CONTROL OF FLEXIBLE SPACECRAFT DURING A MINIMUM-TIME MANEUVER

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(ABSTRACT)

The problem of simultaneous maneuver and vibration control of a flexible spacecraft can be solved by means of a perturbation approach whereby the slewing of the spacecraft regarded as rigid represents the zero-order problem and the control of elastic vibration, as well as of elastic perturbations from the rigid-body maneuver, represents the first-order problem. The zero-order control is to be carried out in minimum time, which implies on-off control. On the other hand, the perturbed model is described by a high-order set of linear time-varying ordinary differential equations subjected to persistent, piecewise-constant disturbances caused by inertial forces resulting from the maneuver. This dissertation is concerned primarily with the control of the perturbed model during maneuver.

On-line computer limitations dictate a reduced-order compensator, thus only a reduced-order model (ROM) is controlled while the remaining states are regarded as residual. Hence, the problem reduces to 1) control in a short time period of a linear time-varying ROM subject to constant disturbances and 2) mitigation of control and observation spillover effects, as well as modeling errors, in a way that the full

modeled system remains finite-time stable.

The control policy is based on a compensator, which consists of a Luenberger observer and a controller. The main features of the control design are: (1) the time-varying ROM is stabilized within the finite-time interval by an optimal linear quadratic regulator, (2) a weighted norm spanning the full modeled state is minimized toward the end of the time interval, and (3) the supremum "time constant" of the full modeled system is minimized, while (1) serves as a constraint, thus resulting in a finite-time stable modeled system. The above developments are illustrated by means of a numerical example.

To my parents

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LIST OF ABBREVIATIONS

ROM	Reduced-Order Model
LQR	Linear Quadratic Regulator
TPBVP	Two-Point Boundary-Value Problem
FTS	Finite-Time Stability
ROC	Reduced-Order Compensator
LTV & FT	Linear Time-Varying and Finite-Time
LTI & IT	Linear Time-Invariant and Infinite-Time
LM	Lyapunov Matrix
LDM	Lyapunov Derivative Matrix
SFTS	Strictly Finite-Time Stabilizable
STC	Supremum Time Constant
ECS	Exponentially Contractively Stable
QCS	Quasi-Contractively Stable

CHAPTER 1

INTRODUCTION

1.1 MANEUVERING OF FLEXIBLE STRUCTURES - PROBLEM FORMULATION

This dissertation is concerned with the problem of vibration control during a minimum-time maneuver of a flexible spacecraft in space (Fig.1). Fast maneuver of highly flexible structures tends to excite elastic vibration, where the vibration is caused by the inertial forces resulting from the maneuver. The amplitude of the vibration increases as the ratio between the maneuver angular velocity and the natural frequencies of the structure increases. Quite often, the ratio is sufficiently high for the elastic motions to exceed the linear range, and even to cause damage to the structure. If the structure is to perform a minimum-time maneuver as closely as possible, with tight accuracy demands at the end point, then vibration control during the maneuver is essential.

The main purpose of this dissertation is to synthesize an on-line control law capable of minimizing the elastic perturbations of the structure from a rigid-body minimum-time path, especially toward the end of the maneuver.

The equations of motion describing the slewing of a flexible spacecraft constitute a hybrid set in the sense that the rigid body motions of the spacecraft are described by nonlinear ordinary differential equations and the elastic motion by linear partial differential equations, where the elastic motion is assumed to remain in

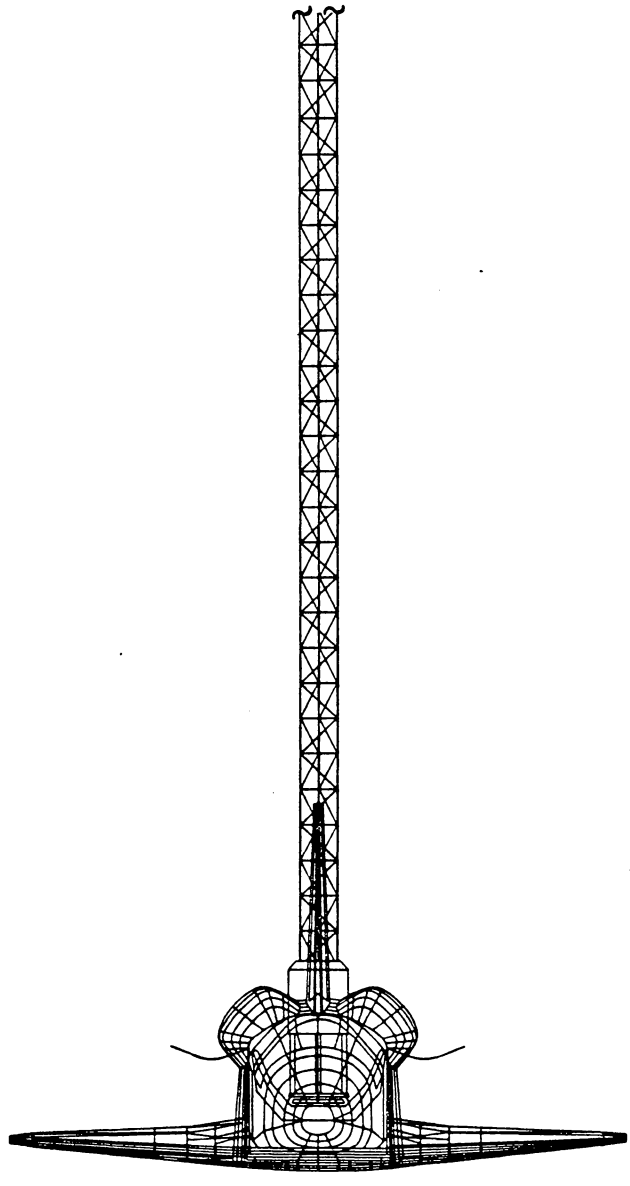


Figure 1. A Flexible Spacecraft

the linear range. The on-line control must be implemented by means of a relatively low-order compensator. Hence, the original model, which is nonlinear and infinite-dimensional, must be controlled by a finite-order compensator. One approach is to synthesize a finite-order compensator whose gains converge to the representation of the optimal LQR feedback operator in infinite-dimensional space when the order of the compensator is large enough. Although positive results were published in the case of a system described by partial differential equations in [1] and [2], no solution seems to exist for the nonlinear case. A second approach is to design an optimal controller for a truncated model. The discretization of the model is ordinarily carried out by representing the elastic motions as linear combinations of space-dependent admissible functions multiplied by time-dependent generalized coordinates [3]. By taking the linear combinations as finite, the system is truncated. The outcome is a set of nonlinear ordinary differential equations which can still be of relatively high order. Then, the problem consists of designing a reduced-order controller to control the motion of the high order, nonlinear model from one rest point to another in minimum time.

1.2 CONTROL OF MANEUVERING STRUCTURES - LITERATURE SURVEY

In recent years, there has been considerable interest in control of maneuvering flexible structures. Single-axis slew maneuver of a flexible spacecraft consisting of a rigid hub and a number of flexible appendages has been investigated in [4], [5], [6], and [7]. The cost considered in these references is a linear quadratic combination of control effort and

system energy and the nonlinear system must satisfy prescribed final conditions. The solution of the nonlinear two-point boundary-value problem (TPBVP) encountered in the minimum-time problem of a nonlinear system is approximated by means of the continuation method and yield a closed-loop solution. In [7], the cost is defined as the post-maneuver elastic energy and optimal closed-form expressions for the switching points are obtained. This is on-off open-loop control. The problem in which the cost is a combination of fuel and transition time was investigated in [8]. The model is described by the nonmaneuvering linear model and an approximate method yields closed-form expressions for the switching points. This also is on-off open-loop control. Near time-optimal control minimizing the time and a certain measure of residual energy was investigated in [9]. The control is again open-loop. The slewing problem in which the cost is the transition time only was investigated in [10], [11], and [12]. In these investigations, a time-optimal problem is solved for the nonmaneuvering model, resulting in a linear model. Then, this solution serves as an initial guess for the solution of the nonlinear TPBVP describing the maneuvering problem of the actual model. The solution represents on-off open-loop control where the switching curve is determined by a TPBVP solver.

The design of pure minimum-time control for a nonlinear model involves the solution of a nonlinear TPBVP, which in this case has the added complexity of high dimensionality. The solution is very involved even for off-line computers, and is certainly not implementable on an on-line computer. Furthermore, in general the solution represents

open-loop control, which is highly sensitive to parameter inaccuracies ordinarily encountered in structures. To circumvent these difficulties, a perturbation approach was developed in [13], [14] and [15]. The motion of the structure consists of six rigid-body degrees of freedom of a set of body axes plus the elastic vibration relative to the body axes. Because the nonlinearity enters through the rigid-body motions, while the elastic motions tend to be small, the rigid-body maneuvering can be regarded as a zero-order problem, and the elastic motions and deviations from the rigid-body maneuvering can be regarded as a first-order problem. The zero-order problem is generally nonlinear, but of relatively low order, whereas the first-order problem is linear, time-varying, albeit of high order. The control law for the zero-order model is according to minimum-time policy, which results in on-off control [16]. The switching curve is then determined according to a solution of a nonlinear TPBVP of low order. In the special case in which the origin of the body axes coincides with the center of mass of the structure and the axis of rotation is a principal axis, the zero-order model turns out to be linear with simple poles. For this low order case the switching curve can be determined without solving a TPBVP. The first-order model is subjected to kinematical disturbances caused by inertial forces produced by the zero-order, on-off maneuver. The problem can be solved within the framework of linear time-varying feedback control. Hence, the method is less sensitive to parameter uncertainty and more able to cope with the effects of the residual model.

This dissertation is concerned mainly with the synthesis of linear

feedback control for the first-order model. The problem of synthesizing feedback control for the perturbed model during maneuver has been considered in [14] and [17]. According to [14], the maneuver period can be divided into several time intervals. A time-invariant control law based on pole placement or the independent modal-space control (IMSC) method is then carried out for each time interval. The control law is based on a nondisturbed time-invariant system, where the final time in the performance measure is taken as infinite. As a result, although the control is carried out during the maneuver, the bulk of the vibration suppression takes place after the termination of the maneuver. In [17], an optimal LQR including integral feedback is designed to accommodate the disturbances resulting from the minimum-time maneuver of the rigid-body and at the same time to shape the dynamic characteristics of the system within the maneuver period. The performance index is the finite-time performance index and the perturbed model is the time-varying linear model.

1.3 CONTROL STRATEGY AND PROBLEM FORMULATION

This dissertation adopts the perturbation approach to the maneuvering problem of flexible spacecraft. The minimum-time problem is well documented in the technical literature (see, for example, [16]). Hence, the controls switch between two saturation values according to a known switching curve. In our case, the general characterization of the angular acceleration of the structure, regarded as rigid, is that of a piecewise-constant function. On the other hand, the switching curve

depends on the problem at hand. The main research topic is, therefore, the design of a suitable control law for the perturbed model, which is subjected to persistent disturbances generated by the zero-order maneuver described above.

To focus on that problem, we consider a spacecraft in which the origin of the body axes coincides with the center of mass of the structure and the axis of rotation coincides with a principle axis. As a mathematical model, we consider a spacecraft consisting of a rigid hub with a flexible appendage, where the appendage is in the form of a beam with one end attached to the hub and the other end free (Fig.2). The mass of the hub is assumed to be much larger than the mass of the appendage, so that the center of mass of the structure lies on the hub.

The perturbation equations represent a set of high-order linear differential equations with time-varying coefficients and subject to persistent disturbances. The time-dependent terms are proportional to the slewing angular velocity squared. Due to the configuration of the spacecraft under consideration, the inertial disturbances are proportional to the slewing angular acceleration, so that they can be described by a piecewise-constant function. This perturbed model is defined over a short time interval, coinciding with the minimum-time maneuver period of the structure regarded as rigid. We assume that the order of the perturbed model is sufficiently high to represent the dynamic characteristics of the entire infinite-order model accurately. Hence, we assume that a finite-time stable (FTS) closed-loop perturbed model also guarantees an FTS closed-loop infinite-order actual model.

On-line computer limitations dictate a reduced-order compensator (ROC). Hence, only a ROM can be controlled, while the remaining states and disturbance components are regarded as residual. The contamination by the residual states gives rise to the so-called control and observation spillovers. Moreover, there are modeling errors. Both effects can cause a divergent response, in spite of a well-controlled ROM. Hence, the problem of perturbed model control reduces to :(1) optimal control in a relatively short-time period of a linear time-varying ROM subjected to piecewise-constant disturbances and (2) mitigation of control and observation spillover effects, modeling errors and truncated disturbance vector, such that the overall response will remain as close as possible to the controlled response of the ROM.

1.4 CONTROL TOPICS ENCOUNTERED IN THE CONTROL PROBLEM OF THE PERTURBED MODEL - LITERATURE SURVEY.

The control problem for the perturbed model was described at the end of the previous section. The problem is not discussed in the literature. Hence, we propose to survey the main topics encountered in this problem and identify methods that can be adapted to suit our specific control problem.

The control problem can be divided into three general areas :

(1) optimal control of a linear time-varying system defined over a finite-time period. We refer to such a system as LTV & FT,

(2) accommodation of persistent piecewise-constant disturbances in LTV & FT ROM and

(3) finite-time stability in LTV & FT systems controlled by a reduced-order compensator (ROC).

We will also be concerned with linear time-invariant systems defined over an infinite-time period and refer to them as LTI & IT.

First we consider optimal control in LTV & FT systems. The intent here is to design a control that minimizes a performance index and at the same time causes the impulse response to decay to zero at a rate not slower than a desired rate. A related topic, developed for LTI & IT systems is the optimal control with prescribed degree of stability [18]. Inclusion of an exponential convergence term $e^{2\alpha t}$ in the performance index forces the real part of the controlled system poles to be smaller than $-\alpha$. An upper bound on the impulse response of a linear time-varying system can be developed in terms of a proper Lyapunov function and its derivative. This is a well known concept (see, for example, [19]). We propose to unify these two concepts and provide a sufficient condition ensuring a decay time of the impulse response in LTV & FT systems lower than an arbitrary small number within the finite-time of operation.

Secondly we survey methods for accommodation of piecewise-constant disturbances in a LTV & FT ROM. Disturbance accommodation is discussed in the technical literature in connection with fully controlled systems, i.e, without considering effects due to reduction in the model order. There are basically three methods available. In the first method [18], a term involving the time derivative of the control vector is included in

the performance index. This term gives rise to integral feedback counteracting the effect of the constant disturbances. The method was developed in connection to LTI & IT systems in which the number of actuators is equal to at least half the order of the controlled model. A second way of incorporating integral feedback control in LTI & IT systems is to augment the state vector with the integral of selected linear combinations of state variables [20]. The main advantage of this approach is that it allows the designer to choose the set of output variables for which integral feedback control is desired. The third method, developed mainly in [21], [22] and [23], is to use a disturbance model and to estimate the disturbance by means of an on-line observer. Then, a control force is applied to counteract the effect of the observed disturbances, as well as to stabilize the system. The number of sensors must be equal to at least the number of disturbance states estimated so as to keep the system completely observable. This approach is concerned with time-varying systems for which the disturbance can be described as the solution of linear differential equations of any order.

Next, we propose to examine how the above methods relate to the LTV & FT ROM for flexible structures. It is important to limit the number of actuators in controlling flexible structures because of their relatively large weight. By contrast, the number of sensors can be large, because sensors are light and can be handled with ease. In this regard, the third method has a considerable advantage. Next, let us consider the effects of controlling a ROM instead of the full model. According to the first two methods discussed above, the best one can do is to absorb the

disturbance associated with the controlled state only, without consideration of the residual state. On the other hand, in the third method, by treating the dynamic compensation and disturbance accommodation separately, one can minimize the effect of the disturbances on the steady-state response of the full modeled state.

The subject of finite-time stability (FTS) in LTV & FT systems controlled by a ROC is considered next. The concept of FTS [24], is the counterpart of the regular stability concept in systems operating over a finite-time period. Depending on the type of FTS, different upper bounds on the system transient response are developed, along with corresponding sufficiency conditions in terms of Lyapunov type functions. The main drawback of this approach is the absence of guidelines as to how to choose a proper Lyapunov function yielding the less conservative bound. Usually, a quadratic form determined by a symmetric, differentiable and positive definite matrix is considered. We refer to this matrix as a Lyapunov matrix (LM). Then the derivative of the Lyapunov function along solutions of the system equation is defined by a matrix solving the differential Lyapunov equation. We refer to this matrix as the Lyapunov derivative matrix (LDM). The infimum of the ratio between the derivative and its function over some region of the state space can then be interpreted as the reciprocal of the system's largest time constant [19]. Some relief can be found if the real parts of the instantaneous eigenvalues are negative at all times (which is not a sufficient condition for stability in time-varying systems). In this case, the

algebraic Lyapunov equation yields the proper LDM [25]. In [26], a method applicable to LTI & IT systems whereby the LDM is the matrix minimizing the largest time constantis proposed , where the algebraic Lyapunov equation is a constraint in the minimization process. A different approach to choosing a Lyapunov function is suggested in [27], in which the LM is the inverse of the product of the instantaneous matrix of eigenvectors and its complex conjugate transpose.

The main reasons for divergent response are control and observation spillover and model errors, which can be traced to the off-diagonal submatrices in the coefficient matrix. Three approaches can be considered for handling these effects in LTV & FT systems. The first is to regard the off-diagonal matrices as bounded uncertainties. A technique described in [28],[29] and [30] applies the constrained Lyapunov method to design a robust system in the presence of uncertainties. The uncertainties must meet, or almost meet, certain structural conditions called "matching conditions". Unfortunately, the off-diagonal matrices in our case seldom meet these conditions. The second approach is based on the theory of large scale systems [26]. According to this approach, an aggregation matrix is constructed based on the off-diagonal submatrices in conjunction with the LM and the LDM of the main diagonal submatrices, where the aggregation matrix is of much lower order then the coefficient matrix. Then, a positive determinant of this matrix guarantees an exponentially stable system. Moreover, an expression for the largest time constant of the system is developed in terms of the aggregation matrix. This approach yields quite

conservative results because of its complete reliance on Lyapunov functions. The third approach combines several methods for reducing the spillover effect. A proper transformation of the system input matrix [31] or output matrix [32] frees several input or output variables from control or observation spillover, respectively. However, the original controllability or observability is degraded, thus requiring higher gains. Another method uses a linear transformation to achieve a transformed system free of both control and observation spillover [33]. However, the natural characteristics of the open-loop system are altered. As a result, the uncontrolled model can become unstable and the transformed residual disturbance vector can reach high values, as opposed to the "natural" residual disturbance components which tend to decrease. Finally, a method whereby observation spillover is eliminated from several states by increasing the observed model is proposed in [34].

1.5 A CONTROL POLICY FOR THE PERTURBED MODEL

The control policy proposed in this dissertation is based on a ROC designed to achieve two goals. The first goal is to control a ROM according to a finite-time quadratic performance index in a way that the impulse response of the closed-loop controlled model will decay below any desired threshold within the finite-time interval. We refer to such a system as strictly finite-time stable system (SFTS system). Moreover, the finite-time stability of the perturbed model must be preserved in a way that the largest time constant of the perturbed model should be

minimized to produce the smallest supremum time constant (STC). The second goal is to minimize the steady-state response of the perturbed model to the persistent disturbances.

The ROM includes the parts of the perturbed model corresponding to the lowest eigenvalues of the nonmaneuvering structure, which are also subjected to the highest disturbances.

The ROC consists of a Luenberger observer and a controller. The observer estimates the controlled states and the accommodated part of the disturbance vector. The controller is divided into two parts, according to the two goals mentioned above. The first goal is achieved by means of an optimal, finite-time LQR designed to stabilize the ROM within one half of the maneuver period. An SFTS controlled ROM is achieved by including proper convergence factors in the Riccati equations for the controller and the observer. It turns out that the steady-state solutions of both Riccati equations are quasi-constant and converge to the corresponding algebraic solutions as both convergence factors increase. These steady-state solutions can be reached within the finite-time interval if the convergence factors are sufficiently large. The second part of the controller accommodates the observed part of the disturbance vector. Using quasi-constant optimal gains, a weighted norm of the entire perturbed model is minimized toward the end of the time interval. Hence, a suboptimal disturbance accommodation is achieved in steady-state. The above results are obtained, provided both convergence factors are sufficiently large. On the other hand, large convergence factors increase the gains, and thus increase the spillover effect.

This, like modeling error, can cause divergent response of the closed-loop perturbed model. Alleviation of the observation spillover effect is achieved by increasing the number of observed but uncontrolled states, thus shifting the spillover to more robust parts of the perturbed model.

The FTS characteristics of the closed-loop perturbed model are investigated by means of a Lyapunov function based on the instantaneous eigensolution of the perturbed model. A perturbation technique permits an eigensolution based on constant matrices, rather than a time-varying matrices. Moreover, the STC of the perturbed model is minimized by increasing the value of the convergence factors beyond that needed to achieve an SFTS controlled model.

The dissertation is organized as follows: Chapter 2 contains the description of the pseudo-modal equations of motion. In chapter 3, the general control strategy is described. Chapter 4 is devoted to a description of the control policy for the perturbed model, including definitions of FTS and the corresponding sufficient conditions. In chapter 5, a reduced-order controller is designed and explicit sufficiency conditions for SFTS are developed. In chapter 6, the equations of a reduced-order observer and a full-order observer designed to estimate the controlled state and the accommodated part of the disturbance vector are developed. In chapter 7, we analyze the controllability and the observability of the controlled model during maneuver. Chapter 8 describes the disturbance accommodation policy. In chapter 9, the analysis of the FTS of the modeled system and the

development of the STC are described. Chapter 10 presents a numerical example revealing important facts concerning the behavior of the solution. Finally, chapter 11 presents a summary of the research, as well as conclusions. In addition, there is an appendix containing definitions and theorems based on [35], [36] and [37] and concerning controllability and observability of linear time-varying systems.

1.6 SCOPE OF INVESTIGATION

There are two main contributions of this research. The first is in the problem of vibration suppression during a minimum-time maneuver of a flexible spacecraft based on the perturbation approach. The second is in the problem of stabilization and constant disturbance accommodation in a linear time-varying ROM over a short time period in a way that the full model remains finite-time stable. These two problems are not considered elsewhere in the technical literature.

As a preliminary to the first problem, the pseudo-modal equations of motion for a structure in which the principal axes coincide with the geometric axes and the axis of rotation is a principal axis were developed. These equations permitted us to determine that the coupling terms between the perturbed translational model and the rest of the perturbed model are very small, so that the translational motion can be controlled independently from the rotational and elastic motions. Then, it is verified numerically that the steady-state solutions of both the full order observer and the controller Riccati equations, based on the class of systems consisting of the perturbed rotational and elastic

motions, converge to the corresponding algebraic solutions if the convergence factors are large enough. Hence, the optimal gains are quasi-constant. Moreover, using the fact that the system is analytic (see Def. A.4 in the Appendix), we prove that the perturbed model is totally controllable and observable during the maneuver.

In the second problem, a design approach is developed for the stabilization of a LTV & FT ROM subjected to piecewise-constant disturbances in a way so as not to jeopardize the FTS of the full model. The approach is for systems exhibiting quasi-constant Riccati solutions for the controller and the observer. The main features of the design approach are as follows:

(1) Concepts of FTS as well as sufficiency conditions for FTS are defined in terms of convergence factors and are used as design criteria.

(2) The ROM is stabilized during a finite-time interval according to a quadratic performance measure, and a sufficiency condition is developed guaranteeing the SFTS of the controlled model. The equations of the closed-loop modeled system are developed for a reduced-order and a full-order observer. It is shown that the reduced-order observer is more likely to cause a divergent response of the modeled system than the full-order observer.

(3) The disturbance accommodation control minimizes a weighted norm spanning the full state in steady-state conditions provided that a time invariant coefficient matrix is exponentially stable.

(4) The coefficient matrix of the closed-loop full model and the one just mentioned are highly correlated. Hence, a Lyapunov function based

on the eigensolution of the system [27], and a perturbation technique [3] are combined to yield a sufficiency condition guaranteeing the FTS of the full model, as well as the exponential stability of the coefficient matrix associated with the disturbance accommodation process. Moreover, an expression for the supremum time constant of the full model is developed and the convergence factors are determined so as to minimize this supremum. This yields a robust FTS system.

The effectiveness of the above approach is demonstrated by means of a numerical example.

CHAPTER 2

EQUATIONS OF MOTION

2.1 THE NONLINEAR EQUATIONS OF MOTION

The equations of motion describing the slewing of a flexible spacecraft constitute a hybrid set, in the sense that the rigid-body motions of the spacecraft are described by nonlinear ordinary differential equations and the elastic motions of the flexible parts by partial differential equations. Practical considerations dictate discretization in space of the distributed flexible parts, which is ordinarily carried out by representing the elastic motions as linear combinations of space-dependent admissible functions multiplied by time-dependent generalized coordinates [3]. By taking the linear combination as finite, the system is thereby truncated. The net result is a set of nonlinear differential equations, which can still be of relatively high order.

We consider a flexible spacecraft consisting of a rigid hub and a flexible appendage, as shown in Fig. 2. For convenience, we introduce an inertial reference frame XYZ and a set of axes, xyz, embedded in the spacecraft in undeformed state, so that x coincides with the axis of the undeformed appendage. We shall refer to xyz as body axes. The origin O of the body axes coincides with the mass center of the spacecraft in undeformed state. The general motion of the spacecraft can be described in terms of the translation and rotation of the body axes relative to the inertial frame and the elastic motion of the appendage relative to

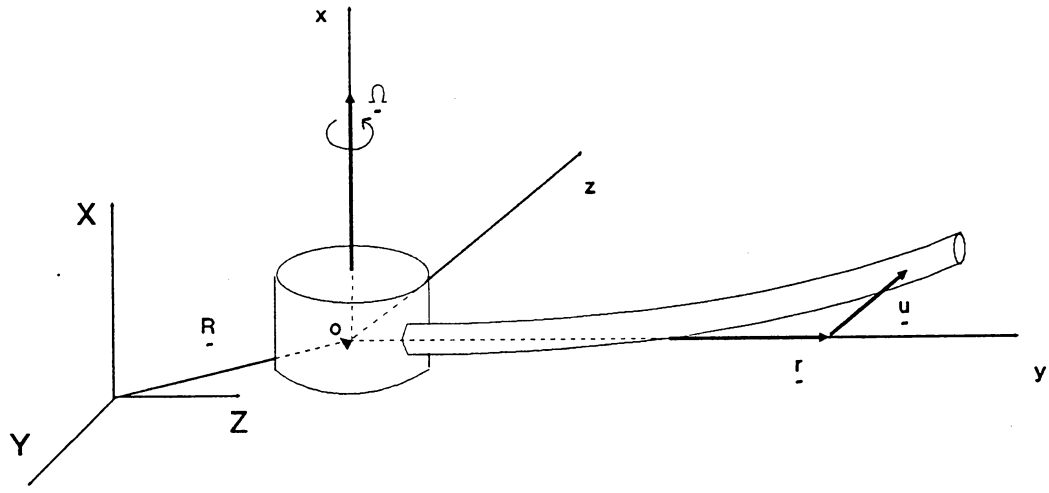


Figure 2. A Mathematical Model for the Flexible Appendage

the body axes. To this end, we denote by $\underset{\sim}{R}_s$ the position vector of O relative to the inertial space, by $\underset{\sim}{r}$ the nominal position vector of a point in the spacecraft relative to the body axes and by $\underset{\sim}{u}(\underset{\sim}{r},t)$ the elastic displacement vector of a typical point in the appendage. Moreover, we denote by $\underset{\sim}{\Omega}(t)$ the angular velocity of the body axes relative to the inertial space. The equations of motion represent a hybrid set of differential equations consisting of six ordinary differential equations for the rigid body motion and three partial differential equations for the elastic motion relative to to the body axes. The partial differential equations for $\underset{\sim}{u}$ are replaced by 3N ordinary differential equations, where N is the number of degrees of freedom used to represent each component of $\underset{\sim}{u}$. The complete set of equations is presented in [13]. Our interest is in a special type of motion, so that the general equations are not listed here.

Let us consider the single-axis maneuver of the spacecraft and in particular the slewing of the spacecraft in the yz-plane, which amounts to the rotation about the x-axis through a given angle. Ideally, the maneuver is to be carried out as if the spacecraft were rigid. In practice, a pure rigid-body maneuver is difficult to achieve, so that the maneuver will excite elastic motions which, in turn, will cause the spacecraft to deviate from the rigid body maneuver. Due to the configuration of the spacecraft under consideration (Fig. 2), in which the axis of rotation is a principal axis, the equations for the rigid-body and elastic displacements in the yz-plane and the rotation about the x axis are decoupled from the remaining equations of motion.

Introducing the necessary simplifications in Eqs. (18) of [13], the pertinent equations are

$$m\ddot{\underset{\sim}{R}}_s + \underset{\sim}{c}_2\bar{\underset{\sim}{\phi}}^T\ddot{\underset{\sim}{q}} - 2\Omega_x\underset{\sim}{c}_1\bar{\underset{\sim}{\phi}}^T\dot{\underset{\sim}{q}} - (\dot{\Omega}_x\underset{\sim}{c}_1 + \Omega_x^2\underset{\sim}{c}_2)\bar{\underset{\sim}{\phi}}^T\underset{\sim}{q} = C^T\underset{\sim}{F} \tag{2.1a}$$

$$I_C\dot{\Omega}_x + \underset{\sim}{\psi}^T\ddot{\underset{\sim}{q}} + \cos\alpha\,\ddot{R}_y\bar{\underset{\sim}{\phi}}^T\underset{\sim}{q} = M \tag{2.1b}$$

$$M_A\ddot{\underset{\sim}{q}} + \dot{\Omega}_x\underset{\sim}{\psi} + D\dot{\underset{\sim}{q}} + (K_A - \Omega_x^2M_A)\underset{\sim}{q} + \bar{\underset{\sim}{\phi}}\,\underset{\sim}{c}_2^T\ddot{\underset{\sim}{R}}_s = \underset{\sim}{Q} \tag{2.1c}$$

where m is the total mass of the spacecraft, $\underset{\sim}{R}_s = [R_y \;\; R_z]^T$ is the position vector of the mass center and

$$C = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \underset{\sim}{c}_1^T \\ \underset{\sim}{c}_2^T \end{bmatrix} \tag{2.2}$$

is the rotation matrix from axes XYZ, in which α is the maneuver angle. Of course, $\Omega_x = \dot{\alpha}$, $\dot{\Omega}_x = \ddot{\alpha}$, are the maneuver angular velocity and acceleration, respectively. In addition, $\underset{\sim}{u}(\underset{\sim}{r},t) = [\,0 \;\vdots\; u_z(y,t)\,]^T$ is the elastic displacement vector, where

$$u_z(y,t) = \underset{\sim}{\phi}^T(y)\underset{\sim}{q}(t) \tag{2.3}$$

in which $\underset{\sim}{\phi}(y)$ is a vector of admissible functions and $\underset{\sim}{q}(t)$ is a vector of generalized coordinates [3]. Moreover,

$$\bar{\underset{\sim}{\phi}} = \int_m \underset{\sim}{\phi}\, dm_A \,, \qquad \underset{\sim}{\psi} = \int_m y\underset{\sim}{\phi}\, dm_A \tag{2.4a,b}$$

where m_A is the mass of the appendage. Other quantities entering into Eqs. (2.1) are the total mass moment of inertia I_C of the undeformed spacecraft about x and the appendage mass and stiffness matrices

$$M_A = \int_{m_A} \underset{\sim}{\phi}(y)\underset{\sim}{\phi}^T(y)\, dm_A \,, \qquad K_A = [\, \underset{\sim}{\phi} \,,\, \underset{\sim}{\phi} \,] \tag{2.5a,b}$$

where [,] represents an energy inner product [3] , and we note that M_A and K_A are real symmetric and positive definite. In addition, D is a damping matrix. Finally, we assume that there are two force actuators F_y and F_z and one torquer M_x acting at the mass center of the spacecraft in undeformed state. It is assumed here that the inertia of the hub is much larger than that of the flexible appendage, so that the mass center lies on the hub. Moreover, there are p torque actuators M_{Ai} acting on the elastic appendage and mounted at the points $y = y_i$ $(i = 1,2,\ldots,p)$ of the appendage. The torque actuators acting on the appendage can be expressed as a distributed actuator torque in the form

$$m_{Ax} = \sum_{i=1}^{p} M_{Ai}\,\delta(y-y_i) \tag{2.6}$$

where $\delta(y-y_i)$ are spatial Dirac delta functions. Then, the forcing terms appearing in Eqs. (2.1) can be written as follows

$$\underset{\sim}{F} = F_y \underset{\sim}{j} + F_z \underset{\sim}{k}, \qquad M = M_x + \int_{D_A} m_{Ax}\, dD_A = M_x + \sum_{i=1}^{p} M_{Ai} \tag{2.7a,b}$$

$$\underset{\sim}{Q} = \int_{D_A} m_{Ax} \underset{\sim}{\phi}'\, dD_A = \sum_{i=1}^{p} M_{Ai} \underset{\sim}{\phi}'(y_i) = E^* \underset{\sim}{M}_A \tag{2.7c}$$

where $E^* = [\, \underset{\sim}{\phi}'(y_1) \;\; \underset{\sim}{\phi}'(y_2) \ldots\ldots \underset{\sim}{\phi}'(y_p) \,]$ is an $N \times p$ modal participation matrix, in which primes denote derivatives with respect to y, and $M_A = [\, M_{A1} \;\; M_{A2} \ldots\ldots M_{Ap} \,]^T$ is a p-vector of actuator torques acting on the elastic appendage.

2.2 THE PERTURBATION APPROACH TO THE MANEUVERING PROBLEM

We consider the case in which the equations of motion consist of some large terms associated with the ideal rigid-body maneuvering and some small terms associated with the elastic motions and the perturbations in the rigid body motions caused by the elastic motions. Consistent with terminology used in perturbation analysis, we refer to the large terms as zero-order terms and to the small terms as first-order terms, and denote them by subscripts 0 and 1, respectively. Hence, we write

$$\underset{\sim}{R}_s = \underset{\sim}{R}_0 + C_0^T \underset{\sim}{R}_1 , \qquad \alpha = \alpha_0 + \alpha_1 \tag{2.8a,b}$$

where $\underset{\sim}{R}_1$ is expressed in terms of components along the body axes, $\underset{\sim}{R}_1 = [\, R_y \;\; R_z \,]^T$, and we note that $C_0 = C(\alpha_0)$. Moreover, $\underset{\sim}{q}$ is regarded as a first-order quantity. It can be verified that

$$\dot{\underset{\sim}{R}}_s = \dot{\underset{\sim}{R}}_0 + C_0^T(\dot{\underset{\sim}{R}}_1 + \tilde{\Omega}_0 \underset{\sim}{R}_1) = \dot{\underset{\sim}{R}}_0 + C_0^T(\dot{\underset{\sim}{R}}_1 + \Omega_0 P \underset{\sim}{R}_1) \tag{2.9a}$$

$$\begin{aligned} \ddot{\underset{\sim}{R}}_s &= \ddot{\underset{\sim}{R}}_0 + C_0^T(\ddot{\underset{\sim}{R}}_1 + \dot{\tilde{\Omega}}_0 \underset{\sim}{R}_1 + 2\tilde{\Omega}_0 \dot{\underset{\sim}{R}}_1 + \tilde{\Omega}_0^2 \underset{\sim}{R}_1) \\ &= \ddot{\underset{\sim}{R}}_0 + C_0^T[\ \ddot{\underset{\sim}{R}}_1 + 2\Omega_0 P \dot{\underset{\sim}{R}}_1 + (\dot{\Omega}_0 P - \Omega_0^2 I)\underset{\sim}{R}_1] \end{aligned} \tag{2.9b}$$

$$\Omega_x = \Omega_0 + \dot{\alpha}_1 , \qquad \dot{\Omega}_x = \dot{\Omega}_0 + \ddot{\alpha}_1 \tag{2.9c,d}$$

where $\tilde{\Omega}_0 = \Omega_0 P$, in which

$$P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.10}$$

Similarly, the forcing terms can be divided as follows

$$\underset{\sim}{F} = \underset{\sim}{F}_0 + \underset{\sim}{F}_1 , \qquad M = M_0 + M_1 \tag{2.11a,b}$$

whereas $\underset{\sim}{Q}$ is assumed to be of first order.

Inserting Eqs. (2.8)-(2.11) into Eqs. (2.1) and neglecting second-order terms in the perturbations, we obtain the zero-order equations

$$m\ddot{\underset{\sim}{R}}_0 = C_0^T \underset{\sim}{F}_0 , \qquad I_c \dot{\Omega}_0 = M_0 \tag{2.12a.b}$$

where, considering Eqs. (2.7), $\underset{\sim}{F}_0 = \underset{\sim}{F}_{y0}\underset{\sim}{j} + \underset{\sim}{F}_{z0}\underset{\sim}{k}$, $M_0 = M_{x0}$. From Eqs. (2.12), it is obvious that, owing to the fact that the motion is referred to the mass center O, the translational and rotational

equations are independent of each other. Because the interest lies in a slewing maneuver, there is no loss of generality in assuming that $\dot{\underset{\sim}{R}}_0 = \ddot{\underset{\sim}{R}}_0 = \underset{\sim}{0}$. This permits us to dispense with Eq. (2.12a), so that we are left with a single zero-order equation, Eq. (2.12b). Moreover, we obtain the first-order equations, which can be written in the compact form

$$M\ddot{\underset{\sim}{x}} + G\dot{\underset{\sim}{x}} + K\underset{\sim}{x} = \underset{\sim}{X} \tag{2.13a}$$

$$\underset{\sim}{y}_d = \tilde{C}\underset{\sim}{x} \ , \qquad \underset{\sim}{y}_v = \tilde{C}\dot{\underset{\sim}{x}} \tag{2.13b}$$

where

$$x = \left[\underset{\sim}{R}_1^T \quad \alpha_1 \quad \underset{\sim}{q}^T \right]^T , \qquad \underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{F}_1 \\ M_1 \\ \underset{\sim}{Q} - \Omega_0 \dot{\underset{\sim}{\psi}} \end{bmatrix} \tag{2.14a,b}$$

are (3+N)-dimensional displacement and generalized force vectors, respectively, in which

$$\underset{\sim}{F}_1 = F_{y1}\underset{\sim}{j} + F_{z1}\underset{\sim}{k} \ , \qquad M_1 = M_{x1} + \sum_{i=1}^{p} M_{Ai} \tag{2.15a,b}$$

and $\underset{\sim}{Q}$ is given by Eq. (2.7c). Moreover,

$$M = \left[\begin{array}{c:c:c} mI_{2\times2} & \underset{\sim}{0}_2 & \underset{\sim}{e}_2\bar{\underset{\sim}{\phi}}^T \\ \hdashline \underset{\sim}{0}_2^T & I_c & \underset{\sim}{\psi}^T \\ \hdashline \bar{\underset{\sim}{\phi}}\,\underset{\sim}{e}_2^T & \underset{\sim}{\psi} & M_A \end{array} \right] , \qquad G = \left[\begin{array}{c:c:c} 2m\Omega_0 P & \underset{\sim}{0}_2 & -2\Omega_0 \underset{\sim}{e}_1\bar{\underset{\sim}{\phi}}^T \\ \hdashline \underset{\sim}{0}_2^T & 0 & \underset{\sim}{0}_N^T \\ \hdashline 2\Omega_0\bar{\underset{\sim}{\phi}}\,\underset{\sim}{e}_1^T & \underset{\sim}{0}_N & D \end{array} \right] \tag{2.16a,b}$$

$$K = \left[\begin{array}{c|c|c} m(\dot{\Omega}_0 P - \Omega_0^2 I) & \underset{\sim}{0}_2 & -(\dot{\Omega}_0 \underset{\sim}{e}_1 + \Omega_0^2 \underset{\sim}{e}_2)\bar{\underset{\sim}{\phi}}^T \\ \hline \underset{\sim}{0}_2^T & 0 & \underset{\sim}{0}_N^T \\ \hline \bar{\underset{\sim}{\phi}}(\dot{\Omega}_0 \underset{\sim}{e}_1^T - \Omega_0^2 \underset{\sim}{e}_2^T) & \underset{\sim}{0}_N & K_A - \Omega_0^2 M_A \end{array}\right] \tag{2.16c}$$

are coefficient matrices, where $\underset{\sim}{e}_1 = [\ 1 \quad 0\ ]^T$ and $\underset{\sim}{e}_2 = [\ 0 \quad 1\ ]^T$ are standard unit vectors and $\underset{\sim}{0}_2$ and $\underset{\sim}{0}_N$ are null vectors of dimensions 2 and N, respectively. Note that G and K depend on time through Ω_0 and $\dot{\Omega}_0$.

The vectors $\underset{\sim}{y}_d$ and $\underset{\sim}{y}_v$ represent the displacement and velocity measurement vectors. We assume that there are two pairs of translational displacement and velocity sensors and one pair of angular displacement and velocity sensor collocated with the actuators F_y, F_z and M_x. In addition, there are m/2 pairs of angular displacement and velocity sensors located throughout the flexible appendage. The perturbed motion is the difference between the sensor data and the commanded motion expressed in body axes. Hence, the output matrix $\tilde{C}$ is

$$\tilde{C} = \left[\begin{array}{c|c} I_{3\times3} & 0_{3\times N} \\ \hline \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{matrix} & C^* \end{array}\right] \tag{2.17}$$

where $C^{*T} = [\ \underset{\sim}{\phi}'(y_1) \quad \underset{\sim}{\phi}'(y_2) \ldots\ldots\ldots \underset{\sim}{\phi}'(y_{m/2})\ ]$ is an $m/2 \times N$ modal participation matrix. We notice that, in the case in which $m/2 = p$ and the sensors are collocated with the actuators $\tilde{C}^T = E$.

It will prove convenient to express the generalized force vector in terms of actual actuator forces and torques. To this end, we introduce the notation

$$F_{y1} = T_1 \ , \quad F_{Z1} = T_2 \ , \quad M_{x1} = T_3 \ , \quad M_{Ai} = T_{3+i}, \quad i=1,2,\ldots,p \tag{2.18a}$$

$$E = \left[\begin{array}{c|c} I_{3\times3} & \begin{matrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 \end{matrix} \\ \hline O_{N\times3} & E^* \end{array}\right] , \qquad \underset{\sim}{\Psi} = \left[\, \underset{\sim}{0}_3^T \ \vdots \ \underset{\sim}{\psi}^T \,\right]^T \tag{2.18b,c}$$

and consider Eqs. (2.7c) and (2.15) to obtain

$$\underset{\sim}{X} = E\underset{\sim}{T} - \dot{\Omega}_0 \underset{\sim}{\psi} \tag{2.19}$$

where $\underset{\sim}{T}$ is the (3+p) vector of actuator forces.

For future reference, we wish to express the matrices G and K in the form

$$G = 2\Omega_0 G^* + D^* \ , \qquad K = K_c + \dot{\Omega}_0 G^* - \Omega_0^2 K_s \tag{2.20a,b}$$

where

$$G^* = \left[\begin{array}{c|c|c} mP & \underset{\sim}{0}_2 & -\underset{\sim}{e}_1 \underset{\sim}{\bar{\phi}}^T \\ \hline \underset{\sim}{0}_2^T & 0 & \underset{\sim}{0}_N^T \\ \hline \underset{\sim}{\bar{\phi}}\, \underset{\sim}{e}_1^T & \underset{\sim}{0}_N & O_{N\times N} \end{array}\right] = -(G^*)^T, \quad D^* = \left[\begin{array}{c|c|c} O_{2\times2} & \underset{\sim}{0}_2 & O_{2\times N} \\ \hline \underset{\sim}{0}_2^T & 0 & \underset{\sim}{0}_N^T \\ \hline O_{N\times N} & \underset{\sim}{0}_N & D \end{array}\right] \tag{2.21a,b}$$

$$K_c = \left[\begin{array}{c:c:c} 0_{2\times2} & \underset{\sim}{0}_2 & 0_{2\times N} \\ \hdashline \underset{\sim}{0}_2^T & 0 & \underset{\sim}{0}_N^T \\ \hdashline 0_{N\times2} & \underset{\sim}{0}_N & K_A \end{array}\right] = K_c^T, \quad K_s = \left[\begin{array}{c:c:c} mI_{2\times2} & \underset{\sim}{0}_2 & \underset{\sim}{e}_2\bar{\underset{\sim}{\phi}}^T \\ \hdashline \underset{\sim}{0}_2^T & 0 & \underset{\sim}{0}_N^T \\ \hdashline \bar{\underset{\sim}{\phi}}\,\underset{\sim}{e}_2^T & \underset{\sim}{0}_N & M_A \end{array}\right] = K_s^T \qquad (2.21c,d)$$

and we recognize that G^* is skew symmetric, D^* is symmetric, $\bar{K}$ is the constant part of K, a symmetric matrix, and K_s is symmetric as well.

2.3 PSEUDO-MODAL EQUATIONS OF MOTION

Equation (2.13) represents a set of 3+N linear ordinary differential equations with time-dependent coefficients. The equation can be simplified by means of a linear transformation involving the matrix of eigenvectors of the undamped nonmaneuvering spacecraft. To this end, we observe that by letting $\Omega_0 = \dot{\Omega}_0 = 0$ Eq. (2.13) reduces to a time-invariant set of equations. The eigenvalue problem for the time-invariant system has the form $K_c U = MU\Lambda$, where U is a (3+N)×(3+N) matrix of eigenvectors and $\Lambda = [\,0\ 0\ 0\ \omega_1^2\ \omega_2^2\ \ldots\ldots\omega_N^2]$ is a diagonal matrix of eigenvalues, in which ω_i(i=1,2.....,N) are distinct natural frequencies of the nonmaneuvering spacecraft. Because M and K_c are real and symmetric and, moreover, M is positive definite and K_c is positive semidefinite, the eigenvectors are real and orthogonal with respect to M and K_c and the nonzero eigenvalues are real and positive [3]. The eigenvectors are assumed to be normalized so as to satisfy

$$U^T M U = I\ , \qquad U^T K_c U = \Lambda \qquad (2.22a,b)$$

Next, let us introduce the linear transformation

$$\underset{\sim}{x}(t) = U\underset{\sim}{v}(t) \tag{2.23}$$

into Eq. (2.13), multiply on the left by U^T, recall Eq. (2.18) and obtain

$$\ddot{\underset{\sim}{v}} + (\, 2\Omega_0\bar{G} + \bar{D}\,)\dot{\underset{\sim}{v}} + (\, \Lambda + \dot{\Omega}_0\bar{G} - \Omega_0^2\bar{K}\,)\underset{\sim}{v} = \bar{E}\,\underset{\sim}{T} - \dot{\Omega}_0\bar{\underset{\sim}{\Psi}} \tag{2.24a}$$

$$\underset{\sim}{y}_d = \bar{C}\underset{\sim}{v}\,, \qquad \underset{\sim}{y}_v = \bar{C}\dot{\underset{\sim}{v}} \tag{2.24b}$$

where

$$\bar{G} = U^TG^*U,\ \bar{D} = U^TD^*U,\ \bar{K} = U^TK_sU,\ \bar{E} = U^TE,\ \bar{C} = \tilde{C}U,\ \bar{\underset{\sim}{\Psi}} = U^T\underset{\sim}{\Psi} \tag{2.25}$$

Equations (2.24) represent a set of pseudo-modal equations.

Due to the nature of the coefficient matrices and of the excitation vector, Eqs,(2.24) possesses a relatively simple structure. To show this, we express the matrix M, as given by Eq. (2.16a), in the partitioned form

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{12}^T & M_{22} \end{array}\right] \tag{2.26}$$

where

$$M_{11} = \left[\begin{array}{c|c} mI_{2\times2} & \underset{\sim}{0}_2 \\ \hline \underset{\sim}{0}_2^T & I_c \end{array}\right], \qquad M_{12} = \left[\begin{array}{c} \underset{\sim}{e}_2\bar{\phi}^T \\ \hline \underset{\sim}{\psi}^T \end{array}\right], \qquad M_{22} = M_A \tag{2.27}$$

are 3×3 ,3×N and N×N matrices, respectively. Accordingly, we partition the matrix of eigenvectors as follows

$$U = \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] \tag{2.28}$$

Inserting Eqs. (2.26)-(2.28) into Eq. (2.22) and recalling Eq. (2.20c), we obtain

$$U_{11} = \left[\begin{array}{c|c} m^{-1/2} I_{2\times 2} & \underset{\sim}{0}_2 \\ \hline \underset{\sim}{0}_2^T & I_c^{-1/2} \end{array}\right], \quad U_{12} = -\left[\begin{array}{c} m^{-1} \underset{\sim}{e}_2 \underset{\sim}{\bar{\phi}}^T \\ \hline I_c^{-1} \underset{\sim}{\psi}^T \end{array}\right] U_{22}, \quad U_{21} = 0 \tag{2.29a,b,c}$$

Moreover, the matrix U_{22} and the nonzero eigenvalues in Λ are the solution of the eigenvalue problem

$$K_A U_{22} = \left[M_A - (m^{-1} \underset{\sim}{\bar{\phi}}\, \underset{\sim}{\bar{\phi}}^T + I_c^{-1} \underset{\sim}{\psi}\, \underset{\sim}{\psi}^T) \right] U_{22} \Lambda_e \tag{2.30}$$

where $\Lambda_e = \text{diag}\ (\omega_1^2\ \ \omega_2^2 \ldots \omega_N^2)$. Then, introducing Eq. (2.28) in conjunction with Eqs. (2.29) into the first four of Eqs. (2.25) and considering Eqs. (2.20a,b,c) and (2.17b), we obtain

$$\bar{G} = \left[\begin{array}{ccc|c} 0 & -1 & 0 & \\ 1 & 0 & 0 & 0_{3\times N} \\ 0 & 0 & 0 & \\ \hline & 0_{N\times 3} & & 0_{N\times N} \end{array}\right], \quad \bar{D} = \left[\begin{array}{c|c} 0_{3\times 3} & 0_{3\times N} \\ \hline 0_{N\times 3} & \bar{D}_e \end{array}\right] \tag{2.31a,b}$$

$$\bar{K} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & O_{3\times N} \\ 0 & 0 & 0 & \\ \hline & O_{N\times 3} & & \bar{K}_{22} \end{array}\right] , \quad \bar{E} = \left[\begin{array}{ccc|c} e_1 & 0 & 0 & \\ 0 & e_1 & 0 & O_{2\times p} \\ 0 & 0 & e_2 & e_2 \underset{\sim}{1}_p^T \\ \hline \underset{\sim}{O}_N & \underset{\sim}{\bar{E}}_2 & \underset{\sim}{\bar{E}}_3 & \bar{E}_e \end{array}\right] \tag{2.31c,d}$$

$$\bar{C} = \left[\begin{array}{ccc|c} e_1 & 0 & 0 & \underset{\sim}{O}_{1\times N} \\ 0 & e_1 & 0 & \underset{\sim}{\bar{E}}_2^T \\ 0 & 0 & e_2 & \underset{\sim}{\bar{E}}_3^T \\ \hline 0 & 0 & \vdots & \\ \vdots & \vdots & e_2 \underset{\sim}{1}_{m/2} & \bar{C}_e \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \vdots & \end{array}\right] \tag{2.31e}$$

where we assumed that the off-diagonal terms of the damping matrix are sufficiently small that they can be ignored. Thus, we obtain

$$\bar{D}_e = \mathrm{diag}(\ 2\zeta_1\omega_1 \quad 2\zeta_2\omega_2 \ldots\ldots\ldots 2\zeta_N\omega_N\ ) \tag{2.32}$$

in which ζ_i (i=1,2,.....,N) are viscous damping factors. Other terms entering into Eqs. (2.31) are as follows

$$\bar{K}_{22} = I_{N\times N} + I_C^{-1} U_{22}^T \underset{\sim}{\psi}\, \underset{\sim}{\psi}^T U_{22} \tag{2.33a}$$

$$e_1 = m^{-1/2}, \quad e_2 = I_C^{-1/2} \tag{2.33b,c}$$

$$\underset{\sim}{1}_p^T = [\ 1\ 1 \ldots\ldots 1\ ] \tag{2.33d}$$

$$\bar{\underset{\sim}{E}}_2 = - m^{-1}U_{22}^T\bar{\underset{\sim}{\phi}} \ , \quad \bar{\underset{\sim}{E}}_3 = - I_C^{-1}U_{22}^T\underset{\sim}{\psi} \tag{2.33e,f}$$

$$\bar{E}_e = U_{22}^T(E^* - I_C^{-1}\underset{\sim}{\psi}\ \underset{\sim}{1}_p^T) \tag{2.33g}$$

$$\bar{C}_e = (C^* - I_C^{-1}\underset{\sim}{1}_{m/2}\underset{\sim}{\psi}^T) U_{22} \tag{2.33h}$$

Considering Eqs. (2.31)-(2.33), we conclude that the pseudo-modal equations, Eqs. (2.24), can be separated into

$$\ddot{\underset{\sim}{v}}_t + 2\Omega_0(t)P\dot{\underset{\sim}{v}}_t + \left[\dot{\Omega}_0(t)P - \Omega_0^2(t)I_2\right]\underset{\sim}{v}_t = e_1[\ T_1 \quad T_2\ ]^T \tag{2.34a}$$

$$\ddot{v}_r = e_2\sum_{j=3}^{p+3} T_j \tag{2.34b}$$

$$\ddot{\underset{\sim}{v}}_e + \bar{D}_e\dot{\underset{\sim}{v}}_e + \left[\Lambda_e - \Omega_0^2(t)\bar{K}_{22}\right]\underset{\sim}{v}_e = \bar{E}_e\underset{\sim}{T}_e + T_2\bar{\underset{\sim}{E}}_2 + T_3\bar{\underset{\sim}{E}}_3 - \dot{\Omega}_0(t)\bar{\underset{\sim}{\psi}} \tag{2.34c}$$

$$\underset{\sim}{y}_d = \bar{C}\underset{\sim}{v} \ , \quad \underset{\sim}{y}_v = \bar{C}\dot{\underset{\sim}{v}} \tag{2.34d}$$

where $\underset{\sim}{v}_t = [\ v_1 \ v_2\ ]^T$ is the vector of perturbations in the rigid body translations, $v_r = v_3$ is the perturbation in the rigid- body rotation, $\underset{\sim}{v}_e = [\ v_4 \ v_5 \ldots\ldots v_{N+3}]^T$ is an N-vector of pseudo-modal coordinates corresponding to the elastic motion and $\underset{\sim}{T}_e = [\ T_4 \ T_5 \ldots\ldots T_{3+p}\ ]^T$ is a vector of torques on the elastic appendage.

The bottom components of $\underset{\sim}{\psi}$ tend to decrease rapidly relative to the top components. It follows that the bottom components of $\underset{\sim}{v}_e$ are less affected by the maneuver velocity $\Omega_0(t)$ and acceleration $\dot{\Omega}_0$, so that

these components tend to behave as if the structure were not slewing. Furthermore, the inertia and the moment of inertia of the hub are relatively large. Therefore, considering Eqs. (2.33e,f), it can be concluded that the components of $\bar{\underset{\sim}{E}}_2$ and $\bar{\underset{\sim}{E}}_3$ are relatively small. This was actually verified numerically.

CHAPTER 3

GENERAL CONTROL STRATEGY

3.1 RIGID-BODY SLEWING

The rigid-body slewing is controlled according to a minimum-time policy, which implies the so called on-off control [16]. Assuming an ideal actuator, Eq. (2.12b) represents a linear time-invariant system with two real eigenvalues. As a result, there is only one switching time, namely halfway through the maneuver time interval. Hence, denoting by t_0 the initial time and by t_f the final time, the switching time is simply $t_1 = (t_f - t_0)/2$, so that the slewing angular acceleration and velocity are

$$\dot{\Omega}_0(t) = \begin{cases} c & \text{for} \quad t_0 \le t < t_1 \\ -c & \text{for} \quad t_1 \le t \le t_f \end{cases} \tag{3.1a}$$

$$\Omega_0(t) = \begin{cases} c(t - t_0) & \text{for} \quad t_0 \le t < t_1 \\ -c(t - t_f) & \text{for} \quad t_1 \le t \le t_f \end{cases} \tag{3.1b}$$

3.2 CONTROL OF THE PERTURBED MODEL

Equations (2.34) describe a linear time-varying system subjected to persistent disturbances. The design of a reduced-order compensator, according to the methods yet to be described (Chapters 5 and 6 and Sections 8.2 and 8.3), can be based on this set of equations. It turns out, however, that Eq. (2.34a) precludes the achievement of quasi-constant gain matrices. To achieve sub-optimal steady-state

disturbance accommodation, it is necessary to control the translational motion independently. Furthermore, to decrease the control gains, so as to reduce the spillover effect, it is desirable to control the rotational motion independently. To this end, we let

$$T_3 = T_r - \sum_{j=4}^{p+3} T_j \tag{3.2}$$

so that Eqs. (2.34b,c) can be rewritten as

$$\ddot{v}_r = e_2 T_r \tag{3.3a}$$

$$\ddot{\underset{\sim}{v}}_e + \bar{D}\dot{\underset{\sim}{v}}_e + \left[\Lambda_e - \Omega_0^2(t)\bar{K}_{22} \right]\underset{\sim}{v}_e = \bar{E}^*_e \underset{\sim}{T}_e + T_2\underset{\sim}{\bar{E}}_2 + T_r\underset{\sim}{\bar{E}}_3 - \dot{\Omega}_0(t)\underset{\sim}{\bar{\psi}} \tag{3.3b}$$

where

$$\bar{E}^*_e = \bar{E}_e - \underset{\sim}{\bar{E}}_3 \underset{\sim}{1}_p^T = U_{22}^T E^* \tag{3.4}$$

in which $\underset{\sim}{1}^T = [\ 1\ 1 \ldots . 1\ ]$. Clearly, the rigid-body perturbations $\underset{\sim}{v}_t$ and v_r are not subjected to input disturbances caused by the maneuver and can be controlled independently of the elastic motions. This can be done according to a finite-time optimal control law, such as one encountered in a linear quadratic regulator. In this design process, the small coupling terms $\underset{\sim}{\bar{E}}_2^T \underset{\sim}{v}_e$ and $\underset{\sim}{\bar{E}}_3^T \underset{\sim}{v}_e$ contaminating the output of the rigid-body sensors will be treated as measurement noise. The control gains are designed so as to produce fast decay in $\underset{\sim}{v}_t$ and v_r, which implies that the controls T_1, T_2 and T_r are of relatively short duration. Then, the elastic motions can be controlled independently of the rigid-body

perturbations by regarding the terms $T_2\bar{E}_2$ and $T_r\bar{E}_3$ as transient disturbances. In view of this, we focus our attention on the control of the elastic motions under persistent disturbances caused by the maneuver.

Introducing the state vector $\underset{\sim}{z}_e(t) = [\, \underset{\sim}{v}_e^T(t) \,\vdots\, \dot{\underset{\sim}{v}}_e^T(t) \,]^T$, as well as the coefficient matrices

$$A(t) = \left[\begin{array}{c:c} 0_{N\times N} & I_{N\times N} \\ \hdashline -\Lambda_e + \Omega_0^2(t)\bar{K}_{22} & -\bar{D}_e \end{array}\right], \qquad B = \left[\begin{array}{c} 0_{N\times p} \\ \hdashline \bar{E}_e^* \end{array}\right] \tag{3.5a,b}$$

and the vectors

$$\underset{\sim}{R} = [\, \underset{\sim}{0}_N^T \,\vdots\, \bar{\underset{\sim}{\psi}}^T \,]^T, \qquad \underset{\sim}{d}(t) = [\, \underset{\sim}{0}_N^T \,\vdots\, (\, T_2\bar{E}_2 + T_r\bar{E}_3)^T \,]^T \tag{3.5c,d}$$

we can rewrite Eq. (3.3b) in the state form

$$\dot{\underset{\sim}{z}}_e = A(t)\underset{\sim}{z}_e + B\underset{\sim}{T}_e - \dot{\Omega}_0\underset{\sim}{R} + \underset{\sim}{d}(t) \tag{3.6}$$

Moreover, we write the output equation as

$$\underset{\sim}{y}_m = C\underset{\sim}{z}_e \tag{3.7}$$

where $\underset{\sim}{y}_m$ is an m-dimensional measurement vector, in which m/2 measurements are of angular displacements and m/2 measurements are of angular velocities, such that

$$C = \left[\begin{array}{c|c} \bar{C}_e^* & O_{m/2\times N} \\ \hline O_{m/2\times N} & \bar{C}_e^* \end{array}\right] \tag{3.8}$$

where $\bar{C}_e^* = C^* U_{22}$. Hence, $\underset{\sim}{y}_m$ corresponds to the difference between the appendage angular motion and the rigid-body rotational motion.

The coefficient matrix corresponding to the nonmaneuvering structure, namely

$$A_0 = \left[\begin{array}{c|c} 0 & I \\ \hline -\Lambda_e & -\bar{D}_e \end{array}\right] \tag{3.9}$$

is exponentially stable.

CHAPTER 4

CONTROL OF THE ELASTIC MODEL - PROBLEM FORMULATION

4.1 THE REDUCED-ORDER MODEL

The actual elastic displacement vector, which in general implies an infinite number of degrees of freedom, is approximated by a finite-dimensional vector $\underset{\sim}{u}(y,t)$ defined by Eq. (2.3). It is typical of structures that computed higher modes are inaccurate [3]. Moreover, inherent damping improves the robustness of higher modes and causes them to decay faster then lower modes. In addition, higher modes are more difficult to excite, unless the excitation has a high frequency content, and the components of the disturbance vector $-\dot{\Omega}_0(t)\bar{\underset{\sim}{\psi}}$ corresponding to the higher states tend to have lower values. Hence, it is reasonable to retain the first N components of $\underset{\sim}{v}_e$ in the perturbed elastic model. The state representation of this model is defined by Eq. (3.6). We denote the order of the elastic model by n_M, where $n_M = 2N$, and refer to it as the "modeled system". It is assumed that the dynamic characteristics of the modeled system are known fairly well and the value of n_M is such that 1) it is unlikely that the ignored degrees of freedom will destabilize the actual closed-loop system and will participate significantly in the motion and 2) an off-line computer is capable of carrying out any computations involving matrices of order n_M.

Control implementation considerations dictate further truncation. We retain the first $n_C/2$ components of $\underset{\sim}{v}_e$ for control, and denote the corresponding state vector by $\underset{\sim}{z}_C$. If we were to use an observer to

estimate the controlled state only, we could experience observation spillover which in conjunction with control spillover might cause divergence [38]. This effect can be eliminated from the significant part of the modeled system by increasing the order of the estimated state [34], thus shifting the observation spillover to more robust parts of the modeled system. Hence, we propose to estimate $n_O/2$ additional components without controlling them, and denote the corresponding part of the state vector by $\underset{\sim}{z}_O$. Finally, we refer to the remaining $n_R/2$ components as residual, and denote the corresponding part of the state vector by $\underset{\sim}{z}_R$. Hence, $\underset{\sim}{z}_e = [\, \underset{\sim}{z}_C^T \,\vdots\, \underset{\sim}{z}_O^T \,\vdots\, \underset{\sim}{z}_R^T \,]^T$ and $n_C + n_O + n_R = n_M$. Considering the above definitions, Eqs. (3.6) and (3.7) can be partitioned as follows

$$\frac{d}{dt}\begin{bmatrix} \underset{\sim}{z}_C \\ \hdashline \underset{\sim}{z}_O \\ \hdashline \underset{\sim}{z}_R \end{bmatrix} = \left[\begin{array}{c:c:c} A_{CC} & A_{CO} & A_{CR} \\ \hdashline A_{OC} & A_{OO} & A_{OR} \\ \hdashline A_{RC} & A_{RO} & A_{RR} \end{array}\right] \begin{bmatrix} \underset{\sim}{z}_C \\ \hdashline \underset{\sim}{z}_O \\ \hdashline \underset{\sim}{z}_R \end{bmatrix} + \begin{bmatrix} B_C \\ \hdashline B_O \\ \hdashline B_R \end{bmatrix} \underset{\sim}{T}_e - \dot{\Omega}_O \begin{bmatrix} \underset{\sim}{R}_C \\ \hdashline \underset{\sim}{R}_O \\ \hdashline \underset{\sim}{R}_R \end{bmatrix}$$

$$+ \begin{bmatrix} \underset{\sim}{d}_C(t) \\ \hdashline \underset{\sim}{d}_O(t) \\ \hdashline \underset{\sim}{d}_R(t) \end{bmatrix} \tag{4.1a}$$

$$\underset{\sim}{y}_m = [\, C_C \,\vdots\, C_O \,\vdots\, C_R \,] \begin{bmatrix} \underset{\sim}{z}_C \\ \hdashline \underset{\sim}{z}_O \\ \hdashline \underset{\sim}{z}_R \end{bmatrix} \tag{4.1b}$$

where

$$A_{ii} = \left[\begin{array}{c:c} 0_{i\times i} & I_{i\times i} \\ \hdashline -\Lambda_{ii} + \Omega_0^2(t)\bar{K}_{ii} & -\bar{D}_{ii} \end{array}\right], \quad A_{ij} = \Omega_0^2(t)\left[\begin{array}{c:c} 0_{i\times j} & 0_{i\times j} \\ \hdashline \bar{K}_{ij} & 0_{i\times j} \end{array}\right] \qquad (4.1c,d)$$

in which $i, j = c, o, R$.

4.2 FINITE-TIME STABILITY

The ordinary stability definitions relate to systems operating over an infinite-time interval. Hence, for the purpose of this dissertation, we must consider different definitions.

Under consideration is the time-varying system

$$\dot{\underset{\sim}{z}} = A(t)\underset{\sim}{z} \ , \quad z(t_0) = \underset{\sim}{z}_0 \qquad (4.2)$$

defined over the time interval $\nu = [\ t_0\ ,\ t_f\ )$, where $A(t)$ is an $n\times n$ matrix continuous on $R^n\times\nu$ and bounded for all $t \in \nu$. Then, we denote by $\bar{\underset{\sim}{z}}(t^*, t, \underset{\sim}{z})$ a trajectory of (4.2) evaluated at time t^*, which takes on the value $\underset{\sim}{z}$ at time t, and by $\|\ \|$ a norm on $R^n\times\nu$. This permits us to advance the following :

Definition 4.1 : The system (4.2) is exponentially contractively stable (ECS) with respect to $[\delta,\ \psi,\ \alpha,\ \mu(t),\ \nu,\ \|\ \|]$, where α is a real-valued positive constant, $\mu(t)$ is a real-valued continuous positive function and $\psi\geq\delta$, if every trajectory $\bar{\underset{\sim}{z}}(t^*,\ t_0,\ \underset{\sim}{z}_0)$, for which $\|\underset{\sim}{z}_0\|<\delta$, is such that

$$\| \bar{\underset{\sim}{z}}(t, t_0, \underset{\sim}{z}_0) \| < \psi \exp [-\int_{t_0}^{t} \mu(s)\, ds] \exp [- \alpha (t - t_0)], \quad t \in \tau \qquad (4.3)$$

Definition 4.2 [24] : The system (4.2) is quasi-contractively stable (QCS) with respect to $[\delta, \rho, \nu, \| \ \|]$ if for every trajectory $\bar{\underset{\sim}{z}}(t^*, t_0, \underset{\sim}{z}_0)$, where $\| \underset{\sim}{z}_0 \| < \delta$, there exists a $T_1 \in (t_0, t_f)$ such that

$$\| \bar{\underset{\sim}{z}}(t^*, t_0, \underset{\sim}{z}_0) \| < \rho , \quad \rho < \delta, \qquad t^* \in (T_1, t_f) \qquad (4.4)$$

Definition 4.3 : The system (4.2) is strictly finite-time stabilizable (SFTS) with respect to $[\delta, \varepsilon, \nu, \| \ \|]$ if for any positive ε there exists a T_1 such that $t_0 < T_1 < t_f$ and every trajectory $\bar{\underset{\sim}{z}}(t^*, t_0, \underset{\sim}{z}_0)$ for which $\| \underset{\sim}{z}_0 \| < \delta$ has the property

$$\| \bar{\underset{\sim}{z}}(t^*, t_0, \underset{\sim}{z}_0) \| < \varepsilon \quad \text{for all} \quad t^* \in (T_1, t_f) \qquad (4.5)$$

The most obvious difference between the definitions of finite-time stability and the usual stability definitions is that, in the former case, there are fixed prespecified bounds on the trajectories. The meaning of the above definitions is as follows (see Fig.3): A system is ECS if its impulse response is bounded by an exponentially decaying function, it is QCS if its impulse response decays below a threshold value lower than its initial value and it is SFTS if its impulse response can be forced to decay below any arbitrarily small threshold.

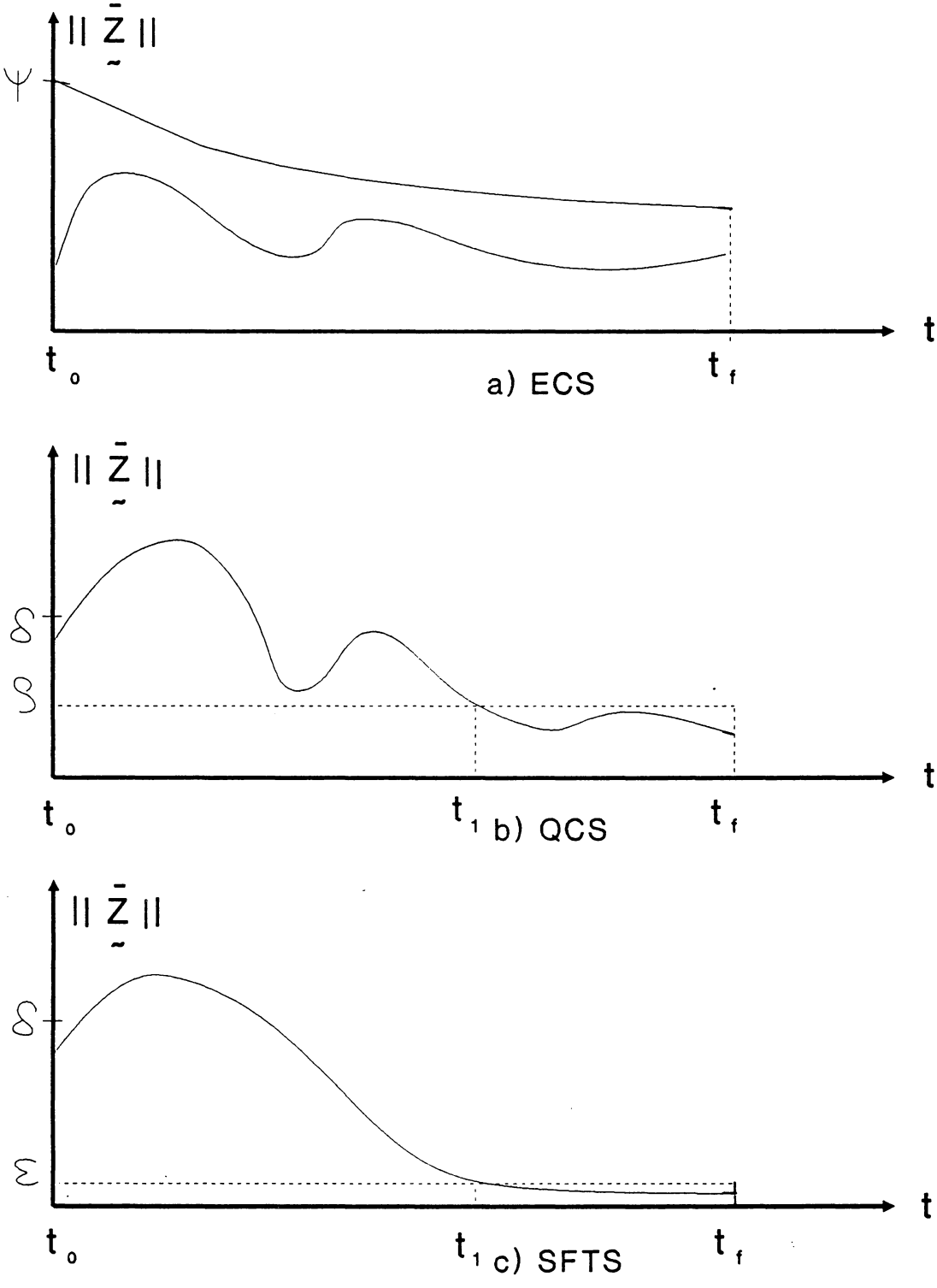


Figure 3. Finite-Time Stability Definitions

At this point, we wish to formalize sufficiency conditions for ECS and SFTS systems corresponding to a specific bound, which is a slight modification of the regular bound suggested elsewhere [19]. With reference to Eq. (4.2), we define the new variable

$$\underset{\sim}{z}^* = e^{\gamma t}\underset{\sim}{z}, \quad \gamma > 0 \tag{4.6}$$

so that Eq. (4.2) yields

$$\dot{\underset{\sim}{z}}^* = A^*(t)\underset{\sim}{z}^* \quad , \quad \underset{\sim}{z}^*(t_0) = \underset{\sim}{z}_0^* \tag{4.7a,b}$$

where

$$A^*(t) = A(t) + \gamma I \tag{4.8}$$

Next, we introduce the Lyapunov function

$$V(\underset{\sim}{z}^*, t) = \underset{\sim}{z}^{*T}B(t)\underset{\sim}{z}^* \tag{4.9}$$

where B(t) is a symmetric and differentiable matrix such that

$$0 \le c_1 \le \|B(t)\| \le c_2 < \infty, \quad t \in \nu \tag{4.10}$$

The implication of the above is that $V(\underset{\sim}{z}^*, t)$ is a positive definite function for all $t \in \nu$. Differentiating Eq. (4.9) with respect to time and evaluating along a trajectory of Eq. (4.7), we obtain

$$\dot{V}(\underset{\sim}{z}^*, t) = - \underset{\sim}{z}^{*T} C(t) \underset{\sim}{z}^* \tag{4.11}$$

where C(t) is a symmetric differentiable matrix such that

$$\dot{B}(t) = - B(t)A^*(t) - A^{*T}(t)B(t) - C(t) \tag{4.12}$$

We introduce now the following definition

$$\mu(t) = \frac{\min_i \lambda_i(C(t))}{\max_i \lambda_i(B(t))} \tag{4.13}$$

so that, considering Eqs. (4.9) and (4.11) we obtain

$$\dot{V}(\underset{\sim}{z}^*, t) \le - \mu(t) V(\underset{\sim}{z}^*, t) \qquad \text{for all } t \in \tau. \tag{4.14}$$

Then, integrating both sides of the above inequality, we obtain

$$V(\underset{\sim}{z}^*, t) \le V(\underset{\sim}{z}_0^*, t_0) \exp \left[- \frac{1}{2} \int_{t_0}^{t} \mu(s)\, ds \right] \tag{4.15}$$

We consider now the following property of positive definite quadratic forms [19]

$$\min_i \lambda_i(B(t)) \, \| \underset{\sim}{z}^* \|^2 \le \| \underset{\sim}{z}^{*T} B(t) \underset{\sim}{z}^* \| \le \max_i \lambda_i(B(t)) \, \| \underset{\sim}{z}^* \|^2 \tag{4.16}$$

so that, considering Eqs. (4.6), (4.9), (4.15) and (4.16) we obtain

$$\| \underset{\sim}{z}(t) \| < \| \underset{\sim}{z}_0 \| \, r_B^{1/2}(t) \exp \left[- \frac{1}{2} \int_{t_0}^{t} \mu(s) \, ds \right] \exp \left[- \gamma \, (t-t_0) \right] \tag{4.17}$$

where

$$r_B(t) = \frac{\max_i \lambda_i(B(t_0))}{\min_i \lambda_i(B(t))} \tag{4.18}$$

so that, considering inequality (4.10) and Eqs. (4.13), we conclude that $\mu(t) > 0$ for all $t \in \tau$ if and only if $C(t) > 0$ for $t \in \nu$. The above developments, in conjunction with Definitions 4.1 and 4.3, permits us to state the following sufficiency conditions:

1. If $C(t) > 0$ for $t \in \nu$, Eq. (4.2) represents an ECS system.
2. If $C(t) > 0$ for $t \in \nu$, and in addition $r_B^{1/2}(t) \exp [-\gamma \, (t-t_0)] < \varepsilon$ for all $t \in (T_1, t_f)$, $t_0 < T_1 < t_f$ then Eq. (4.2) represents an SFTS system.

Finally, defining

$$\bar{r}_B = \max_{t \in \tau} (r_B(t)) \tag{4.19}$$

we can rewrite inequality (4.17) in the form

$$\| \underset{\sim}{z}(t) \| < \| \underset{\sim}{z}_0 \| \, \bar{r}_B^{1/2} \exp \left[- \frac{1}{2} \int_{t_0}^{t} \mu(s) \, ds \right] \exp \left[- \gamma \, (t-t_0) \right] \tag{4.20}$$

It is clear that inequality (4.20) gives an estimate of how fast equilibrium is approached. In fact, if $\mu(t) > 0$ for all $t \in \tau$, then $\|z(t)\|$ converges to zero not slower than $e^{-\gamma t}$. Therefore, $1/\gamma$ may be interpreted as an upper bound on the time constant of system (4.2). We denote by π the least upper bound of $1/\gamma$ and refer to π as the supremum time constant (STC). It is clear that the STC corresponds to the value of γ for which $\mu(\gamma, t) = 0$ for some $t \in \tau$. Hence, we can define π as

$$\pi = 1/\bar{\gamma} \tag{4.21a}$$

where $\bar{\gamma}$ is the maximum value of γ, for which

$$\min_{t \in \tau} |\lambda_m[C(\bar{\gamma}, t)]| = 0 \tag{4.21b}$$

4.3 OBJECTIVE AND DESIGN POLICY

Our objective is to design a compensator exhibiting the following characteristics :

(i) The controlled model should be stabilized according to a finite-time quadratic performance measure such that the closed-loop controlled model is SFTS.

(ii) The disturbance accommodation control law should minimize a weighted norm of the entire modeled state by the end of the maneuver period, provided that steady-state condition is reached. Due to on-line computer limitations, the gain matrix should be a constant matrix.

(iii) The closed-loop modeled system should remain an ECS system.

Moreover, the STC of the modeled system should be minimized, where the minimization problem is constrained by (i) and (ii) above.
Clearly, the elastic motions must remain in the linear range during maneuver.

Equations (3.6) and (3.7) represent a linear, time-varying system subjected to a piecewise-constant disturbance and a transient disturbance. The open-loop coefficient matrix A(t) is assumed to be a QCS matrix through the maneuver. The commanded angular velocity $\Omega_0(t)$ and angular acceleration $\dot{\Omega}_0(t)$ are known a priori. On the other hand, we shall regard $\underset{\sim}{R}$ as insufficiently accurate for direct compensation. Hence, we shall treat the persistent disturbance as unknown, except that it is piecewise constant. Moreover, we assume that the number of sensors is not sufficient to measure all the controlled states, so that we must use an observer to estimate the controlled state $\underset{\sim}{z}_C$ and part of the disturbance, where that part is denoted by $\underset{\sim}{f}_E$.

Due to the nature of the disturbance vector $\dot{\Omega}_0\underset{\sim}{R}$, as described by Eq. (3.1a), we propose to carry out the control and estimation over one half of the maneuver period at a time. Hence, we denote that interval by $\tau = [\ t_i,\ t_h)$, where $t_i = t_0$ and $t_h = t_1$ for the first half of the maneuver period and $t_i = t_1$ and $t_h = t_f$ for the second half.

Considering the three approaches for disturbance accommodation discussed in Sec. 1.4, the preferred method in case of a L.T.V & F.T ROM is to divide the control into two parts as follows

$$\underset{\sim}{T}_e = \underset{\sim}{u}_C + \underset{\sim}{u}_D \tag{4.22}$$

where $\underset{\sim}{u}_C$ represents feedback control in the absence of persistent disturbances and is designed so as to achieve objectives (i) and (iii) and steady-state condition within the finite-time interval. On the other hand, $\underset{\sim}{u}_D$ represents the disturbance accommodation part of the control according to objective (ii).

CHAPTER 5

CONTROLLER DESIGN

To design the feedback control according to objective (i), we consider the controlled model in absence of disturbances, so that Eq. (4.1) reduces to

$$\dot{\underset{\sim}{z}}_c(t) = A_{cc}(t)\, \underset{\sim}{z}_c(t) + B_c\, \underset{\sim}{u}_c(t), \qquad \underset{\sim}{z}_c(t_o) = \underset{\sim}{z}_{co}, \quad t\epsilon\tau \tag{5.1}$$

Considering Eq. (3.1b), we conclude that $A_{cc}(t)$ and all its derivatives depend continuously on t and are bounded for all $t\epsilon\tau$. The dimension of $\underset{\sim}{z}_c$ is n_c and that of $\underset{\sim}{u}_c$ is p, where $p \leq n_c/2$ .It is shown in Chapter 7 that the pair $[A_{cc}(t),\ B_c]$ is totally controllable (Def. A.3 in the Appendix) for $t\epsilon\tau$ if several conditions are satisfied. To achieve the first control objective described above, we consider the performance measure

$$J = \int_{t_i}^{t_h} e^{2\alpha t} (\underset{\sim}{z}_c^T\, Q\, \underset{\sim}{z}_c + \underset{\sim}{u}_c^T\, R\, \underset{\sim}{u}_c)dt + e^{2\alpha t_h}\, \underset{\sim}{z}_c^T\, S_1\, \underset{\sim}{z}_c \tag{5.2}$$

where $\alpha > 0$ is a convergence factor, Q and R are real symmetric positive definite matrices and S_1 is a real symmetric and positive semi definite matrix. Moreover, Q can be decomposed according to $Q = LL^T$, where L is such that the pair $[A_{cc}(t),L]$ is totally observable for all $t\epsilon\tau$. The problem defined by Eqs. (5.1) and (5.2) can be reduced to a standard LQR

problem by introducing the transformations [18]

$$\underset{\sim}{z}_c^*(t) = e^{\alpha t}\underset{\sim}{z}_c(t) \ , \qquad \underset{\sim}{u}_c^*(t) = e^{\alpha t}\underset{\sim}{u}_c(t) \qquad \alpha \geq 0 \tag{5.3a,b}$$

Inserting Eqs. (5.3) into Eq. (5.1) and rearranging, we obtain

$$\dot{\underset{\sim}{z}}_c^*(t) = A_{cc}^*(t)\ \underset{\sim}{z}_c^*(t) + B_c\underset{\sim}{u}_c^*(t) \tag{5.4}$$

where

$$A_{cc}^*(t) = A_{cc}(t) + \alpha I \tag{5.5}$$

It is shown in Chapter 7 that the total controllability of the original system, Eq. (5.1), also implies the total controllability of the modified system defined by Eq. (5.4). Moreover, using Eqs. (5.3) the performance measure becomes

$$J^* = \int_{t_i}^{t_h} (\underset{\sim}{z}_c^{*T}\ Q\ \underset{\sim}{z}_c^* + \underset{\sim}{u}_c^{*T} R\ \underset{\sim}{u}_c^*)dt + \underset{\sim}{z}_c^*\ S_1\ \underset{\sim}{z}_c^* \tag{5.6}$$

The control law minimizing J^* is

$$\underset{\sim}{u}_c^*(t) = - R^{-1}B_c^T S(t)\underset{\sim}{z}_c^*(t) \tag{5.7}$$

where S(t) is the solution of the matrix Riccati equation

$$\dot{S}(t) = -Q - S(t)\ A_{cc}^*(t) - A_{cc}^{*T}(t)S(t) + S(t)B_cR^{-1}B_c^T S(t), \tag{5.8a}$$

where

$$S(t_h)=S_1 \tag{5.8b}$$

Introducing Eqs. (5.3) into Eq. (5.7), the control law for the original system can be written in the form

$$\underset{\sim}{u}_c(t) = K_c(t)\underset{\sim}{z}_c(t) \tag{5.9}$$

where

$$K_c(t) = - R^{-1}B_c^T S(t) \tag{5.10}$$

is the control gain matrix.

Next, we wish to develop an explicit sufficiency condition guaranteeing the SFTS of the closed-loop system described by Eqs. (5.1),(5.9) and (5.10).To this end we consider the second sufficiency condition (Sec.4.2) and define the Lyapunov function

$$V(\underset{\sim}{z}_c^*, t) = \underset{\sim}{z}_c^{*T} S(t)\underset{\sim}{z}_c^* \tag{5.11}$$

Differentiating Eq. (5.11) with respect to time, evaluating the result along solutions of Eq. (5.4) and considering Eq. (5.8), we obtain

$$\dot{V}(\underset{\sim}{z}_c^*, t) = - \underset{\sim}{z}_c^{*T} C(t)\underset{\sim}{z}_c^* \tag{5.12}$$

where

$$C(t) = Q + S(t)B_c^T R^{-1} B_c S(t) \tag{5.13}$$

is a positive definite matrix for all $t\epsilon\tau$, because $Q>0$ and $S(t)B_c^T R^{-1} B_c S(t) \geq 0$ for all $t\epsilon\tau$. This proves the first part of the sufficiency condition.

To satisfy the second part of the sufficiency condition , we must show that, for sufficiently large α,

$$\Pi(t,\alpha) < \varepsilon \quad \text{for all } t\epsilon\tau_1 \text{ and } \alpha \geq \bar{\alpha}(\varepsilon) > 0 \tag{5.14a}$$

where

$$\tau_1 = (T_1, t_h) \quad , \quad t_i < T_1 < t_h \tag{5.14b}$$

and

$$\Pi(t,\alpha) = r_s(t,\alpha) \exp[-2\alpha(t - t_i)] \tag{5.14c}$$

in which

$$r_s(t,\alpha) = \frac{\lambda_M[S(t_i,\alpha)]}{\lambda_m[S(t,\alpha)]} \tag{5.14d}$$

where $\varepsilon > 0$ is an arbitrary small number and λ_M, λ_m denote maximum and minimum eigenvalues of the argument matrix, respectively. Condition (5.14) is satisfied, for α large enough, if

$$\frac{\Pi(t,\alpha + \Delta\alpha)}{\Pi(t,\alpha)} < 1 \quad \text{for all } t\epsilon\tau_1 , \alpha > \bar{\alpha}(\varepsilon) \text{ and } \Delta\alpha > 0 \tag{5.15}$$

Inserting Eq. (5.14) into Eq. (5.15), we can rewrite condition (5.15), as

$$\chi < \exp[2\Delta\alpha(t - t_i)] \quad \text{for all } t\epsilon\tau_1 , \alpha > \bar{\alpha}(\varepsilon) \tag{5.16}$$

where

$$\chi = \frac{\lambda_M[S(t_i,\alpha + \Delta\alpha)]}{\lambda_M[S(t_i,\alpha)]} \; \frac{\lambda_m[S(t,\alpha)]}{\lambda_m[S(t,\alpha + \Delta\alpha)]} \tag{5.17}$$

Next, we define

$$\Delta S(t) = S(t,\alpha + \Delta\alpha) - S(t,\alpha) \tag{5.18}$$

and recognize that

$$\Delta S(t) = \Delta S^T(t) \qquad \text{for all } t\epsilon\tau \tag{5.19}$$

Furthermore, differentiating Eq. (5.8) with respect to α, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial \alpha}\right) = -\frac{\partial S}{\partial \alpha}(A^*_{CC} + B_C K_C) - (A^*_{CC} + B_C K_C)^T \frac{\partial S}{\partial \alpha} - 2S \tag{5.20}$$

Hence,

$$\frac{\partial S}{\partial \alpha} = 2\int_t^{t_h} \bar{\Phi}^T(\sigma,t)\, S(\sigma)\, \bar{\Phi}(\sigma,t)\, d\sigma \tag{5.21}$$

where $\bar{\Phi}(\ ,\ )$ denotes the transition matrix of $(A^*_{CC} + B_C K_C)$. Moreover,

$$S(t) > 0 \quad \text{for all } t\epsilon\tau \tag{5.22}$$

so that

$$\frac{\partial S}{\partial \alpha} > 0 \qquad \text{for all } t\epsilon\tau \tag{5.23}$$

Considering Eqs. (5.19), (5.23) and definition (5.18), we obtain

$$\lambda_i[\Delta S(t)] > 0 \quad \text{for all } t\epsilon\tau \quad \text{and } i=1,2,\ldots,n_c \tag{5.24}$$

Recalling the well-known bounds on the eigenvalues of the sum of two Hermitian matrices [3], we further obtain

$$\lambda_i[S(t,\alpha)]+\lambda_m[\Delta S(t)] \le \lambda_i[S(t,\alpha+\Delta\alpha)] \le \lambda_i[S(t,\alpha)]+\lambda_M[\Delta S(t)] \tag{5.25}$$

Considering Eq. (5.24), we have

$$\lambda_m[S(t,\alpha+\Delta\alpha)] \ge \lambda_m[S(t,\alpha)] \tag{5.26}$$

so that, inserting inequality (5.26) into Eq. (5.7), we obtain

$$\chi < \frac{\lambda_M[S(t_i,\alpha+\Delta\alpha)]}{\lambda_M[S(t_i,\alpha)]} \tag{5.27}$$

Hence, the second part of the sufficiency condition for an SFTS system is

$$\frac{\lambda_M[S(t_i,\alpha+\Delta\alpha)]}{\lambda_M[S(t_i,\alpha)]} < \exp[2\,\Delta\alpha\,(t-t_i)] \text{ for all } t\epsilon\tau_i \text{ and } \alpha > \bar{\alpha}(\varepsilon) \tag{5.28}$$

and we recognize that the above condition is expressed in terms of the Riccati solution S(t) evaluated at the initial time t_i where the solution is likely to reach its steady-state value if α is sufficiently large. Hence, we can finally state that, if inequality (5.28) is

satisfied, then, for sufficiently large α, the system represented by Eqs. (5.1), (5.9) and (5.10) is SFTS.

It turns out that condition (5.14) is satisfied for the system considered here, even for moderate values of α. This quality can be attributed to the controllability and observability characteristics of the system. Indeed, following Theorem A-6, we can conclude that

(i) If $[A_{cc}, B_c]$ is totally controllable for $t \in \tau$, then

$$S(t) \le b_c(t, t_2) I \quad \text{for all} \quad t < t_2 \le t_h \tag{5.29}$$

(ii) If $[A_{cc}, L]$ is totally observable for $t \in \tau$, then

$$S^{-1}(t) \le b_o(t, t_2) \quad \text{for all} \quad t < t_2 \le t_h \tag{5.30}$$

in which $b_c(\ ,\ )$ and $b_o(\ ,\ )$ are defined in Theorem A-6 (Appendix). Hence, recalling Eq. (5.14), $r_s(t, \alpha)$ is bounded from above by

$$r_s(t, \alpha) \le b_c(t_i, t_h - \sigma)\, b_o(t_i, t_h - \sigma) \quad \text{for all} \quad t \in [t_i, t_h - \sigma) \tag{5.31}$$

where $\sigma \ge 0$.

CHAPTER 6

OBSERVER DESIGN

6.1 INTRODUCTION

The observer plays a major role in the proposed solution. It is used to estimate the controlled state vector, the accommodated part of the disturbance vector and, if necessary, to augment the observed state vector to reduce observation spillover effects. Because of on-line computer limitations, there are limitations on the order of the observer. On the other hand, the number of sensors on the flexible appendage can be relatively large because sensors are light and easy to handle. Hence, a reduced-order observer is very appealing, considering that the order of a reduced-order observer is equal to the difference between the dimension of the observed vector and the number of sensors. In this chapter, we consider both types of observers and examine them with regard to the effect of the residual states.

6.2 PLANT EQUATIONS IN TERMS OF THE OBSERVED VECTOR

The observer is expected to estimate not only part of the state vector but also part of the persistent disturbance vector. Hence, let us introduce the notation

$$- \dot{\Omega}_{0} \underset{\sim}{R} = \underset{\sim}{W} = \left[\underset{\sim}{W}_C^T \;\vdots\; \underset{\sim}{W}_O^T \;\vdots\; \underset{\sim}{W}_R^T \right]^T \tag{6.1}$$

where $\underset{\sim}{W}_C$, $\underset{\sim}{W}_O$ and $\underset{\sim}{W}_R$ are the parts of $-\dot{\Omega}_{0} \underset{\sim}{R}$ corresponding to the

controlled and observed, uncontrolled but observed and residual states, respectively. Considering the disturbance dynamics, Eqs. (3.1), and the structure of $\underset{\sim}{R}$, Eq. (3.5c), we denote the part of $-\dot{\Omega}_0\bar{\underset{\sim}{\psi}}$ to be estimated by $\underset{\sim}{f}_E$ and introduce the relations

$$\dot{\underset{\sim}{f}}_E = \underset{\sim}{\delta}_E(t), \qquad \underset{\sim}{W}_E = H_E\underset{\sim}{f}_E \tag{6.2}$$

where $\underset{\sim}{f}_E$ is an $n_W/2$-vector, $\underset{\sim}{\delta}_E(t)$ is a vector of Dirac delta functions of the same dimension, $\underset{\sim}{W}_E$ is an n_W-vector, $n_W \le n_C + n_O$, and H_E is an $(n_C + n_O) \times n_W/2$ matrix having the form

$$H_E = \left[\begin{array}{c:c} O_1 & O_2 \\ \hdashline I_C & O_3 \\ \hdashline O_4 & O_5 \\ \hdashline O_6 & \begin{array}{c} I_L \\ \hdashline O_7 \end{array} \end{array}\right] \tag{6.3}$$

in which I_C and I_L are $n_C/2 \times n_C/2$ and $n_L/2 \times n_L/2$ identity matrices, respectively, where $n_L = n_W - n_C$. Furthermore, O_i $(i = 1,\ldots,7)$ are $n_C/2 \times n_C/2$, $n_C/2 \times n_L/2$, $n_C/2 \times n_L/2$, $n_O/2 \times n_C/2$, $n_O/2 \times n_L/2$, $n_O/2 \times n_C/2$ and $(n_O - n_L)/2 \times n_L/2$ null matrices. For convenience, we rewrite Eq. (6.1) as

$$-\dot{\Omega}_0\underset{\sim}{R} = \left[\ (\underset{\sim}{W}_E^{**})^T \,\vdots\, \underset{\sim}{W}_R^T\ \right]^T + \left[\begin{array}{c} H_E \\ \hdashline O \end{array}\right] \underset{\sim}{f}_E \tag{6.4}$$

where O denotes a null matrix. Defining the vector to be estimated as $\underset{\sim}{\eta}_E = [\, \underset{\sim}{z}_C^T \,\vdots\, \underset{\sim}{z}_O^T \,\vdots\, \underset{\sim}{f}_E^T \,]^T$, Eqs. (3.6), (6.2a) and (6.4) can be combined into

$$\begin{bmatrix} \dot{\underset{\sim}{\eta}}_E \\ \hline \dot{\underset{\sim}{z}}_R \end{bmatrix} = \left[\begin{array}{c|c} A_{EE} & A_{ER} \\ \hline A_{RE} & A_{RR} \end{array}\right] \begin{bmatrix} \underset{\sim}{\eta}_E \\ \hline \underset{\sim}{z}_R \end{bmatrix} + \begin{bmatrix} B_E \\ \hline B_R \end{bmatrix} \underset{\sim}{T}_e + \begin{bmatrix} I_E^* \\ \hline O \end{bmatrix} \underset{\sim}{\delta}_E(t)$$

$$+ \begin{bmatrix} \underset{\sim}{W}_E^* \\ \hline \underset{\sim}{W}_R \end{bmatrix} + \begin{bmatrix} \underset{\sim}{d}_E^*(t) \\ \hline \underset{\sim}{d}_R(t) \end{bmatrix} \tag{6.5}$$

where

$$A_{EE} = \left[\begin{array}{c:c:c} A_{CC} & A_{CO} & \\ \hline A_{OC} & A_{OO} & H_E \\ \hline O & O & O \end{array}\right], \quad A_{ER} = \left[\begin{array}{c} A_{CR} \\ \hline A_{OR} \\ \hline O \end{array}\right], \quad A_{RE} = [\, A_{RC} \,\vdots\, A_{RO} \,\vdots\, O \,]$$

$$B_E = \begin{bmatrix} B_C \\ \hline B_O \\ \hline O \end{bmatrix}, \quad I_E^* = \begin{bmatrix} O \\ \hline O \\ \hline I_E \end{bmatrix}, \quad \underset{\sim}{W}_E^* = \begin{bmatrix} \underset{\sim}{W}_E^{**} \\ \hline O \end{bmatrix}, \quad \underset{\sim}{d}_E^* = \begin{bmatrix} \underset{\sim}{d}_C \\ \hline \underset{\sim}{d}_O \\ \hline \underset{\sim}{O} \end{bmatrix} \tag{6.6}$$

in which I_E is an $n_W/2 \times n_W/2$ identity matrix; the various submatrices in Eqs. (6.6) can be deduced from Eqs. (3.5). Consistent with the above, the output vector can be written in the form

$$\underset{\sim}{y}_m = C_E \underset{\sim}{\eta}_E + C_R \underset{\sim}{z}_R \tag{6.7}$$

where, in accordance with Eq. (3.8),

$$C_E = [\ C_P \ \vdots\ 0\ ] \tag{6.8a}$$

$$C_P = \left[\begin{array}{cc|cc} \bar{C}_C & 0 & \bar{C}_O & 0 \\ 0 & \bar{C}_C & 0 & \bar{C}_O \end{array}\right], \quad C_R = \begin{bmatrix} \bar{C}_R & 0 \\ 0 & \bar{C}_R \end{bmatrix} \tag{6.8b,c}$$

6.3 A FULL-ORDER OBSERVER

Introducing the estimated vector $\hat{\underset{\sim}{\eta}}_E = [\ \hat{\underset{\sim}{z}}_C^T \ \vdots\ \hat{\underset{\sim}{z}}_O^T \ \vdots\ \hat{\underset{\sim}{f}}_E^T\ ]^T$ and recalling Eq. (4.22), we consider an observer described by

$$\dot{\hat{\underset{\sim}{\eta}}}_E = A_{EE}\hat{\underset{\sim}{\eta}}_E + B_E(\underset{\sim}{u}_C + \underset{\sim}{u}_D) + K_E(\hat{\underset{\sim}{y}}_m - \underset{\sim}{y}_m) \tag{6.9}$$

where $K_E = K_E(t)$ is the observer gain matrix, $\underset{\sim}{y}_m$ is the measurement vector and

$$\hat{\underset{\sim}{y}}_m = C_E\hat{\underset{\sim}{\eta}}_E \tag{6.10}$$

is the observer output vector. Defining the observer error vector as

$$\underset{\sim}{e} = \underset{\sim}{\eta}_E - \hat{\underset{\sim}{\eta}}_E \tag{6.11}$$

and considering Eqs. (6.5), (6.7), (6.9) and (6.10), we obtain the differential equation of the observer error

$$\dot{\underset{\sim}{e}} = (A_{EE} + K_E C_E)\underset{\sim}{e} + (A_{ER} + K_E C_R)\underset{\sim}{z}_R + I_E^*\underset{\sim}{\delta}_E(t) + \underset{\sim}{W}_E^* + \underset{\sim}{d}_E^*(t) \tag{6.12}$$

Next, we wish to derive the closed-loop equations for the modeled

system incorporating the observer described above. To this end, we write the dynamic control and the persistent disturbance control in the form

$$\underset{\sim}{u}_C(t) = K_C(t)\hat{\underset{\sim}{z}}_C(t) \ , \ \underset{\sim}{u}_D(t) = K_D\hat{\underset{\sim}{f}}_E \tag{6.13a,b}$$

Then, denoting the uncontrolled state vector by $\underset{\sim}{z}_U = \left[\underset{\sim}{z}_O^T \ \vdots \ \underset{\sim}{z}_R^T \right]^T$ and considering Eqs. (6.3) and (6.12), we obtain

$$\begin{bmatrix} \dot{\underset{\sim}{z}}_C \\ \hline \dot{\underset{\sim}{e}} \\ \hline \dot{\underset{\sim}{z}}_U \end{bmatrix} = \left[\begin{array}{c:c:c} \tilde{A}_{CC} & \tilde{A}_{CE} & \tilde{A}_{CU} \\ \hline 0 & \tilde{A}_{EE} & \tilde{A}_{EU} \\ \hline \tilde{A}_{UC} & \tilde{A}_{UE} & \tilde{A}_{UU} \end{array}\right] \begin{bmatrix} \underset{\sim}{z}_C \\ \hline \underset{\sim}{e} \\ \hline \underset{\sim}{z}_U \end{bmatrix} + \begin{bmatrix} \underset{\sim}{W}_C + B_C K_D \underset{\sim}{f}_E \\ \hline \underset{\sim}{W}_E^* \\ \hline \underset{\sim}{W}_U + B_U K_D \underset{\sim}{f}_E \end{bmatrix} + \begin{bmatrix} 0 \\ \hline I_E \\ \hline 0 \end{bmatrix} \underset{\sim}{\delta}_E(t)$$

$$+ \begin{bmatrix} \underset{\sim}{d}_C(t) \\ \hline \underset{\sim}{d}_E^*(t) \\ \hline \underset{\sim}{d}_U(t) \end{bmatrix} \tag{6.14}$$

where

$$\tilde{A}_{CC} = A_{CC} + B_C K_C \ , \ \tilde{A}_{CE} = - \left[B_C K_C \ \vdots \ 0 \ \vdots \ B_C K_D \right] \ , \ \tilde{A}_{CU} = \left[A_{CO} \ \vdots \ A_{CR} \right]$$

$$\tilde{A}_{EE} = A_{EE} + K_E C_E, \ \tilde{A}_{EU} = \left[\begin{array}{c:c} 0 & A_{CR} + K_{EC} C_R \\ \hline 0 & A_{OR} + K_{EO} C_R \\ \hline 0 & K_{Ef} C_R \end{array}\right], \ \tilde{A}_{UC} = \left[\begin{array}{c} A_{OC} + B_O K_C \\ \hline A_{RC} + B_R K_C \end{array}\right] \tag{6.15}$$

$$\tilde{A}_{UE} = - \left[\begin{array}{c:c:c} B_O K_C & 0 & B_O K_D \\ \hline B_R K_C & 0 & B_R K_D \end{array}\right] \ , \ \tilde{A}_{UU} = \left[\begin{array}{c:c} A_{OO} & A_{OR} \\ \hline A_{RO} & A_{RR} \end{array}\right]$$

in which we introduced the notation

$$K_E = \left[K_{EC}^T \,\vdots\, K_{EO}^T \,\vdots\, K_{Ef}^T \right]^T , \quad B_U = \begin{bmatrix} B_O \\ \hdashline B_R \end{bmatrix} , \quad \underset{\sim}{W}_U = \begin{bmatrix} \underset{\sim}{W}_O \\ \hdashline \underset{\sim}{W}_R \end{bmatrix} , \quad \underset{\sim}{d}_U = \begin{bmatrix} \underset{\sim}{d}_O \\ \hdashline \underset{\sim}{d}_R \end{bmatrix} \tag{6.16}$$

Concentrating on the coefficient matrix in Eq. (6.14), we note that the diagonal submatrices $\tilde{A}_{CC}(t)$ and $\tilde{A}_{EE}(t)$ are SFTS matrices by design, whereas $\tilde{A}_{UU}(t)$ is a QCS matrix by assumption (see Sec. 4.3). From the structure of the matrix $\tilde{A}_{EU}(t)$, we conclude that the observation spillover is eliminated not only from the controlled model, but from the observed model as well. Also from Eq. (6.14), we see that $\underset{\sim}{W}_E$ should contain the bulk of the disturbance, so that the residual disturbance vector has only a minor effect on the system. This implies that $\underset{\sim}{f}_E$ resembles a vector containing the first $n_W/2$ components of $-\dot{\Omega}_O \underset{\sim}{\bar{\psi}}$.

6.4 A REDUCED-ORDER OBSERVER

The idea behind a reduced-order observer is to combine an n_rth-order observer with an m-dimensional sensor output vector to produce an n_E-dimensional observed vector, where

$$n_E = n_r + m \tag{6.17}$$

We define the part of the output vector corresponding to the observed output vector as

$$\underset{\sim}{y}_E = C_E \underset{\sim}{\eta}_E \tag{6.18}$$

and represent the estimated vector $\underset{\sim}{\eta}_E$ by

$$\underset{\sim}{\eta}_E = \left[L_1 \vdots L_2 \right] \begin{bmatrix} \underset{\sim}{y}_E \\ \hdashline \underset{\sim}{y}_1 \end{bmatrix} \tag{6.19}$$

where $\underset{\sim}{y}_1$ is an n_r- dimensional vector to be determined and L_1 and L_2 are $n_E \times m$ and $n_E \times n_r$ matrices, respectively, such that

$$\text{rank } [L_1] = m \quad , \quad \text{rank } [L_2] = n_r \tag{6.20a,b}$$

Partitioning $[L_1 \vdots L_2]$ into

$$\left[L_1 \vdots L_2 \right] = \left[\begin{array}{c:c} L_{11} & L_{12} \\ \hdashline L_{21} & L_{22} \end{array} \right] \tag{6.21}$$

where L_{11} and L_{21} are $n_E \times m$ matrices and L_{12} and L_{22} are $n_E \times n_r$ matrices, and introducing Eq. (6.19) into Eq. (6.18), we obtain

$$\underset{\sim}{y}_E = C_E L_{11} \underset{\sim}{y}_E + C_E L_{12} \underset{\sim}{y}_1 \tag{6.22}$$

To satisfy condition (6.20) and Eq. (6.22), the following relations must hold

$$C_E L_{11} = I_{m \times m} \ , \quad C_E L_{12} = 0_{m \times n_r} \ , \quad L_{21} = 0_{n_E \times m} \tag{6.23a,b,c}$$

while L_{22} is such that condition (6.20b) holds.

The matrix L_{11} is not unique and can be determined according to a desired optimization criterion using Eq. (6.23a) as a constraint, or simply as the pseudo-inverse

$$L_{11} = C_E^T(C_E C_E^T)^{-1} \tag{6.24}$$

On the other hand, L_{12} and L_{22} can be determined by singular value decomposition [39] of C_E, which yields

$$V^T C_E W = \left[\begin{array}{c|c} \Sigma & 0 \\ \hline 0 & 0 \end{array}\right] \tag{6.25}$$

where V and W are unitary matrices and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$, in which σ_i (i = 1, , m) are the nonzero, square-root of the eigenvalues of $C_E^T C_E$. The singular values $\sigma_1, \sigma_2, \ldots, \sigma_{n_q}$ in Σ are ordered in descending order. Considering that C_E is a full rank matrix, we have $\sigma_{m+1} = \sigma_{m+2} = \ldots = \sigma_{n_q} = 0$. Next, let $\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_{n_q}$ be a set of orthonormal eigenvectors belonging to $\sigma_1^2, \sigma_2^2, \ldots, \sigma_{n_q}^2$ and $W_1 = [\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_m]$ and $W_2 = [\underset{\sim}{w}_{m+1}, \ldots, w_{n_q}]$, where $n_q = n_c + n_o$. From the above definitions we have

$$W_1^T C_E^T C_E W_1 = \Sigma^2 \tag{6.26}$$

$$\Sigma^{-1}W_1^T C_E^T C_E W_1 \Sigma^{-1} = I \tag{6.27}$$

$$W_2^T C_E^T C_E W_2 = 0 \tag{6.28}$$

so that

$$C_E W_2 = 0 \tag{6.29}$$

it follows that

$$L_{12} = [\ W_2 \ \vdots\ 0\ ]\ , \quad L_{22} = [\ 0 \ \vdots\ I_{n_w/2 \times n_w/2}\ ] \tag{6.30a,b}$$

Further computations involving the previous definitions of $\underset{\sim}{y}_1$ and $\underset{\sim}{y}_E$ yield an input to the observer that includes the term $\dot{\underset{\sim}{y}}(t)$. To avoid using a derivative of a noisy output as an input to the observer, we introduce the change of variables [40]

$$\underset{\sim}{y}_r = \underset{\sim}{y}_1 + K_E(t)\underset{\sim}{y}_E \tag{6.31}$$

where $K_E(t)$ is the observer gain matrix. Inserting Eq. (6.31) into Eq. (6.19) and using Eqs. (6.23c) and (6.30), we obtain

$$\underset{\sim}{\eta}_E = \begin{bmatrix} \underset{\sim}{z}_c \\ \underset{\sim}{z}_o \\ \hline \underset{\sim}{f}_E \end{bmatrix} = \left[\begin{array}{cc:c} W_2 & 0 & L_{11} - W_2 K_{EZ} \\ \hline 0 & I & -K_{Ef} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{y}_r \\ \hline \underset{\sim}{y}_E \end{array}\right] \tag{6.32}$$

where, in view of the first of Eqs. (6.16), we denoted $K_{EZ}=[K_{EC}^T \vdots K_{EO}^T]^T$. Using the inverse to the transformation defined by Eq. (6.32), we obtain

$$\begin{bmatrix} \underset{\sim}{y}_r \\ \hdashline \underset{\sim}{y}_E \end{bmatrix} = \left[\begin{array}{c:c} K_E C_E + Q_1 & Q_2 \\ \hdashline C_E & 0 \end{array}\right] \begin{bmatrix} \underset{\sim}{z}_C \\ \underset{\sim}{z}_O \\ \hdashline \underset{\sim}{f}_E \end{bmatrix} \tag{6.33a}$$

where

$$Q_1 = \begin{bmatrix} W_2^T \\ \hdashline 0 \end{bmatrix} \quad , \quad Q_2 = \begin{bmatrix} 0 \\ \hdashline I_{n_W/2 \times n_W/2} \end{bmatrix} \tag{6.33b}$$

Differentiating $\underset{\sim}{y}_r$ in Eq. (6.33a) and introducing Eqs. (6.32) and (6.5), we have

$$\dot{\underset{\sim}{y}}_r = [\, F_q(t) + K_E(t)G_q(t)]\underset{\sim}{y}_r + E_y(t)\underset{\sim}{y}_E + L_R B_q \underset{\sim}{T}_E + L_R A_{CR}\underset{\sim}{z}_R \tag{6.34a}$$

where

$$F_q(t) = \left[\begin{array}{c:c} W_2^T A_q W_2 & W_2^T H_E \\ \hdashline 0 & 0 \end{array}\right] , \quad G_q(t) = [\, C_E A_q W_2 \vdots C_E H_E \,]$$

$$A_q = \left[\begin{array}{c:c} A_{CC} & A_{CO} \\ \hdashline A_{OC} & A_{OO} \end{array}\right] , \quad B_q = \begin{bmatrix} B_C \\ \hdashline B_O \end{bmatrix} \tag{6.34b}$$

$$E_y(t) = \dot{K}_E(t) - [\, F_q(t) + K_E(t)G_q(t)]K_E(t) + [\, Q_1 + K_E(t)C_E]A_q L_{11}$$

$$L_R = [\, Q_1 + K_E(t)C_E \,]$$

Hence, the reduced-order observer equation is

$$\dot{\hat{\underset{\sim}{y}}}_r = [\, F_q(t) + K_E(t)G_q(t)]\hat{\underset{\sim}{y}}_r + E_y\underset{\sim}{y}_m + L_R B_q \underset{\sim}{T}_E \tag{6.35}$$

Defining the observer error as

$$\underset{\sim}{e}_r(t) = \underset{\sim}{y}_r(t) - \hat{\underset{\sim}{y}}_r(t) \tag{6.36}$$

and introducing Eqs. (6.34) and (6.35) into Eq. (6.36), we obtain

$$\dot{\underset{\sim}{e}}_r = [\, F_q(t) + K_E(t)G_q(t)\,]\underset{\sim}{e}_r(t) + L_{ER}(t)\underset{\sim}{z}_R(t) \tag{6.37a}$$

where

$$L_{ER} = L_R A_{CR} - E_y C_R \tag{6.37b}$$

Next, we wish to derive the closed-loop equations for the modeled system incorporating the observer described above. To this end, we use Eq. (6.32) and write the observed state as

$$\begin{bmatrix} \hat{\underset{\sim}{z}}_C \\ \hat{\underset{\sim}{z}}_O \\ \hline \hat{\underset{\sim}{f}}_E \end{bmatrix} = \left[\begin{array}{cc|c} W_2 & O & L_{11} - W_2 K_{EZ} \\ \hline O & I & -K_{Ef} \end{array}\right] \begin{bmatrix} \hat{\underset{\sim}{y}}_r \\ \hline \underset{\sim}{y}_E \end{bmatrix} \tag{6.38}$$

Introducing Eqs. (6.14), (6.37) and (6.38) into Eq. (6.5), denoting the uncontrolled state vector by $\underset{\sim}{z}_U = [\, \underset{\sim}{z}_O^T \,\vdots\, \underset{\sim}{z}_R^T \,]^T$ and partitioning W_2 into $W_2^T = [\, W_2^{*T} \,\vdots\, W_2^{**T} \,]^T$, where W_2^* and W_2^{**} consists of the first n_C rows and the n_R rows of W_2, respectively, we finally obtain

$$\begin{bmatrix} \dot{\underset{\sim}{z}}_C \\ \hline \dot{\underset{\sim}{e}}_\Gamma \\ \hline \dot{\underset{\sim}{z}}_U \end{bmatrix} = \left[\begin{array}{c|c|c} \tilde{R}_{CC} & \tilde{R}_{CE} & \tilde{R}_{CU} \\ \hline 0 & \tilde{R}_{EE} & \tilde{R}_{EU} \\ \hline \tilde{R}_{UC} & \tilde{R}_{UE} & \tilde{R}_{UU} \end{array}\right] \begin{bmatrix} \underset{\sim}{z}_C \\ \hline \underset{\sim}{e}_\Gamma \\ \hline \underset{\sim}{z}_U \end{bmatrix} + \begin{bmatrix} \underset{\sim}{W}_C + B_C K_D \underset{\sim}{f}_E \\ \hline \underset{\sim}{W}^*_E \\ \hline \underset{\sim}{W}_U + B_U K_D \underset{\sim}{f}_E \end{bmatrix} + \begin{bmatrix} 0 \\ \hline I_E \\ \hline 0 \end{bmatrix} \underset{\sim}{\delta}_E(t)$$

$$+ \begin{bmatrix} \underset{\sim}{d}_C(t) \\ \hline \underset{\sim}{d}^*_E(t) \\ \hline \underset{\sim}{d}_U(t) \end{bmatrix} \tag{6.39}$$

where

$$\tilde{R}_{CC} = A_{CC} + B_C K_C \ , \ \tilde{R}_{CE} = - \left[B_C K_C W_2^* \,\vdots\, 0 \,\vdots\, B_C K_D \right] \ , \ \tilde{R}_{CU} = \left[A_{CO} \,\vdots\, L_{CR} \right]$$

$$\tilde{R}_{EE} = F_q + K_E G_q, \ \tilde{R}_{EU} = \left[0 \,\vdots\, L_{ER} \right], \ \tilde{R}_{UC} = \begin{bmatrix} A_{OC} + B_O K_C \\ \hline A_{RC} + B_R K_C \end{bmatrix} \tag{6.40}$$

$$\tilde{R}_{UE} = - \left[\begin{array}{c|c|c} B_O K_C W_2^* & 0 & B_O K_D \\ \hline B_R K_C W_2^* & 0 & B_R K_D \end{array}\right] , \qquad \tilde{R}_{UU} = \left[\begin{array}{c|c} A_{OO} & L_{OR} \\ \hline A_{RO} & L_{RR} \end{array}\right]$$

in which

$$L_{CR} = A_{CR} + B_C S_R, \quad L_{OR} = A_{OR} + B_O S_R, \quad L_{RR} = A_{RR} + B_R S_R \tag{6.41}$$

$$S_R = \left[K_C L_{11} - K_C W_2 K_{EZ} - K_D K_{Ef} \right] C_R \tag{6.42}$$

Concentrating on the closed-loop coefficient matrix $\tilde{R}(t)$ in Eq. (6.39), we note that the diagonal submatrices $\tilde{R}_{CC}(t)$ and $\tilde{R}_{EE}(t)$ are SFTS matrices by design. But, $\tilde{R}_{UU}$ is a matrix that can be not QCS, because the term L_{RR} can jeopardize the natural QCS character of the matrix A_{RR}.

6.5 DESIGN OF A GAIN MATRIX

The controlled part of the estimation error, which is represented by Eq. (6.12) in case of a full-order observer and by Eq. (6.35) in case of a reduced-order observer must converge to zero faster then $\underset{\sim}{z}_C(t)$. Hence, the matrices $(A_{EE} + K_E C_E)$, or $(F_q + K_E G_q)$, should be SFTS with faster dynamics then the dynamics of $(A_{CC} + B_C K_C)$. Considering the drawback of the reduced-order observer mentioned at the end of the previous section, we concentrate on the full-order observer. The basic equation is

$$\dot{\underset{\sim}{e}}(t) = [A_{EE}(t) + K_E(t)C_E]\underset{\sim}{e}(t) \tag{6.43}$$

It is shown in Chapter 7 that the pair $[A_{EE}(t),\ C_E]$ is totally observable for all $t \in \tau$ provided that several conditions are met. Introducing the transformation

$$\underset{\sim}{e}^*(t) = e^{\beta t}\underset{\sim}{e}(t), \qquad \beta \geq 0 \tag{6.44}$$

and inserting Eq. (6.44) into Eq. (6.43), we obtain

$$\underset{\sim}{e}^*(t) = [A_{EE}(t) + \beta I + K_E(t)C_E]\underset{\sim}{e}^*(t) \tag{6.45}$$

Following [41], we propose the following gain matrix

$$K_E(t) = -\ 1/2\ P(t)C_E^T \tag{6.46}$$

where P(t) is the solution of the Riccati equation

$$\dot{P} = (A_{EE} + \beta I)P + P(A_{EE}^T + \beta I) - PC_E^T C_E P + Q_E \ , \quad P(t_i) = P_0 \qquad (6.47a)$$

where

$$Q_E = Q_E^T > 0, \quad P_0 = P_0^T > 0 \qquad (6.47b)$$

Furthermore, Q_E can be decomposed according to $Q_E = L_E L_E^T$ where L_E is such that the pair $[A_{EE}(t), L_E]$ is totally controllable for all $t \epsilon \tau$.

Next, we wish to develop a sufficiency condition guaranteeing the SFTS of the system defined by Eqs. (6.45), (6.46) and (6.47). Following the developments of Sec. 4.2, we define the Lyapunov function

$$V(\underset{\sim}{e}^*, t) = \underset{\sim}{e}^{*T} P^{-1}(t) \underset{\sim}{e}^* \qquad (6.48)$$

Differentiating Eq. (6.48) with respect to time along solutions of Eq. (6.45) and considering Eq. (6.47) we obtain

$$\dot{V}(\underset{\sim}{e}^*, t) = - \underset{\sim}{e}^{*T} P^{-1}(t) Q_E P^{-1}(t) \underset{\sim}{e}^* \qquad (6.49)$$

Thus, considering Eq. (6.47b), it is obvious that [41]

$$P^{-1}(t) > 0 \quad \text{for all } t \epsilon \tau. \qquad (6.50)$$

This completes the first part of the sufficiency condition. The second part is completely dual to the corresponding part in Chapter 5, where P, β and $r_P(t^*, \beta)$ replace S, α and $r_S(t^*, \alpha)$, respectively, in which

$$r_p(t,\beta) = \frac{\lambda_M[P(t)]}{\lambda_m[P(t_i)]} \tag{6.51}$$

The sufficiency condition thus obtained is

$$\frac{\lambda_M[P(t,\beta+\Delta\beta)]}{\lambda_M[P(t,\beta)]} < \exp\,[2\,\Delta\beta\,(t-t_i)] \text{ for all } t\epsilon\tau_1 \text{ and } \beta \geq \bar{\beta}(\varepsilon) > 0 \tag{6.52}$$

where $\tau_1 = (T_1, t_h)$, $t_i < T_1 < t_h$. Then, by increasing β, the second part of the condition for a SFTS system is satisfied. It is worth mentioning at this point that the above analysis is also applicable to the reduced-order observer, where F_q and G_q replace A_{EE} and C_E.

CHAPTER 7

CONTROLLABILITY AND OBSERVABILITY DURING MANEUVER

7.1 CONTROLLABILITY DURING MANEUVER

We wish to investigate the controllability of the controlled model during both halves of the maneuver time period. The translational and the rotational models, represented by Eqs. (3.34a,b) are uniformly completely controllable for all $t \in \tau$. This is guaranteed by the assumption of an actuator for each axis, as described in Secs. 2.1 and 2.2. Hence, we focus our attention on the elastic model as described by Eq. (2.34c). For control purposes, we have to investigate the controllability of the pair $[A^*_{cc}(t), B_c]$ defined by Eqs. (5.4), (5.1) and (3.5a,b). For convenience, we omit the damping matrix $\bar{D}_e$ from Eq. (3.5a). Because of its extremely low value, the inclusion of damping is not expected to affect the controllability of the system, which is preserved under small perturbations, [42] and [43].

Lemma 7.1 : The pair $[A^*_{cc}(t), B_c]$ is totally controllable (Def. A.1), for all $t \in \tau$, if :

(1) $p \geq 1$, and the actuators are placed such that

$$\| \bar{E}^*_{cj} \| > 0 \quad \text{for all } j=1,2,\ldots,n_c/2 \tag{7.1}$$

where p is the dimension of the control vector and $\bar{E}^*_c$ is derived from $\bar{E}^*_e$, Eq. (3.4), by retaining the entries corresponding to the controlled

model. Moreover, j denotes the jth row of the matrix.

(2) $c \neq c_j \quad j = 1,2,\ldots,j_T; \quad j_T \leq n_c$

where c is the commanded acceleration and c_j are the solutions of

$$\det\,[Q_C^*(t_B)Q_C^{*T}(t_B)] = 0 \tag{7.2}$$

in which Q_C^* is the controllability matrix of $[A_{CC}^*(t),\ B_C]$ and $t_B = t_i$ in the first half of the maneuver and $t_B = t_f$ in the second half. Moreover, for p = 1, the total controllability holds if and only if:

(3) condition (1) is satisfied and

(4) $c \neq c_j\ , \quad j = 1,2,\ldots,j_T\ ;\ j_T \leq n_c - 4$

where c_j are the solutions of

$$\det\,[\ Q_C^*(t_B)] = 0 \tag{7.3}$$

Proof :

Considering Eq. (3.1b), it is obvious that $\Omega_0(t)$ is an analytic function and therefore $A_{CC}^*(t)$ is an analytic matrix for all $t\epsilon\tau$. Hence, according to the conclusion following Theorem A.3 (Appendix), if $[A_{CC}^*(t),\ B_C]$ are controllable for any $t\epsilon\tau$, then they are controllable for every $t\epsilon\tau$. We can, therefore, examine the controllability at $t = t_B$, where $t_B = t_0$ in the first half of the maneuver and $t_B = t_f$ in the second half. Considering Eq. (3.1), we obtain

$$\Omega_0(t_B) = 0, \quad \dot{\Omega}_0(t_B) = c, \quad \Omega_0^{(i)}(t_B) = 0\ ; \quad i \geq 2 \tag{7.4a,b c}$$

where i denotes the order of the derivative.

The controllability matrix is defined by Eqs. (A.7a,b). Rewriting $A^*_{CC}(t)$ as

$$A^*_{CC}(t) = \bar{A}^*_{CC} + \Omega^2_o(t)\tilde{A}_{CC} \tag{7.5}$$

where $\bar{A}^*_{CC}$ denotes the time-invariant part of the matrix, and inserting Eqs. (7.4) and (7.5) into Eqs. (A.7), we obtain the $n_c \times pn_c$ controllability matrix at $t = t_B$

$$Q^*_c(t_B) = [\, B_c \,\vdots\, -\bar{A}^*_{CC}B_c \,\vdots\, \bar{A}^{*2}_{CC}B_c \,\vdots\, -\bar{A}^{*3}_{CC}B_c \,\vdots\, (\bar{A}^{*4}_{CC} + 2c^2\tilde{A}_{CC}\bar{A}^*_{CC})B_c \,\vdots$$

$$- [\bar{A}^{*5}_{CC} + 2c^2(3\bar{A}^*_{CC}\tilde{A}_{CC}\bar{A}^*_{CC} + \tilde{A}_{CC}\bar{A}^{*2}_{CC})]B_c \,\vdots\, \ldots\ldots\ldots \,\vdots\, \ldots] \tag{7.6}$$

We conclude from the structure of the above controllability matrix that, when $(n_c - 4)p < n_c$, a necessary condition for full rank matrix, is

$$\text{rank } [Q^*_c(t_B, c = 0)] = n_c \tag{7.7}$$

i.e. the pair $[\bar{A}^*_{CC}, B_c]$ must be completely controllable. To this end, we consider Theorem A.5 and conclude that the complete controllability of $[\bar{A}^*_{CC}, B_c]$ is equivalent to the complete controllability of $[\bar{A}_{CC}, B_c]$, where $\bar{A}_{CC} = \bar{A}^*_{CC}(\alpha=0)$. But the latter pair is simple the pair corresponding to the nonmaneuvering structure. Hence, recalling our basic assumption that all the eigenvalues are distinct, we conclude [42] that the pair $[\bar{A}_{CC}, B_c]$ is completely controllable if and only if

inequality (7.1) is satisfied. Then, $Q_C^*(t_B)$ is a full rank matrix if condition (2) is satisfied.

In the case in which p = 1, $Q_C^*(t_B)$ becomes a square matrix. As a result, rank $[Q_C^*(t_B)] = n_C$ if and only if inequalities (7.8) and (7.9) are satisfied and Eq. (7.10) is replaced by

$$\det [Q_C^*(t_B)] = 0 \tag{7.8}$$

Moreover, according to Theorem A.3, the condition for $Q_C^*(t_B)$ to have full rank is also a necessary condition for total controllability of $[A_{CC}^*(t), B_C]$ for all $t \in \tau$.

7.2 OBSERVABILITY DURING MANEUVER

We have to analyze the observability of the pair $[A_{EE}^*(t), C_E]$, Eq. (6.19), where

$$A_{EE}^* = A_{EE} + \beta I \tag{7.9}$$

Lemma 7.2 : The pair $[A_{EE}^*(t), C_E]$ is totally observable for all $t \in \tau$, if :

(1) $m \geq \text{maximum } [\ 1,\ n_w/2\ ]$ (7.10)

and the sensors are placed such that

$$\| \bar{C}_{Pi}^* \| > 0 \ , \quad i = 1,2,\ldots,(n_C + n_O)/2 \tag{7.11}$$

where i denotes a column number of the matrix.

(2) rank $[C_P(\bar{A}_E + K_1C_P)^{-1}H_E] = n_W/2$ (7.12)

where $\bar{A}_E = A_{EE}(\Omega_0 = 0,\ H_E = 0)$ and K_1 is any matrix for which $(\bar{A}_E + K_1C_P)$ is nonsingular.

(3) $c \neq c_k,\ k = 1,2,\ldots,k_T;\ k_T \leq n_M$ (7.13)

where c_k are the solutions of

$$\det\ [Q_O^{*T}(t_B)\ Q_O^{*}(t_B)] = 0 \tag{7.14}$$

where $Q_O^{*}(t_B)$ is the observability matrix of the pair $[A_{EE}^{*}(t), C_E]$.

Proof :

The proof is completely dual to the proof of Lemma 7.1 where $Q_O^{*}(t)$, defined by Eqs. (A.8), replaces $Q_C^{*}(t)$. By analogy to [44], it can be concluded that the pair $[\bar{A}_{EE}, C_E]$ is completely observable if and only if condition (7.12) is satisfied and the pair $[\bar{A}_E, C_P]$ is completely observable, where $\bar{A}_{EE} = A_{EE}^{*}(\Omega_0 = 0,\ \beta = 0)$. A necessary condition to satisfy Eq. (7.12) is that the number of measurements (displacement or velocity) will be at least equal to $n_W/2$. The pair $[\bar{A}_E, C_P]$ is just the observability pair of a nonmaneuvering structure. Hence, according to [42], $[\bar{A}_E, C_P]$ is completely observable if and only if inequality (7.11) is satisfied. Finally, condition (7.13) is necessary to prevent the rank

of the observability matrix from being less then n_M, due to $\dot{\Omega}_0$.

CHAPTER 8

DISTURBANCE ACCOMMODATION

8.1 SYSTEM EQUATIONS RELATED TO DISTURBANCE ACCOMMODATION

The goal of disturbance accommodation is to minimize the response of an output vector associated with the disturbance. To this end, we define an l-dimensional output vector $\underset{\sim}{y}_l(t)$ and propose to design a disturbance accommodation control so as to minimize $\|\underset{\sim}{y}_l(t)\|$. The output vector $\underset{\sim}{y}_l(t)$ is in general different from any output vector considered earlier, and its purpose is to provide a design criterion. We assume that the output equation has the form

$$\underset{\sim}{y}_l(t) = L_y \underset{\sim}{z}_e(t) \tag{8.1}$$

where L_y is an $l \times n_M$ matrix of rank l, $l \le n_M$, and $\underset{\sim}{z}_e = [\, \underset{\sim}{z}_C^T \,\vdots\, \underset{\sim}{z}_O^T \,\vdots\, \underset{\sim}{z}_R^T \,]^T$.

For the minimization process contemplated, it will prove convenient to rearrange the state vector in Eq. (6.14) and to retain only the terms related to the observed disturbance vector, $\hat{\underset{\sim}{f}}_E$. Hence, we rewrite Eq. (6.14) in the form

$$\begin{bmatrix} \dot{\underset{\sim}{z}}_e \\ \hline \dot{\underset{\sim}{e}} \end{bmatrix} = \left[\begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array}\right] \begin{bmatrix} \underset{\sim}{z}_e \\ \hline \underset{\sim}{e} \end{bmatrix} + \begin{bmatrix} H_F + BK_D \\ \hline 0 \end{bmatrix} \hat{\underset{\sim}{f}}_E \tag{8.2}$$

where

$$F_{11} = \left[\begin{array}{c:c} \tilde{A}_{CC} & \tilde{A}_{CU} \\ \hdashline \tilde{A}_{UC} & \tilde{A}_{UU} \end{array}\right], \quad F_{12} = [-BK_C \;\vdots\; 0 \;\vdots\; H_F\,], \quad F_{21} = [\,0 \;\vdots\; \tilde{A}_{EU}] \tag{8.3}$$

$$F_{22} = \tilde{A}_{EE}, \quad H_F = \left[\begin{array}{c} H_E \\ \hdashline 0 \end{array}\right], \quad B = \left[\begin{array}{c} B_C \\ \hdashline B_U \end{array}\right]$$

Moreover, we extend the output vector so that the output equation reads

$$\underset{\sim}{y}_o(t) = [\,L_y \;\vdots\; 0\,] \left[\begin{array}{c} \underset{\sim}{z}_e \\ \hdashline \underset{\sim}{e} \end{array}\right] \tag{8.4}$$

Next, let us write the output vector in the form

$$\underset{\sim}{y}_o(t) = \underset{\sim}{y}_{tr}(t) + \underset{\sim}{y}_f(t) \tag{8.5}$$

where $\underset{\sim}{y}_{tr}(t)$ represents the transient part of the response and $\underset{\sim}{y}_f(t)$ is the response to $\hat{\underset{\sim}{f}}_E$. The latter can be expressed as

$$\underset{\sim}{y}_f(t) = \int_{t_i}^{t} [L_y \vdots 0]\; \Phi_F(t,\sigma) \left[\begin{array}{c} H_F + BK_D \\ \hdashline 0 \end{array}\right] \hat{\underset{\sim}{f}}_E(\sigma)\, d\sigma\,, \quad t\in\tau. \tag{8.6}$$

where $\Phi_F(t,\sigma)$ is the transition matrix corresponding to the coefficient matrix $F(t)$ in Eq. (8.2).

We can distinguish between three approaches to the minimization of $\|\underset{\sim}{y}_f(t)\|$ by a constant gain matrix. In the following, we describe all

three approaches and focus on the third approach because it has most of the advantages.

8.2 PARTIAL ANNIHILATION OF THE DISTURBANCE VECTOR

According to this method a p-dimensional observed disturbance vector is being annihilated where p is the number of actuators. To this end, we consider Eqs. (3.5) and (8.6) and recognize that the input vector to be annihilated is $[I_p + \bar{E}_p^* K_D]\ \hat{\underset{\sim}{f}}_p$, where I_p is a p×p identity matrix, $\bar{E}_p^*$ is a p×p matrix derived from $\bar{E}_e^*$ and $\hat{\underset{\sim}{f}}_p$ is the p-dimensional observed disturbance vector. The control vector annihilating the input vector mentioned above is

$$\underset{\sim}{u}_D = - \bar{E}_p^{*-1} \hat{\underset{\sim}{f}}_p \tag{8.7}$$

where the corresponding gain matrix is

$$K_D = - \bar{E}_p^{*-1} \tag{8.8}$$

where we assumed that the actuators are placed such that the above inverse exists. We recognize that this method does not minimize the entire output vector $\|\underset{\sim}{y}_f\|$, nor the entire input vector.

8.3 LEAST-SQUARES MINIMIZATION OF THE INPUT

Here, the entire observed disturbance vector is minimized according

to the least squares minimization approach. The control vector $\underset{\sim}{u}_D$ is computed by minimizing the performance measure

$$J_D = \|(H_F + BK_D)\hat{\underset{\sim}{f}}_E\|^2 \tag{8.9}$$

From the above, we obtain

$$u_D = -(B^TB)^{-1}B^TH_F\hat{\underset{\sim}{f}}_E \tag{8.10}$$

and the corresponding gain matrix is

$$K_D = -(B^TB)^{-1}B^TH_F \tag{8.11}$$

This solution does not minimize $\|y_f\|$, and actually can even increase $\|y_f\|$ as compared to a noncontrolled system.

8.4 LEAST-SQUARES MINIMIZATION OF A WEIGHTED NORM SPANNING THE MODELED STATE

The objective here is to minimize $\|\underset{\sim}{y}_f(t)\|$, $t\in\tau$, especially toward the end of the time interval. Hence, in accordance with objective (ii) in Sec. 4.3, we seek a constant gain matrix K_D minimizing the performance measure

$$J_D = \lim_{t\to t_h} \|\underset{\sim}{y}_f(t)\|^2 \tag{8.12}$$

The solution of the above problem has meaning only if quasi-time invariant conditions exist for $t \in \tau$. Hence, we assume now, and justify later, that the coefficient matrix is such that

$$F(t) = F_0 + \varepsilon_1 F_1(t), \quad \varepsilon_1 << 1, \quad t \in \tau \tag{8.13}$$

where F_0 is a constant matrix. Then, the transition matrix can be shown [45] to have the form

$$\Phi_F(t,t_i) = e^{F_0(t-t_i)} + \varepsilon_1 e^{F_0 t} \int_{t_i}^{t} e^{-F_0\tau} F_1(\tau)\, e^{F_0\tau} d\tau + O(\varepsilon_1^2) \tag{8.14}$$

At this point, we assume that F_0 is a stable matrix, i.e.,

$$\mathrm{Re}\ \lambda_i(F_0) < 0, \ i = 1,2,\ldots,n_A; \ n_A = n_M + n_C + n_O + n_W/2 \tag{8.15}$$

Moreover, it is not difficult to verify that

$$\int_{t_i}^{t} e^{F_0(t-\tau)} d\tau = - F_0^{-1} \left[I - e^{F_0(t-t_i)} \right] \tag{8.16}$$

where the existence of F_0^{-1} is guaranteed by inequalities (8.15). Then, using Eqs. (8.6), (8.14) and (8.16) and retaining the zero-order term only, we can write

$$\underset{\sim}{y}_{fo}(t) = - [L_y \vdots 0]\, F_0^{-1} \left[I - e^{F_0(t-t_i)} \right] \begin{bmatrix} H_F + BK_D \\ \hline 0 \end{bmatrix} \hat{\underset{\sim}{f}}_E , \quad t \in \tau \tag{8.17}$$

Next, let us separate $\underset{\sim}{y}_{fo}(t)$ into a constant part and a time-varying part, or

$$\underset{\sim}{y}_{fo}(t) = \bar{\underset{\sim}{y}}_{fo} + \overset{*}{\underset{\sim}{y}}_{fo}(t) \tag{8.18}$$

and determine the constant gain matrix K_D by minimizing $\|\bar{\underset{\sim}{y}}_{fo}\|^2$. The error in the minimization caused by $\overset{*}{\underset{\sim}{y}}_{fo}(t)$ will remain small as long as

$$\lim_{t \to t_h} \| e^{F_0 t} \| < \rho << 1 \tag{8.19}$$

To produce the gain matrix K_D, we partition F_0 in the same way as in Eq. (8.2) and introduce the notation

$$F_0 = \left[\begin{array}{c|c} \bar{F}_C & \bar{F}_{12} \\ \hline \bar{F}_{21} & \bar{F}_E \end{array}\right] \tag{8.20}$$

Inserting Eq. (8.20) into Eq. (8.17), letting $t_i = 0$ and retaining the constant part only, we obtain

$$\bar{\underset{\sim}{y}}_{fo} = - L_y \Gamma^{-1} (H_F + BK_D) \hat{\underset{\sim}{f}}_E = - L_y \Gamma^{-1} (H_F \hat{\underset{\sim}{f}}_E + B\underset{\sim}{u}_D) \tag{8.21}$$

where we recalled Eq. (6.13b). Moreover, we used the notation

$$\Gamma = \bar{F}_C - \bar{F}_{12} \bar{F}_E^{-1} \bar{F}_{21} \tag{8.22}$$

The above requires that $\bar{F}_E$ be nonsingular, which implies the assumption

$$\mathrm{Re}\ \lambda_i(\bar{F}_E) < 0, \quad i = 1,2,\ldots,n_A \tag{8.23}$$

In addition, Γ itself must be nonsingular. Using Eqs. (8.20), (8.22) and (8.23), it can be shown easily that $\Gamma^{-1} = (F_0^{-1})_{11}$, where F_0^{-1} is partitioned in the same way as F_0 in Eq. (8.20). Hence, the existence of Γ^{-1} is guaranteed by inequalities (8.15) and (8.23).

At this point, we have all the prerequisites for the minimization problem. To this end, we introduce the notation

$$\bar{J}_D = \| \bar{\underset{\sim}{y}}_{f0} \|^2 \tag{8.24}$$

where $\bar{\underset{\sim}{y}}_{f0}$ is given by Eq. (8.21), and seek the value of $\underset{\sim}{u}_D$ that minimizes $\bar{J}_D$. This value must satisfy

$$\frac{\partial \bar{J}_D}{\partial \underset{\sim}{u}_D} = B^T\Gamma^{-T}W\Gamma^{-1}(H_F\hat{\underset{\sim}{f}}_E + B\underset{\sim}{u}_D) = \underset{\sim}{0} \tag{8.25}$$

where $\Gamma^{-T} = (\Gamma^{-1})^T$ and

$$W = L_y^T L_y \tag{8.26}$$

Equation (8.25) has the solution

$$\underset{\sim}{u}_D = -(B^T\Gamma^{-T}W\Gamma^{-1}B)^{-1}B^T\Gamma^{-T}W\Gamma^{-1}H_F\hat{\underset{\sim}{f}}_E \tag{8.27}$$

so that, recalling Eq. (6.13b) once again, we conclude that the desired gain matrix has the expression

$$K_D = -(B^T\Gamma^{-T}W\Gamma^{-1}B)^{-1}B^T\Gamma^{-T}W\Gamma^{-1}H_F \tag{8.28}$$

A sufficient condition for the existence of the above minimum is

$$B^T\Gamma^{-T}W\Gamma^{-1}B > 0 \tag{8.29}$$

Considering the nonsingularity of Γ^{-1}, this condition is guaranteed by choosing $W > 0$, i.e., l should be equal to n_M in Eq. (8.1), and by placing the actuators so as to ensure a full-rank matrix B.

In the following we propose to validate two of the necessary conditions implied by Eq. (8.28), namely assumptions (8.13) and (8.23). The other necessary condition is assumption (8.15), which represents a design requirement and is analyzed in Chapter 9.

The control gain matrix $K_C(t)$ and observer gain matrix $K_E(t)$, which define the optimal solution for the controlled model, involve solutions of dual Riccati equations, Eqs. (5.8) and (6.47), respectively. As such, the characteristics of one imply the characteristics of the other. In view of this , we concentrate our attention on the characteristics of $K_C(t)$ and invoke the duality to infer the characteristics of $K_E(t)$.

It was demonstrated in Chapter 5 that a sufficiently large convergence factor is capable of causing the controlled model to become

an SFTS system within $t\epsilon\tau$. It turns out that the specific structure of the coefficient matrix $A^*_{CC}(t)$, in conjunction with the system total controllability and observability for all $t\epsilon\tau$, leads to a Riccati solution possessing certain characteristics that can be used to shed some light on the nature of the Riccati matrix. These characteristics were verified numerically.

Let us consider the system described by Eqs. (5.1), (5.8), (5.9) and (5.10). The discussion can be generalized to some extent by replacing the matrix $A^*_{CC}(t)$ by

$$A^*(t) = \left[\begin{array}{c:c} \alpha I & I \\ \hdashline \varphi(t)\bar{K}_{22} - \Lambda_e & \alpha I - \bar{D}_e \end{array}\right] \tag{8.30}$$

where $0 \le \alpha \le \alpha_M$ and $\varphi(t) = \varphi_0 t^k$, $k \ge 2$. We assume that φ_0 is such that $\varphi(t)\| \bar{K}_{22}\| >> \| \Lambda_e \|$ for $t\epsilon\tau$, where t is sufficiently large. As in the case of the original matrix $A^*_{CC}(t)$, the pair $[A^*(t),B_C]$ is totally controllable and the pair $[A^*(t),L]$ is totally observable for every $t\epsilon\tau$. Then, the Riccati matrix S(t) possesses the following characteristics:

(i) by increasing the parameter α, the steady-state solution of the Riccati equation, Eq. (5.8), can be reached within the time τ_2, where $\tau_2 = [t_i, T_2]$, $t_i < T_2 < t_h$,

(ii) the above steady-state solution can be expressed as

$$\bar{S}(t) = \bar{S}_0 + \varepsilon_1(\alpha)\, S_1(t)\ , \quad \text{for all } t\epsilon\tau_2 \tag{8.31}$$

where $\bar{S}_0$ is the solution of the algebraic Riccati equation using the matrix A_0^*, obtained from Eq. (8.30) by setting $\varphi(t) \equiv 0$, and $\varepsilon_1(\alpha)$ is a positive scalar decreasing monotonically with increasing α and

(iii) choosing the end condition $S_f = \bar{S}_0$ for Eq. (5.8), we obtain the solution

$$S(t) = \bar{S}(t) + \varepsilon_2(\alpha)S_2(t) \ , \quad \text{for all} \quad t\epsilon\tau \tag{8.32}$$

where $\varepsilon_2(\alpha)$ is a positive scalar. Then, in view of Eq. (8.31),

$$S(t) = \bar{S}_0 + \varepsilon_3(\alpha)S_3(t) \ , \quad \text{for all} \quad t\epsilon\tau. \tag{8.33}$$

in which $\varepsilon_3(\alpha) > 0$. Again, $\varepsilon_3(\alpha)$ decreases monotonically with increasing α.

The above characteristics hold for the solution of the observer Riccati equation, Eq. (6.47). Indeed, choosing the initial condition $P_i = \bar{P}_0$, we obtain

$$P(t) = \bar{P}_0 + \varepsilon_4(\beta)P_1(t) \ , \quad \text{for all} \quad t\epsilon\tau. \tag{8.34}$$

where $\varepsilon_4(\beta)$ is a positive scalar decreasing monotonically with increasing β. We refer to solutions (8.33) and (8.34) as quasi-constant.

For the structure considered, convergence factors α and β for which Eqs. (8.33) and (8.34) hold yield an SFTS controlled system also. In general, larger values of the convergence factors are needed to

achieve quasi-constant Riccati solutions for all $t \in \tau$ than to achieve a SFTS system.

Next, let us introduce Eq. (8.33) into Eq. (5.10) and obtain the quasi-constant optimal control gain matrix

$$K_C(t) = \bar{K}_C + \varepsilon K_{C1}(t), \quad \varepsilon << 1 \tag{8.35}$$

where

$$\bar{K}_C = - R^{-1}B_C^T\bar{S}_0 , \quad K_{C1}(t) = - R^{-1}B_C^T S_1(t) \tag{8.36a,b}$$

Similarly, inserting Eq. (8.34) into Eq. (6.46), we obtain the quasi-constant observer gain matrix

$$K_E(t) = \bar{K}_E + \varepsilon K_{E1}(t), \quad \varepsilon << 1 \tag{8.37}$$

where

$$\bar{K}_E = - 1/2\ \bar{P}_0 C_E^T , \quad K_{E1}(t) = - 1/2\ P_1(t) C_E^T \tag{8.38a,b}$$

Introducing Eqs. (8.35) and (8.37) into Eq. (8.2) and recalling Eqs. (6.15) and (8.3), we conclude that the modeled system closed-loop matrix has the quasi-constant form given by Eq. (8.13). The time-invariant matrix in Eq. (8.13) is defined by Eq. (8.20), where

$$\bar{F}_C = \bar{A} + B\ [\ \bar{K}_C \ \vdots\ 0 \ \vdots\ 0\ ], \quad \bar{F}_{12} = \left[\begin{array}{c:c:c} -B_C\bar{K}_C & 0 & \\ \hdashline -B_0\bar{K}_C & 0 & H_E \\ \hdashline -B_R\bar{K}_C & 0 & 0 \end{array}\right] \tag{8.39}$$

$$\bar{F}_{21} = \bar{K}_E \; [\; 0 \;\vdots\; 0 \;\vdots\; C_R \;], \quad \bar{F}_E = \bar{A}_{EE} + \bar{K}_E C_E \tag{8.40}$$

in which

$$\bar{A} = \left[\begin{array}{c:c:c} \bar{A}_{CC} & 0 & 0 \\ \hdashline 0 & \bar{A}_{OO} & 0 \\ \hdashline 0 & 0 & \bar{A}_{RR} \end{array}\right], \quad \bar{A}_{EE} = \left[\begin{array}{c:c:c} \bar{A}_{CC} & 0 & \\ \hdashline 0 & \bar{A}_{OO} & H_E \\ \hdashline 0 & 0 & 0 \end{array}\right], \quad \bar{A}_{jj} = \left[\begin{array}{c:c} 0 & I \\ \hdashline -\Lambda_{ej} & -\bar{D}_{ej} \end{array}\right] \tag{8.41}$$

where j = c, o, R. Moreover, the time-varying matrix has the form

$$F_1(t) = \left[\begin{array}{c:c} \tilde{F}_C(t) & \tilde{F}_{12}(t) \\ \hdashline \tilde{F}_{21}(t) & \tilde{F}_E(t) \end{array}\right] \tag{8.42}$$

where

$$\tilde{F}_C(t) = \tilde{A}_F(t) + B \; [\; K_{C1}(t) \;\vdots\; 0 \;\vdots\; 0 \;], \quad \tilde{F}_{12}(t) = -B \; [\; K_{C1}(t) \;\vdots\; 0 \;\vdots\; 0 \;] \tag{8.43}$$

$$\tilde{F}_{21}(t) = \left[\begin{array}{c:c:c} 0 & 0 & A_{CR}(t) + K_{EC1}(t)\, C_R \\ \hdashline 0 & 0 & A_{OR}(t) + K_{EO1}(t)\, C_R \\ \hdashline 0 & 0 & K_{Ef1}(t)\, C_R \end{array}\right], \quad \tilde{F}_E(t) = \tilde{A}_E(t) + K_{E1}(t)\, C_E$$

in which

$$\tilde{A}_F(t) = \left[\begin{array}{c:c:c} \tilde{A}_{CC}(t) & \tilde{A}_{CO}(t) & \tilde{A}_{CR}(t) \\ \hdashline \tilde{A}_{OC}(t) & \tilde{A}_{OO}(t) & \tilde{A}_{OR}(t) \\ \hdashline \tilde{A}_{RC}(t) & \tilde{A}_{RO}(t) & \tilde{A}_{RR}(t) \end{array}\right], \quad \tilde{A}_E = \left[\begin{array}{c:c:c} \tilde{A}_{CC}(t) & \tilde{A}_{CO}(t) & 0 \\ \hdashline \tilde{A}_{OC}(t) & \tilde{A}_{OO}(t) & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right] \tag{8.44}$$

$$\tilde{A}_{ij}(t) = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline \Omega_0^2(t)\bar{K}_{ij} & 0 \end{array}\right], \quad i, j = c,\ o,\ R$$

Considering Eqs. (8.35) and (8.37) and our assumption that $\| A_{ij}(t) \|$ is small compared to $\bar{K}_C, \bar{K}_E$ and Λ_{ei} $(i = n_C/2 + 1,\ n_C/2 + 2, \ldots\ldots, n_M/2)$, the factor ε_1 in Eq. (8.13) must be regarded as small.

Next, we propose to verify that the matrix $\bar{F}_E$ is exponentially stable, as implied in inequalities (8.23). To this end, we introduce Eq. (8.38a) into the last of Eqs. (8.40) and obtain

$$\bar{F}_E = \bar{A}_{EE} - 1/2\ \bar{P}_0 C_E^T C_E \tag{8.45}$$

where $\bar{P}_0$ is the solution of the algebraic matrix Riccati equation

$$(\bar{A}_{EE} + \beta I)\bar{P}_0 + \bar{P}_0(\bar{A}_{EE} + \beta I) - \bar{P}_0 C_E^T C_E \bar{P}_0 + Q_E = 0 \tag{8.46}$$

Inserting $\bar{F}_E$, Eq. (8.45), into Eq. (8.46), we obtain the Lyapunov equation

$$(\bar{F}_E^T + \beta I)\bar{P}_0 + \bar{P}_0(\bar{F}_E + \beta I) = - Q_E \tag{8.47}$$

In view of the fact that $\bar{P}_0$ and Q_E are symmetric and positive definite, we have [19]

$$\mathrm{Re}\ \lambda_i(\bar{F}_E + \beta I) < 0, \quad i = 1,2,\ldots,n_C + n_0 + n_W/2 \tag{8.48}$$

which validates assumption (8.23). From (8.48), we conclude that

$$\mathrm{Re}\ \lambda_i(\bar{F}_E) < -\beta, \quad i = 1.2,\ldots\ldots,n_C + n_0 + n_W/2 \tag{8.49}$$

Using the same approach, it can be shown that

$$\mathrm{Re}\ \lambda_i(\bar{A}_{CC} + B_C\bar{K}_C) < -\alpha, \quad i = 1,2,\ldots,n_C \tag{8.50}$$

so that, by using a controlled model of proper dimension, the eigenvalues of $\bar{A}_{RR}$ will lie sufficiently deep in the left half of the complex plane that

$$\lim_{t\to t_h} \|e^{\bar{F}_C t}\| < \rho_1, \quad \lim_{t\to t_h} \|e^{\bar{F}_E t}\| < \rho_2, \quad \rho_1,\rho_2 << 1 \tag{8.51}$$

CHAPTER 9

FINITE-TIME STABILITY OF THE MODELED SYSTEM

9.1 DESIGN STRATEGY

The closed-loop modeled system is defined by Eq. (6.14). Denoting the n_A-dimensional modeled state vector by $\underset{\sim}{\eta}_A = [\underset{\sim}{z}_C^T \vdots \underset{\sim}{e}^T \vdots \underset{\sim}{z}_U^T]^T$, where $n_A = n_M + n_C + n_O + n_W/2$, and denoting the coefficient matrix in Eq. (6.14) by $\tilde{A}(t)$ we can rewrite the homogeneous part of Eq. (6.14) as

$$\dot{\underset{\sim}{\eta}}_A(t) = \tilde{A}(t)\underset{\sim}{\eta}_A(t) \ , \qquad \underset{\sim}{\eta}_A(t_0) = \underset{\sim}{\eta}_{A0} \qquad \text{for all} \quad t\epsilon\tau. \tag{9.1}$$

The convergence factors α and β should be determined so as to achieve objectives (i) and (ii) for the controlled model, as outlined in the beginning of sec.4.3, and at the same time to preserve the following characteristics of the modeled system :

(1) The system described by Eq. (9.1) should remain ECS and the factors α and β should minimize the Supremum Time Constant (STC), (Eq. 4.19), without jeopardizing the characteristics (i) and (ii) of the controlled model.

(2) The matrix F_0, defined by Eq. (8.20), should remain exponentially stable so as to agree with inequalities (8.15).

The above goals are easily achieved if the convergence characteristics of system (9.1), as well as those of F_0, are dominated by the diagonal submatrices of the closed-loop matrices $\tilde{A}(t)$ and F_0, respectively. Indeed, as mentioned in Secs.6.3 and 8.4, the diagonal

submatrices $\tilde{A}_{CC}(t)$ and $\tilde{A}_{EE}(t)$ of $\tilde{A}(t)$ are SFTS by design and the open-loop submatrix $\tilde{A}_{UU}(t)$ is QCS by assumption (Sec. 4.3). Similarly, the diagonal submatrices $\bar{F}_C$ and $\bar{F}_E$ of F_0 are exponentially stable. Unfortunately, the convergence characteristics are affected by the off-diagonal submatrices which incorporate model errors and spillover effects. As a result, there is a degradation in the convergence characteristics of the closed-loop modeled system, and even a divergent response.

A reduction in the degradation can be achieved by shifting the observation spillover to more "robust parts" of the model. For structures this implies shifting the spillover to the higher states, as they possess higher inherent damping. This upward shift can be achieved in part by proper placement of the actuators and sensors, but mainly by including critical states in the observed model. Hence, given the maximum order of the observer, as imposed by on-line computer limitations, the division of the observer between observed states and observed disturbance components is a design tool capable of alleviating the spillover effect. Moreover, in the case of extreme maneuvers, one can substitute during the second half of the maneuver observed disturbance components by observed states, where the former are opposite in sign but otherwise identical to the corresponding components observed during the first half.

At this point, we propose the following procedure for determining the factors α and β so as to achieve the goals stated above:

(1) Initial values for α and β, say α_0 and β_0, are determined as the

minimum values for which objective (i) and conditions (8.33) and (8.34) are satisfied. As mentioned in Sec.8, values of α and β for which quasi-constant gains are achieved usually imply a SFTS controlled model.

(2) The ECS of $\tilde{A}(t,\alpha_0,\beta_0)$ and the exponential stability of $F_0(\alpha_0,\beta_0)$ are checked according to the first sufficiency condition in Sec. 4.2 and inequalities (8.15), respectively. If either one of the two matrices fails to be "stable", then the observed state must be increased.

(3) The convergence factors α and β are increased gradually until the minimal STC is achieved. This minimization provides the optimal ECS robustness for the modeled system, but not necessarily the best disturbance response. Thus, the final tuning of the convergence factors remains a design tool.

9.2 THE FINITE-TIME STABILITY TEST AND THE SUPREMUM TIME CONSTANT IN CASE OF A GENERAL SYSTEM

We wish to determine a Lyapunov function for the ECS test and an explicit expression for the STC. Rearranging the state vector in Eq. (6.14) to correspond to the state vector in Eq. (8.2), the constant part of $\tilde{A}(t)$, denoted by $\tilde{A}_0$, can be rewritten as

$$\tilde{A}_0^f = \left[\begin{array}{c:c} \bar{F}_C & \bar{A}_{12} \\ \hdashline \bar{F}_{21} & \bar{F}_E \end{array}\right] \tag{9.2a}$$

where

$$\bar{A}_{12} = -\left[\begin{array}{c:c:c} BK_C & 0 & BK_D \end{array}\right] \tag{9.2b}$$

and $\bar{F}_C$, $\bar{F}_E$ and $\bar{F}_{21}$ are defined by Eqs. (8.40). Then, comparing between the matrix F_0, defined by Eqs. (8.39), and the matrix $\tilde{A}_0^f$, defined by Eq. (9.2a), we can conclude that the eigenvalues of these two matrices differ only slightly. This difference is due to the spillover matrix $\bar{A}_{12}$, defined by Eq. (9.2b), which involves the disturbance accommodation gain matrix K_D. The small difference can be explained by the fact that the eigenvalues of a matrix are continuous functions of the entries of the matrix [39]. Indeed, considering Eq. (8.28), it can be shown that the values of K_D are much smaller than those of $\bar{K}_C$ and $\bar{K}_E$. Moreover, the spillover effect can be reduced by increasing the order of the observed state. Hence, in view of the close correlation between the eigenvalues of F_0 and $\tilde{A}_0^f$, which are identical to those of $\tilde{A}_0$, it is essential to develop an ECS test which also guarantees that

$$\mathrm{Re}\ \lambda_i(\tilde{A}_0) < 0, \quad i=1,2,\ldots\ldots,n_A \tag{9.3}$$

To this end, we introduce the following Lyapunov function [27]

$$V(\underset{\sim}{\eta}_A, t) = \underset{\sim}{\eta}_A^T(t) B_a(t) \underset{\sim}{\eta}_A(t) \tag{9.4a}$$

where $B_a(t)$ is a Hermitian matrix given by

$$B_a(t) = [U_a(t) U_a^H(t)]^{-1} \tag{9.4b}$$

in which the superscript H denotes the complex conjugate transpose of a

matrix and $U_a(t)$ is the matrix of instantaneous eigenvectors of $\tilde{A}(t)$. It is assumed that the eigenvalues of $\tilde{A}(t)$ are distinct for all $t \epsilon \tau$, so that the matrix $U_a(t)$ is always nonsingular. It follows that

$$B_a(t) > 0 \quad \text{for all} \quad t \epsilon \tau. \tag{9.5}$$

Then, it can be shown that [27]

$$\dot{V}(\underset{\sim}{\eta}_A, t) = \underset{\sim}{\eta}_A^T(t) U_a^{-H}(t) C_a(t) U_a^{-1}(t) \underset{\sim}{\eta}_A(t) \tag{9.6}$$

where $[\ ]^{-H}$ is the inverse of $[\ ]^H$, and

$$C_a(t) = 2 \text{ Re } [\Lambda_a(t)] + T(t) \tag{9.7}$$

in which $\Lambda_a(t)$ is the diagonal matrix of the eigenvalues of $\tilde{A}(t)$, and

$$T(t) = U_a^H(t) \dot{B}_a(t) U_a(t) \tag{9.8}$$

and we recognize that both matrices $T(t)$ and $C_a(t)$ are Hermitian.

Considering the first sufficiency condition for an ECS system (Sec. 4.2) and the nonsingularity of $U_a(t)$ together with Eqs. (9.6) and (9.7) and inequality (9.5), we can conclude that

(1) The system defined by Eq. (9.1), is ECS if

$$\lambda_m [\ C_a(t)\ ] < 0 \quad \text{for all } t \epsilon \tau \tag{9.9}$$

where $\lambda_m[\ ]$ denotes the minimum eigenvalue of [ ].

(2) Provided that condition (9.9) is satisfied, the STC of system (9.1) can be expressed as

$$\pi = 1/\ \bar{\gamma} \tag{9.10}$$

where

$$\bar{\gamma} = 1/2 \left\{ \min_{t\in\tau} |\lambda_m[C_a(t)]| \right\} \tag{9.11}$$

Proof :

The supremum time constant of the linear time-varying system, described by Eq. (4.2), is given by Eq. (4.21). To obtain an explicit expression for the modeled system, Eq. (9.1), we follow the lines of Sec. 4.2 and define the modified system as

$$\dot{\underset{\sim}{\eta}}_A^*(t) = \tilde{A}^*(t)\ \underset{\sim}{\eta}_A^*(t) \tag{9.12a}$$

where

$$\underset{\sim}{\eta}_A^*(t) = e^{\gamma t}\ \underset{\sim}{\eta}_A(t)\ , \qquad \tilde{A}^*(t) = \tilde{A}(t) + \gamma I \tag{9.12b,c}$$

Then, by analogy with Eq. (9.7), we obtain

$$C_a^*(t) = 2\ \mathrm{Re}\ [\Lambda_a^*(t)] + T^*(t) \tag{9.13}$$

where all the terms denoted with an asterisk are obtained from the corresponding terms without asterisk, except that $\tilde{A}(t)$ is replaced by $\tilde{A}^*(t)$. Considering Eq. (9.13b) and the definitions of $\Lambda_a(t)$ and $T(t)$

given by Eqs. (9.7) and (9.8), we have

$$\Lambda_a^*(t) = \gamma I + \Lambda_a(t) \quad , \quad T^*(t) = T(t) \tag{9.14a,b}$$

so that

$$C_a^*(t) = 2\gamma I + C_a(t) \tag{9.15}$$

Finally, using Eq. (4.21b) with C(t) replaced by $C_a^*(t)$, we obtain

$$\bar{\gamma} = 1/2 \min_{t\in\tau} |\lambda_m[C_a(t)]| \tag{9.16}$$

We recognize, from Eq. (9.7), that condition (9.9) also implies condition (9.3), as required. The STC is a function of the convergence parameters, α and β. We define, therefore, the minimal STC as

$$\pi_m = \min_{\alpha,\beta} [\pi(\alpha,\beta)] \tag{9.17}$$

Considering Eqs. (9.17), (9.16) and (9.14) we can conclude that the minimization of π also decreases the parameter ρ in Eq. (8.19), thus providing a better approximation to the optimal, steady-state, disturbance accommdation.

9.3 THE FINITE-TIME STABILITY TEST AND THE SUPREMUM TIME CONSTANT IN CASE OF A PERTURBED SYSTEM.

The computations involved in Eqs. (9.9) and (9.11) can be simplified by considering the quasi-constant values of $K_C(t)$ and $K_E(t)$, as given by Eqs. (8.35) and (8.37) respectively. Then, an analysis of system (9.1)

based upon $\bar{K}_C$ and $\bar{K}_E$, Eqs. (8.36a) and (8.38a), respectively, yields an approximated ECS condition and STC expression depending on the eigensolution for constant matrices. Hence, inserting Eqs. (8.36a) and (8.38a) into Eq. (6.14) and considering Eq. (3.5a), we can rewrite the matrix $\tilde{A}(t)$ as

$$\tilde{A}(t) = \tilde{A}_0 + \Omega_0^2(t)A_1 \tag{9.18}$$

where A_1 is also a constant matrix. Then, recalling the relatively high norm of $\bar{K}_C$ and $\bar{K}_E$, we can assume that

$$\| A_0 \| >> \Omega_0^2(t) \| A_1 \| \quad \text{for all} \quad t\epsilon\tau. \tag{9.19}$$

At this point we can obtain the eigensolution of $\tilde{A}(t)$ defined by Eq. (9.18), according to a perturbation technique [3]. Retaining first-order terms only, we obtain

$$\Lambda_a(t) = \Lambda_0 + \Omega_0^2(t)\Lambda_1, \quad U_a(t) = U_0 + \Omega_0^2(t)U_1 \tag{9.20a,b}$$

where Λ_0 is a diagonal matrix of eigenvalues of $\tilde{A}_0$ and U_0 is a matrix of right eigenvectors of $\tilde{A}_0$. Similarly, $\Lambda_1 = \mathrm{diag}(\lambda_{1i})$, where λ_{1i} ($i = 1,2,\ldots,n_A$) are perturbations in the eigenvalues and $U_1 = [\underset{\sim}{u}_{1i}]$, where $\underset{\sim}{u}_{1i}$ ($i = 1,2,\ldots,n_A$) are perturbations in the right eigenvectors. The perturbations in the eigensolutions can be expressed as

$$\Lambda_1 = \mathrm{diag}(e_{Pii}), \quad U_1 = U_0\bar{E}_P \tag{9.21a,b}$$

where the entries of the matrix $\bar{E}_P$ are

$$\bar{e}_{Pij} = \frac{e_{Pij}}{e_{Pjj} - e_{Pii}} \, , i \neq j \; ; \; \bar{e}_{Pii} = 0 \; ; \quad i,j = 1,2,\ldots,n_A \tag{9.22}$$

in which e_{Pij} are the entries of the matrix E_P defined by

$$E_P = V_0^T A_1 U_0 \tag{9.23}$$

where V_0 is a matrix of left eigenvectors of $\tilde{A}_0$. Inserting Eq. (9.20b) into Eq. (9.8) and considering Eqs. (3.1b) and (9.21b), we obtain

$$T(t) = -2(t - t_A)\dot{\Omega}_0^2 \, [\, M_a + M_a^H \,] \tag{9.24a}$$

in which

$$M_a = [\, I_{n_A \times n_A} + (t - t_A)^2 \dot{\Omega}_0^2 \bar{E}_P \,]^{-1} \bar{E}_P \tag{9.24b}$$

provided that the inverse matrix exists, where $t_A = t_0$ for the first half

of the maneuver and $t_A = t_f$ for the second half. Then, recognizing that

$$\| U_0 \| >> (t-t_A)^2 \dot{\Omega}_0^2 \| U_1 \| \quad \text{for all} \quad t \in \tau. \tag{9.25}$$

we can conclude from Eq. (9.21b) that

$$\| I_{n_A \times n_A} \| >> (t-t_A)^2 \dot{\Omega}_0^2 \| \bar{E}_P \| \quad \text{for all} \quad t \in \tau. \tag{9.26}$$

Thus, we can further simplify Eq. (9.24) by ignoring the second-order term in Eq. (9.24b) and writing

$$M_a \cong \bar{E}_P \tag{9.27}$$

Inserting Eqs. (9.27) and (9.24) into Eq. (9.7), we obtain finally

$$\bar{C}_a(t) = 2 \text{ Re } [\Lambda_0] + 2\dot{\Omega}_0^2 \left\{ (t - t_A)^2 \text{Re } [\Lambda_1] - (t - t_A)[\bar{E}_P + \bar{E}_P^H] \right\} \tag{9.28}$$

It follows that the ECS sufficiency condition and the STC expression for system (9.1), where $\tilde{A}(t)$ is defined by Eq. (9.18), can be deduced from Eqs. (9.9) and (9.11) by replacing $C_a(t)$ with $\bar{C}_a(t)$.

CHAPTER 10

NUMERICAL EXAMPLE

The above developments were applied to the flexible spacecraft shown in Fig. 2. The mathematical model consists of a rigid hub and a flexible appendage in the form of a uniform beam 24 ft long. The mass of the hub is m_h = 27 slugs, the mass density of the beam is ρ_b = 0.01 slug ft^{-1} and the mass moment of inertia of the hub about its own mass center is 264 slug ft^2. The perturbed model, described by Eqs. (2.34a) and (3.3), is assumed to possess two rigid-body translational, one rigid-body rotational and seven elastic degrees of freedom, for a total of ten degrees of freedom. The first ten natural frequencies of the structure are: 0, 0, 0, 0.32, 1.8, 5.0, 9.8, 16.2, 24.2 and 33.8 (Hz). All damping factors are assumed to have the same value, ζ_i = 0.01 (i = 4,5,...,10).

The "rigid-body" slewing of the spacecraft is carried out according to a minimum-time control policy, which implies on-off control. A 180° maneuver is shown in Fig. 4. The maneuver is relatively fast, so that in the absence of vibration control the elastic deformations tend to be large. Indeed, Fig. 5 shows that for uncontrolled vibration the slope of the elastic deflection curve at the tip of the beam can reach 19°.

The controlled degrees of freedom consist of up to seven degrees of freedom, three representing rigid-body motions and up to four representing elastic motions. As suggested in Sec. 3.2, the rigid-body perturbations are controlled independently of the elastic motions by three sets of collocated actuators and sensors, one for each degree of

freedom. All actuators and sensors are located at the mass center of the spacecraft, assumed to lie on the hub. The elastic motions are controlled by up to four sets of collocated torque actuators and angular displacement and angular velocity sensors, where the collocation is advised for spillover reduction. Obviously, the number of sensors must be so as to maintain the observability condition, Eq. (7.13), i.e., it is equal to or it exceeds the number of actuators. These actuators and sensors are placed on the flexible beam at $y_i = iL/4$ $(i = 1,2,3,4)$. The feedback control $\underset{\sim}{u}_C$ is applied during and after the maneuver, while the disturbance accommodation $\underset{\sim}{u}_D$ operates only during the maneuver.

The observer is designed to estimate the controlled state and a 2-dimensional disturbance vector; additional observed states are not needed in this example. During the maneuver the full observer model, Eq. (6.9), is operational. After the maneuver is completed, $\underset{\sim}{f}_E$ must be removed from the observer state, as the persistent disturbance no longer exists. Otherwise, significant transient estimation errors arising from the disturbance estimation errors can plague the controlled state, causing an increase in the settling time after the termination of the maneuver. Hence, the observer model used after the completion of the maneuver is the one defined by Eq. (6.9), in which H_E is omitted.

From now on, we focus our attention on the 14th-order modeled system, which consists of the elastic degrees of freedom, although the simulation results consists of all the controlled degrees of freedom, i.e., Eqs. (2.34a), (3.2) and (3.3a) and the controlled part of Eq. (3.3b). In every case, the response represents the deflection at the tip

of the beam, at which point the deflection is the largest.

The controlled model response, is bounded according to inequality (4.17). As argued in Chapter 5, the controlled system can be forced to become an SFTS system by increasing the convergence factor α. This process is demonstrated in Fig.6, where it is shown that, for sufficiently large α, the term $r_B^{1/2}(t)\ e^{-\alpha t}$ is dominated by the exponential part for $t \in (T_1, t_h)$, $t_i < T_1 < t_h$. Figure 7 shows the response of an 8th-order controlled model, i.e., $n_c = 8$, without the residual effect where $m/2 = p = 4$. As $t \to t_h$, the angularerror, shown in part a, drops below 0.26×10^{-5} deg. and the displacement error, shown in part b, drops below 0.8×10^{-6} ft.

To obtain a satisfactory steady-state disturbance accommodation, it is necessary that the controller and observer Riccati equation possess quasi-constant solution according to Eqs. (8.33) and (8.34). To demonstrate this, we consider an extreme case. With reference to Eq. (8.30), we use a 2-nd order model where $\varphi(t) = 0.25\ t^2\ (\text{rad/s})^2$, $t \in [0\ ,\ 2.5]$, $\omega_1 = 0.2$ rad/s, $\| \bar{K}_{22} \| \cong 1$, so that $\varphi(t) \| \bar{K}_{22} \| > w_1^2$, for $t > 0.4$ s. Figures 8 and 9 show plots of $S_{11}(t)\ /\ \bar{S}_{011}$ vs t for $\alpha = 4$ and $\alpha = 8$, respectively, where $S_{11}(t)$ is the entry of S(t), Eq. (5.8), exhibiting the largest deviation relative to the corresponding entry, $\bar{S}_{011}$, of the algebraic Riccati matrix $\bar{S}_0$. Of course, $\bar{S}_0$ satisfies the algebraic version of Eq. (5.8). It is obvious that convergence rate is decreased for increasing α. Figure 10 demonstrates the decrease in $\varepsilon_1(\alpha)$, Eq. (8.31), for increasing α. Finally, Fig.11 shows that, by choosing $S(t_f) = \bar{S}_0$, the transient Riccati solution approaches the algebraic solution

for $t \in \tau$, as indicated by Eq. (8.33). From Fig.11, we see that the maximum deviation from the algebraic solution is smaller than 5% for α = 4 and smaller than 0.5 % for α = 12.

Figure 12 shows the deviation of the optimal control gains from the algebraic values for α = 3, for the case considered in this example. For n_c = 2, the gain matrix reduces to the vector $[k_1(t)\ k_2(t)]^T$. We observe that the maximum deviation corresponds to $k_2(t)/\ k_{02}$ and is smaller than 1 % for $t \in [t_i,\ 2.75]$ and less that 4 % for $t \in [2.75,\ t_f]$.

The design policy yielding an exponentially stable matrix F_0 and the minimal STC is demonstrated next. The gain matrices $K_C(t)$, $K_E(t)$ and K_D are calculated according to Eqs. (5.10), (6.46) and (8.28), respectively, the convergence factor β is calculated according to $\beta = 3\alpha$ and the weighting matrices Q and R in Eq. (5.6) and Q_E in Eq. (6.47a) are taken as unit matrices. The matrix L_y in Eq. (8.1) consists of slopes at the locations $y_j = jL$ (j = 1.0, 0.85, 0.75, 0.65, 0.55, 0.35, 0.25). A 4th-order, 6th-order and 8th-order controlled models are examined. The locations of the actuators are $y_i = iL/4$ where i = 2 for p = 1, i = 2 and 3 for p = 2 and i = 1, 2 and 4 for p = 3. The achievement of the "stability" goals for both matrices by a single Lyapunov function, Eq. (9.4a), is based on the small difference between the eigenvalues of $\tilde{A}_0^f$ and F_0. Tables 1 to 4 demonstrate this correlation for the cases in which n_c = 8 and p = 1, n_c = 4 and p = 2, n_c = 6 and p = 3 and n_c = 8 and p= 4, respectively. The convergence factor is taken as α = 2 to highlight the difference in the eigenvalues in cases of relatively small gain matrices $\bar{K}_C$ and $\bar{K}_E$. We conclude, as should be

expected, that the correlation generally improves as the spillover effect decreases.

The minimal STC is achieved by increasing the convergence factor α beyond the value necessary to achieve quasi-constant optimal gain matrices $K_C(t)$ and $K_E(t)$. Table 5 and Figure 13 show the value of $\bar{\gamma}$ vs α in the case in which $n_C = 6$ and $p = 3$ for the original modeled system and the perturbed modeled system. The STC is calculated according to Eqs. (9.10), (9.11) and (9.17), where $C_a(t)$ is defined by Eqs. (9.7) and (9.8) for the first case and by Eq. (9.28) for the second case. The minimal STC is achieved with $\alpha = 3.5$ and is equal to 0.49 s in the case of the original modeled system, whereas $\alpha = 3.0$ and $\pi_m = 0.506$ s are the optimal values in the case of the perturbed system. It is obvious that the differences are small while the reduction in computation effort is significant.

The closed-loop response of the modeled system is shown next. Figures 14 to 16 show the response in the case of a 4th-order, 6th-order and 8th-order controlled model, respectively, where $p = n_C/2$. The optimal values of α yielding the minimal STC are 2.2, 3.0 and 5.0, respectively. As $t \to t_h$, the angular errors drop below 0.04°, 0.09° and 0.018°, respectively, and the displacement errors drop below $0.8\ 10^{-2}$ft, $0.6\ 10^{-2}$ft and $0.8\ 10^{-2}$ ft, respectively. It can be concluded that, as the order of the controlled model increases, higher values of α and β are possible, without causing a divergent response. Moreover, the stabilization during maneuver also improves as a result of the lower

states being controlled. The deflection errors in the steady-state generally decreases when the order of the controlled model increases, although they are highly sensitive to actuator and sensor locations. Moreover, the deflections during maneuver remain relatively small, which guarantees a linear elastic model.

The response of the closed-loop modeled system, corresponding to $n_c = 8$ and a single actuator is shown in Fig. 17. As $t \rightarrow t_h$, the angular error drops below 0.24° and the displacement error drops below 0.017 ft. It is obvious that the error in steady-state is much larger compared to the case in which $n_c = 8$ and $p = 4$. Moreover, due to the reduced degree of controllability, which forces higher gains, the convergence factors are limited to lower values so as not to cause divergent response and the FTS of the modeled system is more sensitive to actuator and sensor locations. Finally, Fig. 18 represents the angular displacement of the spacecraft at the tip of the beam for an 180° maneuver corresponding to the case in which $n_c = 8$ and $p = 4$. Deviations from the ideal on-off maneuver are well below the resolution of the graph.

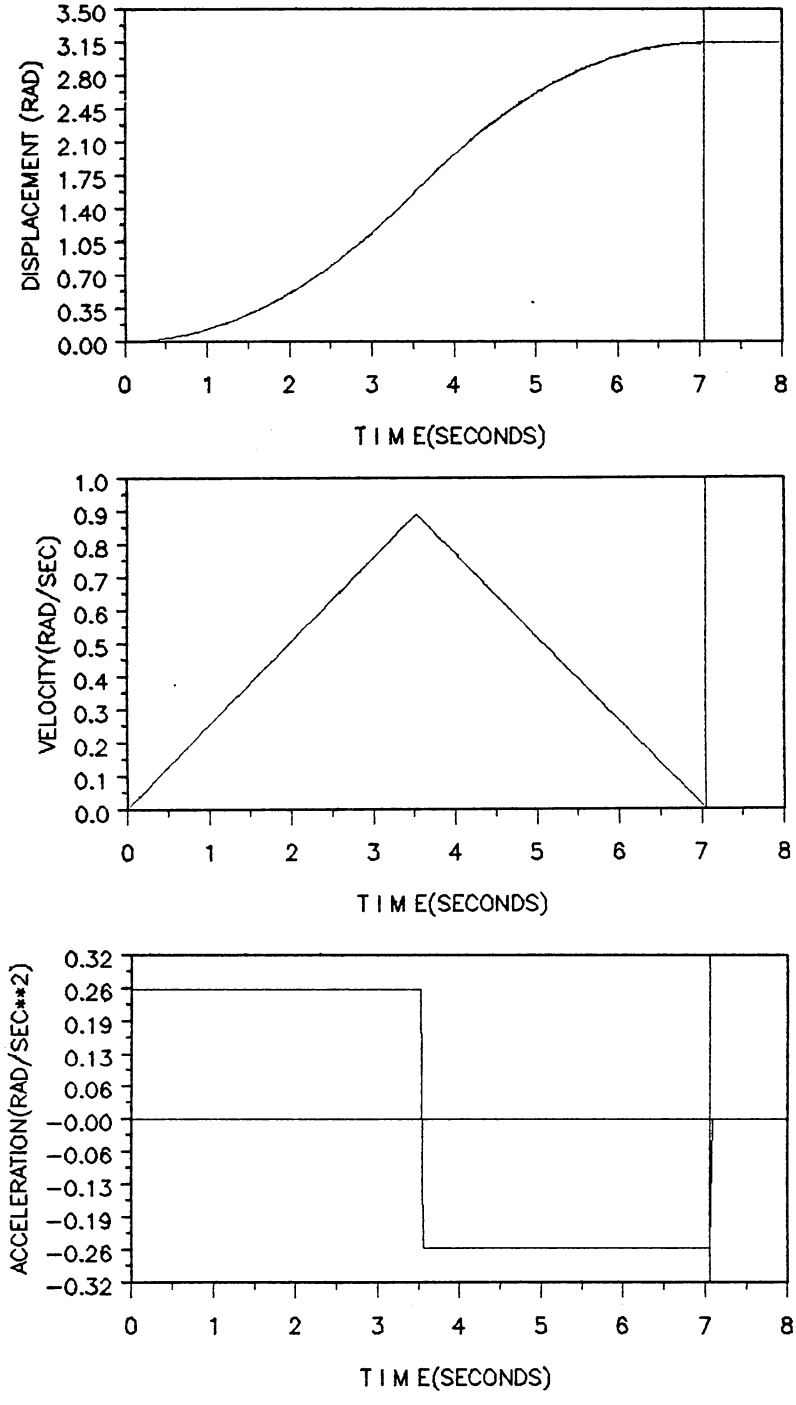


Figure 4. On-Off Control
Maneuver Angle = 180°

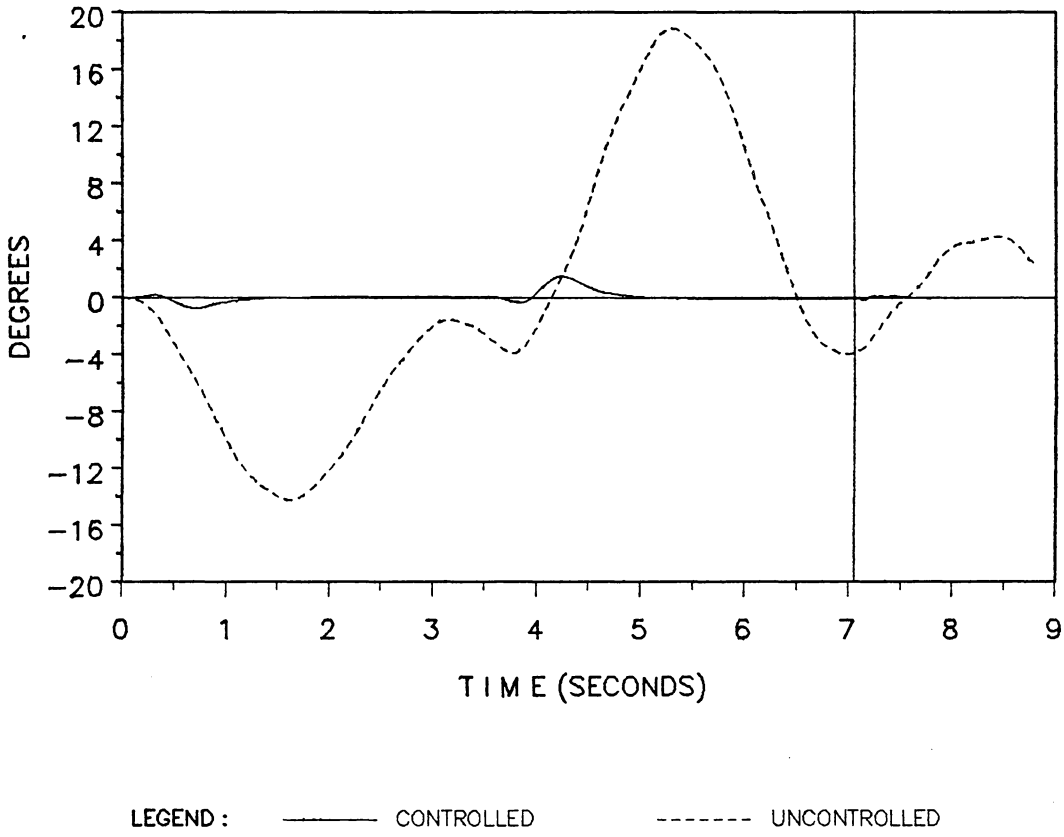


Figure 5. Controlled vs Uncontrolled Disturbance Response
Elastic Slope Error at the Tip
Maneuver Angle = 180°

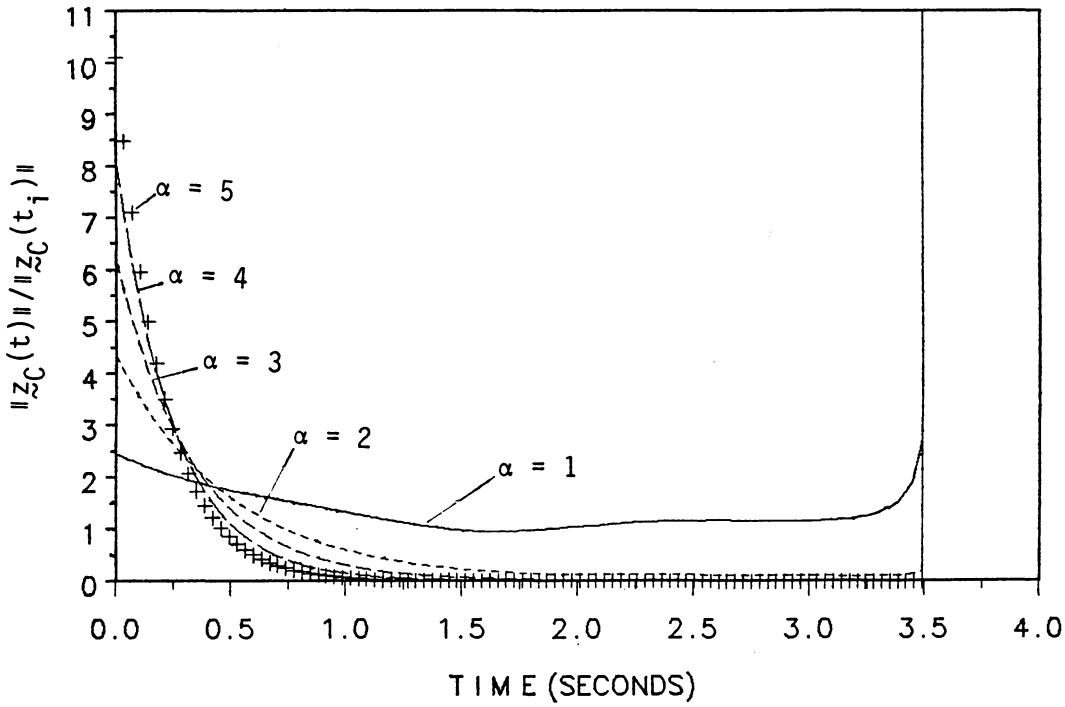
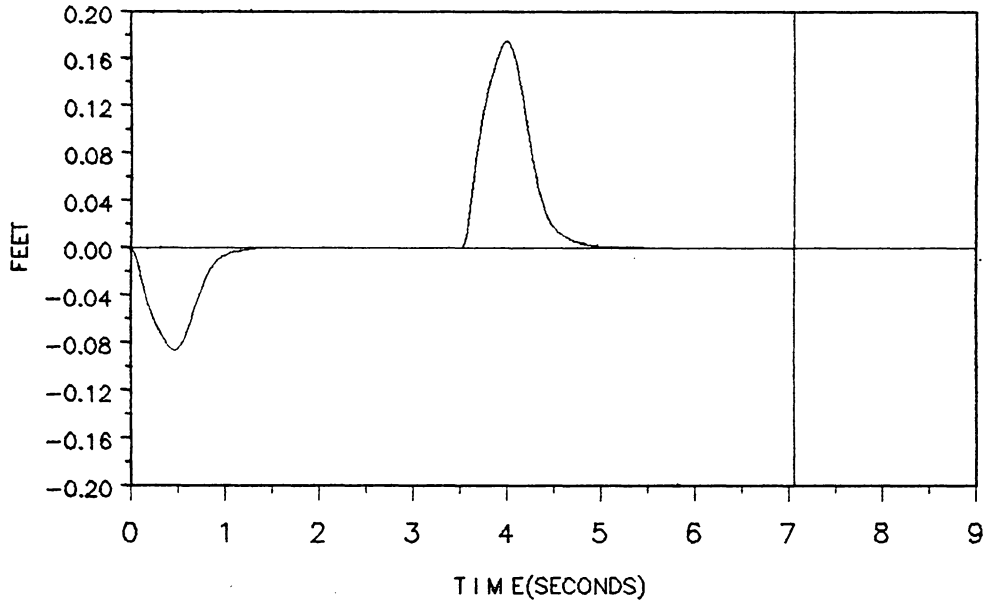
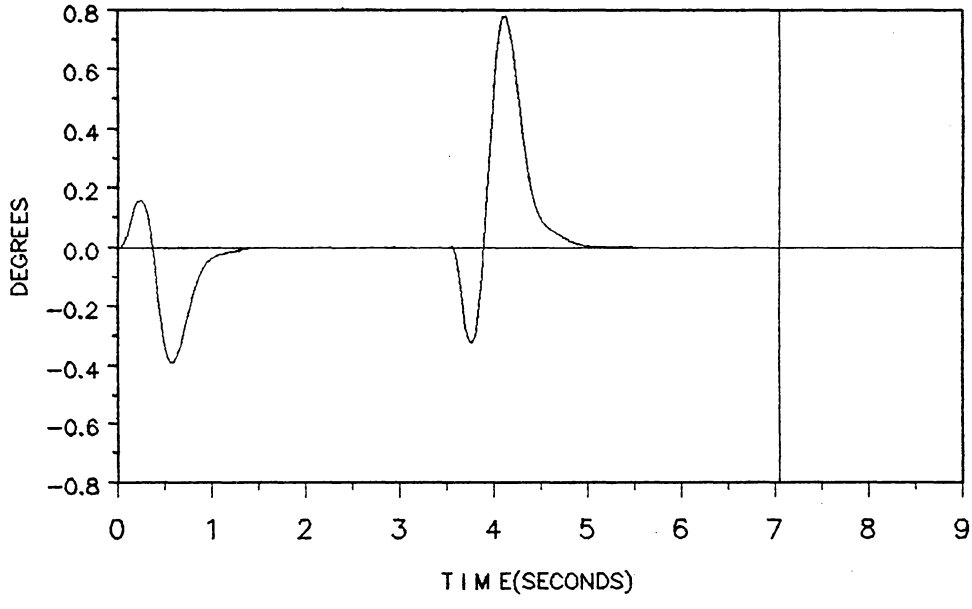


Figure 6. An Upper Bound of the Controlled Model Impulse Response



a) Displacement Error



b) Slope Error

Figure 7. Error at the Tip
8th-Order Modeled System, 8th-Order Controlled Model
Maneuver Angle = 180°

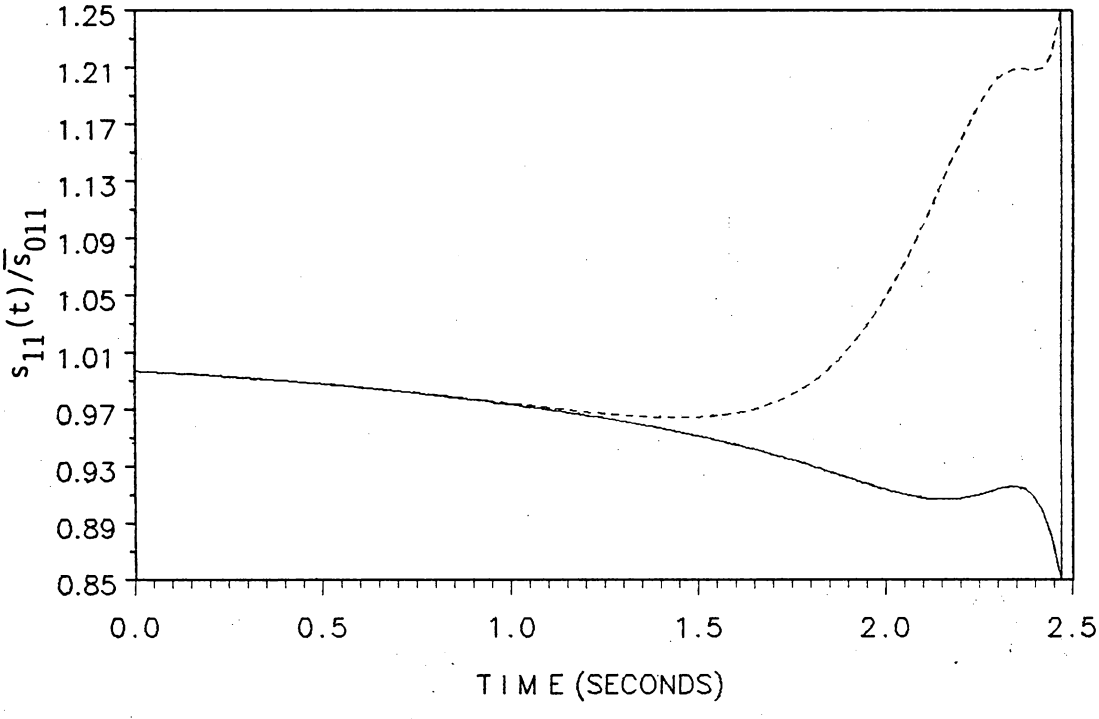


Figure 8. Convergence of Riccati Solution to the Steady-State Solution
$\alpha = 4$

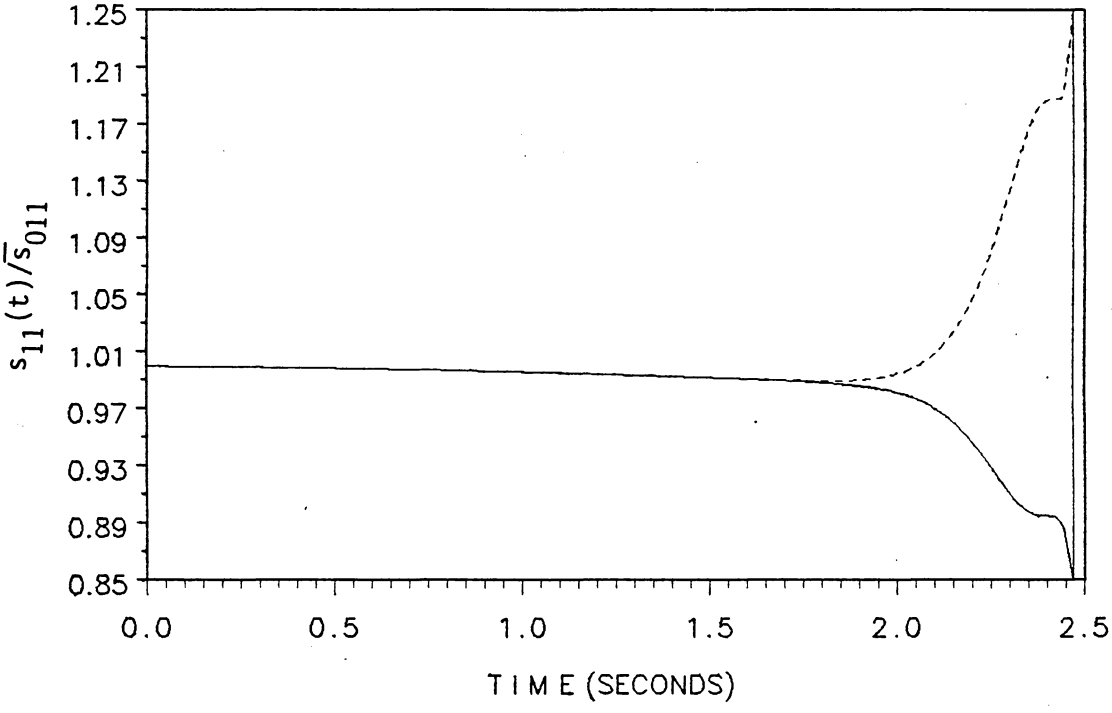


Figure 9. Convergence of Riccati Solution to the Steady-State Solution
$\alpha = 8$

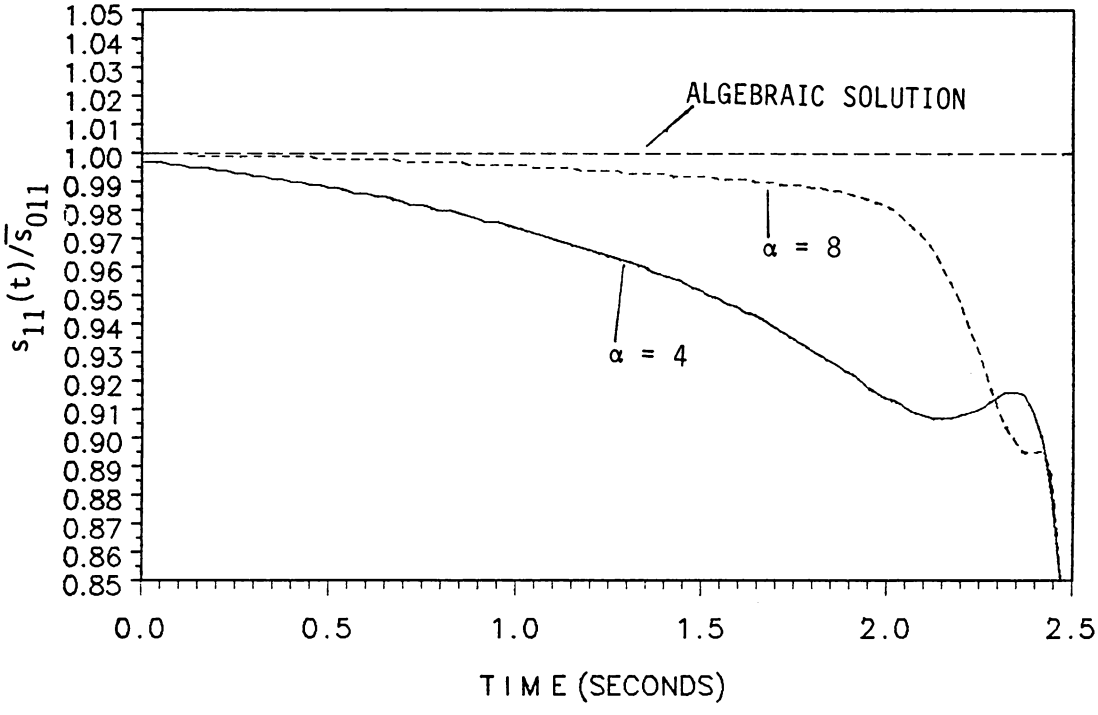


Figure 10. Deviation of Riccati Solution from the Algebraic Solution $S(t_f) = 0.85\ \bar{S}_0$

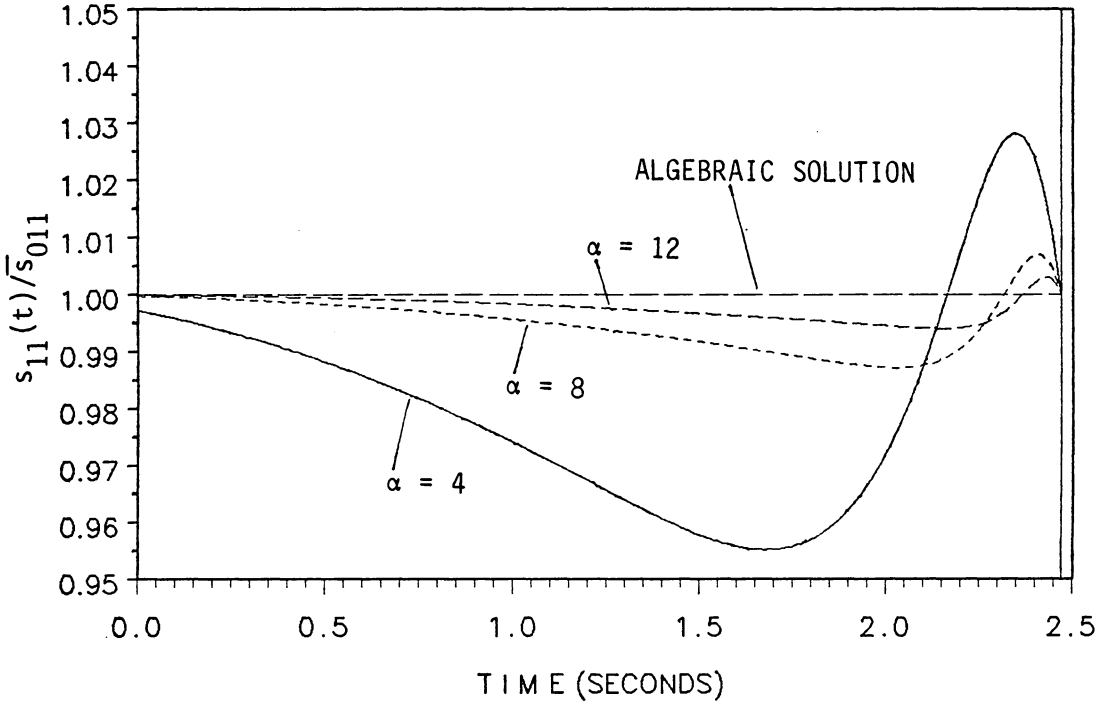


Figure 11. Deviation of Riccati Solution from the Algebraic Solution $S(t_f) = \bar{S}_0$

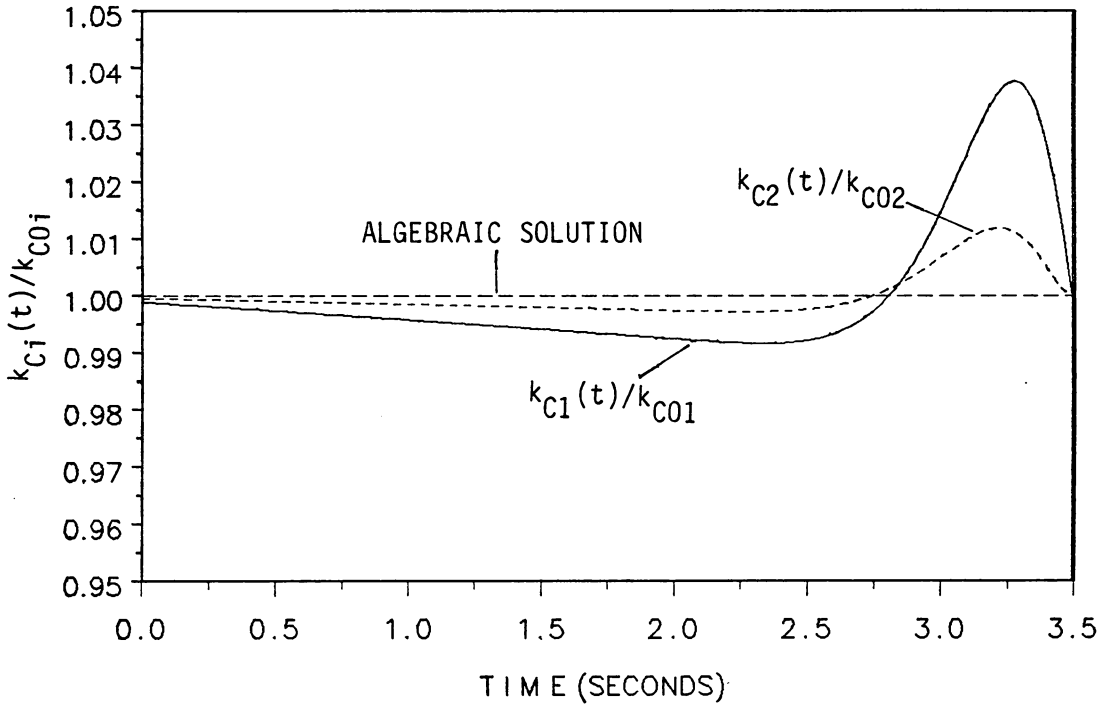


Figure 12. Deviation of the Optimal Gains from the "Algebraic Gains" $S(t_f) = \bar{S}_o$, $\alpha = 3$

Table 1. A Comparison of the Eigenvalues of $\tilde{A}_0^f$ and F_0

$n_M = 14,\ n_C = 8,\ p = 1,\ \alpha = 2$

$\tilde{A}_0^f$	F_0
$-4.196 \pm j233.238$	$-1.009 \pm j205.876$
$-1.520 \pm j152.186$	$-1.521 \pm j152.189$
$-5.755 \pm j104.903$	$-6.192 \pm j127.722$
$-18.836 \pm j72.676$	$-22.895 \pm j71.443$
$-0.689 \pm j61.648$	$-0.663 \pm j61.651$
$-12.118 \pm j34.644$	$-14.874 \pm j33.716$
$-3.279 \pm j31.698$	$-2.355 \pm j32.351$
$-2.860 \pm j12.741$	$-2.001 \pm j14.312$
$-9.004 \pm j9.185$	$-5.447 \pm j7.381$
$-5.069 \pm j6.524$	$-6.487 \pm j7.256$
$-3.872 \pm j1.834$	$-3.801 \pm j1.826$
$-7.311 \pm j0.0$	$-7.246 \pm j0.0$
$-6.006 \pm j0.0$	$-5.978 \pm j0.0$

Table 2. A Comparison of the Eigenvalues of $\tilde{A}_0^f$ and F_0

$n_M = 14,\ n_C = 4,\ p = 2,\ \alpha = 2$

$\tilde{A}_0^f$	F_0
$-3.451 \pm j228.307$	$-2.792 \pm j219.379$
$-2.145 \pm j164.772$	$-1.787 \pm j155.198$
$-4.440 \pm j145.172$	$-4.455 \pm j129.836$
$-0.868 \pm j63.364$	$-0.772 \pm j62.245$
$-1.190 \pm j35.441$	$-1.409 \pm j34.616$
$-4.128 \pm j13.575$	$-4.540 \pm j13.392$
$-4.479 \pm j10.660$	$-4.692 \pm j11.105$
$-1.520 \pm j5.834$	$-1.990 \pm j6.810$
$-3.067 \pm j1.704$	$-3.112 \pm j1.585$
$-6.820 \pm j0.0$	$-6.656 \pm j0.0$
$-6.544 \pm j0.0$	$-6.189 \pm j0.0$

Table 3. A Comparison of the Eigenvalues of $\tilde{A}_0^f$ and F_0

$n_M = 14,\ n_C = 6,\ p = 3,\ \alpha = 2$

$\tilde{A}_0^f$	F_0
$-2.701 \pm j222.974$	$-2.365 \pm j216.739$
$-5.284 \pm j184.416$	$-4.651 \pm j174.236$
$-2.890 \pm j122.214$	$-2.850 \pm j113.098$
$-1.796 \pm j69.950$	$-2.094 \pm j68.885$
$-12.760 \pm j35.702$	$-12.798 \pm j35.757$
$-3.720 \pm j31.979$	$-3.490 \pm j32.031$
$-2.902 \pm j13.253$	$-3.000 \pm j13.436$
$-5.109 \pm j10.950$	$-5.834 \pm j11.292$
$-2.224 \pm j6.380$	$-2.554 \pm j6.756$
$-3.143 \pm j1.733$	$-3.138 \pm j1.639$
$-6.420 \pm j0.0$	$-6.140 \pm j0.0$
$-6.899 \pm j0.0$	$-6.688 \pm j0.0$

Table 4. A Comparison of the Eigenvalues of $\tilde{A}_0^f$ and F_0

$n_M = 14,\ n_C = 8,\ p = 4,\ \alpha = 2$

$\tilde{A}_0^f$	F_0
$-2.042 \pm j225.973$	$-2.009 \pm j217.950$
$-3.864 \pm j161.532$	$-3.431 \pm j158.688$
$-2.113 \pm j106.848$	$-2.546 \pm j104.214$
$-22.557 \pm j70.630$	$-22.033 \pm j70.677$
$-4.489 \pm j61.999$	$-4.334 \pm j61.981$
$-12.062 \pm j35.660$	$-12.036 \pm j35.688$
$-3.926 \pm j31.454$	$-3.945 \pm j31.436$
$-6.224 \pm j13.409$	$-6.490 \pm j13.608$
$-3.775 \pm j10.704$	$-3.706 \pm j10.719$
$-5.961 \pm j5.833$	$-6.599 \pm j5.794$
$-3.817 \pm j2.027$	$-3.816 \pm j2.029$
$-6.352 \pm j0.703$	$-6.385 \pm j0.0$
	$-6.090 \pm j0.0$

Table 5. The Supremum Time-Constant of the Modeled System

$n_M = 14,\ n_C = 6,\ p = 3$

α	$\bar{\gamma}$ Original System	$\bar{\gamma}$ Perturbed System
1.5	1.6121	1.5456
2.0	1.8221	1.7732
2.5	1.9613	1.9094
3.0	2.0238	1.9759
3.5	2.0360	1.9641
4.0	1.8026	1.7346
4.5	1.4901	1.4236
5.0	1.1917	1.1269
5.5	0.9302	0.8715
6.0	0.7195	0.6573
6.5	0.5377	0.4783
7.0	0.3893	0.3289
7.5	0.2637	0.2041
8.0	0.1589	0.1000
8.5	0.0692	0.0135

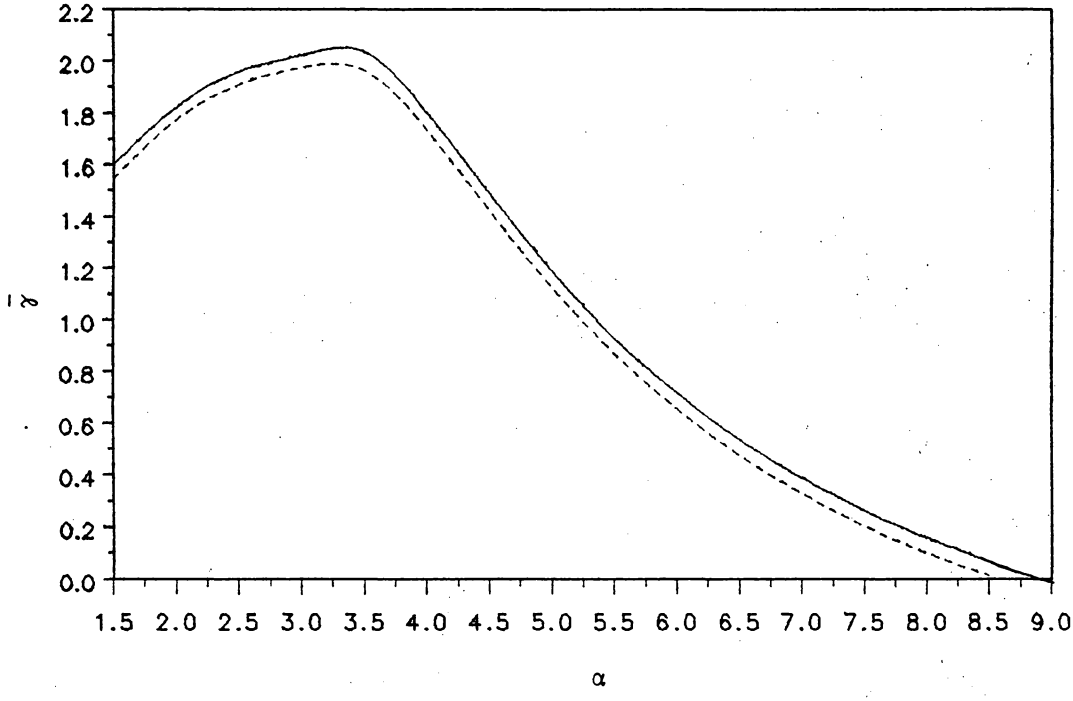
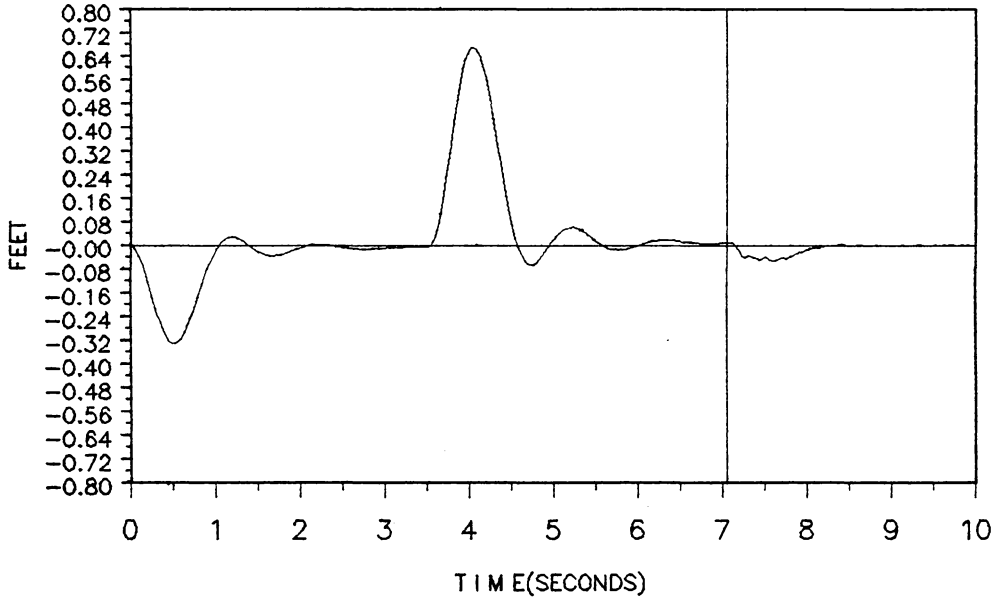
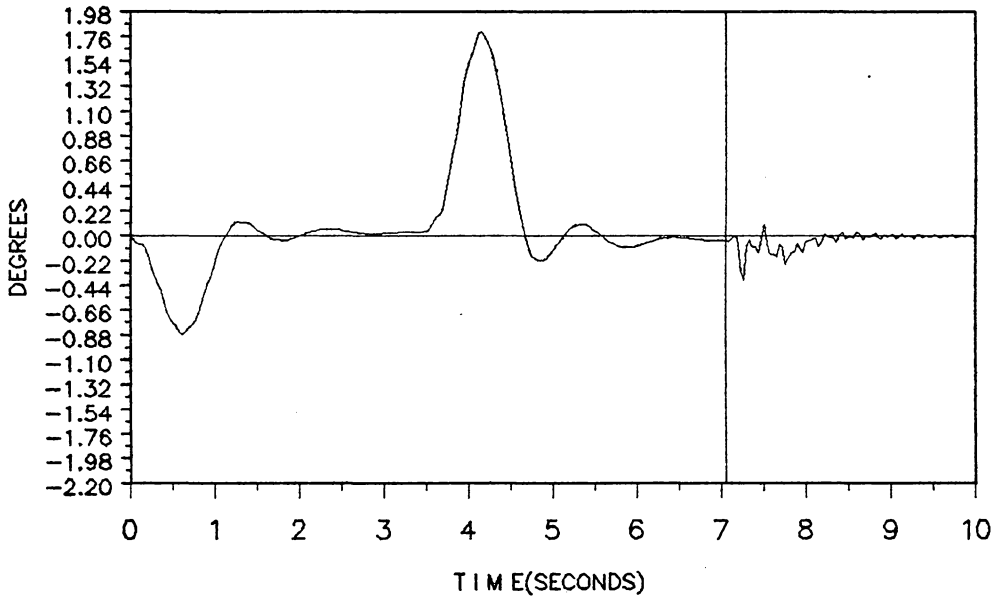


Figure 13. The Supremum Time-Constant of the Modeled System
14th-Order Modeled System, 6th-Order Controlled Model, p = 3

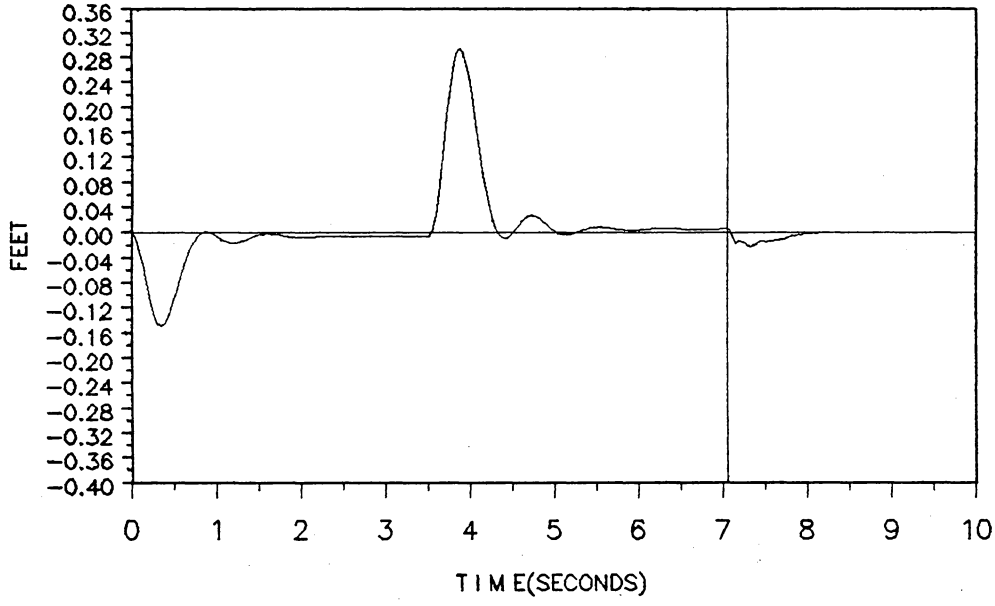


a) Displacement Error

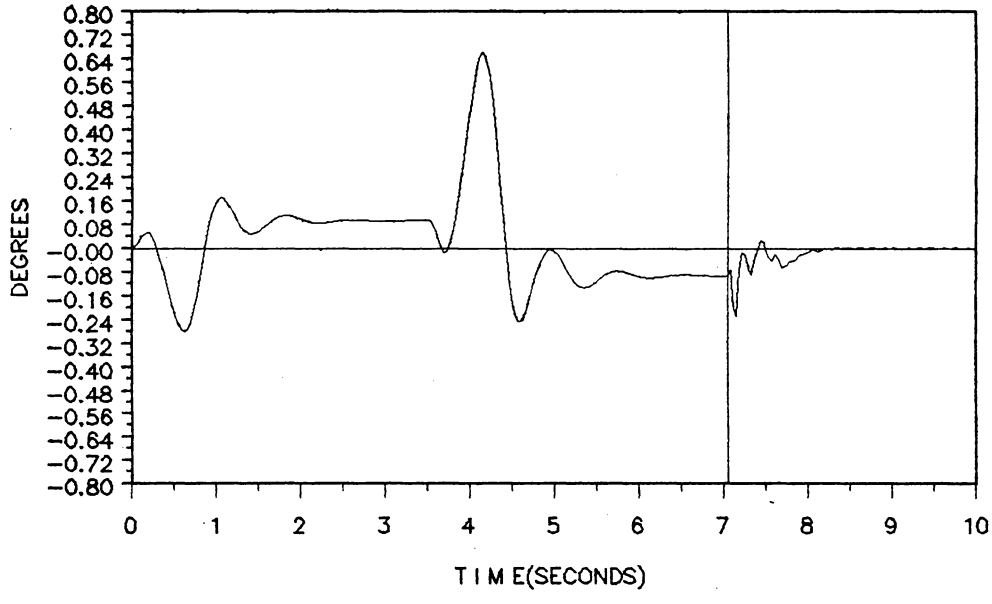


b) Slope Error

Figure 14. Error at the Tip
14th-Order Modeled System, 4th-Order Controlled Model, $p = 2$, $\alpha = 2.2$
Maneuver Angle = 180°

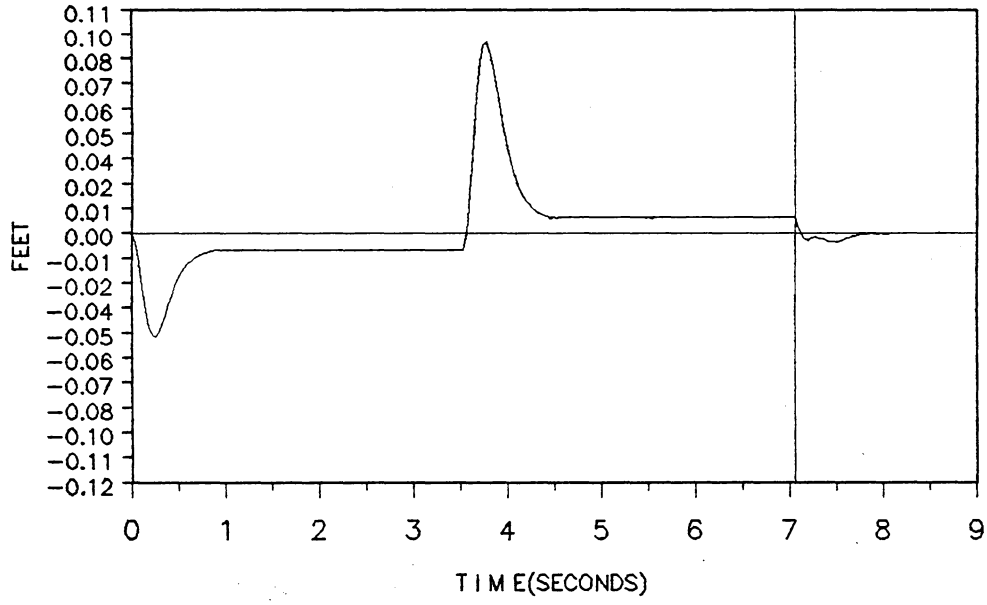


a) Displacement Error

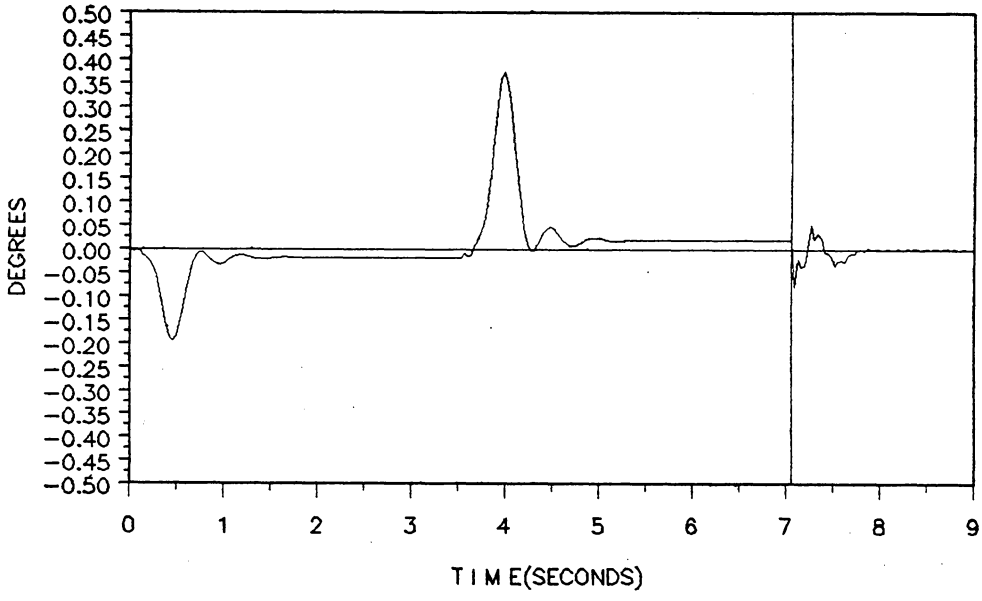


b) Slope Error

Figure 15. Error at the Tip
14th-Order Modeled System, 6th-Order Controlled Model, $p = 3$, $\alpha = 3$
Maneuver Angle = 180°

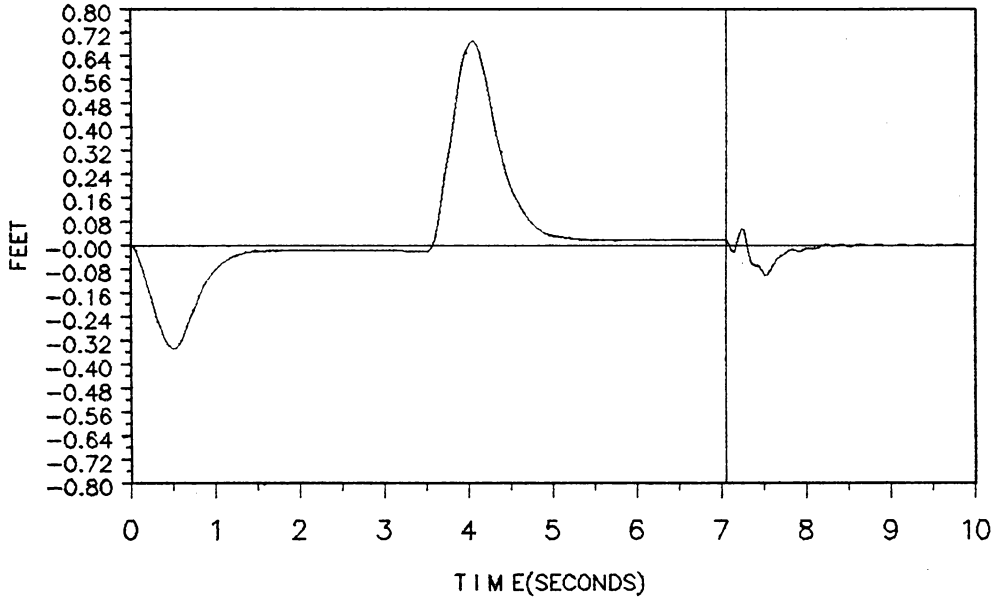


a) Displacement Error

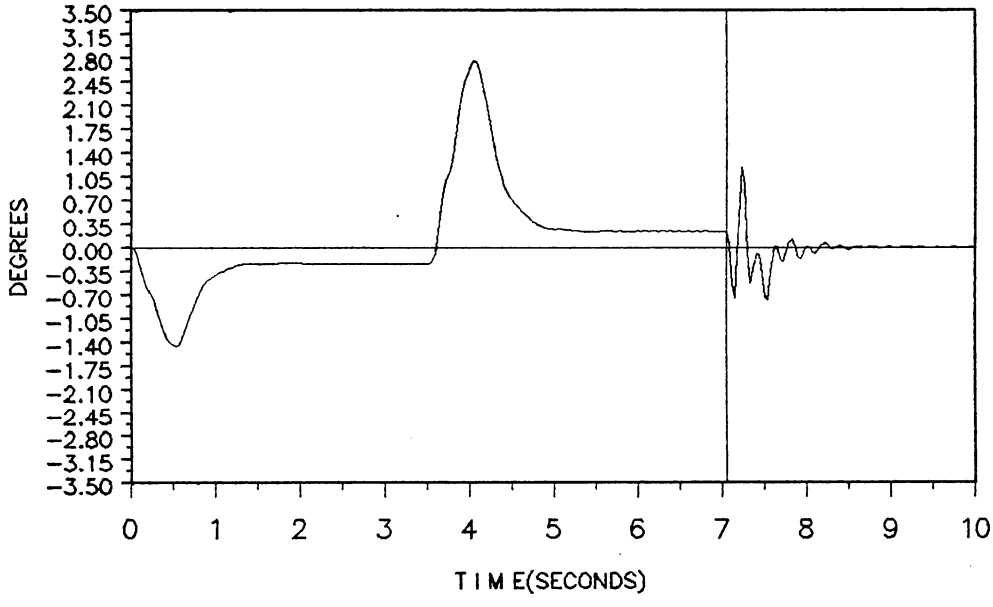


b) Slope Error

Figure 16. Error at the Tip
14th-Order Modeled System, 8th-Order Controlled Model, $p = 4$, $\alpha = 5$
Maneuver Angle = 180°



a) Displacement Error



b) Slope Error

Figure 17. Error at the Tip
14th-Order Modeled System, 8th-Order Controlled Model, $p = 1$, $\alpha = 2$
Maneuver Angle = 180°

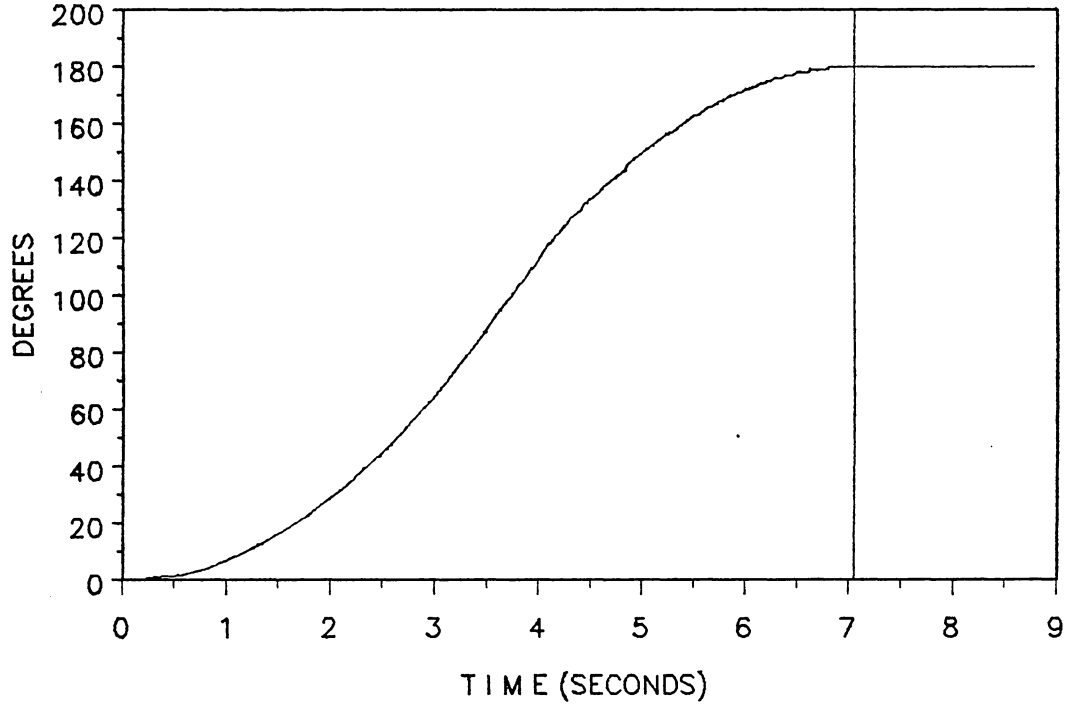


Figure 18. Angular Displacement at the Tip

Maneuver Angle = 180°

CHAPTER 11

SUMMARY AND CONCLUSIONS

The equations of motion describing the slewing of a flexible spacecraft represent a hybrid set consisting of nonlinear ordinary differential equations and partial differential equations. For practical reasons, the equations are discretized in space, resulting in a set of nonlinear ordinary differential equations of high order. Then, the problem consists of designing controls capable of transferring the system from one rest point to another in minimum time. To circumvent the difficulties encountered with on-line solutions of high-order nonlinear two-point boundary-value problems, a perturbation approach was developed [13]. This approach yields a zero-order nonlinear problem of low order describing the "rigid-body" slewing maneuver of the spacecraft and a first-order linear time-varying problem representing the elastic vibrations of the flexible parts and deviations of the spacecraft as a whole from the "rigid-body" maneuver. The control law for the zero-order problem is according to a minimum-time policy, which results in on-off control. This dissertation is concerned with the first-order problem, and in particular with the elastic vibration suppression during the minimum-time maneuver.

The perturbed model is subjected to persistent piecewise-constant disturbances caused by inertial forces resulting from the maneuver. The control is to be carried out in finite time and is divided into two parts, one designed to enhance the closed-loop dynamic characteristics

and the other to accommodate persistent disturbances. Because of on-line computer limitations, not all the modeled states are controlled, resulting in a reduced-order compensator consisting of a controller and a Luenberger observer. The observer estimates part of the modeled state, which includes the controlled states and the observed but uncontrolled states, as well as part of the disturbance vector. The decision as to the quantities to be included in the observer state is dictated by the desire to reduce the spillover effect. The first part of the controller, concerned with the closed-loop dynamic characteristics, represents a linear quadratic regulator designed to stabilize the reduced-order model (ROM) within one half of the maneuver period. Convergence within this time interval is achieved by including convergence factors in the Riccati equations for both the controller and the observer. The equations of the closed-loop modeled system are developed for a reduced-order observer and a full-order observer. It is shown that a reduced-order observer can jeopardize the finite-time stability of the main diagonal submatrix corresponding to the uncontrolled states and therefore is more likely to cause divergence of the modeled system.

The second part of the controller is based on a constant gain matrix designed to minimize a weighted norm spanning the full modeled state in steady state. The minimization is meaningful as long as the optimal gain matrices are quasi-constant. To achieve this characteristic, the translational motion, which is not subjected to disturbances caused by the maneuver, is controlled independently. The rest of the model, which consists of the rotational and the elastic motions, is characterized by

the fact that, as the convergence factor increases, the steady-state solution of both the controller and the observer Riccati equations converges to the corresponding algebraic solution. This fact is actually verified numerically. The steady-state solutions can be reached within the finite time of operation if the convergence factors are sufficiently large. To keep these factors as low as possible, to reduce the spillover effect, it is desired that the rotational motion be controlled independently of the elastic motions.

The spillover effect, as well as modeling errors caused by the truncation of the modeled system, tend to induce divergence in the response of the full modeled system. Therefore the process of increasing the convergence factors is carried out so as to minimize the supremum "time constant" of the full modeled system, thus resulting in a finite time stable modeled system. The supremum time constant and the finite-time stability test are expressed in terms of a Lyapunov function consisting of a perturbed eigensolution of the modeled system.

The above developments are illustrated by means of a numerical example that demonstrates the design process and verifies various assumptions made throughout this research.

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APPENDIX

CONTROLLABILITY, OBSERVABILITY AND BOUNDS ON RICCATI SOLUTIONS IN LINEAR TIME-DEPENDENT SYSTEMS

Linear time-dependent systems can be described by

$$\dot{\underset{\sim}{x}}(t) = A(t)\underset{\sim}{x}(t) + B(t)\underset{\sim}{u}(t), \qquad \underset{\sim}{x}(t_0) = \underset{\sim}{x}_0 \tag{A.1a,b}$$
$$\underset{\sim}{y}(t) = C(t)\underset{\sim}{x}(t)$$

where $\underset{\sim}{x}\in R^n$, $\underset{\sim}{u}\in R^p$ and $\underset{\sim}{y}\in R^m$. It is assumed that A(t), B(t) and C(t) together with their first n - 2, n - 1 and n - 1 derivatives, respectively, are continuous in time. The system is controlled according to the standard quadratic performence index, i.e., $\underset{\sim}{u}$ minimizes

$$J = \int_{t_0}^{t_f} (\underset{\sim}{x}^T Q(t)\underset{\sim}{x} + \underset{\sim}{u}^T R(t)\underset{\sim}{u}) \, dt + \underset{\sim}{x}^T(t_f)F\underset{\sim}{x}(t_f) \tag{A.2a,b}$$

where $Q = L^T L \geq 0$, $R = R^T > 0$, $F = F^T \geq 0$. The control law has the form

$$\underset{\sim}{u} = - R^{-1}(t)B^T(t)P(t)\underset{\sim}{x} \tag{A.3}$$

where P(t) is the solution of the Riccati equation

$$- \dot{P}(t) = P(t)A(t) + A^T(t)P(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t) + Q(t)$$
$$P(t_f) = F \tag{A.4a,b}$$

In the following we present definitions and theorems concerned with controllability, observability and boundness of Riccati solutions, based on [18], [35], [36] and [37].

Definition A.1 [35]. The system defined by Eq. (A.1) is totally controllable (observable) on an interval (t_0, t_f) if it is completely controllable (observable) on every arbitrary subinterval of (t_0, t_f).

Definition A.2 [36]. The (i) controllability and (ii) observability Gramians of (A.1) are defined as

$$\text{(i)} \quad W(t,t_1) = \int_t^{t_1} \Phi(t,\tau)B(\tau)B^T(\tau)\Phi^T(t,\tau)\, d\tau \tag{A.5}$$

$$\text{(ii)} \quad M(t,t_1) = \int_t^{t_1} \Phi^T(\tau,t)B^T(\tau)B(\tau)\Phi(\tau,t)\, d\tau \tag{A.6}$$

respectively, where $\Phi(\ ,\ )$ is the transition matrix of $A(t)$.

Definition A.3 [35]. The (i) controllability (ii) observability matrices of (A.1) are defined, respectively, as

$$\text{(i)} \quad Q_C(t) = [\ N_0(t) \vdots\ N_1(t) \vdots \ldots\ldots \vdots\ N_{n-1}(t)] \tag{A.7a}$$

where

$$N_{k+1}(t) = -A(t)N_k(t) + \dot{N}_k(t)\ , \quad N_0(t) = B(t) \tag{A.7b}$$

(ii) $Q_0(t) = [\, R_0(t) \,\vdots\, R_1(t) \,\vdots\, \ldots\ldots \,\vdots\, R_{n-1}(t)]$ (A.8a)

where

$$R_{k+1}(t) = -A^T(t)R_k(t) + \dot{R}_k(t) \;, \quad R_0(t) = C^T(t) \tag{A.8b}$$

Definition A.4 [36]. A function of a real variable, f( ), is said to be analytic on D if f is an element of C^∞ and if for each t_0 in D there exists a positive real number ε_0 such that, for all t in $(t_0-\varepsilon_0, t_0+\varepsilon_0)$, f(t) is representable by a Taylor series about the point t_0.

Theorem A.1 [37]. The system defined by (A.1) is (i) completely controllable, (ii) completely observable over (t_0, t_f) if and only if

(i) $W(t_0, t_f) > 0$ (A.9a)

(ii) $M(t_0, t_f) > 0$ (A.9b)

Theorem A.2 [35]. The system defined by (A.1) is (i) totally controllable, (ii) totally observable on the interval (t_0, t_f) if and only if

(i) $Q_c(t)$ does not have rank less than n on any subinterval of (t_0, t_f).

(ii) $Q_0(t)$ does not have rank less than n on any subinterval of (t_0, t_f).

Theorem A.3 [36]. If the matrices A(t) and B(t) are analytic for every t then (A.1) is totally controllable (observable) for every t if and only if $Q_c(t)$ ($Q_0(t)$) has rank n almost everywhere.

Conclusion based on Theorem A.3 [36]. If A(t) and B(t) are analytic for every t and (A.1) is controllable (observable) at any point at all, it is totally controllable (observable) at every t.

Theorem A.4 [37]. If (i) the system defined by Eq. (A.1) is completely controllable, with $t^* \le t_f$, and (ii) the pair [A(t),L(t)] is completely obsevable, with$t^* \le t_f$, then

(i) $P(t) \le b_c(t,t^*)I$ for all $t < t^* \le t_f$ (A.10a)

(ii) If $P^{-1}(t)$ exists, then:

$$P^{-1}(t) \le b_o(t,t^*)I \quad \text{for all} \quad t < t^* \le t_f \tag{A.10b}$$

where

$$b_c(t,t^*) = \left\{ \frac{\lambda_M[W(t,t^*-)]}{\lambda_m[W(t,t^*)]} \right\}^2 \text{tr } [M_L(t,t^*)] + \frac{1}{\lambda_m[W(t,t^*)]} \tag{A.11a}$$

$$b_o(t,t^*) = \left\{ \frac{\lambda_M[M_L(t,t^*-)]}{\lambda_m[M_L(t,t^*)]} \right\}^2 \text{tr } [W(t,t^*)] + \frac{1}{\lambda_m[M_L(t,t^*)]} \tag{A.11b}$$

in which M_L is defined as M, Eq. (A.9b), where C is replaced by L and λ_m and λ_M denote minimum and maximum eigenvalues, respectively.

Theorem A.5 [18]. Consider the time-invariant version of (A.1), where A(t), B(t) and C(t) are replaced by $\bar{A}$, $\bar{B}$ and $\bar{C}$, respectively. Then, the complete controllability of $[\bar{A} + \alpha I, \bar{B}]$ follows from that of

$[\bar{A}, \bar{B}]$ (and vice-versa) and the complete observability of $[\bar{A} + \alpha I, \bar{C}]$ follows from that of $[\bar{A}, \bar{C}]$ (and vice-versa). To prove that, we observe the equivalence of the following four statements:

(1) $[\bar{A}, \bar{B}]$ is completely controllable.

(2) For constant $\underset{\sim}{w}$ and all t, $\underset{\sim}{w}^T e^{\bar{A}t}\bar{B} = \underset{\sim}{0}^T$ implies $\underset{\sim}{w} = \underset{\sim}{0}$.

(3) For constant $\underset{\sim}{w}$ and all t, $\underset{\sim}{w}^T e^{\alpha t} e^{\bar{A}t} = \underset{\sim}{w}^T e^{(\bar{A}+\alpha I)t}\bar{B} = \underset{\sim}{0}^T$ implies $\underset{\sim}{w} = \underset{\sim}{0}$.

(4) $[\bar{A} + \alpha I, \bar{B}]$ is completely controllable. By duality, we obtain the proof for the observability equivalence.

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