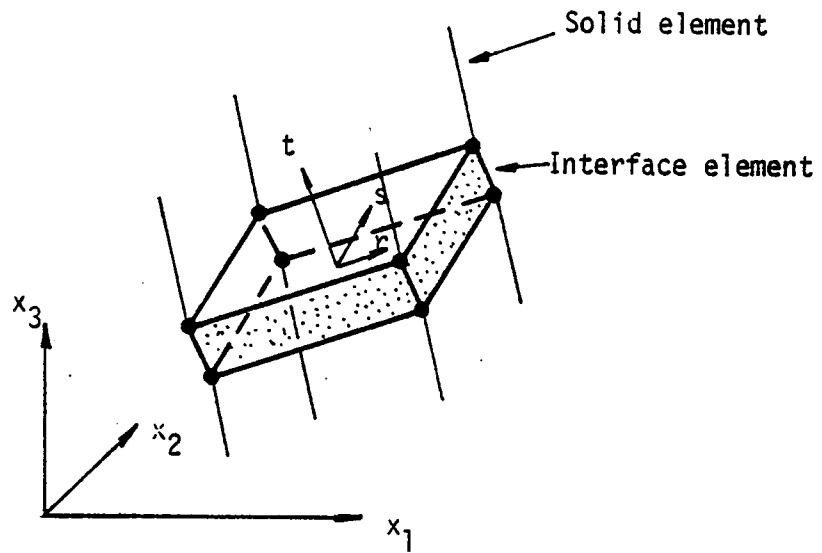


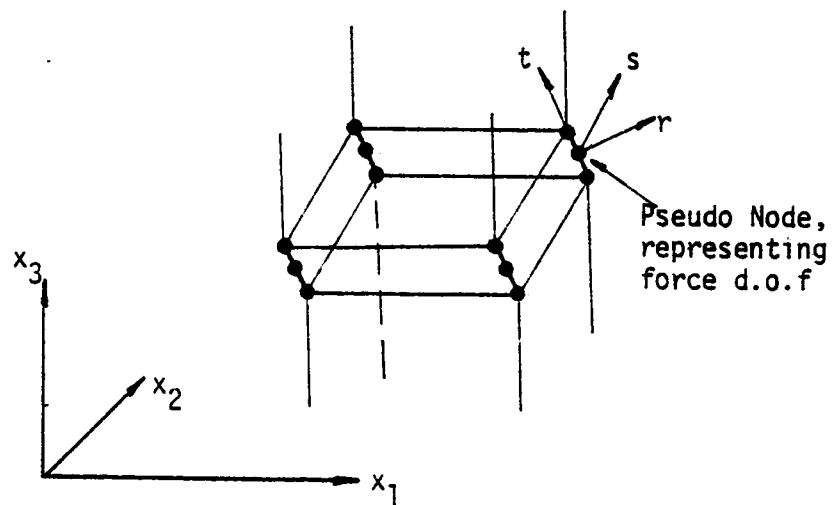
Figure 5.4. Tool in Soil Body with Interface Elements

It is attempted in this research to use two types of elements:

1) Goodman's type as developed for three-dimensional analysis by Desai et al [18], and 2) Kantona's type [42]. A typical mesh layout is shown in Figures (5.5a) and (5.5b) for the two models. It is found that by using a small value of normal stiffness, Equation (5.3), whenever separation occurs, model 1 can lead to approximately the similar results as those from model 2; if a small value of the zero-diagonal term in the latter model is used in the conventional solving subroutine. This could be acceptable for the type of soil-structure interaction problem under lateral load in the current research. For a general problem, further research would be needed.



a. 3-D Interface Model, Compatible Type



b. 3-D Interface Model, Equilibrium Type

Figure 5.5. Interface Models used in Current Research

CHAPTER VI

THREE-DIMENSIONAL FINITE ELEMENTS AND NONLINEAR ANALYSIS

VI.1. Variable-Node, Hexahedral Finite Element

There is a variety of three-dimensional finite elements that can be chosen. The shape of an element can be either tetrahedral or hexahedral. The choice of a specific type of element may depend upon some factors such as the modes of deformation that the element is required to provide and the computer time to calculate the stiffness of the element.

A survey of various types of three-dimensional finite elements varying from first order to second order in displacement mode, i.e., with or without midside nodes, was given by Clough [11]. Through this survey, it has been accepted that the hexahedral isoparametric elements are better than the tetrahedral elements in terms of accuracy and computer time. Another advantage of the isoparametric elements is that they are capable of describing the nodal displacements in nonlinear modes, which correspond to the similar description of the element geometry with curved shapes. In this respect, it is possible to include additional nodes on the side of an element to provide improved simulation of the displacement mode on that side, provided that the interpolation functions to describe those nodes are used to describe their coordinates [6, 70, 76].

For a brick element with isoparametric formulation, the number of nodes can vary from 8 to 21 nodes [5, 6], as shown in Figure (6.1). Such a variable-node element system can provide a more flexible dis-

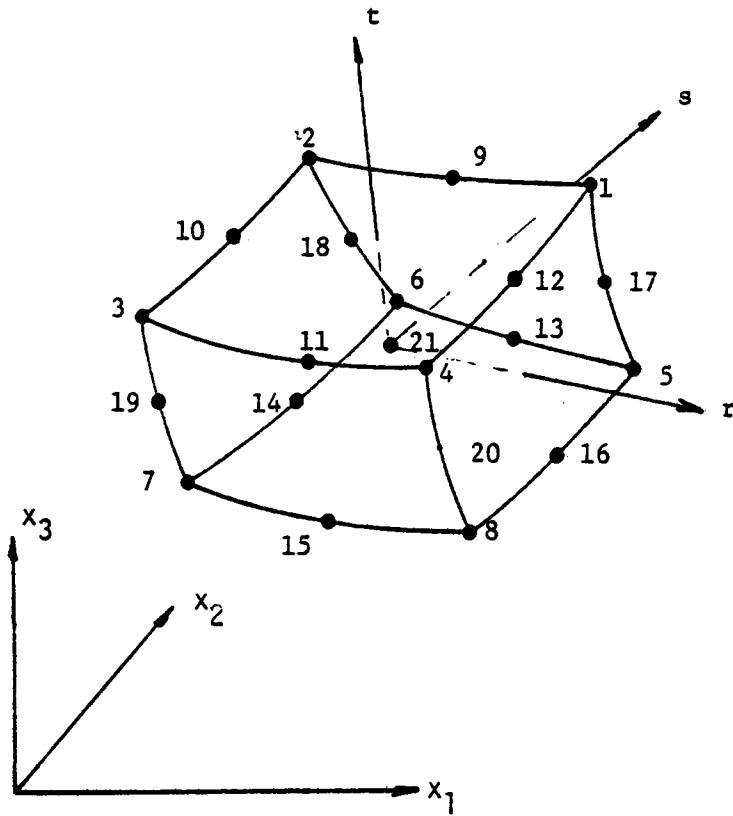


Figure 6.1. 8 to 21-node Isoparametric Element

placement formulation. This aspect can be illustrated in the example of a problem of semi-infinite medium subjected to a point load. In this problem, high stress concentration would exist around the point of load application; hence a finite element mesh should have more nodes (or elements) around this area in order to catch the deformation pattern and to compute the stresses more accurately. The regions far from the point of load application, on the other hand, can be coarser and can have 8-node elements. A typical example of such a mesh is shown in Figure (6.2).

Another advantage of the variable-node element is its ease in computer implementation. The computation of the stiffness of each element is performed by using the numerical integration. The stiffness computed at each integration point can be easily modified to account for the contribution of the mid-side nodes [6]. And, provided that a proper bookkeeping system is available in the computer program, the inclusion of additional nodes in any element can be handled easily and straightforward as in the 8-node elements.

The use of variable-node element, however, is dependent on the problem under consideration. This aspect was discussed by Clough [11], where two cantilever beams, one deep and one shallow, were analyzed by using 8-and 20-node three-dimensional isoparametric elements. It was found that for a shallow beam, an 8-node element gives better prediction of the deflection at the far end of the beam.

Although the finite element code developed in the current research is capable of solving problems with variable-node elements, most of

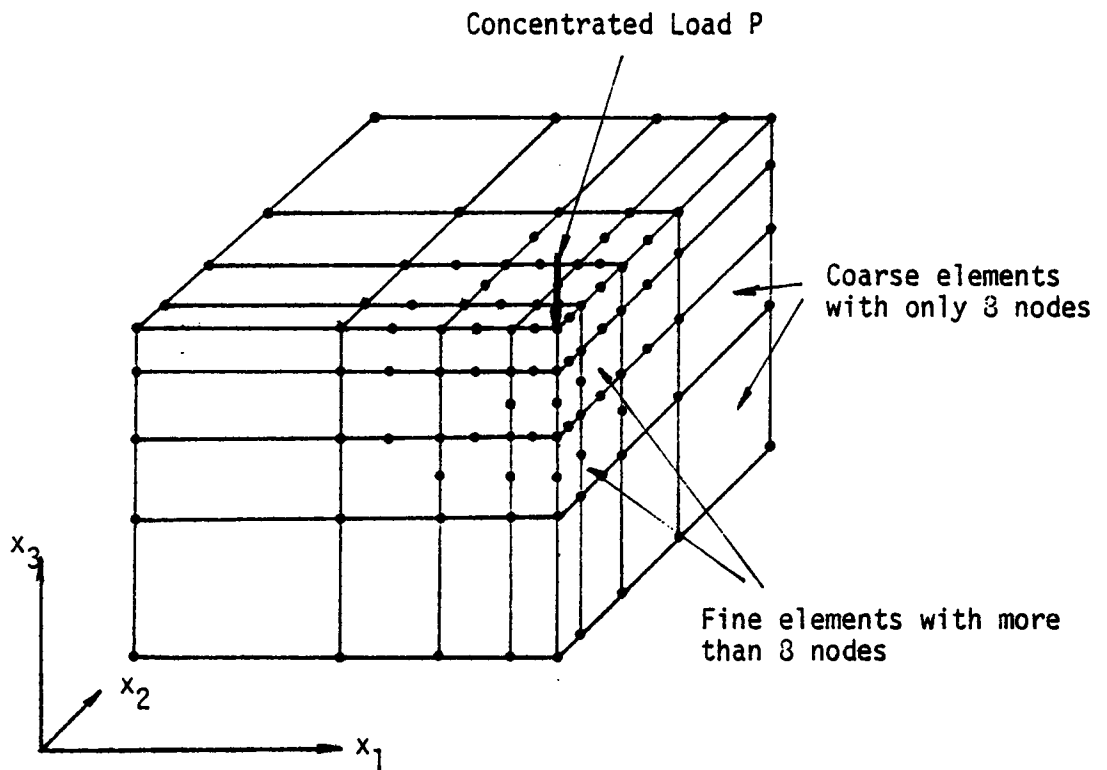


Figure 6.2. Typical Mesh with Variable-Node Elements for a Problem of Point Load on a Semi-Infinite Medium

the analyses performed use only conventional 8-node isoparametric elements. This is because for the soil-moving tool interaction problem, the tool is driven only at a shallow depth into the soil with respect to the size of the tool, and it is found that the analyses with 8-node elements are more suitable and accurate.

The formulation of variable-node three-dimensional element and the interpolation functions have been described in References [5, 6], hence will not be repeated here.

VI.2. Nonlinear Numerical Technique

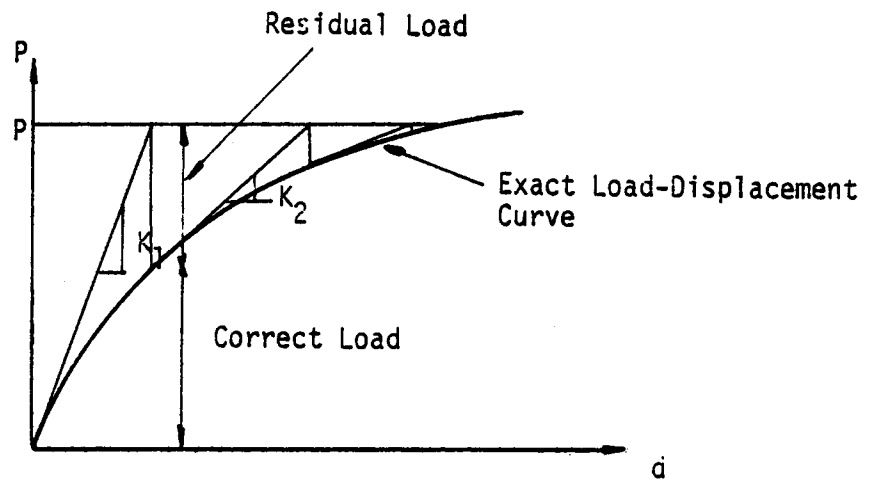
An important aspect in numerical analysis is the computational scheme in nonlinear problems. There are two schemes which are used widely in the finite element method, namely incremental scheme and iterative scheme [15]. Each technique has its own merits and its usage is dependent on the specific problems. In large deformation analysis with plasticity, a combination of incremental and iterative techniques is found most suitable. This is due to the nature of the analysis where load is applied incrementally and the growth of plastic deformation at a load step is highly dependent on the values of stresses and strains at the previous load step. The geometry of the body is also load dependent. For such a problem, an incremental scheme alone is not sufficient to maintain the condition of equilibrium throughout the loading history. Hence, a combination of these two techniques needs to be employed.

The formulation and discussion of the incremental technique as well as the iterative technique have been given elsewhere [17, 34, 50,

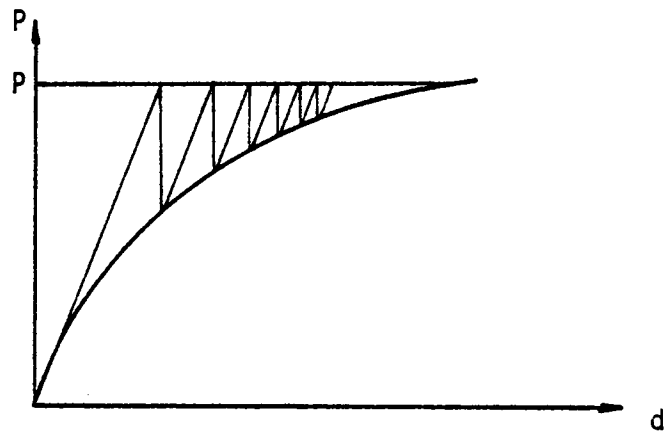
76]. Only the technique used in this research is presented next.

The most widely accepted approach to handle the nonlinear characteristics is probably the Newton-Raphson approach [34]. This is an iterative technique to solve a system of nonlinear differential equations. The steps of using the Newton-Raphson technique are shown in Figure (6.3a). If instead of using the tangent stiffness at each iteration, the initial stiffness at the beginning of the load is used throughout the iterative process, one obtains the modified Newton-Raphson technique, Figure (6.3b). The Newton-Raphson technique, when combined with the incremental technique, Figure (6.4), gives a complete description of the numerical scheme.

For nonlinear material behavior, Zienkiewicz et al [79] have proposed an approach called "initial stress" method, which can be treated as a modified Newton-Raphson approach with the elastic stiffness kept throughout the analysis; this is shown in Figure (6.3b). Often, in elastic-plastic small strain analyses, the initial stress method coupled with the incremental procedure is considered economical with respect to computer time and storage requirements. However, in the real three-dimensional problems with large strains and highly nonlinear material, this may not be the case. One of the main factors in numerical analysis is that of convergence. This factor may be difficult to obtain with the initial stress method, unless a large number of iterations are performed within each incremental load step. Convergence can be obtained if the size of each load increment is reduced, but then the number of increments must be increased. Hence, keeping the initial



a. Original Newton-Raphson Method



b. Modified Newton-Raphson Method

Figure 6.3. Newton-Raphson Iterative Methods

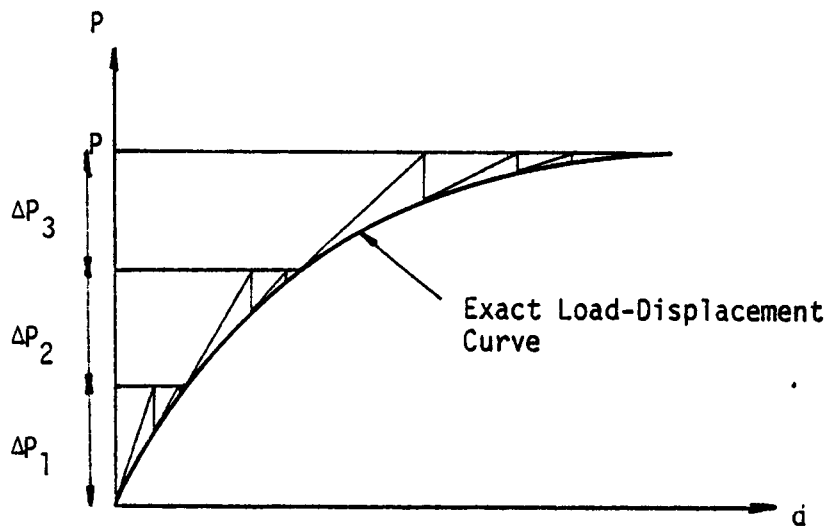


Figure 6.4. Combination of Incremental and Iterative Techniques

elastic stiffness during the entire process could be time consuming and expensive.

On the other hand, for the problems under this research, it is found convenient and economical to use the original Newton technique, Figure (6.4). The stiffness is changed at each load step and each iteration, corresponding to the changes of geometry and material properties. In this respect, the usage of the tangent stiffness could reflect more realistically the path dependent characteristics of the problems. Furthermore, when the body configuration is updated continuously as in the updated Lagrangian approach, the finite-element formulation appears to be more consistent with the original Newton-Raphson numerical scheme.

VI.3. Appropriate Stresses and Strains in Nonlinear Analysis

In analysis of the geometric and material nonlinearities, care must be performed in using the right stress to define the material law; and the right strain referred to proper geometry in order that the breaking of elastic and plastic parts could be reasonable. The following discusses in detail a numerical technique which can be used for large strain and large rotation analyses with elasto-plasticity.

Suppose at the end of load step n , equilibrium is maintained in the continuum and the total displacements u_i and the current stress tensor σ_{ij}^n are found, Figure (6.5). The current constitutive tensor C_{ijrs}^n , which is computed from σ_{ij}^n and the total plastic strain tensor ϵ_{ij}^p , is then used as the tangent 'stiffness' for next load. Now an incremental load is applied to deform the body further, and to change

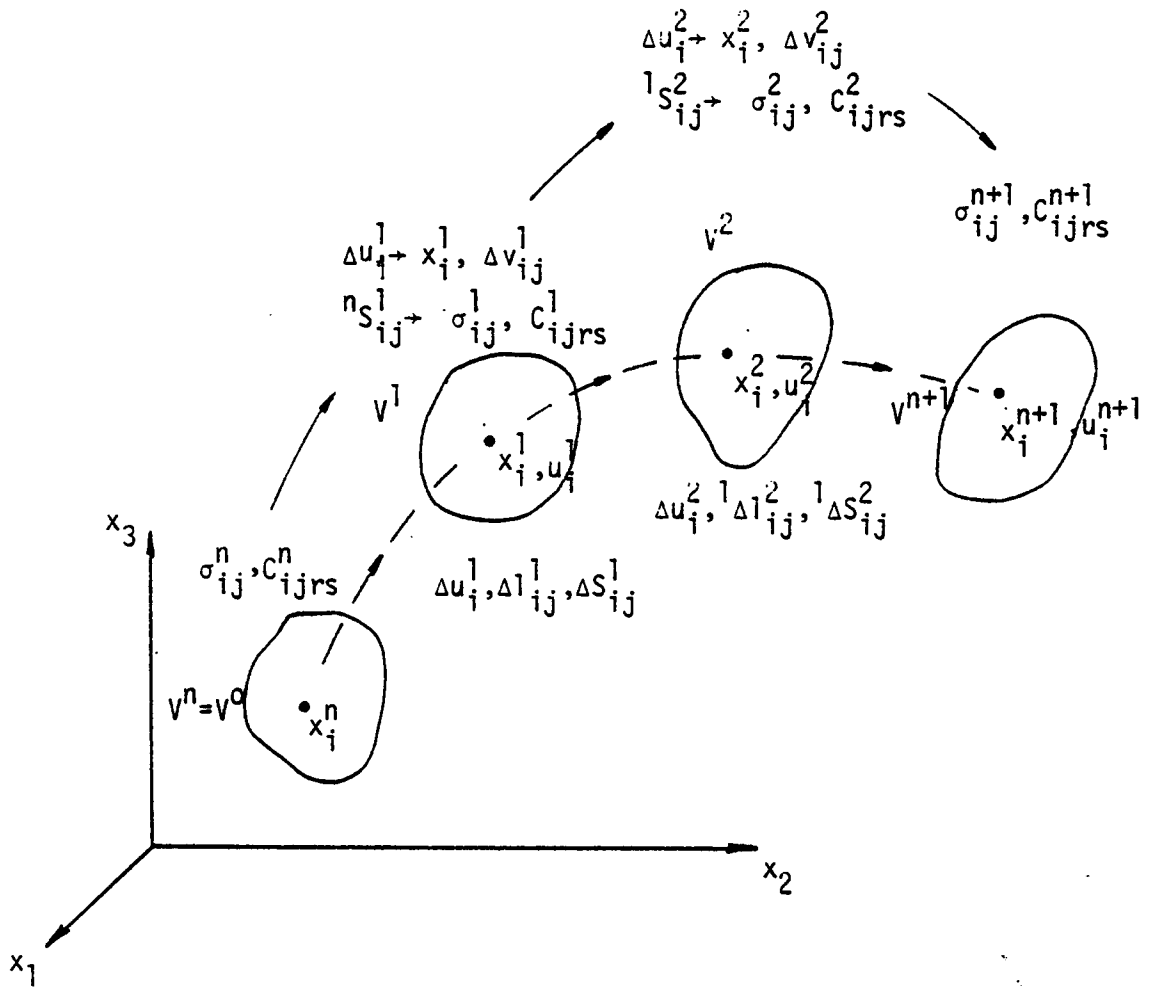


Figure 6.5. Transformation of Static and Kinematic Variables during Application of INcremental Load

its configuration from step n to step $n + 1$, Figure (6.5).

To find out the first approximation of the incremental displacements, the updated coordinates x_i^n , the current stress tensor σ_{ij}^n and the current constitutive tensor C_{ijrs}^n are used in the Equations (2.47) and (2.54). With this incremental displacement Δu_i^1 , where superscript 1 denotes first iteration, the incremental second Piola-Kirchoff stress tensor ${}^n\Delta S_{ij}^1$ and the incremental Green-Lagrange strain tensor ${}^n\Delta l_{ij}^1$ can be obtained from Equations (2.26) and (2.9), respectively. In ${}^n\Delta S_{ij}^1$ and ${}^n\Delta l_{ij}^1$, the right superscript denotes the value as computed at the end of iteration number 1, and the left superscript denotes the value is referred to the configuration of the body at step n . Adding ${}^n\Delta S_{ij}^1$ to the Cauchy stress tensor σ_{ij}^n , Equation (2.25), gives the total second Piola-Kirchoff stress tensor ${}^nS_{ij}^1$. ${}^nS_{ij}^1$ is then transformed to Cauchy (current) stress tensor σ_{ij}^1 by using Equation (2.23). The incremental Green-Lagrangian strain tensor ${}^n\Delta l_{ij}^1$ is also transformed to the infinitesimal Almansi strain tensor (or rate-of-deformation tensor) Δv_{ij}^1 . The latter transformation is necessary since the computation of the new constitutive tensor C_{ijrs}^1 requires the proper incremental strain which must be referred to the current configuration.

If the material is nonlinear plastic, the Cauchy stress tensor σ_{ij}^1 is used to check the yield function, Equation (3.1). If this yield criterion is violated, i.e., $f \geq 0$, plastic behavior of the element must be considered; and a new constitutive tensor C_{ijrs}^1 is formulated by using σ_{ij}^1 .

The Cauchy stress tensor σ_{ij}^n is transformed to the second Piola-Kirchoff stress tensor ${}^1S_{ij}^n$. With Δv_{ij}^1 and the constitutive tensor C_{ijrs}^1 , an incremental stress $\Delta \sigma_{ij}^1$ can be found from Equation (2.27). This incremental stress tensor $\Delta \sigma_{ij}^1$ is then added to the stress tensor ${}^1S_{ij}^n$, which results to a new stress tensor σ_{ij}^1 . σ_{ij}^1 just calculated is the current stress at the end of iteration 1 which satisfied the condition of equilibrium in the element corresponding to the incremental displacement Δu_i^1 .

An internal load Q_0^1 is then computed by using σ_{ij}^1 , Equation (2.58). An unbalance load could be found from Equation (2.55), which is then used in the finite element equations (2.59) or (2.60) for next iteration. In these latter equations, σ_{ij}^1 and C_{ijrs}^1 must be used for consistency.

The process of calculation is repeated following the above procedure for each iteration until convergence is obtained. The convergence criterion is considered satisfactory if a previously chosen tolerance of the unbalance load is violated.

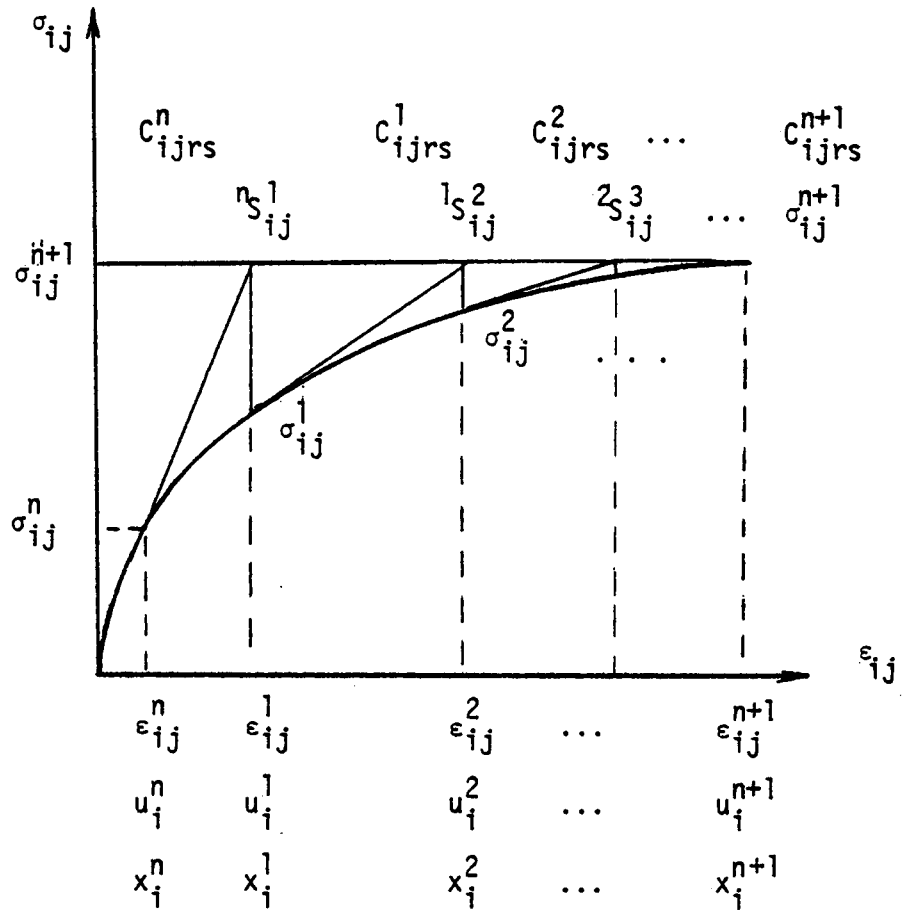
The scheme discussed above is further illustrated in Figures (6.6) and (6.7) where the appropriate stresses and strains are fully mobilized to make the geometric and material nonlinear analysis consistent.

Coordinate Independence of Material

In case of some problems where the independence of material law with respect to the change of the coordinates at the end of each load step is necessary, the Jaumann's rate of stress must be used as dis-

$$\sigma_{ij}^{k+1} = f(s_{ij}^{k+1}, s_{ij}^{k+1}, \Delta l_{ij}^{k+1})$$

$$c_{ijrs}^{k+1} = f(\sigma_{ij}^{k+1})$$



k : kth iteration

Fig. 6.6. Computational Algorithm to handle Large Plastic Deformation

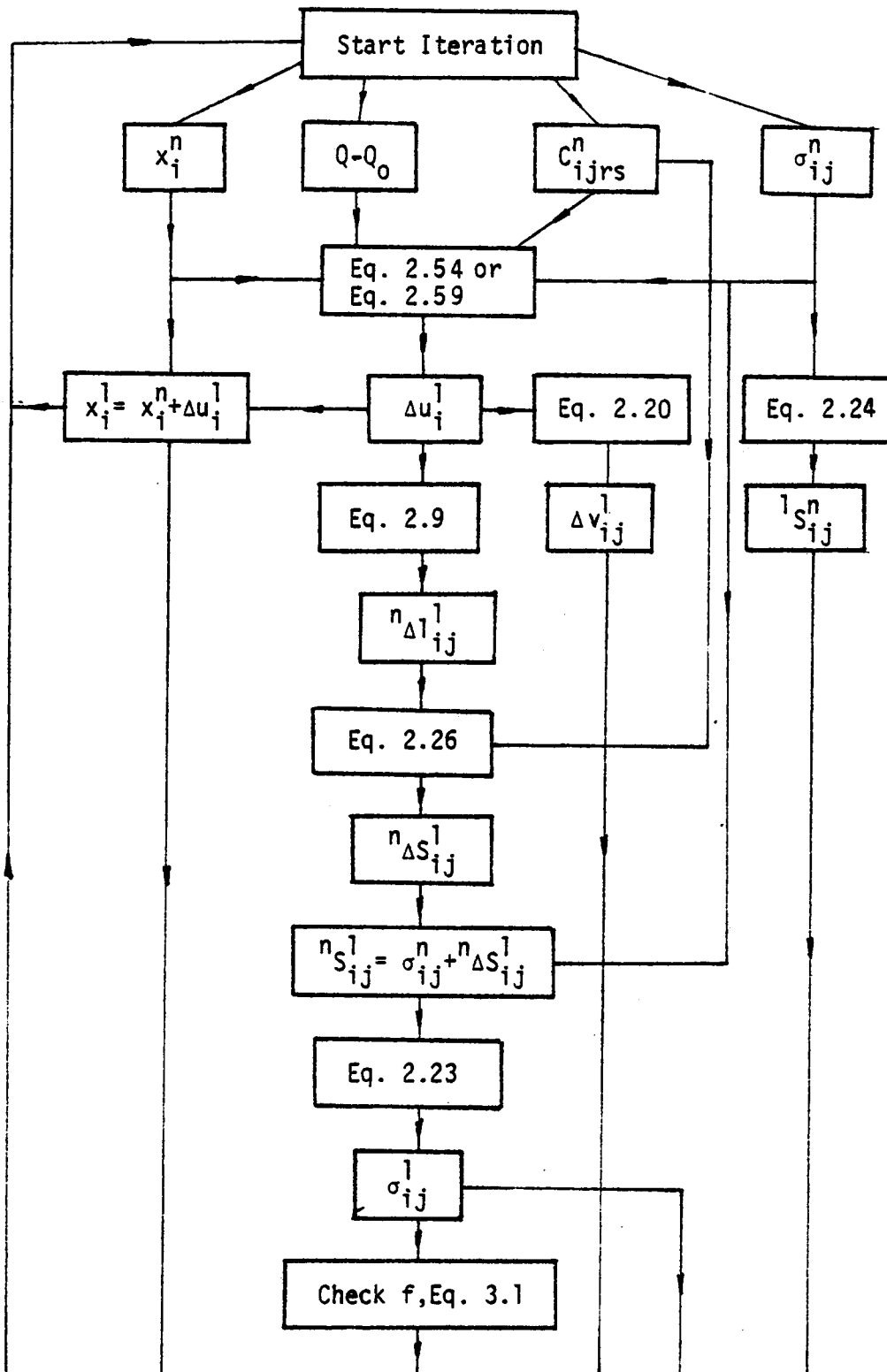


Figure 6.7. Flow Chart of Numerical Scheme to handle Large Deformation , First Iteration

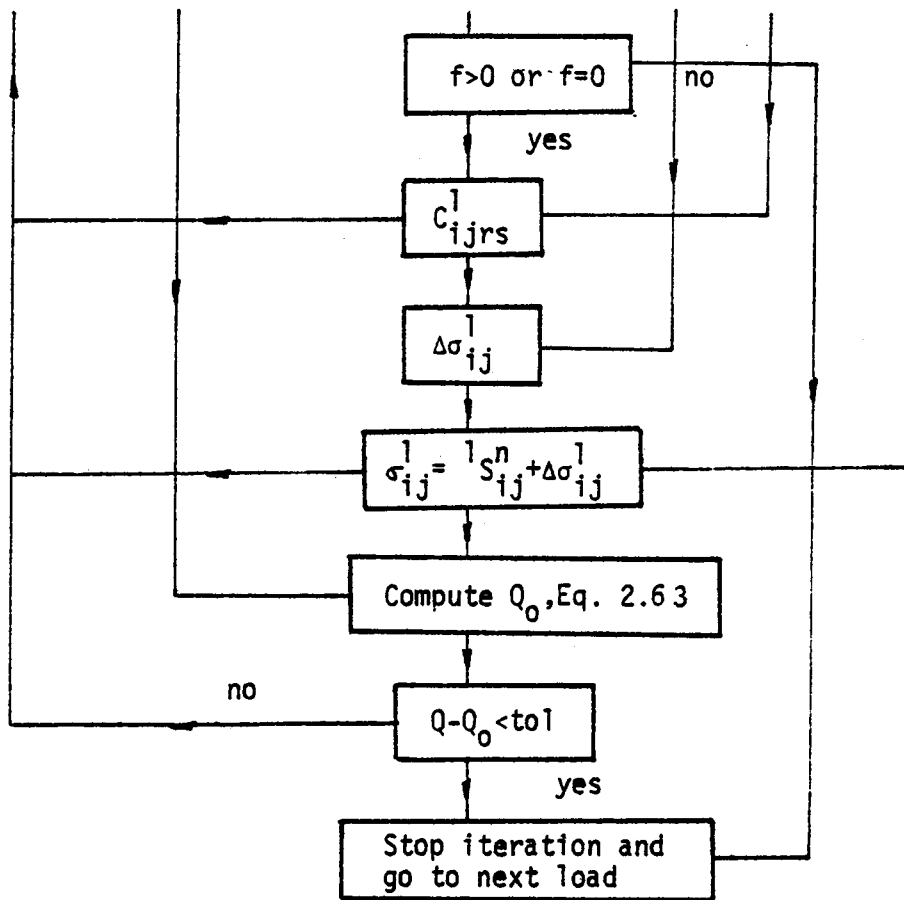


Figure 6.7. (continued)

cussed in Chapter II. The rate of deformation tensor Δv_{ij}^1 is used in Equation (2.64) to find the Jaumann's rate of stress $\Delta \sigma_{ij}^J$. The incremental spin tensor $\Delta \Omega_{ij}^1$ is calculated from Equation (2.21) which, together with C_{ijrs}^1 , $\Delta \sigma_{ij}^J$ and σ_{ij}^1 , can be used to calculate $\Delta \sigma_{ij}^1$, Equation (2.65).

Work-Hardening Material

If the material model is of work-hardening plastic type, the total plastic strain must be retained at any load level in order that the yield function could be properly applied, Equation (3.1). This aspect can be performed in the above scheme in a straightforward manner. At the end of each incremental load, by using the current stress tensor σ_{ij}^1 and the flow rule in Equation (3.2), the incremental Almansi plastic strain tensor $\Delta \epsilon_{ij}^{1(p)}$ can be computed. Superscript p denotes the plastic part of the incremental Almansi strain $\Delta \epsilon_{ij}^1$. This plastic strain tensor $\Delta \epsilon_{ij}^{1(p)}$ is added to the previous total plastic strain, and the analysis can be performed for the next iteration.

The computational scheme with proper definitions of stresses and strains as discussed, together with the original Newton-Raphson technique have been adopted in the current research. Examples of the application of such a 'combined' nonlinear technique will be given in the next chapter.

VI.4. Stress Transfer Technique

Soil is a geologic material usually with very small tensile strength. All material laws are often written for soil behavior under

the compressive state of stress. The tensile stress, if it exists in some area of the soil medium, is physically absorbed by the surrounding areas which are in compression. In the finite element analysis, the occurrence of the tensile stress could be possible at some load level. Special consideration, hence, must be utilized to transfer this stress to other zones of the medium. Such a numerical scheme was proposed by Zienkiewicz et al [80] and is called the stress transfer technique.

The stress transfer technique was implemented in the three-dimensional finite element code, and the necessary steps are discussed subsequently.

At the end of each load step, the first invariant of the current stress σ_{ij}^1 (Section VI.3), representing the mean pressure, is calculated. A check is made to see if the element is in tension or compression by using this first invariant. If the element is in tension, the principal stresses are then formulated. The tensile part of the principal stresses are saved for a residual load computation. This residual load, after assembled for all elements, is added to the load vector, and the finite element analysis is performed again for additional deformation of the continuum. Thus, the excess tensile stresses in that particular element are redistributed to other adjacent elements. The system is assumed to be in equilibrium if after some iterations, convergence is obtained and the newly computed tensile stresses are negligible.

Chapter VII

VERIFICATIONS: TWO-DIMENSIONAL PROBLEMS SOLVED BY USING THE THREE-DIMENSIONAL PROCEDURE

The theoretical formulations involving geometric and material nonlinearities with interface effect, and the numerical technique, which were discussed in the previous chapters, have been implemented in a general three-dimensional finite element code. This code has a variety of options which can be selected depending on each specific problem. In this chapter some problems, varying from simple to rather complex problems, are solved to verify the accuracy and the applicability of the procedure. However, due to the general lack of available observed test data for three-dimensional analyses, the problems in this chapter are restricted to plane strain condition in order to compare with previous solutions. The plane strain condition can be modelled by constraining the degrees-of-freedom in the direction which is perpendicular to the plane, and by assuming the thickness of the body in that direction to be unity. Comparisons and verifications of the predictions, with respect to actual laboratory prototype test results obtained in a large soil-bin test facility involving structures moving in an actual three-dimensional soil body, are given in Chapter VIII.

VII.1. Constrained Footing in Compression

A piston-like problem of a material compressed in uniaxial strain condition is solved to investigate the effect of geometric nonlinearity. This problem has been solved by Davidson et al [12],

where a two-dimensional plane strain idealization of a quadrilateral element composed of four constant strain triangular elements was used. The finite element mesh using three-dimensional elements is shown in Figure (7.1a). The material is assumed to be linear elastic and its properties (for any consistent unit), are

$$E = 9000.0$$

$$\nu = 0.2$$

The exact solution for this problem, in case of small deformation, is

$$\sigma = M \frac{w}{L} \quad (7.1)$$

where w is the vertical displacement, L is the original length, σ is the vertical stress which is equal to the uniform pressure, p , i.e.

$$\sigma = p, \quad (7.2)$$

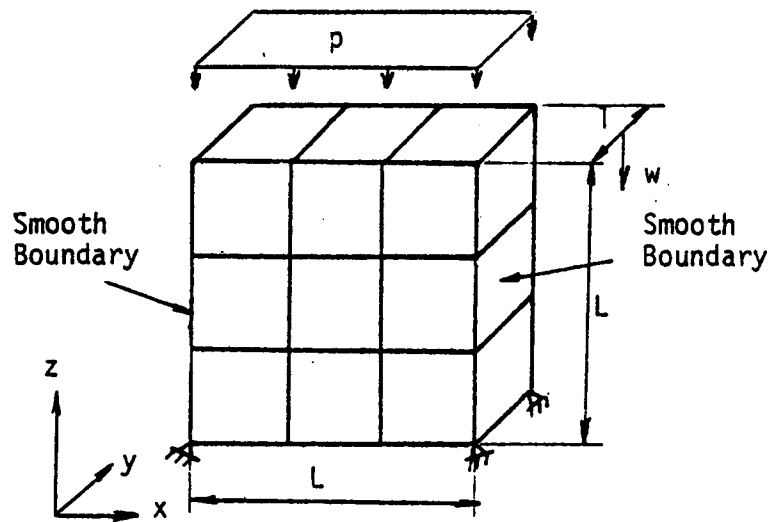
and M is the constrained modulus and is defined as

$$M = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \quad (7.3)$$

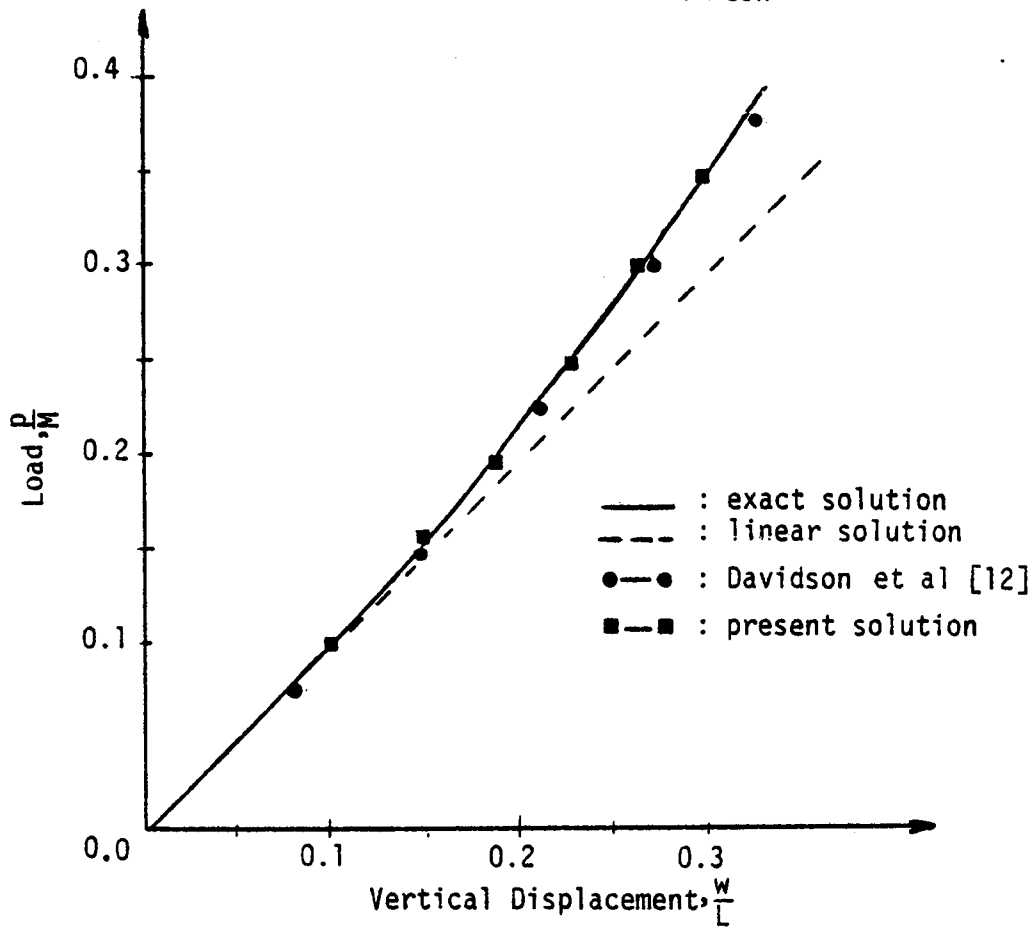
In case of large deformation, the exact solution is computed from the definition of the logarithmic strain in one-dimension. This strain is defined as

$$\epsilon = \ln \left(\frac{L}{L-w} \right) \quad (7.4)$$

and is related to σ by



a. Finite Element Mesh



b. Load vs. Displacement Plot

Figure 7.1. Linear Elastic, Large Deformation Problem under Uniform Load

$$\sigma = M\epsilon \quad (7.5)$$

or

$$\sigma = M \ln \left(\frac{L}{L-w} \right) \quad (7.6)$$

The result obtained from the analysis, utilizing six load increments, is shown in Figure (7.1b), and compares very well with the closed-form, large deformation solution and the solution given by Davidson et al. [12].

VII.2. Solid Block under Uniaxial Load

In order to study the effect of plasticity on the deformation pattern, a soil block subjected to axial load is analyzed. Figure (7.2a) shows the finite element mesh where the degrees-of-freedom in the x-direction are released so the body can deform laterally. This problem was also solved by Davidson et al [12], but only for the case of small deformation. The material is assumed to follow the extended Von-Mises or Drucker-Prager yield criterion which results in an elasto-perfectly plastic material response.

The properties of the soil are

$$E = 500,000.0 \text{ psf } (23,950.0 \text{ kN/m}^2)$$

$$\nu = 0.0$$

$$c = 500.0 \text{ psf } (23.95 \text{ kN/m}^2)$$

$$\phi = 30 \text{ degrees}$$

where c is the cohesion and ϕ is the angle of internal friction.

As expected, a value of zero for the Poisson's ratio, ν , shows

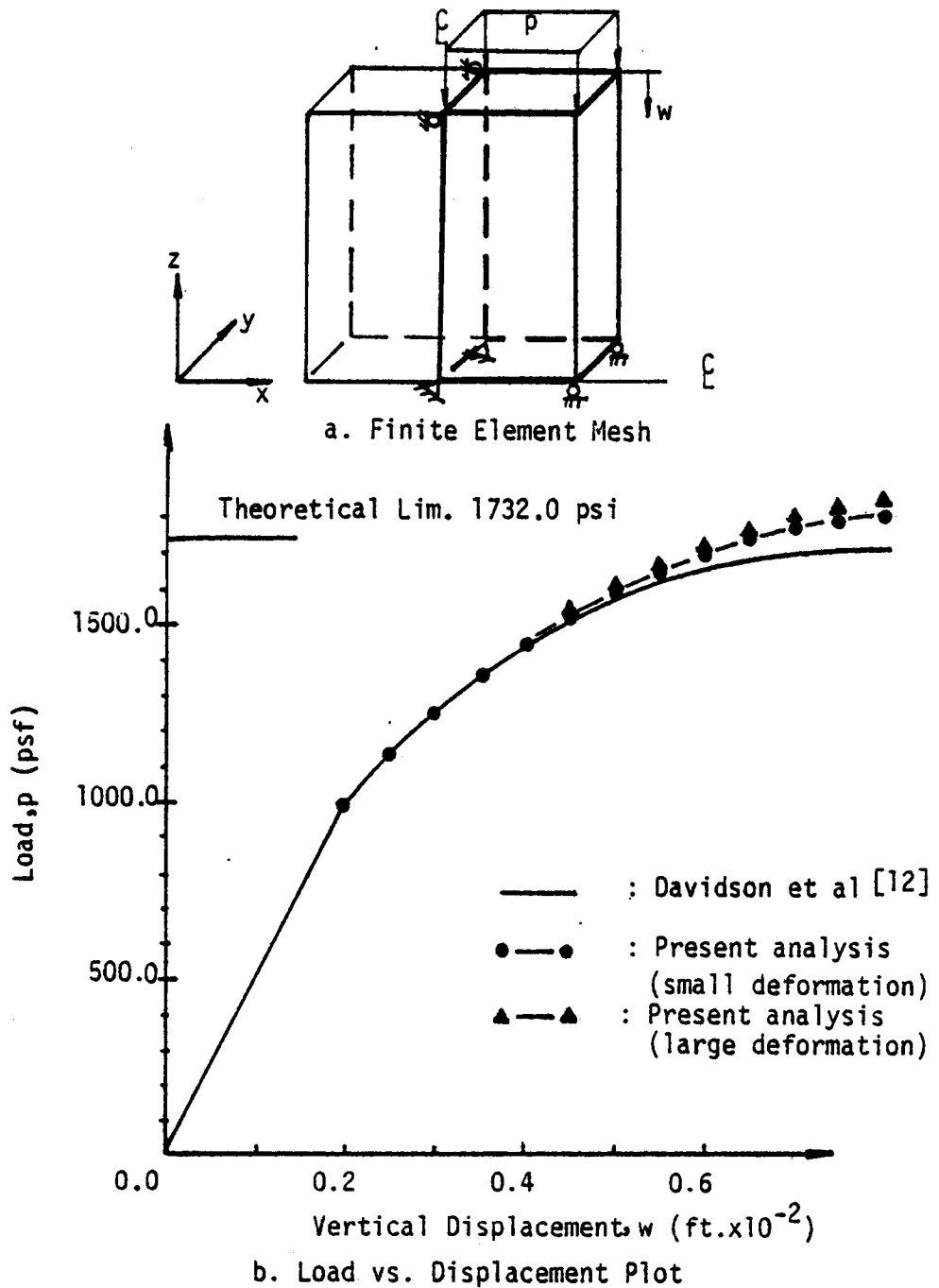


Figure 7.2. Plane strain, Uniaxial Behavior of Drucker-Prager Material.

no lateral movement in the x-direction during elastic response, until the material starts yielding at load p approximately equal to 1000.0 psf (47.9 kN/m²), Figure (7.2b). The displacement in the x-direction is significant as the body continues deforming plastically until ultimate conditions are reached. The maximum ultimate stress estimated in the present analysis for small deformation case is about 1800.0 psf (86.22 kN/m²) whereas the maximum yield stress given by Davidson et al [12] is 1725.0 psf (82.62 kN/m²). The exact solution for this problem can be obtained if we assume the soil is of Mohr-Coulomb type, i.e. the failure condition satisfy the equation

$$\tau_n = c + \sigma_n \tan \phi \quad (7.7)$$

for one-dimensional compression. The adoption of zero value of Poisson's ratio results to zero stress in x and y-directions for the elastic behavior. Hence σ_n is the maximum stress in the z-direction which defines the ultimate condition of the soil. A plot of Mohr circle with $\sigma_1 = 0.0$ psf, $c = 500.0$ psf (23.95 kN/m²) and $\phi = 30$ degrees yields $\sigma_3 = \sigma_n = 1732.0$ psf (82.96 kN/m²). This value could be considered approximately the closed-form solution.

The comparisons among the three solutions at ultimate limit are satisfactory as shown in Figure (7.2b). The large deformation solution obtained in the present analysis shows a similar response to the small deformation analysis. This could be due to the soil body is always in compression at subsequent loads, and the deformation of the body considered here is dominantly governed by the material

nonlinearity and not by the change of geometry.

VII.3. Passive Translation of a Retaining Wall

In an attempt to analyze the vertical movement of a soil body when load is applied horizontally, a problem of retaining wall in passive translation is solved. The wall is pushed toward the soil body to model the passive translation situation. This can be performed by applying prescribed displacements at various load steps. A finite element mesh of 18 elements is used as shown in Figure (7.3a). This problem has been solved by Matzuzaki [46] by using a quadrilateral element with linear displacement approximation.

The soil is considered to be a Drucker-Prager elasto-perfectly plastic material with properties given by

$$E = 7000.0 \text{ psi } (48,300.0 \text{ kN/m}^2)$$

$$\nu = 0.3$$

$$c = 0.0 \text{ psi } (0.0 \text{ kN/m}^2)$$

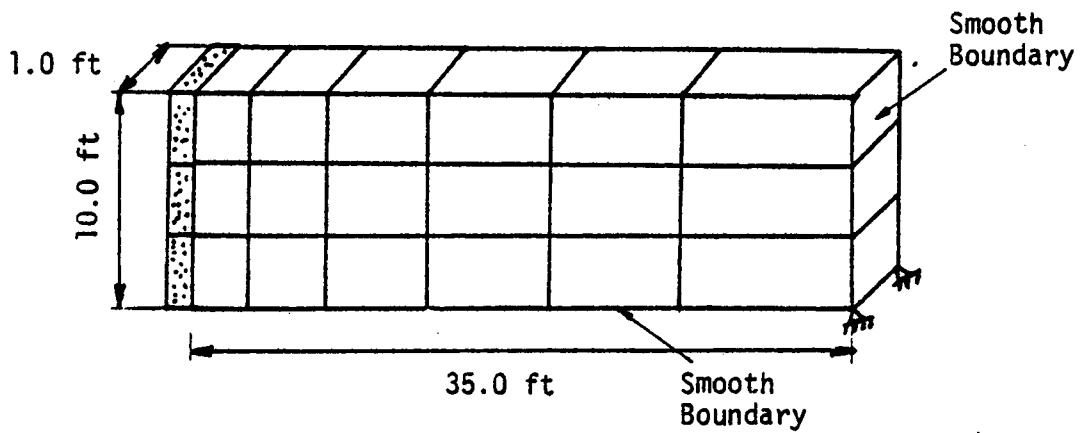
$$\phi = 35 \text{ degrees}$$

$$\gamma = 0.064 \text{ pci } (17.5 \text{ kN/m}^3)$$

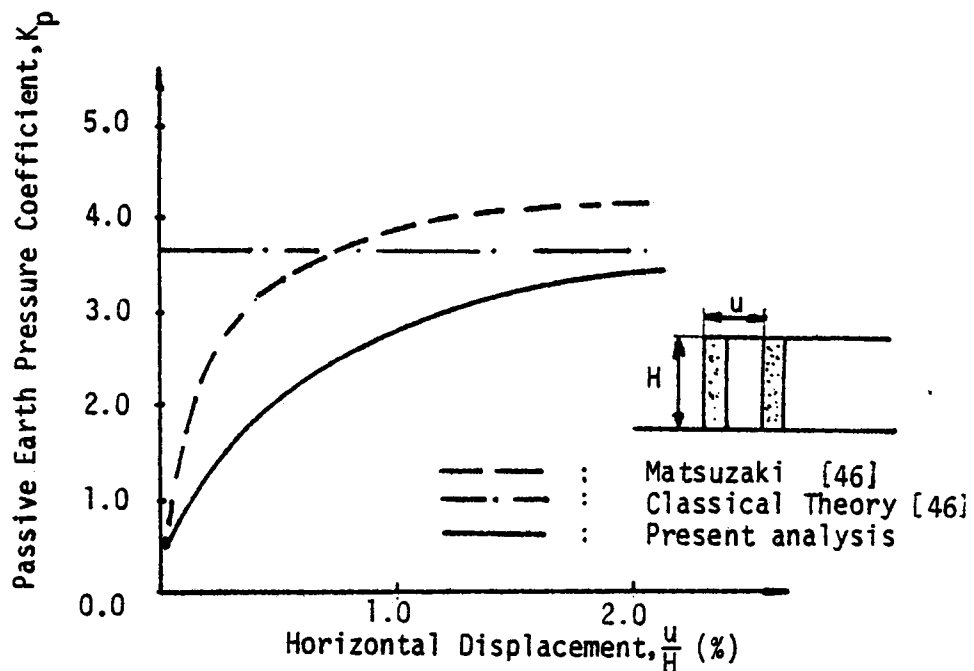
$$k = 0.43$$

where γ is the density of the soil, and k is the coefficient of earth pressure. These two parameters are used in initial stress computation.

Only small deformation analysis is performed in this problem. The solution obtained from the present analysis and the solution given by Matzuzaki [46] are shown in Figure (7.3b), together with



a) Finite Element Mesh



b. Passive Earth Pressure Coefficient vs. Wall Horizontal Movement

Figure 7.3. Retaining Wall Problem - Passive Translation

the passive earth pressure coefficient solution given by the classical theory [46]. In this figure, K_p is a non-dimension earth pressure coefficient and is defined as

$$K_p = \frac{2p}{\gamma H^2}$$

The solution given by Matzuzaki show an ultimate load which is higher than the load limit given by the classical solution, whereas the solution obtained in the present analysis shows lower values which approach the limit solution. The difference in the results can be due to different elements used, and due to the use of the original Newton-Raphson procedure in the present analysis. Overall, the correlations appear to be satisfactory.

VII.4. Loading on Half Space

In order to verify the accuracy of the three-dimensional procedure in solving plane strain, two-dimensional geomechanical problems, a problem of a soil mass subjected to a strip load is analyzed. The soil is of clay-type in undrained condition, and its behavior is assumed to follow the Von Mises, elasto-perfectly plastic, criterion. The parameters of the soil are

$$E = 30,000.0 \text{ psi (207,000.0 kN/m}^2\text{)}$$

$$\nu = 0.30$$

$$c = 17.5 \text{ psi (120.75 kN/m}^2\text{)}$$

$$\phi = 0.0 \text{ degrees}$$

The finite element mesh is shown in Figure (7.4a), where the

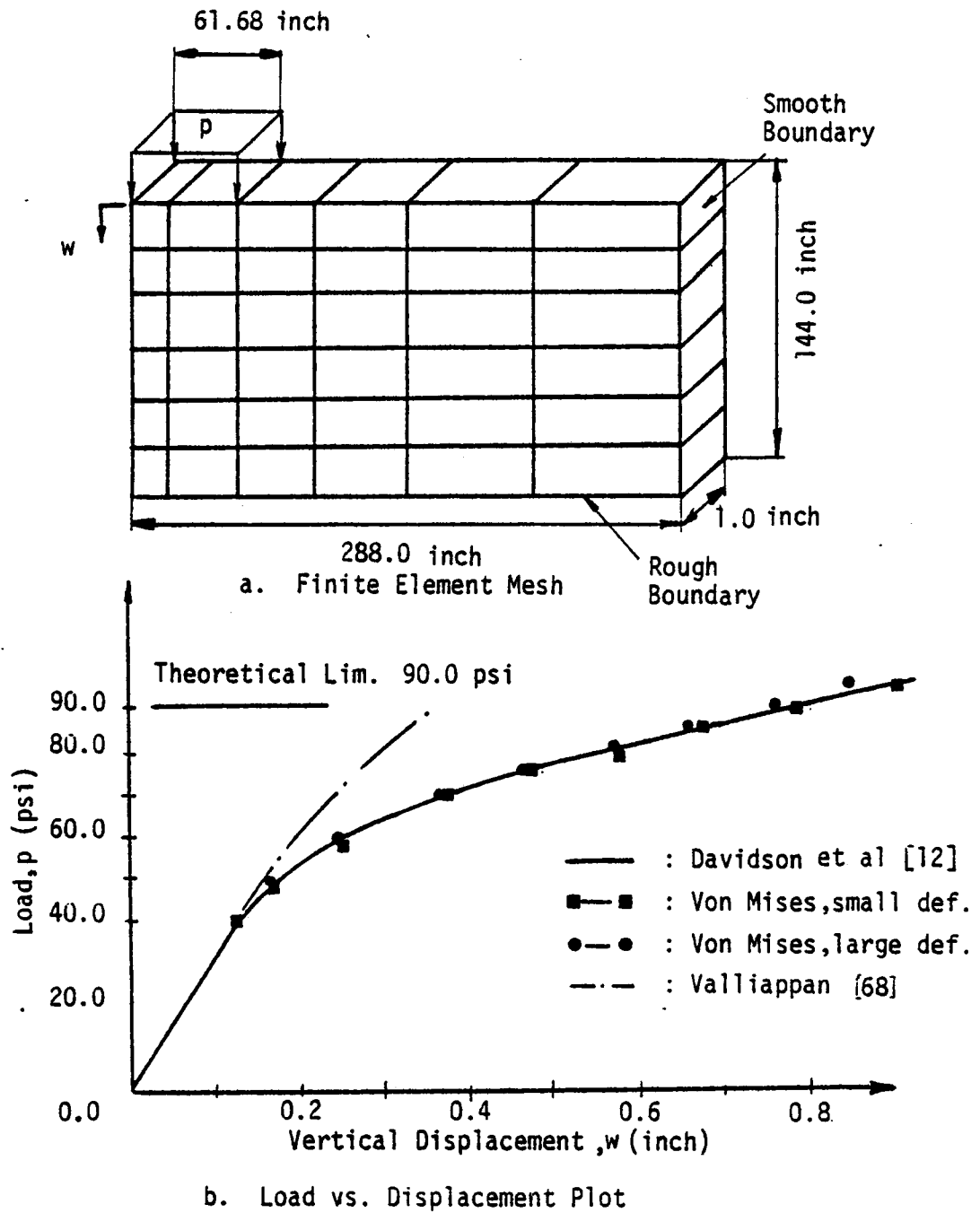


Figure 7.4. Deformation Analysis of A Strip Footing

bottom of the soil is totally fixed and the soil is allowed to move freely in the vertical direction on the side boundary. This problem has been solved by Davidson et al [12] and Valliappan [68]; both for the case of small deformation.

Davidson et al used a two-dimensional uniform mesh consisting of 120 nodes and 98 quadrilateral elements. Each quadrilateral element was defined by four constant strain elements. Valliappan used a coarser mesh with 94 nodes and 150 constant strain triangular elements. The mesh used in the present analysis is nonuniform and consists of 98 nodes and 36 three-dimensional brick elements.

Figure (7.4b) shows the solutions of the displacement at the center of the footing. The solution given by Davidson et al was based on a total number of 16 load increments and the numerical scheme used was the mid-point integration rule [12]. No iterative steps were performed in this scheme. Valliappan used the initial stress scheme [79] where the total load was applied and iterations were performed with constant stiffness until convergence was obtained. The load at which the iteration stopped was 78.0 psi (538.2 kN/m^2) [66]. In the present analysis, the original Newton-Raphson technique coupled with an incremental procedure is used with a total number of 10 load steps.

The comparisons are excellent for the solution obtained in the present analysis and the solution given by Davidson et al. The solution given by Valliappan, however, is significantly different

from the solutions in the present analysis. The latter solutions also compare very well with the ultimate limit given by the limit equilibrium formula.

The predictions with small and large strain assumptions yield again similar results. This is, in addition to the reason given in Example VII.2., perhaps due to the characteristics of the problem involving load which causes confined plastic flow in a half-space medium. This aspect is evident in other footing problems analyzed by Davidson et al [12].

VII.5. Strip Footing versus Laboratory Test

In order to verify the capacity of the cap models in predicting the behavior of the artificial soil in the present research, a problem of strip footing on a half-space medium is solved. This particular problem is attempted in order to investigate the correlation between numerical solutions and experimental data from a real laboratory test.

The laboratory test was conducted in an open 'rigid' container with the dimensions shown in Figure (7.5). The container was constrained on y-direction in order to model the plane strain condition. The artificial soil was placed in layers which were compacted until the desired depth was reached. A piece of stiff plastic, simulating the footing, was laid on the center of the soil surface, shown in Figure (7.5a), and then the load increments were applied. The vertical displacement of the plastic footing was measured with respect to subsequent load increments. The solid

line in Figure (7.5c) shows the measured load-displacement curve.

In the finite element analysis, a mesh of 21 three-dimensional elements is used. The boundary condition adopted is shown in Figure (7.5b). The parameters of the artificial soil are derived from a comprehensive series of laboratory tests described in Appendix C and are given below

$$\begin{aligned}
 E &= 4000.0 \text{ psi } (27600.0 \text{ kN/m}^2), & A &= 5.6 \text{ psi } (38.64 \text{ kN/m}^2) \\
 \nu &= 0.35 & \beta &= 0.11 \\
 c &= 0.0 \text{ psi } (0.0 \text{ kN/m}^2) & C &= 5.6 \text{ psi } (38.64 \text{ kN/m}^2) \\
 \phi &= 33 \text{ degrees} & B &= 0.062 \\
 \lambda &= 0.11 & R &= 2.0 \\
 \kappa &= 0.001 & W &= 0.18 \\
 \gamma &= 0.072 \text{ pci } (20.0 \text{ kN/m}^3) & D &= 0.05 \\
 e_0 &= 0.65
 \end{aligned}$$

where λ , the compression index, κ , the rebound index, and e_0 , the initial void ratio, are parameters for the critical state model, and A , β , C , B , R , W and D are parameters for the cap models.

Figure (7.5c) shows the solutions using various constitutive models. From this figure, it is seen that the Drucker-Prager elasto-perfectly plastic model does not yield satisfactory results. This is because in this model, the elastic behavior is assumed until the stress reaches the ultimate strength of the soil, then the soil starts experiencing plastic deformation. During plastic deformation, the model does not account approximately for the

volumetric deformations that accompany shear loading, where the latter factor is significant for many geological media including the artificial soil, hence the predictions from the Drucker-Prager model show less satisfactory results.

The critical state model shows a better prediction for the displacement at various loading steps. The allowance of yielding of the soil at even a small load in this model is found suitable in controlling the volumetric plastic strain, which occurs as the soils at the area surrounding the rigid footing displace vertically.

The cap model used in current research and the Sandler's cap model give excellent solutions in comparison to the experimental solution. As the load increases, the soil reaches its ultimate strengths, and the volume change decreases until the soil becomes incompressible. This behavior can be captured by the cap models, especially when the stress in the soil body is always in compressive state as observed in this type of problem. This capability of the cap models could probably account for the good predictions.

Figure (7.5c) also shows the large deformation solution using the cap model developed in the present research. As expected, for the constrained strip-footing problem, the difference between the small deformation solution and the large deformation solution is not significant.

Figure (7.5d) and (7.5e) show the displacement fields of the soil medium for two solutions with Drucker-Prager and cap models, respectively. The Drucker-Prager model predicts a significant

vertical displacement in the soil regions which are away from the footing, whereas the cap model predicts very small movements in these regions. This can be due to the fact that in these regions, the soil is in compression with only small levels of stress, and the Drucker-Prager model predicts only elastic deformation. The elastic deformation would give rise to a certain amount of volume change, which results in a significant movement beyond these regions.

On the other hand, the capability of the cap model to control the plastic flow, by limiting the plastic volumetric change at various load levels, would yield small displacements at the areas away from the footing. As such, for the type of footing analysis as in this problem, the cap models are more accurate than the classical elasto-perfectly plastic model in predicting the real behavior of the artificial soil.

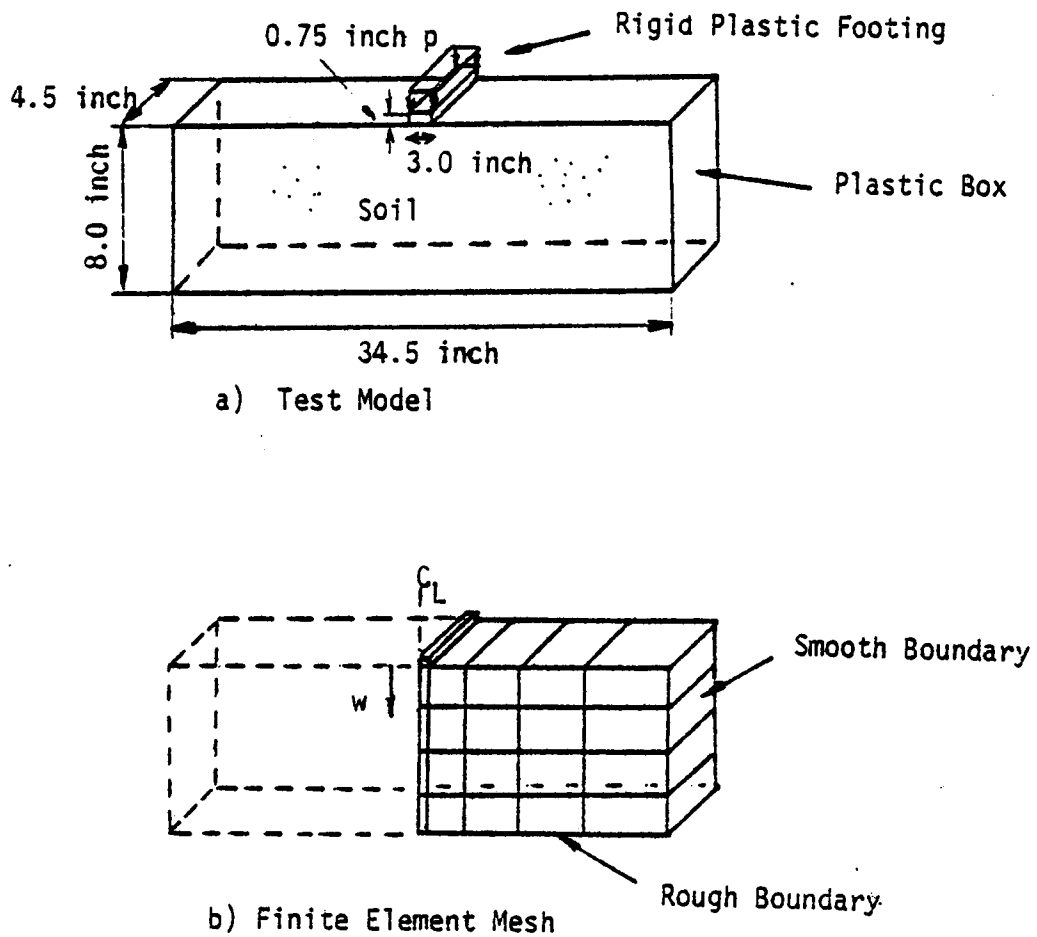
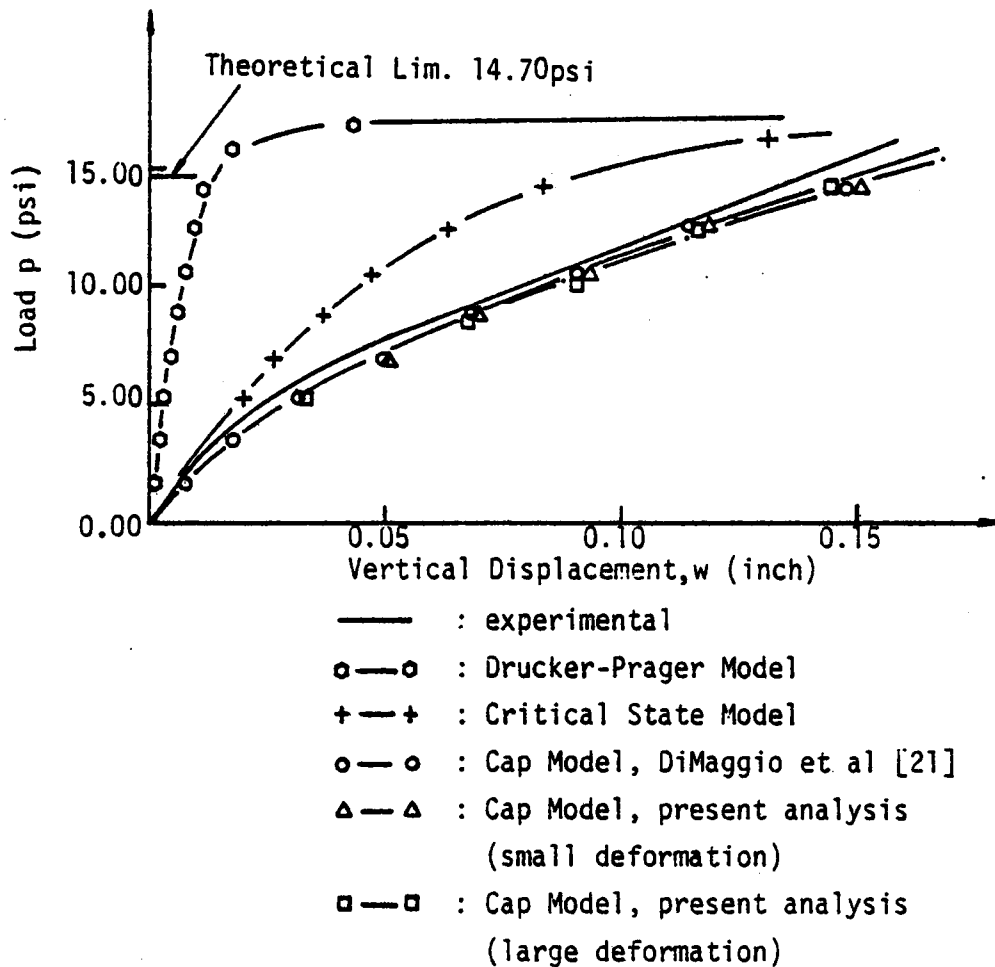


Figure 7.5. Nonlinear Material Results vs. Experimental Result for A Plane Strain Test



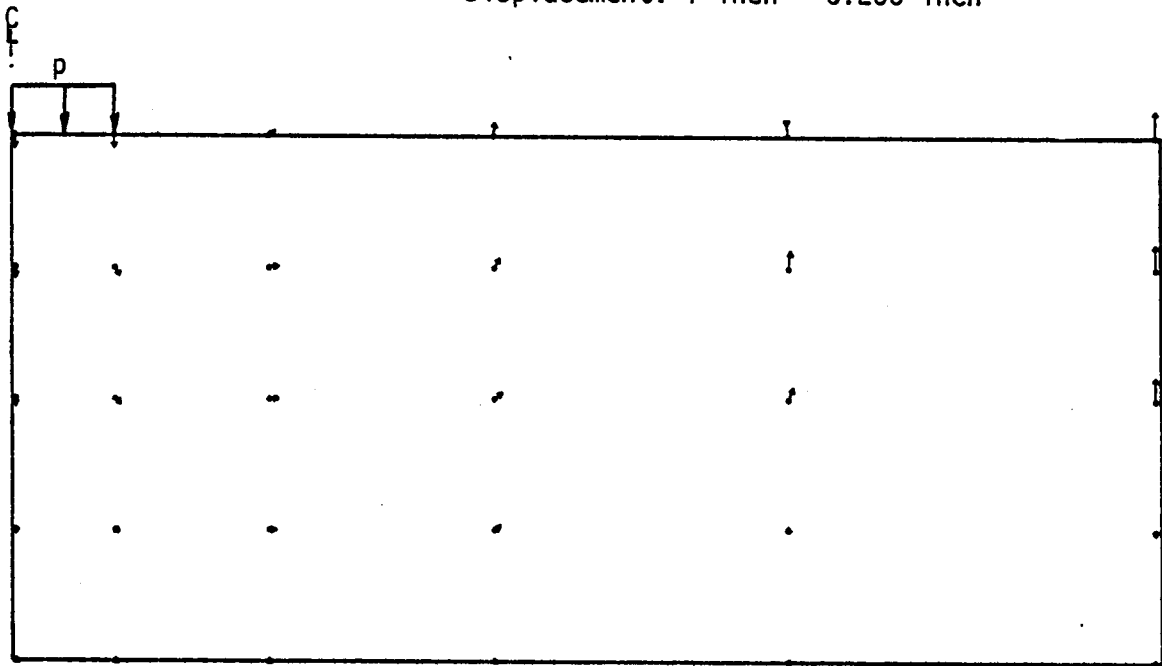
c. Vertical Displacement vs. Load Plots

Figure 7.5. (continued)

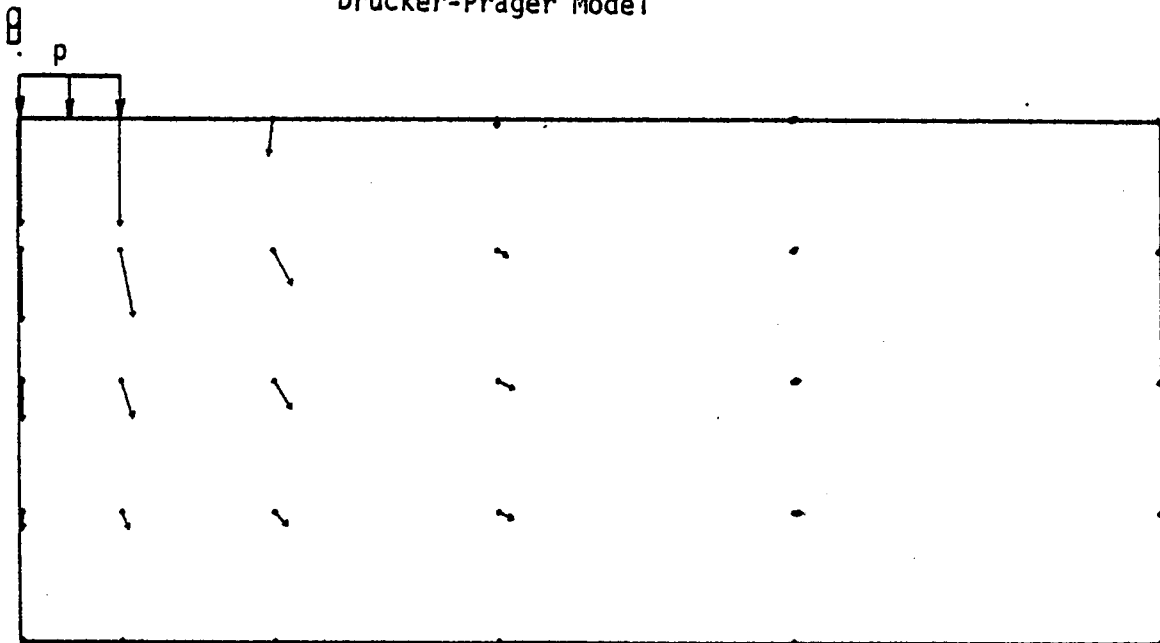
SCALES

Coordinate : 1 inch = 2.875 inch

Displacement: 1 inch = 0.266 inch



d. Displacement Field at $p = 16.0$ psi
Drucker-Prager Model



e. Displacement Field at $p = 15.0$ psi
Cap Model, present analysis
(small deformation)

Chapter VIII

VERIFICATIONS AND APPLICATIONS: THREE-DIMENSIONAL SOIL-STRUCTURE INTERACTION

As discussed previously, the purpose of the research is to study the behavior of a structure (tool) moving in soft soil. This chapter shows the applications of the finite element procedure and code in solving such a soil-structure interaction problem. Since no closed-form solutions are available, for this problem the numerical solutions obtained are compared with the experimental data from a series of tillage tool tests.

Prototype tests are conducted in a soil-bin facility, to determine the draft force (force acting opposite to the direction of movement of the tool) as a function of a resultant displacement of the tool. The horizontal force on the tool is recorded with the help of a multiple degrees-of-freedom dynamometer consisting of six load cells. Tests are conducted with tools of different widths (1-in, 2-in, 3-in, and 4-in) at different depths and at different tool angles. All tools used for the test are made of cold-rolled steel 0.5 in. thick. During tests, the tool displacement is applied manually with a cranking mechanism and the tool displacement is recorded with a dial gage. Hydrostatic drive system available with the soil-bin facility is not utilized for these tests because the desired tool speed is extremely slow. A complete explanation of the experimental facility is given by Durant [26] and Durant et al [27].

In the finite element analysis, two three-dimensional meshes are used as shown in Figures (8.1a) and (8.1b) where the dimensions of the bin and the tool are appropriately retained.

In Figure (8.1a), the tool is placed on the side of a half-infinite soil medium, and in Figure (8.1b), it is embedded in an infinite soil medium. The length of the meshes away from the tool in the x-direction is about 9 times the width of the tool, which is found to be acceptable in modeling the real situation in the experiments.

The boundary condition is also shown in Figures (8.1a) and (8.1b), where the degrees-of-freedom in the x-direction at the far ends of the mesh are constrained. It is found in the research, however, that the effect of fixing the degrees-of-freedom in the x-direction at the right end of the mesh, that is, in the direction of the load, do not yield significantly different results compared to those from releasing these degrees-of-freedom. This could be due to the possibility that significant soil-tool interaction occurs only in the vicinity of the tool. This point will be discussed again in this chapter.

The experimental load-displacement curves shown in Figures (8.2) to Figure (8.9) are taken as an average from different test results for each specific problem. For each problem, various tests are conducted at different locations along the bin. It is found that the data are not unique, and the values of the measured loads scatter significantly at different tool locations. Hence this

aspect should be taken into account when comparisons between numerical solutions and experimental solutions are made. Figures (8.5), (8.7) and (8.9) show some typical examples of different experimental solutions for each problem.

Figure (8.2) shows the solutions of a vertical tool placed on the side of the half-infinite soil medium. The comparisons between the numerical solutions and the experimental solution are satisfactory. The small deformation analysis without interface element gives a good prediction within the range of small load. However, at load of 60.0 lbs. (0.267 kN) and higher, the soil still retains some strength, and the displacement of the tool increases uniformly and not abruptly as observed in the experiment.

The analysis with interface elements shows a 'softer' response of the soil, but the distinction is not significant as compared to the solution without interface elements. This could be due to the interface elements used in the former analysis are only at the junction of the tool and the soil. When the horizontal load is applied, these elements would always be in the non-slip mode, and the interaction between the tool and the soil is not much different with the case where no interface element is considered. The slip mode in z-direction, although is detected in the analysis, only changes the vertical movement slightly.

The solution of large deformation problem without interface behavior shows a stiffening response. This could be due to the effect of the geometric stiffness in the finite element equations.

The nonlinear terms of the Green-Lagrange strain tensor contribute some increase to the geometric stiffness of the elements in the front of the tool. The soil elements on the side of the tool, on the other hand, are in a shearing condition and the inclusion of the nonlinear term could only cause large rotation. The result is, for such a continuous process, the large deformation analysis shows a 'stiffer' response than the small deformation analysis.

Figures (8.3) to (8.9) show a series of analyses for the real situation of the tool embedded in a three-dimensional soil space. In each analysis, the tool is inclined at different angles and the tool width varies. Figures (8.3), (8.4), (8.5) and (8.6) show the load-displacement curves for the problems of vertical (90-degree) tools of 2-in., 1-in., 3-in., and 4-in. width, respectively. Figures (8.7), (8.8) and (8.9) show the load-displacement curves for the problems of 75-degree, 45-degree, and 30-degree inclined tool of 2-in. width, respectively.

Figure (8.3) gives the predictions of the small deformation analyses with and without interface elements, and the large deformation analysis with interface elements. The comparisons are fair. As can be seen, the numerical solution of the case involving interface element shows a significant improvement in comparison to the case without interface element. This could be due to the fact that in the latter case, the deformation process is continuous, that is compatibility is maintained at the tool and the soil behind it. This has the effect of causing restraints on the movement of

the tool. The analysis with interface consideration, however, would 'disconnect' the effect of the soil elements behind the tool, and separation occurs as tension develops in the interface elements. This results to such an improvement in the load-deformation response.

The large deformation analysis with interface elements shows 'softer' response than the response obtained from the small deformation analysis at the low load level, but the differences are not significant. At the high load level, the response of the former analysis becomes 'stiffer', but again the differences are negligible. The reason could be that the strains and rotations in the soil regions close to the tool are small, that is, in the order of 10^{-3} , hence the effect of large deformation in the global system could be small. Perhaps a measurement of volume change for small and large displacement cases would better illustrate this aspect. Shown below is the volume change for a soil element in the front of the tool, at different load levels.

Volume Change Measured by
Incremental Volumetric Strain, $\Delta\epsilon_{ij} \times 10^{-3}$

(Negative sign for compression)

Load, lbs. (1bs = 4.45 N)	Small Deformation Analysis	Large Deformation Analysis
12.0	-0.54	-0.57
24.0	-1.53	-1.47
36.0	-1.31	-1.98
48.0	-1.19	-2.88
60.0	-2.70	-2.58
72.0	-3.03	-4.87
84.0	-3.18	-3.34

From the above table, it can be seen that the volume changes at various load levels do not differ much between the small deformation analysis and the large deformation analyses. Hence, for this type of soil-structure interaction problem, the analysis with small deformation consideration is found to be acceptable.

Due to the above discussion, the analyses for deformation cases in Figures (8.4), (8.5), and (8.6), only the load-displacement curves of the case with interface elements are plotted. The trends of solutions are similar to those in the case of the solution involving 2-inch wide tool. Improvements of the results are seen as the front face of the tool becomes wider. The tool with smaller width is capable of punching through the soil at even a small load, and the numerical model may not account fully for this punching phenomenon. On the other hand, the tool with larger width could activate much higher restraint from the soil in the front of it, hence the continuous flow of the soil at subsequent loads until failure could be captured in the numerical model developed herein.

Figures (8.7), (8.8) and (8.9) show the responses of the tool with inclinations of 75, 45 and 30 degrees, respectively. The tool used is 2-inch wide. In these problems, the comparisons are very good. Again the small deformation analysis without interface elements gives unrealistic results. The analysis with interface elements, however, improved significantly. The soil shows a 'softer' response at small load levels, but still retains some strength as the load increases. Such a behavior was not observed

from the experimental data. The differences between the experimental solutions and the finite element solutions could be due to the fact that in the former solution, particularly when the tool is nearer to the soil surface (small inclination), the front soil starts cracking at lower load levels. This fracture effect could reduce the strength of the soil to smaller values, which in turn could lead to the failure of the soil. In the finite element solutions, the tensile stress developed in the front of the tool is transferred to the other regions by the stress transfer technique, which could result to 'softer' response at the small load levels. As the load increases, however, the deformation process is continuous and from moving freely, and as such, prevents the body from total collapse. In this respect, the present finite element procedure is limited, and a more comprehensive fracture criterion could improve the situation.

Displacement Patterns

In order to understand the mechanism of soil-structure interaction, the displacements of the soil body and the tool are plotted at various sections during subsequent load levels. Figures (8.10), (8.11) and (8.12) show the displacement fields for the problem of the 2-inch wide tool inclined at 90 degrees (vertical) with and without interface elements. The applied loads are 36.0 lbs., 72.0 lbs., and 108.0 lbs., respectively.

Since a three-dimensional graphic plotting package is not available at the present time, only two-dimensional plots are given at

3 typical sections on the finite-element mesh. These sections are shown in Figure (8.1).

From Figures (8.10) to (8.12) it can be deduced that the soil-structure interaction occurs only at the vicinity of the tool. The soil regions which are away from the tool experience only small displacements in the order of 10^{-2} in comparison to those of the tool. The displacements of the soil at the bottom of the bin also are very small. This could be due to the fact that since the stiffness of the tool is significantly higher than the stiffness of the soil, the tool could penetrate into the soil body without much resistance of the soil. When the load is reasonably small, only the strength of the soil close to the tool is motivated to contribute to this resistance, and the soil away from the tool is hardly affected. As the load is increasing, the soil at the far regions starts to activate to resist the tool penetration, which results to significant displacements, Figure (8.12). This aspect again explains the continuous movement of the tool as shown in Figure (8.3), where a total collapse of the soil could not be predicted.

Figures (8.12a) and (8.12b) show the difference between the small deformation analysis with and without interface elements. Displacements of the soil behind the tool are detected in the latter case, whereas they are not seen in the former case. As discussed previously, the compatibility condition in the finite element procedure results to such a displacement. The interface elements, on the other hand, could 'separate' the tool and the soil behind it,

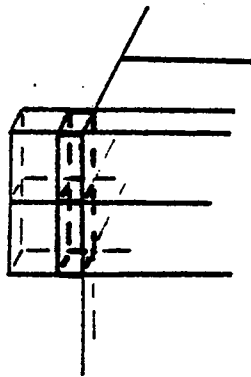
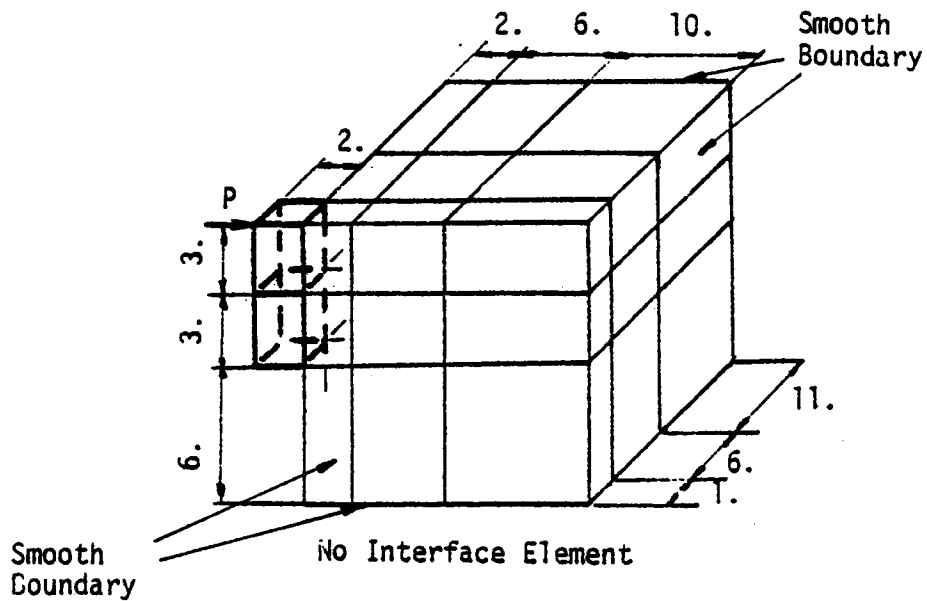
Figure (8.12b). Hence, the analysis with interface elements is again proved to be more realistic than the analysis without interface elements in comparison to the observations from the experiments.

Figures (8.10e), (8.10f), (8.11e), (8.11f), (8.12e) and (8.12f) show the pattern of the soil-deformations on the bin surface at various load levels. As the tool moves, the soil at the side of the tool tends to move toward the former, whereas the soil in the front of the tool moves around and directs to the side of the bin. This aspect, although it is not shown clearly in these figures due to small displacements of the soil away from the tool, could be captured in the numerical analyses, and compares reasonably good with the observations.

Figures (8.10a), (8.10b), (8.10c), (8.10d), (8.11a), (8.11b), (8.11c), (8.11d), (8.12a), (8.12b), (8.12c), and (8.12d) show the deformations of the soil at the front of the tool and beneath the tool. The soil has the tendency to move upward in the x- and y- directions, which is again compared good with the observations.

From the above discussions, it is seen that the numerical analyses predict fairly well the complex behavior of the soil-moving tool interaction. The limit of the procedure at the current level of research, of course, is its inability to account for other phenomena such as fracture and punching. Overall, the finite element formulation and the code developed herein could be adopted to solve a general three-dimensional soil-structure problem.

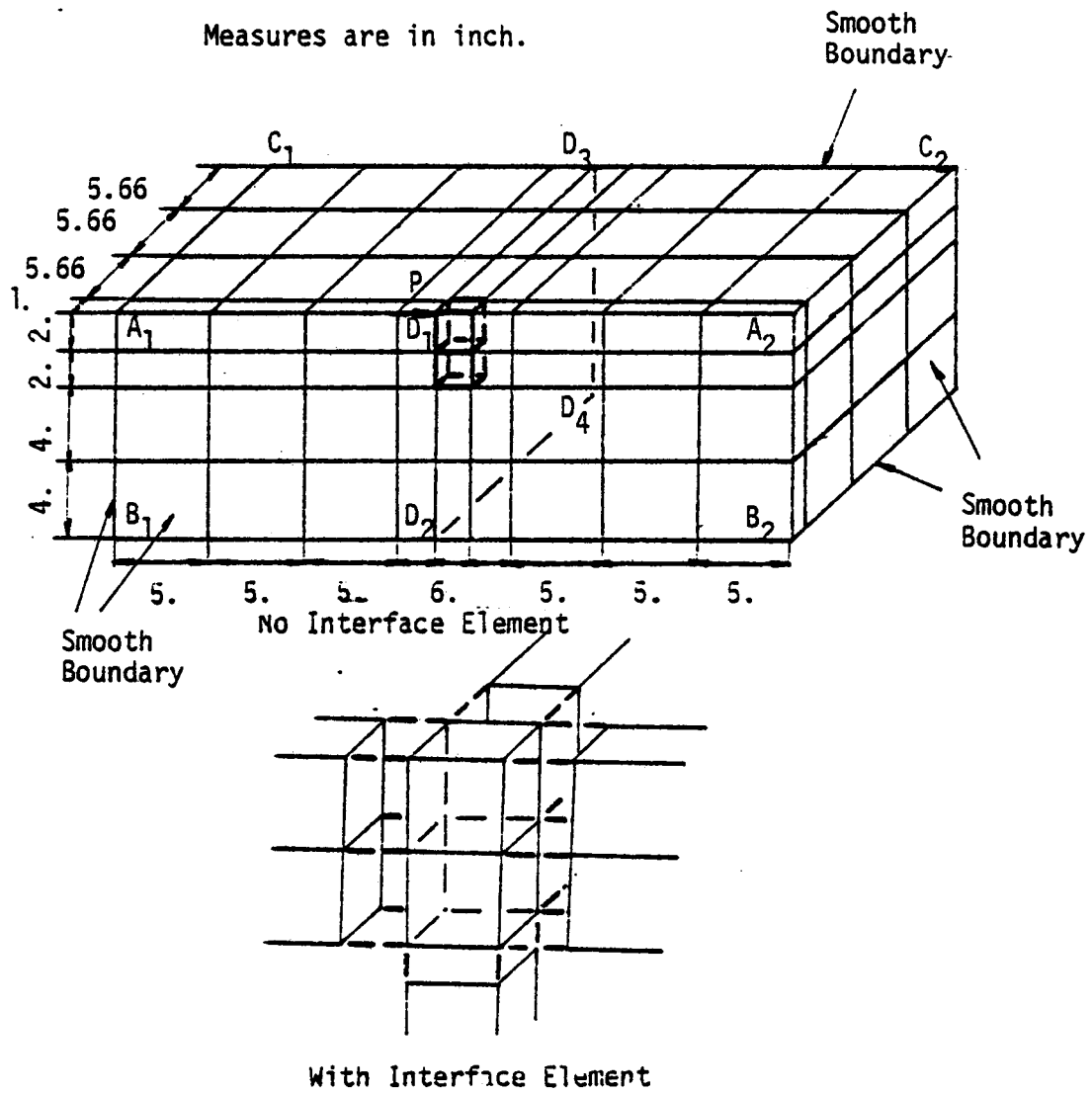
Measures are in inch.



With Interface Element

a. Half-Infinite Soil Medium

Figure 8.1. Three-Dimensional Finite Element Meshes for Soil-Tool Analyses



b. Infinite Soil Medium

Figure 8.1. (continued)

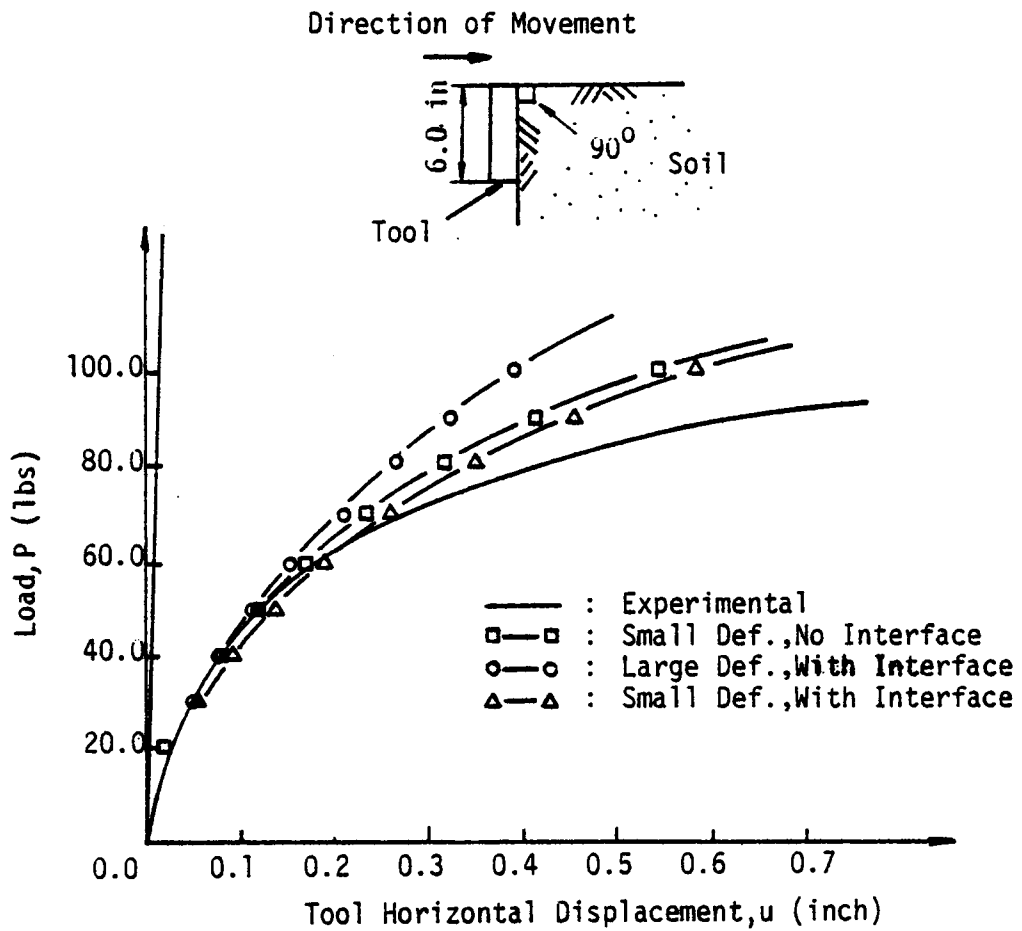


Figure 8.2. Load vs. Displacement Plot, Half-Space Case
2 inch tool, 90 degrees

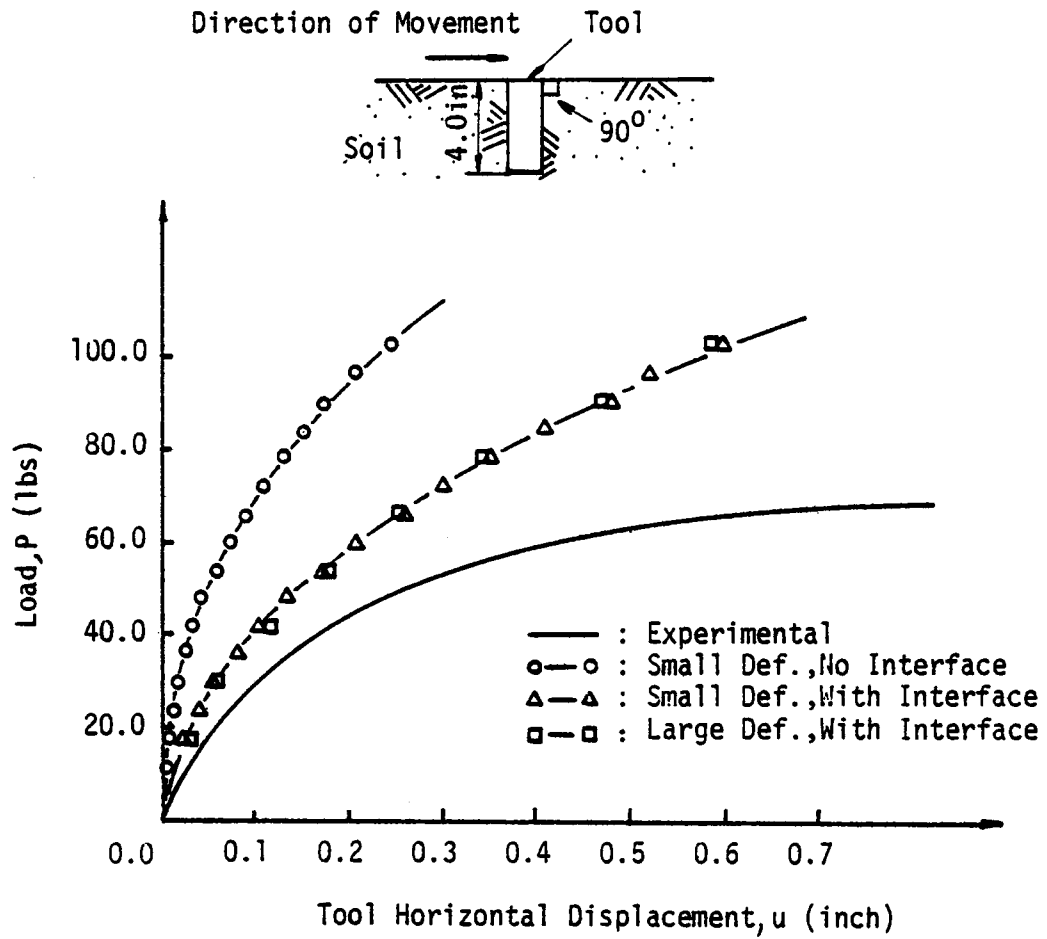


Figure 8.3. Load vs. Displacement Plot, Full-Space Case
2 inch tool, 90 degrees

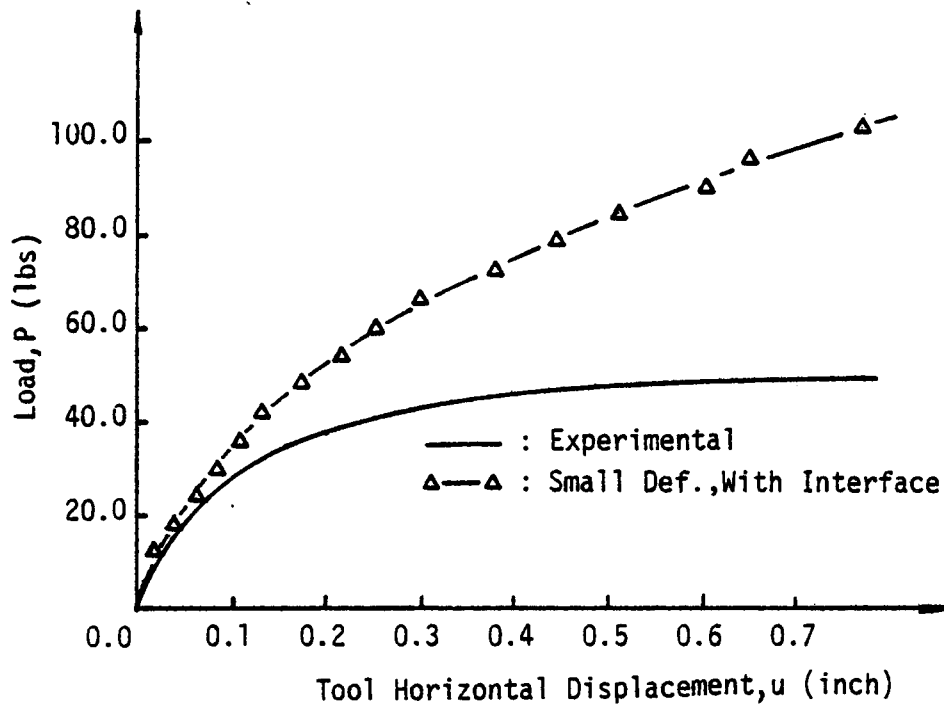


Figure 8.4. Load vs. Displacement Plot
1 inch tool, 90 degrees

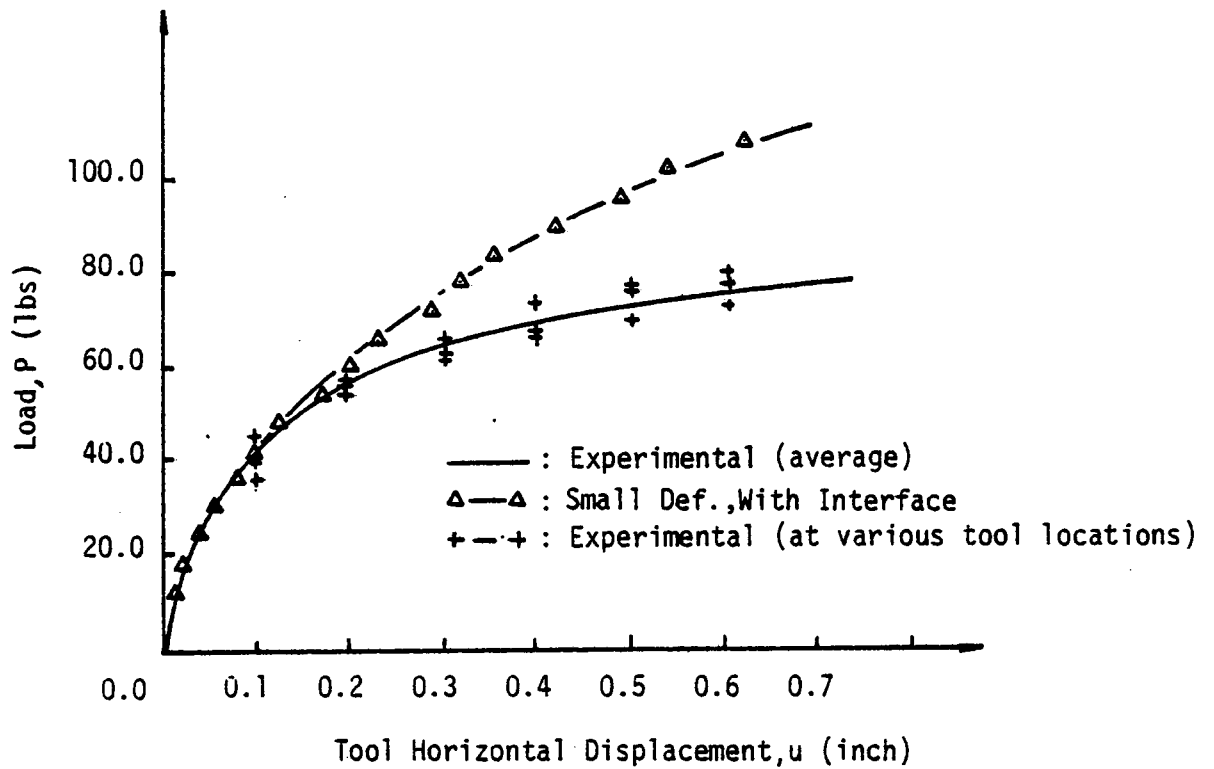


Figure 8.5. Load vs. Displacement Plot
3 inch tool, 90 degrees

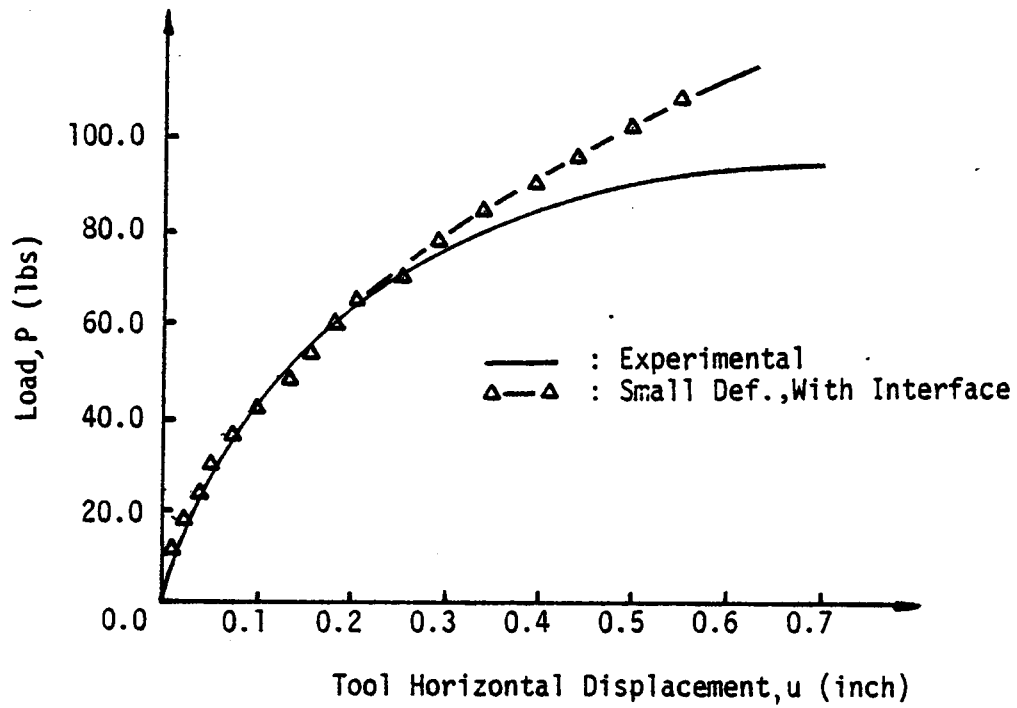


Figure 8.6. Load vs. Displacement Plot
4 inch tool, 90 degrees

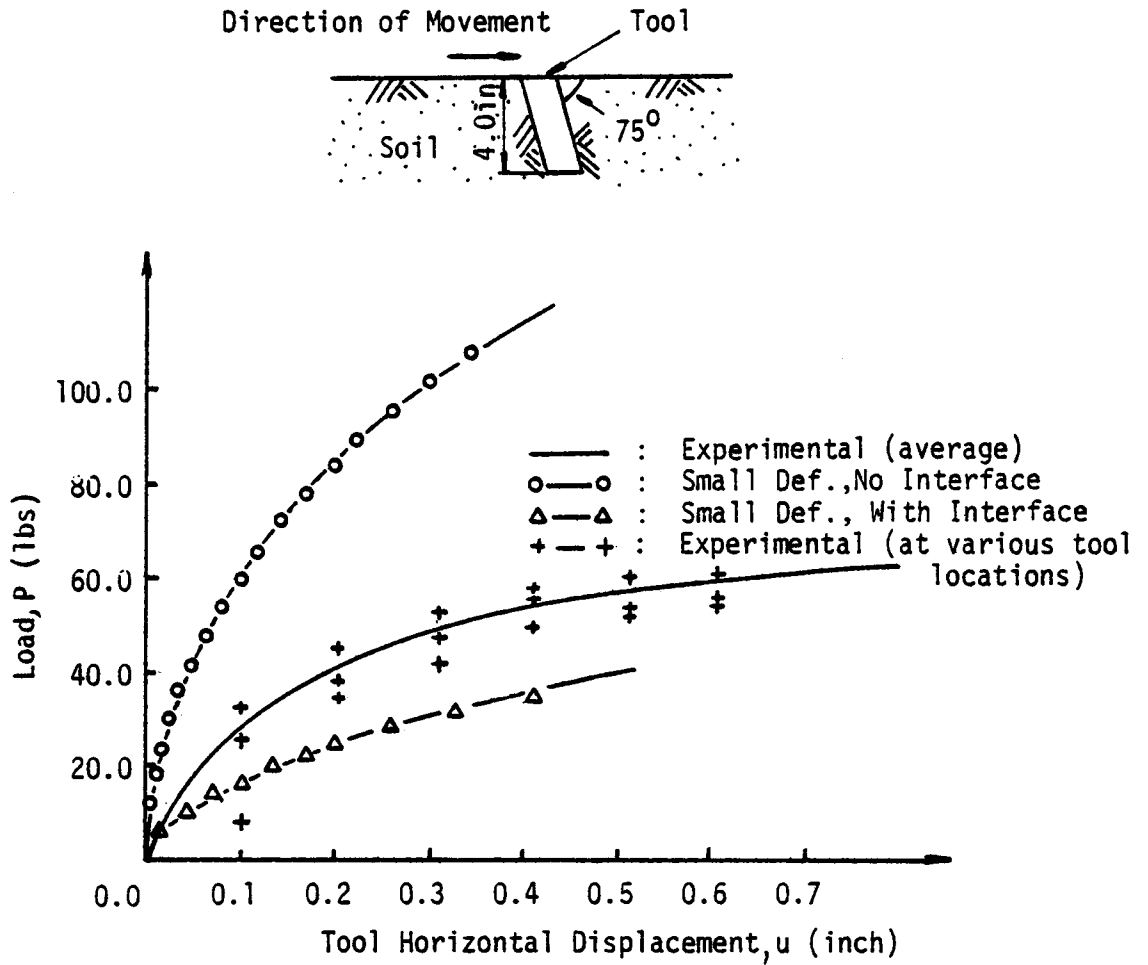


Figure 8.7. Load vs. Displacement Plot
2 inch tool, 75 degrees

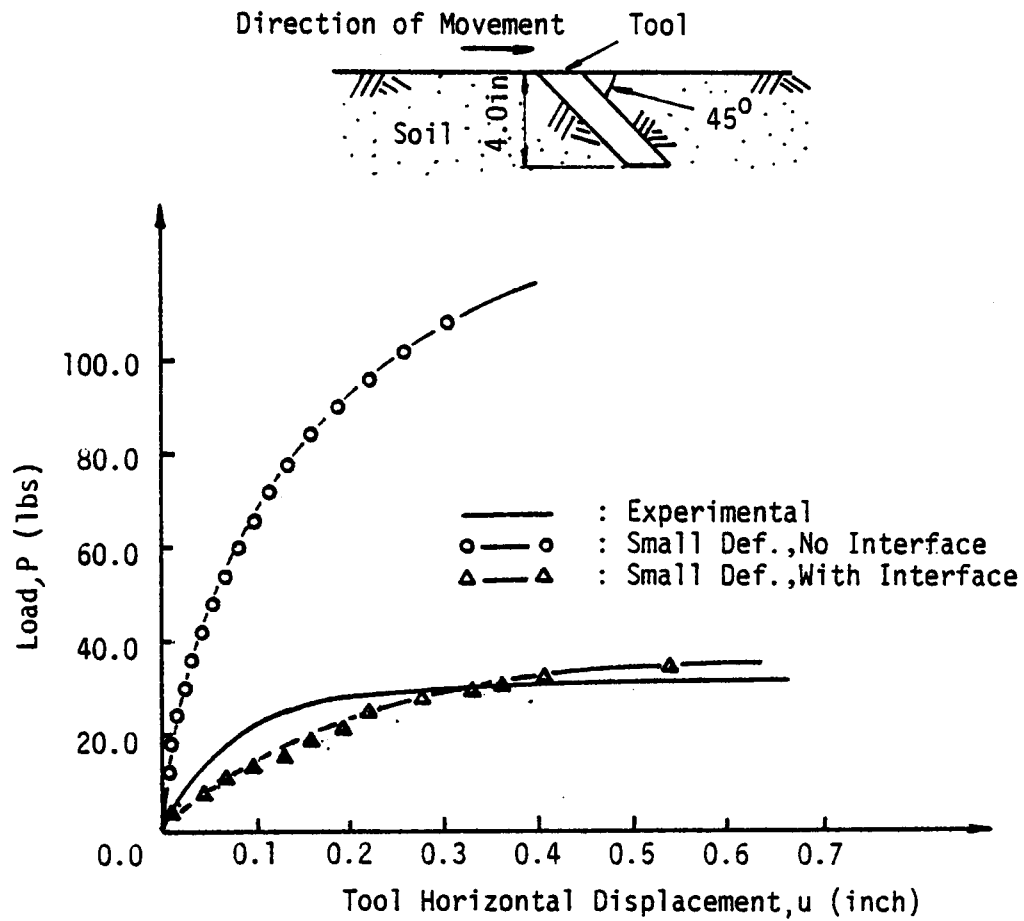


Figure 8.8. Load vs. Displacement Plot
2 inch tool, 45 degrees

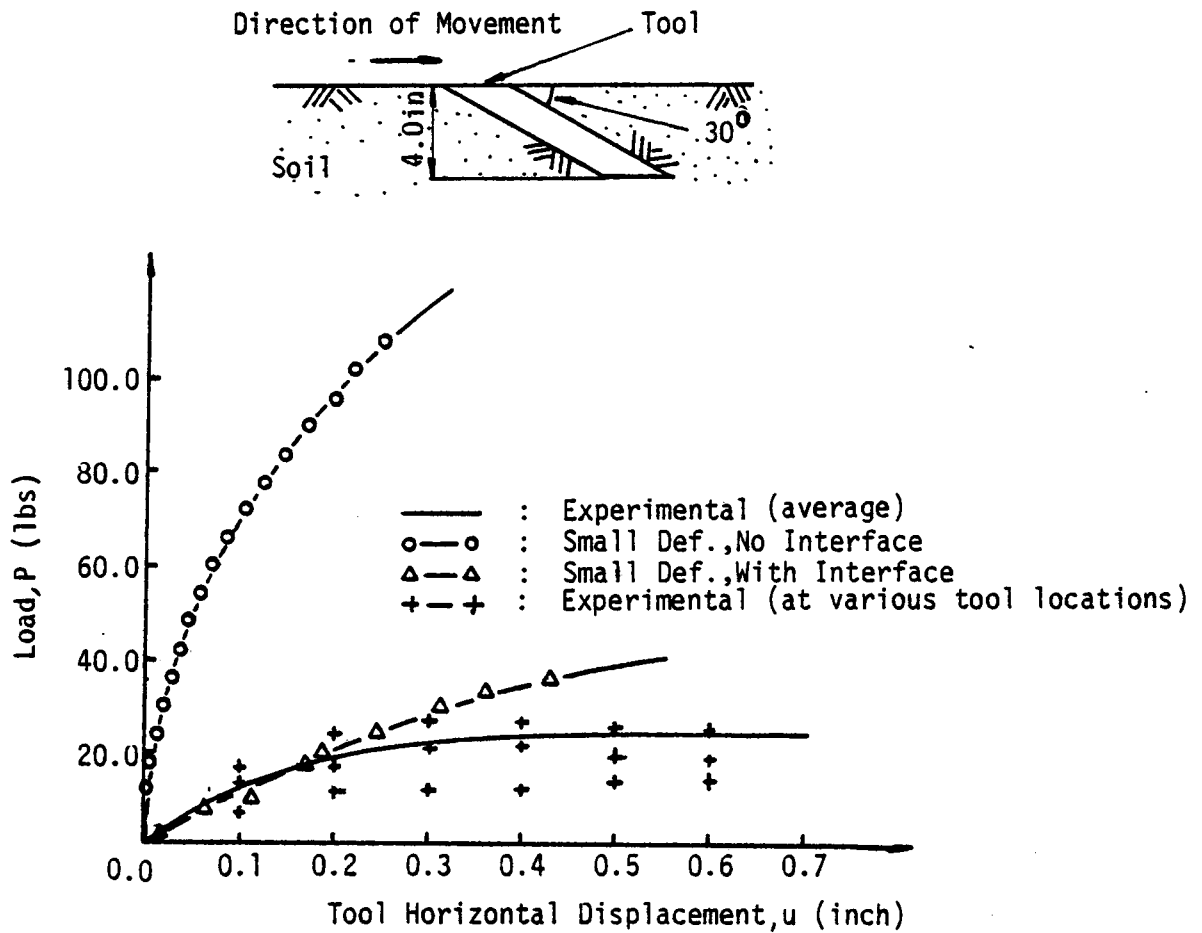
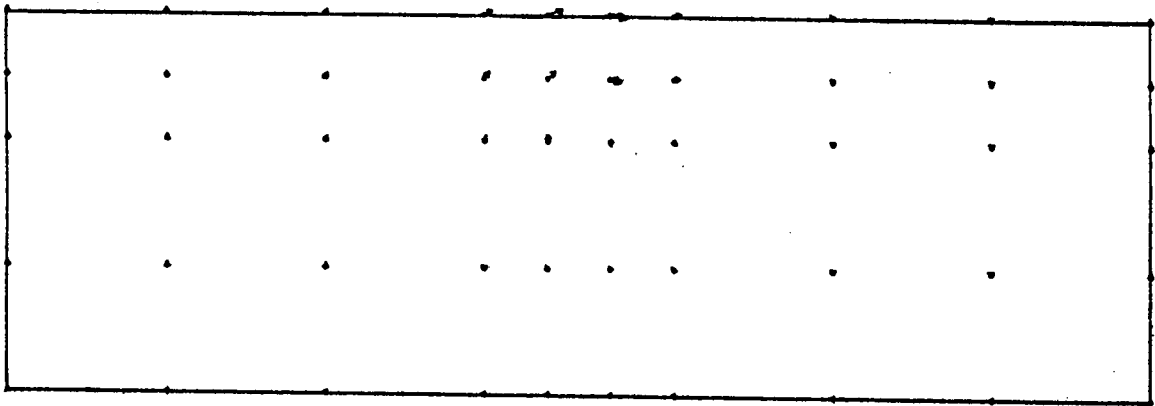


Figure 8.9. Load vs. Displacement Plot
2 inch tool, 30 degrees

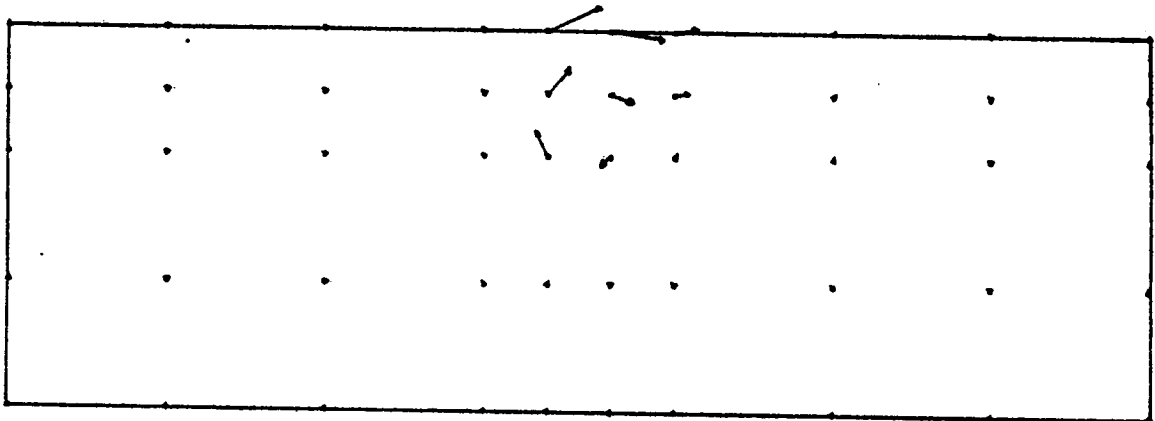


- a. At Section $A_1-A_2-B_1-B_2$
No Interface Element

SCALES

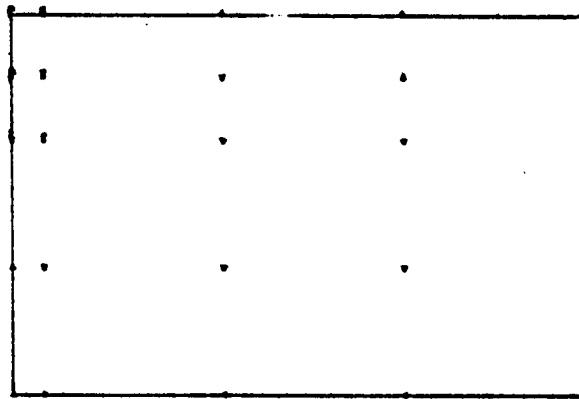
Coordinate : 1 inch = 6.0 inch

Displacement: 1 inch = 0.266 inch

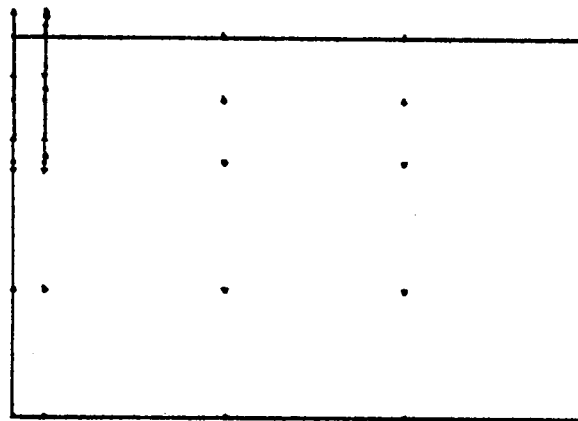


- b. At Section $A_1-A_2-B_1-B_2$
With Interface Element

Figure 8.10. Displacement Fields at Various Sections
at Load $P = 36.0$ lbs
Problem of 2-Inch Wide Tool, 90 Degrees

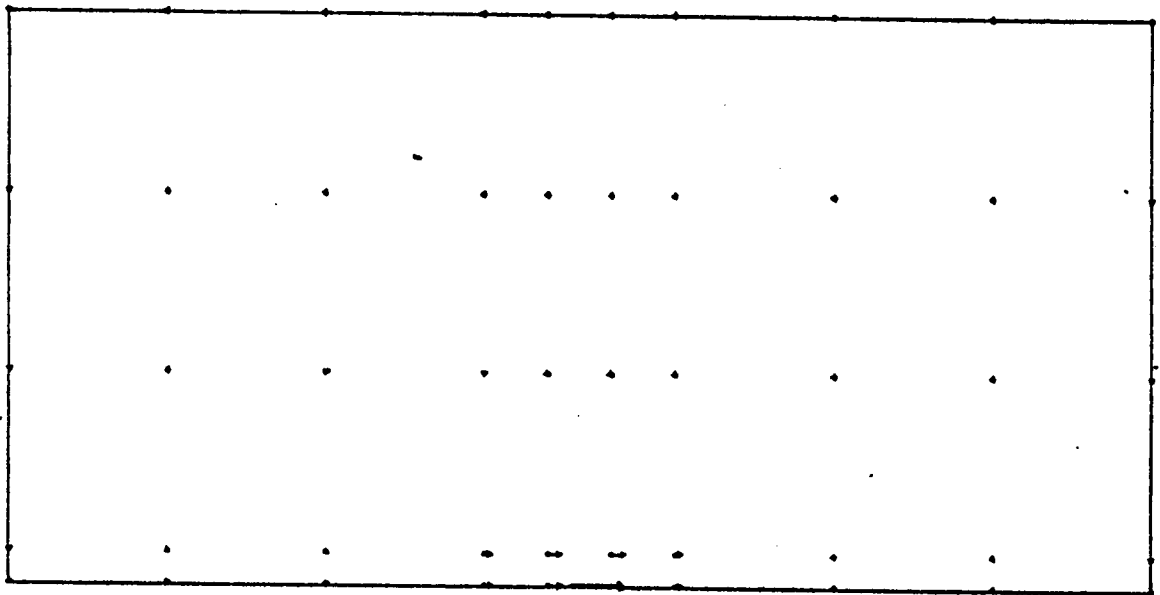


c. At Section $D_1-D_2-D_3-D_4$
No Interface Element



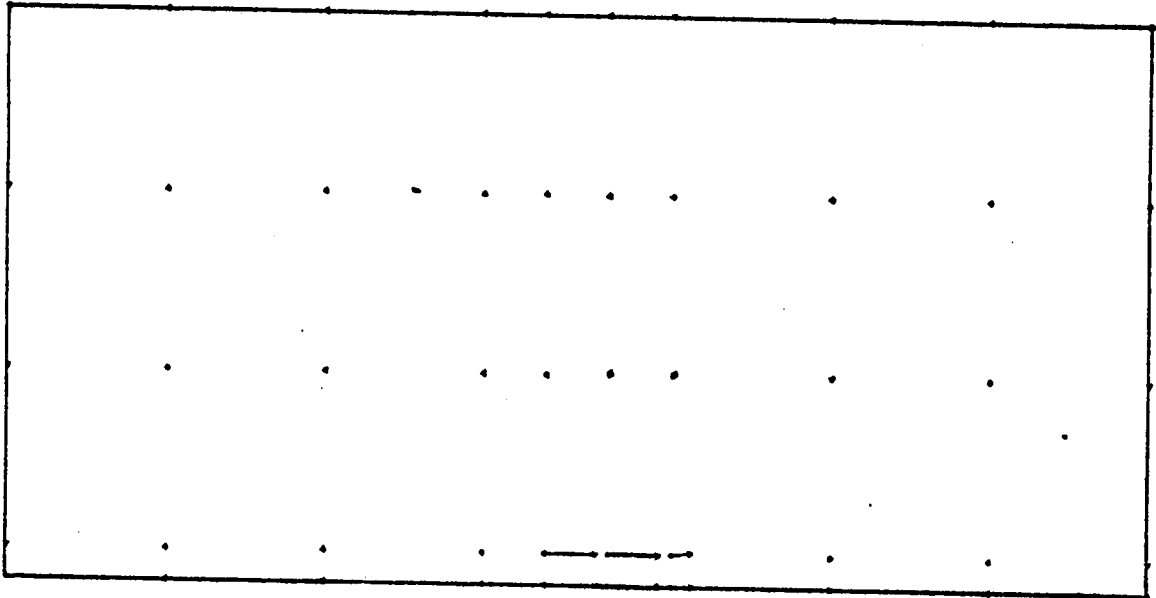
d. At Section $D_1-D_2-D_3-D_4$
With Interface Element

Figure 8.10. (continued)



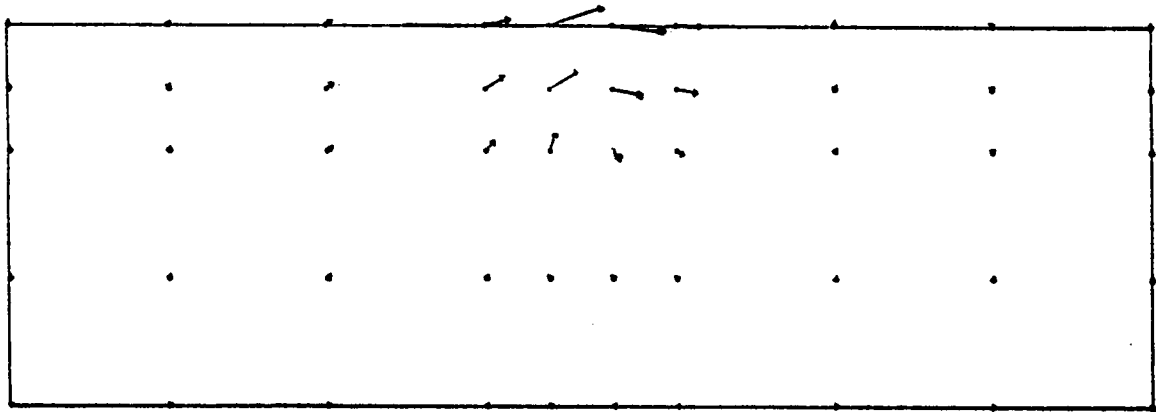
e. At Section $A_1-A_2-C_1-C_2$
No Interface Element

Figure 8.10. (continued)



f. At Section $A_1-A_2-C_1-C_2$
With Interface Element

Figure 8.10. (continued)

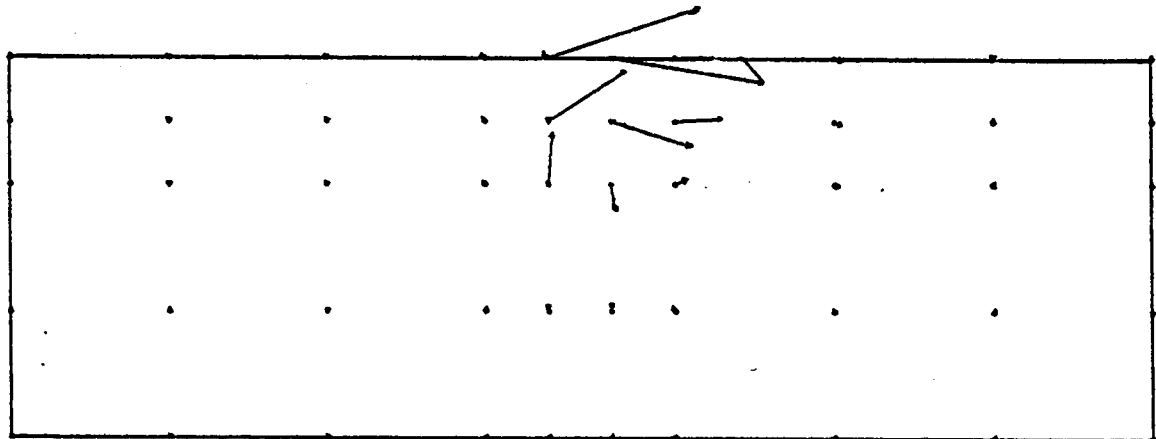


- a. At Section $A_1-A_2-B_1-B_2$
No Interface Element

SCALES

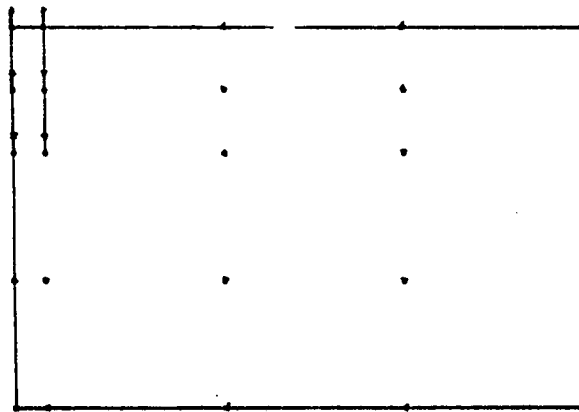
Coordinate : 1 inch = 6.0 inch

Displacement: 1 inch = 0.266 inch

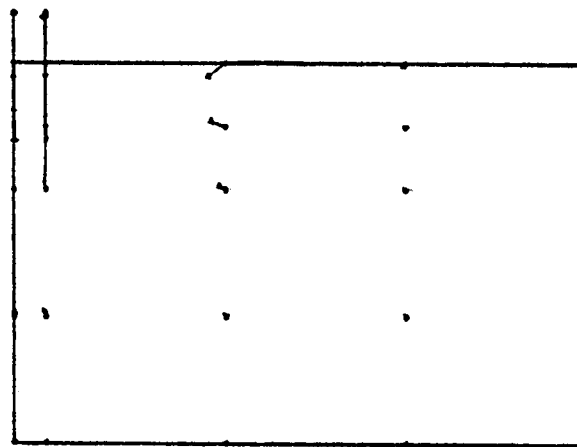


- b. At Section $A_1-A_2-B_1-B_2$
With Interface Element

Figure 8.11. Displacement Fields at Various Sections
at Load $P = 72.0$ lbs
Problem of 2-Inch Wide Tool, 90 Degrees

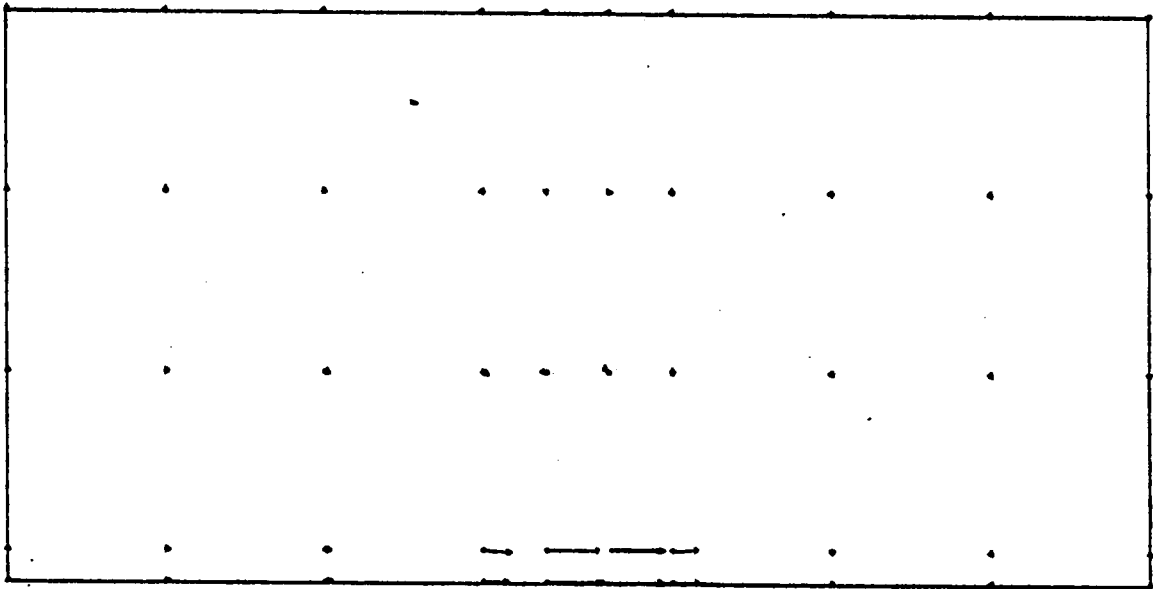


c. At Section $D_1-D_2-D_3-D_4$
No Interface Element



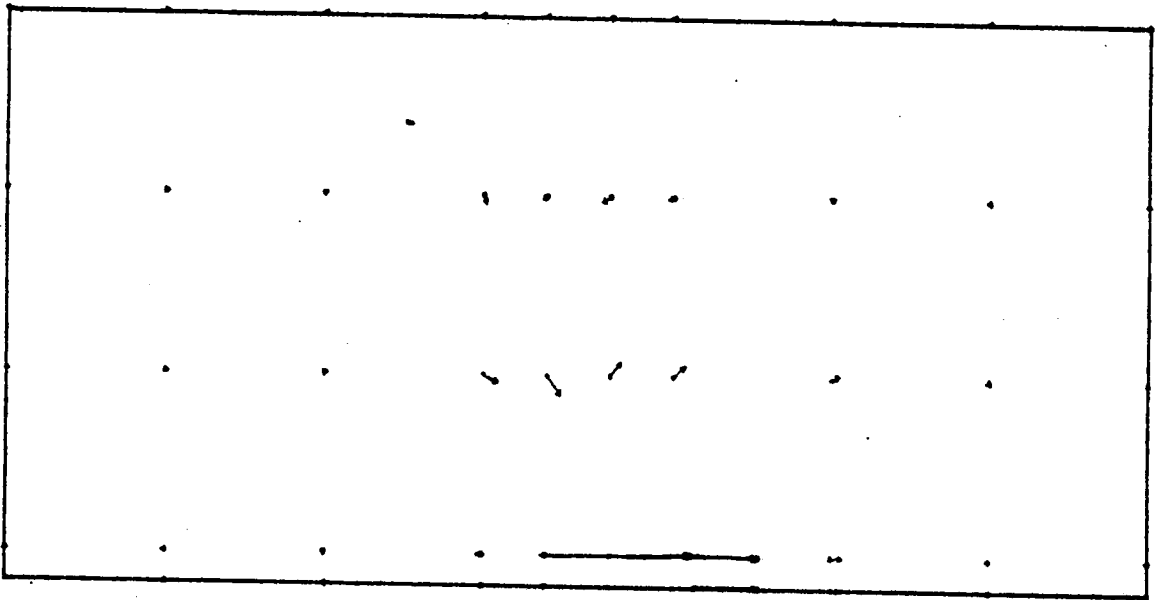
d. At Section $D_1-D_2-D_3-D_4$
With Interface Element

Figure 8.11. (continued)



e. At Section $A_1-A_2-C_1-C_2$
No Interface Element

Figure 8.11. (continued)



f. At Section $A_1-A_2-C_1-C_2$
With Interface Element

Figure 8.11. (continued)

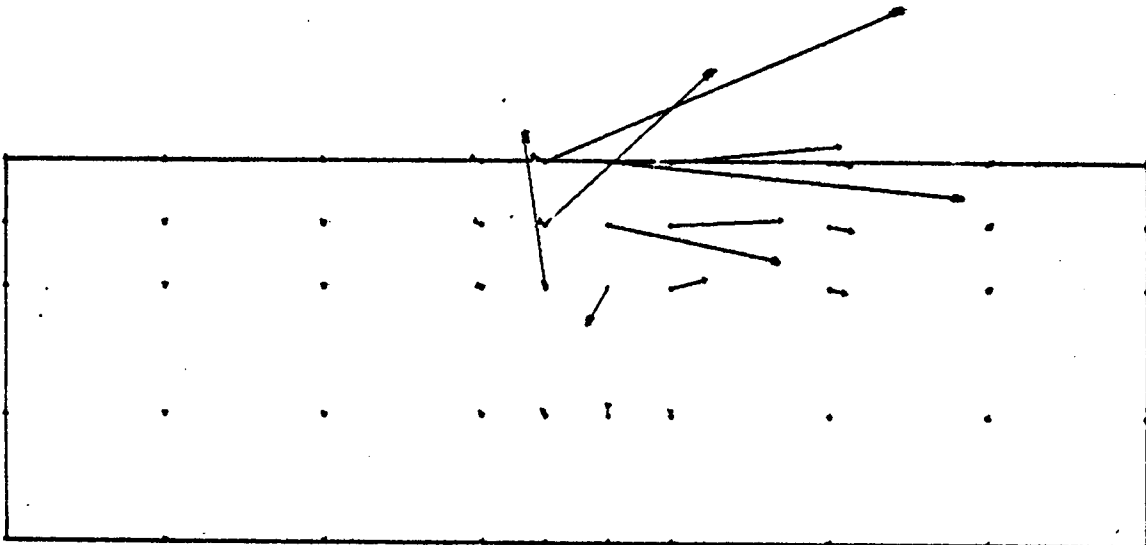


a. At Section $A_1-A_2-B_1-B_2$
No Interface Element

SCALES

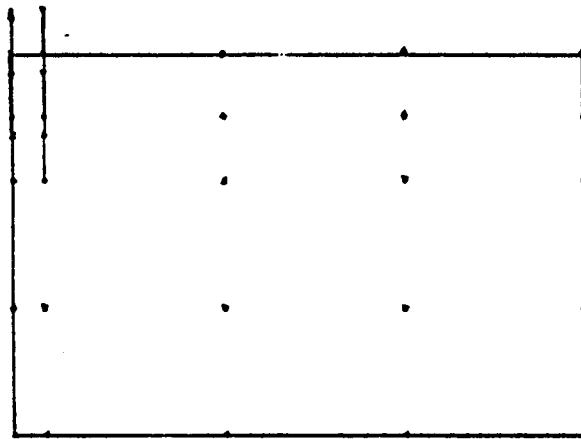
Coordinate : 1 inch = 6.0 inch

Displacement: 1 inch = 0.266 inch

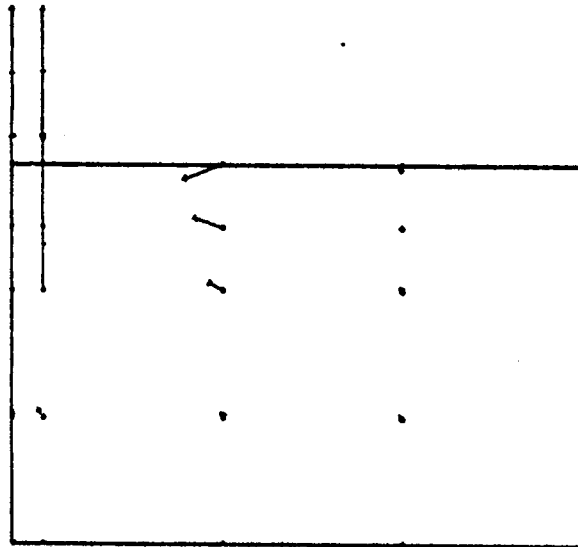


b. At Section $A_1-A_2-B_1-B_2$
With Interface Element

Figure 8.12. Displacement Fields at Various Sections
at Load $P = 108.0$ lbs
Problem of 2-Inch Wide Tool, 90 Degrees

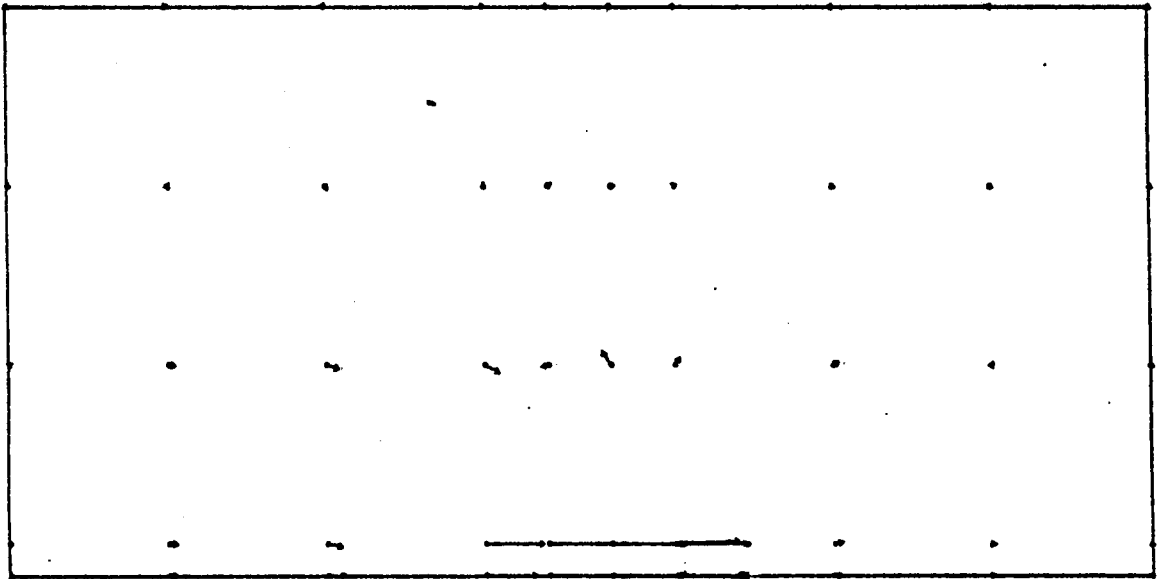


c. At Section $D_1-D_2-D_3-D_4$
No Interface Element



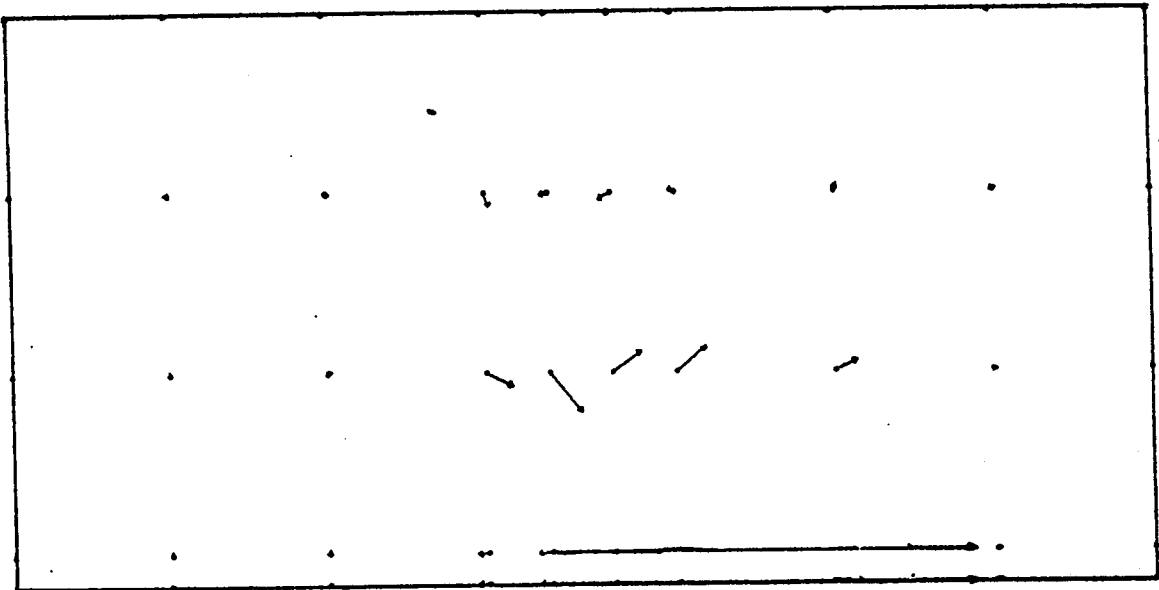
d. At Section $D_1-D_2-D_3-D_4$
With Interface Element

Figure 8.12. (continued)



e. At Section $A_1-A_2-C_1-C_2$
No Interface Element

Figure 8.12. (continued)



f. At Section $A_1-A_2-C_1-C_2$
With Interface Element

Figure 8.12. (continued)

CHAPTER IX

SUMMARY AND CONCLUSION

IX.1. Summary

Throughout various chapters, the formulation and implementation of geometric and material nonlinearities have been discussed. In the formulation of geometric nonlinearity, the updated Lagrangian approach was adopted. A scheme of transforming the stresses and strains to the current configuration of the body was used in this approach. This scheme was found consistent with the determination of the constitutive law, since in the laboratory, the material parameters for the constitutive model were computed from the real stress at the current volume of the test sample.

In the implementation of material nonlinearity, the original Newton-Ralphson technique was utilized in which the stiffness of the domain was updated at every incremental load step and at every iteration within this load step. This technique, coupled with the updated Lagrangian approach, showed a consistent formulation for a problem with large strain and large displacement. In addition, the consideration of Jaumann's rate of stress, which retained the coordinate independence characteristics of the constitutive law, was appropriate for this kind of problems.

Since the material behavior is an essential factor in the analysis of soil-structure interaction, various constitutive models, ranging from elasto-perfectly plastic to work-hardening plastic, were implemented. A new work-hardening plastic model was developed from

various laboratory tests for the artificial soil. A problem of strip footing was solved to check the solutions of different models in predicting this type of soil. It was found that the cap models, one developed by DiMaggio et al [21] and one developed in the current research [66], gave excellent solutions.

The interface behavior also contributed significantly to the analysis of soil-structure interaction, especially the analysis of tool moving in soil. This effect was modelled by the inclusion of a three-dimensional interface element, compatible type. The equilibrium interface element which was believed to be better than the compatible element, yielded a global non-positive definite stiffness matrix and required a special solving scheme. It was found inappropriate in the current computer code, hence, not used throughout the analysis.

A series of analyses of tools moving in the artificial soil were solved, where all the effects of nonlinearities were included. The cap models were satisfactory in predicting the behavior of the artificial soil when the applied load was reasonably small.

These models, however, predicted that the soil still retained strength at high load levels, which did not compare very well with the observation from experiments. It was found that the consideration of geometric nonlinearity did not affect very much the behavior of the system, since the overall deformation of the soil body was mainly governed by the nonlinear effect of the soil properties.

IX.2. Conclusion and Recommendations

The formulation of geometric and material nonlinearities developed herein can solve a general problem in soil-structure interaction. The interface behavior between soil and structure can be handled by a special interface element. The finite element method is a very convenient and efficient technique in incorporating these nonlinear effects in the analysis. However, for the problem of soil-moving tool interaction as in the current research, a tremendous work is still needed to properly predict the behavior of the system, and research is currently being performed in some aspects described below.

As the tool is moving in the soil body, cracks would occur in the front of the tool. The fractures, after initiating, would propagate rapidly when an increase of load is applied. This results to a total collapse of the soil. The observations from the experiments showed that even the soil at the side of the tool exhibited some cracks due to shearing. This reduces the strength of the soil significantly and the assumption of the continuity of the soil is no longer valid. The finite element procedure should be modified to handle this effect. A crack element should be introduced to replace the continuous element whenever tension is found to exceed a limit in that particular continuous element. The singularities occurring at the tip of the crack can be accounted for by introducing another term in the interpolation function of the regular element.

Another factor which affects the soil deformation at the start of the load application is the initial stress condition. As observed in the experiments, the density of the soil changes at various depth levels. The soil becomes loose as the depth of the soil mass increases. This results to the soil assumes less strength away from the surface or the top of the soil mass. The coefficient of earth pressure is also not uniform at different depth levels. At the region close to the surface of the soil, a high coefficient is recorded which leads to large initial horizontal stress. As the depth increases, the coefficient of earth pressure becomes smaller and leads to small stress. The modification of the initial stress condition should be performed to handle this effect.

X. REFERENCES

1. Banerjee, P. K., and Fathallah, R. C., "An Eulerian Formulation of Finite Element Method for Predicting the Stress and Pore-water Pressures Around a Driven Pile," Proc. 3rd Int. Conf. Num. Meth. in Geomechanics, Aachen, W. Germany, 1979.
2. Banerjee, P. K., and Stipho, A. S., "Associated and Non-Associated Constitutive Relations for Undrained Behavior of Isotropic Soft Clays," Int. J. for Num. and Anal. Meth. in Geomechanics, V. 2. No. 2., 1978.
3. Banerjee, P. K., and Stipho, A. S., "An Elasto-Plastic Model for Undrained Behavior of Heavily Overconsolidated Clays," Int. J. for Num. and Anal. Meth. in Geomechanics, V. 3, No. 1, 1979.
4. Bathe, K. J., "ADINA, Static and Dynamic Geometric and Material Nonlinear Analysis," Rep. No. 82448-2, MIT, Cambridge, Mass., 1976.
5. Bathe, K. J., Ramm, E., and Wilson, E. L., "Finite Element Formulation for Large Deformation Dynamic Analysis," Int. J. for Num. Meth. in Eng., V. 9, 1975, pp. 353-386.
6. Bathe, K. J. and Wilson, E. L., Numerical Methods in Finite Element Analysis, Prentice-Hall, Englewood Cliffs, N. J., 1976.
7. Biot, M. H., Mechanics of Incremental Deformation, John Wiley and Sons, N. Y., 1965.
8. Carter, J. P., Booker, J. R., and Davis, E. H., "Finite Deformation of an Elasto-Plastic Soil," Int. J. for Num. and Anal. Meth. in Geomechanics, V.1, 1977.
9. Carter, J. P., Booker, J. R., and Small, J. C., "The Analysis of Finite Elasto-Plastic Consolidation," Int. J. for Num. and Anal. Meth. in Geomechanics, V. 3, 1979.
10. Chen, W. F., and Davidson, H. L., "Large Deformation Response of Clay to Loads," Proc. Int. Conf. on Finite Elements in Nonlin. Solid and Struct. Mech., Geilo, Norway, 1977.
11. Clough, R. W., "Comparison of Three-Dimensional Finite Elements," Proc. Symp. on Appl. of Finite Element Method in Civil Eng., Knoxville, Tenn., 1959.

12. Davidson, H. L., and Chen, W. F., "Elastic-Plastic Large Deformation Response of Clay to Footing Loads," Rep. No. 355-18, Fritz Eng. Lab., LeHigh University, Pa., 1974.
13. Desai, C. S., Elementary Finite Element Method, Prentice-Hall, Englewood Cliffs, N. J., 1979.
14. Desai, C. S., "Soil-Structure Interaction and Simulation Problems," Finite Elements in Geomechanics, Ed. by Gudehus, G., John Wiley, N. Y., 1977.
15. Desai, C. S., "Deep Foundations," Numerical Methods in Geotechnical Engineering, Ed. by Desai, C.S., and Christian, J.T., McGraw-Hill, N.Y., 1977.
16. Desai, C. S., "Some Aspects of Constitutive Models for Geologic Media," Proc. 3rd Int. Conf. Num. Meth. in Geomechanics, Aachen, W. Germany, 1979.
17. Desai, C. S. and Abel, J., Introduction to the Finite Element Method, Van Nostrand Reinhold, N.Y., 1972.
18. Desai, C. S., and Appel, G. C., Jr., "Three-Dimensional Analysis of Laterally Loaded Structures," Proc. 2nd Int. Conf. on Num. Meth. in Geomechanics, VPI & SU, Blacksburg, VA, 1976.
19. Desai, C. S. and Christian, J., Numerical Methods in Geotechnical Engineering, McGraw-Hill, N. Y., 1977.
20. Desai, C. S., and Phan, H. V., "Three-Dimensional Finite Element Analysis Including Material and Geometric Nonlinearities," 2nd Int. Conf. Comp. Meth. in Nonlinear Mechanics, Austin, Texas, 1979.
21. DiMaggio, F. L., and Sandler, I. S., "Material Model for Granular Soil," J. of Eng. Mech. Div., ASCE, V. 97, No. 3, 1971.
22. Drucker, D. C., "Some Implications of Work-Hardening and Ideal Elasticity," Quarterly of Appl. Math., V. 7, No. 4, 1950, pp. 411-418.
23. Drucker, D. C., "On Uniqueness in the Theory of Plasticity," Quarterly of Appl. Math., V. 14, 1956.
24. Drucker, D. C., Gibson, R. E., and Henckel, D. J., "Soil Mechanics and Work-Hardening Theories of Plasticity," Trans. ASCE, V. 122, 1957, pp. 338-346.

25. Drucker, D. C. and Prager, W., "Soil Mechanics and Plastic Analysis of Limit Design," Quarterly of Appl. Math., V. 10, No. 2, 1952.
26. Durant, D. M., "Soil-Bin Test Facility for Soil-Tillage Tool Interaction Studies," M.S. Thesis, VPI&SU, Blacksburg, VA., 1979.
27. Durant, D. M., Perumpral, J. V. and Desai, C. S., "Soil-Bin Test Facility for Soil-Tillage Tool Interaction Studies," Paper presented at the 1979 ASAE Winter Meeting, New Orleans, LA., 1979.
28. Frederiksson, B. Rydholm, G. and Sjoblom, P., "Variational Inequalities in Structural Mechanics with Emphasis on Contact Problems," Proc. Int. Conf. on Finite Elements in Nonlin. Solids and Struct. Mech., Geilo, Norway, 1977.
29. Fung, Y. C., Foundations of Solid Mechanics, Prentice-Hall, Englewood Cliffs, N. J., 1965.
30. Gallagher, R. H., Finite Element Analysis: Fundamentals, Prentice-Hall, Englewood Cliffs, N. J., 1972.
31. Ghaboussi, J., Wilson, E. L., and Isenberg, J., "Finite Element for Rock Joints and Interfaces," J. Soil Mech. Found Div., ASCE, V. 99, No. 10, 1973.
32. Goodman, R. E., Taylor, R. L., and Brekke, T. L., "A Model for the Mechanics of Jointed Rock," J. Soil Mech. Found Div., ASCE, V. 94, No. 3, 1958.
33. Green, A. E., and Nagdhi, P. M., "A General Theory of an Elastic-Plastic Continuum," Arch. Rat. Mech. Anal., V. 18, 1965.
34. Haisler, W. E., Stricklin, J. A., and Stebbins, F. J., "Development and Evaluation of Solution Procedures for Geometrically Nonlinear Structural Analysis," J. AIAA, V. 10, No. 3, 1972.
35. Herrmann, L. R., "Nonlinear Finite Element Analysis of Frictional Systems," Proc. Int. Conf. On Finite Elements in Nonlin. Solids and Struct. Mech., Geilo, Norway, 1977.
36. Herrmann, L. R., "Finite Element Analysis of Contact Problems," J. Eng. Mech. Div., ASCE, V. 104, No. 5, 1978.
37. Hibbitt, H. D., Marcal, P. V., and Rice, J. R., "A Finite Element Formulation for Problems of Large Strain and Large Displacements," Int. J. Solids and Structures, V. 6, 1970.

38. Hill, R., The Mathematical Theory of Plasticity, Oxford Press, London, 1950.
39. Hofmeister, L. D., Greenbaum, G. A., and Evensen, D. A., "Large Strain, Elasto-Plastic Finite Element Analysis," J. AIAA, V. 9, No. 7, 1971.
40. Kachanov, L. M., Foundations of the Theory of Plasticity, North Holland Publishing, Amsterdam, London.
41. Katona, M. G., Smith, J. M., Odello, R. S., and Allgood, J. R., "CANDE, A Modern Approach for the Structural Design and Analysis of Buried Culverts," Rep. FHWA-RD-77 prepared for Federal Highway Administration, Washington, D. C., 1976.
42. Lee, E. H., "Elastic-Plastic Deformation at Finite Strains," J. Appl. Mech., Trans., ASME, V. 38, 1959.
43. Lee, E. H., and Liu, D. T., "Finite Strain Elastic-Plastic Theory with Application to Plane Wave Analysis," J. Appl. Physics, V. 8, No. 1, 1967.
44. Mahtab, M. A., and Goodman, R. E., "Three-Dimensional Finite Element Analysis of Jointed Rock Slopes," Proc. 2nd Conf. of Int. Soc. for Rock Mech., Belgrade, Yugoslavia, 1970.
45. Martin, H. C., "Derivation of Stiffness Matrices for the Analysis of Large Deflection and Stability Problems," Proc. 1st Conf. on Matrix Meth. in Struct. Mech., Wright-Patterson AFB, Ohio, 1966.
46. Matsuzaki, K., "Prediction of Earth Pressure in Retaining Structures," Ph.D. Thesis, University of New South Wales, 1978.
47. McMeeking, R. M. and Rice, J. R., "Finite Element Formulation for Problems of Large Elastic-Plastic Deformation," Int. J. of Solids and Structures, V. 11, 1975.
48. Murray, P. W., and Wilson, E. L., "Finite Element Large Deflection Analysis of Plates," J. Eng. Mech. Div., ASCE, V. 94, 1965.
49. Oden, J. T., Finite Elements of Nonlinear Continua, McGraw-Hill, N. Y. 1971.
50. Oden, J. T., "Finite Element Applications in Nonlinear Structural Analysis," Proc. Symp. on Application of Finite Element Method in Civil Eng., Knoxville, Tenn., 1969.

51. Oden, J. T., "Finite Element Formulation of Problems of Finite Deformation and Irreversible Thermodynamics of Nonlinear Continua - A Survey and Extension of Recent Developments," Proc. Recent Advances in Matrix Meth. of Struct. Anal. and Design, Univ. of Alabama Press, Al., 1971.
52. Osias, J. R., and Swedlow, J. L., "Finite Elasto-Plastic Deformation: Theory and Numerical Examples," Int. J. of Solids and Structures, V. 10, 1974.
53. Peterson, H., and Popov, E. P., "Substructuring and Equation System Solutions in Finite Element Analysis," Computers & Structures, V. 7, 1977.
54. Peterson, H., "Application of the Finite Element Method in the Analysis of Contact Problems," Proc. Int. Conf. on Finite Elements in Nonlin. Solids and Struct. Mech., Geilo, Norway, 1977.
55. Phan, H. V., Desai, C. S., Sture, S., and Perumpral, J., "Three-Dimensional Geometric and Material Nonlinear Analysis of Some Problems in Geomechanics," Proc. 3rd Int. Conf. on Num. Meth. in Geomechanics, Aachen, W. Germany, 1979.
56. Prevost, J. H., "Soil Stress-Strain-Strength Models based on Plasticity Theory," Ph.D. Dissertation, Stanford Univ., 1974.
57. Roscoe, K. H., and Burland, J. B., "On the Generalized Stress-Strain Behavior of Wet Clay," Engineering Plasticity, Ed. by Heyman and Leckis, Cambridge Univ. Press, 1958.
58. Roscoe, K. H., Schofield, A. N., and Wroth, C. P., "On Yielding of Soils," Geotechnique, V. 8, No. 1, London, 1958.
59. Sandler, L. S., DiMaggio, F. L., and Baladi, G. Y., "Generalized Cap Model for Geologic Materials," J. Geotech. Eng. Div., ASCE, V. 102, No. 2, 1976.
60. Sandler, L. S., and Rubin, D., "An Algorithm and a Modular Subroutine for the Cap Model," Int. J. for Num. and Anal. Meth. in Geomechanics, V. 3, 1979.
61. Schofield, A. N., and Wroth, C. P., Critical State Soil Mechanics McGraw-Hill, London, 1968.
62. Singh, R. D., "Mechanical Characterization of Finite Element Analysis of Elastic-Plastic, Work-Hardening Soils," Ph.D. Dissertation, Ohio State Univ., Columbus, Ohio, 1972.

63. Stricklin, J. A., Haisler, W. E., and Von Riesenmann, W. A., "Geometrically Nonlinear Structural Analysis by Direct Stiffness Method," J. Struct. Div., ASCE, V. 97, No. 9, 1971.
64. Stricklin, J. A., Von Riesenmann, W. A., Tillerson, J. A., and Haisler, W. E., "Static Geometric and Material Nonlinear Analysis," Proc. Advances in Comput. Meth. in Struct. Mech. and Design, 2nd US-Japan Seminar on Matrix Meth. Struct. Anal. Design, Univ. of Alabama Press, Ala., 1972.
65. Sture, S., "Development of a Multiaxial Cubical Test Device with Porewater Pressure Monitoring Facilities," Rep. No. VPI-E-79.18, prepared for National Science Foundation, VPI&SU, Blacksburg, VA., 1979.
66. Sture, S., Desai, C. S., and Janardhanam, R., "Development of a Constitutive Law for an Artificial Soil," Proc. 3rd Int. Conf. on Num. Meth. in Geomechanics, Aachen, W. Germany, 1979.
67. Toh, C. T., and Sloan, S. W., "Finite Element Analyses of Isotropic and Anisotropic Cohesive Soils with a View to Correctly Predict Impending Collapse," to be published in Int. J. for Num. and Anal. Meth. in Geomechanics.
68. Valliappan, S., Discussion, J. Soil Mech. & Found Div., ASCE, V. 95, No. 2, 1969.
69. Washizu, K., Variational Methods in Elasticity and Plasticity, Pergamon Press, N. Y., 1975.
70. Wilson, E. L., "Finite Elements for Foundations, Joints and Fluids," Finite Elements in Geomechanics, Ed. by Gudehus, G., John Wiley & Sons, N. Y., 1977.
71. Yaghmai, S., and Popov, E. P., "Incremental Analysis of Large Deflections of Shells of Revolution," Int. J. Solids and Structures, V. 7, No. 10, 1971.
72. Yamada, Y., "Incremental Formulation for Problems with Geometric and Material Nonlinearities," Proc. Advances in Comput. Meth. in Struct. Mech. and Design, 2nd US-Japan Seminar on Matrix Meth. Struct. Anal. Design, Univ. of Alabama Press, Ala, 1972.
73. Yamada, Y., Takatsuka, K., and Iwata, K., "Nonlinear Analysis by the Finite Element Method and Some Expository Examples," Proc. Theory and Practice in Finite Element Struct. Anal., Univ. of Tokyo Press, Japan, 1973.

74. Yamada, Y., and Wifi, A., "Large Strain Analysis of Some Geomechanics Problems by the Finite Element Method," Int. J. for Num. and Anal. Meth. in Geomechanics, V. 1, 1977.
75. Yamada, Y., Yoshimura, N., and Sakurai, T., "Plastic Stress-Strain Matrix and Its Application for the Solution of Elastic-Plastic Problems by the Finite-Element Method," Int. J. of Mech. Sciences, V. 10, 1968.
- ✓ 76. Zienkiewicz, O. C., The Finite Element Method, McGraw-Hill, N. Y., 1977.
- ✓ 77. Zienkiewicz, O. C., and Nayak, G. C., "A General Approach to Problems of Large Deformation and Plasticity Using Isoparametric Elements," Proc. 3rd Conf. on Matrix Meth. in Struct. Mech., Wright-Patterson AFB, Ohio, 1971.
78. Zienkiewicz, O. C., and Pande, G. N., "Some Useful Forms of Isotropic Yield Surfaces for Soil and Rock Mechanics," Finite Elements in Geomechanics, Ed. by Gudehus, G., John Wiley & Sons, N. Y., 1977.
79. Zienkiewicz, O. C., Valliappan, S., and King, I. P., "Elasto-Plastic Solutions of Engineering Problems, Initial Stress, Finite Element Approach," Int. J. of Num. Meth. in Eng., V. 1, 1969.
80. Zienkiewicz, O. C., Valliappan, S., and King, I. P., "Stress Analysis of Rock as a No-Tension Material," Geotechnique, V. 18, No. 1, 1968.
81. Zienkiewicz, O. C., Valliappan, S., and Dullage, C., and Stagg, K. G., "Analysis of Nonlinear Problems in Rock Mechanics with Particular Reference to Jointed Rock Systems," Proc. 2nd Conf. Int. Soc. for Rock Mech., Belgrade, Yugoslavia, 1970.

APPENDIX A

Appendix A

For an isoparametric hexahedral element, the matrices given in the finite element equation, Equation (2.59), are evaluated on the current configuration at load step n . Their explicit forms are shown as follows:

a. Conventional (small) strain-displacement matrix

$$[B_L^n] = \begin{bmatrix} N_{1,1} & 0 & 0 & \dots & N_{m,1} & 0 & 0 \\ 0 & N_{1,2} & 0 & \dots & 0 & N_{m,2} & 0 \\ 0 & 0 & N_{1,3} & \dots & 0 & 0 & N_{m,3} \\ N_{1,2} & N_{1,1} & 0 & \dots & N_{m,2} & N_{m,1} & 0 \\ 0 & N_{1,3} & N_{1,2} & \dots & 0 & N_{m,3} & N_{m,2} \\ N_{1,3} & 0 & N_{1,1} & \dots & N_{m,3} & 0 & N_{m,1} \end{bmatrix}$$

$(6 \times 3m)$

where

$$N_{i,j} = \frac{\partial N_i}{\partial x_j^n}$$

where i is from 1 to m , j from 1 to 3, and m is the total number of nodes of the element.

b. Nonlinear strain-displacement matrix:

$$[B_{NL}^n] = \begin{bmatrix} N_{1,1} & 0 & 0 & \dots & N_{m,1} & 0 & 0 \\ N_{1,2} & 0 & 0 & \dots & N_{m,2} & 0 & 0 \\ N_{1,3} & 0 & 0 & \dots & N_{m,3} & 0 & 0 \\ 0 & N_{1,1} & 0 & \dots & 0 & N_{m,1} & 0 \\ 0 & N_{1,2} & 0 & \dots & 0 & N_{m,2} & 0 \\ 0 & N_{1,3} & 0 & \dots & 0 & N_{m,3} & 0 \\ 0 & 0 & N_{1,1} & \dots & 0 & 0 & N_{m,1} \\ 0 & 0 & N_{1,2} & \dots & 0 & 0 & N_{m,2} \\ 0 & 0 & N_{1,3} & \dots & 0 & 0 & N_{m,3} \end{bmatrix}$$

(9 × 3m)

c. Constitutive matrix for a linear, isotropic, elastic material

$$[C] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ \text{sym.} & & & \frac{1-2\nu}{2} & 0 & 0 \\ & & & & \frac{1-2\nu}{2} & 0 \\ & & & & & \frac{1-2\nu}{2} \end{bmatrix}$$

The constitutive matrices for elasto-plastic materials are given by Equation (3.16).

d. Current stress matrix

$$[\sigma^n] = \begin{bmatrix} [\sigma_{ij}^n] & 0 & 0 \\ 0 & [\sigma_{ij}^n] & 0 \\ 0 & 0 & [\sigma_{ij}^n] \end{bmatrix}$$

(9 × 9)

where

$$[\sigma_{ij}^n] = \begin{bmatrix} \sigma_{11}^n & \sigma_{12}^n & \sigma_{13}^n \\ \sigma_{21}^n & \sigma_{22}^n & \sigma_{23}^n \\ \sigma_{31}^n & \sigma_{32}^n & \sigma_{33}^n \end{bmatrix}$$

where

$$\sigma_{ij}^n = \sigma_{ji}^n$$

3. Current stress vector:

$$\{\sigma^n\} = \begin{Bmatrix} \sigma_{11}^n \\ \sigma_{22}^n \\ \sigma_{33}^n \\ \sigma_{12}^n \\ \sigma_{23}^n \\ \sigma_{31}^n \end{Bmatrix}$$

APPENDIX B

Appendix B

The differences among various elasto-plastic constitutive laws are the yield criteria and the work-hardening rules to define these laws. For the laws currently used in research, there are different parameters which must be properly extracted from various tests in the laboratory.

1. Von Mises Model

To define this model the yield criterion is given as:

$$f = J_{2D}^{1/2} - k = 0 \quad (B.1)$$

where k is a constant yield stress and is determined from the cohesion of the material. The constitutive matrix for this model is given by Equation (3.16) where the term $\frac{\partial f}{\partial \epsilon_{kk}^p} \frac{\partial f}{\partial \sigma_{kk}}$ is dropped out (elasto-perfectly plastic model).

2. Drucker-Prager Model

This is a modification of Mohr-Coulomb law for granular material, and the yield criterion is defined by

$$f = \alpha J_1 + J_{2D}^{1/2} - k = 0 \quad (B.2)$$

where α is a constant parameter and is a function of the angle of internal friction of the material.

In soil mechanics, α and k are determined from c and ϕ , the cohesion and the angle of friction, for conventional triaxial test and direct shear test. They are given as [10].

	Axisymmetric Case	Plane Strain Case
α	$\frac{2 \sin \phi}{\sqrt{3}(3-\sin \phi)}$	$\frac{\tan \phi}{\sqrt{9+12 \tan^2 \phi}}$
k	$\frac{6c \cos \phi}{\sqrt{3}(3-\sin \phi)}$	$\frac{3c}{\sqrt{9+12 \tan^2 \phi}}$

3. Critical State Models

There are two types of critical state models, namely the Cam-Clay model and the modified Cam-Clay model. The parameters used to determine these models are often obtained from the conventional tri-axial compression test in drained and undrained conditions. The critical state or failure criterion is defined as

$$f_F = q - Mp = 0 \quad (\text{B.3})$$

where

$$q = \sigma_1 - \sigma_3 \quad (\text{B.4})$$

is the deviatoric stress, and

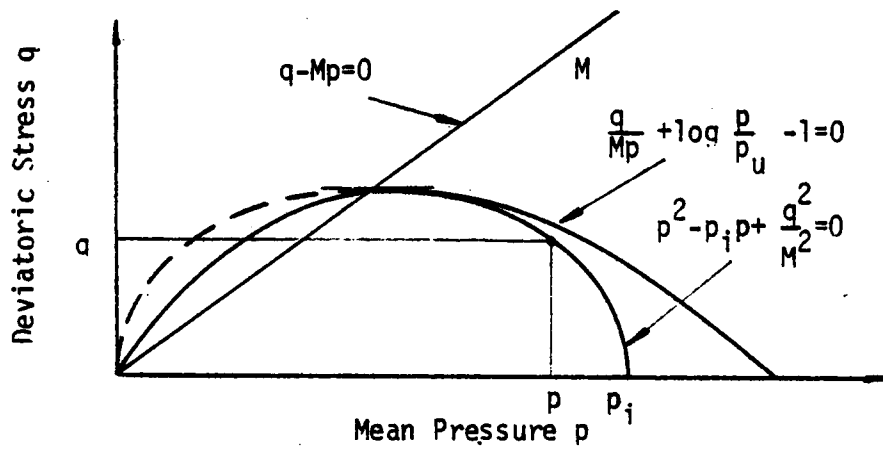
$$p = \frac{\sigma_1 + 2\sigma_3}{3} \quad (\text{B.5})$$

is the effective mean pressure.

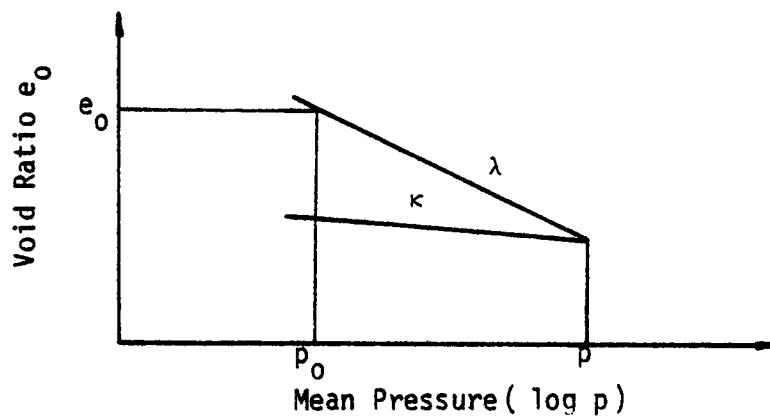
M is the slope of the failure line, and is given by the Mohr-Coulomb criterion as

$$M = \frac{6 \sin \phi}{3 - \sin \phi}$$

The yield surface for the cap is [57]



a. Critical State Models



b. Volumetric Strain defined from Consolidation Test

Figure B.1. Critical State Models and Parameters

$$f = \frac{q}{Mp} + \log \frac{p}{p_u} - 1 = 0 \quad (\text{B.6})$$

for Cam-Clay model, and

$$f = p^2 - p_i p + \frac{q^2}{M^2} = 0 \quad (\text{B.7})$$

for modified Cam-Clay model.

p_i is the initial mean pressure, Figure (B.1a), and $p_u = p_i/2.7182$.

The hardening rule is defined as a function of only the plastic volumetric strain, and is given by

$$\epsilon_{ii}^p = - \frac{\lambda - \kappa}{(1 + e_0)} \log\left(\frac{p}{p_0}\right) \quad (\text{B.8})$$

where λ and κ are, respectively, the compression index and the swelling index obtained from a consolidation test, Figure (B.1b), e_0 is the initial void ratio, and p_0 is the initial mean pressure corresponding to e_0 before the load is applied.

4. Cap Models

Cap model proposed by DiMaggio et al [21] has the failure surface defined by

$$f_F = J_{2D}^{1/2} - A + C e^{\frac{BJ_1}{2}} = 0 \quad (\text{B.9})$$

and the cap surface defined by

$$f_C = \left(\frac{p_e - J_1}{p_e - p_i}\right)^2 + \left(\frac{J_{2D}^{1/2}}{A - C e^{\frac{BJ_1}{2}}}\right)^2 - 1 = 0 \quad (\text{B.10})$$

where A, B, and C are determined from triaxial test data, and p_e and p_i are shown in Figure (B.2).

The cap is assumed to have an elliptical shape with a constant eccentricity R. The plastic volumetric strain is a function of the initial mean pressure and defined as

$$\epsilon_{ii}^p = W(e^{Dp_i} - 1) \quad (B.11)$$

where W and D are obtained from the hydrostatic compression test data.

The cap model used in current research differs from the DiMaggio et al's model in the definition of the failure surface [66]. The new failure surface is defined as

$$f_F = J_{2D}^{1/2} - A - \beta J_1 + Ce^{\beta J_1} = 0 \quad (B.12)$$

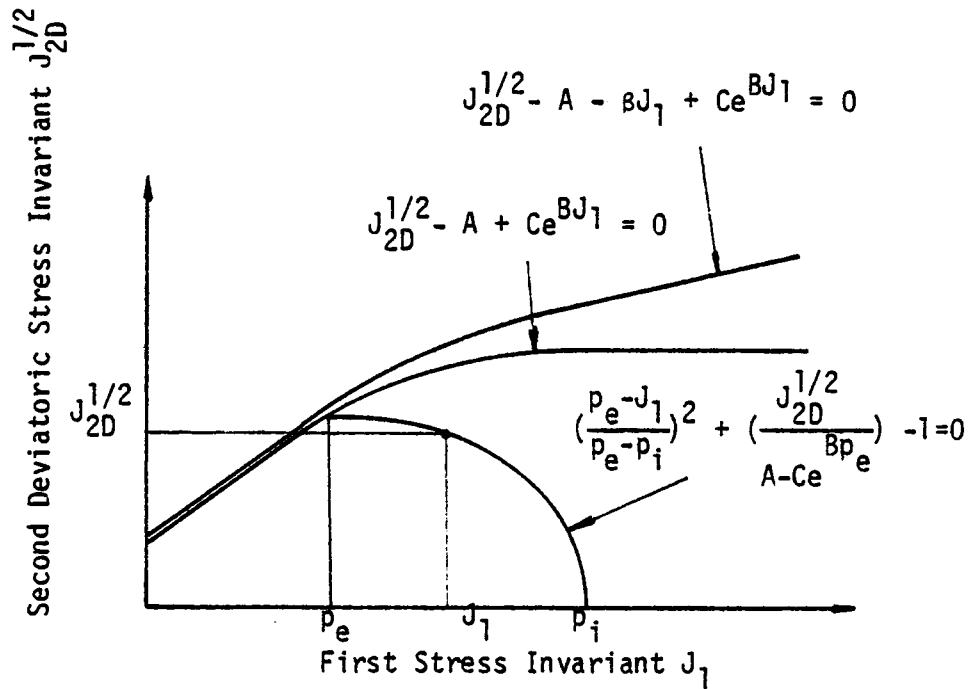
where β is a new parameter which defines the failure line at high load level as a Drucker-Prager line.

A detailed description of the parameters used in the cap models and the algorithms to incorporate these models in a finite element program have been given elsewhere [21, 60, 66], hence will not be repeated here.

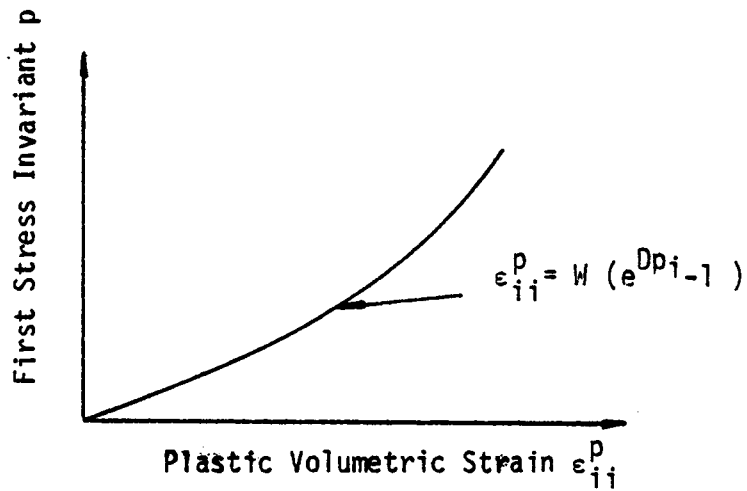
5. Interface Constitutive Model

The constitutive behavior of the interface element is often defined by the Mohr-Coulomb criterion, in which the slip mode is to occur if

$$\tau = c + \sigma_n \tan \phi \quad (B.13)$$



a. Cap Models

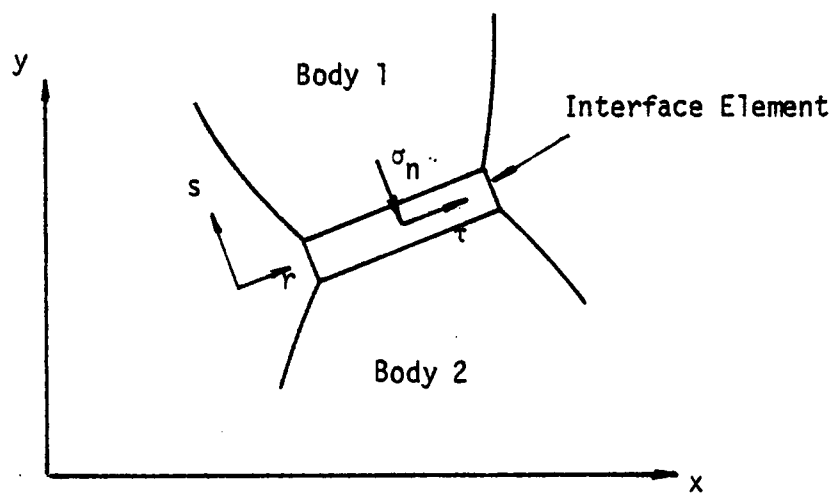


b. Plastic Volumetric Strain defined from Hydrostatic Compression Test

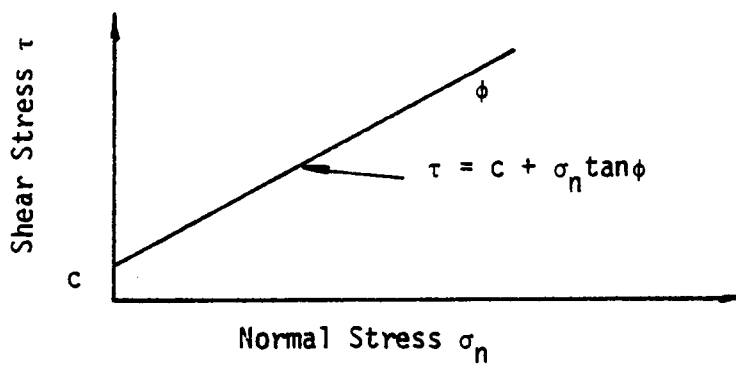
Figure B.2. Cap Models and Parameters

where τ is the shearing stress, σ_n is the normal stress, c is the internal cohesion or frictional resistance of the two solid bodies and ϕ is the angle of friction.

Equation (B.13) is further illustrated in Figure (B.3). σ_n must be in compressive state in order that Equation (B.13) is held, otherwise there exists a gap separating the solid body to the other adjacent body.



a. Stresses in Interface Element



b. Interface Constitutive Parameters defined from Direct Shear Test

Figure B.3. Constitutive Behavior of Interface Element

APPENDIX C

Appendix C

In order to compare the solutions predicted by the capped work-hardening model and the results obtained from different tests on a cubical testing device [65], some finite element analyses were performed. The trial mesh has only one cubical element with the dimensions and boundary conditions as shown in Figure (C.1).

The purpose of using this simple mesh is to approximately model the behavior of the sample of dimension 4in x 4in x 4in used in the device. The boundary conditions are appropriate under any test conditions.

Figure (C.2a) shows the prediction of the finite element analysis and the experimental solution for a hydrostatic compression test. As expected, the comparisons are excellent. This is due to the fact that the parameters used to define the plastic volumetric strain, Appendix B, are taken from the data of this kind of test. The stress path is marching along the hydrostatic axis, Figure (C.2a) where the volumetric change is basically computed on; and there is no effect of shearing.

Figure (c.2b) shows the numerical prediction and the experimental data for a conventional triaxial compression test using the cubical device. In the test, the sample is uniformly compressed to a certain hydrostatic stress level, then the load in a particular direction is increased until failure occurs. In the finite element analysis, the element is assumed to be in a compressive state by specifying the position of the initial cap is at that state. Incremental prescribed displacements are then applied in the negative z direction,

Figure (C.1). The stress path for this test is shown in Figure (C.2b). The comparisons are fairly well. The expansion of the cap following subsequent increases of the compressive stress is monotonic, and the volumetric change approximately resembles the volumetric change in a hydrostatic compression condition. The effect of shearing in determining the plastic strain vector, on the other hand, cannot be well controlled by the cap model. Hence the prediction results in such a difference to the observed solution.

Figure (c.3a) shows the results for a conventional triaxial extension test. The stress path of this test is given in Figure (C.3a). To model this type of test, displacements are incrementally applied in the positive z-direction. The comparisons are very poor. Although the cap model is capable of handling unloading situation by contraction, it hardly predicts the real behavior of the artificial soil in the similar situation for this type of test. The soil fails at small loads as predicted by the cap model, whereas in reality, the strength of the soil is significantly higher.

Figure (C.3b) shows the results for a simple shear test. The stress path is given again in Figure (C.3b). The comparisons are fair. The difference could be due to in shearing mode, the parameters used in determining the plastic volumetric change become insignificant.

The artificial soil is a very complicated material. It is highly dependent on the state of stress and the stress path. Hence the prediction of the cap model is also dependent on the condition of loading.

The cap ratio or the ellipse eccentricity R , as well as other parameters, varies in different types of tests. The values taken from specific tests might not be accurate in predicting other tests. But overall, the cap model is the best choice among other models in predicting the behavior of the artificial soil under any load conditions. Further works about modifying the present cap model are in progress. Additional details can be found in References [16, 65, 66].

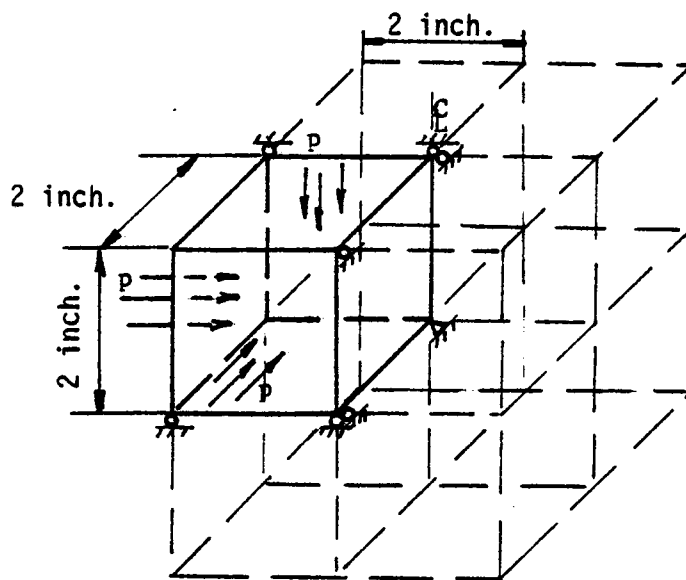
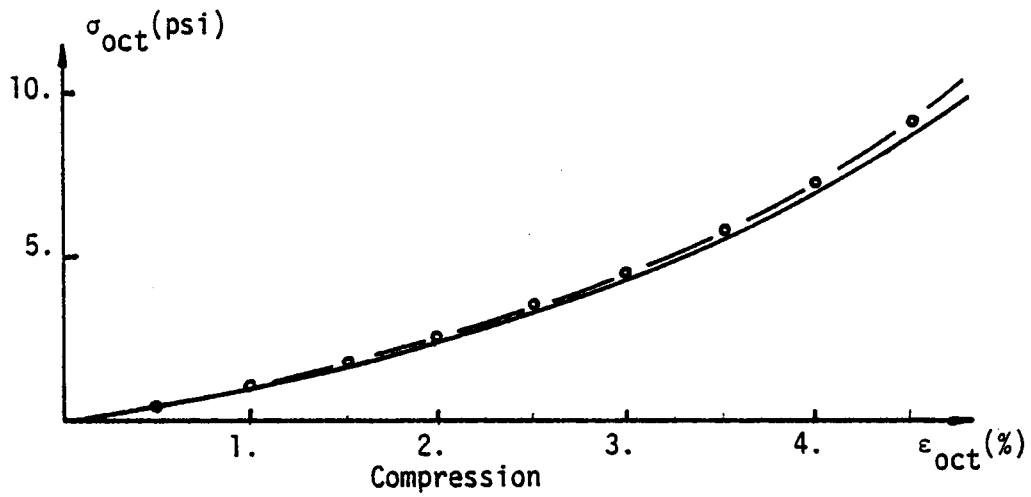
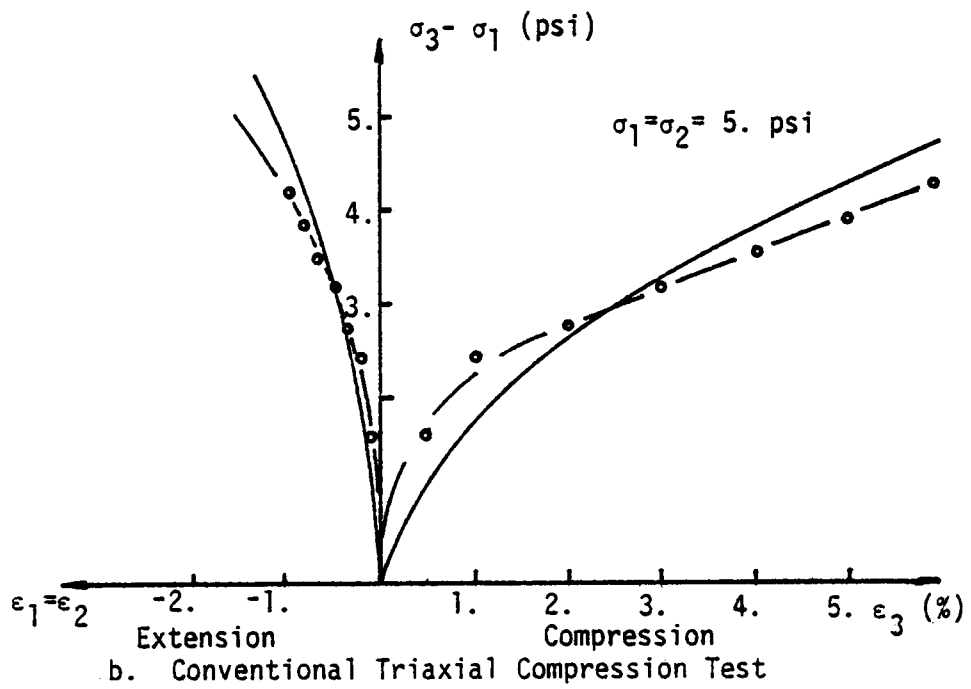


Figure C.1. Finite Element Mesh to Model the Cubical Sample

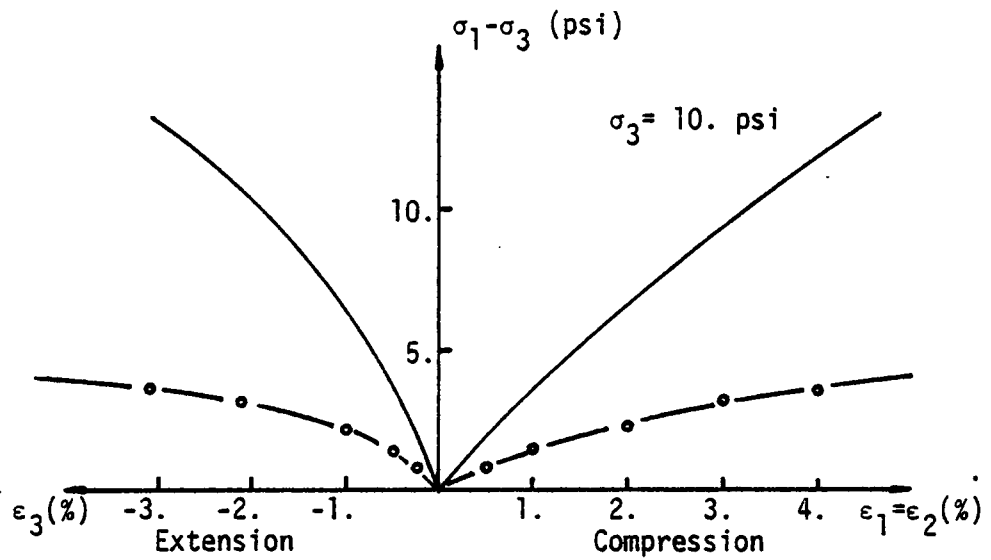


a. Hydrostatic Compression Test

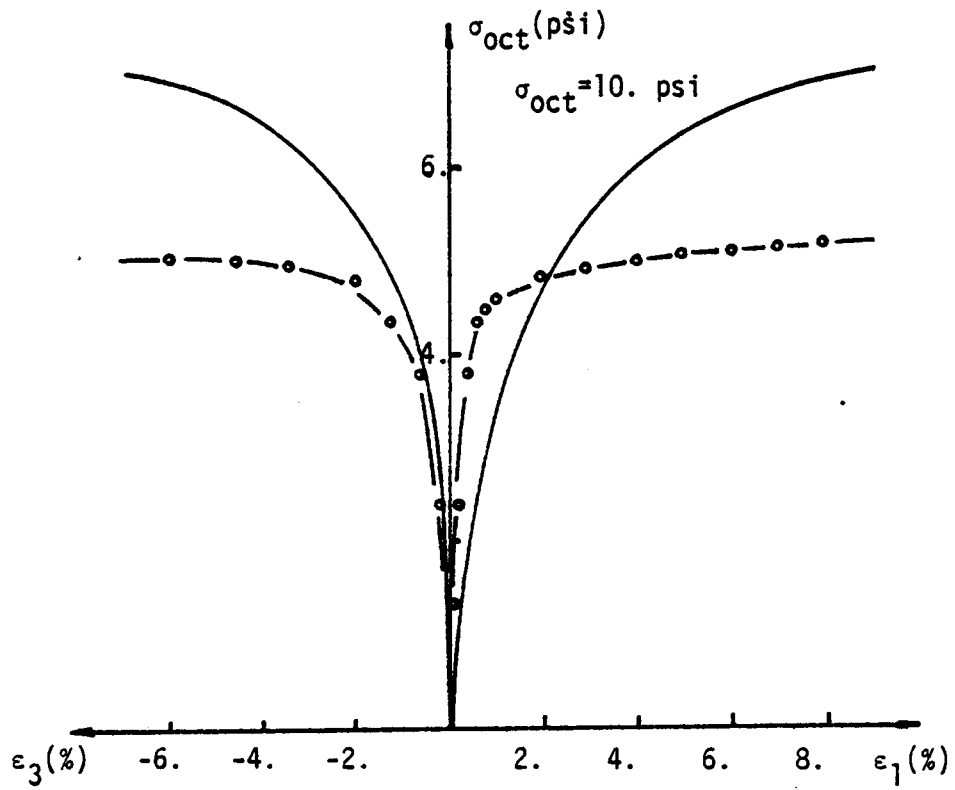


b. Conventional Triaxial Compression Test

Figure C.2. Stress-Strain Relationship



a. Conventional Triaxial Extension Test



b. Simple Shear Test

Figure C.3. Stress-Strain Relationship

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GEOMETRIC AND MATERIAL NONLINEAR ANALYSIS
OF THREE-DIMENSIONAL SOIL-STRUCTURE INTERACTION

by

Hoang Viet Phan

(ABSTRACT)

A finite element procedure is developed for stress-deformation analysis of three-dimensional solid bodies including geometric and material nonlinearities. The formulation also includes the soil-structure interaction effect by using an interface element. A scheme is formulated to allow consistent definitions of stress, stress and strain rates, and constitutive laws. The analysis adopts the original Newton-Raphson technique coupled with incremental approach. Different elasto-plastic laws based on Von-Mises, Drucker-Prager, critical state, and cap criteria are incorporated in the formulation and computer code, and they can be used depending on the geological material involved. A special cap model is also incorporated to predict the behavior of the artificial soil used in current research. Examples are given to verify the formulation and the finite element code. Examples of the problems of soil-moving tool are also shown to compare to the experimental solutions observed in a prototype soil-bin test facility.