AXISYMMETRIC AND NON-AXISYMMETRIC MODES OF INSTABILITY FOR FLOW BETWEEN ROTATING CYLINDERS WITH A TRANSVERSE PRESSURE GRADIENT

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A	constant defined by equation (9)
a	non-dimensional longitudinal wave number
a_c	critical value of the non-dimensional longitudinal wave number
B	constant defined by equation (10)
C	constant defined by equation (15)
C_i	coefficient vector of equations (73)
D	differential operator $\frac{d}{d\zeta}$
d	gap width of annulus $=R_2-R_1$
E	constant defined by equation (16)
$F(\zeta)$	function defined by equations (31) and (47)
H	differential operator defined by equation (23)
h	step size on independent variable ζ in numerical integration process
k	longitudinal wave number
L	differential operator defined by equation (48)
m	circumferential wave number
m_c	critical value of the circumferential wave number
N	differential operator defined by equation (24)
p	pressure
R_1	radius of inner surface of annulus
R_2	radius of outer surface of annulus

r, θ, z	cylinderical coordinates
T	Taylor number
T_c	critical value of the Taylor number
t	time
u_r	radial component of fluid velocity
$u(r)$	component of radial velocity perturbation due to variation of r
u_θ	circumferential component of fluid velocity
u_z	axial component of fluid velocity
V	initial steady state fluid velocity
V_m	function defined by equation (37)
$v(r)$	component of circumferential velocity perturbation due to variation of r
$w(r)$	component of axial velocity perturbation due to variation of r
Y_i	variables defined by equation (67)
Y_i^j	variables defined by equation (72)
δ	ratio of gap width to inner radius $(\frac{d}{R_1})$
δ_{ij}	Kronecker delta
ζ	independent variable in integration process defined by equation (32)
η	ratio of the radius of the inner cylinder to the radius of the outer cylinder $(\frac{R_1}{R_2})$

λ	function defined by equation (58)
Λ	function defined by equation (87)
μ	ratio of angular velocity of the outer cylinder to angular velocity of the inner cylinder $(\frac{\Omega_2}{\Omega_1})$
ν	kinematic viscosity
ρ	density
σ_i	non-dimensional frequency of perturbations
σ_r	non-dimensional growth ratio of perturbations
σ	$\sigma=\sigma_r+i\sigma_i$
σ_c	critical value of σ
ϕ	Rayleigh discriminant defined by equation (84)
Ω	angular velocity
ω_i	frequency of perturbations
ω_r	growth rate of perturbations
ω	$\omega=\omega_r+i\omega_i$
$\tilde{\omega}$	pressure perturbation divided by density $(\frac{\delta p}{\rho})$

V. INTRODUCTION

Basic Concepts of Fluid Stability

The equations of hydrodynamics, in spite of their complexity allow some simple patterns of flow as time independent or stationary solutions. In these flows every fluid particle moves along a given streamline. The flow is well-ordered and particles travel along neighboring layers (laminar flow). These patterns of flow can, however, be realized only for certain ranges of the parameters characterizing the system. Outside these ranges these simple patterns of flow may cease to exist with a strong mixing of all the particles occurring (turbulent flow) or the flow may change to another more complex stationary flow. The reason for this lies in the inherent instability of the flow, i.e., the inability of the flow to sustain itself against small perturbations to which any physical system is subject. It is in the differentiation of the stable from the unstable patterns of permissible flows that the problem of hydrodynamic stability originates.

Mathematically, the problem is as follows; suppose that the system of hydrodynamic equations has a stationary solution. Let X_1, X_2,----X_j be a set of parameters which define the system. These parameters will include geometrical parameters such as the dimensions of the system; parameters characterizing the velocity field which may prevail in the system; the magnitudes of the forces which may be acting on the system, such as pressure gradients, temperature gradients, magnetic fields and rotation; and others.

In considering the stability of such a system (with a given set of parameters) we essentially seek to determine the reaction of the system when subjected to small disturbances. The decisive question to be answered is whether the disturbances increase or decrease in magnitude with the passing of time. If the disturbances decay with time so that the system approaches the original steady state condition as time $\rightarrow \infty$ then the flow is considered stable. On the other hand, if the disturbances increase in magnitude with time so that the flow progressively departs from the steady state condition and never reverts to it, the flow is considered unstable. It should be pointed out that instability does not necessarily lead to turbulent flow; it can lead to another state of laminar flow.

If all initial states are classified as stable or unstable according to the criteria stated, then the locus of conditions which separates the two classes of states defines the state of marginal stability of the system. It is reasonable to assume that a marginal state is a state of neutral stability. The determination of the locus of the marginal states is one of the prime objects of an investigation of hydrodynamic instability.

States of marginal stability can be one of two kinds. The two kinds correspond to the two ways in which the amplitude of a small disturbance can grow or be damped. The disturbances can grow (or be damped) aperiodically, or they can grow (or be damped) by oscillations of increasing (or decreasing) amplitude. In the former case, the transition from stability to instability takes place via a marginal state exhibiting a stationary pattern of flow. In the latter case, the transition takes place via a marginal state exhibiting oscillatory motion with a definite characteristic frequency. Accordingly, in this case, the theory must yield not only what mode of disturbance will be manifest at the onset of instability but also the characteristic frequency of the oscillation.

If at the onset of instability a stationary pattern of motion prevails, then one says that the principle of the exchange of stabilities is valid and that instability sets in as stationary cellular convection or secondary flow. On the other hand, if at the onset of instability, oscillatory motion prevails, then one says that a case of overstability exists.

In classifying marginal states into two classes --- stationary and oscillatory --- a dissipative system has been assumed. In a non-dissipative, conservative system the situation is somewhat different. In this case the stable states, when perturbed, execute undamped oscillations with certain definite characteristic frequencies; while in the unstable states small perturbations tend to grow exponentially with time; and the marginal states themselves are stationary.

The mathematical treatment of a problem in stability generally proceeds along the following lines. We start from an initial flow which represents a stationary state of the system. By supposing that the various physical variables describing the flow suffer small (infintesimal) perturbations, we first obtain the equations governing these perturbations. In obtaining these equations all products and powers of the disturbances and their derivatives are neglected and only the terms linear in them are retained. The theory is thus a linear stability theory. Solutions of these equations must then be sought which satisfy certain necessary boundary conditions. In general the equations will not allow non-trivial solutions for an arbitrary assigned set of system parameters. Indeed, the requirement that the equations allow non-trivial solutions satisfying the various boundary conditions leads directly to a characteristic value problem for the parameters characterizing the system.

Stability means stability with respect to all possible (infinitesimal) disturbances. Accordingly, for an investigation to be complete, it is necessary that the reaction of the system to all possible disturbances be examined. These disturbances may be symmetrical with respect to some axis of symmetry of the problem or it may be necessary to consider the more general case of non-axisymmetric disturbances. In practice this is accomplished by expressing an arbitrary disturbance as a superposition of certain basic modes and examining the stability of the system with respect to each of these modes. In accordance with the theory, when an unstable flow condition exists small disturbances which fall within a certain range of frequency and wavelengths are amplified, whereas, disturbances of smaller or larger wavelengths are damped.

Problem to be Investigated

The stability of the flow of a viscous incompressible fluid between concentric cylinders is analyzed, for the case in which the basic velocity distribution is the sum of a velocity distribution due to the rotation of the cylinders and a "pumping" velocity distribution due to a pressure gradient acting round the cylinders. It is assumed that the spacing between the cylinders is small. Previous theoretical investigations of this problem have assumed axisymmetric disturbances and that the critical mode of instability was of a stationary cellular motion. Experimental results indicate that the stationary axisymmetric modes are probably the critical modes for most combinations of system parameters; however, some recent results suggest the possibility of axisymmetric oscillatory modes or non-axisymmetric modes being the critical mode of instability for some regions of parameters.

The instability for this problem results from a potentially unstable arrangement of flow resulting from an adverse gradient of angular momentum. For the case of an inviscid fluid, Rayleigh [16] has shown that the necessary and sufficient condition for a distribution of angular velocity $\Omega(r)$ to be stable is

$$\frac{d}{dr}(r^2\Omega)^2 \; > \; 0$$

everywhere in the interval, and further, that the distribution is unstable if $(r^2\Omega)^2$ should decrease anywhere inside the interval.

No such simple conclusion can be obtained for the viscous fluid. For the viscous fluid it might be expected that the effect of viscosity will be to postpone the onset of stability beyond the point predicted by the Rayleigh criterion. The extent to which Rayleigh's criterion can be violated before instability will manifest itself presents a formidable problem which has been the subject of much investigation.

This investigation will essentially consist of three parts: (1) An extension of the present results for axisymmetric stationary modes to include a wider range of system parameters, (2) An investigation of the possibility of axisymmetric oscillatory modes which might be the critical mode of instability, and (3) An investigation of non-axisymmetric modes of instability. Most of the work will be restricted to the physically most interesting case where the outer cylinder is at rest. There are a number of applications, particularly in chemical engineering (see [2]), which essentially consist of a large cylinder which rotates inside a container while the fluid in the annulus is also subjected to pressure gradients. In many of these cases the onset of instability within the annulus may adversely affect the operation of the system or the quality of the final product. Thus, a knowledge of the onset of instability for a wide range of system parameters is highly desirable.

VI. REVIEW OF PERTINENT LITERATURE

The stability of the flow of a Newtonian fluid between concentric rotating cylinders was first considered experimentally and theoretically by Taylor [19]. Taylor assumed that the critical mode of instability would result from axisymmetric disturbances imposed on the original steady flow and that the marginal state exhibited a stationary form. Using linearized theory and the additional assumption that the spacing between the cylinders is small compared to the mean radius (small gap assumption), he obtained a criterion for the onset of instability which could be attributed to a stationary secondary motion with a cellular form (cellular vortex). The theoretical results were verified by his experiments. The linearized problem for the stability of Couette flow with respect to axisymmetric disturbances leads to an eigenvalue problem for the determination of the critical speed of the inner cylinder; the latter appearing in the form of the Taylor number T (based on the speed of the inner cylinder and containing a geometric factor representing the curvature effect), which is a function of the parameters $\mu=\Omega_2/\Omega_1$ $\eta=R_1/R_2$, and the dimensionless longitudinal wave number of the disturbances. Here Ω_1, Ω_2 and R_1, R_2 are the angular velocities and the radii of the inner and outer cylinder respectively.

This problem has been discussed by numerous authors (see for example [3] and [10]) attempting to develop a more elegant and practical technique for solving the eigenvalue problem and with computations for a wider range of parameters than originally considered by Taylor. The natural assumption that axisymmetric disturbances lead to the critical mode of instability has been well verified by experimental observations for positive values of μ but recent analytical results in the literature ([14] and [17]) indicate that for μ sufficiently negative the critical speed for Couette flow may occur for non-axisymmetric disturbances. The experimental observations in regard to this point are somewhat contradictory. Taylor's observations for the three sets of cylinder η=0.74, 0.88 and 0.94 and for a wide range of positive and negative values of μ indicate that the original instability leads to a symmetric stationary flow. However, he does point out that while for μ numerically less than a certain positive number (which appears to vary with η) the vortex motion is stable with increasing speed of the inner cylinder; for $\eta \geq 0.88$ and $\mu < -1$, on the other hand, the symmetric vortex flow was found to break up shortly after formation. The observations of Donnelly and Fultz [9] for the case η=0.5 are in general agreement with those of Taylor. They note that at values of μ between -0.2 and -1.0 the symmetric vortex cells break up spontaneously within less than a minute of formation.

Nissan [17] using a set of cylinders with η=0.85 has measured the critical speed for Taylor vorticies and the second critical speed at which this flow breaks down into a non-axisymmetric motion. With decreasing μ the value of the second critical speed approaches that for Taylor vorticies, the two points coinciding at μ=-0.73. Below this the non-axisymmetric motions appear to be the critical mode of instability. The recent analytical results of Krueger, Gross and

DiPrima [14] indicate that the critical value of μ is approximately -0.78, above this the critical disturbance is axisymmetric and below this it is non-axisymmetric.

Coles [5], using an apparatus with η=0.88 (but with a short axial length compared to that of most experimenters), has observed the onset of instability with a stationary flow (Taylor vorticies periodic in the axial direction) and a second critical speed at which there appears travelling waves in the circumferential direction (also periodic in the axial direction). In contrast to the observations of Nissan, Coles observes that the onset of instability with an oscillatory form lies above the Taylor instability for $\mu \approx -1$, though extrapolating his results it is possible that the boundaries may cross at $\mu \approx 3$. However he noted that for opposite rotation of the two cylinders a weak spiral configuration appears to be superimposed on the Taylor instability except at low Reynolds numbers (based on the speed of the outer cylinder). Coles' work contains some very interesting photographs of the onset of instability of the two different forms. The visual observations were obtained by observing the motion of fine aluminum particles suspended in the fluid.

In part these differences in experimental observations may be due to the different geometries, and to the different methods of visualization or measurement. Additional experimental work is necessary in this area to resolve these apparent contradictions.

A similar type of instability occurs when a viscous fluid flows in the curved channel formed by concentric cylinders under a pressure gradient acting round the cylinders. This problem was considered originally by Dean [6]. Using assumptions similar to those made by Taylor, he obtained a criterion for the onset of instability due to a stationary cellular motion similar to that obtained by Taylor. The required assumptions are linearized theory, axisymmetric disturbances, small gap assumption and a stationary marginal state.

The linearized problem for the stability of this flow through a curved channel under a pressure gradient acting round the channel leads to an eigenvalue problem for the critical mean velocity in the channel; the latter appearing in the form of the Reynolds number of the mean flow. This problem has also been studied by other authors with results verifying those of Dean. The possibility of the onset of instability being other than as a stationary secondary motion has not been considered in the literature.

The combination of the problem of Taylor and that of Dean has been of recent interest. In this problem a viscous fluid between concentric rotating cylinders is also subjected to a pressure gradient round the annulus. Most of the work has been restricted to the case of a small gap between cylinders, linearized theory, axisymmetric disturbance and a stationary marginal state. The primary interest has also been restricted to the case where the outer cylinder is at rest ($\mu=0$). The theoretical conclusions of Brewster, Grosberg and Nissan [1] are based on the use of the necessary condition for instability that the square of the circulation should decrease outward from the inner cylinder. The eigenvalue problem is explicitly solved only in the case when the average velocity of rotation is equal to, but opposite in direction to the average pumping.

DiPrima [7] has solved the combined problem by a method originally suggested by Chandrasekhar [3] in connection with his treatment of the Taylor problem. In this method one chooses a representation for the solution in the form of an infinite series of orthogonal functions which automatically satisfy two of the boundary conditions; the remaining boundary conditions are then satisfied by integrating a related lower order equation. In this way the required characteristic equation is obtained in the form of a determinate of infinite order. In practice only a small number of terms are retained in the expansion, but even so the amount of labor involved, both analytical and numerical, can become quite large. This method can in principle be used to obtain the eigenfunctions in addition to the eigenvalues of the problem; however, the labor involved is very great and this has not been done by DiPrima. These results were also not extended to include the condition where the velocity distribution is much larger (both negative and positive) than the contribution due to the rotation of the cylinders. The problem was solved subject to the small gap assumption, axisymmetric disturbances, linearized theory and a stationary marginal state.

Hughes and Reid [13] have applied the method of direct numerical integration described by Duty and Reid [10] to the combined problem. However, their investigation is confined to a very small range of system parameters and the same general assumptions as made by DiPrima.

The results of their calculations do, however, imply the possible existance of regions where the critical mode of instability is not of a stationary form.

Very few experimental results are available in the literature for this problem. Brewster, Grosberg and Nissan [1] have investigated the problem experimentally with satisfactory agreement with the theoretical calculations, although the experimental apparatus used does not exactly reproduce the analytical model. There were also some combinations of parameters for which difficulty was encountered in the determination of the onset of instability.

The possibility of oscillatory or non-axisymmetric modes of instability have not been previously investigated for this problem. Such an investigation is necessary to complete the mathematic analysis of the problem and possibly to suggest areas for additional experimental work.

VII. THE INVESTIGATION

Governing Equations

Historically the problem to be investigated developed as a combination of the problem of flow between concentric rotating cylinders (the Taylor problem) and that of flow between concentric (but non-rotating) cylinders with a pressure gradient acting round the annulus (the Dean problem). The governing equations for this problem will be derived from the viewpoint of this natural development. The basic equations required in the consideration of this problem are the Navier-Stokes equations and the continuity equation (four equations and four unknowns).

Let (r, θ, z) be a set of cylindrical coordinates, with the z axis coinciding with the axis of the cylinders, and let R_1, R_2, Ω_1 and Ω_2 denote the radii and the angular velocities of the inner and outer cylinders, respectively. Denote the components of velocity in the increasing r, θ, and z direction by u_r, u_θ and u_z and let p denote the pressure.

In cylindrical polar coordinates the Navier-Stokes equations for viscous incompressible fluids take the form

$$\frac{\partial u_r}{\partial t} + \left(u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}\right)u_r - \frac{u_\theta^2}{r} = -\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) + \nu\left[\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right], \qquad (1)$$

$$\frac{\partial u_\theta}{\partial t} + (u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z})\, u_\theta + \frac{u_r u_\theta}{r} = - \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{p}{\rho}) +$$

$$\nu\, [\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta}{r^2}], \qquad (2)$$

and

$$\frac{\partial u_z}{\partial t} + (u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}) u_z = - \frac{\partial}{\partial z}(\frac{p}{\rho}) + \nu \nabla^2 u_z, \qquad (3)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

The additional governing relationship is the equation of continuity which reduces to

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \qquad (4)$$

The assumptions for the Taylor problem are

$$u_r = u_z = 0$$

and

$$u_\theta = V(r).$$

Using these conditions in the governing equations, the equations are satisfied and a stationary solution is possible if p is not a function of θ and if

$$\frac{d}{dr}(\frac{p}{\rho}) = \frac{V^2}{r} \qquad (5)$$

and

$$\nu[\nabla^2 V - \frac{V}{r^2}] = \nu\frac{d}{dr}(\frac{d}{dr} + \frac{1}{r})\ V = 0. \quad (6)$$

Solving equation (6), it is found that the most general form for the velocity distribution is

$$V = Ar + \frac{B}{r}, \quad (7)$$

where A and B are constants related to the angular velocities Ω_1 and Ω_2 with which the two cylinders are rotating. These constants can be determined by applying the boundary conditions which require that the fluid and cylinder have the same velocity at the contact surface; or

$$V = \Omega_1 R_1 \qquad \text{at } r = R_1$$

and (8)

$$V = \Omega_2 R_2 \qquad \text{at } r = R_2.$$

Applying the boundary conditions and solving the resulting system of algebraic equations yields

$$A = -\frac{\Omega_1 \eta^2 [1 - \frac{\mu}{\eta^2}]}{1-\eta^2} \quad (9)$$

and

$$B = \frac{\Omega_1 R_1^2 (1-\mu)}{1-\eta^2}, \quad (10)$$

where

$$\eta = \frac{R_1}{R_2} \quad \text{and} \quad \mu = \frac{\Omega_2}{\Omega_1}. \quad (11)$$

The assumptions used in Dean's problem are that

$$u_r = u_z = 0, \quad u_\theta = V(r),$$

and

$$\frac{\partial p}{\partial \theta} = \text{constant}.$$

Substituting these conditions into the governing equations, a stationary solution is allowed provided that

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{V^2}{r} \tag{12}$$

and

$$\nu\left[\nabla^2 V - \frac{V}{r^2}\right] = \nu\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rV)\right] = \frac{1}{\rho r}\frac{\partial p}{\partial \theta}. \tag{13}$$

The general solution of equation (13) is

$$V = \frac{1}{2\rho\nu}\left(\frac{\partial p}{\partial \theta}\right) r \ln r + Cr + \frac{E}{r}. \tag{14}$$

Upon applying the boundary conditions which require that V vanishes at the inner and outer cylinder, C and E can be evaluated in terms of the physical parameters of the system:

$$C = -\frac{1}{2\rho\nu}\left(\frac{\partial p}{\partial \theta}\right)\frac{R_2^2 \ln R_2 - R_1^2 \ln R_1}{R_2^2 - R_1^2} \tag{15}$$

and

$$E = \frac{1}{2\rho\nu}\left(\frac{\partial p}{\partial \theta}\right)\frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\ln\frac{R_2}{R_1}. \tag{16}$$

The problem to be investigated can be thought of physically as a combination of the problems of Taylor and Dean. The concentric cylinders are allowed to rotate causing a velocity distribution and there is also a pressure gradient round the annulus between the cylinders introducing an additional velocity distribution. This problem imposes the conditions that

$$u_r = u_z = 0, \quad u_\theta = V(r)$$

and

$$\frac{\partial p}{\partial \theta} = \text{constant.}$$

By substitution into the governing equations, it can be verified that the velocity distribution obtained from the sum of the velocity distributions obtained in the Taylor and Dean problem is also a possible steady state solution. Thus, the most general form of the velocity distribution is

$$V = \frac{1}{2\rho\nu}\left(\frac{\partial p}{\partial \theta}\right) r \ln r + Cr + \frac{E}{r} + Ar + \frac{B}{r} . \qquad (17)$$

Using the stationary solution for the problem represented by Couette flow with a transverse pressure gradient, equation (17), superimpose on this steady motion a small infintesimal disturbance of the form

$$u_r = u(r) \cos(kz) e^{\omega t + m\theta},$$
$$u_\theta = v(r) \cos(kz) e^{\omega t + m\theta},$$
$$u_z = w(r) \sin(kz) e^{\omega t + m\theta}, \qquad (18)$$

and

$$\frac{\delta p}{\rho} = \tilde{\omega}(r) \cos(kz) e^{\omega t + m\theta}.$$

Here, k is the wave number of the disturbance in the axial direction and ω and m are constants which can be complex. The motion will be stable if the real part of ω is less than zero, and unstable if it is greater than zero. The states of neutral stability are characterized by the real part of ω, ω_r, equal to zero. The distinction between the two types of marginal states (stationary and oscillatory) corresponds to whether the imaginary part of ω, ω_i, vanishes when ω_r does. If $\omega_r=0$ implies $\omega_i=0$ then the principle of the exchange of stabilities will be valid; otherwise, we will have overstability. Accordingly, in case of oscillatory modes, the theory must yield not only what the mode of disturbance is that will be manifest but also the characteristic frequency of the oscillation. The velocity distributions obtained from the combination of the steady state flow and the infintesimal disturbances are now substituted into the governing equations, equations (1) through (4). Restricting our attention to the linearized theory (neglecting products of perturbation quantities and their derivatives) and making use of the previously derived relationships between the steady state velocity and pressure gradients, these equations can be simplified to the form:

$$\nu[Hu - \frac{u}{r^2} - \frac{2mv}{r^2}] - \frac{d\tilde{\omega}}{dr} = (\omega + \frac{mV}{r})u - \frac{2V}{r}v. \tag{19}$$

$$\nu[Hv - \frac{v}{r^2} + \frac{2m}{r^2}u] - \frac{m\tilde{\omega}}{r} = (\frac{dV}{dr} + \frac{V}{r})u + (\omega + \frac{mV}{r})v, \tag{20}$$

$$\nu[Hw] + k\tilde{\omega} = [\omega + \frac{mV}{r}]\, w, \tag{21}$$

and

$$\frac{du}{dr} + \frac{u}{r} + \frac{m}{r}\, v + kw = 0, \tag{22}$$

where

$$H = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{m^2}{r^2} - k^2, \tag{23}$$

and

$$N = (\omega + \frac{mV}{r}) - \nu H. \tag{24}$$

To simplify this system of equations, equation (22) can be solved for w and then substituted into equation (21) to get $\tilde{\omega}$ in terms of u and v:

$$w = -\frac{1}{k}[\frac{du}{dr} + \frac{u}{r} + \frac{m}{r}\, v], \tag{25}$$

and

$$\nu[H][-\frac{1}{k}(\frac{du}{dr} + \frac{u}{r} + \frac{m}{r}\, v] + k\tilde{\omega} = [\omega + \frac{mV}{r}][-\frac{1}{k}(\frac{du}{dr} + \frac{u}{r} + \frac{m}{r}\, v)] \tag{26}$$

or

$$\tilde{\omega} = -\frac{1}{k^2}[(\omega + \frac{mV}{r})(\frac{du}{dr} + \frac{u}{r} + \frac{m}{r}\, v)] + \frac{\nu H}{k^2}(\frac{du}{dr} + \frac{u}{r} + \frac{m}{r}\, v). \tag{27}$$

This result (equation (27)) for $\tilde{\omega}$ can be substituted into equation (19) and equation (20) to get two equations relating u and v:

$$\frac{d}{dr} N (\frac{d}{dr} + \frac{1}{r})u - k^2(N + \frac{\nu}{r^2})u = -2k^2(\frac{V}{r} - \frac{m\nu}{r^2})v - m\frac{d}{dr} N\{\frac{v}{r}\} \tag{28}$$

and

$$-k^2(N + \frac{\nu}{r^2})v + \frac{m^2}{r} N(\frac{v}{r}) = k^2(\frac{dV}{dr} + \frac{V}{r} - \frac{2\nu m}{r^2})u - \frac{m}{r} N(\frac{du}{dr} + \frac{u}{r}). \quad (29)$$

The requirement of no slip at the boundaries gives the boundary conditions

$$u = v = w = 0 \quad \text{at} \quad r = R_1 \text{ and } r = R_2.$$

The perturbation quantities w and $\tilde{\omega}$ have effectively been eliminated from the actual determination of the critical flow condition except for the boundary condition involving w. This boundary condition restriction will be eliminated in subsequent work. If the quantities w or $\tilde{\omega}$ are desired they can be evaluated separately once the critical flow condition has been obtained.

It is now convenient to introduce a change of variables. Let

$$r = R_1 + \zeta d \quad (30)$$

and

$$\frac{V}{r} = \Omega_r = \Omega_1 F(\zeta) \quad (31)$$

where

$$\zeta = \frac{r - R_1}{R_2 - R_1} = \frac{r - R_1}{d}. \quad (32)$$

The gap width d is defined by

$$d = R_2 - R_1. \quad (33)$$

With this notation, equation (28) can be written in a more convenient form for introducing the small gap assumption;

$$\frac{1}{d}\frac{d}{d\zeta}\{[(\omega + m\Omega_r) - \frac{\nu}{d^2}(\frac{d^2}{d\zeta^2} + \frac{d}{r}\frac{d}{d\zeta} + \frac{m^2d^2}{r^2} - k^2d^2)][\frac{1}{d}(\frac{d}{d\zeta} + \frac{d}{r})u]\}$$

$$-k^2[(\omega + m\Omega_r) - \frac{\nu}{d^2}(\frac{d^2}{d\zeta^2} + \frac{d}{r}\frac{d}{d\zeta} + \frac{m^2d^2}{r^2} - k^2d^2) + \frac{\nu}{r^2}]u$$

$$= -2k^2(\Omega_r - \frac{m\nu}{r^2})v - \frac{m}{d}\frac{d}{d\zeta}\{[\omega + m\Omega_r) - \frac{\nu}{d^2}(\frac{d^2}{d\zeta^2} + \frac{d}{r}\frac{d}{d\zeta} + \frac{m^2d^2}{r^2}$$

$$-k^2d^2)](\frac{v}{r})\}. \qquad (34)$$

After completing the indicated differentiation, r is approximated by R_1 and the new non-dimensional variables

$$\delta = \frac{d}{R_1}, \quad \text{and} \quad a = kd \qquad (35)$$

are introduced. For this problem (see Appendix A) under the small gap assumption

$$\frac{dV}{dr} + \frac{V}{r} = 2A + \frac{6V_m}{d}(1-2\zeta) \qquad (36)$$

where

$$V_m = -\frac{d^2}{12\rho\nu R_1}(\frac{\partial p}{\partial \theta}). \qquad (37)$$

Equation (34) can then be written as

$$-\frac{d^2}{\nu}(\omega + m\Omega_r)(\frac{d^2u}{d\zeta^2} + \delta\frac{du}{d\zeta} - \delta^2 u) - \frac{md^2\delta}{\nu}(\frac{du}{d\zeta} + \delta u)[2A - \frac{6V_m}{d}(1-2\zeta)]$$

$$+ (\frac{d^4u}{d\zeta^4} + 2\delta\frac{d^3u}{d\zeta^3} - 3\delta^2\frac{d^2u}{d\zeta^2} + 3\delta^3\frac{du}{d\zeta} - 3\delta^4 u + m^2\delta^2\frac{d^2u}{d\zeta^2}$$

(equation continued on next page)

$$- m^2\delta^3 \frac{du}{d\zeta} - 3m^2\delta^4 u - a^2 \frac{d^2u}{d\zeta^2} - a^2\delta \frac{du}{d\zeta} + a^2\delta^2 u) + \frac{a^2d^2}{\nu}$$

$$[(\omega + m\Omega_r)u - \frac{\nu}{d^2}(\frac{d^2u}{d\zeta^2} + \delta \frac{du}{d\zeta} + m^2\delta^2 u - a^2 u - \delta^2 u)]$$

$$= \frac{2a^2d^2}{\nu}(\Omega_r - \frac{m\nu}{r^2}) v + \frac{md^2}{\nu} \{(\omega + m\Omega_r)(\delta\frac{dv}{d\zeta} - \delta^2 v)$$

$$+ m\delta^2 v[2A - \frac{6V_m}{d}(1-2\zeta)] - \frac{\nu}{d^2}[\delta \frac{d^3v}{d\zeta^3} - 2\delta^2 \frac{d^2v}{d\zeta^2} + 3\delta^3 \frac{dv}{d\zeta}$$

$$-3\delta^4 v + m^2\delta^3 \frac{dv}{d\zeta} - 3\delta^4 m^2 v - a^2\delta \frac{dv}{d\zeta} + a^2\delta^2 v]\}. \tag{38}$$

The additional parameters

$$\sigma = \frac{\omega d^2}{\nu}, \tag{39}$$

$$n = (-\frac{\Omega_1}{4A})^{1/2} m, \tag{40}$$

and

$$T = -\frac{4A\Omega_1 d^4}{\nu^2} \tag{41}$$

are introduced. Observe that since

$$A = -\frac{\Omega_1(1-u)}{2\delta} \tag{42}$$

plus terms of O(1), n can be written as

$$n \simeq [\frac{\delta}{2(1-u)}]^{1/2} m. \tag{43}$$

For the small gap problem equation (43) will be taken to define n.

Note also that

$$n(T)^{\frac{1}{2}} = \frac{\Omega_1 d^2}{\nu} m. \tag{44}$$

Equation (38) now reduces to the form

$$- [\sigma + n(T)^{\frac{1}{2}}F(\zeta)](\frac{d^2u}{d\zeta^2} + \delta \frac{du}{d\zeta} - \delta^2 u) - \frac{md^2}{\nu}(\frac{du}{d\zeta} + \delta u)\delta[2A - \frac{6V_m}{d}(1-2\zeta)]$$

$$+ [\frac{d^4u}{d\zeta^4} + 2\delta \frac{d^3u}{d\zeta^3} - 3\delta^2 \frac{d^2u}{d\zeta^2} + 3\delta^3 \frac{du}{d\zeta} - 3\delta^4 u + m^2\delta^2 \frac{d^2u}{d\zeta^2}$$

$$- m^2\delta^3 \frac{du}{d\zeta} - 3m^2\delta^4 u - a^2 \frac{d^2u}{d\zeta^2} - a^2\delta \frac{du}{d\zeta} + a^2\delta^2 u] + a^2[(\sigma + n(T)^{\frac{1}{2}}$$

$$F(\zeta))] - a^2 (\frac{d^2u}{d\zeta^2} + \delta \frac{du}{d\zeta} + m^2\delta^2 u - a^2 u - \delta^2 u) = \frac{2a^2d^2}{\nu} \Omega_1 F(\zeta) v$$

$$- 2a^2 m\delta^2 v + m \{(\sigma + n(T)^{\frac{1}{2}} F(\zeta))(\delta \frac{dv}{d\zeta} - \delta^2 v) + m\delta^2 v[2A$$

$$- \frac{6V_m}{d} (1-2\zeta)] - (\delta \frac{d^3v}{d\zeta^3} - 2\delta^2 \frac{d^2v}{d\zeta^2} + 3\delta^3 \frac{dv}{d\zeta} - 3\delta^4 v + m^2\delta^3 \frac{dv}{d\zeta}$$

$$- 3\delta^4 m^2 v - a^2\delta \frac{dv}{d\zeta} + a^2\delta^2 v]\}. \tag{45}$$

If the so called small gap assumption is made, then the problem is restricted to the case where $(R_2 - R_1)$ is small with respect to $\frac{1}{2}(R_1 + R_2)$. Under this condition if the limit in equation (45) is considered as $\delta \to 0$ with σ, n, a, and T remaining constant, then the equation reduces to

$$-(\sigma + n(T)^{\frac{1}{2}}F(\zeta)) \frac{d^2u}{d\zeta^2} + \frac{d^4u}{d\zeta^4} - a^2 \frac{d^2u}{d\zeta^2} + a^2[(\sigma + n(T)^{\frac{1}{2}}F(\zeta)]$$

$$-a^2 \frac{d^2u}{d\zeta^2} + a^4 u = \frac{2a^2d^2}{\nu} \Omega_1 F(\zeta) v. \tag{46}$$

In the small gap approximation $F(\zeta)$ becomes

$$F(\zeta) = [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)] \tag{47}$$

as is shown in Appendix A. Rearranging the equation, letting

$$L = \left[(D^2 - a^2) - \{ (\sigma + n(T)^{1/2} [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}\right], \tag{48}$$

the equation becomes

$$L\ (D^2 - a^2)u = \frac{2a^2d^2}{\nu} [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]v. \tag{49}$$

With the additional change of variables

$$u = \frac{2\Omega_1 d^2 a^2}{\nu}\ u' \tag{50}$$

and then dropping the primed notation the equation takes the final form

$$L(D^2 - a^2)u = [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]v. \tag{51}$$

In a similar manner, equation (29) can be written as

$$-k^2[(\omega + m\Omega_r) - \nu(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{m^2}{r^2} - k^2) + \frac{\nu}{r^2}]v + \frac{m}{r}[(\omega + m\Omega_r)$$

$$-\nu\ (\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{m^2}{r^2} - k^2)](\frac{v}{r}) = k^2\ (\frac{dV}{dr} + \frac{V}{r} - \frac{2\nu m}{r^2})\ u$$

$$-\frac{m}{r}\ [(\omega + m\Omega_r) - \nu\ (\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \frac{m^2}{r^2} - k^2)](\frac{du}{dr} + \frac{u}{r}). \tag{52}$$

Introducing the change in independent variables from r to ζ used previously this becomes

$$-k^2\ \{(\omega + m\Omega_r) - \frac{\nu}{d^2}\ [\frac{d^2v}{d\zeta^2} + (\frac{d}{r})\ \frac{dv}{d\zeta} + m^2(\frac{d}{r})^2 v - k^2 d^2 v + (\frac{d}{r})^2 v]\}$$

(equation continued on next page)

$$+ \frac{m}{r} (\omega + m\Omega_r)(\frac{v}{r}) - \frac{m\nu}{r} (\frac{1}{d^2} \frac{d^2 v}{d\zeta^2} + \frac{1}{r^2 d} \frac{dv}{d\zeta} - \frac{v}{r^3} + \frac{m^2 v}{r^3} - \frac{k^2 v}{r})$$

$$= k^2 [2\Lambda - \frac{6V_m}{d} (1-2\zeta)] - \frac{2k^2 \nu m}{r^2} u - \frac{m}{r} (\omega + m\Omega_r)(\frac{1}{d} \frac{du}{d\zeta} + \frac{u}{r})$$

$$+ \frac{m\nu}{r} [\frac{1}{d^3} \frac{d^3 u}{d\zeta^3} + \frac{1}{d^2} \frac{d^2}{d\zeta^2} (\frac{u}{r}) + \frac{1}{rd^2} \frac{d^2 u}{d\zeta^2} + \frac{1}{rd} \frac{d}{d\zeta} (\frac{u}{r}) + \frac{m^2}{dr^2} \frac{du}{d\zeta}$$

$$+ \frac{m^2 u}{r^3} - \frac{k^2}{d} \frac{du}{d\zeta} - \frac{k^2 u}{r}]. \qquad (53)$$

Again using the previously defined quantities a, δ , σ, n, and T equation (53) can be simplified to the form

$$a^2 \{(\sigma + n(T)^{\frac{1}{2}} F(\zeta) - [\frac{d^2 v}{d\zeta^2} + \delta \frac{dv}{d\zeta} + m^2 \delta^2 v - a^2 v + \delta^2 v]\}$$

$$+ m\delta^2 (\sigma + n(T)^{\frac{1}{2}} F(\zeta))v - m\delta [d \frac{d^2 v}{d\zeta^2} + \delta^2 \frac{dv}{d\zeta} - v\delta^3 + m^2 \delta^3 v$$

$$-a^2 \delta v] = \frac{a^2 d^2}{\nu} [2A - \frac{6V_m}{d} (1-2\zeta) - \frac{2\nu m}{r^2}]u - m\delta (\sigma + n(T)^{\frac{1}{2}} F(\zeta))$$

$$(\frac{du}{d\zeta} + \delta u) + m\delta \{[\frac{d^3 u}{d\zeta^3} + \delta \frac{d^2 u}{d\zeta^2} - 2\delta^2 \frac{du}{d\zeta} + 2u\delta^3] + \delta \frac{d^2 u}{d\zeta^2}$$

$$+ \delta^2 (\frac{du}{d\zeta} - \delta u) + m^2 \delta^2 \frac{du}{d\zeta} + m^2 \delta^3 u - a^2 \frac{du}{d\zeta} - a^2 \delta u\}. \qquad (54)$$

Letting $\delta \to 0$ with the other parameters remaining constant, then

$$a^2 \{\sigma + n(T)^{1/2} [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)] - \frac{d^2v}{d\zeta^2} - a^2 v\} = a^2 d^2 (\frac{dV}{dr}$$

$$+ \frac{V}{r})u - a^2 \frac{du}{d\zeta} , \tag{55}$$

or

$$\left[\frac{d^2}{d\zeta^2} - a^2 v - a^2 \{\sigma + n(T)^{1/2} [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}\right] v$$

$$= - \frac{a^2 d^2}{\nu} (\frac{dV}{dr} + \frac{V}{r})u. \tag{56}$$

For this problem the relationship

$$\frac{dV}{dr} + \frac{V}{r} = 2A + \frac{6V_m}{d} (1-2\zeta) \tag{36}$$

has already been introduced. Using this result, equation (56) reduces to

$$\left[\frac{d^2}{d\zeta^2} - a^2 - \{(\sigma + n(T)^{1/2} [1-(1-\mu)\zeta + \lambda\zeta(1-\mu)]\}\right] v$$

$$= \frac{2a^2 d^2}{\nu} [A + \frac{3V_m}{d} (1-2\mu)]u. \tag{57}$$

Let

$$\lambda = \frac{6V_m}{R_1 \Omega_1} \tag{58}$$

which is an indication of the relative strength of the pumping velocity and the rotational velocity. As $\lambda \to 0$ the problem reduces to the Taylor problem and as $\lambda \to \infty$ it approaches the Dean problem. For a narrow gap

$$\eta = \frac{R_1}{R_2} \approx 1$$

and

$$\frac{3V_m}{Ad} \approx -\frac{6V_m}{R_1\Omega_1(1-\mu)} \approx \frac{\lambda}{1-\mu} \cdot \qquad (59)$$

Equation (57) can then be written as

$$\left[\frac{d^2}{d\zeta^2} - a^2 - \{\sigma + n(T)^{\frac{1}{2}}\,[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}\right] v$$

$$= \frac{2Aa^2d^2}{\nu}\,[1-\frac{\lambda}{1-\mu}\,(1-2\zeta)]u. \qquad (60)$$

Introducing the Taylor Number, the change of variables introduced previously

$$u = \frac{2\Omega_1 d^2 a^2}{\nu}\,u' \qquad (50)$$

and then again dropping the prime notation we obtain

$$\left[\frac{d^2}{d\zeta^2} - a^2 - \{\sigma + n(T)^{\frac{1}{2}}\,[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}\right] v$$

$$= -a^2T[1 - \frac{\lambda}{1-\mu}\,(1-2\zeta)]u. \qquad (61)$$

Using the operator defined by equation (48) this can be written in the simpler form

$$Lv = -\,a^2T\,[1-\frac{\lambda}{1-\mu}\,(1-2\zeta)]u. \qquad (62)$$

Equations (51) and (62) are the governing equations for the case of flow between concentric rotating cylinders with a transverse pressure gradient. The auxiliary equations for w and $\tilde{\omega}$ reduce to

$$w = -\frac{1}{a}\left(\frac{du}{d\zeta}\right) \tag{63}$$

and

$$\tilde{\omega} = -\frac{1}{a^2}\left[\{\sigma + n(T)^{\frac{1}{2}}[1 - (1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}\frac{du}{d\zeta} - \frac{d^3u}{d\zeta^3}\right] \tag{64}$$

using the small gap assumption. This is worked out in Appendix A. Solutions of these equations must be sought which satisfy the boundary conditions

$$u = v = w = 0 \quad \text{at} \quad \zeta = 0 \text{ and } \zeta = 1.0. \tag{65}$$

Using equation (63), observe that the boundary conditions can also be written as

$$u = Du = v = 0 \quad \text{at} \quad \zeta = 0 \quad \text{and} \quad \zeta = 1.0. \tag{66}$$

Equations (51) and (62) must be solved subject to the boundary conditions imposed by equation (66). The variables w and $\tilde{\omega}$ have thus been eliminated from the actual determination of the critical Taylor number. The problem has been reduced to a boundary value problem involving variables u, v, their derivatives, the physical parameter of the system and the parameters of the assumed disturbances. The quantities w and $\tilde{\omega}$ can be calculated, if desired, from equations (63) and (64) once the boundary value problem has been solved.

Method of Solution

The governing equations for the problem are

$$L(D^2 - a^2)u = [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]v \tag{51}$$

and

$$Lv = - a^2 T \left[1 - \frac{\lambda}{1-\mu}(1-2\zeta)\right]u. \tag{62}$$

with the boundary conditions

$$u = v = Du = 0 \quad \text{at} \quad \zeta = 0 \quad \text{and} \quad \zeta = 1.0. \tag{66}$$

The eigenvalue problem thus defined is a two point boundary value problem. For numerical purposes, however, it is desirable to convert it into an initial value problem and this can be done in the following manner: Rewrite equations (51) and (62) as a system of first order equations. To accomplish this the following new variables are introduced: Let

$$Y_A = u,$$

$$Y_B = v,$$

$$Y_C = Du,$$

$$Y_D = Dv,$$

$$Y_E = (D^2 - a^2)u,$$

and

$$Y_F = D\,(D^2 - a^2)u. \tag{67}$$

Then equations (51) and (62) can be replaced by the following system of first order equations:

$$\frac{dY_A}{d\zeta} = Y_C,$$

$$\frac{dY_B}{d\zeta} = Y_D,$$

$$\frac{dY_C}{d\zeta} = Y_E + a^2 Y_A,$$

$$\frac{dY_D}{d\zeta} = \{a^2 + \sigma + n(T)^{\frac{1}{2}} [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\} Y_B$$

$$- a^2T [1 - \frac{\lambda}{1-\mu} (1-2\zeta)] Y_\Lambda,$$

$$\frac{dY_E}{d\zeta} = Y_F,$$

and

$$\frac{dY_F}{d\zeta} = a^2Y_E + \{\sigma + n(T)^{\frac{1}{2}} [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\} Y_E$$

$$+ [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)] Y_B. \qquad (68)$$

Our original assumption for the disturbances contained the form $\cos(kz)e^{\omega t + m\theta}$. In the general case ω can be a complex number $\omega = \omega_r + i\omega_i$. The term resulting from ω_r determines the growth or decay of the disturbance while the term resulting from ω_i defines the frequency of the oscillation. To insure that the solution is single-valued and bounded, m must either be an imaginary number $m=i\ m_i$ where m_i is an integer for the non-axisymmetric disturbances or m=0 for axisymmetric disturbances. The quantity m_i determines the wavelength of the oscillation in the θ direction. The wavelength in the z direction is determined by k which must be a real number.

If attention is restricted to the case for marginal stability then $\omega=0$for the stationary modes and $\omega=i\omega_i$ for the oscillatory modes. For the oscillatory modes (either axisymmetric or non-axisymmetric disturbances) the problem will develop into a system of twelve first order equations after separating the real and imaginary parts. The problem reduces to a system of 6 first order equations for the stationary modes if axisymmetric disturbances are assumed.

For the case of oscillatory modes (either axisymmetric or non-axisymmetric disturbances) since σ and m are complex it is obvious that the variables Y_A, Y_B, Y_C, Y_D, Y_E and Y_F will have complex solutions.

Then equations (68) become

$$\frac{d}{d\zeta}(Y_1 + iY_2) = Y_5 + iY_6,$$

$$\frac{d}{d\zeta}(Y_3 + iY_4) = Y_7 + iY_8,$$

$$\frac{d}{d\zeta}(Y_5 + iY_6) = (Y_9 + iY_{10}) + a^2 (Y_1 + iY_2),$$

$$\frac{d}{d\zeta}(Y_7 + iY_8) = -a^2T[1-\frac{\lambda}{1-\mu}(1-2\zeta)](Y_1 + iY_2) + \{a^2 + i\sigma_i + in_i(T)^{\frac{1}{2}}[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}(Y_3 + iY_4)$$

$$\frac{d}{d\zeta}(Y_9 + iY_{10}) = Y_{11} + iY_{12},$$

and

$$\frac{d}{d\zeta}(Y_{11} + iY_{12}) = \{a^2 + i\sigma_i + in_i(T)^{\frac{1}{2}}[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}(Y_9 + iY_{10}) + [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)](Y_3 + iY_4). \quad (69)$$

After separating the real and imaginary parts equations (69) reduce to the following system of 12 first order equations:

$$\frac{dY_1}{d\zeta} = Y_5,$$

$$\frac{dY_2}{d\zeta} = Y_6,$$

$$\frac{dY_3}{d\zeta} = Y_7,$$

$$\frac{dY_4}{d\zeta} = Y_8,$$

$$\frac{dY_5}{d\zeta} = Y_9 + a^2 Y_1,$$

$$\frac{dY_6}{d\zeta} = Y_{10} + a^2 Y_2,$$

$$\frac{dY_7}{d\zeta} = -a^2 T \left[1 - \frac{\lambda}{1-\mu}(1-2\zeta)\right] Y_1 + a^2 Y_3 - \{\sigma_i + n_i (T)^{1/2} \left[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)\right]\} Y_4,$$

$$\frac{dY_8}{d\zeta} = -a^2 T \left[1 - \frac{\lambda}{1-\mu}(1-2\zeta)\right] Y_2 + a^2 Y_4 + \{\sigma_i + n_i (T)^{1/2} \left[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)\right]\} Y_3,$$

$$\frac{dY_9}{d\zeta} = Y_{11},$$

$$\frac{dY_{10}}{d\zeta} = Y_{12},$$

$$\frac{dY_{11}}{d\zeta} = a^2 Y_9 - \{\sigma_i + n_i (T)^{1/2} \left[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)\right]\} Y_{10} + \left[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)\right] Y_3,$$

and

$$\frac{dY_{12}}{d\zeta} = a^2 Y_{10} + \{\sigma_i + n_i (T)^{\frac{1}{2}} \lfloor 1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)\rfloor\} Y_9$$

$$+ \lfloor 1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)\rfloor Y_4. \quad (70)$$

The boundary conditions are

$$Y_1 = Y_2 = Y_3 = Y_4 = Y_5 = Y_6 = 0 \quad \text{at} \quad \zeta = 0 \text{ and } 1.0. \quad (71)$$

Let us now define six linearly independent solutions Y^i ($i = 1, 2,$----6) of this system of equations by imposing the initial conditions,

$$\left.\begin{aligned} Y^i_j &= 0 & j &= 1,\text{----}6 \\ Y^i_j &= \delta_{i,j-6} & j &= 7,\text{----}12 \end{aligned}\right\} \quad i = 1,\text{----}6 \quad (72)$$

where δ_{ij} is the Kronecker delta. The boundary conditions are automatically satisfied at ζ=0 and we then seek a linear combination of the solutions which satisfy the boundary conditions at ζ=1.0,

$$\sum_{i=1}^{6} C_i Y^j_i(1) = 0 \qquad j = 1,\text{----}6\ . \quad (73)$$

The condition that the above equations have a non-trivial solution is that the determinant

$$|Y^i_j\ (1)| = 0 \qquad i,\ j = 1,\text{----}6. \quad (74)$$

This is the required characteristic equation from which the curves of neutral stability can be obtained.

The basic solutions at ζ=1.0, $Y^i(1)$, are obtained by integrating the system of first order equations by a Runge-Kutta method and the determinant was evaluated by the condensation of diagonal technique. The effect of finite interval size (h) was investigated by obtaining solutions at different values of h, decreasing h in regular intervals until the successive solutions agreed to within the desired accuracy. For a given value of λ the primary interest is in determining the least positive Taylor number (and the corresponding values of σ_i, a and m_i) which satisfies equation (74). These values define the critical condition at which instability first occurs.

In practice for each value of λ to be investigated, a value for σ_i, a and m_i is chosen and then T is varied until equation (74) is satisfied. This procedure is then repeated for a different value of σ_i, a and m_i until the desired range of values for σ_i, a and m_i has been investigated. From the resulting curves the critical condition for each value of λ can be determined.

Once equation (74) is satisfied the 6 equations represented by equations (73) are no longer independent. Any 5 of these equations are independent and can be used to determine the eigenfunctions. Choosing 5 of the equations we have

$$C_1Y_1^1 + C_2Y_1^2 + C_3Y_1^3 + C_4Y_1^4 + C_5Y_1^5 + C_6Y_1^6 = 0,$$

$$C_1Y_2^1 + C_2Y_2^2 + C_3Y_2^3 + C_4Y_2^4 + C_5Y_2^5 + C_6Y_2^6 = 0,$$

$$C_1Y_3^1 + C_2Y_3^2 + C_3Y_3^3 + C_4Y_3^4 + C_5Y_3^5 + C_6Y_3^6 = 0,$$

(equation continued on next page)

$$C_1Y_4^1 + C_2Y_4^2 + C_3Y_4^3 + C_4Y_4^4 + C_5Y_4^5 + C_6Y_4^6 = 0,$$

and

$$C_1Y_5^1 + C_2Y_5^2 + C_3Y_5^3 + C_4Y_5^4 + C_5Y_5^5 + C_6Y_5^6 = 0. \qquad (75)$$

After dividing by C_6 and simplifying, the system of equations can be written as

$$\frac{C_1}{C_6} Y_1^1 + \frac{C_2}{C_6} Y_1^2 + \frac{C_3}{C_6} Y_1^3 + \frac{C_4}{C_6} Y_1^4 + \frac{C_5}{C_6} Y_1^5 = -Y_1^6,$$

$$\frac{C_1}{C_6} Y_2^1 + \frac{C_2}{C_6} Y_2^2 + \frac{C_3}{C_6} Y_2^3 + \frac{C_4}{C_6} Y_2^4 + \frac{C_5}{C_6} Y_2^5 = -Y_2^6,$$

$$\frac{C_1}{C_6} Y_3^1 + \frac{C_2}{C_6} Y_3^2 + \frac{C_3}{C_6} Y_3^3 + \frac{C_4}{C_6} Y_3^4 + \frac{C_5}{C_6} Y_3^5 = -Y_3^6,$$

$$\frac{C_1}{C_6} Y_4^1 + \frac{C_2}{C_6} Y_4^2 + \frac{C_3}{C_6} Y_4^3 + \frac{C_4}{C_6} Y_4^4 + \frac{C_5}{C_6} Y_4^5 = -Y_4^6,$$

and

$$\frac{C_1}{C_6} Y_5^1 + \frac{C_2}{C_6} Y_5^2 + \frac{C_3}{C_6} Y_5^3 + \frac{C_4}{C_6} Y_5^4 + \frac{C_5}{C_6} Y_5^5 = -Y_5^6. \qquad (76)$$

These equations can be solved simultaneously for the coefficients C_1/C_6, C_2/C_6, C_3/C_6, C_4/C_6 and C_5/C_6. Once these coefficients are determined then the eigenfunctions at any value of ζ can be found to within a constant factor, since

$$\frac{Y_i(\zeta)}{C_6} = \frac{C_1}{C_6} Y_i^1(\zeta) + \frac{C_2}{C_6} Y_i^2(\zeta) + \frac{C_3}{C_6} Y_i^3(\zeta) + \frac{C_4}{C_6} Y_i^4(\zeta) +$$

$$\frac{C_5}{C_6} Y_i^5(\zeta) + Y_i^6(\zeta). \tag{77}$$

For the stationary axisymmetric modes ($\sigma = 0$, $m = 0$) the system of equations reduce to

$$\frac{dY_1}{d\zeta} = Y_5,$$

$$\frac{dY_3}{d\zeta} = Y_7,$$

$$\frac{dY_5}{d\zeta} = Y_9 + a^2 Y_1,$$

$$\frac{dY_7}{d\zeta} = -Ta^2[1- \frac{\lambda}{1-\mu} (1-2\zeta)]\, Y_1 + a^2 Y_3,$$

$$\frac{dY_9}{d\zeta} = Y_{11},$$

and

$$\frac{dY_{11}}{d\zeta} = a^2 Y_9 + [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\, Y_3. \tag{78}$$

The boundary conditions are $Y_1 = Y_3 = Y_5 = 0$ at $\zeta = 0$ and $\zeta = 1.0$. If we reidentify the variables and let

$$Y_1 = Y_1,$$

$$Y_2 = Y_3,$$

$$Y_3 = Y_5,$$

$$Y_4 = Y_7,$$

$$Y_5 = Y_9,$$

and

$$Y_6 = Y_{11}, \tag{79}$$

the governing equations become

$$\frac{dY_1}{d\zeta} = Y_3,$$

$$\frac{dY_2}{d\zeta} = Y_4,$$

$$\frac{dY_3}{d\zeta} = Y_5 + a^2Y_1,$$

$$\frac{dY_4}{d\zeta} = -a^2T\,[1- \frac{\lambda}{1-\mu}\,(1-2\zeta)]\,Y_1 + a^2Y_2,$$

$$\frac{dY_5}{d\zeta} = Y_6,$$

and

$$\frac{dY_6}{d\zeta} = a^2Y_5 + [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\,Y_2 \tag{80}$$

with the boundary conditions

$$Y_1 = Y_2 = Y_3 = 0 \quad \text{at} \quad \zeta = 0 \quad \text{and} \quad \zeta = 1.0. \tag{81}$$

In this case, 3 linearly independent solutions $Y^i(i=1, 2, 3)$ of this system of equations are defined by imposing the initial conditions

$$\left.\begin{aligned} Y^i_j &= 0 & j &= 1, 2, 3 \\ Y^i_j &= \delta_{i,j-3} & j &= 4, 5, 6 \end{aligned}\right\} \quad i = 1, 2, 3. \tag{82}$$

This leads to the requirement that

$$|Y_j^i(1)| = 0. \qquad \begin{cases} i = 1, 2, 3 \\ j = 1, 2, 3 \end{cases} \tag{83}$$

The same general method of solution is followed for the stationary modes as for the oscillatory modes except that three less parameters (σ_i, δ, and m_i) are present in the problem and the required calculations are greatly simplified. In this case, for each value of λ to be investigated, a value of a is chosen and then T is varied until equation (83) is satisfied. This procedure is then repeated for a different value of a until the desired range has been investigated.

VIII. RESULTS AND DISCUSSION

The onset of instability for Couette flow between rotating cylinders and subjected to a transverse pressure gradient has been investigated. This problem permits a wide range of initial steady state velocity distributions between the two cylinders. Some representative velocity distributions are shown in Figure 1. The parameter λ defined by equation (58) characterizes the relative strength of the pumping velocity and the rotational velocity. For $\lambda=0$, the problem reduces to the Taylor problem. As $\lambda \to \pm\infty$ the problem approaches the Dean problem. For positive values of λ the pumping velocity is in the same direction as the rotational velocity, negative values of λ indicate a pumping velocity opposite in direction to the rotational velocity. For $\lambda \equiv \pm\infty$ the problem reduces identically to the Dean problem; however, for large (but finite) values of λ the velocity distribution near the inner cylinder differs significantly from that of the Dean problem. This can be seen in Figure 1. It should be noted also that for a large positive number N, the shape of the initial velocity distribution curve for $\lambda=+N$ differs from that of $\lambda=-N$, thus it would be expected that the critical Taylor numbers for the two conditions would also be different. The average velocity contribution due to the rotation of the cylinder and the pumping contribution are approximately equal for $\lambda=\pm3.0$. For $\lambda=+3$ the velocities are in the same direction, while for $\lambda=-3$ the velocities are in opposite directions. As has been indicated previously, the instability for this problem results

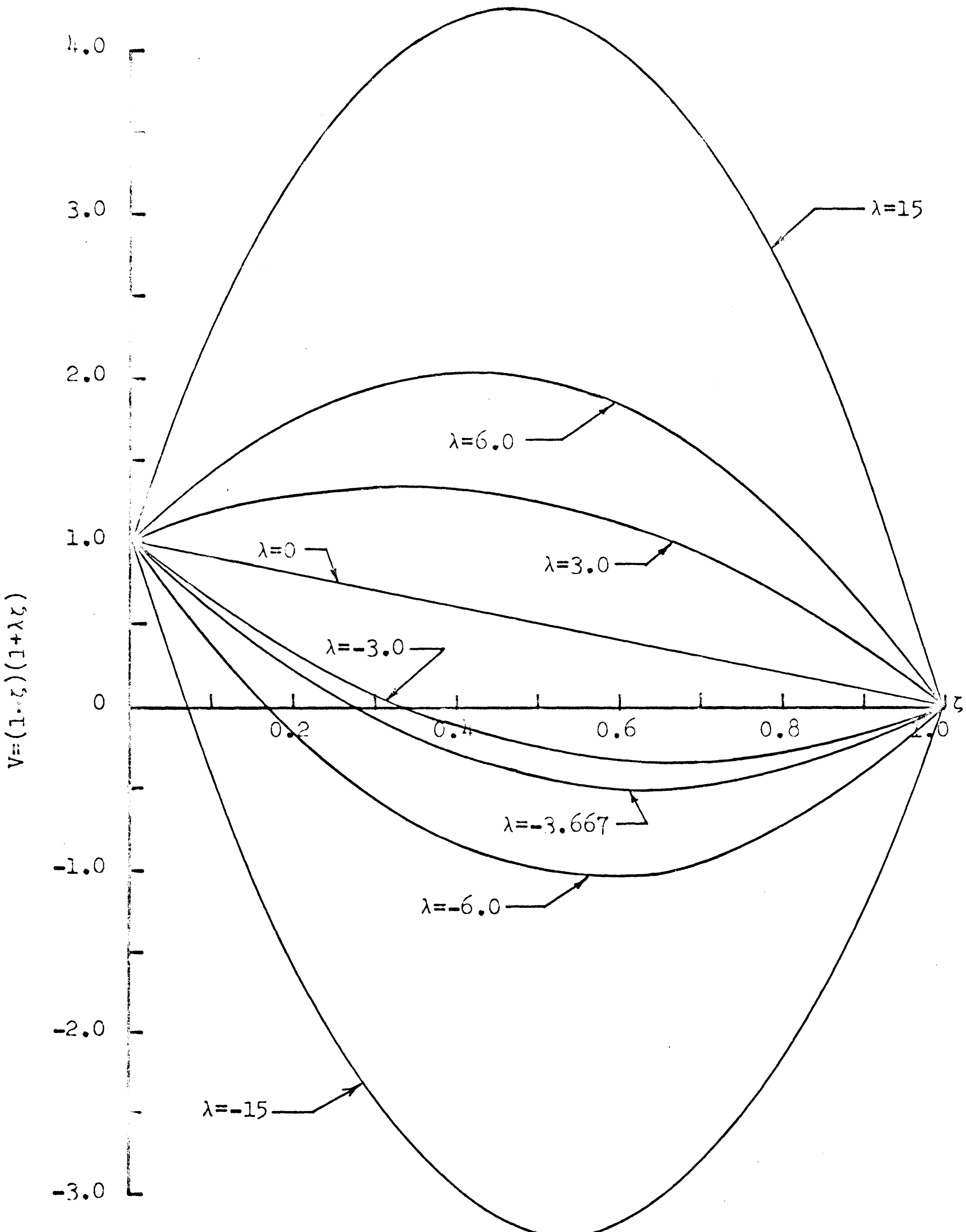


Figure 1. Distribution of the Initial Transverse Velocity for Values of λ for μ=0

from a potentially unstable arrangement of flow resulting from an adverse radial gradient of angular momentum. For the case of an inviscid fluid Rayleigh [16] has shown that the necessary and sufficient conditon for a distribution of angular velocity to be stable is

$$\frac{d}{dr}(r^2\Omega)^2 > 0$$

everywhere in the interval. For this problem with $\mu=0$ Rayleigh's discriminant for the velocity distribution reduces to

$$\phi = (1-\zeta)(1+\lambda\zeta)(\lambda-1-2\lambda\zeta) \tag{84}$$

and Rayleigh's criterion for stability is that ϕ must be positive. The stable and unstable regions are shown in Figure 2.

In the investigation of the onset of instability it is necessary to make assumptions concerning the type of disturbance which leads to the critical stability condition and also to assume the type of marginal state at the onset of instability. The problem was investigated for three conditions: (1) Assuming axisymmetric disturbances imposed on the original flow and a stationary marginal state, (2) Assuming axisymmetric disturbances and an oscillatory marginal state, and (3) Assuming non-axisymmetric disturbances and an oscillatory marginal state. Most of the previous work in the literature has been restricted to the assumption of axisymmetric disturbances and a stationary marginal state. For a given value of λ the primary desired result is to determine the minimum Taylor number and the corresponding longitudinal wave number at the onset of instability. All of the numerical work was restricted to the

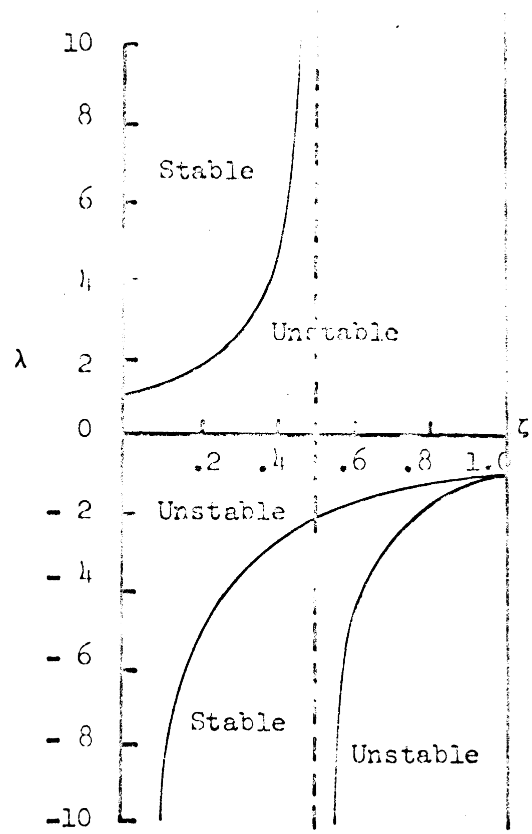


Figure 2. Stable and Unstable Regions of the Fluid According to the Rayleigh Criterion ($\mu=0$)

physically most interesting case where the outer cylinder is stationary (μ=0), although the equations are derived for the general case of arbitrary values of μ.

In the numerical integration of the governing system of equations, the step size (h) for the independent variable in the integration process was systematically reduced by one half to check the effect of step size on the results. The step size was reduced until the results for Y_i (i=1,----6) at ζ=1.0 were identical to six decimal places with the results obtained using the previous step size. In general, a step size of h=0.01 was sufficient to obtain this accuracy. The Taylor number and the corresponding values of the longitudinal wave number (a) and the frequency (σ_i) were then determined to at least ±½%. In some cases greater accuracy was necessary to determine the critical values. For the non-axisymmetric disturbances the calculations were carried out for only one value of δ (the ratio of the annulus gap to the inner radius) which was small enough for the small gap assumption to be valid. Results for other values of δ should be qualitatively the same although they may differ slightly in actual numerical value. Note that while δ does not appear explicitly in equations (51), (62) or (66), it has not been completely eliminated from the problem since it appears in the definition of n as shown in equation (43).

Axisymmetric Disturbances

The results for axisymmetric disturbances are shown in Figures 3 through 32. Figures 3 through 12 show the variation of Taylor number with the longitudinal wave number for a wide range of values of λ. From these curves of neutral stability the critical value of the Taylor number (T_c) and the corresponding value of the longitudinal wave number (a_c) can be found for each value of λ. The critical Taylor number is small for large values of λ, both negative and positive, increasing in magnitude as $|\lambda|$ decreases. The maximum critical Taylor number occurs in the region of $\lambda=-3.667$, decreasing rapidly from this point for both larger and smaller values of λ. These results agree very well with available literature results for the stationary marginal states. The results in the literature are, however, confined to a much smaller range of values of λ and the oscillatory stationary states have not been previously investigated for this problem. For large values of λ (both positive and negative) the only analytical results available in the literature are for an approximate asymptotic analysis as $\lambda \to \pm\infty$. To obtain the asymptotic results, Chandrasekher [3] has simplified equations (51) and (62) by letting $|\lambda| \to \infty$. These equations reduce to

$$(D^2-a^2)^2 u = \lambda\zeta(1-\zeta)v \qquad (85)$$

and

$$(D^2-a^2)\ v = (\frac{Ta^2}{1-\mu})\lambda(1-2\zeta)u. \qquad (86)$$

By the further transformation $u=\lambda u$ these equations become similar to the governing equations derived for the Dean problem. The single difference is that for the equations derived for the Dean problem $\frac{Ta^2}{1-\mu}$ is replaced by Λ. The quantity Λ is the non-dimensional

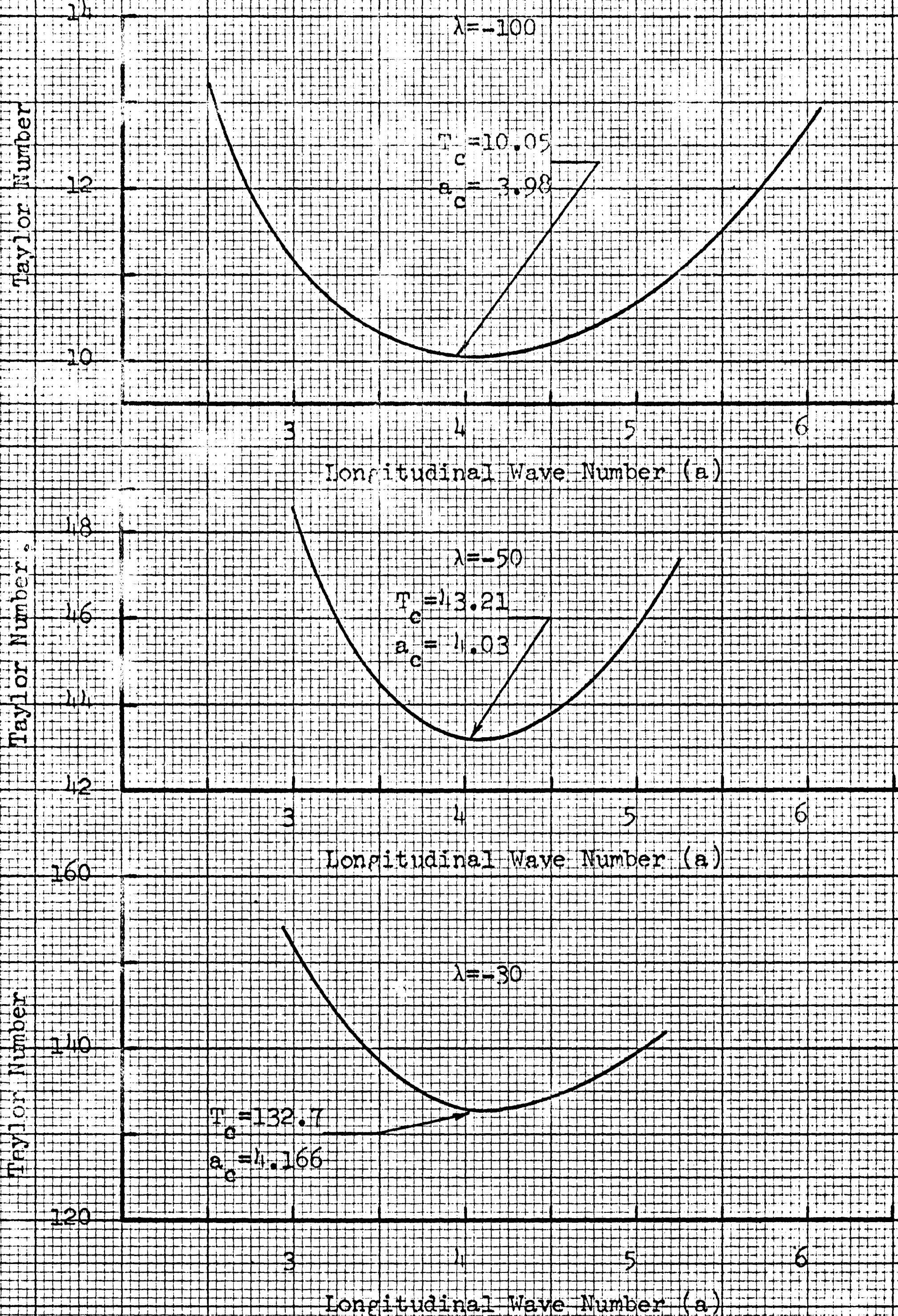
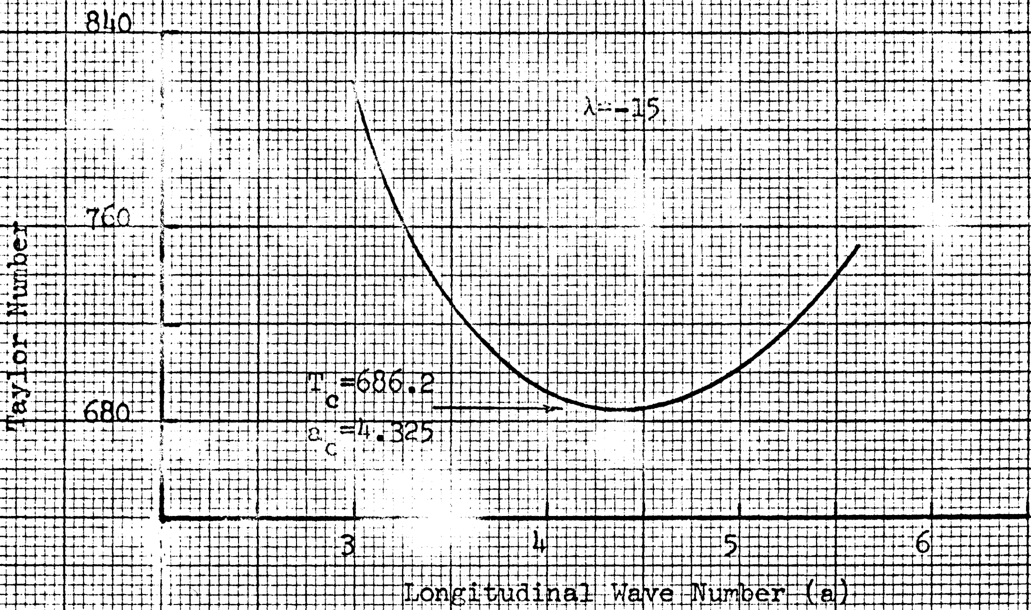


Figure 3. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for $\lambda=-100$, $\lambda=-50$ and $\lambda=-30$ Assuming Axisymmetric Disturbances, a Stationary Marginal State and $\mu=0$



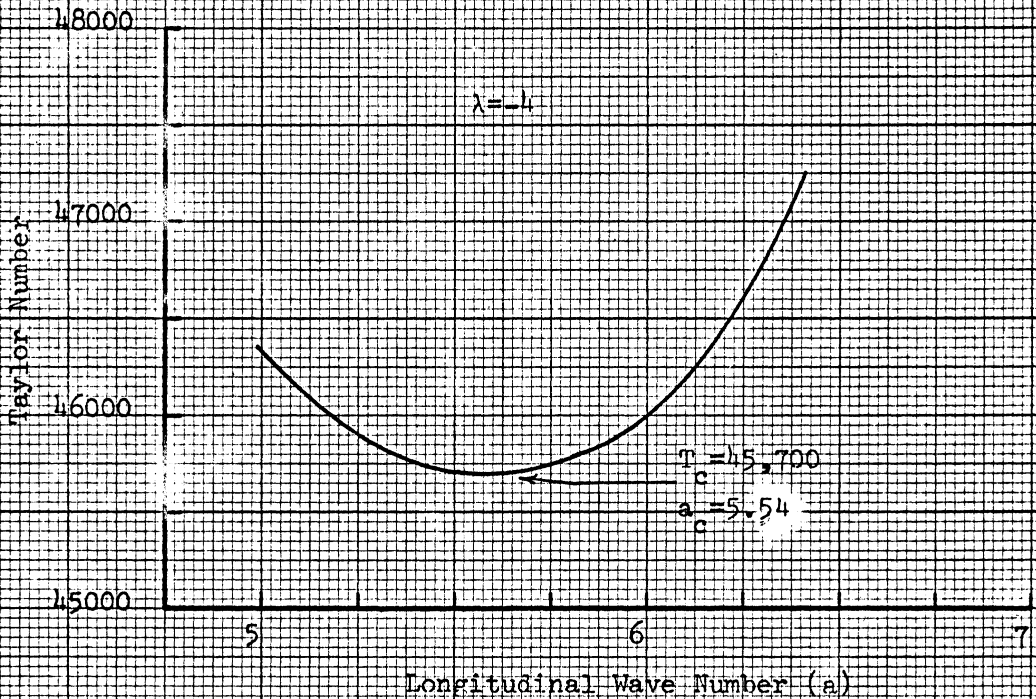


Figure 4. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for λ=-15 and λ=-4 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0

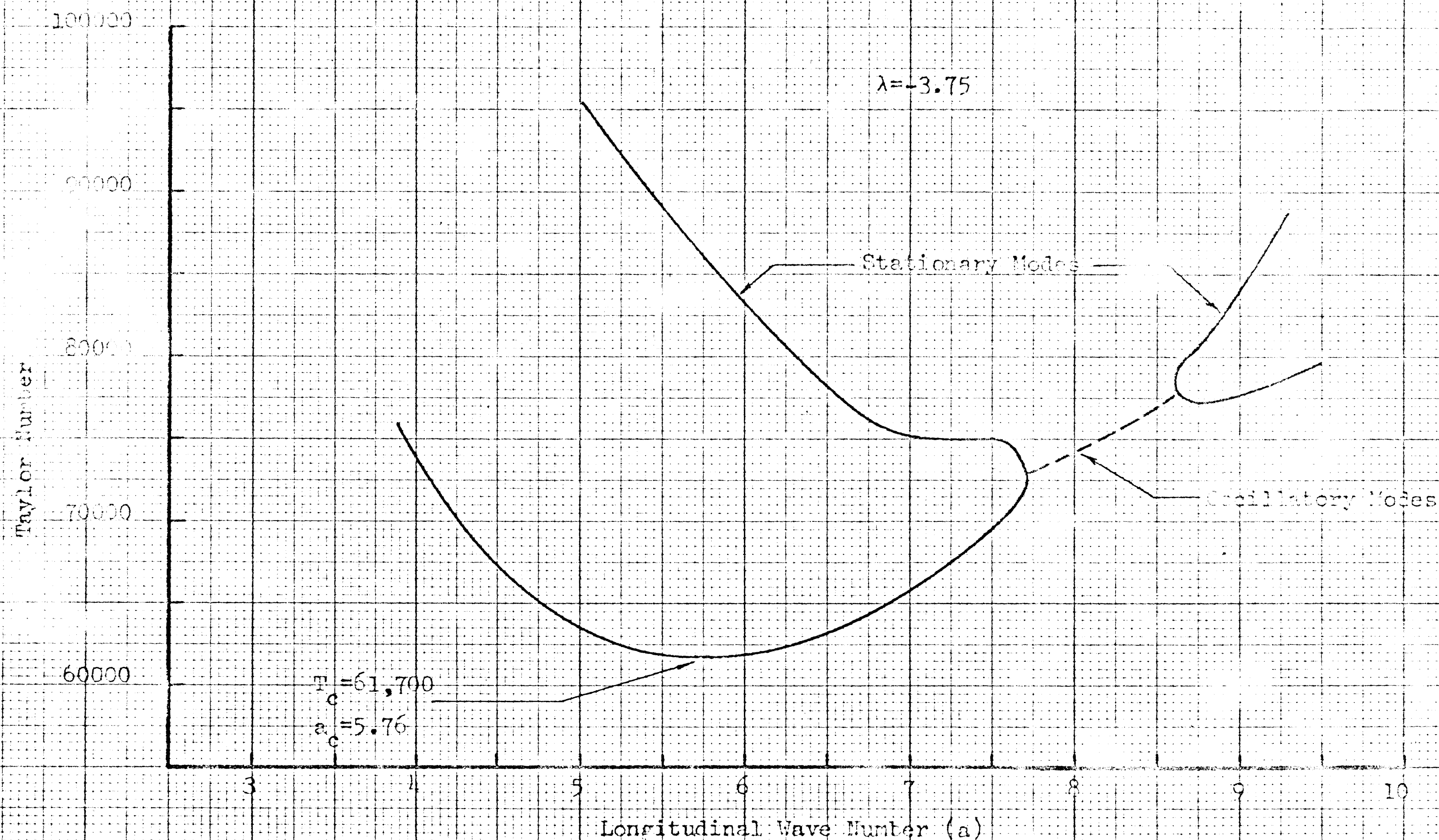


Figure 5. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for $\lambda=-3.75$ Assuming Axisymmetric Disturbances and $\mu=0$. (The Solid Curves Represent Stationary Marginal States and the Broken Curve Represents Oscillatory Marginal States)

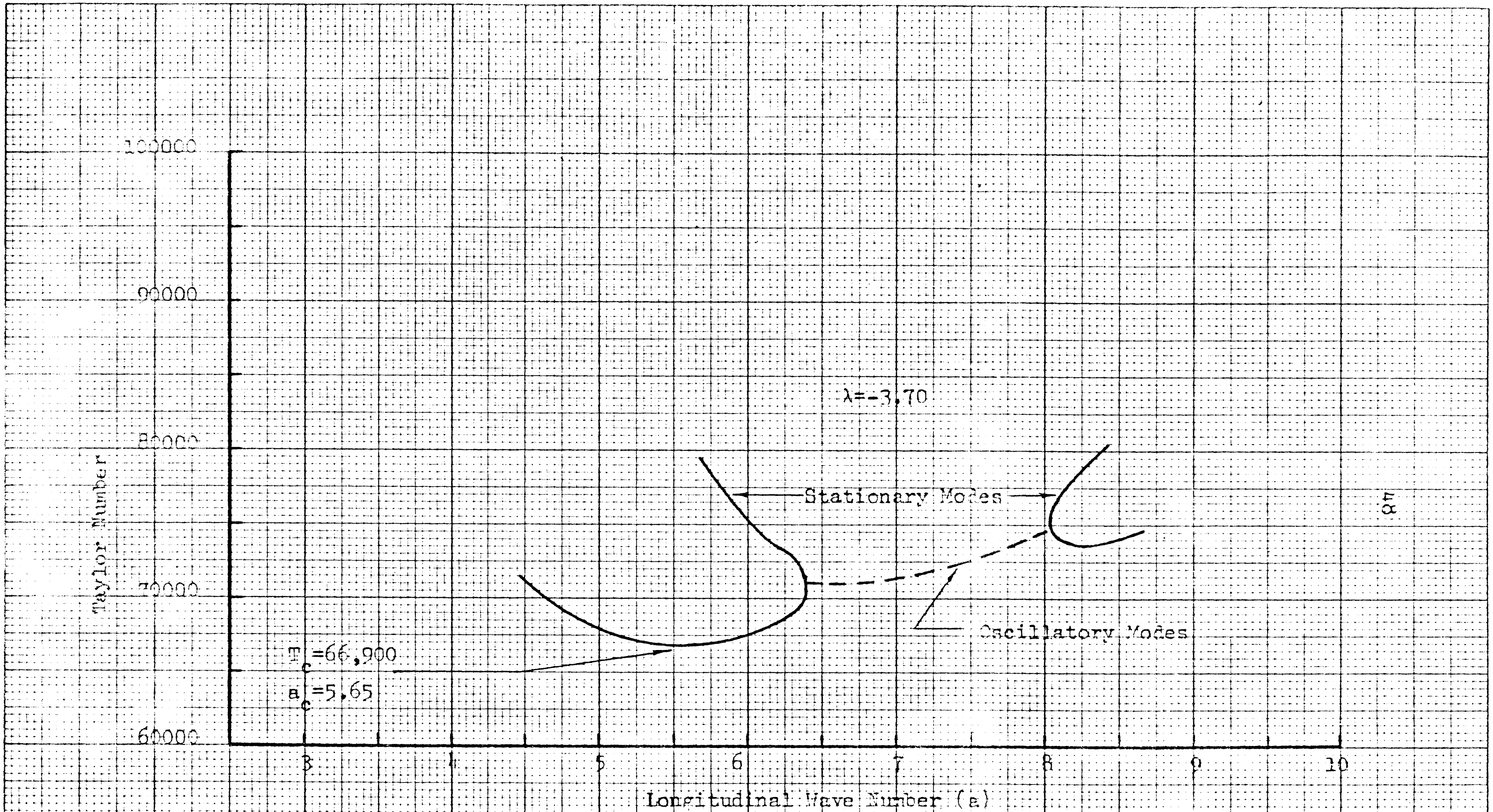


Figure 6. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for λ=-3.70 Assuming Axisymmetric Disturbances and μ=0. (The Solid Curves Represent Stationary Marginal States and the Broken Curve Represents Oscillatory Marginal States)

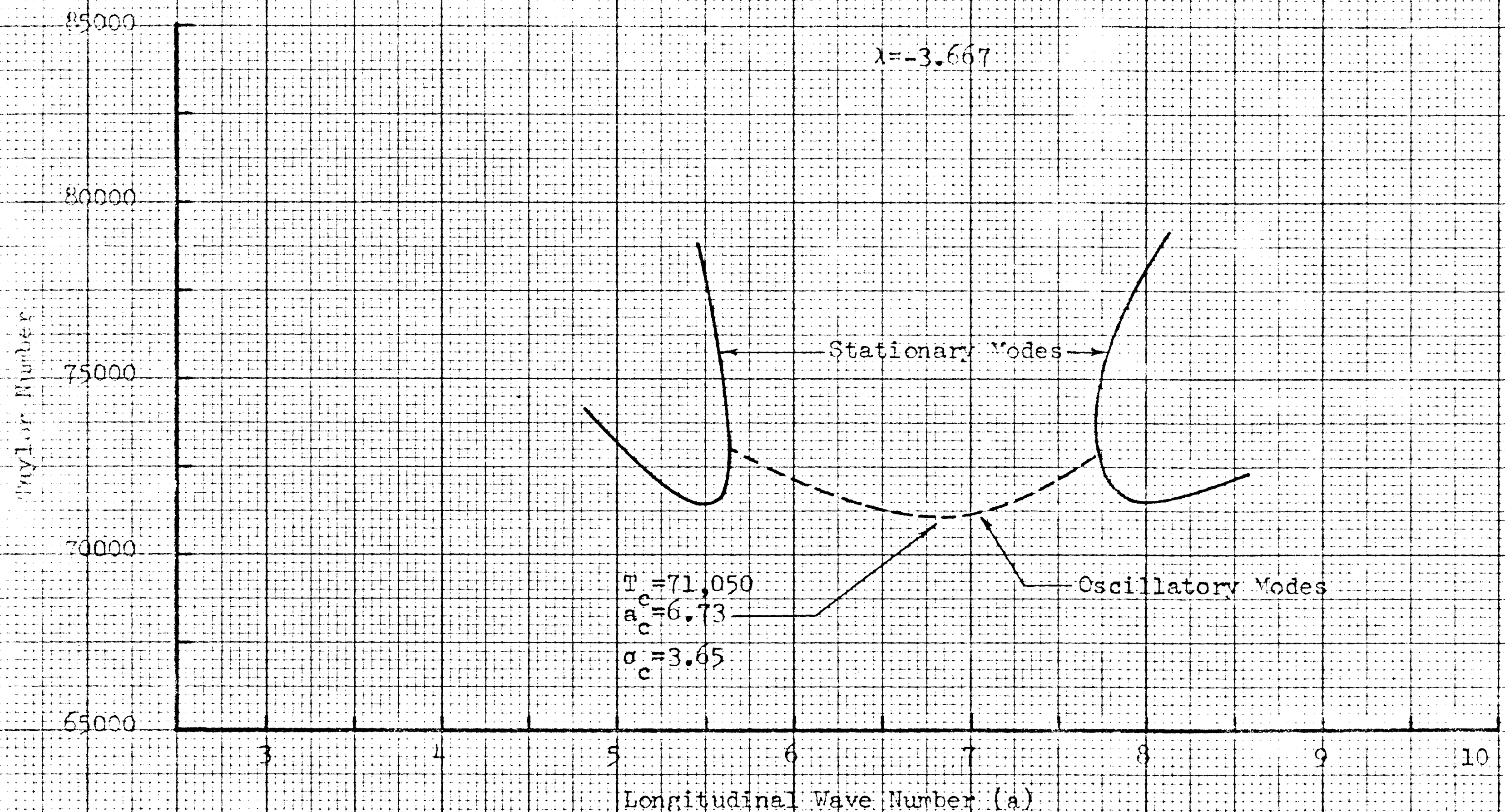


Figure 7. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for $\lambda=-3.667$ Assuming Axisymmetric Disturbances and $\mu=0$. (The Solid Curves Represent Stationary Marginal States and the Broken Curve Represents Oscillatory Marginal States)

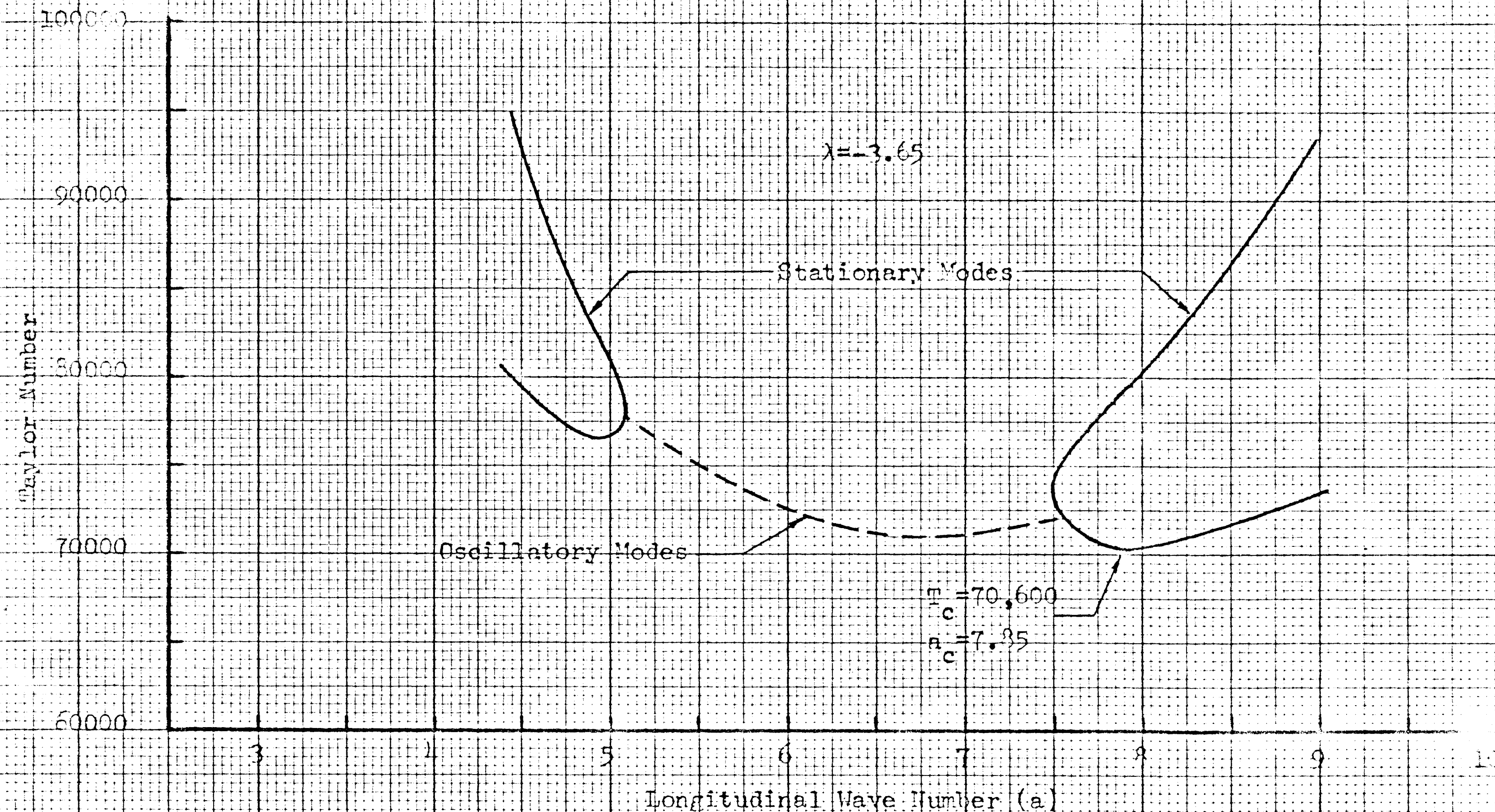


Figure 8. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for $\lambda=-3.65$ Assuming Axisymmetric Disturbances and $\mu=0$. (The Solid Curves Represent Stationary Marginal States and the Broken Curve Represents Oscillatory Marginal States)

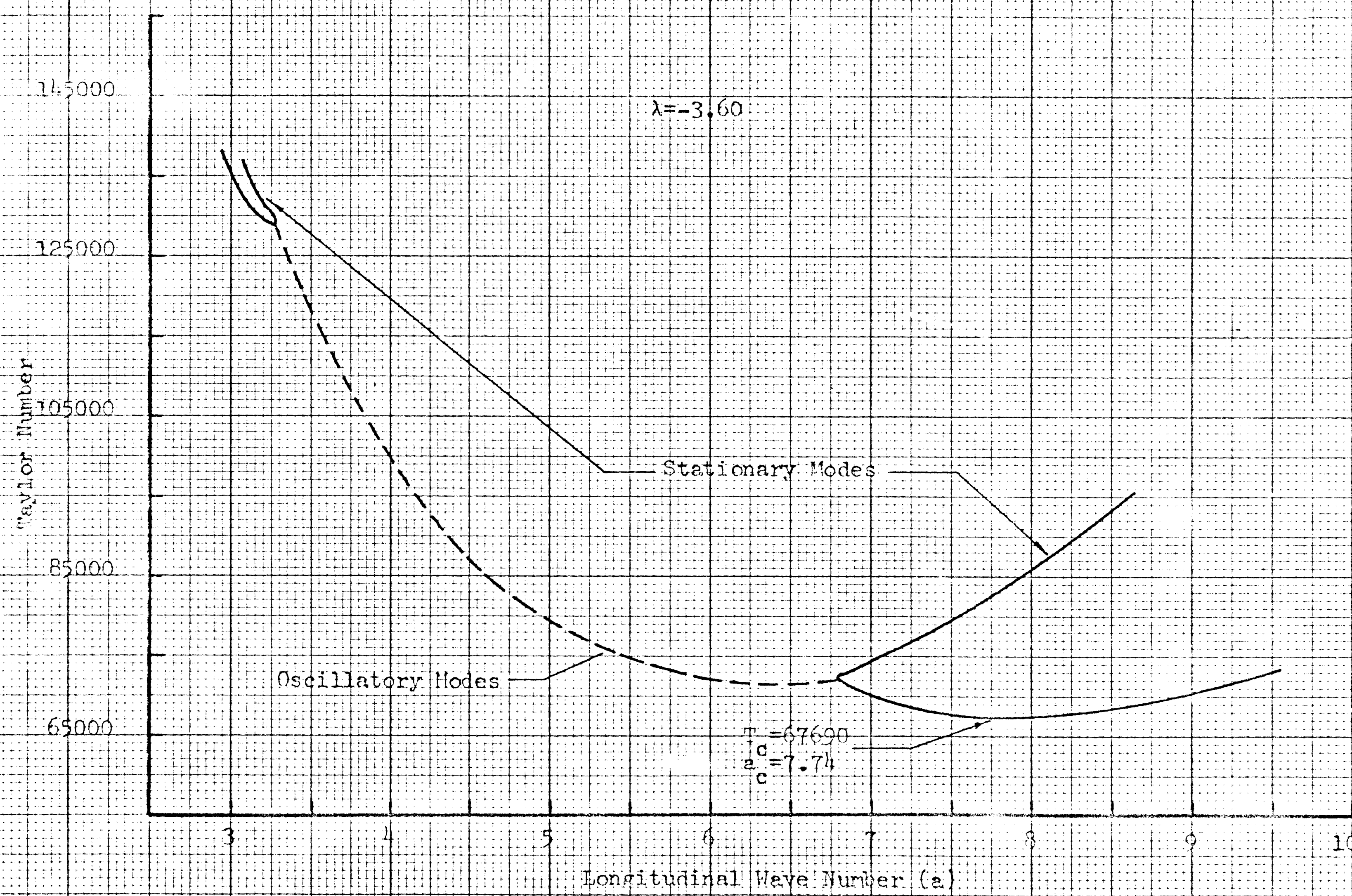
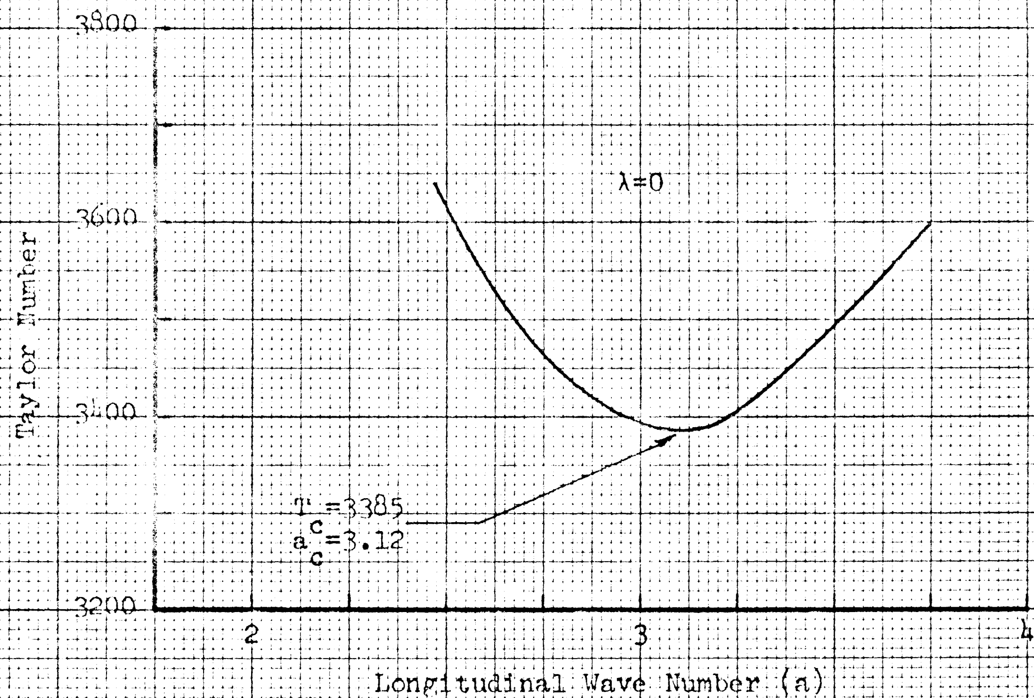


Figure 9. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for $\lambda=-3.60$ Assuming Axisymmetric Disturbances and $\mu=0$. (The Solid Curves Represent Stationary Marginal States and the Broken Curve Represents Oscillatory Marginal States)



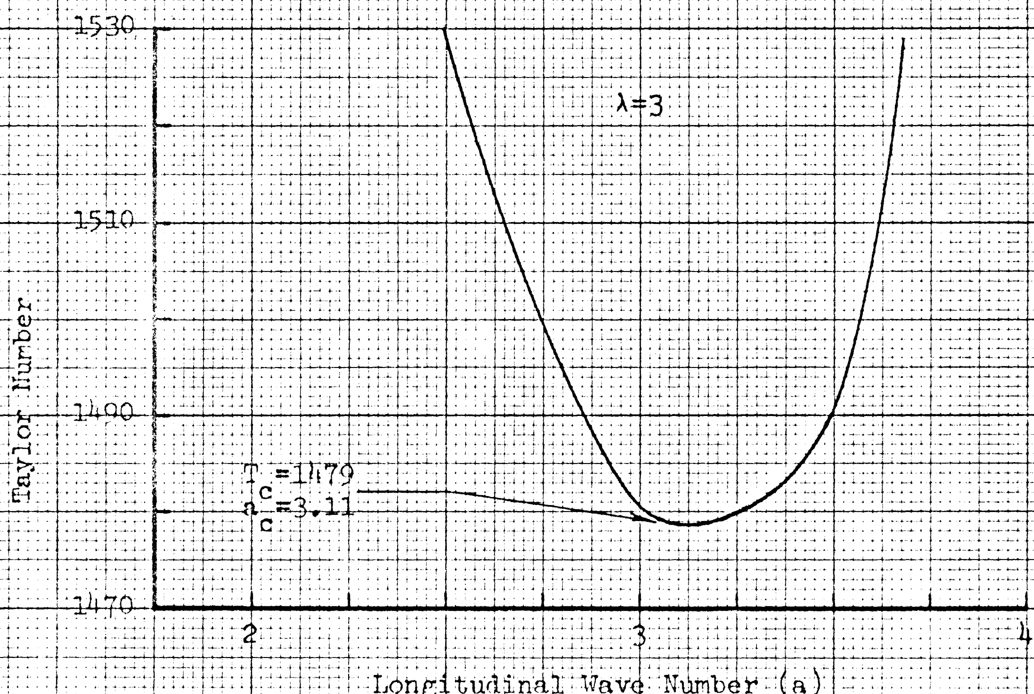


Figure 10. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for λ=0 and λ=3 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0

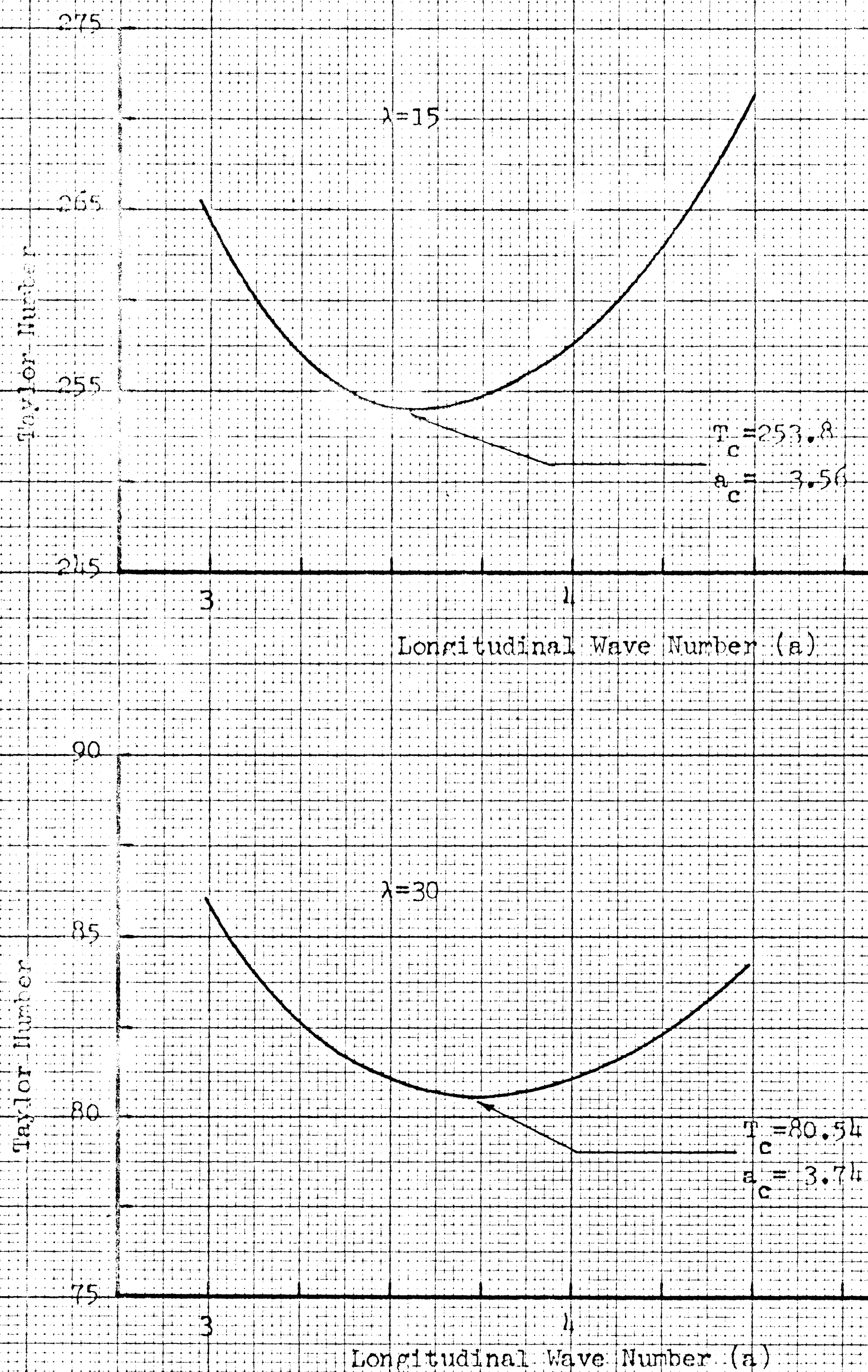
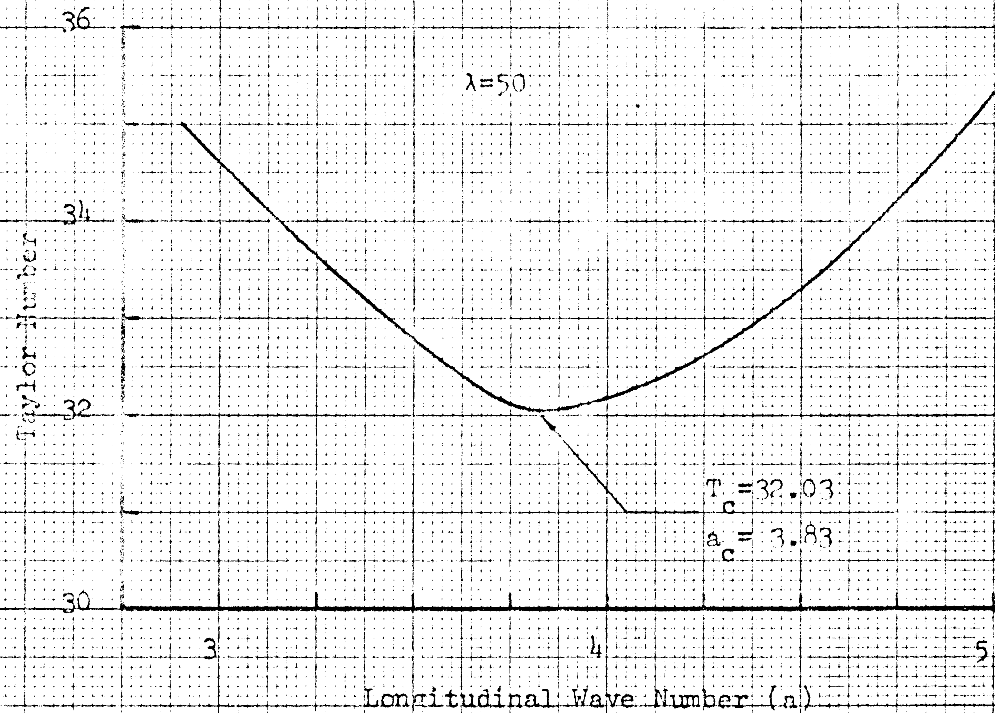


Figure 11. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for λ=15 and λ=30 Assuming Axisymmetric Disturbances, a Stationary Marginal State and u=0



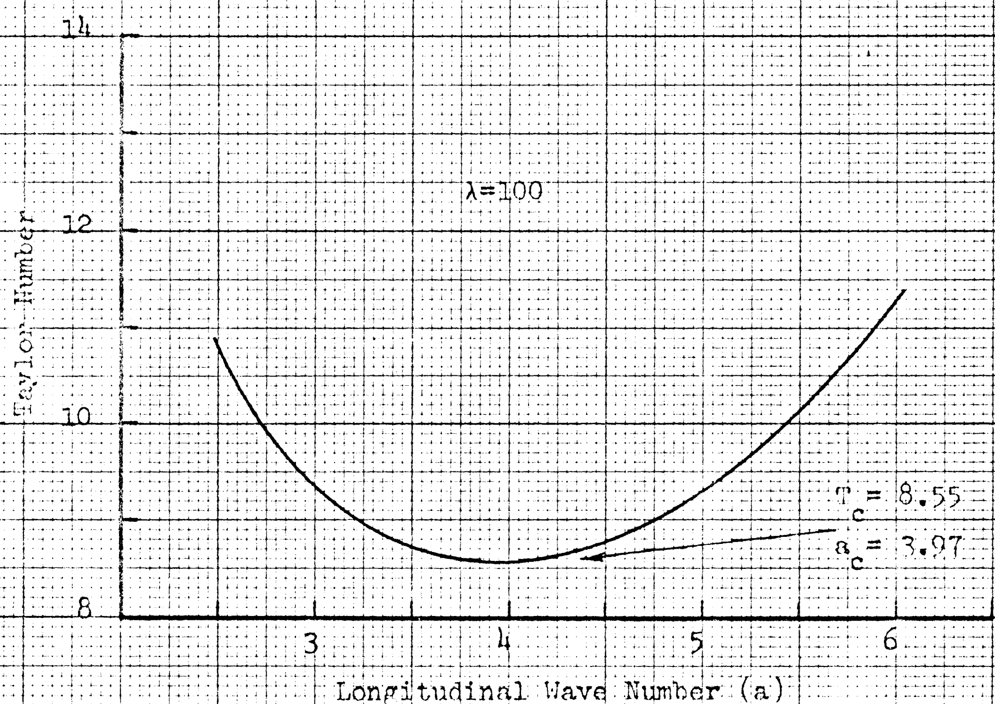


Figure 12. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for λ=50 and λ=100 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0

characteristic parameter for the Dean problem and is defined by

$$\Lambda = \frac{72V_m^2 d^3}{R_1 \nu^2} . \qquad (87)$$

Then based on the analytical results for the Dean problem and this similarity in the two sets of equations, Chandrasekher concluded that for $|\lambda| \rightarrow \infty$ the critical Taylor number for the onset of instability tends asymptotically to the value

$$\frac{T_c \lambda^2}{1-\mu} \quad \Lambda_c = 9.3 \times 10^4 . \qquad (88)$$

For $\Omega_1 \equiv 0$ ($\lambda = \infty$) the combined problem does reduce to the Dean problem, however, for $\Omega_1 = 0$ (even though λ is very large) the velocity distribution near the inner cylinder does differ significantly from that of the Dean problem. Therefore, some error is to be expected in applying the asymptotic results to predict the critical Taylor number for large values of λ. Comparing Chandrasekher's asymptotic results with the results of this study, for $\lambda = \pm 100$ his results indicate $T_c = 9.3$. From the results of this investigation (see Figures 3 and 12) for $\lambda = +100$ then $T_c = 8.55$; while for $\lambda = -100$, $T_c = 10.05$. Thus, the asymptotic results of Chandrasekher for the critical Taylor number are slightly high for large positive values of λ and slightly low for large negative values of λ. This can perhaps be accounted for by considering the shape of the velocity distribution curves for the Dean problem and for large positive and negative values of λ.

These curves are shown in Figure 14. For large positive values of λ the effective gap width would appear to be larger than the actual gap width for the Dean problem while for large negative values of λ the effective gap width is smaller than for the Dean problem. From equations (87) and (88), based on the asymptotic analysis it can be observed that

$$\frac{T_c \lambda^2}{1-\mu} = \frac{72 V_m^2 d^3}{R_1 \nu}$$

and a larger effective gap width (other parameters being constant) will lead to a larger critical Taylor number. On the other hand, a smaller effective gap width would lead to a smaller critical Taylor number. Thus, critical Taylor numbers predicted from this analysis should be higher than the actual values for large positive λ and smaller than the actual values for large negative values of λ.

For most values of λ the onset of instability with an oscillatory marginal state either does not appear possible or it occurs at a much higher Taylor number than does the stationary marginal state. The critical mode of instability due to axisymmetric disturbances thus exhibits a stationary marginal state under most conditions as is shown in these results and verified experimentally in the literature. In general, at a given value of λ, a neutral stability diagram is multiple valued, with one branch for each of

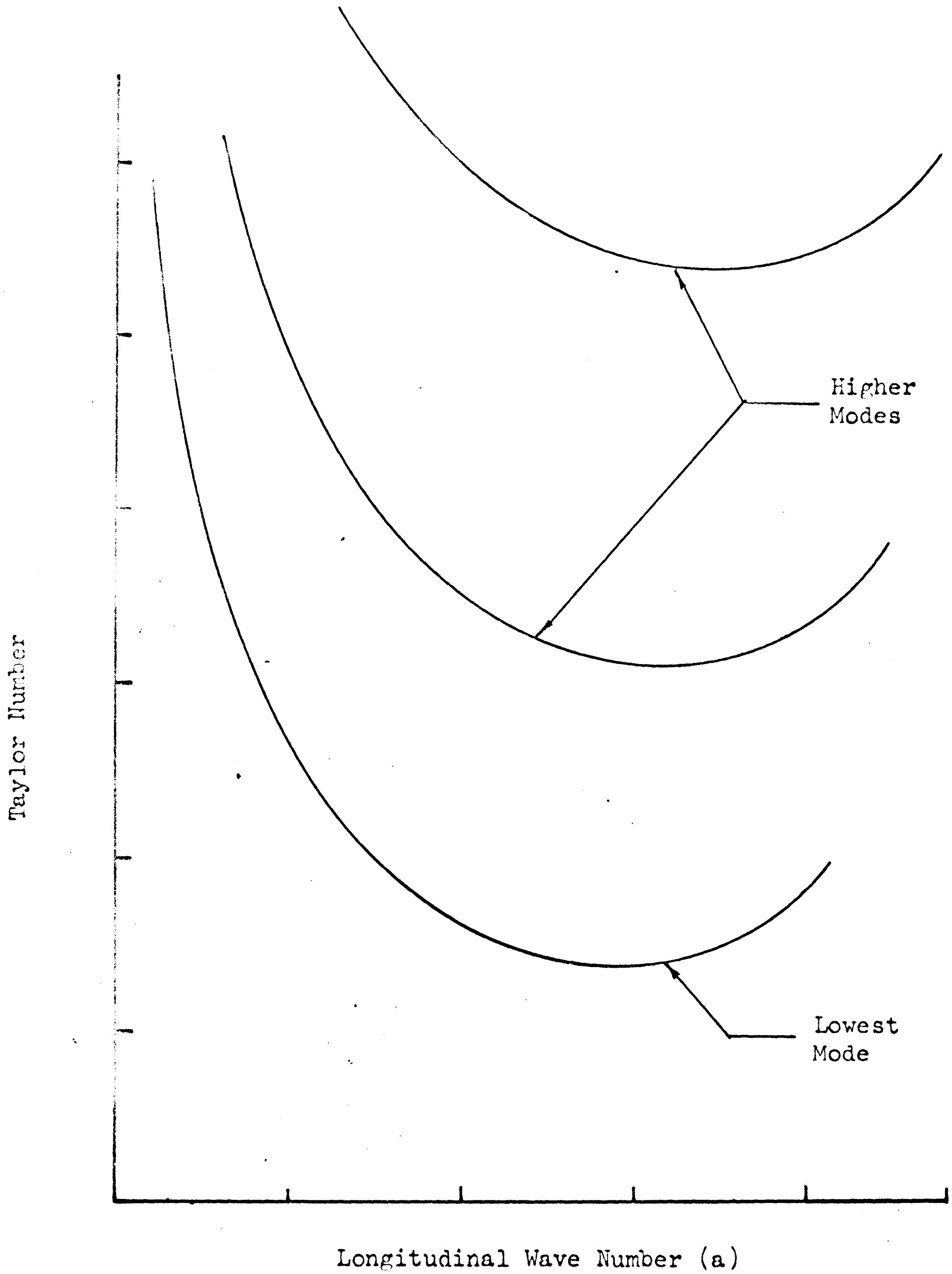


Figure 13. Schematic Representation of the Multiple Valued Neutral Stability Curves

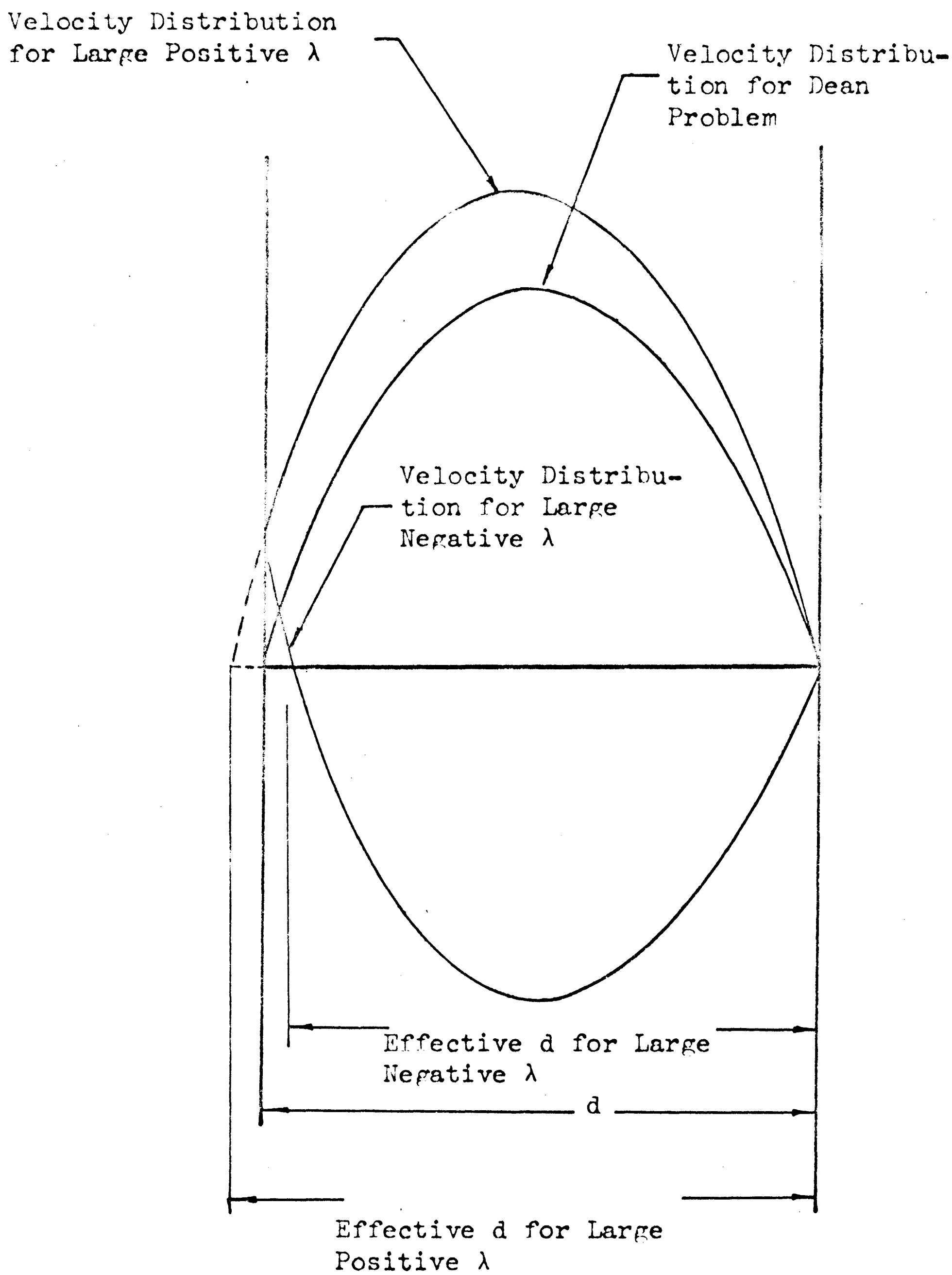


Figure 14. Comparison of Initial Velocity Distributions for Dean Problem and for Large Positive and Negative Values of λ

an infinite sequence of normal modes and a minimum value of T for each branch. This is shown schematically in Figure 13. For most values of λ, however, one branch of the neutral curve corresponds to the lowest minimum value of T and lies well below the others.

In the region between $\lambda=-3.0$ and $\lambda=-4.0$ the lowest stationary critical mode is suppressed to such an extent that the two lowest branches of the neutral curve begin to converge and finally intersect and break into two separate loops with a gap in between. In the gap between the two curves there now appears to be no possible stationary marginal states. Thus, considering only stationary marginal states there results a curve of two loops separated by a region where no stationary marginal states are possible. This is shown in Figures 5 through 9. In the region around $\lambda=-3.667$ the two lowest neutral curves intersect such that the gap between the curves occurs near what would normally appear to be the critical Taylor number. As λ increases or decreases from this region, the intersection occurs further and further from the apparent minimum point on the lowest neutral curve. For λ more negative than -3.667, the left loop of the curve yields the critical Taylor number and corresponding critical wave number while for λ less negative than -3.667 the right loop of the curve yields the critical flow condition. Thus, there is a sudden jump of the critical longitudinal wave number near $\lambda=-3.667$ corresponding to the shift of the critical stationary mode from the left loop to the right loop of the neutral curve. For there to exist values of the longitudinal wave number for which the fluid never becomes unstable (at least due to the two lowest neutral curves) appears mathematically as well as intuitively unreasonable. This point is discussed by Chang [4] in some detail for a similar problem. It then becomes reasonable to ask if there exists a value of the Taylor number which makes the modes neutrally stable, but oscillatory rather than stationary. Such a Taylor number has been found to exist.

Figures 5 through 9 also show the onset of instability with an oscillatory marginal state and resulting from axisymmetric disturbances. These solutions are seen to "tie together" the two loops of the curve obtained from the stationary marginal states. Thus, there are no longitudinal wave numbers at which instability never sets in due to axisymmetric disturbances.

In general the stationary marginal states are still the critical modes of instability, however, there is a very small region near $\lambda=-3.667$ where the oscillatory modes are the critical modes due to axisymmetric disturbances. This is shown in Figure 7. The region is small where the oscillatory modes due to axisymmetric disturbance are critical, being confined to those values of λ where the two lowest neutral curves intersect very near the apparent minimum point on the lowest curve. Of those conditions investigated in detail only $\lambda=-3.667$ actually has an oscillatory marginal state as the critical mode for axisymmetric disturbances.

In Figures 15 through 30 the normalized eigenfunction (u, v and w) curves are shown at the onset of instability for each value of λ and for the critical Taylor number. All of these have stationary marginal states except for λ=-3.667, which has an oscillatory marginal state. The eigenfunction curves (for u, v, and w) are seen to vary with λ in a smooth orderly manner except in the region near λ=-3.667. Between λ=-3.65 and λ=-3.70 the eigenfunction curves completely change in character. This is in the region where the neutral curves for stationary modes consist of two loops and the abrupt change in the eigenfunctions corresponds to the shifting of the location of the critical Taylor number from one loop to the other.

The normalized eigenfunction curves correlate very well with the Rayleigh criterion. As can be seen from Figure 2, there are regions between the cylinders where the fluid is unstable and others where the fluid is stable according to the Rayleigh criterion. It appears reasonable that the onset of instability will be determined, principally, by the unstable zone of widest extent provided that the regions of unstable flow are well separated. If the regions of unstable flow are not well separated, the interaction between the zones may contribute significantly to the onset of instability. The critical Taylor number for the flow would also seem to vary in some inverse fashion with the width of the widest unstable region. A large unstable region would appear to quickly lead to the onset of instability at low Taylor numbers (slow rotation of inner cylinder)

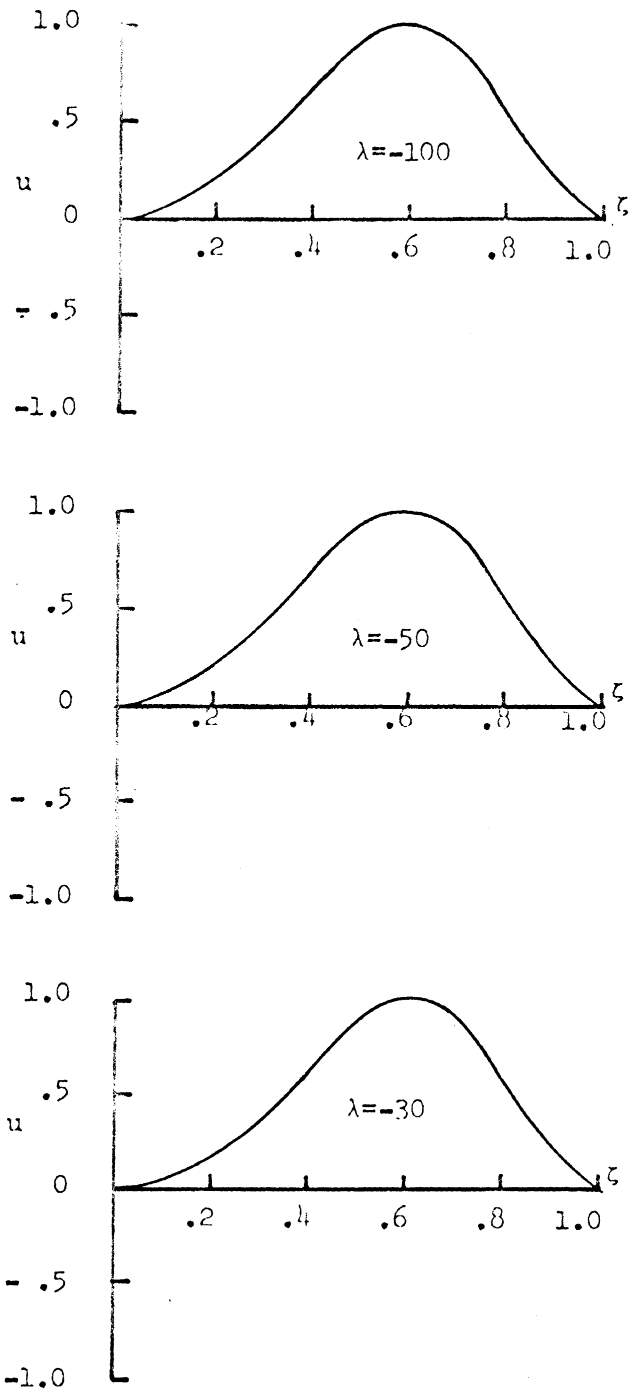


Figure 15. Normalized Eigenfunction u at the Onset of Instability for $\lambda=-100$, $\lambda=-50$ and $\lambda=-30$ Assuming Axisymmetric Disturbances, a Stationary Marginal State and $\mu=0$

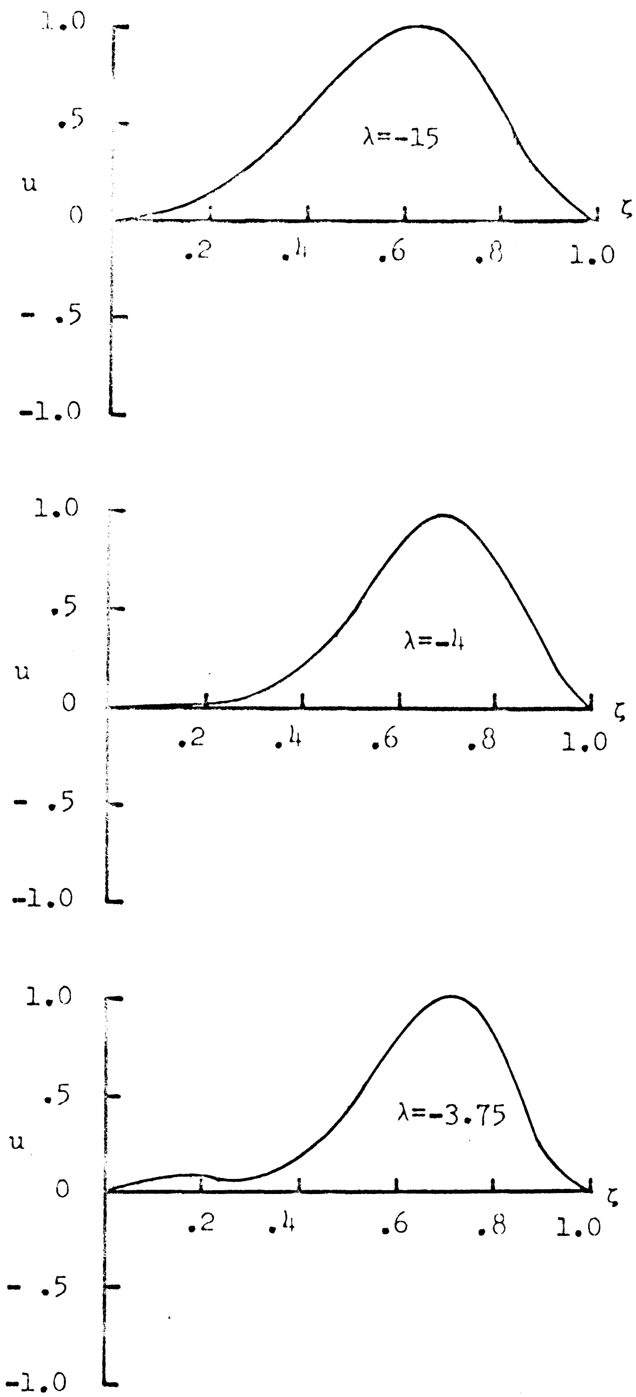


Figure 16. Normalized Eigenfunction u at the Onset of Instability for $\lambda=-15$, $\lambda=-4$ and $\lambda=-3.75$ Assuming Axisymmetric Disturbances, a Stationary Marginal State and $\mu=0$

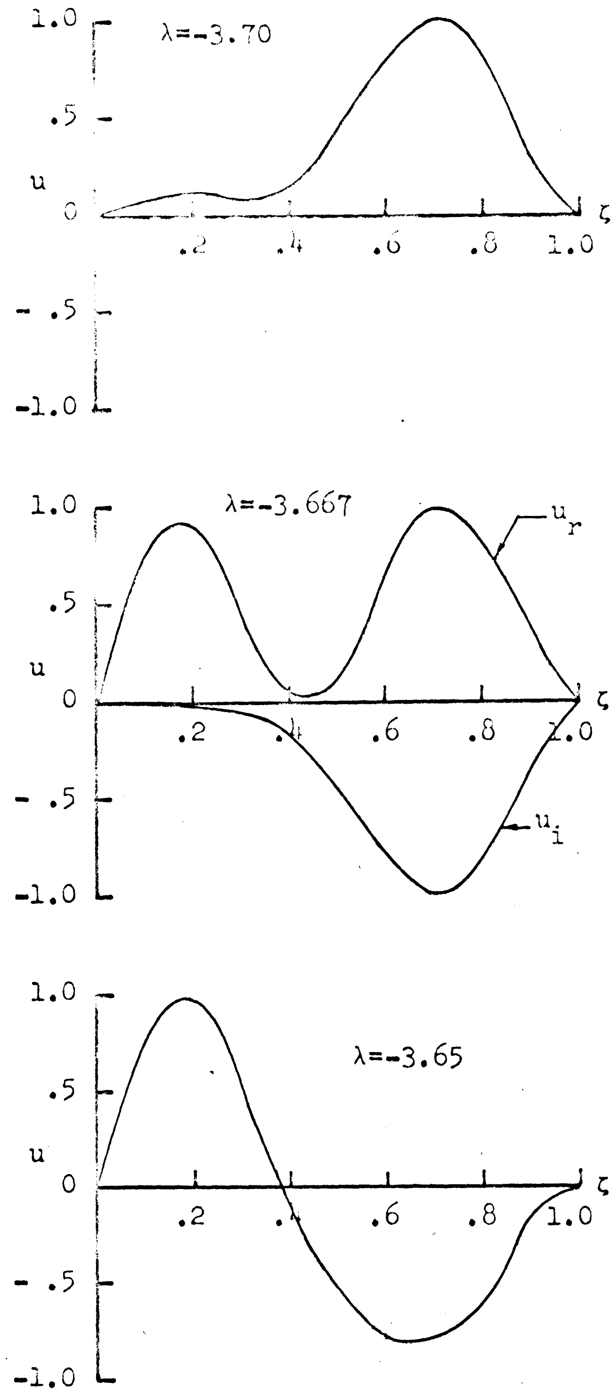


Figure 17. Normalized Eigenfunction u at the Onset of Instability for $\lambda=-3.70$, $\lambda=-3.667$ and $\lambda=-3.65$ Assuming Axisymmetric Disturbances and $\mu=0$ ($\lambda=-3.70$ and $\lambda=-3.65$ are for Stationary Marginal States, $\lambda=-3.667$ is for an Oscillatory Marginal State)

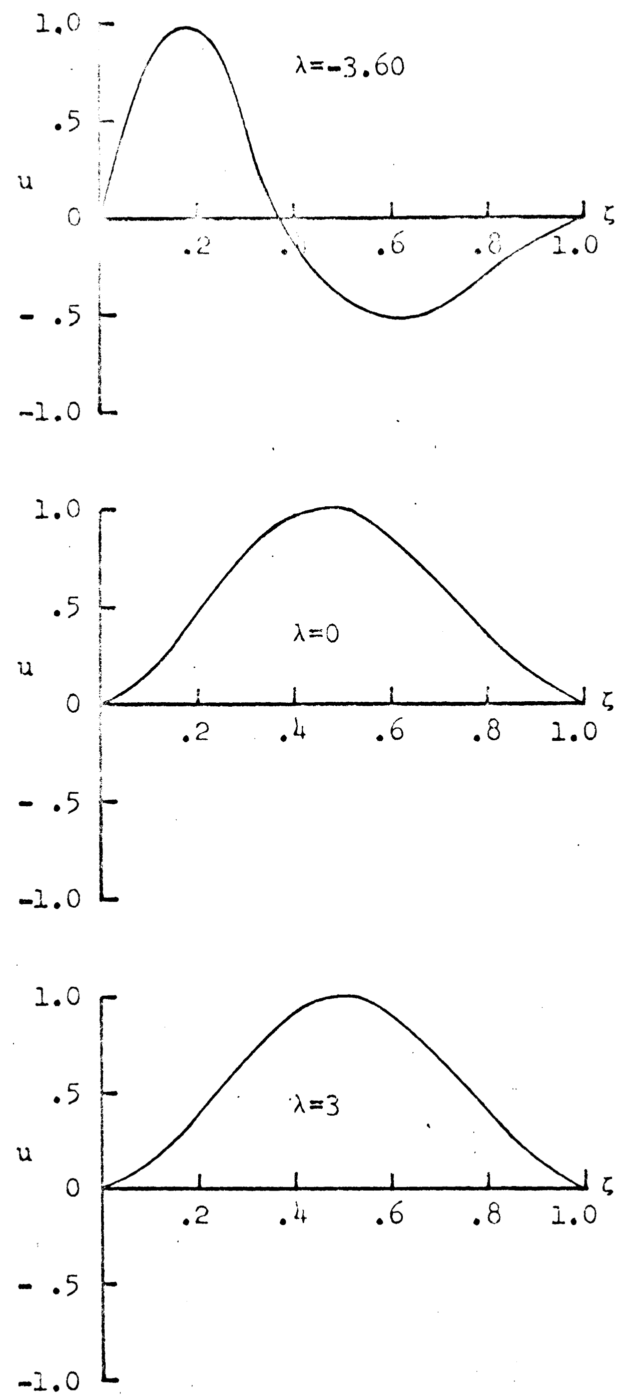


Figure 18. Normalized Eigenfunction u at the Onset of Instability for $\lambda=-3.60$, $\lambda=0$ and $\lambda=3$ Assuming Axisymmetric Disturbances, a Stationary Marginal State and $\mu=0$

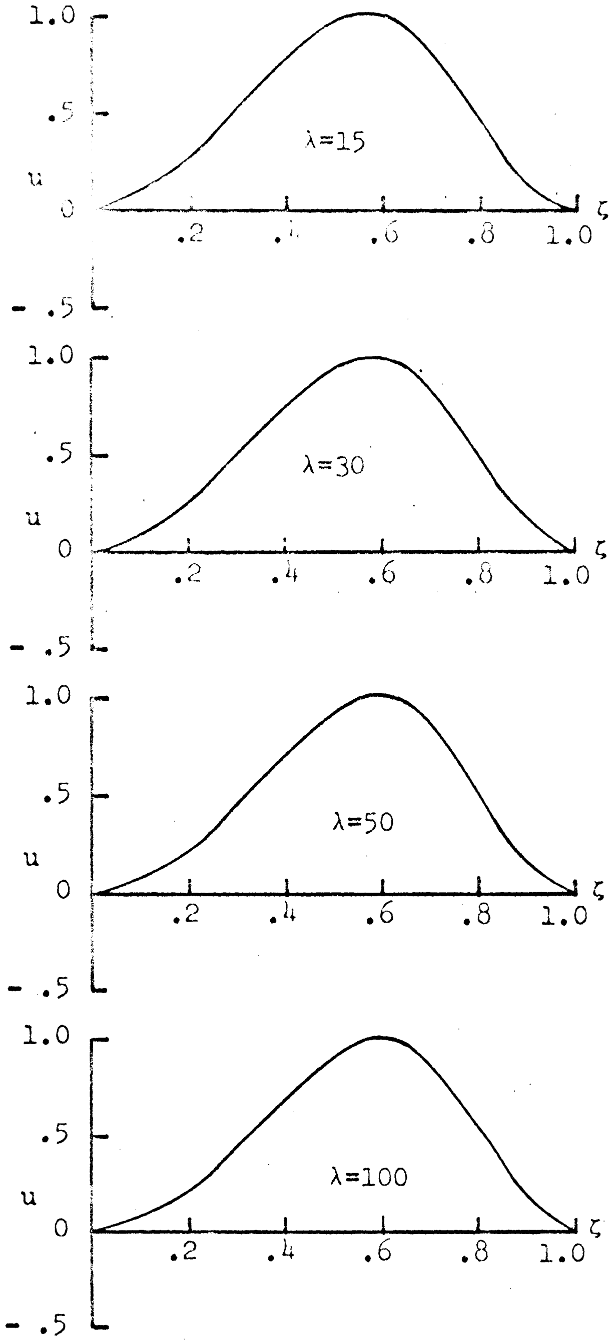


Figure 19. Normalized Eigenfunction u at the Onset of Instability for λ=15, λ=30, λ=50 and λ=100 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0

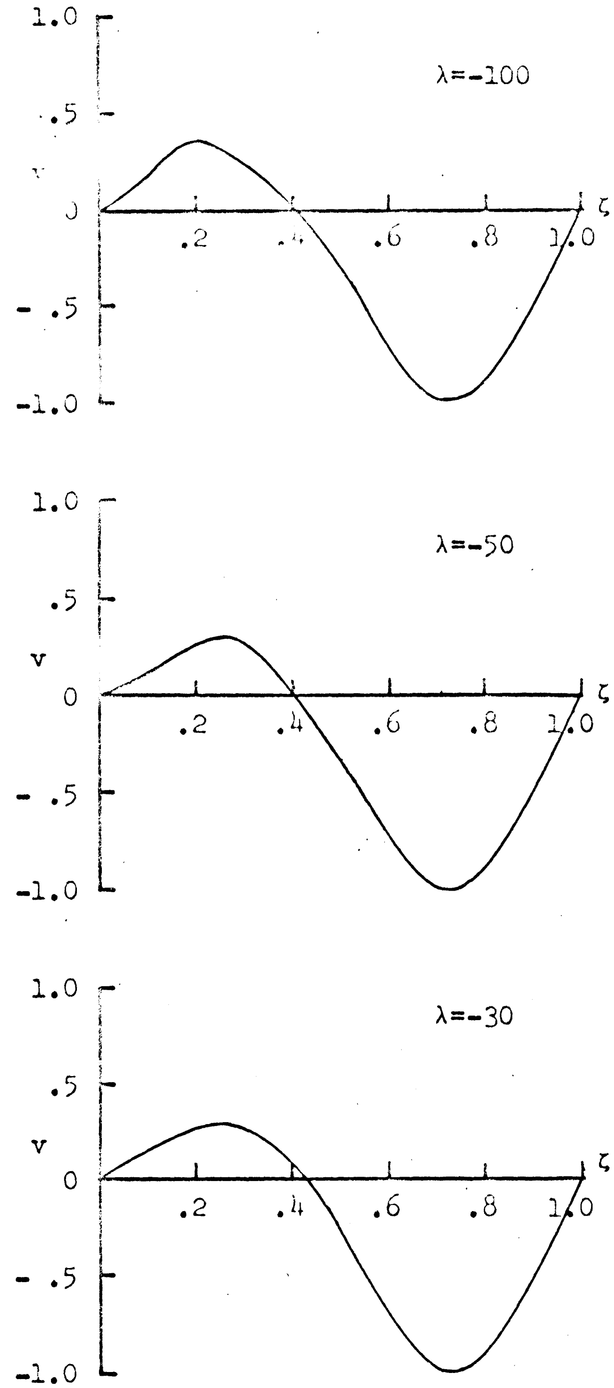


Figure 20. Normalized Eigenfunction v at the Onset of Instability for λ=-100, λ=-50 and λ=-30 Assuming Axisymmetric Disturbances, a Stationary Marginal State and $\mu=0$

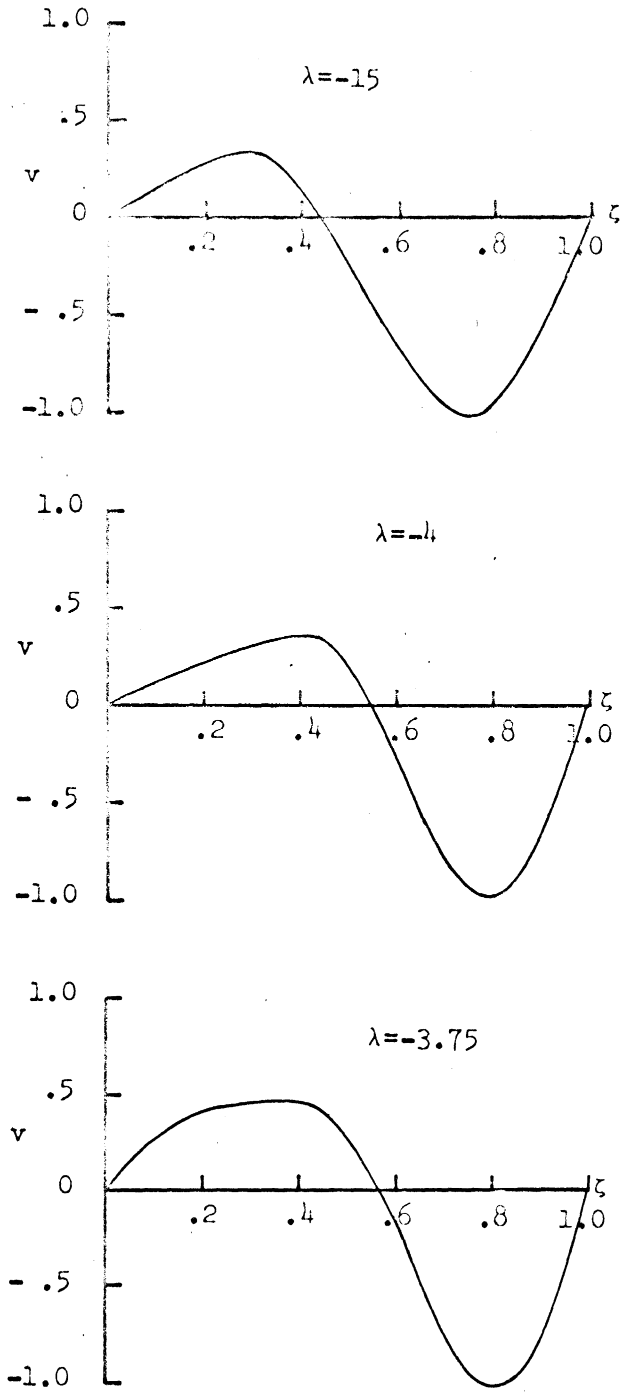


Figure 21. Normalized Eigenfunction v at the Onset of Instability for $\lambda=-15$, $\lambda=-4$ and $\lambda=-3.75$ Assuming Axisymmetric Disturbances, a Stationary Marginal State and $\mu=0$

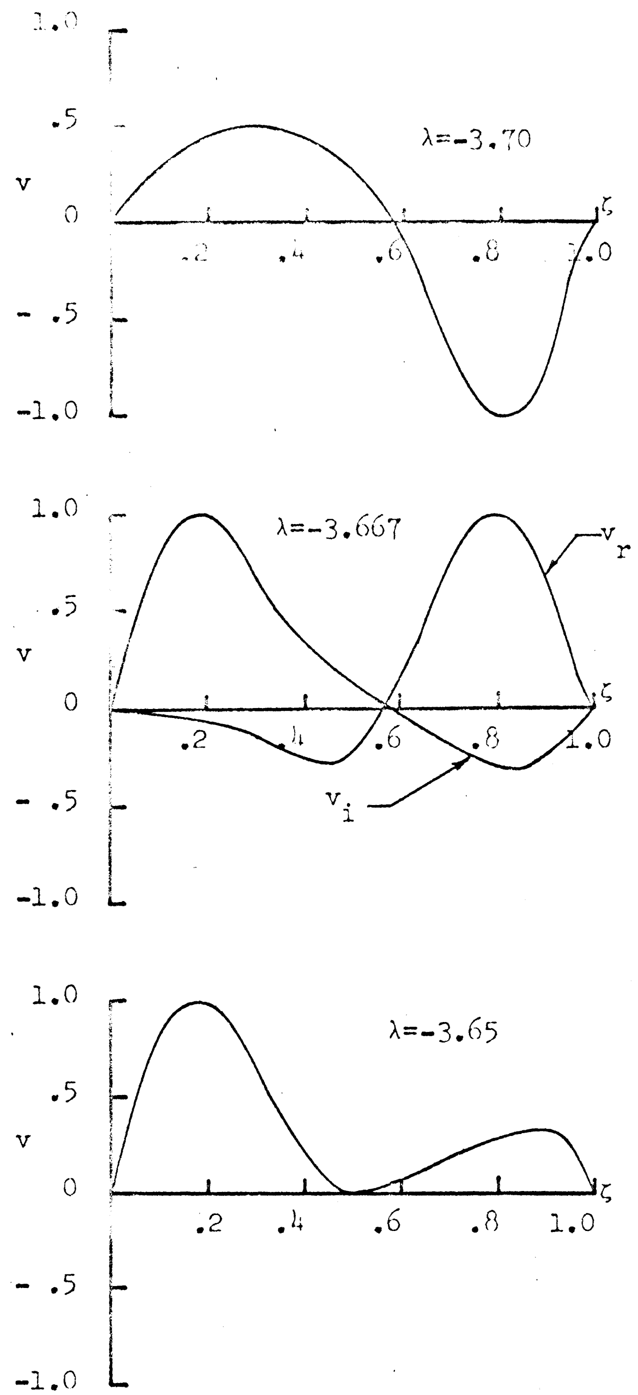


Figure 22. Normalized Eigenfunction v at the Onset of Instability for $\lambda=-3.70$, $\lambda=-3.667$ and $\lambda=-3.65$ Assuming Axisymmetric Disturbances and $\mu=0$ ($\lambda=-3.70$ and $\lambda=-3.60$ are for Stationary Marginal States, $\lambda=-3.667$ is for an Oscillatory Marginal State

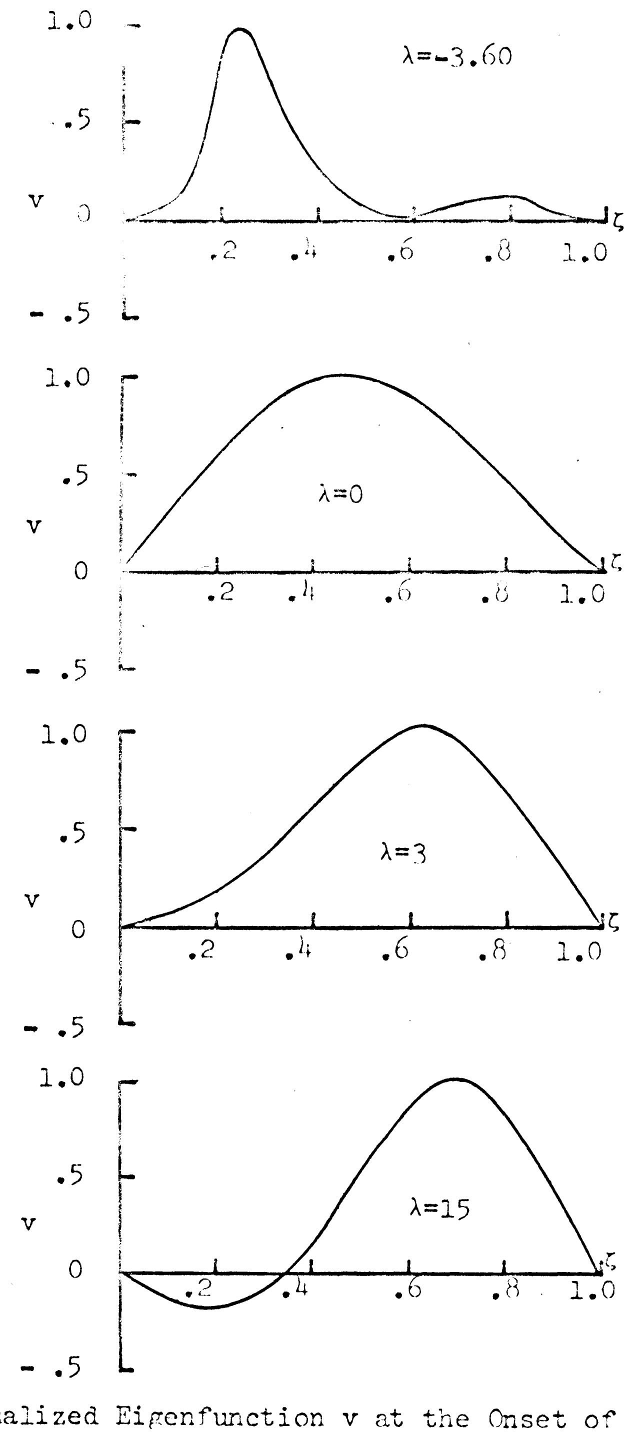


Figure 23. Normalized Eigenfunction v at the Onset of Instability for λ=-3.60, λ=0, λ=3 and λ=15 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0

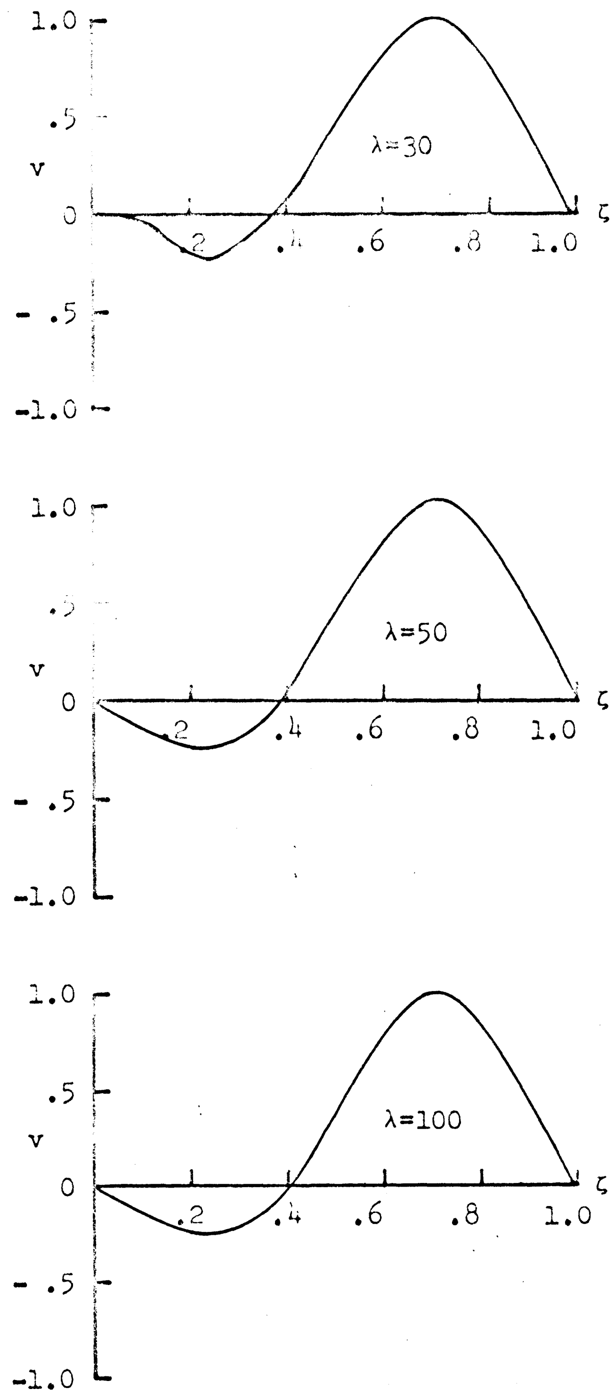


Figure 24. Normalized Eigenfunction v at the Onset of Instability for λ=30, λ=50 and λ=100 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0

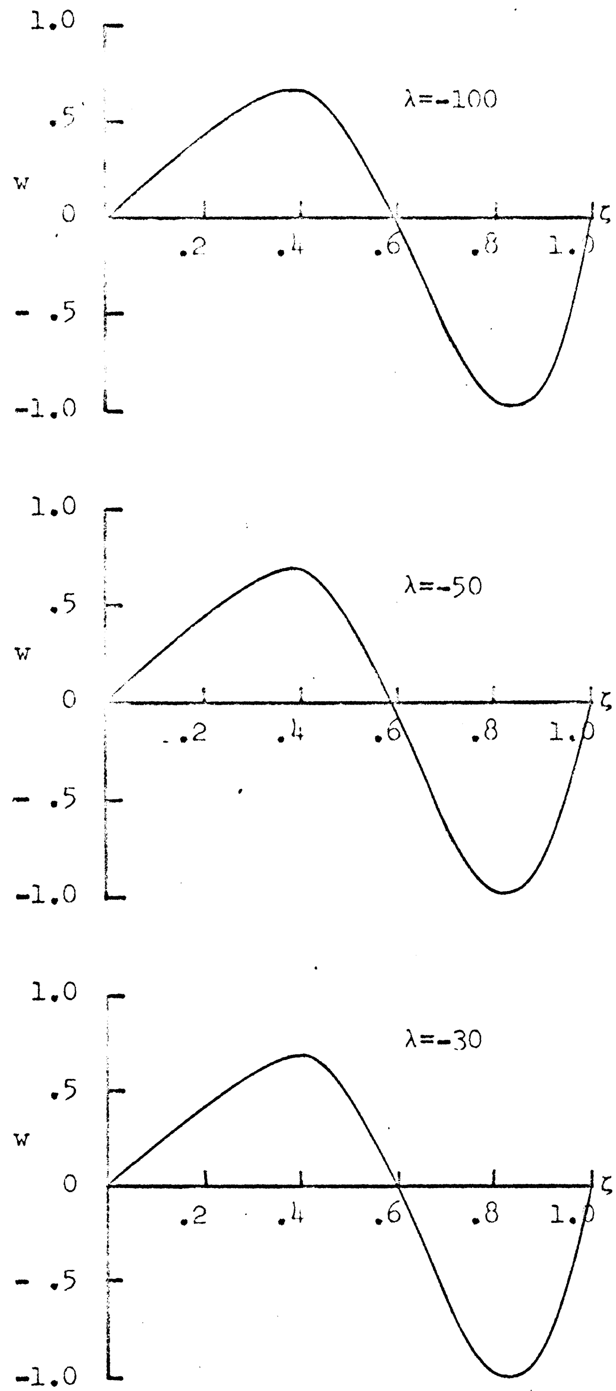


Figure 25. Normalized Eigenfunction w at the Onset of Instability for $\lambda=-100$, $\lambda=-50$ and $\lambda=-30$ Assuming Axisymmetric Disturbances, a Stationary Marginal State and $\mu=0$

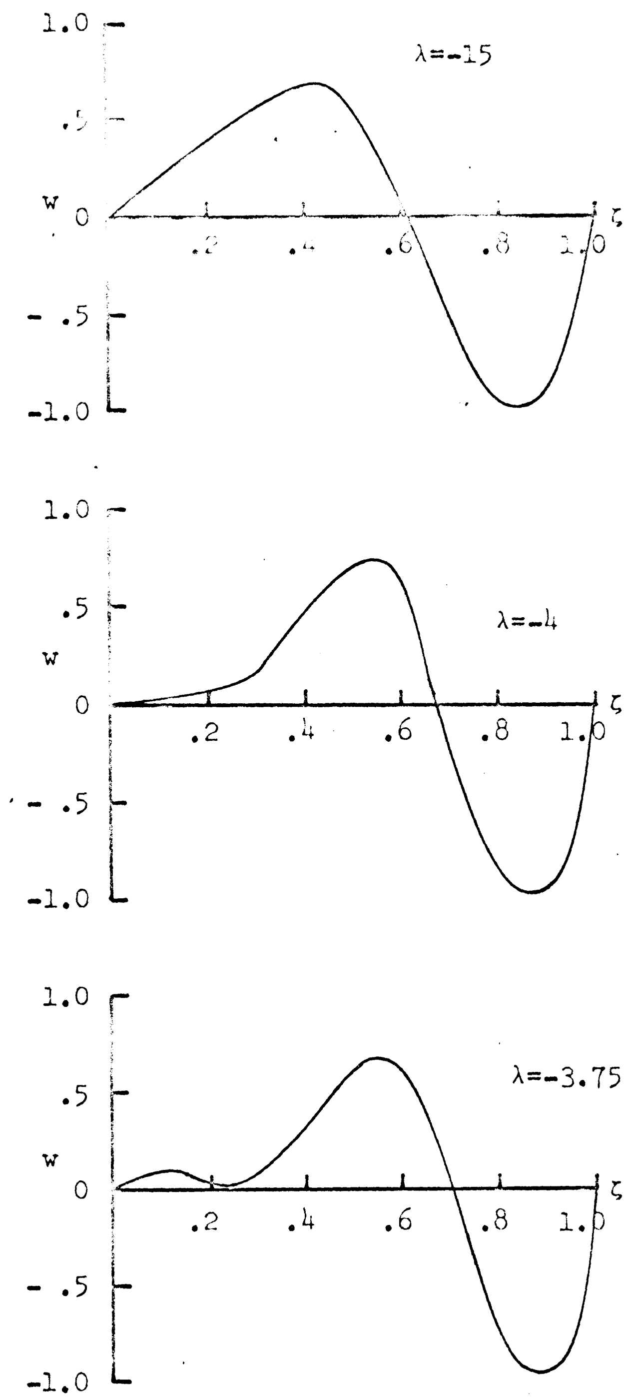


Figure 26. Normalized Eigenfunction w at the Onset of Instability for λ=-15, λ=-4 and λ=-3.75 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0

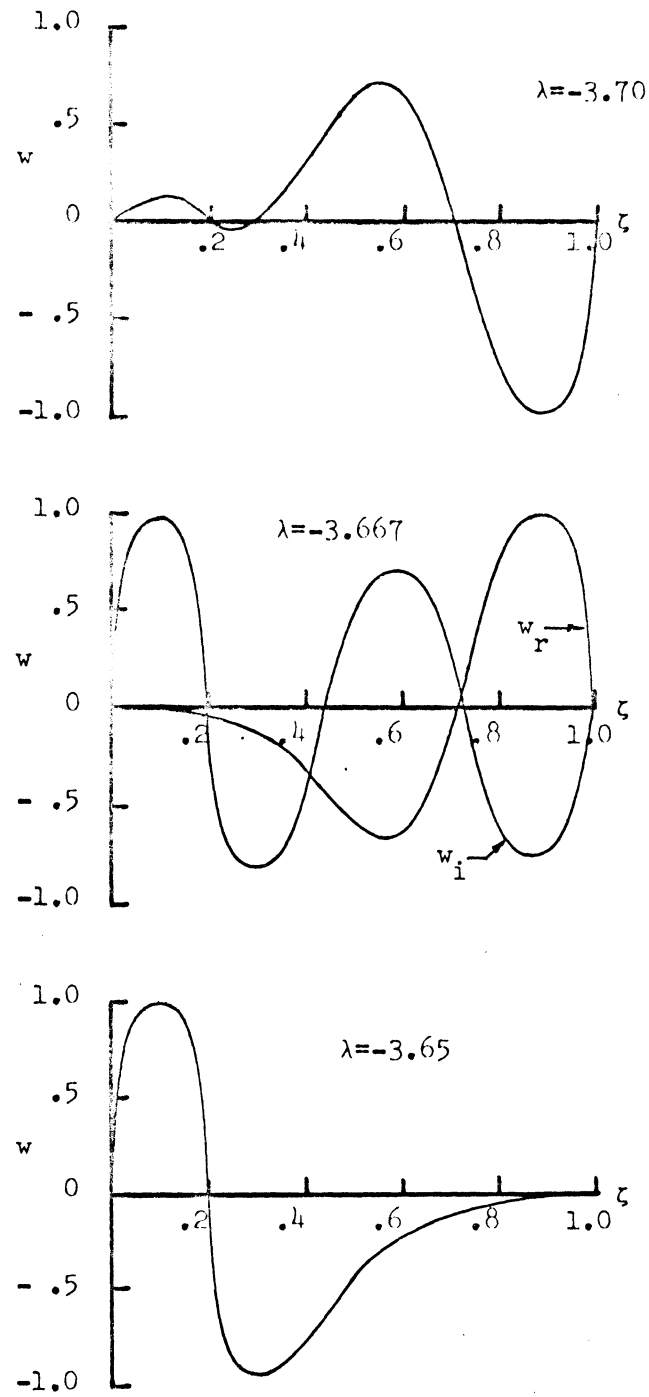


Figure 27. Normalized Eigenfunction w at the Onset of Instability for λ=-3.70, λ=-3.667 and λ=-3.65 Assuming Axisymmetric Disturbances and μ=0 (λ=-3.70 and λ=-3.65 are for Stationary Marginal States, λ=-3.667 is for an Oscillatory Marginal State)

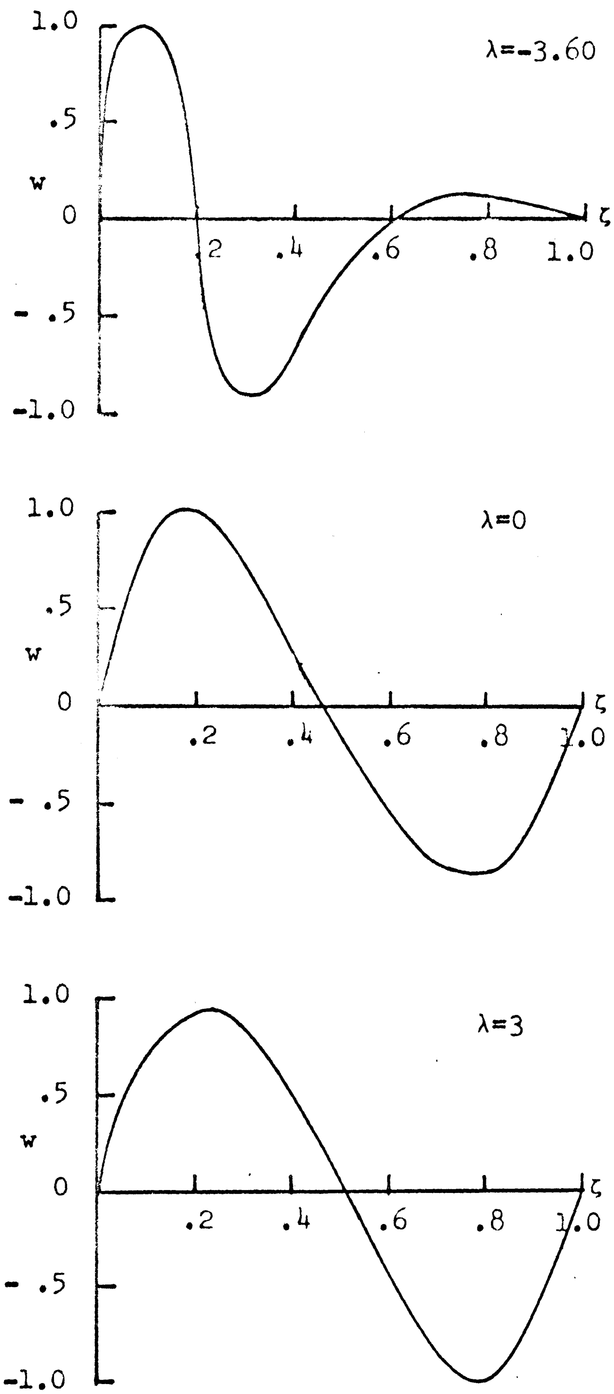


Figure 28. Normalized Eigenfunction w at the Onset of Instability for λ=-3.60, λ=0 and λ=3 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0

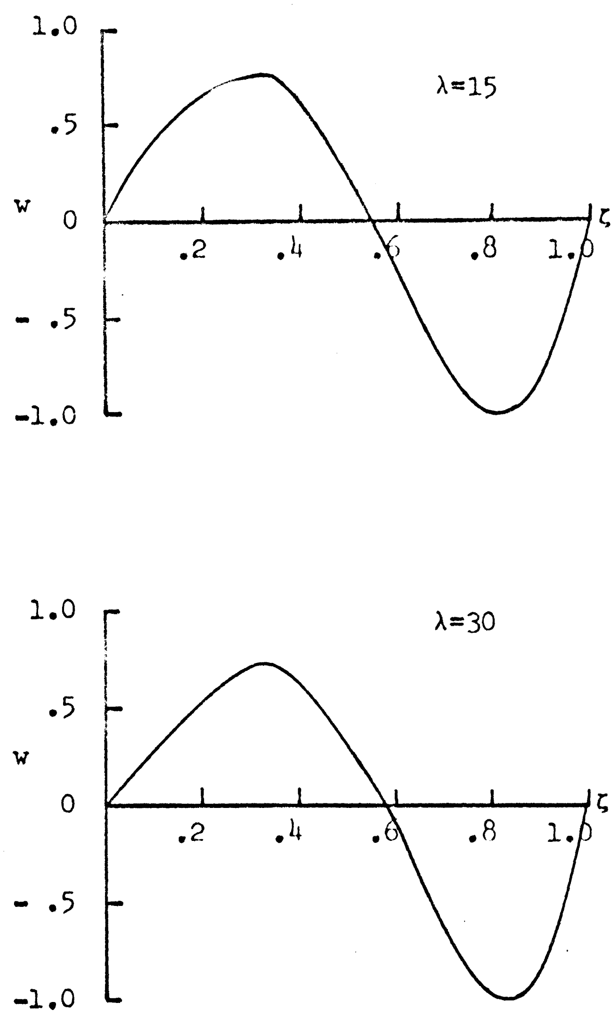
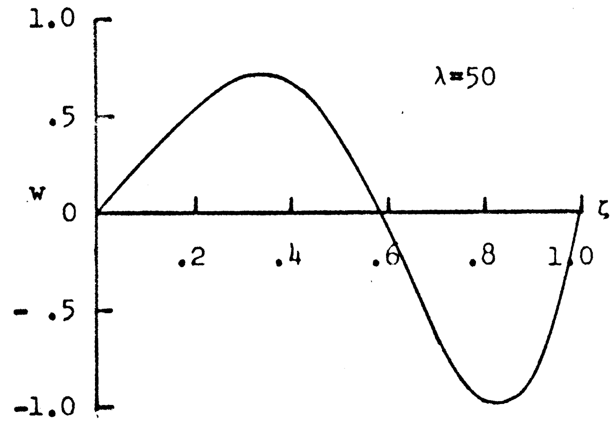


Figure 29. Normalized Eigenfunction w at the Onset of Instability for λ=15 and λ=30 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0



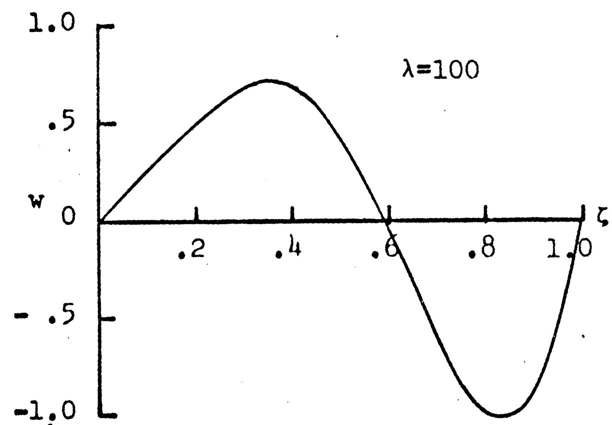


Figure 30. Normalized Eigenfunction w at the Onset of Instability for λ=50 and λ=100 Assuming Axisymmetric Disturbances, a Stationary Marginal State and μ=0

while a small unstable region would require a larger Taylor number (faster rotation of the inner cylinder) before the onset of instability.

Using this intuitive reasoning, observe that for large negative values of λ there is a large region near the outer cylinder and a much smaller region near the inner cylinder where the fluid is unstable according to the Rayleigh criterion. The fluid is stable in the center portion of the gap between the two cylinders, however, it is influenced by the surrounding unstable regions. The eigenfunction curves for large negative values of λ all have their maximum amplitude in the large outer unstable region with smaller amplitudes in the inner unstable region. If the amplitudes of the eigenfunctions in these two unstable regions are in opposite directions, then the stable region is a region where the eigenfunction amplitudes are small and change in direction. If the amplitudes in the two unstable regions are in the same direction, then the stable region is one of gradual change between the amplitudes of the two regions. The width of the larger unstable region is almost the outer one half of the gap width so one would expect the onset of instability for the entire system to occur at a small Taylor number and then increase as λ becomes less negative.

As $\lambda \rightarrow 0$ from the negative direction, the unstable region near the inner cylinder decreases in width. This can be seen in a gradual increase in magnitude of the eigenfunction in the region near the inner cylinder. The two unstable regions become the same width at $\lambda=-3.0$. The sharp increase in the critical Taylor number as $\lambda \rightarrow -3.0$ from the negative side can be predicted since the extent of the widest zone of instability is decreasing rapidly. If the reasoning concerning the onset of instability being governed by the width of the widest unstable region were exact, then the maximum critical Taylor number should occur at $\lambda=-3$. The two zones of unstable fluid are not, however, well separated, as originally supposed, and there is an interaction between the two regions. Also the Rayleigh criterion is not exact since it was derived for an inviscid rather than a viscous fluid. The net result is that the maximum critical Taylor number actually occurs near $\lambda=-3.667$.

The eigenfunctions undergo a complete change of character between $\lambda=-3.65$ and $\lambda=-3.70$. This coincides with the occurrence of the maximum Taylor number and the shifting of the predominant unstable region from the region near the outer cylinder to the region near the inner cylinder. Between $\lambda=-3$ and $\lambda=-1$ the inner unstable region becomes the widest and begins to be the dominant feature. The critical Taylor number decreases in this area since the width of the widest unstable region is increasing.

For λ between -1 and $+1$ the entire fluid region is unstable and the eigenfunction curves become almost perfectly symmetric about the center of the gap between the cylinders. Either the maximum amplitude occurs at $\lambda \approx 0.5$ or there are 2 regions with almost equal amplitudes but of opposite sign with almost zero amplitude at $\lambda=0.5$. For large positive values of λ the unstable region in the fluid is confined essentially to the outer half of the gap between the cylinder. This is reflected in the eigenfunction curves. The maximum amplitudes in all cases occurs in the outer region of the cylinder with decreasing amplitudes near the inner cylinder. The width of the unstable region is increasing as λ increases so the critical Taylor number is decreasing.

In Figures 31 and 32 are shown the variation of the frequency (σ_i) with the longitudinal wave number at the onset of instability for those values of λ investigated which exhibited oscillatory marginal states due to axisymmetric disturbances. These curves are observed to be very regular and almost harmonic across the gap between the stationary marginal states.

Non-axisymmetric Disturbances

In Figures 33 and 34 the variation of Taylor number with longitudinal wave number at the onset of instability is shown where the disturbances imposed on the system are assumed to be non-axisymmetric and the marginal state is assumed to be oscillatory. These curves were calculated for different values of the azimuthal wave number (m_i). In all cases considered, $m_i=1$ leads to the critical condition although it appears that for value of λ slightly more negative that -3.70 then $m_i=2$ might be the critical case since the two curves are getting closer together. Of course, m_i must be an integer in

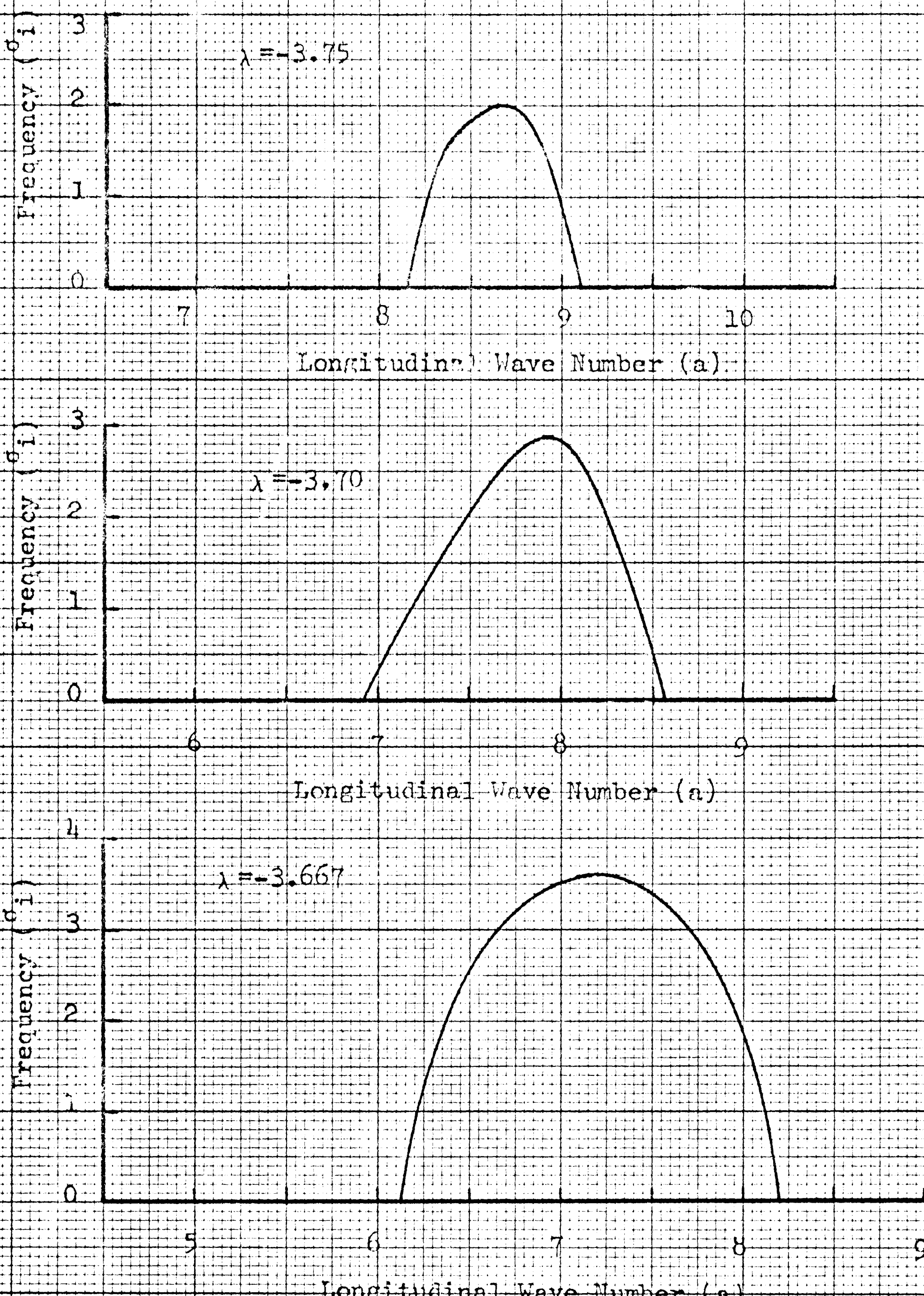
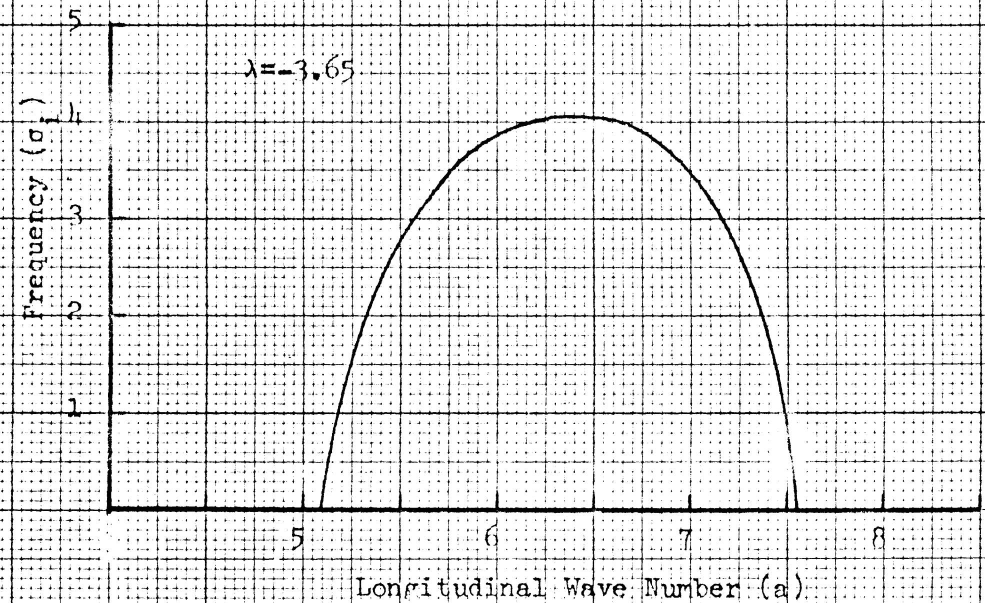


Figure 31. Variation of the Frequency (σ_i) with Longitudinal Wave Number at the Onset of Instability for λ=-3.75, λ=-3.70 and λ=-3.667 Assuming Axisymmetric Disturbances, an Oscillatory Marginal State and μ=0



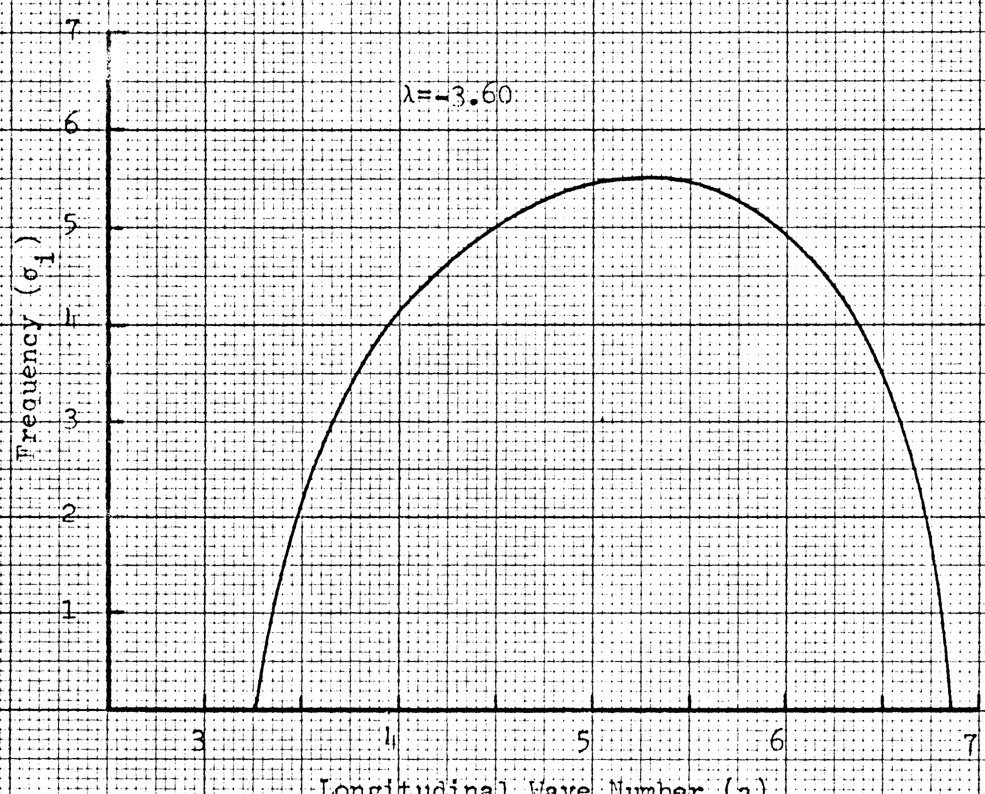
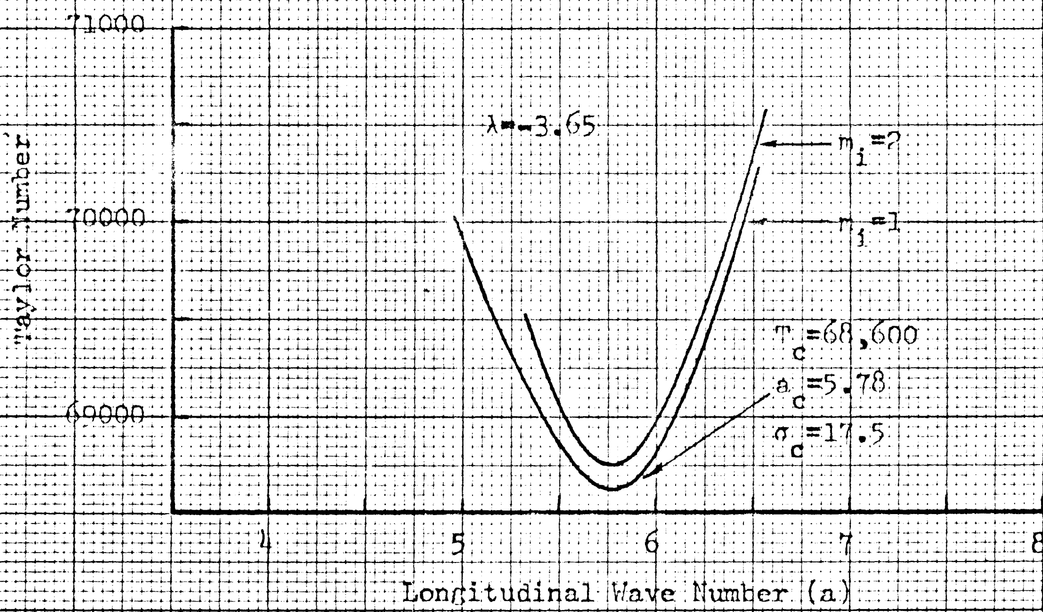


Figure 32. Variation of σ with Longitudinal Wave Number at the Onset of Instability for $\lambda=-3.65$ and $\lambda=-3.60$ Assuming Axisymmetric Disturbances, an Oscillatory Marginal State and $\mu=0$



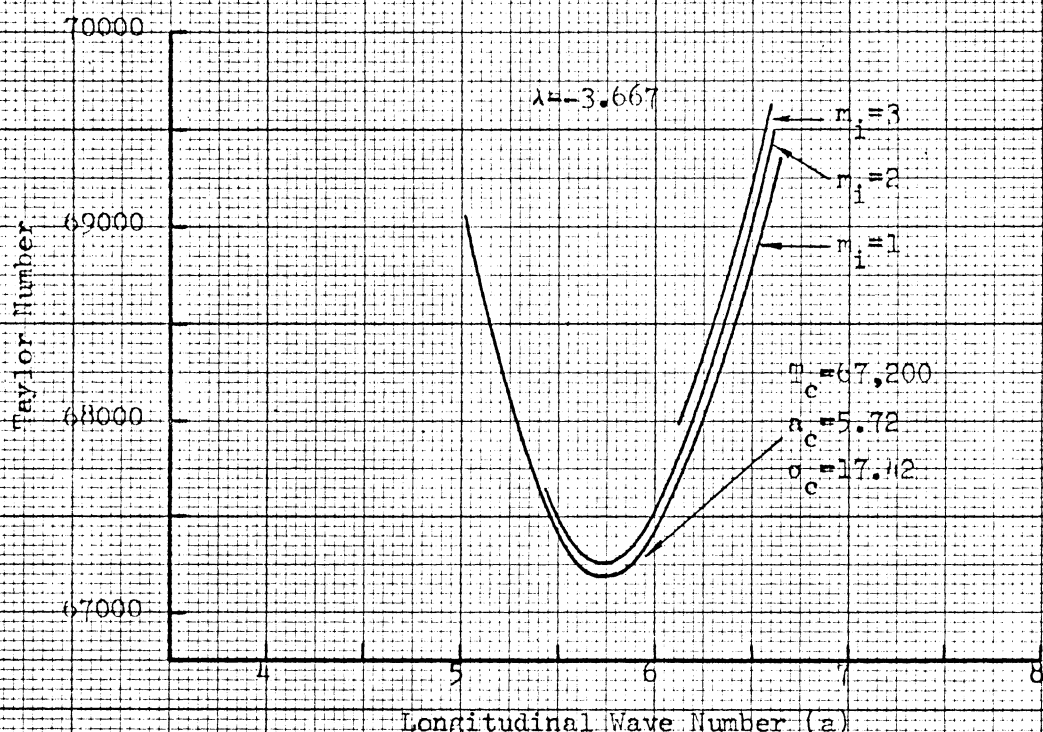


Figure 33. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for λ=-3.65 and λ=-3.667 Assuming Non-axisymmetric Disturbances, an Oscillatory Marginal State, μ=0 and δ=0.05

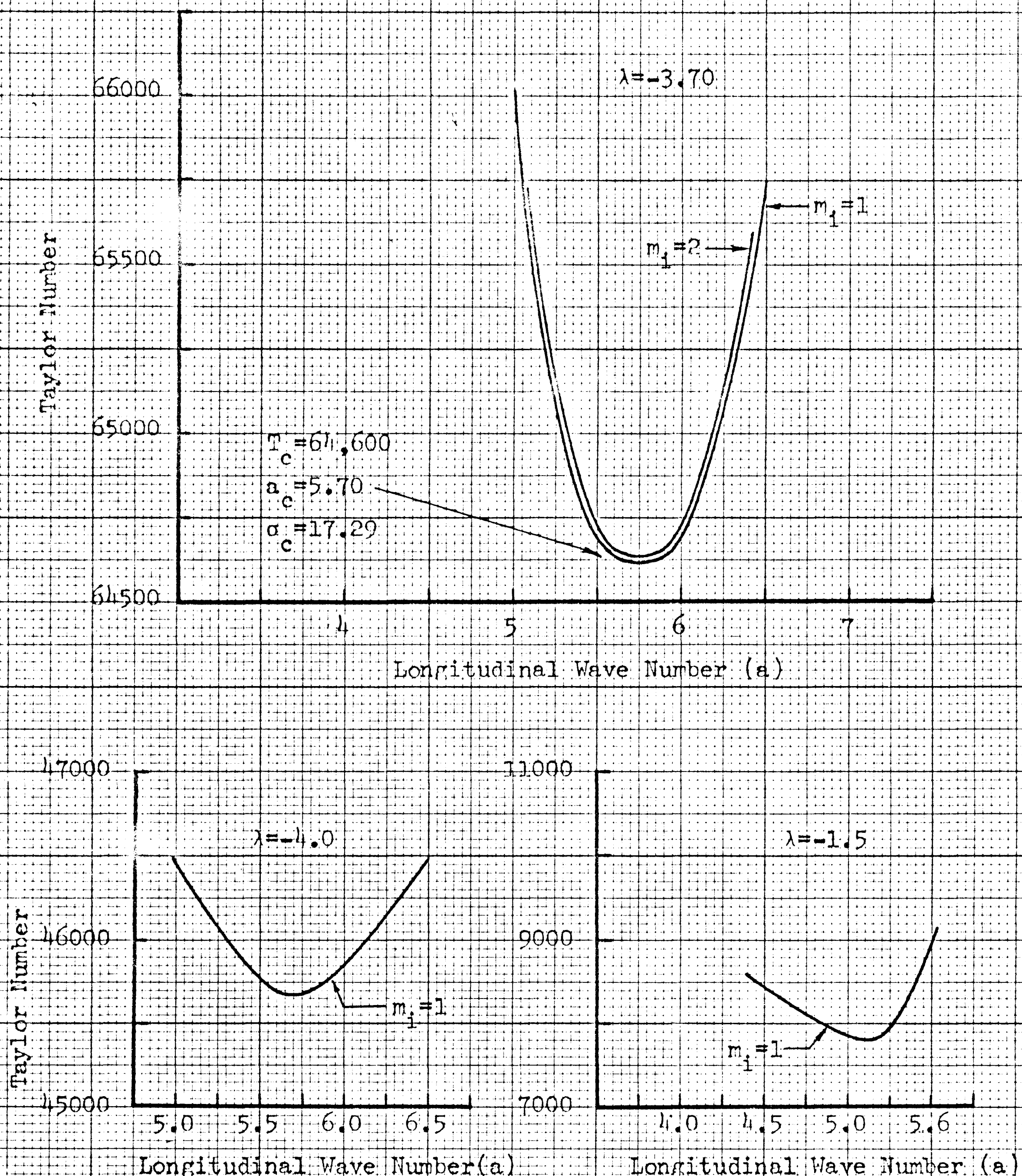


Figure 34. Variation of Taylor Number with Longitudinal Wave Number at the Onset of Instability for λ=-3.70, λ=-4.0 and λ=-1.5 Assuming Non-axisymmetric Disturbances, an Oscillatory Marginal State, μ=0 and δ=0.05

order for the solution to be single-valued. If these results are compared with the previous results for axisymmetric disturbances then for λ=-3.65, λ=-3.667 and λ=-3.70 the critical mode of instability is due to the non-axisymmetric disturbances. Because of the large amount of computer time involved a large number of values of λ were not considered. The primary purpose of the non-axisymmetric investigation was to demonstrate the existance of non-axisymmetric modes which were the critical modes of instability. An attempt was made however, to determine approximately the extent of the region of λ for which non-axisymmetric disturbances are the critical mode of instability. This examination produced the neutral curves for λ=-1.5 and λ=-4.0 shown in Figure 34. If these curves are compared with the results due to axisymmetric disturbances then these values of λ are observed to be the approximate "cross-over" points. In other words, in the region between approximately λ=-1.5 and λ=-4.0, non-axisymmetric disturbances lead to the critical mode of instability. Outside this region the critical mode of instability is due to axisymmetric disturbances and the marginal state is stationary. No values of λ were found where axisymmetric disturbances with an oscillatory marginal state would lead to the critical mode of instability

Figures 38 through 40 show the variation of the frequency (σ_i) with the longitudinal wave number at the onset of instability for the non-axisymmetric disturbances. The values of σ_i at the onset of instability varies only slightly with the value of the longitudinal wave number.

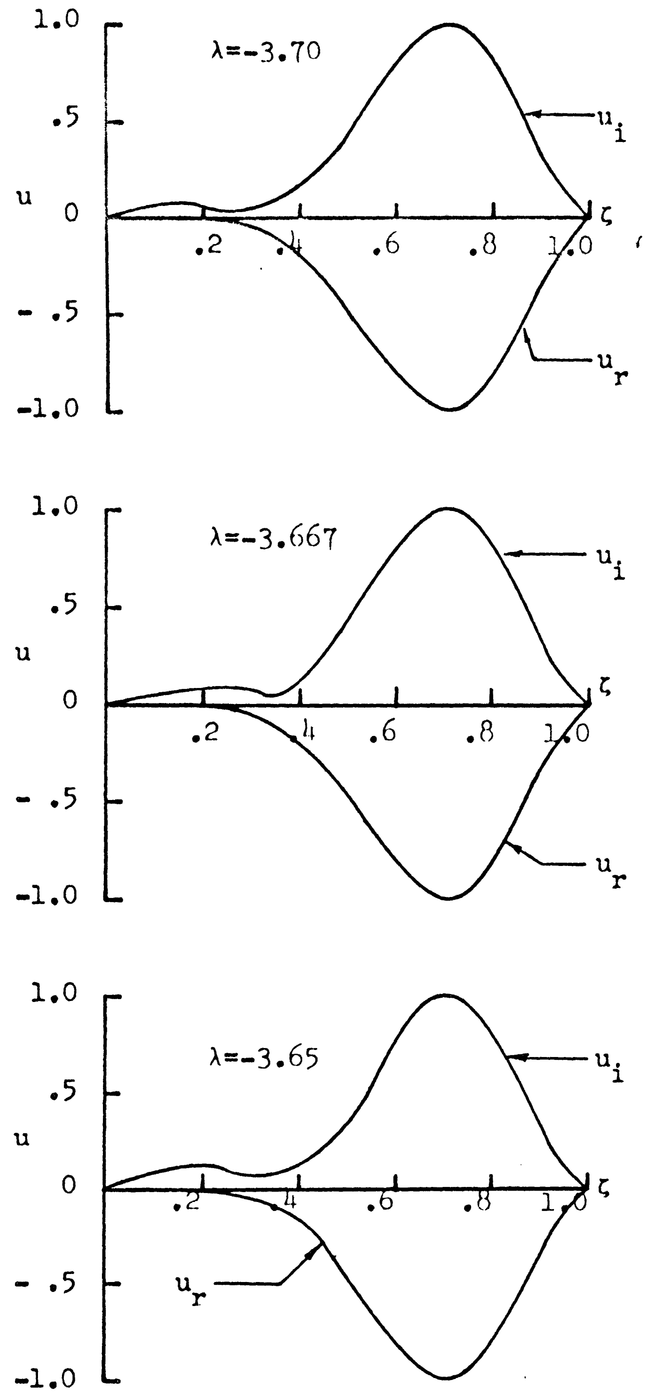


Figure 35. Normalized Eigenfunction u at the Onset of Instability for λ=-3.70, λ=-3.667 and λ=-3.65 Assuming Non-axisymmetric Disturbances, an Oscillatory Marginal State, μ=0 and δ=0.05

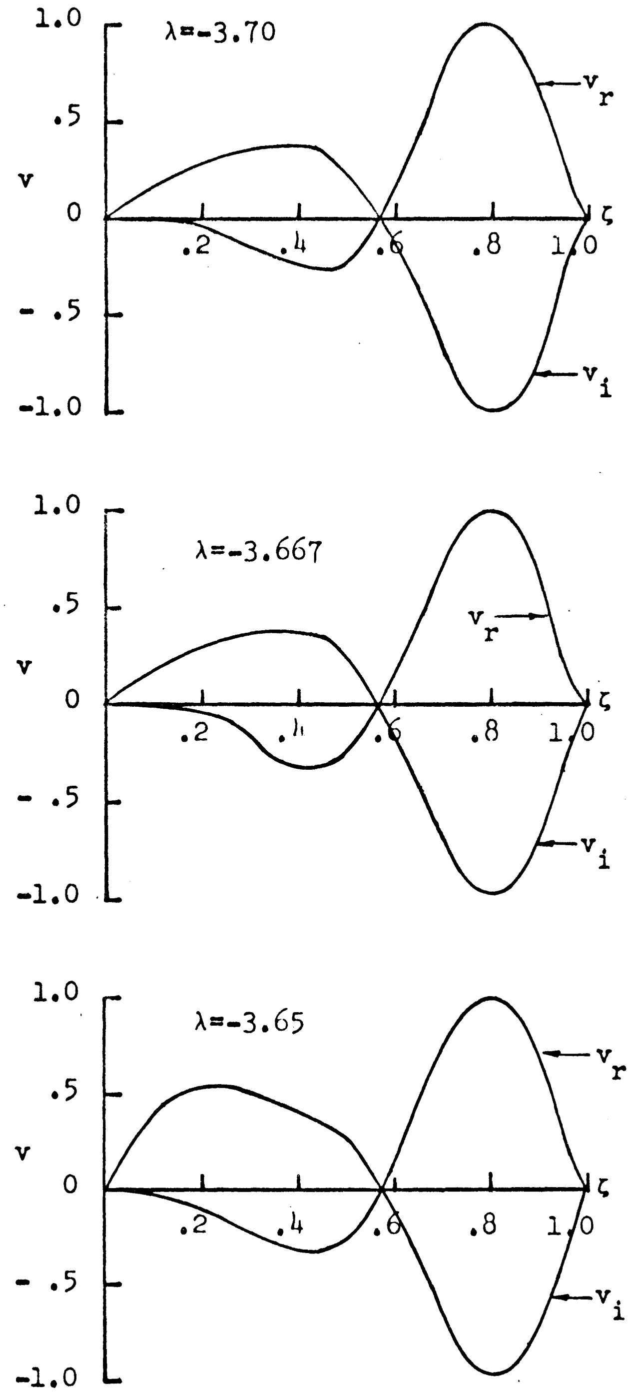


Figure 36. Normalized Eigenfunction v(r) at the Onset of Instability for λ=-3.70, λ=-3.667 and λ=-3.65 Assuming Non-axisymmetric Disturbances, an Oscillatory Marginal State, μ=0 and δ=0.05

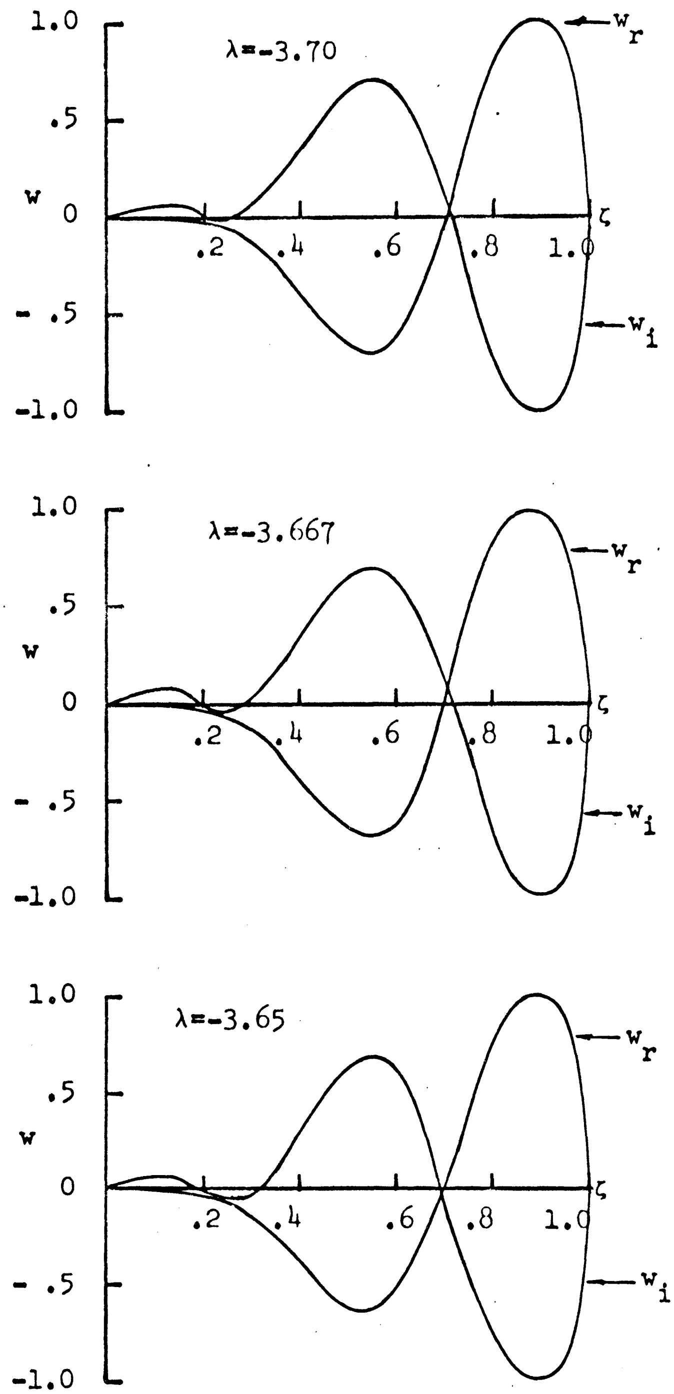
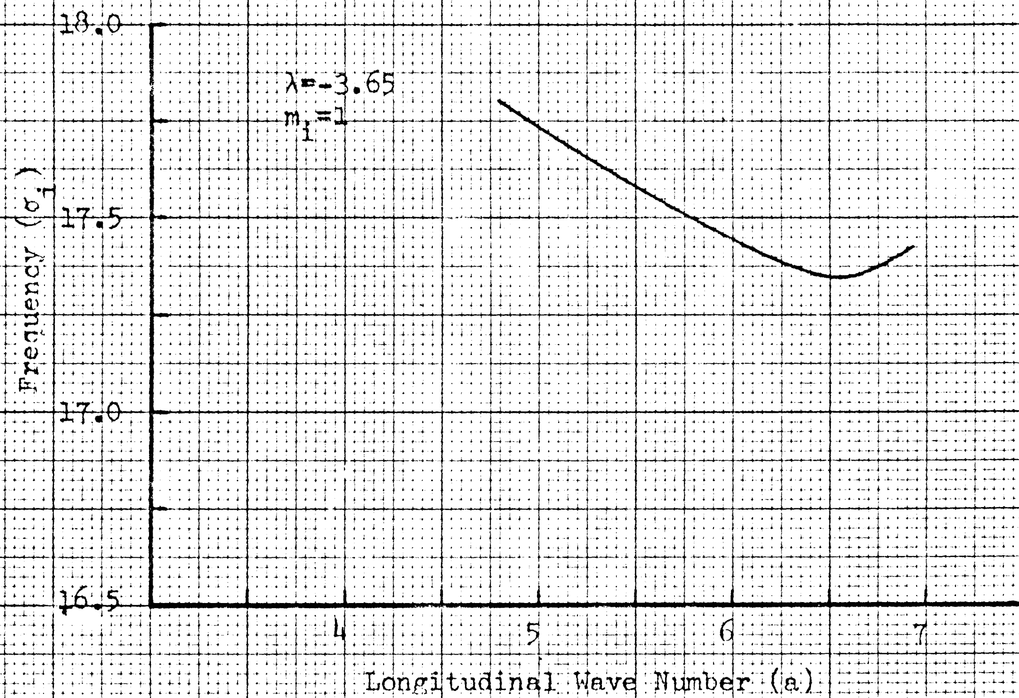


Figure 37. Normalized Eigenfunction w(r) at the Onset of Instability for λ=-3.70, λ=-3.667 and λ=-3.65 Assuming Non-axisymmetric Disturbances, an Oscillatory Marginal State μ=0 and δ=0.05



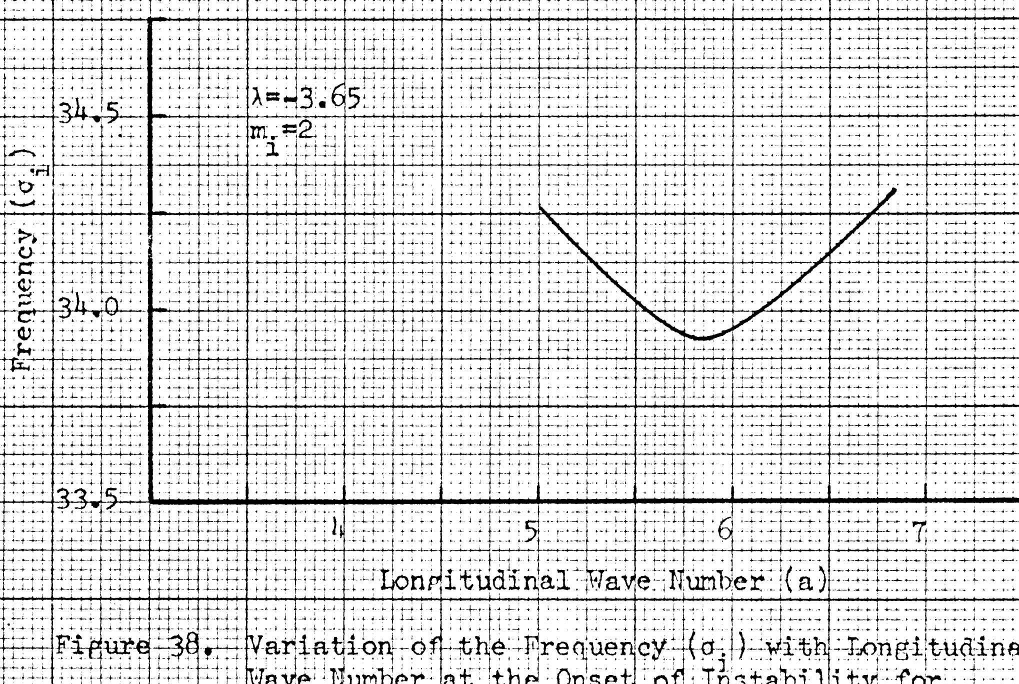
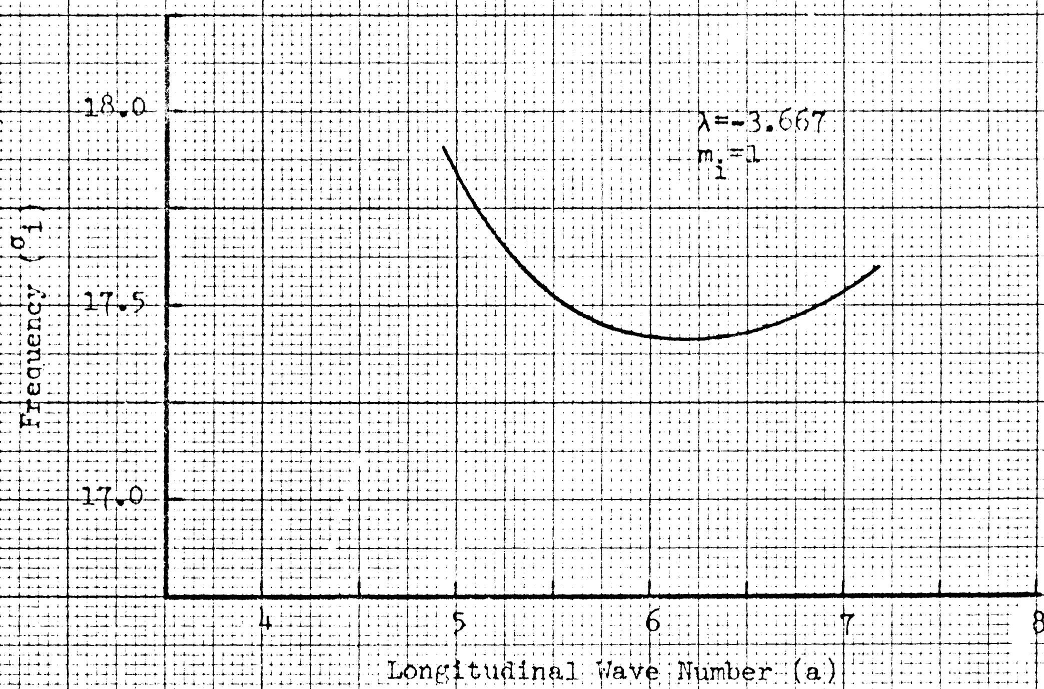


Figure 38. Variation of the Frequency (σ_i) with Longitudinal Wave Number at the Onset of Instability for $\lambda=-3.65$ ($m_i=1$ and $m_i=2$) Assuming Non-axisymmetric Disturbances, an Oscillatory Marginal State $\mu=0$ and $\delta=0.05$



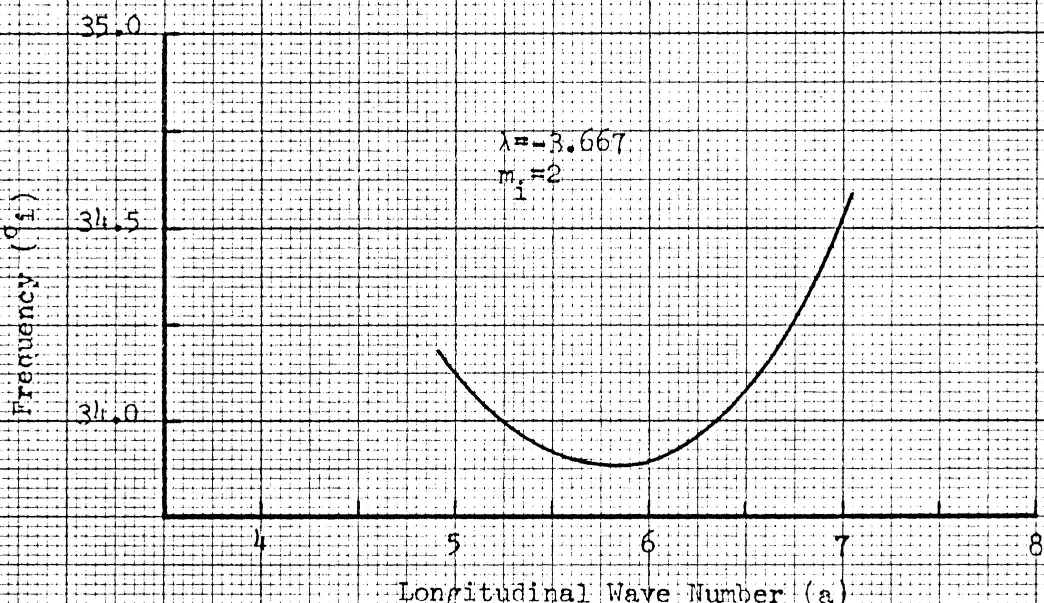
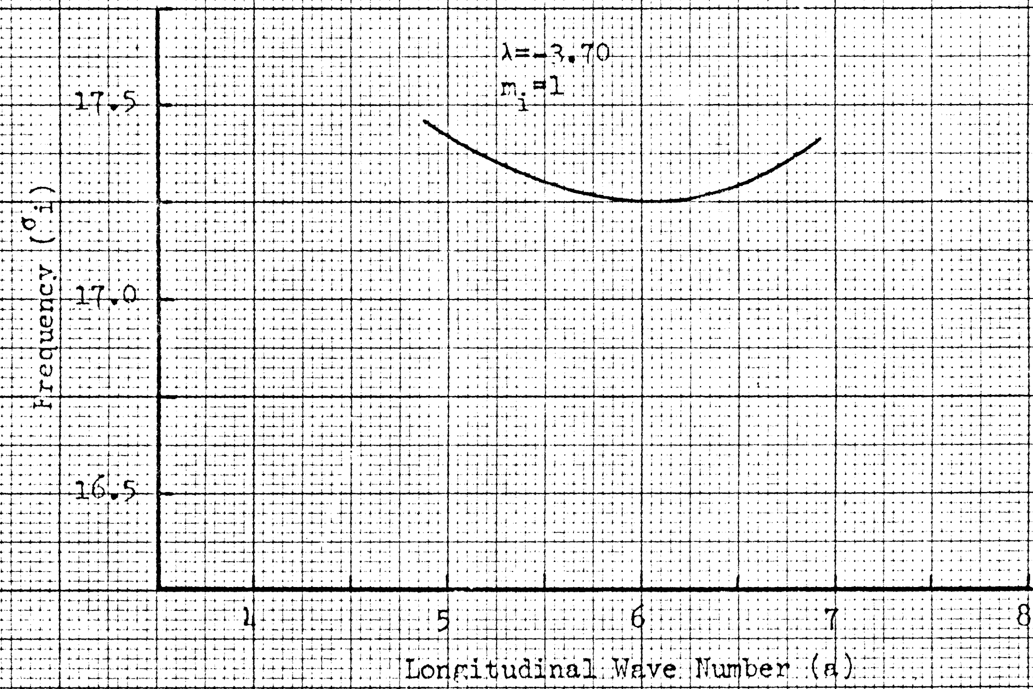


Figure 39. Variation of the Frequency (σ_i) with Longitudinal Wave Number at the Onset of Instability for λ=-3.667 (m_i=1 and m_i=2) Assuming Non-axisymmetric Disturbances, an Oscillatory Marginal State, μ=0 and δ=0.05



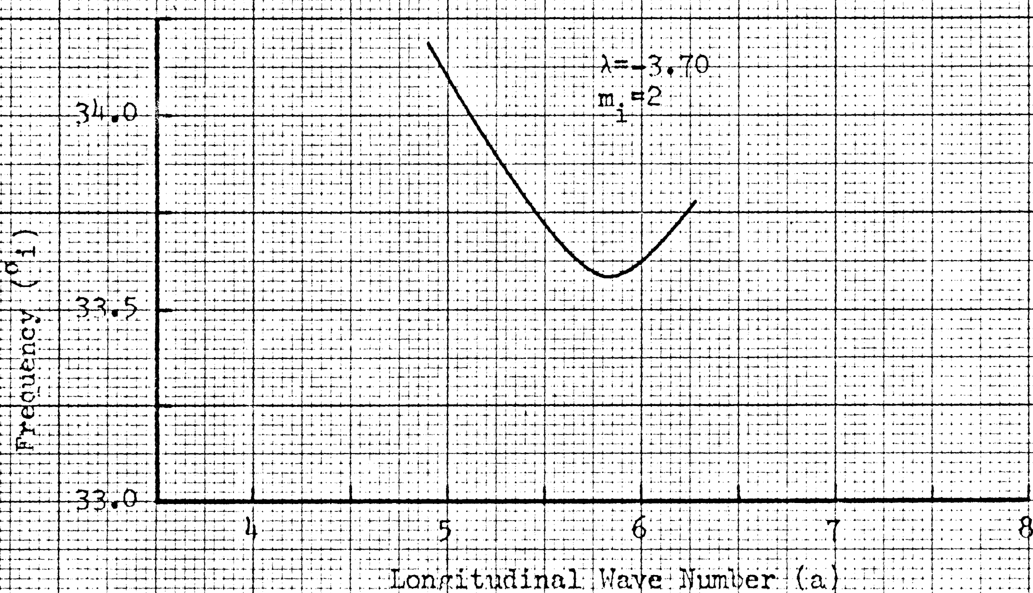


Figure 40. Variation of the Frequency (σ_i) with Longitudinal Wave Number at the Onset of Instability for $\lambda=-3.70$ ($m_i=1$ and $m_i=2$) Assuming Non-axisymmetric Disturbances, an Oscillatory Marginal State, $\mu=0$ and $\delta=0.05$

For the oscillatory modes there are two possible eigenfunctions for u, v and w. It was pointed out previously that for the stationary modes there is a sharp change in the eigenfunction curves in the region near λ=-3.667. An investigation of the non-axisymmetric oscillatory eigenfunctions (see Figures 34 through 36) reveal that one of the two possible solutions offers a more gradual change in the eigenfunction curves in this region. Thus, if the critical flow condition changes from a stationary form to an oscillatory form and then back to a stationary form in the region between λ= -1.0 and -4.0, the eigenfunction curves (u, v and w) undergo only a gradual change rather than the sharp change near λ=-3.667 if only stationary modes are considered. Of course, the actual velocity perturbations (u_r, u_θ, and u_z) for the oscillatory modes are a function of time also rather than being stationary solutions.

Figures 41 and 42 show the variation of the critical Taylor number T_c and critical longitudinal wave number for a wide range of values of λ. The effect of the non-axisymmetric critical modes is to smooth out the curves and reduce the maximum critical Taylor number and longitudinal wave number in the region between λ=-1.5 and λ=-4.0. The results for the stationary modes agree very well with the available results in the literature. The results for large positive and negative λ extend the stationary mode results beyond those values previously available in the literature. The oscillatory modes have not previously been investigated for this problem and the demonstration of their existance is the primary achievement of this investigation.

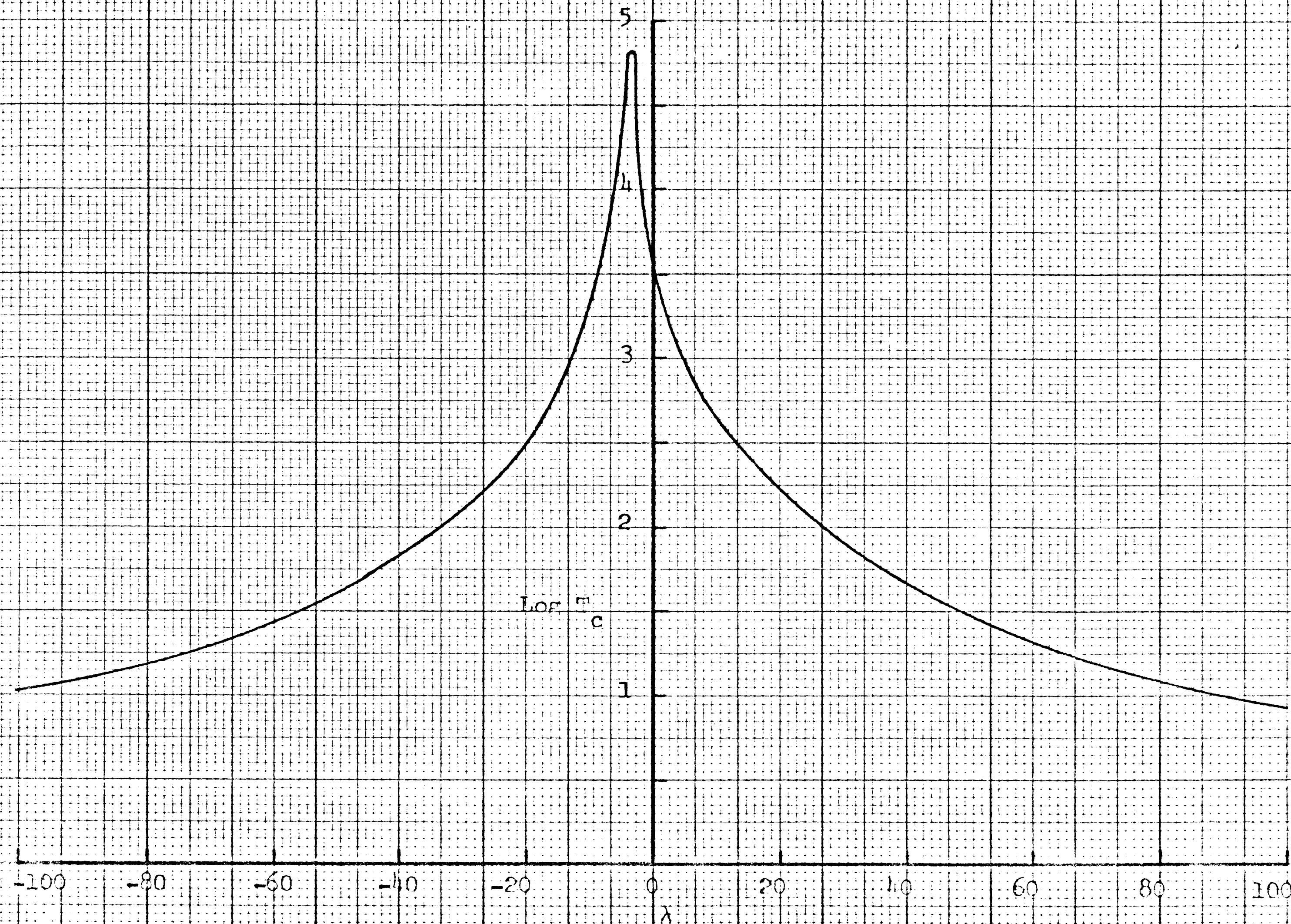


Figure 41. Variation of the Critical Taylor Number T_c as a Function of λ for $\mu=0$

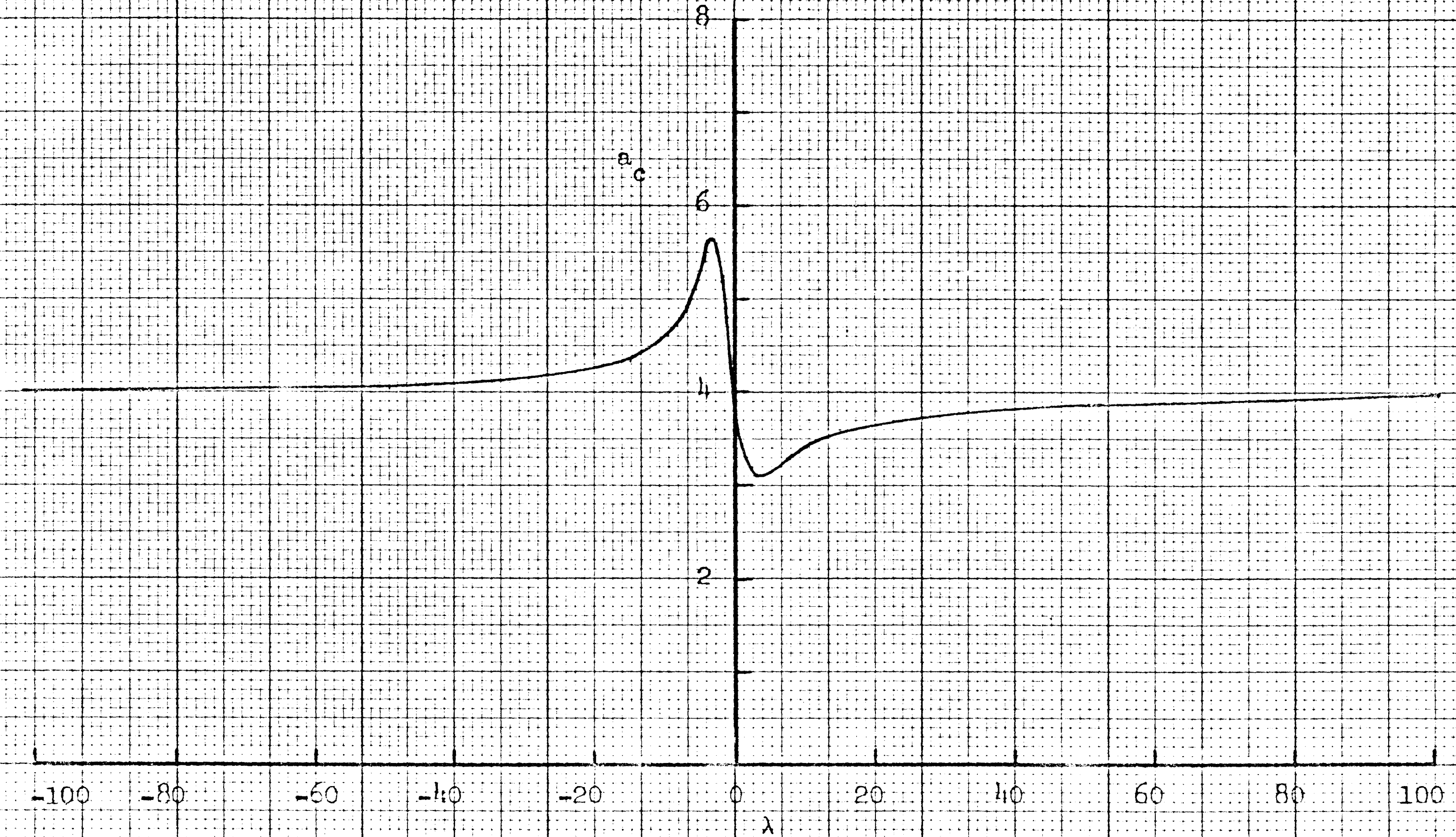


Figure 42. Variation of the Critical Longitudinal Wave Number a_c as a Function of λ (μ=0)

IX. CONCLUDING REMARKS

The onset of instability for flow between rotating cylinders with a transverse pressure gradient has been investigated assuming that the original steady flow is subjected to both axisymmetric and non-axisymmetric disturbances. This investigation extends the existing literature results for axisymmetric disturbances where a stationary marginal state is assumed and demonstrates the existance of oscillatory marginal states for certain system parameters. The existence of oscillatory modes arising from non-axisymmetric disturbance which are the critical mode of instability for certain system parameters has also been demonstrated. The existence of oscillatory modes for this problem had not been previously reported.

The existence of oscillatory modes for this problem occurs in a region where the critical Taylor number attains its maximum value and its maximum rate of change with λ (the parameter which characterizes the problem). This is also the region where some experimental difficulty has been encountered in attempting to visually observe the onset of instability. The effect of the oscillatory modes is to decrease the critical Taylor number and critical longitudinal wave number for a small range of system parameters and to smooth out the curves for the various parameters characterizing the system.

Based on this investigation, additional experimental work is indicated for this problem to determine if the oscillatory marginal states can be observed experimentally. The experimental apparatus to represent this problem presents many difficulties, however, which do not appear to have a simple solution. Previous attempts to develop an experimental model for this problem have produced results which agree fairly well with the analytical results in those regions where stationary marginal states are the critical mode of instability, but have encountered difficulty in the region where this investigation predicts the onset of instability with an oscillatory marginal state. The development of a satisfactory experimental model for this problem in itself appears to present a challenging problem.

X. ACKNOWLEDGMENTS

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APPENDIX A

SMALL GAP APPROXIMATION FOR THE STEADY STATE VELOCITY DISTRIBUTION FOR FLOW BETWEEN ROTATING CYLINDERS WITH A TRANSVERSE PRESSURE GRADIENT

Taylor Problem

If $d=R_2-R_1$ is small compared with $\frac{1}{2}(R_1+R_2)$, the flow between concentric rotating cylinders approaches simple Couette flow. In this case a linear distribution of angular velocity is assumed from the inner to the outer cylinder. This becomes

$$\Omega(r) = \Omega_1 + (\Omega_2-\Omega_1)\left(\frac{r-R_1}{R_2-R_1}\right),$$

$$\Omega(r) = \Omega_1\left[1-(1-\mu)\left(\frac{r-R_1}{R_2-R_1}\right)\right],$$

or

$$V = r\Omega = r\Omega_1\left[1-(1-\mu)\left(\frac{r-R_1}{R_2-R_1}\right)\right],$$

where

$$\mu = \frac{\Omega_2}{\Omega_1}.$$

Replacing the independent variable r by $\zeta = \frac{r-R_1}{R_2-R_1}$ then

$$\Omega(\zeta) = \frac{V}{r} = \Omega_1[1-(1-\mu)\zeta].$$

This relationship will be used for Ω in the small gap approximation.

In some cases it is desirable to go back to the general relationship for the velocity, namely $V = A + \frac{B}{r^2}$. Then

$$\frac{dV}{dr} = A - \frac{B}{r^2}$$

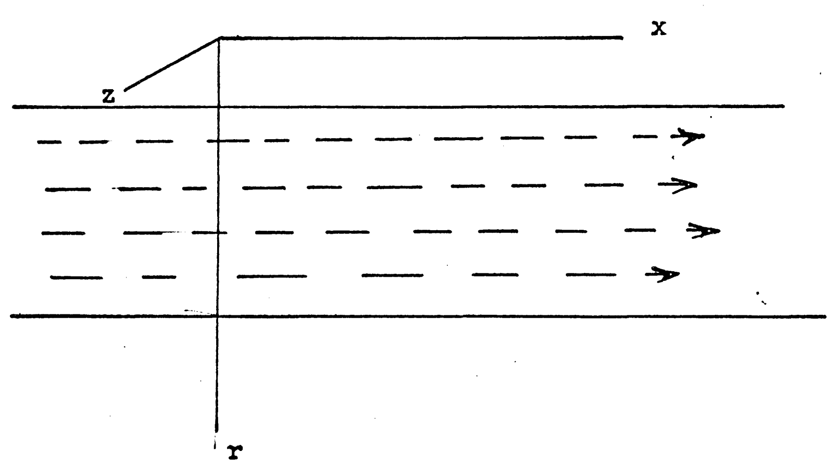
and

$$\frac{dV}{dr} + \frac{V}{r} = 2A.$$

This relationship will be used in the small gap approximation for $(\frac{dV}{dr} + \frac{V}{r})$ because of its simplicity, although it is actually an exact relationship involving no small gap assumption.

Dean Problem

If d is small compared with $\frac{1}{2}(R_1+R_2)$, the problem approaches that of laminar flow between parallel plates. In this case



$$u_r = u_z = 0$$

and

$$u_x = V(r).$$

The governing equations (equations of motion and continuity equation) lead to

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 V}{\partial r^2},\ \frac{\partial p}{\partial r} = 0,\ \frac{\partial p}{\partial z} = 0 \text{ and } \frac{\partial u_x}{\partial x} = 0.$$

Or this becomes

$$\frac{dp}{dx} = \mu\frac{d^2 V}{dr^2}$$

subject to the boundary conditions

$$u_x = 0 \text{ at } r = R_1 \text{ and } r = R_2.$$

Solving this equation

$$V = -\frac{1}{2\rho\nu}\left(\frac{dp}{dx}\right)[(r-R_1)(R_2-r)].$$

Letting

$$\zeta = \frac{r-R_1}{R_2-R_1},$$

$$x \approx R_1\theta,$$

and

$$\frac{dp}{dx} \approx \frac{1}{R_1}\ \frac{dp}{d\theta}$$

then

$$V = 6\left(\frac{-d^2}{12\rho\nu R_1}\right)\left(\frac{dp}{d\theta}\right)\zeta(1-\zeta) = 6V_m\zeta(1-\zeta)$$

or

$$\frac{V}{r} \approx \frac{6V_m\zeta(1-\zeta)}{R_1}$$

where

$$V_m = \frac{-d^2}{12\rho\nu R_1}\left(\frac{dp}{d\theta}\right).$$

The relationship

$$\frac{V}{r} = \frac{6V_m\zeta(1-\zeta)}{R_1}$$

will be used in the small gap approximation. Note also that

$$\frac{dV}{d\zeta} = 6V_m(1-2\zeta)$$

or

$$\frac{dV}{dr} = \frac{dV}{d\zeta}\frac{d\zeta}{dr} = \frac{6V_m(1-2\zeta)}{d}$$

For the Dean problem we observe that

$$\frac{dV}{dr} + \frac{V}{r} = \frac{6V_m(1-2\zeta)}{d} + \frac{6V_m\zeta(1-\zeta)}{R_1}$$

or

$$\frac{dV}{dr} + \frac{V}{r} = \frac{1}{d}\left[6V_m(1-2\zeta) + 6\delta V_m\zeta(1-\zeta)\right].$$

In the limit as $\delta \to 0$ (small gap assumption) then

$$\frac{dV}{dr} + \frac{V}{r} = \frac{6V_m(1-2\zeta)}{d}.$$

This is the relationship which will be used in the small gap approximation.

The Combined Problem

For the combined problem of Couette flow between rotating cylinders with a transverse pressure gradient, the narrow gap assumptions lead to the following velocity relationships;

$$\frac{V}{r} = \Omega = \Omega_1 F(\zeta) = \Omega_1[1-(1-u)\zeta] + \frac{6V_m(1-\zeta)\zeta}{R_1}$$

or

$$\frac{V}{r} = \Omega_1[1-(1-u)\zeta + \lambda\zeta(1-\zeta)]$$

and

$$\frac{dV}{dr} + \frac{V}{r} = 2A + \frac{6V(1-2\zeta)}{a} .$$

For this problem the function $F(\zeta)$ reduces to

$$F(\zeta) = [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)].$$

Small Gap Approximation for the Auxiliary Equations (Equations (25) and (27)

The axuiliary equation for w and $\tilde{\omega}$ are

$$w = -\frac{1}{k}\left[\frac{du}{dr} + \frac{u}{r} + \frac{m}{r}v\right] \tag{25}$$

$$\tilde{\omega} = -\frac{1}{k^2}\left[\left(\omega + \frac{mV}{r}\right)\left(\frac{du}{dr} + \frac{u}{r} + \frac{m}{r}v\right)\right] + \left[\frac{\nu H}{k^2}\right]\left[\frac{du}{dr} + \frac{u}{r} + \frac{m}{r}v\right]. \tag{27}$$

Using the small gap assumption these equations can be reduced to a simpler form.

Introducing the change of independent variables

$$\zeta = \frac{r-R_1}{R_2-R_1} = \frac{r-R_1}{d}$$

equation (25) can be written in the form

$$w = -\frac{1}{k}\left[\frac{1}{d}\frac{du}{d\zeta} + \frac{u}{r} + \frac{m}{r}v\right].$$

Replacing r by R_1 after the integration and introducing $\delta = \frac{d}{R_1}$ and $a = kd$ then this equation can be written as

$$w = -\frac{1}{a}\left[\frac{du}{d\zeta} + \delta u + \delta m v\right].$$

Letting $\delta \to 0$ with the other variables constant, then for the small gap assumption

$$w = -\frac{1}{a}\frac{du}{d\zeta} \tag{63}$$

In a similar manner, observe that equation (22) can be written as

$$\tilde{\omega} = -\frac{1}{k^2 d}\left[(\omega + \frac{mV}{r})(\frac{du}{d\zeta} + \delta u + \delta mv\right] + \frac{\nu H}{k^2 d}(\frac{du}{d\zeta} + \delta u + \delta mv).$$

Introducing the variables σ, n, T, a and again in the limit as $\delta \to 0$

$$\tilde{\omega} = -\frac{1}{a^2}\{\sigma + n(T)^{\frac{1}{2}}[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}\frac{du}{d\zeta} + \frac{1}{a^2}\frac{d^3u}{d\zeta^3}. \tag{64}$$

APPENDIX B

COMPUTER PROGRAM FOR AXISYMMETRIC DISTURBANCES AND ASSUMING A STATIONARY MARGINAL STATE

This computer program integrates the initial value problem defined by the system of first order equations

$$\frac{dY_1}{d\zeta} = Y_3,$$

$$\frac{dY_2}{d\zeta} = Y_4,$$

$$\frac{dY_3}{d\zeta} = Y_5 + a^2 Y_1,$$

$$\frac{dY_4}{d\zeta} = -Ta^2[1- \frac{\lambda}{1-u}(1-2\zeta)]\ Y_1 + a^2 Y_2,$$

$$\frac{dY_5}{d\zeta} = Y_6,$$

and

$$\frac{dY_6}{d\zeta} = a^2 Y_5 + [1-(1-u)\zeta + \lambda\zeta(1-\zeta)]\ Y_2$$

from $\zeta=0$ to $\zeta=1.0$. The system for $Y_j (j=1,----6)$ is integrated 3 times with three different sets of initial conditions defined by

$$\left.\begin{array}{ll} Y_j^i = 0 & j = 1, 2, 3 \\ Y_j^i = \delta_{i,\ j-3} & j = 4, 5, 6 \end{array}\right\} \quad i = 1, 2, 3.$$

The program then evaluates the determinant

$$|Y_j^i(1)| \qquad i, j = 1, 2, 3.$$

If the condition

$$|Y_j^i(1)| = 0$$

is satisfied then a linear combination of the 3 sets of solutions will satisfy the boundary condition at $\zeta=1.0$ and will represent a solution for the problem.

There is one data card required for each case to be investigated. This card has the format 6F10.0. The 6 items of input data in the order listed on the data card are a^2, μ, λ, T, FIN and DT. FIN is a printout control, for FIN=1 only the final results at $\zeta=1.0$ are printed out. For FIN=0, intermediate results are also printed out. DT is the step size on the independent variable ζ.

```
$IBFTC DCR
C       PROGRAM  TO DETERMINE THE EFFECT OF A TRANSVERSE
C       PRESSURE GRADIENT  ON THE STABILITY OF COUETTE FLOW
C       ASSUMING AXISYMMETRIC DISTURBANCES AND A STATIONARY
C       MARGINAL STATE  USING A RUNGE-KUTTA METHOD FOR
C       THE INTEGRATION
        COMMON TIME,D1,D2,D3,D4,D5,D6,Y1,Y2,Y3,Y4,Y5,Y6,
      1Y1D,Y2D,Y3D,Y4D,Y5D,Y6D,DUM1,DT
        COMMON P
        DIMENSION DUM1(19)
        INTEGER DD,CT
        DIMENSION A(10,10)
        DIMENSION T(6),P(6),ANS(25),C(3)
  1     READ (5,6) (P(J), J=1,6)
        DT=P(6)
        RST=DT
        WRITE (6,780) P(1)
  780   FORMAT (9H A SQR = F15.6/)
        WRITE (6,781) P(2)
  781   FORMAT (6H MU = F15.6/)
        WRITE (6,782) P(3)
  782   FORMAT (10H LAMBDA = F15.6/)
        WRITE (6,783) P(4)
  783   FORMAT (13H TAYLOR NO = F15.6/)
        WRITE (6,950) DT
```

```
 950  FORMAT  (6H DT = F15.6)
C     P1= A SQR   P2=MU  P3=LAMBDA  P4=TAYLOR NO. P5=FIN
C     P6=DT
C     FIN=PRINTOUT CONTROL FOR A POSITIVE NUMBER EQUAL TO 1
C     FINAL RESULTS ONLY ARE PRINTED OUT  FOR FIN=0
C     INTERMEDIATE RESULTS ARE ALSO PRINTED OUT
      FIN=P(5)
 6    FORMAT (6F10.0)
      CT=0
      KNT=0
      DO 75  J=1,6,1
 75   T(J)=0.0
      DD=4
      T(DD)=1.0
      Y1=T(1)
      Y2=T(2)
      Y3=T(3)
      Y4=T(4)
      Y5=T(5)
      Y6=T(6)
      TIME =0
      CALL MR
 3    FORMAT (8F10.0)
 7    WRITE(6,784)
 784  FORMAT (20H BOUNDARY CONDITIONS)
```

```
      WRITE(6,796) Y1,Y2,Y3,Y4,Y5,Y6
 796  FORMAT (6F15.6)
 913  JT=1
      CHK=.1
      M=0
      N=6
 11   CALL RK(N,M)
C     EXIT FROM DIFFERENTIAL EQUATIONS
 34   IF (CHK-TIME) 600,600,11
 600  IF (FIN-.5) 598,599,599
 598  WRITE(6,795)
      WRITE(6,85) TIME,Y1,Y2,Y3,Y4,Y5,Y6
 599  IF (JT-9) 601,602,604
 601  CHK=CHK+.1
      JT=JT+1
      GO TO 11
 602  TSFER=1.0
      CHK=1.000-DT/10.0
      JT=JT+1
      GO TO 11
 604  IF (FIN-.5) 597,596,596
 596  WRITE(6,795)
      WRITE(6,85) TIME,Y1,Y2,Y3,Y4,Y5,Y6
 597  KNT=KNT+1
 18   FORMAT (6E15.6//)
```

```
85    FORMAT (7E15.6/)
795   FORMAT (6H ZETA 12X,4H Y1 13X,4H Y2 13X,4H Y3 13X,
     14H Y4 13X,
     24H Y5 13X,4H Y6 13X)
      CT=CT+1
      IF (CT-2) 26,28,30
26    MM=0
      GO TO 41
28    MM=5
      GO TO 41
30    MM=10
      GO TO 41
41    JA=MM+KNT+1
      ANS(JA)=Y1
      JA=MM+KNT+2
      ANS(JA)=Y2
      JA=MM+KNT+3
      ANS(JA)=Y3
      JA=MM+KNT+4
      ANS(JA)=Y4
      JA=MM+KNT+5
      ANS(JA)=Y5
      JA=MM+KNT+6
      ANS(JA)=Y6
      IF (CT-2) 60,62,64
```

```
60    C(1)=TIME
      DD=5
      DO 27  J=1,6,1
27    T(J)=0.0
      T(DD)=1.0
      GO TO 66
62    C(2)=TIME
      DD=6
      DO 31  J=1,6,1
31    T(J)=0.0
      T(DD)=1.0
      GO TO 66
64    C(3)=TIME
      GO TO 38
66    Y1=T(1)
      Y2=T(2)
      Y3=T(3)
      Y4=T(4)
      Y5=T(5)
      Y6=T(6)
      TIME =0
      DT=RST
      CALL MR
38    IF (KNT-3)  7,70,70
C       DETERMINANT BY PIVOTAL CONDENSATION
```

```
70    N=3
      DO 200  J=1,3,1
      JB=J+1
200   A(J,1)=ANS(JB)
      DO 201  J=1,3,1
      JB=J+7
201   A(J,2)=ANS(JB)
      DO 202 J=1,3,1
      JB=J+13
202   A(J,3)=ANS(JB)
      WRITE(6,998) N,((A(I,J),J=1,N),I=1,N)
203   K=2
      L=1
5     DO 10  I=K,N,1
      RATIO=A(I,L)/A(L,L)
      DO 10  J=K,N,1
10    A(I,J)=A(I,J)-A(L,J)*RATIO
      IF (K-N) 14,20,20
14    L=K
      K=K+1
      GO TO 5
20    DETERM=1
      DO 25  L=1,N
25    DETERM=DETERM*A(L,L)
      WRITE(6,997) DETERM
```

```
 997  FORMAT (/ 12X,33HTHE DETERMINANT OF THIS MATRIX IS
     1E15.6)
 998  FORMAT (//27X, 15HMATRIX OF ORDER I2 // (6E15.6))
 1000 GO TO 1
      END
$IBFTC RK
      SUBROUTINE RK (N,M)
C      N=NUMBER OF FIRST ORDER.  M=NUMBER OF SECOND ORDER.
      COMMON TTT,TTT1,DT
      COMMON P
      DIMENSION TTT(19),TTT1(19),C1(6),C2(6),C3(6),P(5)
      J2=N-M+2
      J1=3*N+1
      TTT1(1)=TTT(1)
      DO 14 I=J2,J1
 14   TTT1(I)=TTT(I)
      DTO2=DT/2.
      J=N+1
      TTT(1)=DTO2 +TTT1(1)
      DO 1 I=1,N
      L=I+J
      K=L+N
      C1(I)=TTT1(K)*(DTO2 )
    1 TTT(L)=TTT1(L)+C1(I)
      IF(M)3,3,2
```

```
2 DO 4 I=1,M
  L=I+J
4 TTT(I+1)=TTT1(I+1)+((DTO2 )*(C1(I)/2.+TTT1(L)))
3 CALL MR
  DO 5 I=1,N
  L=I+J
  K=L+N
  C2(I)=TTT(K)*DTO2
5 TTT(L)=TTT1(L)+C2(I)
  CALL MR
  TTT(1)=TTT1(1)+DT
  DO 6 I=1,N
  L=I+J
  K=L+N
  C3(I)=TTT(K)*DTO2
6 TTT(L)=(2.*C3(I))+TTT1(L)
  IF(M)7,7,8
8 DO 9 I=1,M
  L=I+J
9 TTT(I+1)=TTT1(I+1)+(DT*(TTT1(L)+C3(I)))
7 CALL MR
  DO 10 I=1,N
  L=I+J
  K=L+N
  C1(I)=C1(I)+C2(I)+C3(I)
```

```
   10 TTT(L)=TTT1(L)+((TTT(K) *DTO2 )+C3(I)+C2(I)+C1(I))/3.
      IF(M)11,11,12
   12 DO 13 I=1,M
      L=I+J
   13 TTT(I+1)=TTT1(I+1)+(((C1(I)/3.)+TTT1(L))*DT)
   11 CALL MR
      RETURN
      END
$IBFTC MR
      SUBROUTINE MR
      COMMON TIME,D1,D2,D3,D4,D5,D6,Y1,Y2,Y3,Y4,Y5,Y6,
     1Y1D,Y2D,Y3D,Y4D,Y5D,Y6D,DUM1,DT
      COMMON P
      DIMENSION DUM1(19)
      DIMENSION P(6)
      Y1D=Y3
      Y2D=Y4
      Y3D=Y5+P(1)*Y1
      Y4D=-P(4)*P(1)* (1.0-P(3)*(1.0-2.0*TIME)/(1.0-P(2)))
     1Y1+P(1)*Y2
      Y5D=Y6
      Y6D=P(1)*Y5+(1.0-(1.0-P(2))*TIME+P(3)*TIME*
     1(1.0-TIME))*Y2
      RETURN
      END
```

APPENDIX C

COMPUTER PROGRAM FOR AXISYMMETRIC OR NON-AXISYMMETRIC DISTURBANCES AND ASSUMING AN OSCILLATORY MARGINAL STATE

This computer program integrates the initial value problem defined by the system of first order equations

$$\frac{dY_1}{d\zeta} = Y_5,$$

$$\frac{dY_2}{d\zeta} = Y_6,$$

$$\frac{dY_3}{d\zeta} = Y_7,$$

$$\frac{dY_4}{d\zeta} = Y_8,$$

$$\frac{dY_5}{d\zeta} = Y_9 + a^2Y_1,$$

$$\frac{dY_6}{d\zeta} = Y_{10} + a^2Y_2,$$

$$\frac{dY_7}{d\zeta} = -a^2T[1- \frac{\lambda}{1-\mu}(1-2\zeta))]Y_1 + a^2Y_3 - \{\sigma_i + n_i(T)^{\frac{1}{2}}$$

$$[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}\ Y_4,$$

$$\frac{dY_8}{d\zeta} = -a^2T[1- \frac{\lambda}{1-\mu}(1-2\zeta]Y_2 + a^2Y_4 + \{\sigma_i + n_i(T)^{\frac{1}{2}}$$

$$[1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\}\ Y_3,$$

$$\frac{dY_9}{d\zeta} = Y_{11},$$

$$\frac{dY_{10}}{d\zeta} = Y_{12},$$

$$\frac{dY_{11}}{d\zeta} = a^2 Y_9 - \{\sigma_i + n_i (T)^{\frac{1}{2}} [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\} Y_{10} + [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)] Y_3,$$

and

$$\frac{dY_{12}}{d\zeta} = a^2 Y_{10} + \{\sigma_i + n_i (T)^{\frac{1}{2}} [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)]\} Y_9 + [1-(1-\mu)\zeta + \lambda\zeta(1-\zeta)] Y_4,$$

from $\zeta=0$ to $\zeta=1.0$. The system of equations for $Y_j (j=1,----12)$ is integrated 6 times with 6 different sets of initial conditions defined by

$$\left.\begin{array}{ll} Y_j^i = 0 & j = 1,----6 \\ Y_j^i = \delta_{i,j-6} & j = 7,----12 \end{array}\right\} \quad i = 1,----6.$$

The program then evaluates the determinant

$$|Y_j^i(1)| \qquad i, j = 1,----6.$$

If the conditions

$$|Y_j^i(1)| = 0$$

is satisfied then a linear combination of the 6 sets of solutions will satisfy the boundary conditions at $\zeta=1.0$ and will represent a solution for the problem.

There is one data card required for each case to be investigated. This card has the format (7F10.0, 2F5.0). The 9 items of input data in the order required on the data card are a^2, μ, λ, T, FIN, σ, DT, m and δ. FIN and DT are defined in Appendix B and the others are quantities defined and used in the main body of this work.

```
$IBFTC DCR
C      PROGRAM FOR THE DETERMINATION OF THE ONSET OF
C      INSTABILITY OF COUETTE FLOW BETWEEN CONCENTRIC
C      CYLINDERS WITH A TRANSVERSE PRESSURE GRADIENT
C      THE PROGRAM ASSUMES AN OSCILLATORY MARGINAL STATE
C      AND EITHER AXISYMMETRIC OR NON-AXISYMMETRIC
C      DISTURBANCES CAN BE USED
       COMMON TIME,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,D11,D12,Y1,
      1Y2,Y3,
      2Y4,Y5,Y6,Y7,Y8,Y9,Y10,Y11,Y12,Y1D,Y2D,Y3D,Y4D,Y5D,Y6D,
      3Y7D,Y8D,Y9D,Y10D,Y11D,Y12D,DUM1,DT
       COMMON P
       DIMENSION DUM1(37)
       INTEGER DD,CT
       DIMENSION A(30,30)
       DIMENSION T(12),P(9),ANS(140),C(6)
 1     READ(5,6)  (P(J), J=1,9)
       DT=P(7)
       RST=P(7)
 4     FORMAT (3F10.0)
       WRITE (6,780) P(1)
 780   FORMAT (9H A SQR = F15.6/)
       WRITE (6,781) P(2)
 781   FORMAT (6H MU = F15.6/)
       WRITE (6,782) P(3)
```

```
782   FORMAT (10H LAMBDA = F15.6/)
      WRITE (6,783) P(4)
783   FORMAT (13H TAYLOR NO = F15.6/)
      WRITE (6,785) P(6)
785   FORMAT (9H SIGMA = F15.6/)
      WRITE (6,950) DT
950   FORMAT  (6H DT = F15.6)
      WRITE (6,314) P(8)
314   FORMAT (4H M = F5.3)
      WRITE (6,315) P(9)
315   FORMAT (9H DELTA = F5.4)
C     P1= A SQR  P2=MU  P3=LAMBDA  P4= TAY1  P5=FIN
C     P(6)=SIGMA  P(7)=DT  P(8)=M  P(9)=DELTA
C     FIN=PRINTOUT CONTROL  FOR A POSITIVE NUMBER EQUAL TO
C     1 OR LARGER ONLY FINAL RESULTS ARE PRINTED OUT FOR
C     FIN=0 INTERMEDIATE RESULTS ARE ALSO PRINTED OUT
      FIN=P(5)
6     FORMAT (7F10.0,2F5.0)
      CT=0
      KNT=0
      DO 75 J=1,12,1
75    T(J)=0.0
      DD=7
      T(DD)=1.0
      Y1=T(1)
```

```
      Y2=T(2)
      Y3=T(3)
      Y4=T(4)
      Y5=T(5)
      Y6=T(6)
      Y7=T(7)
      Y8=T(8)
      Y9=T(9)
      Y10=T(10)
      Y11=T(11)
      Y12=T(12)
      TIME =0
      CALL MR
 3    FORMAT (8F10.0)
 7    WRITE(6,784)
 784  FORMAT (20H BOUNDARY CONDITIONS)
      WRITE(6,796) Y1,Y2,Y3,Y4,Y5,Y6
      WRITE(6,796) Y7,Y8,Y9,Y10,Y11,Y12
 796  FORMAT (6F15.6)
 913  JT=1
      CHK=.1
      M=0
      N=12
 11   CALL RK(N,M)
C     EXIT FROM DIFFERENTIAL EQUATIONS
```

```
34   IF (CHK-TIME) 600,600,11
600  IF (FIN-.5) 598,599,599
598  WRITE(6,795)
     WRITE(6,85) TIME,Y1,Y2,Y3,Y4,Y5,Y6
     WRITE(6,799)
     WRITE(6,18) Y7,Y8,Y9,Y10,Y11,Y12
599  IF (JT-9) 601,602,604
601  CHK=CHK+.1
     JT=JT+1
     GO TO 11
602  CHK=1.0-DT/10.0
     JT=JT+1
     GO TO 11
604  IF (FIN-.5) 597,596,596
596  WRITE(6,795)
     WRITE(6,85) TIME,Y1,Y2,Y3,Y4,Y5,Y6
     WRITE(6,799)
     WRITE(6,18) Y7,Y8,Y9,Y10,Y11,Y12
597  KNT=KNT+1
18   FORMAT (6E15.6//)
799  FORMAT (4H Y7 13X,4H Y8 13X,4H Y9 13X,5H Y10 13X,
    15H Y11 13X,5H Y12 13X)
85   FORMAT (7E15.6/)
795  FORMAT (6H ZETA 12X,4H Y1 13X,4H Y2 13X,4H Y3 13X,
    14H Y4 13X,4H Y5 13X,4H Y6 13X)
```

```
      CT=CT+1
      IF (CT-2) 26,28,30
26    MM=0
      GO TO 41
28    MM=11
      GO TO 41
30    IF (CT-4) 32,90,36
32    MM=22
      GO TO 41
90    MM=33
      GO TO 41
36    IF (CT-6) 37,38,1000
37    MM=44
      GO TO 41
38    MM=55
41    JA=MM+KNT+1
      ANS(JA)=Y1
      JA=MM+KNT+2
      ANS(JA)=Y2
      JA=MM+KNT+3
      ANS(JA)=Y3
      JA=MM+KNT+4
      ANS(JA)=Y4
      JA=MM+KNT+5
      ANS(JA)=Y5
```

```
      JA=MM+KNT+6
      ANS(JA)=Y6
      JA=MM+KNT+7
      ANS(JA)=Y7
      JA=MM+KNT+8
      ANS(JA)=Y8
      JA=MM+KNT+9
      ANS(JA)=Y9
      JA=MM+KNT+10
      ANS(JA)=Y10
      JA=MM+KNT+11
      ANS(JA)=Y11
      JA=MM+KNT+12
      ANS(JA)=Y12
      IF (CT-2) 60,62,64
60    C(1)=TIME
      DD=8
      DO 27 J=1,12,1
27    T(J)=0.0
      T(DD)=1.0
      GO TO 66
62    C(2)=TIME
      DD=9
      DO 31 J=1,12,1
31    T(J)=0.0
```

```
      T(DD)=1.0
      GO TO 66
64    IF (CT-4) 61,63,65
61    C(3)=TIME
      DD=10
      DO 33 J=1,12,1
33    T(J)=0.0
      T(DD)=1.0
      GO TO 66
63    C(4)=TIME
      DD=11
      DO 35 J=1,12,1
35    T(J)=0.0
      T(DD)=1.0
      GO TO 66
65    IF (CT-6) 67,69,66
67    C(5)=TIME
      DD=12
      DO 42 J=1,12,1
42    T(J)=0.0
      T(DD)=1.0
      GO TO 66
69    C(6)=TIME
      GO TO 91
66    Y1=T(1)
```

```
      Y2=T(2)
      Y3=T(3)
      Y4=T(4)
      Y5=T(5)
      Y6=T(6)
      Y7=T(7)
      Y8=T(8)
      Y9=T(9)
      Y10=T(10)
      Y11=T(11)
      Y12=T(12)
      TIME =0
      DT=RST
      CALL MR
   91 IF (KNT-6) 7,70,70
C       DETERMINANT BY PIVOTAL CONDENSATION
   70 N=6
      DO 72  J=1,6
      JB=J+1
   72 A(J,1)=ANS(JB)
      DO 74  J=1,6
      JB=J+13
   74 A(J,2)=ANS(JB)
      DO 76  J=1,6
      JB=J+25
```

```
76    A(J,3)=ANS(JB)
      DO 78  J=1,6
      JB=J+37
78    A(J,4)=ANS(JB)
      DO 80  J=1,6
      JB=J+49
80    A(J,5)=ANS(JB)
      DO 82  J=1,6
      JB=J+61
82    A(J,6)=ANS(JB)
      WRITE(6,998) N,((A(I,J),J=1,N) ,I=1,N)
203   K=2
      L=1
5     DO 10  I=K,N,1
      RATIO=A(I,L)/A(L,L)
      DO 10  J=K,N,1
10    A(I,J)=A(I,J)-A(L,J)*RATIO
      IF (K-N) 14,20,20
14    L=K
      K=K+1
      GO TO 5
20    DETERM=1
      DO 25  L=1,N
25    DETERM=DETERM*A(L,L)
      WRITE(6,997) DETERM
```

```
 997  FORMAT (/ 12X,33HTHE DETERMINANT OF THIS MATRIX IS
     1E15.6)
 998  FORMAT (//27X, 15HMATRIX OF ORDER I2 // (6E15.6))
 1000 GO TO 1
      END
$IBFTC RK
      SUBROUTINE RK (N,M)
C      N=NUMBER OF FIRST ORDER.  M=NUMBER OF SECOND ORDER.
      COMMON TTT,TTT1,DT
      COMMON P
      DIMENSION TTT(37),TTT1(37),C1(12),C2(12),C3(12),P(9)
      J2=N-M+2
      J1=3*N+1
      TTT1(1)=TTT(1)
      DO 14 I=J2,J1
14    TTT1(I)=TTT(I)
      DTO2=DT/2.
      J=N+1
      TTT(1)=DTO2 +TTT1(1)
      DO 1 I=1,N
      L=I+J
      K=L+N
      C1(I)=TTT1(K)*(DTO2 )
    1 TTT(L)=TTT1(L)+C1(I)
      IF(M)3,3,2
```

```
2 DO 4 I=1,M
  L=I+J
4 TTT(I+1)=TTT1(I+1)+((DTO2 )*(C1(I)/2.+TTT1(L)))
3 CALL MR
  DO 5 I=1,N
  L=I+J
  K=L+N
  C2(I)=TTT(K)*DTO2
5 TTT(L)=TTT1(L)+C2(I)
  CALL MR
  TTT(1)=TTT1(1)+DT
  DO 6 I=1,N
  L=I+J
  K=L+N
  C3(I)=TTT(K)*DTO2
6 TTT(L)=(2.*C3(I))+TTT1(L)
  IF(M)7,7,8
8 DO 9 I=1,M
  L=I+J
9 TTT(I+1)=TTT1(I+1)+(DT*(TTT1(L)+C3(I)))
7 CALL MR
  DO 10 I=1,N
  L=I+J
  K=L+N
  C1(I)=C1(I)+C2(I)+C3(I)
```

```
   10 TTT(L)=TTT1(L)+((TTT(K) *DTO2 )+C3(I)+C2(I)+C1(I))/3.
      IF(M)11,11,12
   12 DO 13 I=1,M
      L=I+J
   13 TTT(I+1)=TTT1(I+1)+(((C1(I)/3.)+TTT1(L))*DT)
   11 CALL MR
      RETURN
      END
$IBFTC MR
      SUBROUTINE MR
      COMMON TIME,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,D11,D12,
     1Y1,Y2,Y3,
     2Y4,Y5,Y6,Y7,Y8,Y9,Y10,Y11,Y12,Y1D,Y2D,Y3D,Y4D,Y5D,
     3Y6D,
     4Y7D,Y8D,Y9D,Y10D,Y11D,Y12D,DUM1,DT
      COMMON P
      DIMENSION DUM1(37)
      DIMENSION P(9)
      G1=(P(9)/(2.0*(1.0-P(2))))
      G2=SQRT(G1)
      G3=G2*P(8)
C     G3 IS N
      G4=SQRT(P(4))
C     G4 IS SQRT T.
      Y1D=Y5
```

```
 Y2D=Y6
 Y3D=Y7
 Y4D=Y8
 Y5D=Y9+P(1)*Y1
 Y6D=Y10+P(1)*Y2
 Y7D= -P(4)*P(1)*(1.0-P(3)*(1.0-2.0*TIME)/(1.0-P(2)))
1*Y1+P(1)*Y3
2 -P(6)*Y4-G3*G4*(1.0-(1.0-P(2))*TIME+P(3)*TIME*
3(1.0-TIME))*Y4
 Y8D= -P(4)*P(1)*(1.0-P(3)*(1.0-2.0*TIME)/(1.0-P(2)))
1*Y2+P(1)*Y4
2 +P(6)*Y3+G3*G4*(1.0-(1.0-P(2))*TIME+P(3)*TIME*
3(1.0-TIME))*Y3
 Y9D=Y11
 Y10D=Y12
 Y11D= P(1)*Y9-P(6)*Y10+(1.0-(1.0-P(2))*TIME+P(3)*TIME
1*(1.0-TIME))
2 *Y3-G3*G4*(1.0-(1.0-P(2))*TIME+P(3)*TIME*(1.0-TIME))
3*Y10
 Y12D= P(1)*Y10+P(6)*Y9+(1.0-(1.0-P(2))*TIME+P(3)*TIME
1*(1.0-TIME))
2 *Y4+G3*G4*(1.0-(1.0-P(2))*TIME+P(3)*TIME*(1.0-TIME))
3*Y9
 CONTINUE
 RETURN
 END
```

AXISYMMETRIC AND NON-AXISYMMETRIC MODES OF INSTABILITY FOR FLOW BETWEEN ROTATING CYLINDERS WITH A TRANSVERSE PRESSURE GRADIENT

by

Donald Clarence Raney

ABSTRACT

The stability of the flow of a Newtonian fluid between concentric cylinders is considered where the flow is also subjected to a transverse pressure gradient. The problem considered is thus a combination of the problem first considered by Taylor (1923) and Dean (1938) for pure rotation and pure pressure flow, respectively. The investigation is restricted to the case where the annulus between the cylinders is small and the numerical results are confined to the case where the outer cylinder is stationary.

In considering the stability of such a system one seeks to determine the reaction of the system when subjected to small disturbances. If the disturbances decay with time so that the system approaches the original steady state condition as time $\to\infty$, the flow is considered stable. On the other hand, if the disturbances increase in magnitude with time so that the flow progressively departs from the steady state condition and never reverts to it, the flow is considered unstable.

States of marginal stability (states separating the stable from unstable flows) can be one of two kinds, depending on whether the amplitude of a small disturbance grow (or are damped) aperiodically or grow (or are damped) by oscillations of increasing amplitude. In the former case, the transition from stability to instability

takes place via a marginal state exhibiting a stationary pattern of flow. In the latter case, the transition takes place via a marginal state exhibiting oscillatory motion with a definite characteristic frequency.

Previous theoretical investigations of this problem have assumed axisymmetric disturbances and that the critical mode of instability was of a stationary cellular motion. This investigation has not been restricted to this assumption but considered three conditions: (1) Assuming axisymmetric disturbances imposed on the original flow and a stationary marginal state, (2) Assuming axisymmetric disturbances and an oscillatory marginal state, and (3) Assuming non-axisymmetric disturbances and an oscillatory marginal state.

The governing equations for the problem must be solved subject to certain necessary boundary conditions. In general the equations will not admit non-trivial solutions for an arbitrary set of system parameters but allow non-trivial solutions only for certain characterestic values. The resulting characteristic value problem has been solved by a direct numerical process which is particularly useful in obtaining the eigenfunctions associated with the various modes of instability.

The critical Taylor number (non-dimensional parameter characterizing the onset of instability for the problem) has been determined for a wide range of values of λ. The parameter λ is defined as the ratio of the average velocity of pumping to average velocity of rotation and thus defines the initial steady state velocity distribution.

This investigation considered a larger range of values for λ (both positive and negative) than had been considered in the past for the case of axisymmetric disturbances and a stationary marginal state. The existence of oscillatory marginal states arising from axisymmetric disturbances has been demonstrated for certain negative values of λ. The existence of oscillatory modes arising from non-axisymmetric disturbances has also been demonstrated and these have been shown to be the critical mode of instability for the region between approximately $\lambda=-1.5$ and $\lambda=-4.0$. Outside this range the critical mode of instability results from axisymmetric disturbances and the marginal state is stationary. The existence of oscillatory modes for this problem had not been previously reported.

Numerous curves are shown relating the various system parameters at the onset of instability and the FORTRAN computer programs used in the numerical calculations have also been included.