APPLICATION TO SUPERSONIC DIFFUSERS OF A ONE-DIMENSIONAL FLUID FLOW EQUATION OF THE PFAFFIAN TYPE

by

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CHAPTER I

INTRODUCTION

The equation which is investigated in this paper is the friction-loss equation for one-dimensional duct flow. The equation investigated is derived in reference 3 and is given by the expression

$$\frac{(1 - M_e^2)}{M_e\left(1 + \frac{\gamma - 1}{2} M_e^2\right)} dM_e + \frac{dA}{A} - 2\gamma M_e^2 f d\left(\frac{l}{D}\right) = 0 \tag{1-1}$$

Equation (1-1) is an equation which is a member of the group of equations of the general form

$$P_a dx + Q_a dy + R_a dz = 0 \tag{1-2}$$

where P_a, Q_a, and R_a are continuous functions of x, y, and z possessing continuous first partial derivatives with respect to x, y, and z.

From reference 1, if an equation of the form given by equation (1-2) is integrable there exists a function $\mu_a(x, y, z)$ such that the expression $\mu_a P_a dx + \mu_a Q_a dy + \mu_a R_a dz$ is exactly the derivative of some function say $f_a(x, y, z)$ and thus

$$\frac{\partial f_a}{\partial x} = \mu_a P_a, \frac{\partial f_a}{\partial y} = \mu_a Q_a, \frac{\partial f_a}{\partial z} = \mu_a R_a \tag{1-3}$$

Since by hypothesis P_a, Q_a, and R_a have continuous first partial derivatives with respect to x, y, and z then $\frac{\partial^2 f_a}{\partial x \partial y} = \frac{\partial^2 f_a}{\partial y \partial x}$ and from equation (1-3)

$$\mu_a \frac{\partial P_a}{\partial y} + P_a \frac{\partial \mu_a}{\partial y} = \mu_a \frac{\partial Q_a}{\partial x} + Q_a \frac{\partial \mu_a}{\partial x} \tag{1-4}$$

$$\mu_a \frac{\partial Q_a}{\partial z} + Q_a \frac{\partial \mu_a}{\partial z} = \mu_a \frac{\partial R_a}{\partial y} + R_a \frac{\partial \mu_a}{\partial y} \tag{1-5}$$

$$\mu_a \frac{\partial R_a}{\partial x} + R_a \frac{\partial \mu_a}{\partial x} = \mu_a \frac{\partial P_a}{\partial z} + P_a \frac{\partial \mu_a}{\partial z} \tag{1-6}$$

Multiply equation (1-4) by R_a, equation (1-5) by P_a, and equation (1-6) by Q_a and add the results to obtain after rearrangement,

$$P_a\left(\frac{\partial Q_a}{\partial z} - \frac{\partial R_a}{\partial y}\right) + Q_a\left(\frac{\partial R_a}{\partial x} - \frac{\partial P_a}{\partial z}\right) + R_a\left(\frac{\partial P_a}{\partial y} - \frac{\partial Q_a}{\partial x}\right) = 0 \,. \tag{1-7}$$

This equation states a necessary condition that equation (1-2) have a solution. Now it remains to prove that equation (1-7) is a sufficient condition.

Since the equation $P_a dx + Q_a dy = 0$ is always integrable if z is considered constant, there will be no loss in generality in assuming that $P_a dx + Q_a dy$ is an exact differential with respect to x and y. The solution of

$$P_a dx + Q_a dy = 0 \tag{1-8}$$

considering z as constant, may now be written in the form

$$f_a(x, y, z) + \varphi_a(z) = 0 \tag{1-9}$$

where φ_a represents an arbitrary function of z. Since $P_a dx + Q_a dy$ has been assumed to be an exact differential

$$\frac{\partial P_a}{\partial y} = \frac{\partial Q_a}{\partial x}, \; P_a = \frac{\partial f_a}{\partial x}, \; Q_a = \frac{\partial f_a}{\partial y} \tag{1-10}$$

Substitute the expressions for P_a and Q_a of equation (1-10) into equation (1-2). After rearranging the expression obtained for equation (1-2) the following equation is obtained:

$$\frac{\partial f_a}{\partial x} dx + \frac{\partial f_a}{\partial y} dy + \frac{\partial f_a}{\partial z} dz + \left(R_a - \frac{\partial f_a}{\partial z}\right) dz = 0 \tag{1-11}$$

Since $df_a = \frac{\partial f_a}{\partial x} dx + \frac{\partial f_a}{\partial y} dy + \frac{\partial f_a}{\partial z} dz$, equation (1-11) can be written as follows:

$$df_a + \left(R_a - \frac{\partial f_a}{\partial z}\right) dz = 0. \tag{1-12}$$

Equation (1-12) can be integrated if there exists a relation independent of x and y between $R_a - \frac{\partial f_a}{\partial z}$ and f_a; that is, if f_a and $R_a - \frac{\partial f_a}{\partial z}$ considered as functions of x and y(z constant) are dependent (for if there is not such a dependence then there will exist three unknowns in the equation). In reference 2, page 422, it says that for two functions of x and y, say f_1 and f_2, to be dependent f_1 and f_2 must satisfy the relation,

$$\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} = 0. \tag{1-13}$$

Let $f_1 = f_a$ and $f_2 = R - \frac{\partial f_a}{\partial z}$ then for there to be a solution of equation (1-12), equation (1-13) requires the functions f_a and $R - \frac{\partial f_a}{\partial z}$ to satisfy the relation

$$\frac{\partial f_a}{\partial x}\left(\frac{\partial R_a}{\partial y} - \frac{\partial^2 f_a}{\partial z\,\partial x}\right) - \frac{\partial f_a}{\partial y}\left(\frac{\partial R_a}{\partial x} - \frac{\partial^2 f_a}{\partial z\,\partial x}\right) = 0 \qquad (1\text{-}14)$$

Differentiate equation (1-10) to obtain

$$\frac{\partial^2 f_a}{\partial z\,\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f_a}{\partial y}\right) = \frac{\partial Q_a}{\partial z},\ \frac{\partial^2 f_a}{\partial z\,\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f_a}{\partial x}\right) = \frac{\partial P_a}{\partial x}\ . \qquad (1\text{-}15)$$

Substitute equation (1-15) into equation (1-14) to obtain

$$P_a\left(\frac{\partial R_a}{\partial y} - \frac{\partial Q_a}{\partial z}\right) - Q_a\left(\frac{\partial R_a}{\partial x} - \frac{\partial P_a}{\partial z}\right) = 0. \qquad (1\text{-}16)$$

Since from equation (1-10), $\frac{\partial P_a}{\partial y} - \frac{\partial Q_a}{\partial x} = 0$ it appears that equation (1-16) is the same as equation (1-7) with the signs changed; hence if equation (1-7) holds equation (1-12) can be expressed in terms of the two variables f_a and z and the result is in a solvable form. Since equations (1-12) and (1-2) are the same equation, this solution with f_a replaced by its value in terms of x and y will be the integral of equation (1-2). Equation (1-7) is thus a necessary and sufficient condition for equation (1-2) to be integrable.

Equation (1-7) was found to be a necessary and sufficient condition for equations of the form given by equation (1-2) to be integrable. Thus, upon applying equation (1-7) to equation (1-1) it is found that for equation (1-1) to be integrable the value of $\left(\frac{4\gamma M_e f}{A}\right)$ must be zero. Only for the trivial cases is the value of $\frac{4\pi M_c f}{A}$ zero.

Let us assume that the equation of the form given by equation (1-2) is not integrable. When an equation of the form given by equation (1-2) is not integrable, the literature gives two general methods which can be applied to obtain a solution.

The first of these methods is that of obtaining particular solutions. Particular solutions are obtained by assuming any second relation of x, y, and z and solving it simultaneously with equation (1-2). In order to obtain particular solutions of equations of the form given in equation (1-2) when equation (1-7) is not satisfied, assume any arbitrary relation

$$\psi(x, y, z) = 0 \tag{1-17}$$

and differentiate the relation to obtain,

$$\frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy + \frac{\partial\psi}{\partial z}\,dz = 0 \tag{1-18}$$

When the form ψ is specified, these two equations will determine z and dz in terms of x, y, dx, and dy (or generally, one of the variables and its differential in terms of the other two and their differentials); when they are substituted in equation (1-2) they make it of the form

$$M_a dx + N_a dy = 0 \tag{1-19}$$

The parameters M_a and N_a are functions of x and y, the values of which will depend upon the form of the chosen function ψ.

Now equation (1-19) may be solved and the result, containing an arbitrary constant will, together with equation (1-17), constitute a particular solution of equation (1-2). By giving all possible forms to ψ, every possible solution of equation (1-2) will be obtained. The fitting of all the examples of interest when applying this method to equation (1-1) would prove to be a long and tedious job.

By eliminating the differential element dz between the two equations (1-2) and (1-18), to obtain the form of equation (1-19), we can see the relationship the equation of condition (eq. (1-7)) imposes on the solution to equation (1-2). The variable z which occurs in equation (1-2) must be replaced by its value derived from equation (1-17) $\left[\psi(x, y, z) = 0\right]$. Now suppose the equation of condition, equation (1-7), is satisfied so that P_a, Q_a, and R_a are proportional to the differential coefficients of x, y, and z of some function; if this function be $\psi(x, y, z)$, then we have

$$\frac{1}{R_a}\frac{\partial\psi}{\partial z} = \frac{1}{Q_a}\frac{\partial\psi}{\partial y} = \frac{1}{P_a}\frac{\partial\psi}{\partial x} \qquad (1\text{-}20)$$

and the equation involving dx and dy is identically satisfied. There will thus, on this condition, be no other equation necessarily associated with the equation $\psi = 0$, or what is equivalent for this case, $\psi = C$; this by itself is sufficient for the solution of the differential equation, and any other equation associated with $\psi = C$ may be perfectly arbitrary (such as $\psi = 0$), for its expression will not enter into the differential equation when formed from these integral equations. If, however, the equation of condition between P_a, Q_a, and R_a is not satisfied, there is no function ψ such that the relations of equation (1-20) hold; and thus, $M_a dx + N_a dy = 0$, is not an identity but leads to an integral, the form of which is affected by the form of the arbitrary equation first written down and which must be associated with that equation in order to constitute the integral.

The second general method of obtaining a solution to equation (1-2) when the equation of condition is not satisfied is given in a memoir by Pfaff. If no single integral equation exists which is equivalent to the differential equation, then the method of determining the integral equivalent is called Pfaff's problem, and the differential equation itself is often called a Pfaffian equation.

Pfaff showed that an integral equivalent of a total differential equation, containing $(2n_a - 1)$ or $2n_a$ variables, can always be constituted by a system of integral equations not more than n_a in number. A full discussion of Pfaff's problem and its result can be found in "Theory of Differential Equations," vol. I by A. R. Forsyth. Here a discussion of only the three variables case will be given.

Assume the equation of condition (eq. (1-7)) is not satisfied for equation (1-2). Then let

$$P_a dx + Q_a dy + R_a dz = du_a + v_a dw_a \qquad (1\text{-}21)$$

identically. Should it be possible that a u_a, v_a, and w_a exist which are functions of x, y, and z then P_a, Q_a, and R_a must be given by

$$\left.\begin{aligned} P_a &= \frac{\partial u_a}{\partial x} + v_a \frac{\partial w_a}{\partial x} \\ Q_a &= \frac{\partial u_a}{\partial y} + v_a \frac{\partial w_a}{\partial y} \\ R_a &= \frac{\partial u_a}{\partial z} + v_a \frac{\partial w_a}{\partial z} \end{aligned}\right\} \qquad (1\text{-}22)$$

Let P'_a, Q'_a, and R'_a be given by

$$\left.\begin{aligned} P'_a &= \frac{\partial Q_a}{\partial z} - \frac{\partial R_a}{\partial y} \\ Q'_a &= \frac{\partial R_a}{\partial x} - \frac{\partial P_a}{\partial z} \\ R'_a &= \frac{\partial P_a}{\partial y} - \frac{\partial Q_a}{\partial x} \end{aligned}\right\} \qquad (1\text{-}23)$$

Upon substitution of equation (1-22) into equation (1-23)

P'_a, Q'_a, and R'_a are expressed as

$$\left.\begin{aligned} P'_a &= \frac{\partial v_a}{\partial z}\frac{\partial w_a}{\partial y} - \frac{\partial v_a}{\partial y}\frac{\partial w_a}{\partial z} \\ Q'_a &= \frac{\partial v_a}{\partial x}\frac{\partial w_a}{\partial z} - \frac{\partial v_a}{\partial z}\frac{\partial w_a}{\partial x} \\ R'_a &= \frac{\partial v_a}{\partial y}\frac{\partial w_a}{\partial x} - \frac{\partial v_a}{\partial x}\frac{\partial w_a}{\partial y} \end{aligned}\right\} \qquad (1\text{-}24)$$

Upon multiplying equations (1-24) by $\frac{\partial w_a}{\partial x}$, $\frac{\partial w_a}{\partial y}$, and $\frac{\partial w_a}{\partial z}$, respectively, and adding the result and by $\frac{\partial v_a}{\partial x}$, $\frac{\partial v_a}{\partial y}$, and $\frac{\partial v_a}{\partial z}$, respectively, and adding the result, the two following partial differential equations involving v_a and v_a are obtained.

$$\left.\begin{aligned} P'_a \frac{\partial w_a}{\partial x} + Q'_a \frac{\partial w_a}{\partial y} + R'_a \frac{\partial w_a}{\partial z} = 0 \\ P'_a \frac{\partial v_a}{\partial x} + Q'_a \frac{\partial v_a}{\partial y} + R'_a \frac{\partial v_a}{\partial z} = 0 \end{aligned}\right\} \tag{1-25}$$

Now

$$\left.\begin{aligned} &\left(P_a - \frac{\partial u_a}{\partial x}\right)P'_a + \left(Q_a - \frac{\partial u_a}{\partial y}\right)Q'_a + \left(R_a - \frac{\partial u_a}{\partial z}\right)R'_a \\ &\qquad = v_a\left(P'_a \frac{\partial w_a}{\partial x} + Q'_a \frac{\partial w_a}{\partial y} + R'_a \frac{\partial w_a}{\partial z}\right) = 0 \end{aligned}\right\} \tag{1-26}$$

and thus the following expression involving u_a is obtained.

$$P'_a \frac{\partial u_a}{\partial x} + Q'_a \frac{\partial u_a}{\partial y} + R'_a \frac{\partial u_a}{\partial z} = P_a P'_a + Q_a Q'_a + R_a R'_a \tag{1-27}$$

As the original equation (eq. (1-2)) was assumed not to satisfy the equation of condition then the right-hand side of equation (1-27) does not vanish and thus u_a does not satisfy the same general type of equation as v_a and w_a that is

$$P'_a \frac{dh_a}{dx} + Q'_a \frac{dh_a}{dy} + R'_a \frac{dh_a}{dz} = 0 \tag{1-28}$$

Every integral of the equation in h_a is a function of two independent functions say,

$$\left.\begin{aligned} \alpha_a(x, y, z) &= \text{constant} \\ \beta_a(x, y, z) &= \text{constant} \end{aligned}\right\} \qquad (1\text{-}29)$$

Accordingly, both v_a and w_a are only functions of α_a and β_a.

Suppose that w_a equal a function of α_a and β_a, say $w_a = \alpha_a$. Then make

$$\alpha_a(x, y, z) = a, \text{ a constant} \qquad (1\text{-}30)$$

which causes equation (1-21) to become

$$P_a dx + Q_a dy + R_a dz = du_a \qquad (1\text{-}31)$$

that is, for this relation among the variables $P_a dx + Q_a dy + R_a dz$ is a perfect differential. Accordingly, we use the relation $\alpha_a(x, y, z) = a$ to remove one of the variables and its differential element, say z and dz from $P_a dx + Q_a dy + R_a dz$; the resulting expression is a perfect differential, say $d\phi_a(x, y, a)$. In $\phi_a(x, y, a)$ we reinsert the variable z and remove the constant (a) by substituting $a = \alpha_a(x, y, z)$; and then $\phi_a(x, y, a)$ becomes u_a. Now v_a can be obtained from any of the equations in equation (1-22) and thus since u_a, v_a, and w_a are known then equation (1-21) holds, that is

$$du_a + v_a dw_a = 0 \qquad (1\text{-}32)$$

Now the solution of equation (1-32) can be represented by at most, three pairs of integral equations

$$\left\{\begin{array}{ll} (1) & u_a = \text{constant},\ v_a = \text{constant} \\ (2) & u_a = \text{constant},\ v_a = 0 \\ (3) & \psi(u_a, v_a) = 0,\ v_a \dfrac{\partial \psi}{\partial u_a} - \dfrac{\partial \psi}{\partial v_a} = 0 \end{array}\right\}$$

The above is very elegant mathematically but, as the expressions for u_a, v_a, and v_a obtained for equation (1-16) are so complicated and awkward to use, the present method was developed to perform the integrated friction losses through a supersonic inlet.

SYMBOLS

a	speed of sound $\left(= \sqrt{\gamma g R t}\right)$
A	area
A^*	total area at $M = 1.0$
B	width of surface
C_F	average friction factor $\left(= \frac{\tau_{Av.}}{q_l}\right)$
C_f	point friction factor
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
D	hydraulic diameter $\left(= \frac{4 \text{ cross sectional area}}{\text{perimeter}}\right)$
f	effective friction factor $\left(= \frac{\tau_{Av.}}{q_o}\right)$
f_a	function of x, y, and z
$f(M)$	Mach number function $\left(= \frac{M\left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/2}}{(1 + \gamma M^2)}\right)$
$f_d(M)$	Mach number function $\left(= \frac{M^2}{1 + \frac{\gamma - 1}{2} M^2} e^{\frac{2}{(\gamma+1)M^2}}\right)$
g	gravity constant $\left(= 32.2 \frac{\text{ft}}{\text{sec}^2}\right)$
H	total pressure
$\bar{H}$	area weighted total pressure
l	axial length in the x direction
m	mass flow per unit area in the boundarylayer

m_b	mass flow in the boundary layer
m_c	mass flow in the core
m_l	mass flow per unit area in the free stream
m_{total}	mass flow in the inlet
M	Mach number
M_a	function of x and y
N	velocity profile parameter $\left[\frac{V}{V_l} = \left(\frac{y}{\delta}\right)^{1/N}\right]$
N_a	function of x and y
p	static pressure
P	perimeter
P_a	continuous function of x, y, and z
q	dynamic pressure $\left(= \frac{\gamma}{2} pM^2\right)$
Q_a	continuous function of x, y, and z
r	adiabatic wall recovery factor (= 0.896 for turbulent flow and 0.851 for laminar flow)
R	gas constant $\left(= 53.3 \frac{^{\circ}F}{ft}\right)$
R_x	Reynolds number based on x
R_θ	Reynolds number based on θ
R_a	continuous function of x, y, and z
S	slope of a line
t	static temperature
T	stagnation temperature
u_a, v_a, w_a	transformed coordinates
V	velocity parallel to the inlet wall
x, y, z	rectangular coordinates

X	distance in x direction
γ	ratio of the specific heats $\left(= \frac{c_p}{c_v}\right)$
δ	boundary-layer thickness
δ^*	displacement thickness $\delta^* = \int_0^\delta \left(1 - \frac{\rho_b V_b}{\rho_l V_l}\right) dy$
α_a, β_a	functions of x, y, and z
η	$= \int_0^1 \frac{\rho_b V_b^3}{\rho_l V_l^3} d\left(\frac{y}{\delta}\right)$
θ	momentum thickness $\theta = \int_0^\delta \frac{\rho_b V_b}{\rho_l V_l}\left(1 - \frac{V_b}{V_l}\right) dy$
θ_ϵ	energy thickness $\theta_\epsilon = \int_0^\delta \frac{\rho_b V_b}{\rho_l V_l}\left(1 - \frac{V_b^2}{V_l^2}\right) dy$
μ_a	function of x, y, and z
ρ	mass density $\left(= \frac{p}{gRt}\right)$
τ	shear stress at the wall
ϕ	total momentum per unit area in the boundary layer
ϕ_l	total momentum per unit area in the free stream
φ	$= \int_0^1 \rho_b V_b^2 d\left(\frac{y}{\delta}\right)$
φ_l	$= \rho_l V_l^2$
φ_a	arbitrary function of z
Φ	total momentum in the inlet $(= p + \gamma p M^2) A_T$
Φ_b	total momentum in the boundary layer
Φ_c	total momentum in the core
ψ	function of x, y, and z

Subscripts:

A	area
Av.	average value
b	boundary layer
c	core
e	effective
end	end
i	incompressible
inlet	conditions at the inlet leading edge
l	at the edge of the boundary layer
lam.	laminar
mid	middle of the element
N	element number
O	stagnation conditions
P	point value
ref.	upstream of the inlet
s	shock
T	total
turb.	turbulent flow
X	based on X or at station X
1	average of the element
2	end of the element
θ	based on the momentum thickness, θ

CHAPTER II

NUMERICAL METHOD DEVELOPED FOR THEORETICAL FRICTION CALCULATIONS

A brief investigation indicated that a continuous integration along a supersonic diffuser's length to obtain the Mach number losses due to simultaneous effects of shocks, area change, and friction is next to impossible due to the nonlinearity of the differential equation, equation (1-1), involved. Thus, a simplified flow model is assumed which permits the accomplishment of these integrations separately in finite steps. For the purpose of one-dimensional analysis, which is also the effect of the size of the steps, the steps or elements are chosen as 0.25 l/D, 0.5 l/D, and 1.0 l/D in length. These stations are located by integrating a curve of l versus $(1/D)$ and plotting the results against l. On the curve thus obtained, the stations are chosen at every 0.25 l/D, 0.5 l/D, or 1.0 l/D and the corresponding axial locations taken. The model used for friction calculations is assumed to be two-dimensional and a shock pattern was obtained using shock tables and the actual inlet walls as the rebounding surface for the shocks (fig. 2 gives model wall outline and shock diagram). Also assumed is the approximate equality of total and shock areas. Upon examining figure 1 it can be seen that $A_T = A_S + \delta_A^*$ and thus if approximate equality of A_T and A_S is not the case then plate flow friction factors developed in equations (A-1) through (A-11), appendix A cannot be used.

Based on the shock diagram assumption on area, an area weighted core total pressure recovery versus distance from the inlet leading edge can be obtained. A one-dimensional shock Mach number (without boundary-layer effects) can be calculated and plotted versus distance from the leading edge of the inlet by using the area weighted total pressure recovery along the inlet, area change along the inlet, and the assumption of constant mass flow along the inlet. Thus having determined the effect of the shocks on the flow, the effects of friction are handled using one-dimensional flow equations of references 5 and 6. Using the combined effects of shocks and friction up to a station, it is possible to determine the boundary layer and displacement areas using assumed values of δ^*/θ, δ^*/δ, and θ_c/δ. The parameters δ^*/θ, δ^*/δ, and θ_c/δ are determined for laminar flow using a method put forth in reference 7, but instead of using the temperature profiles of reference 7 the temperature profiles put forth in reference 8 are used with the laminar recovery factor (0.851). The parameters δ^*/θ, δ^*/δ, and θ_c/δ for turbulent flow are determined using a method put forth in reference 8 (assuming reasonable values for the turbulent boundary-layer velocity profile parameter N). Adiabatic wall curves of (δ^*/θ), (δ^*/δ), and (θ_c/δ) are presented in figures 3, 4, and 5, respectively.

In essence there are two main equations plus shock losses that must be satisfied to obtain the friction losses and area change effects throughout an element. The two equations are (from app. A)

$$f_d(M_2) = \frac{f_d(M_1)}{e^{\left(\frac{2\gamma}{\gamma + 1} f \frac{l}{D}\right)}} \tag{A-12}$$

and

$$\frac{f(M_b) - f(M_c)}{f(M_c)} = 1 - \frac{(1 + \gamma M_c^2) + \left(1 - \gamma M_c^2 \frac{\theta}{\delta^*}\right)\frac{\delta_A^*}{A_s}}{(1 + \gamma M_c^2)\left(1 + \frac{(r - 1)\frac{\gamma - 1}{2} M_c^2 \frac{\theta_c}{\delta}}{\frac{\delta^*}{\delta}\left(1 + \frac{\gamma - 1}{2} M_c^2\right)} \frac{\delta_A^*}{A_s}\right)^{1/2}} \tag{A-39}$$

It remains to determine the values of the parameters which are used in equations (A-12) and (A-39) and the manner these equations are utilized.

The parameters δ^*/θ, θ_c/δ, and δ^*/δ needed in equation (A-39) are determined if the core Mach number and type of flow is known. Thus it remains to determine the values of the core Mach number, M_c, and δ_A^*/A_s that are needed in the equation. Using the fact that the total area at a station equals the shock area plus the displacement area it is concluded that any change in core properties except that caused by shocks and total area change is caused by the displacement area, δ_A^*. If the displacement area was zero the core Mach number would be that of the shock diagram imposed on the total area, A_T, but this not being the case the core Mach number must be altered by the effect of decreasing the total area to that of the shock area. From figure 6 obtain A_T/A^* (= A/A^* in figure 6) and A_s/A^* (= A/A^* in figure 6) corresponding to the core Mach number (with $\delta_A^* = 0$) and the altered core Mach number assumed. Using

A_T/A^* the value of δ_{A^*}/A_c is obtained corresponding to each assumed altered core Mach number.

$$\frac{\delta_{A^*}}{A_c} = \frac{\frac{A_T}{A^*}}{\frac{A_c}{A^*}} - 1 \qquad (2\text{-}1)$$

Substitute the δ_{A^*}/A_c values of equation (2-1) and the corresponding altered M_c into equation (A-39). From these substitutions a curve of M_c versus M_e can be obtained for the station in question using figure 7 which gives curves of $\left[f(M_e) - f(M_c)\right]/f(M_e)$ versus M_c for constant values of $(M_c - M_e)$. The accumulation of shock losses and friction losses up to the station in question must result in a point on the curve of M_c versus M_e obtained corresponding to equation (A-39).

The change in M_c and M_e due to the accumulation of shock and friction losses is expressed by

$$M_c = M_{c_{inlet}} - \Delta M_{c_{\substack{\text{area change}\\ \text{and shock}\\ \text{losses}}}} - \Delta M_{c_{\delta^*_A}} \qquad (2\text{-}2)$$

and

$$M_e = M_{e_{inlet}} - \Delta M_{e_{\substack{\text{area change}\\ \text{and shock loss}}}} - \Delta M_{e_{friction}} \qquad (2\text{-}3)$$

The element effective Mach number losses $(\Delta M_{e_{\text{area change and shock loss}}})$ due to shock losses and area changes are obtained using the flow parameter $\rho V/\rho_0 a_0$ $\left(\text{given by the relationship } \rho V/\rho_0 a_0 = \frac{M\left(\frac{p}{H}\right)}{\sqrt{\frac{t}{T_0}}}\right)$ and the relationship

$$\left(\frac{\rho_e V_e}{\rho_0 a_0}\right)_{\substack{\text{end of}\\ \text{element}}} = \frac{\left[\frac{\rho_e V_e}{\rho_0 a_0}\,\frac{A_T}{A_{T_{inlet}}}\,\frac{\bar{H}}{H_{ref}}\right]_{\text{beginning of element}}}{\left[\frac{A_T}{A_{T_{inlet}}}\,\frac{\bar{H}}{H_{ref}}\right]_{\text{end of element}}} \qquad (2\text{-}4)$$

The value of $(\rho_e V_e/\rho_0 a_0)_{\text{end of element}}$ obtained in equation (2-4) corresponds to the effective Mach number at the end of the element. The effective Mach number at the beginning of the element is assumed known and thus the M_e loss due to area changes and shock losses can be obtained. The resulting Mach number losses calculated due to shock losses are larger than in actuality because the free stream total pressure loss through the shock is imposed all across the boundary layer.

The effective Mach number loss due to friction ($\Delta M_{e_{friction}}$) is obtained from John R. Henry's one-dimensional equation (ref. 6), equation (A-12), appendix A, for effective Mach number loss in a constant area duct. Henry's equation is a solution of the section of the nonlinear differential equation

$$\frac{(1 - M_e^2)}{M_e\left(1 + \frac{\gamma - 1}{2} M_e^2\right)}\, dM_e + \frac{dA}{A} - 2\gamma M_e^2 f d\left(\frac{l}{D}\right) = 0 \qquad (1\text{-}1)$$

which excludes the area change term. The nonlinear differential equation expressed in equation (1-1) if solved would simultaneously take into account the effective Mach number changes due to area change and friction but would exclude shock losses. Thus, the present method of obtaining the combined effects of friction and area changes on the effective Mach number would also give the results that could be obtained from the solution of equation (1-1) if $\bar{H}/H_{ref}$ was assumed equal to one.

In order to apply equation (A-12) the friction coefficient is needed. The friction factor C_F for element one is obtained using station one assumed values of M_c, M_e, and δ_A^*/A_s (obtained from the curve constructed from equation (A-39) for station one) to calculate $R_{\theta_{end}}$.

The momentum thickness, θ_{end}, at the end of the element is obtained from

$$\theta_{end} = \frac{\left(\frac{\delta_A^*}{A_s}\right)_{end}\left(\frac{A_T}{A_{T_{inlet}}}\right)_{end}\frac{A_{T_{inlet}}}{P_{inlet}}}{\left(\frac{\delta_A^*}{\theta_A}\right)_{end}\left[1+\left(\frac{\delta_A^*}{A_s}\right)_{end}\right]\left(\frac{P}{P_{inlet}}\right)_{end}} \qquad (2\text{-}5)$$

The parameter $(\delta_A^*/\theta_A)_{end}$ is assumed equal to δ^*/θ corresponding to the assumed station one M_c and N if the flow is turbulent and just assumed station one M_c if the flow is laminar. The value of $R_{\theta_{end}}$ is then given by

$$R_{\theta_{end}} = \left(\frac{R_\theta}{\theta \bar{H}}\right)_{end} \theta_{end} H_{ref}\left(\frac{\bar{H}}{H_{ref}}\right)_{end} \qquad (2\text{-}6)$$

where $(R_\theta/\theta\bar{H})$ comes from figure 8 using T_t and assumed station one or element one end point M_c.

The value of C_F/C_{F_i} for element one is obtained from Eckert's reference temperature method using for turbulent flow (ref. 9) the element one midpoint values of M_c and T_t along with $R_{\theta_{end}}$

(adiabatic wall curves of C_F/C_{F_i} are given in figure 9 for turbulent flow and figure 10 for laminar flow) and just element midpoint values of M_c and T_l for laminar flow (ref. 10). Using the fact that $C_F/C_{F_i} = \theta/\theta_i$ (for laminar and turbulent flow, app. A) $R_{\theta_{end}}$ can be converted to $R_{\theta_{i_{end}}}$ by

$$R_{\theta_{i_{end}}} = \frac{R_{\theta_{end}}}{\frac{C_F}{C_{F_i}}} \qquad (2\text{-}7)$$

Using $R_{\theta_{i_{end}}}$ an average incompressible friction coefficient for element one can be obtained from figure 11 corresponding respectively as to whether the flow is laminar or turbulent.

For all elements following the initial element, the Reynolds number is obtained from figure 8 and the element midpoint values of M_c, T_l, and $\bar{H}/H_{ref}$, etc. Equations (2-5), (2-6), and (2-7) become, respectively

$$\theta_{mid.} = \frac{\left(\frac{\delta_A^*}{A_s}\right)_{mid.}\left(\frac{A_T}{A_{T_{inlet}}}\right)_{mid.}\frac{A_{T_{inlet}}}{P_{inlet}}}{\left(\frac{\delta_A^*}{\theta_A}\right)_{mid.}\left[1 + \left(\frac{\delta_A^*}{A_s}\right)_{mid.}\right]\left(\frac{P}{P_{inlet}}\right)_{mid.}} \qquad (2\text{-}8)$$

$$R_{\theta_{mid.}} = \left(\frac{R_\theta}{\theta\bar{H}}\right)_{mid.}\theta_{mid.}H_{ref}\left(\frac{\bar{H}}{H_{ref}}\right)_{mid.} \qquad (2\text{-}9)$$

$$Re_{i_{mid.}} = \frac{(Re)_{mid.}}{\left(\frac{C_F}{C_{F_i}}\right)_{mid.}} \qquad (2\text{-}10)$$

The friction coefficient corresponding to the Reynolds number $Re_{i_{mid.}}$ of elements other than the initial element is an incompressible local friction coefficient C_{f_i} (taken from the curve of incompressible local friction coefficient for laminar, or turbulent, figure 11, flows as the case may be). The resulting point friction coefficient is assumed equal to the average incompressible friction coefficient for the element in question.

For any element the resulting incompressible friction coefficient can then be converted to a compressible value by multiplying by the $C_F/C_{F_i} = C_f/C_{f_i}$ obtained during the calculation of $Re_{i_{end}}$ for element one or $Re_{i_{mid.}}$ for elements following element one. Substitute the C_f or C_F thus obtained along with the assumed element midpoint values of M_c, M_e, and δ_A^*/A_s into equation (A-16), appendix A, derived for the conversion of C_F to J. R. Henry's (ref. 6) effective friction factor, f.

Insert the value of f obtained above in equation (A-12) along with the assumed midpoint value of effective Mach number and the effective Mach number loss due to friction in the element can be calculated. The element midpoint value of effective Mach number must be assumed because J. R. Henry's (ref. 6) equation for effective Mach number loss due to friction, equation (A-12), is for a constant area duct and thus an average M_e must be assumed for the element so as to eliminate the effect of the area change in the inlet. Figures 12 and 13, obtained using equation (A-12) appendix A, gives the effective Mach number loss for an element with $l/D = 0.25$, $l/D = 0.5$ and 1.0, versus element inlet M_e (in this case assumed element midpoint M_e) and constant friction factor, f, values.

The value of the effective Mach number, $(M_e)_{end}$, at the end of the element is given by equation (2-3). The $\Delta M_{e_{\text{area change and shock loss}}}$ is obtained corresponding to equation (2-4) and the M_e at the end of the previous element. The $\Delta M_{e_{\text{friction}}}$ is obtained from figures 12 or 13 and the assumed element midpoint value of M_e along with the calculated friction factor, f. Repetition of this procedure gives a curve of $(M_c)_{end}$ assumed versus $(M_e)_{end}$ calculated and the intersection of this curve with the curve of M_c versus M_e obtained from equation (A-39) gives the correct M_c, M_e, and δ_A^*/A_s for the station in question.

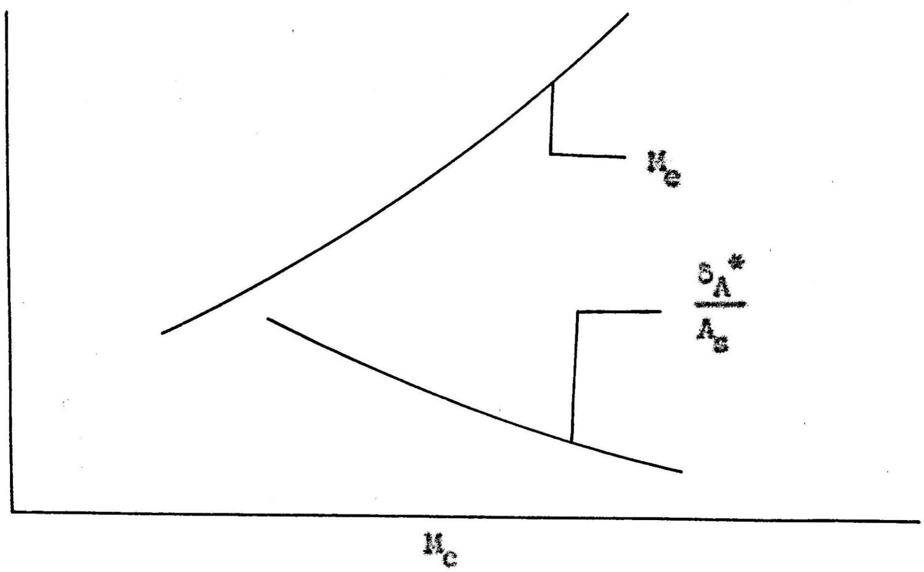
Point transition from laminar to turbulent flow is assumed to occur in conjunction with the assumption of constant effective Mach number M_e, through transition. After transition a new curve of M_c versus M_e can be calculated using equations (A-39) and (2-1) along with a reasonable assumption for the value of the velocity profile parameter N. Using the assumption of constant effective Mach number through transition the turbulent value of M_c (and thus δ_A^*/A_s) can be obtained from the turbulent M_c versus M_e curve.

Appendix B gives a general outline for the engineer to follow in his calculations.

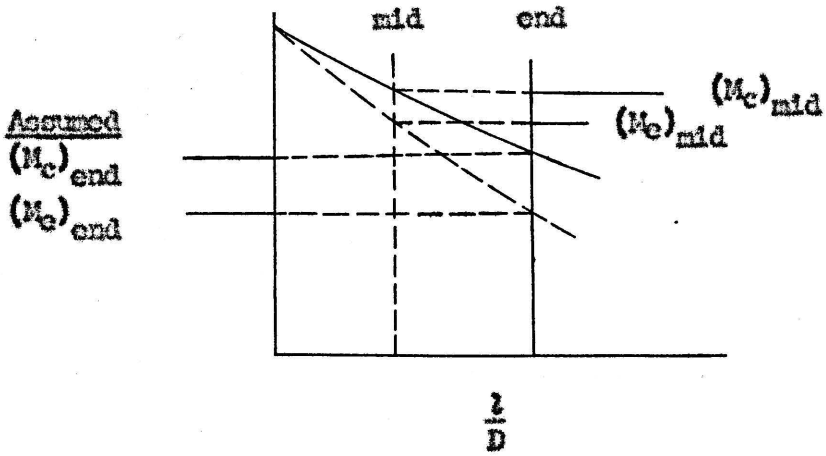
CHAPTER III

ITERATIVE PROCEDURE FOR SOLUTIONS

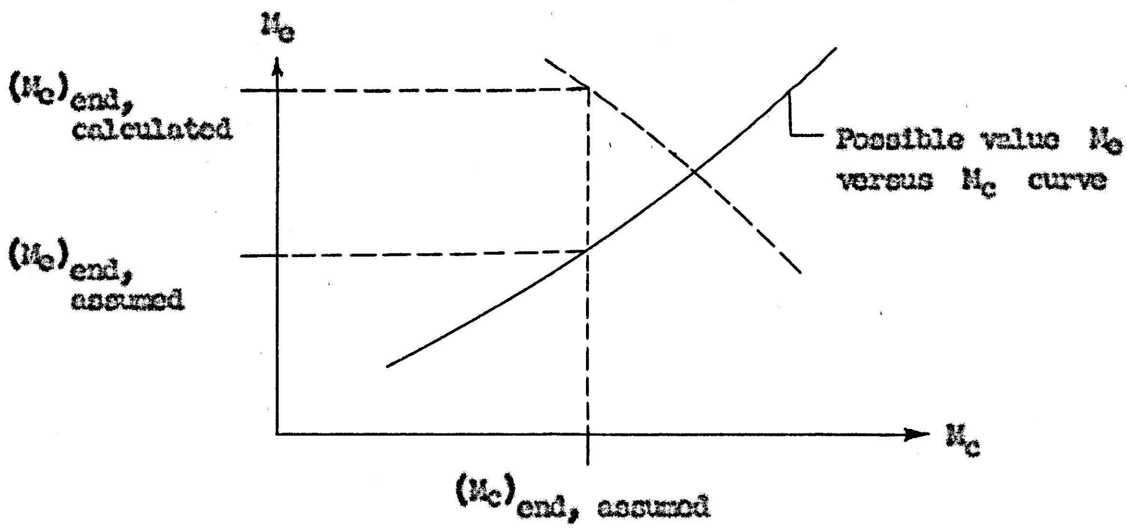
The possible value M_c versus M_e curve is generated, corresponding to equation (A-39), for the end of the element being calculated. At the same time the possible values of (δ_A^*/A_s) versus M_c is generated corresponding to equation (2-1).



Then a value of M_c for the end of the element in question is assumed and a curve of this assumed M_c, and the corresponding δ_A^*/A_s and M_e (taken from the possible value M_c versus M_e curve and the possible value δ_A^*/A_s versus M_c curve) versus integrated l/D is drawn.



The element midpoint values of M_c, M_e, and δ_A^*/A_s are then picked off and the element average friction factor determined. Then using equation (A-12) the ΔM_e due to friction is calculated. The M_e for the end of the element is then given by equation (2-3). The resulting $(M_e)_{end,\ calculated}$ is then plotted versus the assumed $(M_c)_{end}$ on the same plot as that which has the curve of the possible values of M_c and M_e.



If the resulting point plotted on the possible value M_c versus M_e plot does not lie on the possible value M_c versus M_e curve then a new M_c must be chosen and the procedure repeated. The shape of the possible M_c versus M_e curve showed to be (for all cases considered) practically a straight line. Thus by representing the possible value curve by a straight line, a perpendicular can be drawn from the first $(M_c)_{end,\ assumed}$ versus $(M_e)_{end,\ calculated}$ point to the straight line representing the possible value curve. As the slope of the possible value curve is known and thus the slope of its perpendicular then the intersection of the perpendicular line from the calculated point with the representative line is given by

$$(M_c)_2 = (M_e)_1 + \frac{(M_e)_{1\ assumed} - (M_e)_{1\ calculated}}{S_{\substack{perpendicular \\ line}} - S_{\substack{representative \\ line}}} \qquad (3\text{-}1)$$

The representative line is made to pass through the $\left[(M_c)_1, (M_e)_1\right]$ point assumed for the first calculation. The $(M_c)_2$ given by the above formula is the M_c to be assumed for the new calculation.

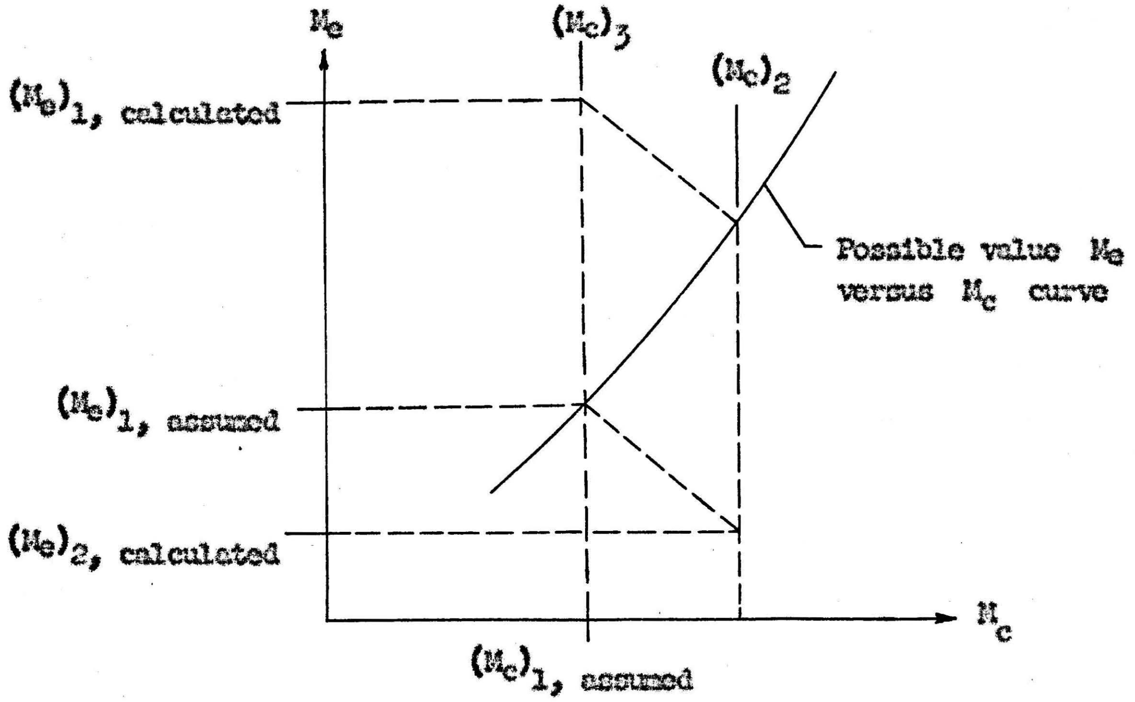
The above iterative method is observed to converge to the intersection of the possible value M_c versus M_e curve and the calculated M_c versus M_e curve. The reason for this is inherent in the shape of the two curves over the range of Mach numbers considered.

(1) The possible value M_c versus M_e curve is always of positive slope.

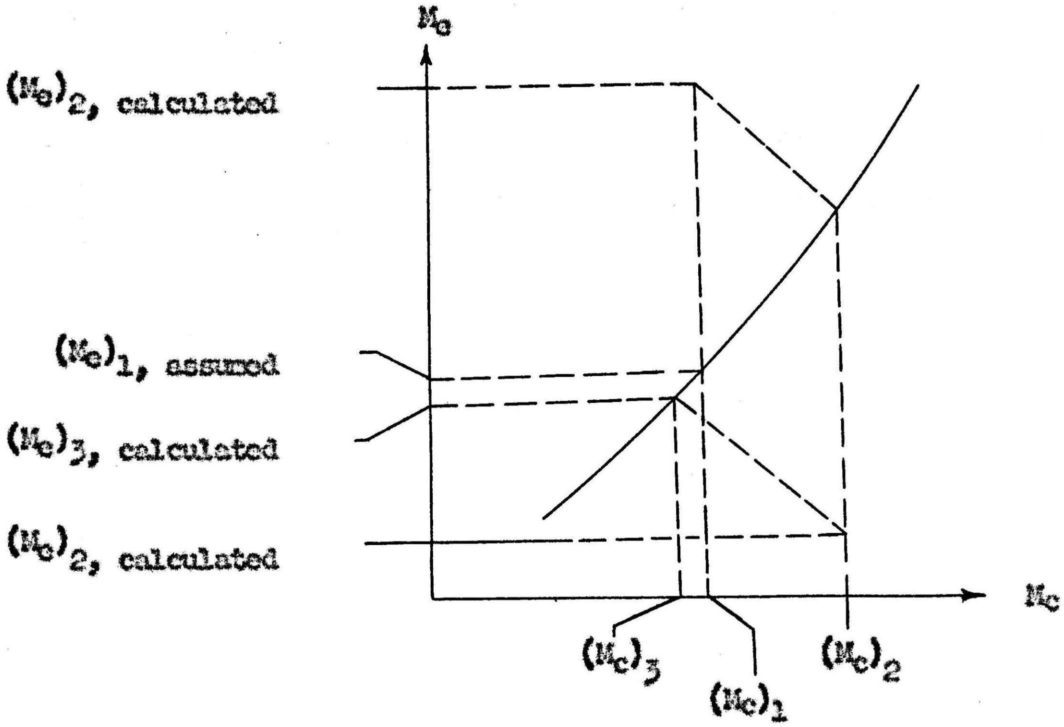
(2) The possible value M_c versus M_e curve can be approximated by a straight line.

(3) The calculated M_c versus M_e curve always has a negative slope and is a smooth curve.

Utilizing facts (1), (2), and (3) about the two curves in question the method discussed converges on the intersection of the two curves if $(M_c)_3 > (M_c)_1$ but



if $(M_c)_3 \leq (M_c)_1$ there is no guarantee of convergence to the intersection.
If $(M_c)_3 < (M_c)_1$ then the point



$[(M_c)_2, (M_e)_{2\text{ cal.}}]$ lies on the opposite side of the M_c versus M_e possible value curve from that of the $[(M_c)_1, (M_e)_{1\text{ cal.}}]$ point. Convergence was then accomplished by the joining of the $(M_e)_{1\text{ cal.}}$ and $(M_e)_{2\text{ cal.}}$ points with a straight line and obtaining its intersection with the straight line representing the possible value curve. The $(M_c)_3$, $(M_c)_4$, etc., would then be given by

$$(M_c)_n = (M_c)_{(n-1)} + \frac{(M_e)_{1\text{ assumed}} - (M_e)_{1\text{ cal.}}}{\dfrac{(M_e)_{2\text{ cal.}} - (M_e)_{1\text{ cal.}}}{(M_c)_2 - (M_c)_1} - S_{\substack{\text{representative}\\ \text{line}}}} \qquad (3\text{-}2)$$

which was observed to converge to the intersection of the two curves.

The previous discussion for choosing the M_c so as to converge to the intersection was for the case in which the first calculated point lay above the possible value M_c versus M_o curve but a similar discussion holds for the case in which the first calculated (M_c, M_o) point lies below the possible value curve. The difference is in that convergence is accomplished if $(M_c)_3 < (M_c)_1$ and if not the second method must be applied.

CHAPTER IV

DISCUSSION AND CONCLUSIONS

Discussion

Figure 2 gives outlines of four inlet models, I, II, III, and IV, which have 1, 2, 3, and 4 shocks, respectively, and give approximately the same Mach number change, 4.0 to 3.0, due to shocks. Friction calculations were conducted on models I, II, III, and IV using the present method but before performing the friction calculations for all four models calculations were conducted for model II to determine the area change effect and thus the element size, on the calculations. Table I gives the necessary information for the friction calculations of all four models.

In order to determine the three-dimensional effects upon the friction calculations, calculations of δ_A^*/A_c, M_c, and M_e were conducted for model II corresponding to increment sizes of $l/D = 0.25$, 0.5, and 1.0. Curves, for the three increment sizes considered for model II, of δ_A^*/A_e versus l are presented in figure 14 and curves of M_e and M_c versus l are presented in figure 15. For all the present calculations transition from laminar to turbulent flow was assumed to occur at the axial station where $l/D = 1.0$. Figures 14 and 15 show the three-dimensional effects near the leading edge of the model to be significant but within the accuracy of the present method they disappear at the axial station where element stations for the three cases considered coincide. As the exit values of δ_A^*/A_c, M_c, and M_e are independent of element size, and thus also the area change effects, the element size chosen for the friction calculations of models I, II, III, and IV was $l/D = 1.0$.

Plots of $\delta_A{}^*/A_s$ versus l for models I, III, and IV are presented in figure 16 and plots of M_o and M_c versus l for models I, III, and IV are presented in figures 17(a) and (b). Figure 18 gives the comparison of the calculated model (I through IV) exit values of $\delta_A{}^*/A_s$ with experimental model exit values of $\delta_A{}^*/A_s$ obtained in the 9- by 9-inch Mach number 4.06 supersonic wind tunnel at the National Aeronautics and Space Administration's Langley Research Center.

Conclusions

Within the accuracy of the present method of calculation the size of the increment or element chosen for the friction calculations is not of importance except close to the leading edge. The importance in this region is due to the large changes in friction factor, f, very close to the leading edge and not to the area change and shock effects upon increment sizes chosen. Thus, the size of increment chosen for the friction calculations depends upon the accuracy wanted in the region of the inlet very close to its leading edge. Additional calculations using element sizes of the order of the total (l/D) of the inlets considered showed that the above conclusions must be restricted in that the element size chosen must be small as to the overall size of the inlet. Figure 18 shows the calculated model III and IV exit values of $(\delta_A{}^*/A_s)_{exit}$ differ by about 10 percent from their corresponding experimental values. As the shock strength becomes larger, in passing from model IV to model I, the difference (percentagewise) between calculated values and experimental values of $(\delta_A^*/A_S)_{exit}$ becomes larger and larger. This deviation between calculated values and experimental values of δ_A^*/A_S is caused by shock boundary-layer interaction effects.

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APPENDIX A

DEVELOPMENT OF THEORY

In reference 10 for laminar flow and reference 9 for turbulent flow, it is shown that the values calculated for C_F/C_{F_i} or C_f/C_{f_i} are found to correspond best to data if Eckert's reference temperature method for laminar flow (ref. 10) and Van Driest's 1954 method for turbulent (ref. 9) flow calculations. However, for consistency as well as expediency, Eckert's reference temperature method for laminar flow and his reference temperature method for turbulent flow (ref. 9) are used in this paper.

Since by definition

$$C_F = \int_0^{\delta} \frac{(\rho_l V_l - \rho_b V_b)}{q_l XB} V_b dy \qquad \text{(A-1)}$$

then

$$C_{F_i} = \int_0^{\delta} \frac{(\rho_l V_l - \rho_b V_b)}{q_l XB} V_b dy \frac{C_{F_i}}{C_F} \qquad \text{(A-2)}$$

where C_F/C_{F_i} values are Eckert's values. Eckert's method gives $C_F/C_{F_i} = C_f/C_{f_i}$ with C_F and C_{F_i} being the average friction coefficients and C_f and C_{f_i} being the point or local friction coefficients. The friction factor C_{F_i} is defined as a function of Reynolds number based on X (the distance of laminar or turbulent flow as the case may be). This is all fine theoretically but in actual flow a portion of the flow at the leading edge of a flat plate is laminar so that to take this into account in the turbulent flow region an effective length X_e, must be derived which in effect says all the flow preceding the station in question by X_e is turbulent. This effective X_e is practically impossible to come by.

Von Karman's momentum equation as stated in equation (226), page 467 in reference 11 (for both laminar and turbulent flows) is

$$\frac{d\theta}{dX} + (H + 2)\theta \frac{dV}{dX} = \frac{\tau_p}{\rho_l V_l^2} \tag{A-3}$$

With no pressure gradient

$$\frac{dV}{dX} = 0$$

and thus

$$\frac{d\theta}{dX} = \frac{\tau_p}{\rho_l V_l^2} = \frac{\tau_p}{2q_l} = \frac{C_f}{2} \tag{A-4}$$

and

$$C_F = \frac{\int_0^X C_f dX}{X} = \frac{\int_0^X 2\frac{d\theta}{dX} dX}{X} = 2\frac{\int_0^\theta d\theta}{X} = \frac{2\theta}{X}$$

Replace X by X_e then for laminar or turbulent flow,

$$C_F = \frac{2\theta}{X_e} \tag{A-5}$$

and

$$C_{F_1} = \frac{2\theta_1}{X_e} \tag{A-6}$$

Using the resulting expression for C_{F_1} it can be shown that C_{F_1} is also a function of Reynolds number based on θ_1 and θ_1 is a property of the station in question.

Reference 12 gives the expression for the average incompressible friction coefficient, C_{F_i}, in turbulent flow as

$$C_{F_i} = \frac{0.472}{\left[L_{n_{10}} R X_c\right]^{2.54}} \qquad \text{(A-7)}$$

and thus using the expression of equation (A-6), equation (A-7) becomes

$$L_{n_{10}} R\theta_i - L_{n_{10}} \frac{C_{F_i}}{2} = \left[0.236 \frac{2}{C_{F_i}}\right]^{0.3876} \qquad \text{(A-8)}$$

From this expression, a curve for $R\theta_i$ versus C_{F_i} can be obtained for turbulent flow. The equation for turbulent incompressible point or local friction factor (ref. 12) after rearrangement to $R\theta_i$ is,

$$C_{f_i} = \frac{0.472}{\left[L_{n_{10}} R\theta_i - L_{n_{10}} \frac{\theta_i}{X_c}\right]^{2.58}} \times \left[1 - \frac{1.12}{L_{n_{10}} R\theta_i - L_{n_{10}} \frac{\theta_i}{X_c}}\right] \qquad \text{(A-9)}$$

Using the H. Blasius expression for incompressible average friction coefficient and the fact that $C_{F_i} = 2C_{f_i}$ in laminar flow, the average incompressible laminar coefficient is

$$C_{F_i} = \frac{0.882}{R\theta_i} \qquad \text{(A-10)}$$

and the point incompressible laminar coefficient is

$$C_{f_i} = \frac{0.441}{R\theta_i} \qquad \text{(A-11)}$$

The above discussion is for plate flow but upon assuming that plate flow exists in the duct until it is filled with boundary layer, the discussion applies to duct flow. The turbulent expression of C_{f_1} does not coincide with the one used in Eckert's reference temperature; however, the difference in numerical values is not of significance.

In reference 6 J. R. Henry dropped the area change term of the one-dimensional flow equation, equation (1-1), and integrated the resulting differential equation. The integral expression obtained by Henry for the one-dimensional equation is for the effective Mach number change due to friction in a constant area duct. Henry's equation is

$$f_d(M_X) = \frac{f_d(M_1)}{e^{\left(\frac{2\gamma}{\gamma+1} \; f \; \frac{l}{D}\right)}} \qquad \text{(A-12)}$$

In J. R. Henry's report the friction factor f is based on q_e and the friction factors C_f and C_F from Rubesin's report (turbulent values) and from the Blasius expression (laminar values) are based on free stream q_1. The derivations of C_F and f are based on the assumptions

$$\tau'_{Av.} = C_F q_c \quad \text{for} \quad C_F \qquad \text{(A-13)}$$

(where $q_c = q_1$ in this case) and

$$\tau_{Av.} = f q_e \quad \text{for} \quad f \qquad \text{(A-14)}$$

The above shear stresses $\tau'_{Av.}$ and $\tau_{Av.}$ are assumed equal and thus,

$$f = C_F \frac{q_c}{q_e} \qquad \text{(A-15)}$$

Substituting $q_c = \frac{\gamma}{2} p_c M_c^2$ and $q_e = \frac{\gamma}{2} p_e M_e^2$ gives $f = \frac{p_c}{p_e}\frac{M_c^2}{M_e^2} C_F$;

whereby Wyatt's report (ref. 5) gives $\frac{p_c}{p_e} = \frac{M_e}{M_c}\frac{\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{1/2}}{\left(1 + \frac{\gamma - 1}{2} M_c^2\right)^{1/2}}\frac{A_T}{A_s}$

and thus assuming $\frac{A_T}{A_s} = 1 + \frac{\delta_A^*}{A_s}$ and $\gamma = 1.4$ then an expression for f is obtained.

$$f = \left(1 + \frac{\delta_A^*}{A_s}\right)\left(\frac{M_e}{M_c}\right)\left(\frac{5 + M_e^2}{5 + M_c^2}\right)^{1/2} C_F \tag{A-16}$$

Corresponding to reference 5, an expression involving total mass flow, total momentum, and effective Mach number is obtained but, in the present paper a constant stagnation temperature across the boundary layer is not assumed. Similar to the equations from reference 5 (which are for constant total temperature across the boundary layer), an expression involving total mass flow, total momentum, effective Mach number, and an effective total temperature (comes about due to total temperature not being constant in the boundary layer even when the wall is adiabatic) is obtained.

$$\frac{m_{total}}{\phi_{total}}\sqrt{\frac{gRT_e}{\gamma}} = \frac{M_e\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{1/2}}{(1 + \gamma M_e^2)} = f(M_e) \tag{A-17}$$

But

$$\frac{m_{total}}{\phi_{total}} = \frac{m_b + m_c}{\phi_b + \phi_c} = f(M_e)\sqrt{\frac{\gamma}{gRT_e}} \tag{A-18}$$

which by algebraic manipulation becomes

$$\frac{m_{total}}{\phi_{total}} = \frac{\left(\dfrac{\frac{m_b}{A_b}}{\frac{m_c}{A_c}}\right)\dfrac{m_c}{A_c}\,A_b + \dfrac{m_c}{A_c}\,A_c}{\left[\left(\dfrac{\frac{\phi_b}{A_b}}{\frac{\phi_c}{A_c}}\right)\right]\dfrac{\phi_c}{A_c}\,A_b + \dfrac{\phi_c}{A_c}\,A_c} \qquad \text{(A-19)}$$

and upon setting

$$m = \frac{m_b}{A_b} \quad \text{and} \quad m_l = \frac{m_c}{A_c}$$

$$\phi = \frac{\phi_b}{A_b} \quad \text{and} \quad \phi_l = \frac{\phi_c}{A_c} \qquad \text{(A-20)}$$

equation (A-19) becomes

$$\frac{m_{total}}{\phi_{total}} = \frac{\dfrac{m}{m_l}\,\dfrac{m_c}{A_c} + \dfrac{m_c}{A_c}\,\dfrac{A_c}{A_b}}{\dfrac{\phi}{\phi_l}\,\dfrac{\phi_c}{A_c} + \dfrac{\phi_c}{A_c}\,\dfrac{A_c}{A_b}} \qquad \text{(A-21)}$$

Substitution of

$$\left.\begin{aligned} \frac{m_c}{A_c} &= \left(\frac{m/A}{\rho_0 a_0}\right)_c M_c \sqrt{\frac{\gamma}{gRT_c}} \\ \frac{\phi_c}{A_c} &= p_c(1 + \gamma M_c^2),\ f(M_c) = \frac{\dfrac{M_c}{p_c}\left(\dfrac{m/A}{\rho_0 a_0}\right)_c}{(1 + \gamma M_c^2)} \end{aligned}\right\} \qquad \text{(A-22)}$$

into equation (A-21) and the result substituted into equation (A-18) gives after rearrangement

$$\left(\frac{T_e}{T_l}\right) = \frac{\left(\frac{A_c}{A_b} + \frac{\phi}{\phi_l}\right)^2}{\left(\frac{A_c}{A_b} + \frac{m}{m_l}\right)^2}\left[\frac{f(M_e)}{f(M_c)}\right]^2 \qquad \text{(A-23a)}$$

The ratio ϕ/ϕ_l is the ratio of the total momentums per unit area and, therefore,

$$\frac{\phi}{\phi_l} = \frac{p_c(1 + \gamma M_b^2)}{p_c(1 + \gamma M_c^2)} = \frac{\frac{1}{\gamma M_c^2} + \frac{\gamma M_b^2}{\gamma M_c^2}}{\frac{1}{\gamma M_c^2} + 1}$$

but

$$\frac{\varphi}{\varphi_l} = \frac{(\rho V^2)_b}{(\rho V^2)_c} = \frac{\gamma M_b^2}{\gamma M_c^2}$$

and, therefore, substituting into the equation for ϕ/ϕ_l gives

$$\frac{\phi}{\phi_l} = \frac{1 + \gamma M_c^2 \frac{\varphi}{\varphi_l}}{\gamma M_c^2 + 1}$$

Substitute the resulting expression for ϕ/ϕ_l in equation (A-23a) to obtain

$$\frac{T_e}{T_l} = \frac{\left(\frac{A_c}{A_b} + \frac{1 + \frac{\varphi}{\varphi_l}\gamma M_c^2}{1 + \gamma M_c^2}\right)^2}{\left(\frac{A_c}{A_b} + \frac{m}{m_l}\right)^2}\left[\frac{f(M_e)}{f(M_c)}\right]^2 \qquad \text{(A-23b)}$$

Figure 1 gives the relationship of A_c and A_b as well as other areas of interest. By definition (m/m_l) is the mass flow per unit area in the boundary layer divided by the mass flow per unit area in the free stream or core flow. If the flow was one dimensional and, therefore, the boundary layer had the same total pressure as the core flow, the amount of core that would be required to carry the mass flow in the boundary layer under these core conditions would be

$$A_s - A_c = \frac{m}{m_l} A_b \qquad \text{(A-24)}$$

Solving for A_c/A_b

$$\frac{A_c}{A_b} = \frac{A_s}{A_b} - \frac{m}{m_l} \qquad \text{(A-25)}$$

The boundary layer is at a reduced total pressure due to friction losses, therefore, in order to pass the same flow as can be passed by the shock area A_s at core conditions, additional area over and beyond the shock diagram area A_s is required and this additional area is by definition the displacement area δ_A^*. Since the fraction of the boundary layer area carried within the shock area is $\left[\left(\frac{m}{m_l}\right) A_b\right]$ the displacement area δ_A^* must be

$$\delta_A^* = \left(1 - \frac{m}{m_l}\right) A_b \qquad \text{(A-26)}$$

or rearranging

$$\frac{m}{m_l} = 1 - \frac{\delta_A^*}{A_b} \qquad \text{(A-27)}$$

Substitute equation (A-27) into equation (A-25) to obtain

$$\frac{A_c}{A_b} = \frac{A_s}{A_b} + \frac{\delta_A^*}{A_b} - 1 \qquad \text{(A-28)}$$

By assumption the following relations are acquired

$$\left.\begin{aligned}
\left(\frac{\delta^*}{\delta}\right) &= \frac{\delta_A^*}{A_b} = \left(1 - \frac{m}{m_l}\right) \\
\left(\frac{\theta}{\delta}\right) &= \frac{\theta_A}{A_b} = \left(\frac{m}{m_l} - \frac{\eta}{\eta_l}\right) \\
\frac{\theta_c}{\delta} &= \frac{\theta_{Ac}}{A_b} = \left(\frac{m}{m_l} - \frac{\eta}{\eta_l}\right) \\
\frac{\theta_H}{\delta} &= \frac{\theta_{AH}}{A_b} = \frac{(mh)}{(mh)_l} - \frac{m}{m_l}
\end{aligned}\right\} \qquad \text{(A-29)}$$

Substituting equation (A-28) for A_c/A_b in equation (A-23b) and utilizing equation (A-29) then equation (A-23b) becomes

$$\frac{T_c}{T_l} = \frac{\left[(1 + \gamma M_c^2) + \left(1 - M_c^2 \frac{\theta}{\delta^*}\right)\frac{\delta_A^*}{A_b}\right]^2}{(1 + \gamma M_c^2)} \left[\frac{f(M_c)}{f(M_c)}\right]^2 \qquad \text{(A-23c)}$$

The energy equation upon assuming adiabatic flow can be expressed as

$$\frac{(mh)_b}{A_b} A_b + \frac{1}{2} \frac{\eta_b}{A_b} + \frac{(mh)_c}{A_c} + \frac{1}{2}\left(\frac{\eta_c}{A_c}\right) A_c = C_p T_c m_{total} \qquad \text{(A-30)}$$

Upon rearranging equation (A-30) becomes

$$\frac{(mh)_c}{A_c}\left[\frac{(mh)}{(mh)_l} - \left(\frac{m}{m_l}\right)\right] A_b + \left(\frac{m}{m_l}\right)\frac{(mh)_c}{A_c} A_b - \frac{1}{2}\left(\frac{\eta_c}{A_c}\right)\left(\frac{m}{m_l} - \frac{\eta}{\eta_l}\right) A_b$$

$$- \frac{1}{2}\frac{\eta_c}{A_c}\left(1 - \frac{m}{m_l}\right) A_b + \frac{1}{2}\frac{\eta_c}{A_c} A_b + \frac{\eta_c}{A_c} A_c + \frac{1}{2}\frac{\eta_c}{A_c} A_c = C_p T_c m_{total} \qquad \text{(A-31a)}$$

Then substitute equation (A-29) into equation (A-31a) to obtain

$$\frac{(\dot{m}h)_c}{A_c}\frac{\theta_{A_H}}{A_b}A_b + \left(1 - \frac{\delta_A^*}{A_b}\right)\left(\frac{(\dot{m}h)_c}{A_c}A_b\right) - \frac{1}{2}\frac{\eta_c}{A_c}\frac{\theta_{A_\epsilon}}{A_b}A_b - \frac{1}{2}\frac{\eta_c}{A_c}\frac{\delta_A^*}{A_b}A_b$$

$$+ \frac{1}{2}\frac{\eta_c}{A_c}A_b + \frac{\eta_c}{A_c}A_c + \frac{1}{2}\frac{\eta_c}{A_c}A_c = C_pT_e\dot{m}_{total} \qquad \text{(A-31b)}$$

Persh's temperature profile (ref. 8) is expressed as

$$t_b = t_l + \frac{\gamma - 1}{2\gamma}\frac{r}{R}(V_l^2 - V_b^2) \qquad \text{(A-32)}$$

Multiply by the local mass flow per unit area and integrate across the boundary layer to obtain

$$\int_0^\delta C_p\frac{\rho_b V_b}{\rho_l V_l}(t_b - t_l)dy = \int_0^\delta C_p\left(\frac{\gamma - 1}{2\gamma}\right)\left(\frac{r}{R}\right)(V_l^2 - V_b^2)dy \qquad \text{(A-33)}$$

For $h = C_pt$ and $R = C_p - C_V$ and $\gamma = \frac{C_p}{C_V}$ equation (A-33) becomes

$$h_l\int_0^\delta \frac{\rho_b V_b}{\rho_l V_l}\left(\frac{h_b}{h_l} - 1\right)dy = \frac{rV_l^2}{2}\int_0^\delta \frac{\rho_b V_b}{\rho_l V_l}\left(1 - \frac{V_b^2}{V_l^2}\right)dy$$

or

$$h_l\theta_H = \frac{rV_l^2}{2}\theta_\epsilon$$

$$\frac{\theta_H}{\delta} = \frac{\gamma - 1}{2}rM_l^2\frac{\theta_\epsilon}{\delta} \qquad \text{(A-34)}$$

Substitute equation (A-34) into equation (A-32) after dividing equation (A-32) by $C_pt_c\frac{\dot{m}_c}{A_c}A_b$ and letting $\frac{V^2}{h} = (\gamma - 1)M^2$. Equation (A-32) then becomes after rearrangement

$$(r - 1)\left(\frac{\gamma - 1}{2}\right)M_c^2 \frac{\theta_{A_c}}{A_b} + \left(1 - \frac{\delta_A^*}{A_b} + \frac{A_c}{A_b}\right)\left(1 + \frac{\gamma - 1}{2} M_c^2\right) = \frac{T_e m_{total}}{t_c \frac{m_c}{A_c} A_b} \quad \text{(A-35)}$$

Since $m_{total} = \frac{m_b}{A_b} A_b + \frac{m_c}{A_c} A_c$, $t_c = \dfrac{T_c}{1 + \frac{\gamma - 1}{2} M_c^2}$, and $T_c = T_l$

equation (A-35) can be expressed as

$$(r - 1) \frac{\gamma - 1}{2} M_c^2 \frac{\theta_{A_c}}{A_b} + \left(1 - \frac{\delta_A^*}{A_b} + \frac{A_c}{A_b}\right)\left(1 + \frac{\gamma - 1}{2} M_c^2\right)$$

$$= \left(\frac{T_e}{T_l}\right)\left(1 - \frac{\delta_A^*}{A_b} + \frac{A_c}{A_b}\right)\left(1 + \frac{\gamma - 1}{2} M_c^2\right)$$

or

$$\frac{T_e}{T_l} = 1 + \frac{(r - 1)\frac{\gamma - 1}{2} M_c^2 \frac{\theta_{A_c}}{A_b}}{\left(1 - \frac{\delta_A^*}{A_b} + \frac{A_c}{A_b}\right)\left(1 + \frac{\gamma - 1}{2} M_c^2\right)} \quad \text{(A-36)}$$

Substitute in equation (A-28) for A_c/A_b and rearrange to obtain

$$\frac{T_e}{T_l} = 1 + \frac{(r - 1)\frac{\gamma - 1}{2} M_c^2 \frac{\theta_{A_c}}{A_b}}{\frac{A_e}{A_b}\left(1 + \frac{\gamma - 1}{2} M_c^2\right)}$$

Upon utilizing equation (A-29) again the expression for (T_e/T_l) becomes

$$\frac{T_e}{T_i} = 1 + \frac{(r - 1)\left(\frac{\gamma - 1}{2}\right) M_c^2 \left(\frac{\theta_c}{\delta}\right)}{\frac{\delta^*}{\delta}\left(1 + \frac{\gamma - 1}{2} M_c^2\right)} \frac{\delta_A^*}{A_s} \tag{A-37}$$

Combine equation (A-23c) and equation (A-37) to obtain

$$\left[\frac{f(M_e)}{f(M_c)}\right]^2 = \frac{(1 + \gamma M_c)^2 \left[1 + \frac{(r - 1)\frac{\gamma - 1}{2} M_c^2 \frac{\theta_c}{\delta}}{\frac{\delta^*}{\delta}\left(1 + \frac{\gamma - 1}{2} M_c^2\right)} \frac{\delta_A^*}{A_s}\right]}{(1 + \gamma M_c^2) + \left[\left(1 - \gamma M_c^2 \frac{\theta}{\delta^*}\right)\frac{\delta_A^*}{A_s}\right]^2} \tag{A-38}$$

or

$$\frac{f(M_e) - f(M_c)}{f(M_e)} = 1 - \frac{\left[(1 + \gamma M_c^2) + \left(1 - \gamma M_c^2 \frac{\theta}{\delta^*}\right)\frac{\delta_A^*}{A_s}\right]}{(1 + \gamma M_c^2)\left(1 + \frac{(r - 1)\frac{\gamma - 1}{2} M_c^2 \frac{\theta_c}{\delta}}{\frac{\delta^*}{\delta}\left(1 + \frac{\gamma - 1}{2} M_c^2\right)} \frac{\delta_A^*}{A_s}\right)^{1/2}} \tag{A-39}$$

APPENDIX B

Following is an outline of an engineering method that can be used in the calculation of effective Mach number loss through a supersonic inlet at a free-stream Mach number of 4.06 and with wall outlines as presented in figure 2.

Outline of Method Used for Friction Calculations

1. Obtain (A_T/A^*) (fig. 6) from curve corresponding to $(12)_{end}$ which is given later in this appendix. Assume different values of M_c after area change due to $\delta_A{}^*$ (for example, $(12)_{end}$ - 0.04, 0.06, 0.08 and greater or smaller as the case may be) and obtain A_s/A^* (fig. 6) for M_c assumed and calculate

$$\frac{\dfrac{A_T}{A^*}}{\dfrac{A_s}{A^*}} = \frac{A_T}{A_s} \quad \text{and} \quad \frac{\delta_A{}^*}{A_s} = \frac{A_T}{A_s} - 1$$

Using this and the M_c's assumed, values of $(M_c - M_e)$ can be obtained using

$$\frac{f(M_e) - f(M_c)}{f(M_e)} = 1 - \frac{(1 + \gamma M_c{}^2) + \left(1 - \gamma M_c{}^2\left(\dfrac{\theta}{\delta^*}\right)\right)\dfrac{\delta_A{}^*}{A_s}}{(1 + \gamma M_c{}^2)\left(1 + \dfrac{(r - 1)\dfrac{\gamma - 1}{2} M_c{}^2 \dfrac{\theta_e}{\delta} \delta_A{}^*}{\dfrac{\delta^*}{\delta}\left(1 + \dfrac{\gamma - 1}{2} M_c{}^2\right) A_s}\right)^{1/2}}$$

and the graph of $\dfrac{f(M_e) - f(M_c)}{f(M_e)}$ (fig. 7) along with $\gamma = 1.4$ and

$r = 0.894$ for turbulent flow and $r = 0.851$ for laminar flow. Correspondingly, a graph of M_c versus M_e can be drawn. The parameters (δ^*/θ), (δ^*/δ), and θ_c/δ (figs. 3, 4, and 5, respectively) for turbulent flow are obtained from graphs of δ^*/θ, δ^*/δ, and θ_c/δ versus $N = (11)_{end}$ and $(M_c)_{end}$ = (assumed M_c's above) if it is the first element of the model and $N = (11)_{end}$ and $(M_c)_{end}$ = (assumed M_c's above) if it is elements following the first element. If it is laminar flow, use (δ^*/θ), (δ^*/δ), and θ_c/δ values from the laminar curve corresponding to the assumed M_c's.

2. Adiabatic wall.

3. T_t = constant.

4. $(\bar{H}/H_{ref})_{mid,end} = (4)_{mid,end}$ - area weighted or other means of obtaining the average total pressure recovery at a station (loss due to shocks only).

5. $(M_e)_{mid,end} = \left[(5)_{mid,end} - (5)_{end}\right] = (M_e)_{end}$ is obtained from plot of M_c versus M_e obtained in (1) and $(5)_{mid} = (M_e)_{mid}$ comes from plotting $(M_e)_{end}$ on curve versus l and reading off value of M_e from the center of the element.

6. Identification of the middle and the end of the element.

7. Increment or element number.

8. $(L)_{mid,end} = X = (8)_{mid,end}$ - The location of the middle and the end of the element (constant for each element).

9. $\left(\frac{A_T}{A_{T_{inlet}}}\right)_{mid,end} = (9)_{mid,end}$ - the total area ratio of the element middle and end divided by the inlet total area $A_{T_{inlet}}$ (constant for each element).

10. $\left(\frac{P}{P_{inlet}}\right)_{mid,end} = (10)_{mid,end}$ - the total perimeter ratio of the element middle and end divided by the inlet total perimeter, P_{inlet} (constant for each element).

11. $(N)_{mid,end} = (11)_{mid,end}$ - the velocity profile parameter, N, at the middle and end of the element for turbulent elements.

12. $(M_c)_{mid,end} = (12)_{mid,end}$ - the core Mach number which includes only shock losses (no friction loss effect).

13. $(M_c)_{mid,end} = \left[(13)_{mid,end} - (M_c)_{end}\right] = (13)_{end}$ is obtained corresponding to $(M_c)_{end}$ chosen for $(5)_{end}$ and $(M_c) = (13)_{mid}$ is obtained by plotting $(M_c)_{end}$ against $l\left[= (8)_{end}\right]$ and picking off the element middle value of M_c.

14. Obtain $(A_T/A^*)_{end}$ (fig. 6) for $(12)_{end}$ and correspondingly $(A_c/A^*)_{end}$ (fig. 6) for $(13)_{end}$. Then

$(14)_{end} = (\delta_A^*/A_s)_{end} = \dfrac{\left(\dfrac{A_T}{A^*}\right)}{\left(\dfrac{A_s}{A^*}\right)_{end}} - 1$ and $(14)_{mid} = (\delta_A^*/A_s)_{mid}$ is

obtained in a similar manner using $(12)_{mid}$ and $(13)_{mid}$.

15. $(R_\theta/\theta\bar{H})_{mid,end} = (11)_{mid,end}$ is obtained from figure 8, using $(13)_{mid,end}$ and $(3) = T_l$ in °F.

16. $(\delta^*/\theta)_{mid,end} = (16a)_{mid,end}$, $(\delta^*/\delta)_{mid,end} = (16c)_{mid,end}$, and $(\theta_c/\delta) = (16b)_{mid,end}$ are obtained from figures 3, 4, and 5, respectively, using $(13)_{mid,end}$ and $(11)_{mid,end}$.

$$17.\quad \theta_{mid,end} = (17)_{mid,end} = \frac{\left(\dfrac{\delta_A^*}{A_s}\right)\left(\dfrac{A_T}{A_{T_0}}\right)\dfrac{A_{T_{inlet}}}{P_{inlet}}}{\left(\dfrac{\delta_A^*}{\theta_A}\right)\left(1 + \dfrac{\delta_A^*}{A_s}\right)\dfrac{P}{P_{inlet}}}$$

$$= \frac{(14)_{mid,end}(9)_{mid,end}\dfrac{A_{T_{inlet}}}{P_{inlet}}}{(16)_{mid,end}(1 + (14)_{mid,end})(10)_{mid,end}}$$

$$18.\quad R_{\theta_{mid,end}} = (18)_{mid,end} = \left(\frac{R_\theta}{\theta\bar{H}}\right)_{mid,end}\theta_{mid,end}H_{ref}\left(\frac{\bar{H}}{H_{ref}}\right)_{mid,end}$$

$$= (15)_{mid,end}(17)_{mid,end}(4)_{mid,end}H_{ref}$$

19. $(C_F/C_{F_i})_{end} = (19)_{end}$ - it is assumed to be the average value for the whole element when obtained (from fig. 9, if turbulent or fig. 10 if laminar) using $(M_c)_{mid} = (13)_{mid}$, $T_l = (3)$ and if turbulent $(R_\theta)_{end} = (18)_{end}$. The (C_F/C_{F_i}) obtained is really an average value of C_F/C_{F_i} relative to the variation of M_c, through the element.

20. $(R_{\theta_1})_{end} = (R_\theta)_{end}/(C_F/C_{F_1})_{end} = (18)_{end}/(19)_{end} = (20)_{end}$

21. $C_{F_1} = (21)$ - is obtained from figure 11 corresponding to $(R_{\theta_1}) = (20)_{end}$ and whether the flow is laminar or turbulent.

22. (a) $C_F = C_{F_1}\dfrac{C_F}{C_{F_1}} = (21)(19)_{end}$

(b) - - - - - - - - - - - - -

(c) $(M)_{end} = (12)_{end}$

(d) $(\Delta M_e)_A = \left[M_{inlet} - (M)_{end}\right] = \left[M_{inlet} - (12)_{end}\right]$ where M_{inlet} is the inlet Mach number.

23. $(M_e,1)_{mid} = (23)_{mid}$ - average effective Mach number $(M_e,1)_{mid} = (23)_{mid} = (5)_{mid}$.

24. $f = (24) = \left[1 + (14)_{mid}\right]\dfrac{(13)_{mid}}{(23)_{mid}}\sqrt{\dfrac{5 + (23)^2_{mid}}{5 + (13)^2_{mid}}}\ (22a)$ or

$$f = \left[1 + \left(\frac{\delta_A^*}{A_s}\right)_{mid}\right]\frac{(M_e)_{mid}}{(M_e,1)_{mid}}\sqrt{\frac{5 + (M_e,1)^2_{mid}}{5 + (M_e)^2_{mid}}}\,C_F$$

The middle values are used in order to obtain an average f for the whole element, as this f is to be used in the one-dimensional calculation of ΔM_e due to friction.

25. $(\Delta M_e)_f = (25)$ - using (24) and $(23)_{mid}$ the $(\Delta M_e)_f$ is obtained from figure 12 or 13 depending on whether the integrated (l/D) length of the element is 0.25, 0.5, or 1.0. These curves are obtained corresponding to the one-dimensional flow equation.

26. $(M_{e,2})_{end} = (26)_{end} = \left[M_{inlet} - (25) - (22)\right]$ or $M_{e,2}$

$= \left[M_{inlet} - (\Delta M_e)_f - (\Delta M_e)_A\right]$

27. Plot $M_{e,2} = (26)_{end}$ on a graph versus the M_c assumed in $(13)_{end}$.

Note: Repeat the procedure until a curve of $(13)_{end} = (M_c)_{end}$ assumed versus $M_{e,2} = (26)_{end}$ is obtained. The point of intersection of this curve and the curve of No. 1 gives the final $(M_c)_{end}$ and $(M_{e,2})$ and correspondingly the δ_A^*/A_s for the element. For δ_A^*/A_s can be calculated in the same manner as (1) using $(12)_{end}$ and the final $(M_c)_{end}$.

28. Obtain $(\rho_e V_e/\rho_0 A_0)_{end}$ from $\rho V/\rho_0 a_0 = \dfrac{M(p/H)}{\sqrt{t/T_0}}$ corresponding to $M_{e,2}$ of (27).

29. (a) $M_{e,2}$ final, (b) M_c final, (c) δ_A^*/A_s final

Treatment of Transition

29. Assume $(M_c)_{turb.} = (M_e)_{lam.} = (29)_E$ = final $M_{e,2}$ of (29a)

30. Calculate a curve of M_c versus M_e in the same manner as (1) using the H for the station of $(M_e)_{lam.}$ to obtain (δ^*/θ).

31. Enter the curve of M_c versus M_e obtained for (30) with $(29)_{end} = M_{e,2}$ and obtain $(M_c)_{turb.\ end} = (31)_{end}$.

32. Calculate $(\delta_A^*/A_s)_{end} = (32)_{end}$ in the same manner as (1) using $(12)_{end}$ and $M_{c_{turb.\ end}} = (31)_{end}$.

Elements Following the First Element or in General,

the N^{th} Element for $N > 1$

Calculations No. (1) through No. (18) remain unchanged.

19. $(C_f/C_{f_i})_{mid} = (15)_{mid} = (C_f/C_{f_i})_{mid}$ is obtained by picking off a value of C_f/C_{f_i}, corresponding to $(R_\theta)_{mid} = (18)_{mid}$ and $(M_c)_{mid} = (13)_{mid}$ and $T_i = (3)$, from figure 9 which is the turbulent flow plot of (C_f/C_{f_i}) or figure 10 corresponding to $(M_c)_{mid} = (13)_{mid}$ and $T_i = (3)$ if it is laminar flow. This is really an average value of C_f/C_{f_i} relative to the variation of M_c and R_θ through the element.

20. $(R_{\theta_i})_{mid} = \dfrac{R_{\theta_{mid}}}{C_f/C_{f_i}} = (18)_{mid}/(19)_{mid} = (20)_{mid}$

21. $C_{f_i} = (21)_{mid}$ - obtained from the point value curve (fig. 11) of C_{f_i} versus $(R_{\theta_i})_{mid} = (20)_{mid}$ for laminar or turbulent flow as the case may be. This point value of the incompressible, friction factor is assumed to be equal to the average incompressible value of the friction factor for the element because it was chosen corresponding to average element factors.

22. (a) $C_f = \dfrac{C_f}{C_{f_i}} C_{f_i} = (19)_{mid}(21)_{mid} = (22a)_{mid}$. The compressible point friction factor is assumed to be equal to the compressible average friction factor of the element because average element factors were used in its determination.

(b) $$\left(\frac{\rho_e V_e}{\rho_0 a_0}\right)_{\substack{end\\N}} = (22b)_{\substack{end\\N}} = \frac{\left(\frac{\rho_e V_e}{\rho_0 a_0}\right)_{\substack{end\\(N-1)}}\left(\frac{A_T}{A_{T_{inlet}}}\right)_{\substack{end\\(N-1)}}\left(\frac{\overline{H}}{H_{ref}}\right)_{\substack{end\\(N-1)}}}{\left(\frac{A_T}{A_{T_{inlet}}}\right)_{\substack{end\\N}}\left(\frac{\overline{H}}{H_{ref}}\right)_{\substack{end\\N}}}$$

or

$$(22b)_{\substack{end\\N}} = \frac{(28)_{\substack{end\\(N-1)}}(9)_{\substack{end\\(N-1)}}(4)_{\substack{end\\(N-1)}}}{(9)_{\substack{end\\N}}(4)_{\substack{end\\N}}}$$

The total pressure loss through a shock is less as the Mach number becomes less and the value obtained for $(22b)_{\substack{end\\N}}$ does not take this into account. The method of obtaining $(22b)_{\substack{end\\N}}$ imposes the free-stream shock loss all across the duct, and therefore, takes into account, to some extent, the momentum loss in the boundary layer due to shock boundary-layer interaction.

(c) $(M_eA)_{\substack{end\\N}} = (22c)_{\substack{end\\N}}$ - is obtained from a curve of $\rho V/\rho_0 a_0$ versus M $\left(\text{generated from } \rho V/\rho_0 a_0 = \frac{M(p/H)}{\sqrt{t/T_0}}\right)$ using $(22b)_{\substack{end\\N}}$ and this value of M_e also includes the shock boundary-layer interaction losses pointed out in (22b).

(d) $$(22d) = (\Delta M_e)_{A_N} = \left[(M_e)_{\substack{end\\(N-1)}} - (M_eA)_{\substack{end\\N}}\right] = \left[(29)_{\substack{end\\(N-1)}} - (22c)_{\substack{end\\N}}\right]$$

23. $(23)_{mid} = (M_{e,1})_{\substack{mid\\N}} = (5)_{mid}$

24. $f = \left(1 + \frac{\delta_A^*}{A_s}\right)\frac{M_c}{M_{e,1}}\sqrt{\frac{5 + M_{e,1}^2}{5 + M_c^2}}$ (22a) = (24)

or

$$f = (24) = \left[1 + (8)_{mid}\right]\frac{(13)_{mid}}{(23)_{mid}}\sqrt{\frac{5 + (23)_{mid}^2}{5 + (13)_{mid}^2}}\ (22a)$$

where f is the average effective friction factor for the entire element.

25. $(\Delta M_e)_f = (25) = (\Delta M_e)_f$ is obtained from figure 12 or 13 using (24) and $(23)_{mid}$ and choosing the figure to use as to the size of element chosen, that is, whether l/D is chosen to be 0.25, 0.5, or 1.0.

26. (a) $(M_{e,2})_{\substack{end\\N}} = \left[(M_e)_{\substack{end\\(N-1)}} - (\Delta M_e)_A - (\Delta M_e)_f\right]$

or

$$(26)_{\substack{end\\N}} = \left[(20)_{\substack{end\\(N-1)}} - (22d)_N - (25)_N\right]$$

Calculations 27 through 29 hold along with the note inserted between 17 and 18.

TABLE 1

Calculations are for $T_o = 100^oF$ or 56^o R, H_o = 220 psi, $\frac{AT_o}{P_o} = 0.0833$, and $\left(\frac{m/A}{\rho_o a_o}\right)_{inlet} = 0.0512$. Transition was assumed to occur at $\left(\frac{l}{D}\right) = 1.0$ for all models except model I.

MODEL I

l (inches)			Element no.	Av. shock "M"		
Mid.	Beg.	End	For $\left(\frac{l}{D}\right) = 1.0$	Beg.	Mid.	End
0	1.88	3.475	1	4.06	3.68	3.288
3.475	0	4.011	Part element	3.288	----	3.098

N		$\left(\frac{P}{P_o}\right)_{end}$	$\left(\frac{\bar{H}}{H_o}\right)_{end}$	$\left(\frac{AT}{AT_o}\right)$	
Mid.	End			Mid.	End
Laminar	Laminar	0.782	0.8565	0.764	0.572
Laminar	Laminar	.758	.8123	.528	.504

TABLE 1.- Continued

MODEL II

l (inches)			Element no.	Av. shock "M"		
Beg.	Mid.	End	For $\left(\frac{l}{D}\right) = 0.25$	Beg.	Mid.	End
0	0.5575	0.970	1	4.06	4.015	3.980
.970	1.450	1.93	2	3.980	3.939	3.896
1.93	2.385	2.84	3	3.896	3.8525	3.812
2.84	3.284	3.73	4	3.812	3.770	3.727
3.73	4.085	4.44	5	3.727	3.690	3.652
4.44	4.890	5.34	6	3.652	3.6025	3.551
5.34	5.73	6.12	7	3.551	3.501	3.452
6.12	6.48	6.84	8	3.452	3.401	3.351
6.84	7.19	7.54	9	3.351	3.301	3.248
7.54	7.865	8.19	10	3.248	3.198	3.144
8.19	8.344	8.498	Part element	3.144	3.123	3.099

TABLE 1.- Continued

Model II - Continued

Element no.	N		$\left(\frac{P}{P_o}\right)_{end}$	$\left(\frac{H}{H_o}\right)_{end}$	$\left(\frac{A_T}{A_{T_o}}\right)$	
	Mid.	End			Mid.	End
1	Laminar	Laminar	0.970	0.990	0.962	0.939
2	Laminar	Laminar	.940	.982	.906	.878
3	Laminar	Laminar	.910	.973	.847	.820
4	Laminar	7.5	.883	.969	.790	.7621
5	7.25	7.0	.859	.960	.738	.7109
6	6.75	6.5	.833	.955	.684	.658
7	6.25	6.0	.811	.948	.630	.6064
8	5.75	5.5	.788	.9415	.580	.5549
9	5.25	5	.766	.935	.531	.5097
10	5	5	.746	.931	.488	.461
Part element	----	5	.736	.929	.450	.443

TABLE 1.- Continued

MODEL III

l (inches)			Element no.	Av. shock "M"		
Beg.	Mid.	End	For $\left(\frac{l}{D}\right) = 1.0$	Beg.	Mid.	End
0	1.75	3.41	1	4.06	3.975	3.883
3.41	5.01	6.61	2	3.883	3.789	3.686
6.61	8.17	9.66	3	3.686	3.582	3.470
9.66	11.07	12.39	4	3.470	3.345	3.215
0	0	12.649	Part element	-----	-----	3.190

N		$\left(\frac{P}{P_o}\right)_{end}$	$\left(\frac{\bar{H}}{H_o}\right)_{end}$	$\left(\frac{A_T}{A_{T_o}}\right)$	
Mid.	End			Mid.	End
Laminar	Laminar	0.933	0.995	0.927	0.857
8.0	7.5	.870	.987	.789	.721
7.0	6.5	.809	.978	.654	.590
6.0	5.5	.752	.970	.529	.473
0	5.5	.749	.969	-----	.462

TABLE 1.- Concluded

MODEL IV

l (inches)			Element no.	Av. shock "M"		
Beg.	Mid.	End	For $\left(\frac{l}{D}\right) = 1.0$	Beg.	Mid.	End
0	1.95	3.85	1	4.06	3.99	3.930
3.85	5.70	7.50	2	3.930	3.820	3.788
7.50	9.23	10.90	3	3.788	3.710	3.620
10.90	12.45	13.87	4	3.620	3.550	3.468
13.87	15.26	16.58	5	3.468	3.350	3.271
-----	-----	17.07	Part element	-----	-----	3.238

N		$\left(\frac{P}{P_o}\right)_{end}$	$\left(\frac{\bar{H}}{H_o}\right)_{end}$	$\left(\frac{A_T}{A_{T_o}}\right)$	
Mid.	End			Mid.	End
Laminar	Laminar	0.947	0.9975	0.945	0.983
8.0	7.5	.900	.9930	.837	.784
7.0	6.5	.8575	.990	.732	.676
6.0	5.5	.817	.989	.635	.588
5.0	5.0	.780	.983	.539	.492
---	5.0	.775	.982	-----	.478

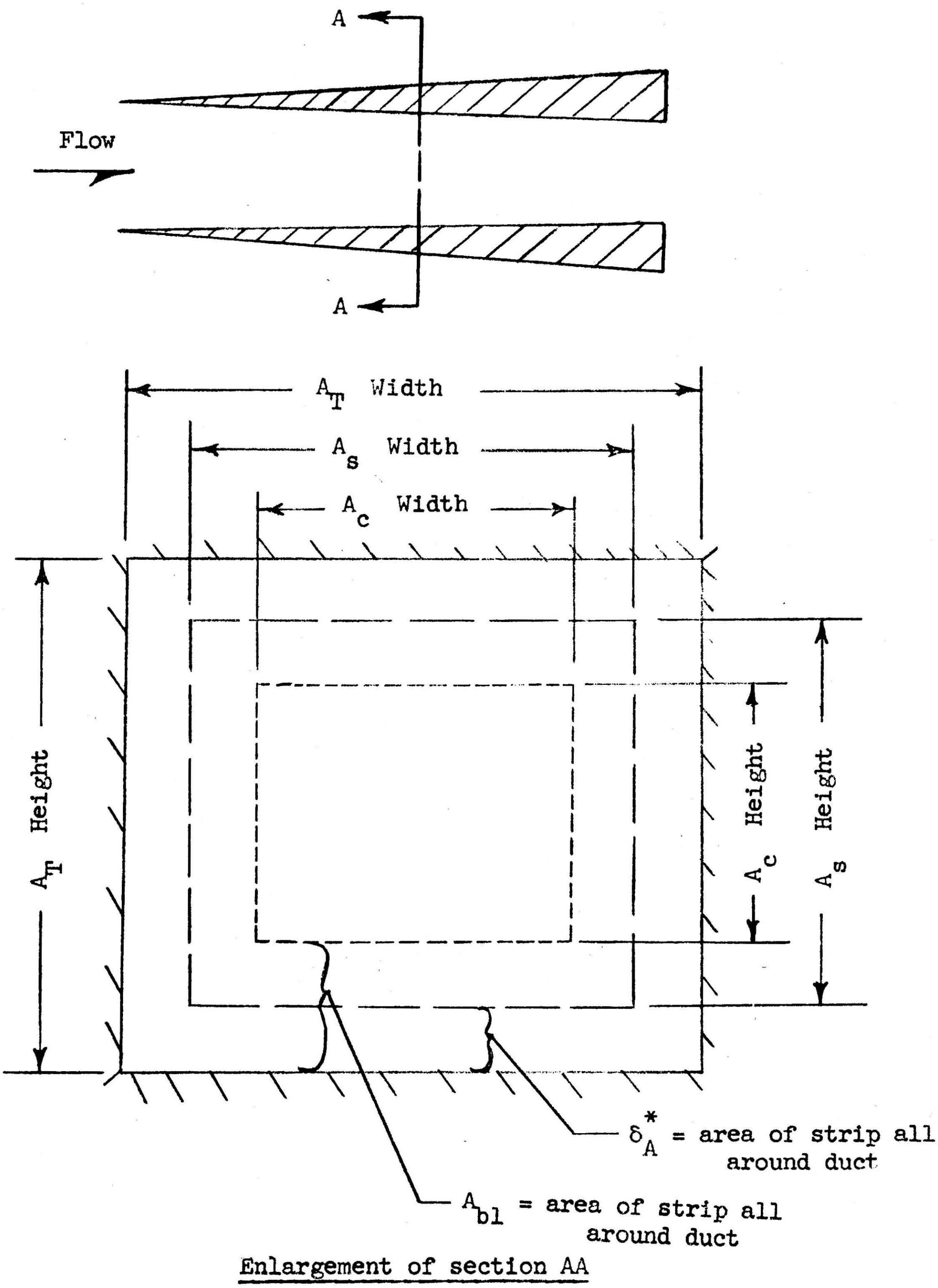


Figure 1.- Relative relationship of flow area parameters.

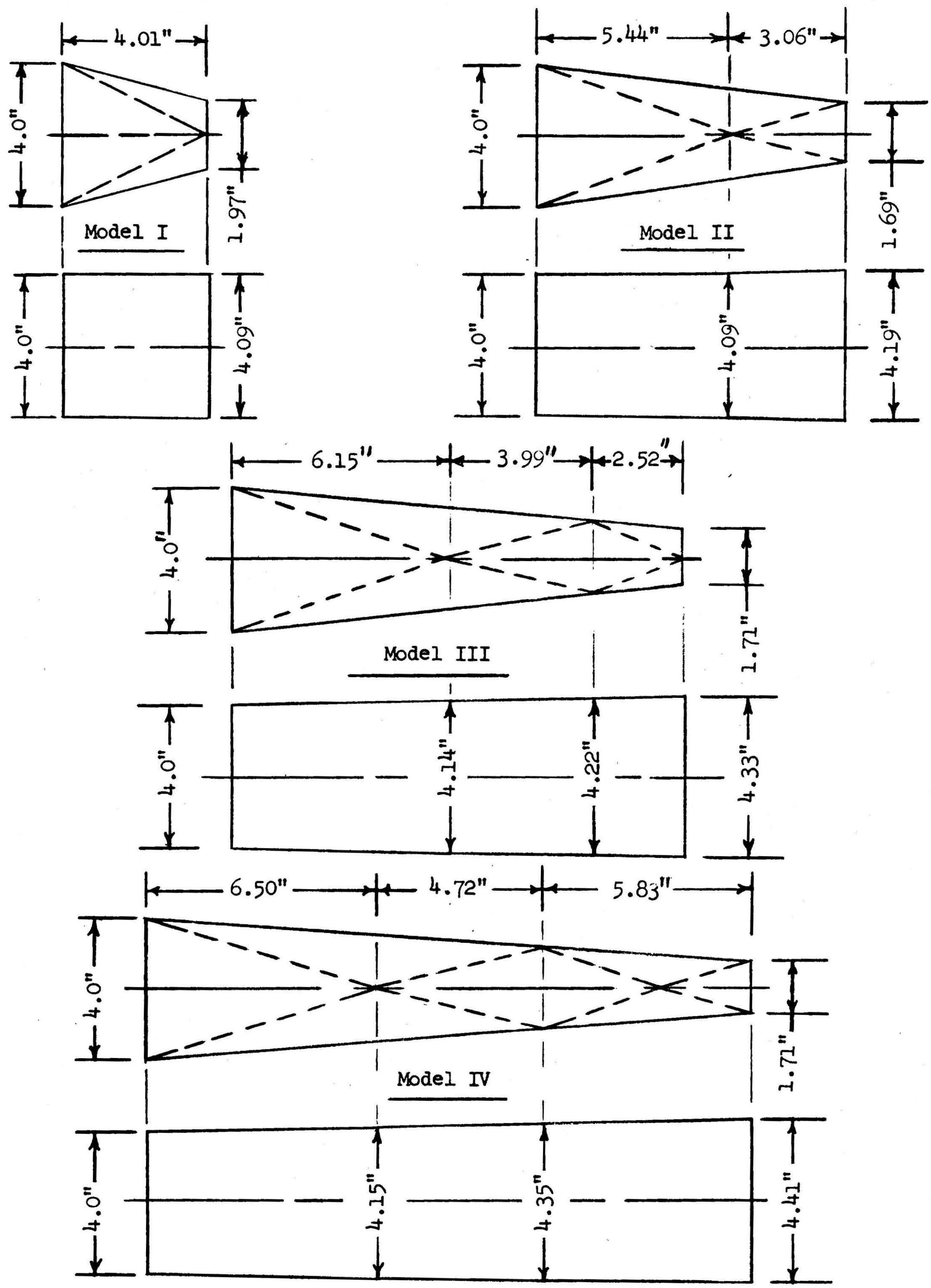


Figure 2.- Shock diagrams for models I through IV.

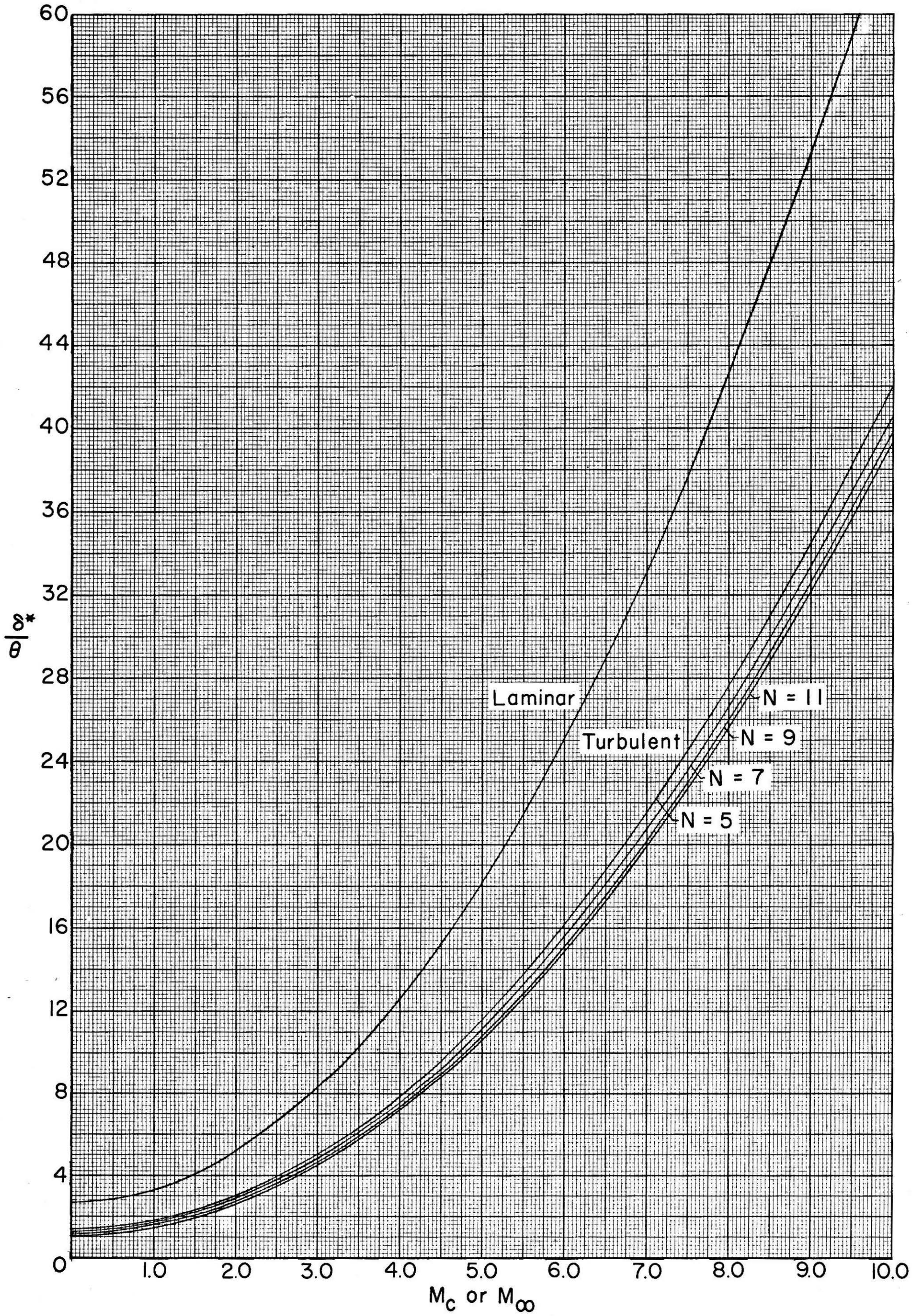


Figure 3.- Adiabatic wall values of shape factor $\left(\frac{\delta^*}{\theta}\right)$ for laminar and turbulent flow.

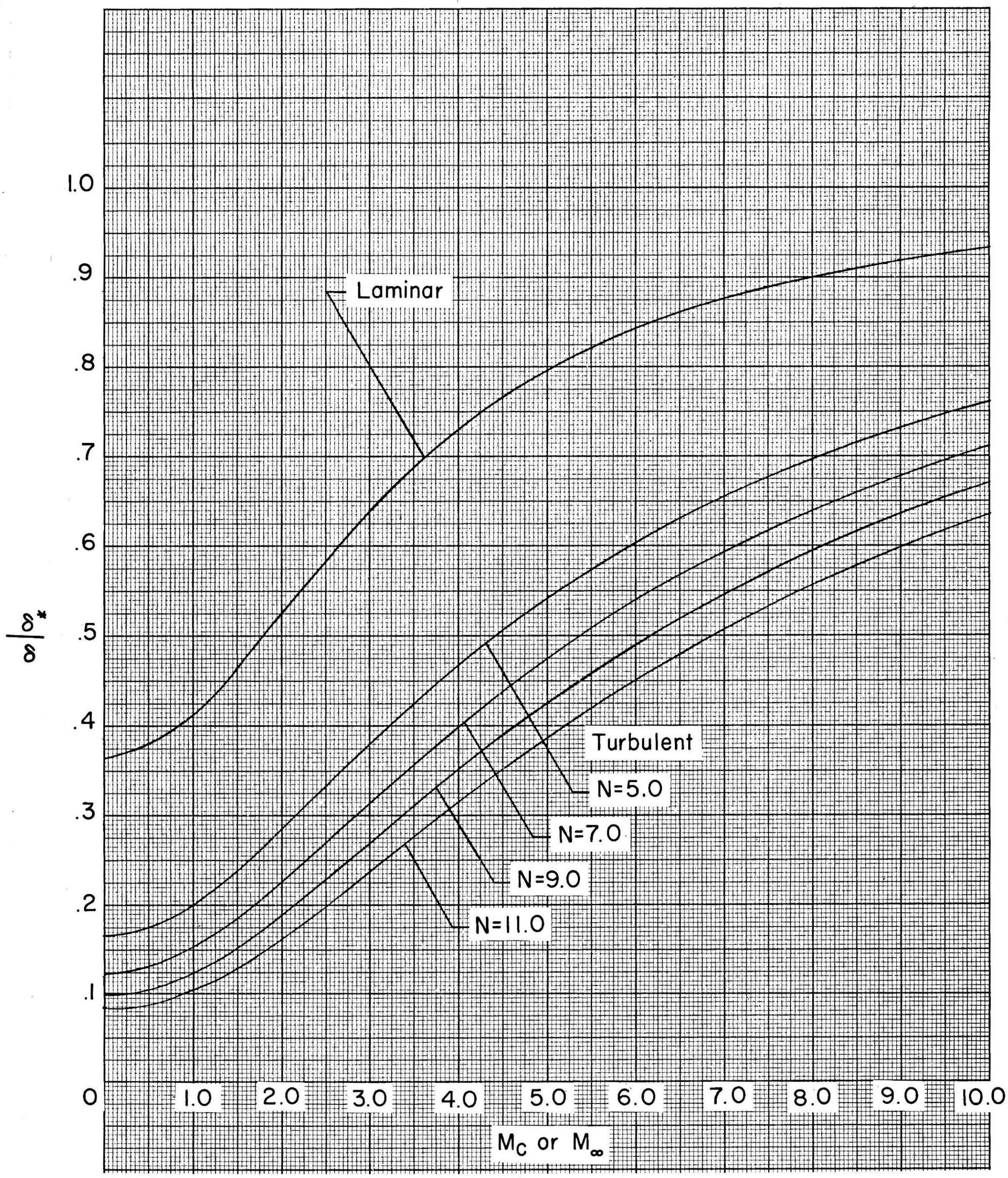


Figure 4.- Adiabatic wall $\frac{\delta^*}{\delta}$ versus free-stream Mach number M_∞ or core Mach number M_c.

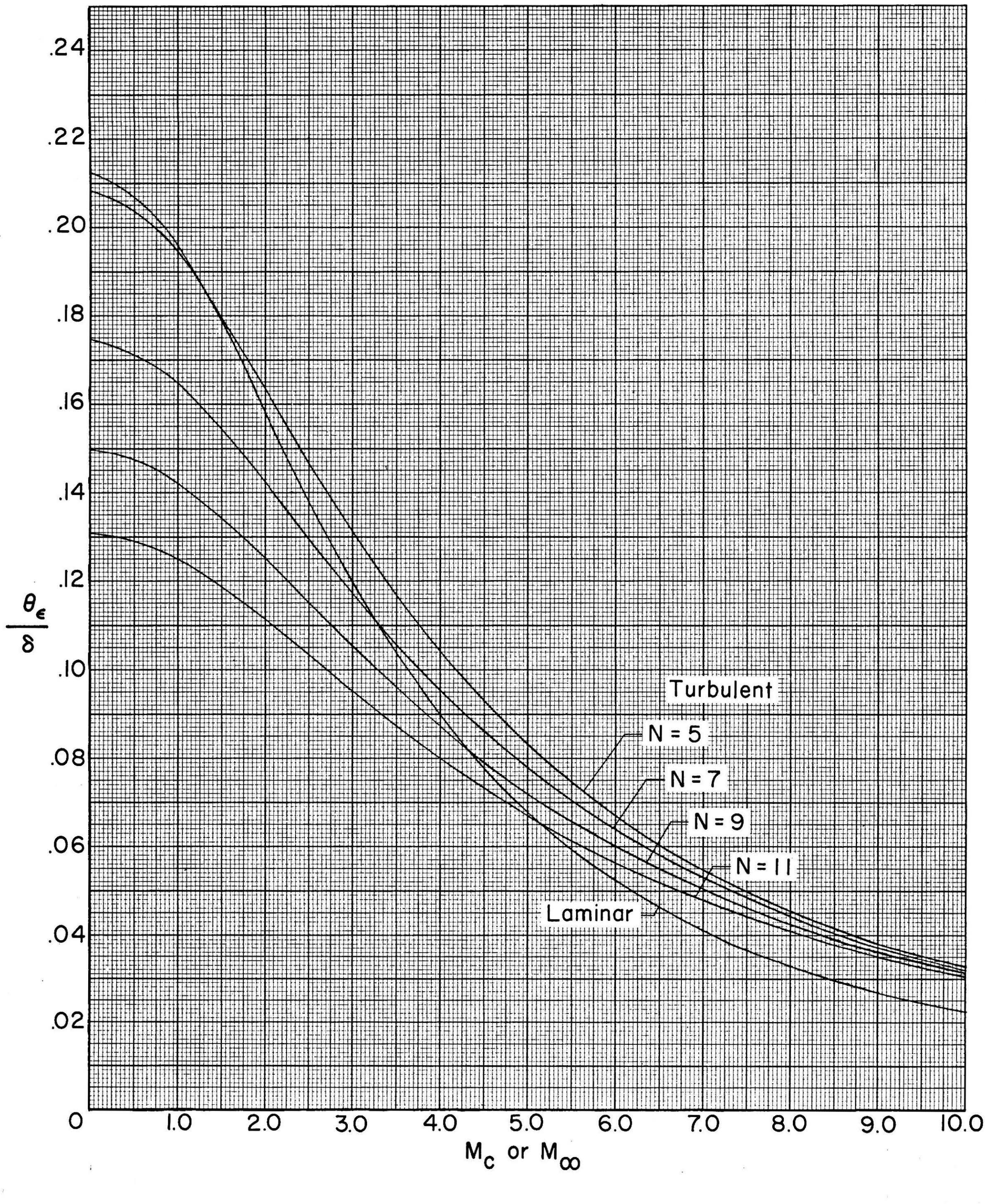


Figure 5.- Adiabatic wall values of $\left(\frac{\theta_\epsilon}{\delta}\right)$ for laminar and turbulent flow.

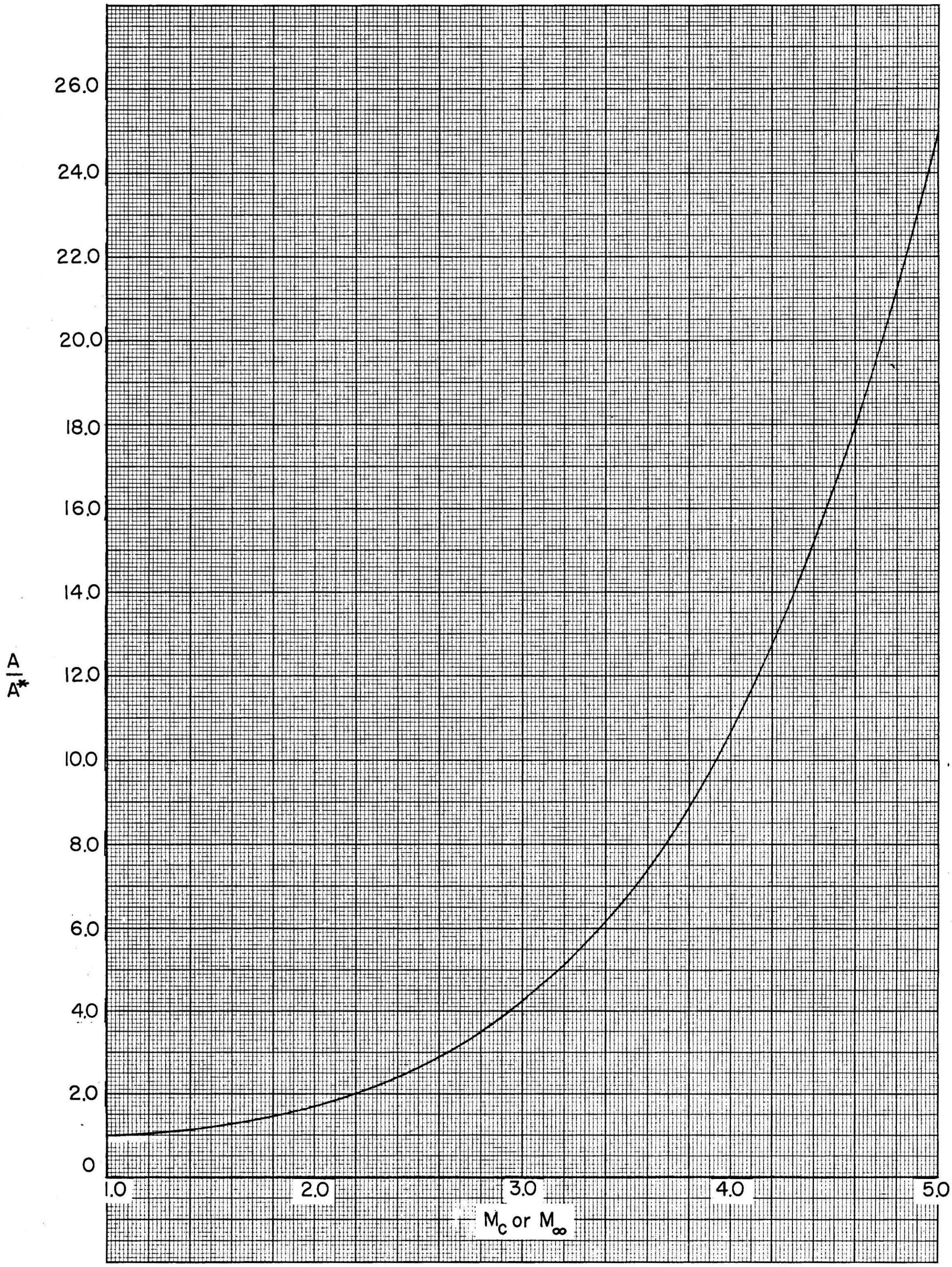


Figure 6.- Ratio of local area, A, to the area, A*, which would occur if the flow was contracted isentropically to a Mach number of 1.0.

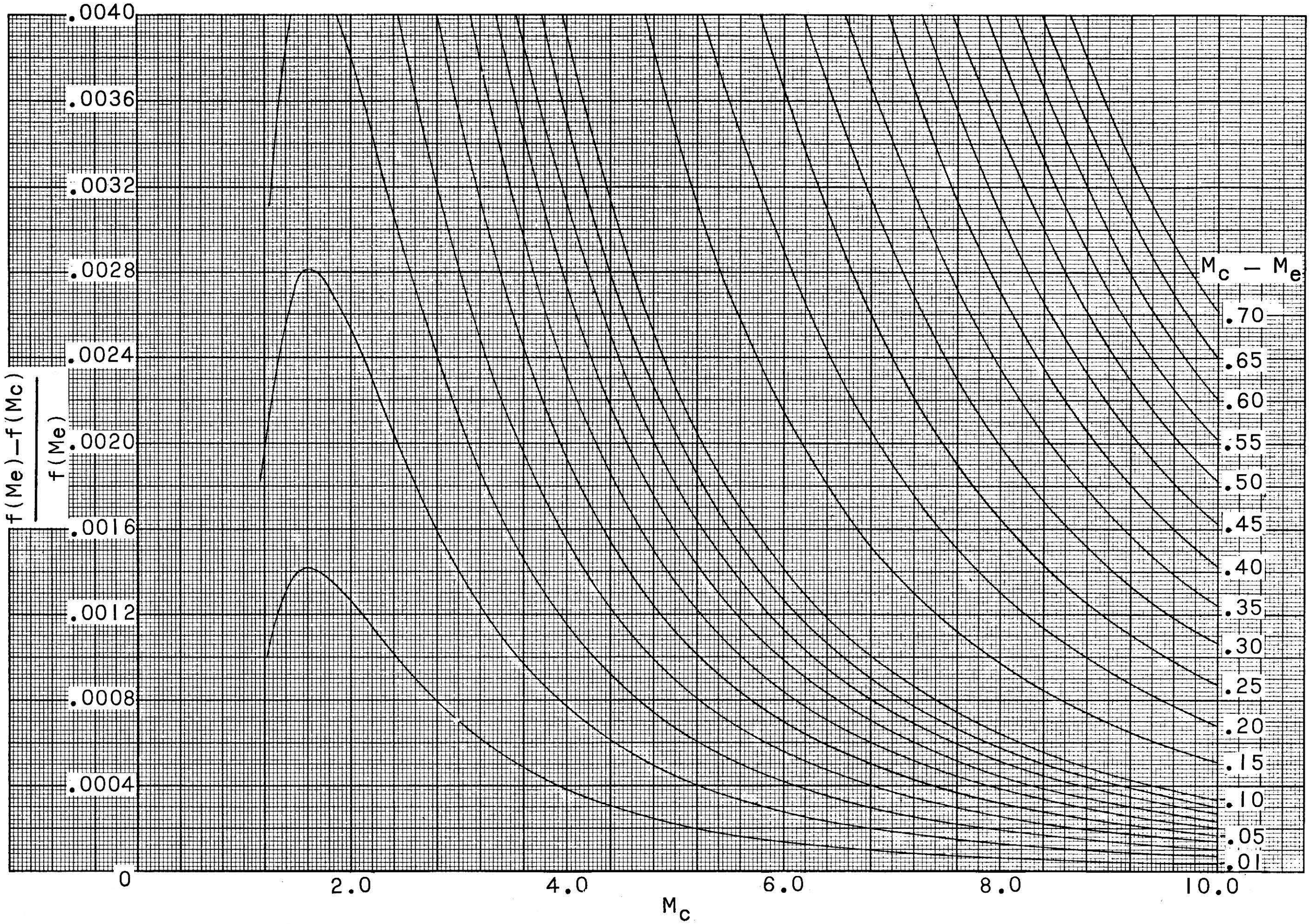


Figure 7(a).- Curves of $\left[\frac{f(M_e) - f(M_c)}{f(M_e)}\right]$ versus core Mach number M_c and constant $(M_c - M_e)$ used in friction calculations.

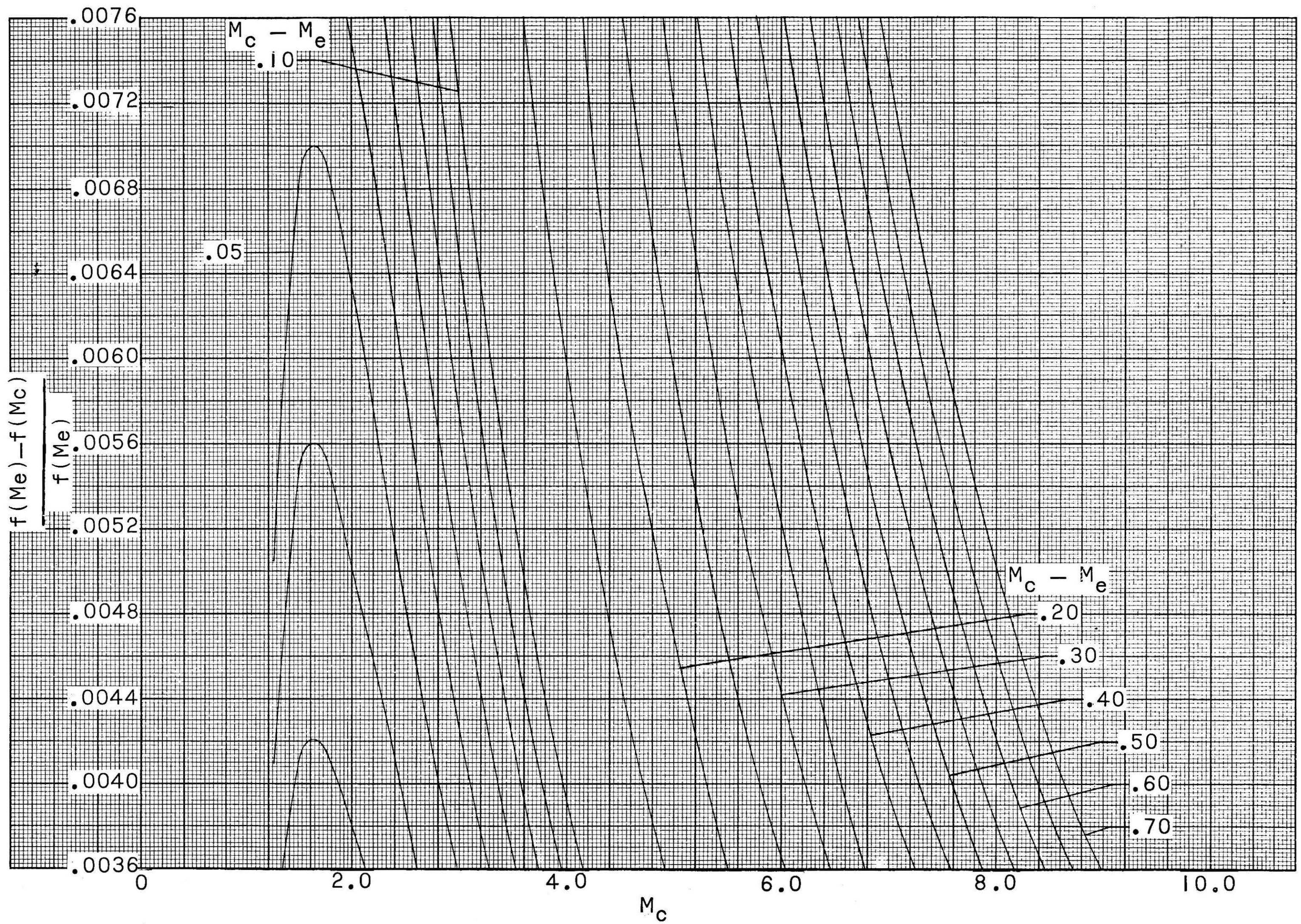

.0076
.0072
.0068
.0064
.0060
.0056
.0052
.0048
.0044
.0040
.0036
$\frac{f(Me)-f(Mc)}{f(Me)}$
0
2.0
4.0
6.0
8.0
10.0
M_c
$M_c - M_e$
.10
.05
$M_c - M_e$
.20
.30
.40
.50
.60
.70

Figure 7(b).

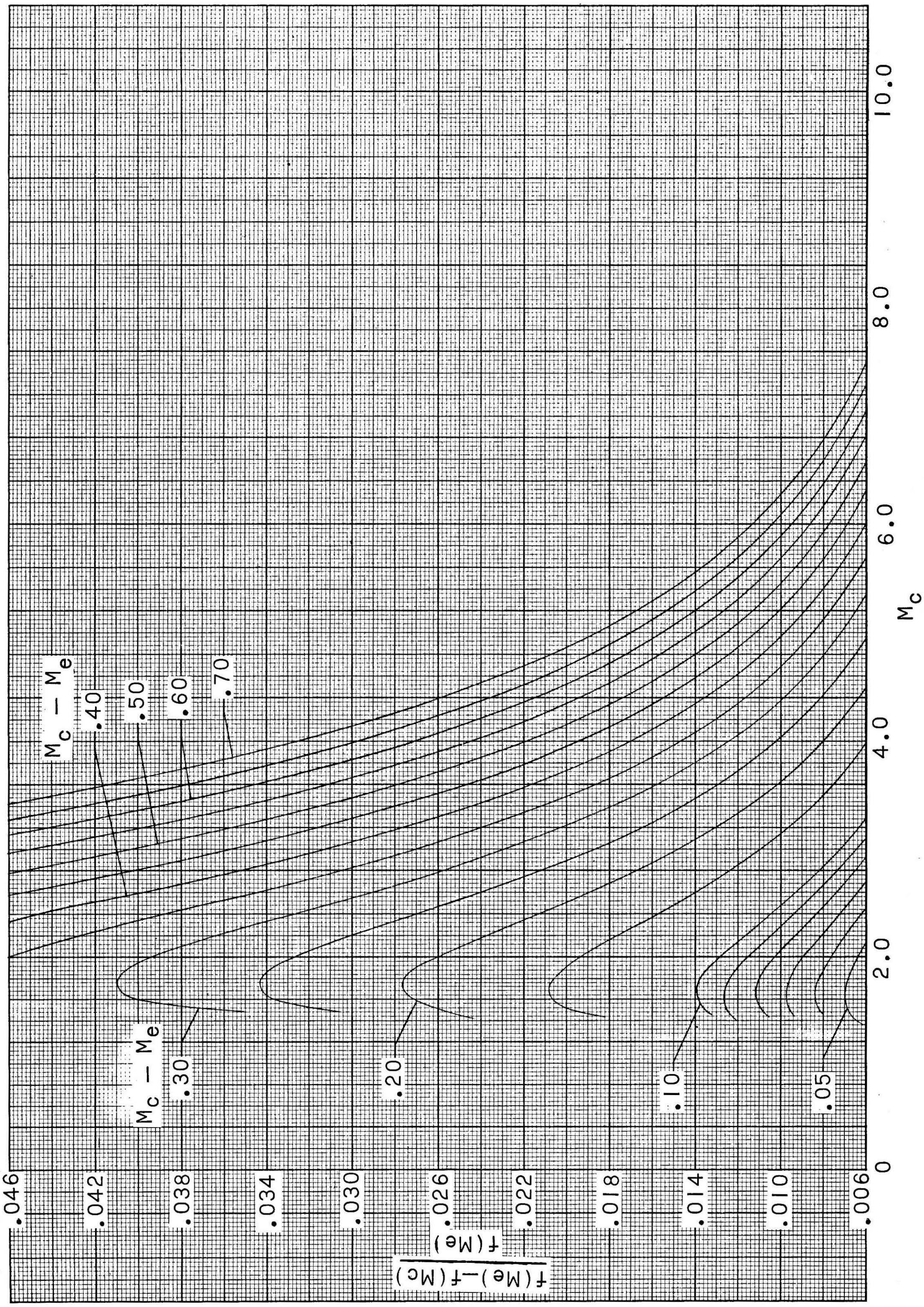

f(Me)—f(Mc) / f(Me)
.046
.042
.038
.034
.030
.026
.022
.018
.014
.010
.006
0
2.0
4.0
6.0
8.0
10.0
M_c
M_c — M_e
.40
.50
.60
.70
M_c — M_e
.30
.20
.10
.05

Figure 7(c).

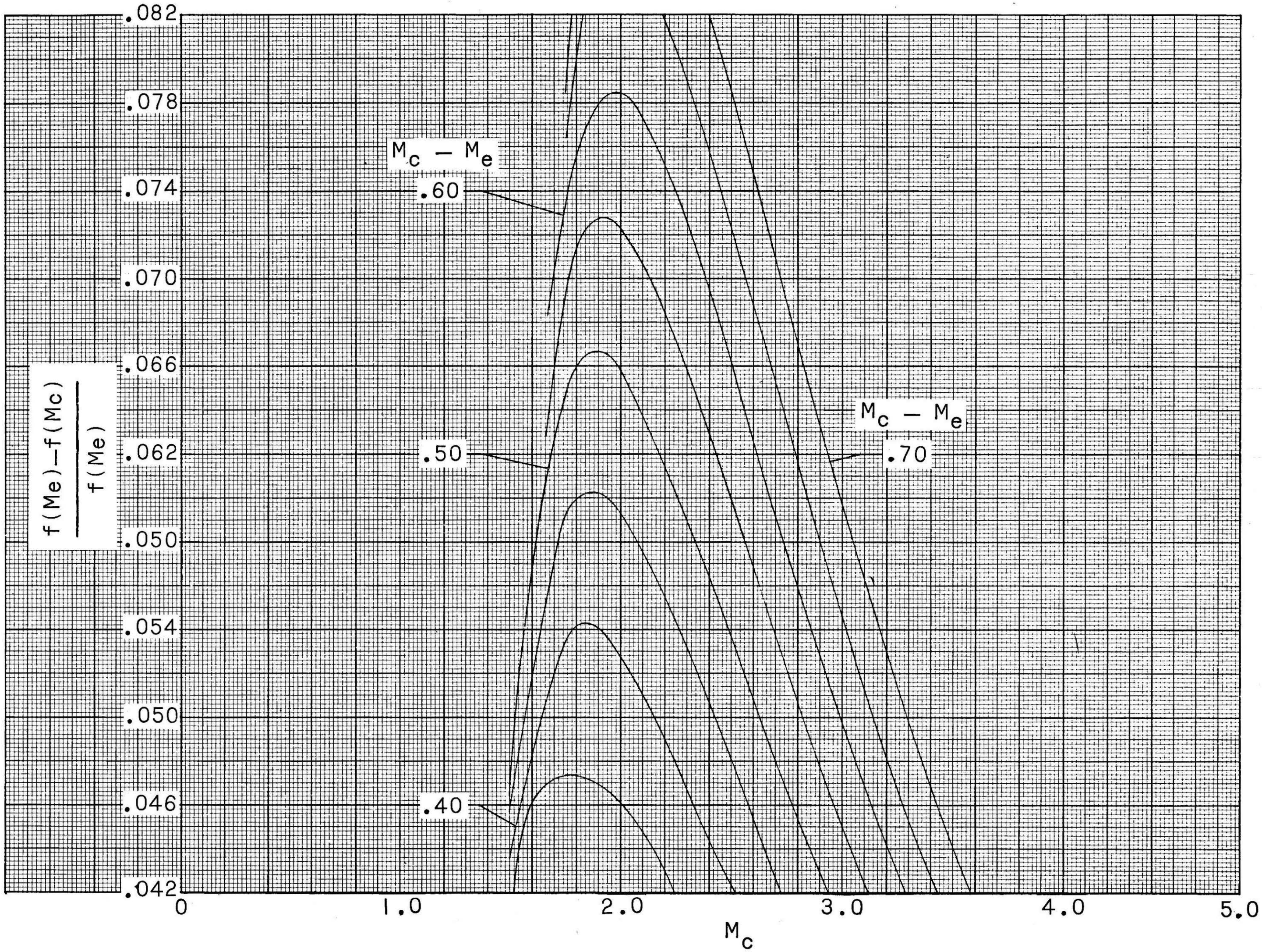

.082
.078
.074
.070
.066
.062
.050
.054
.050
.046
.042
f(Me)—f(Mc) / f(Me)
0
1.0
2.0
3.0
4.0
5.0
Mc
Mc — Me
.60
.50
.40
Mc — Me
.70

Figure 7(d).

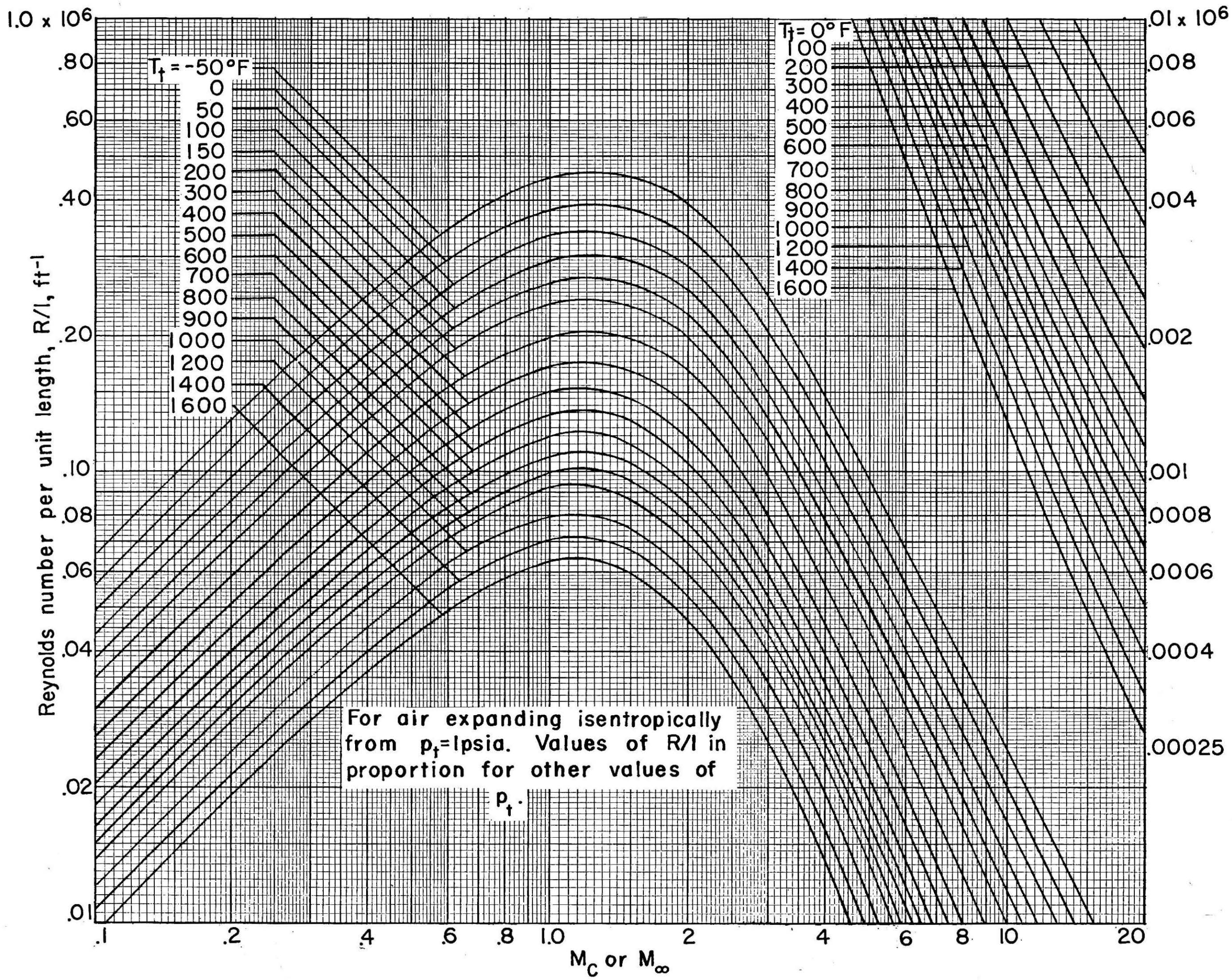


Figure 8.- Variation of Reynolds number per unit length with Mach number for various total temperatures. Perfect gas, $\gamma = 7/5$.

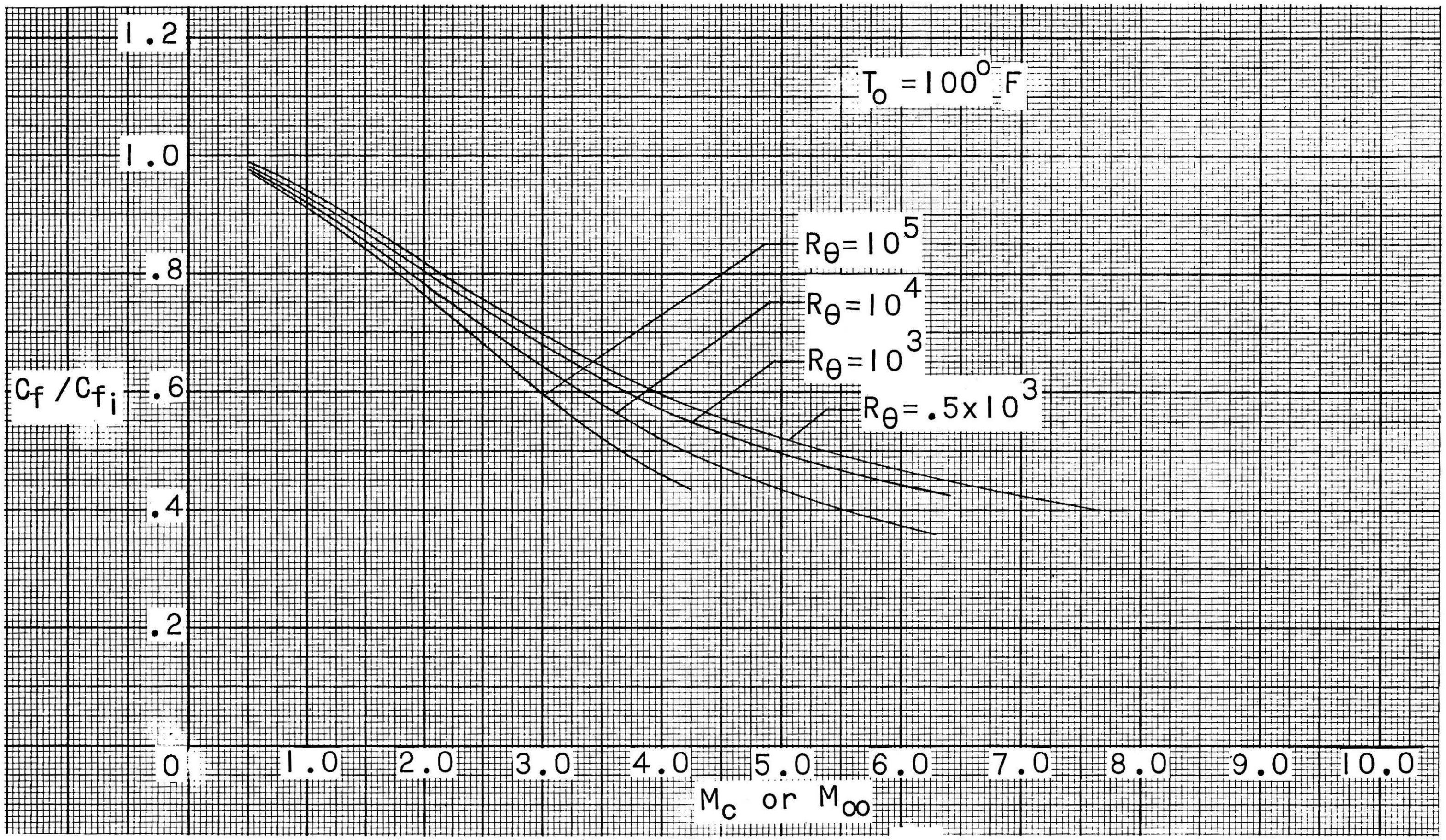
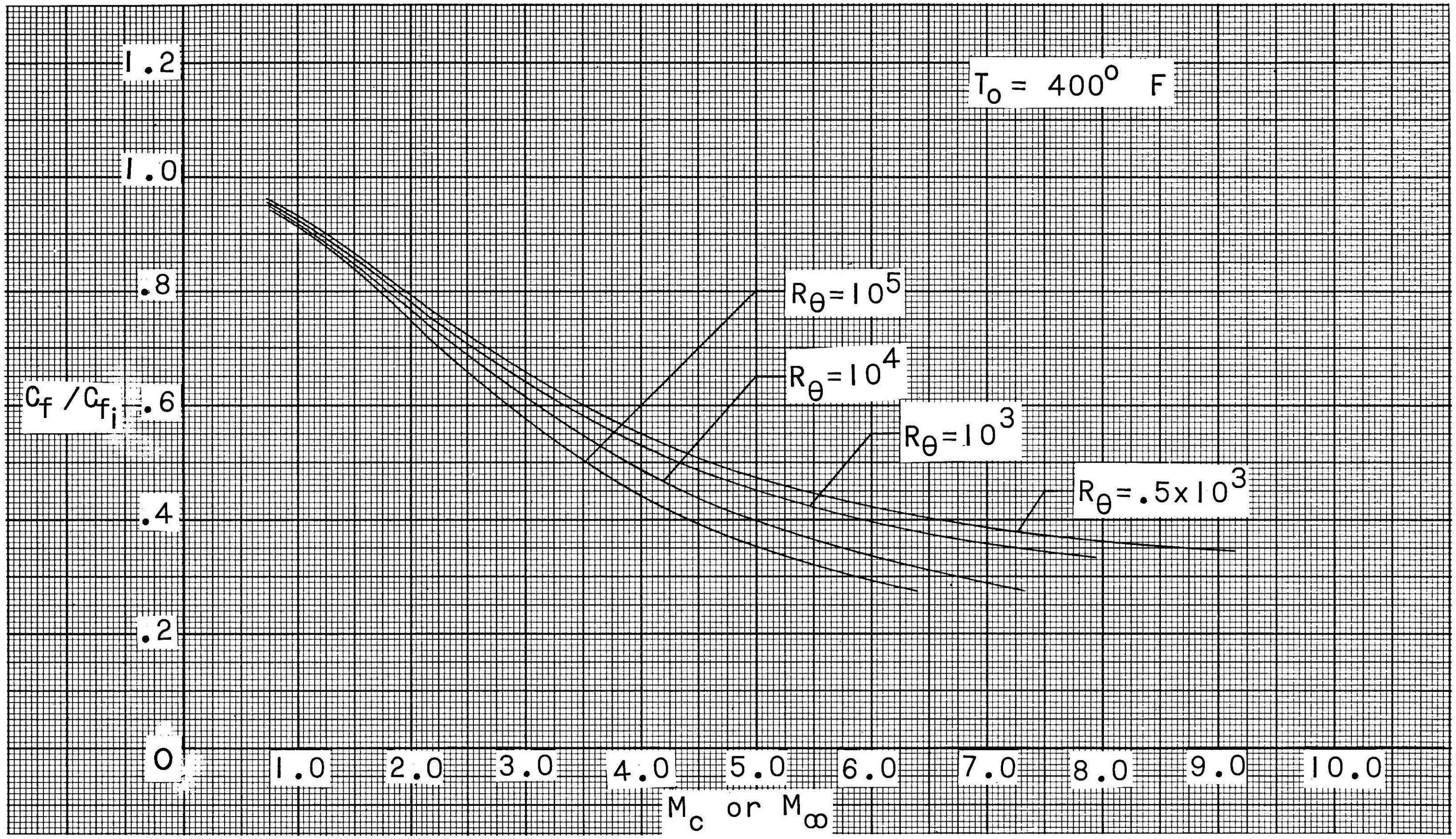
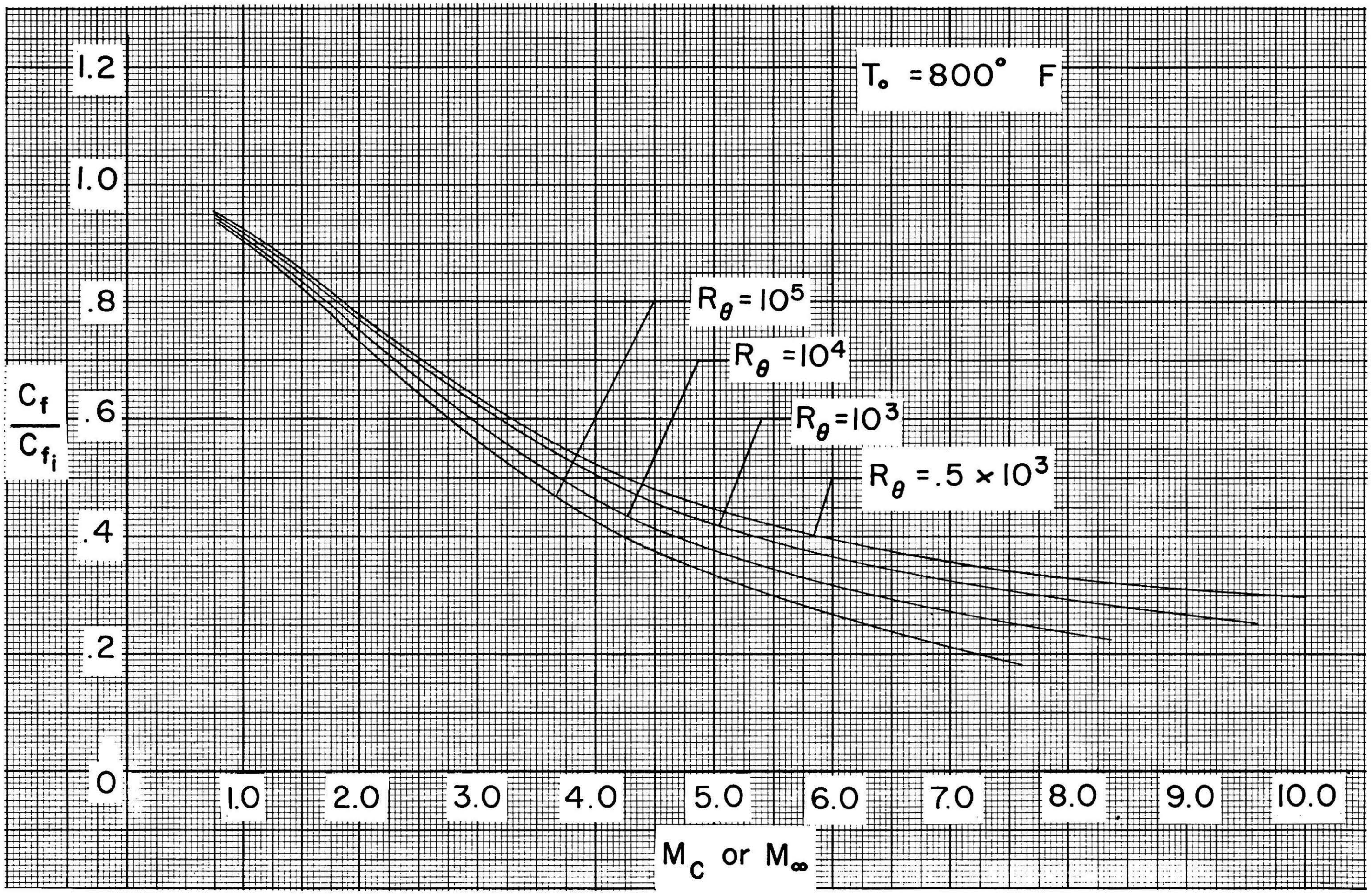


Figure 9(a).- Adiabatic wall values of $\frac{C_f}{C_{fi}}$ for turbulent flow at a stagnation temperature $T_0 = 100^o$ F.



(b) Adiabatic wall values of $\frac{C_f}{C_{fi}}$ for turbulent flow at a stagnation temperature T_0 = 400° F.

Figure 9.- Continued.



(c) Adiabatic wall value of $\frac{C_f}{C_{fi}}$ for turbulent flow at a stagnation temperature $T_0 = 800^\circ$ F.

Figure 9.- Concluded.

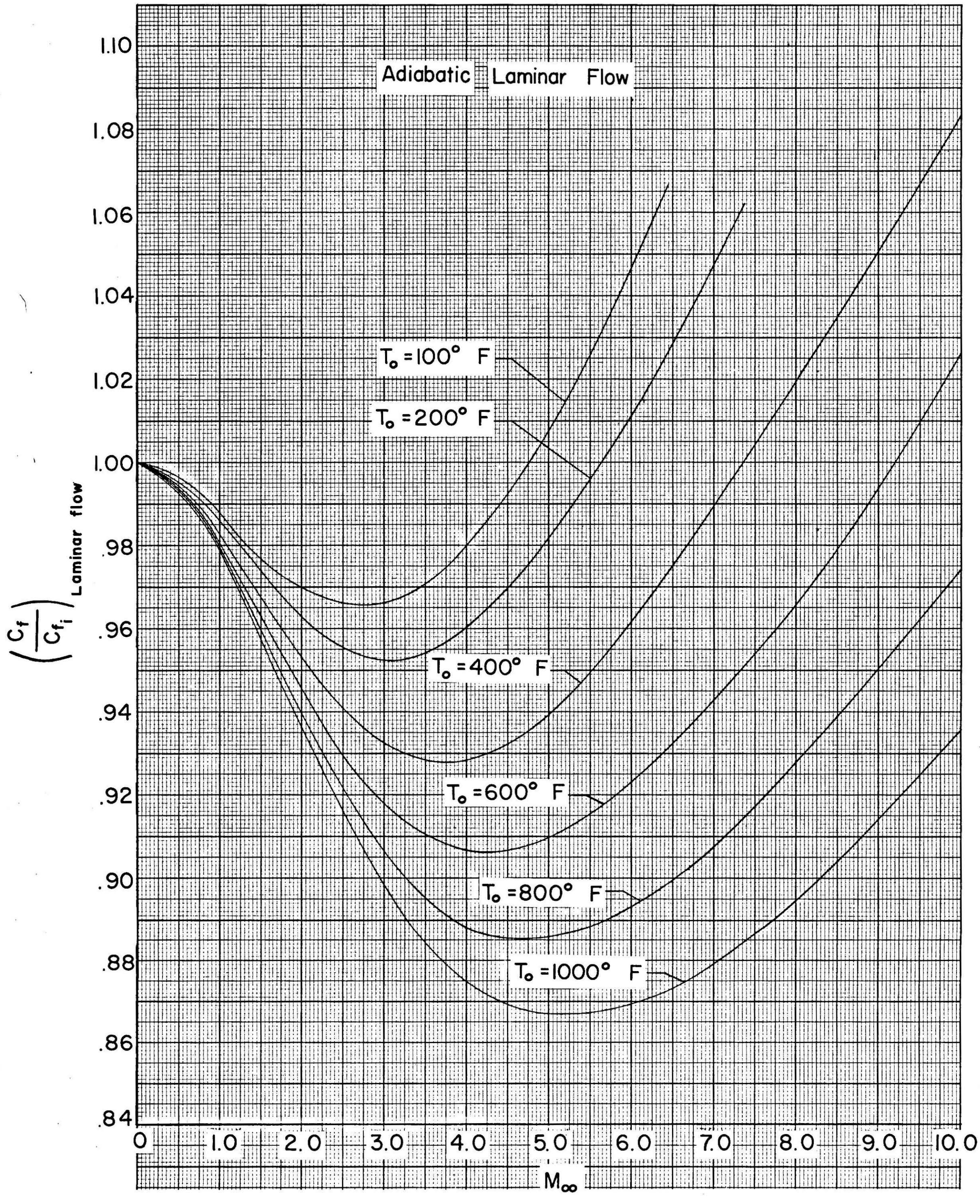


Figure 10.- Adiabatic wall values of $\frac{C_f}{C_{fi}}$ for laminar flow.

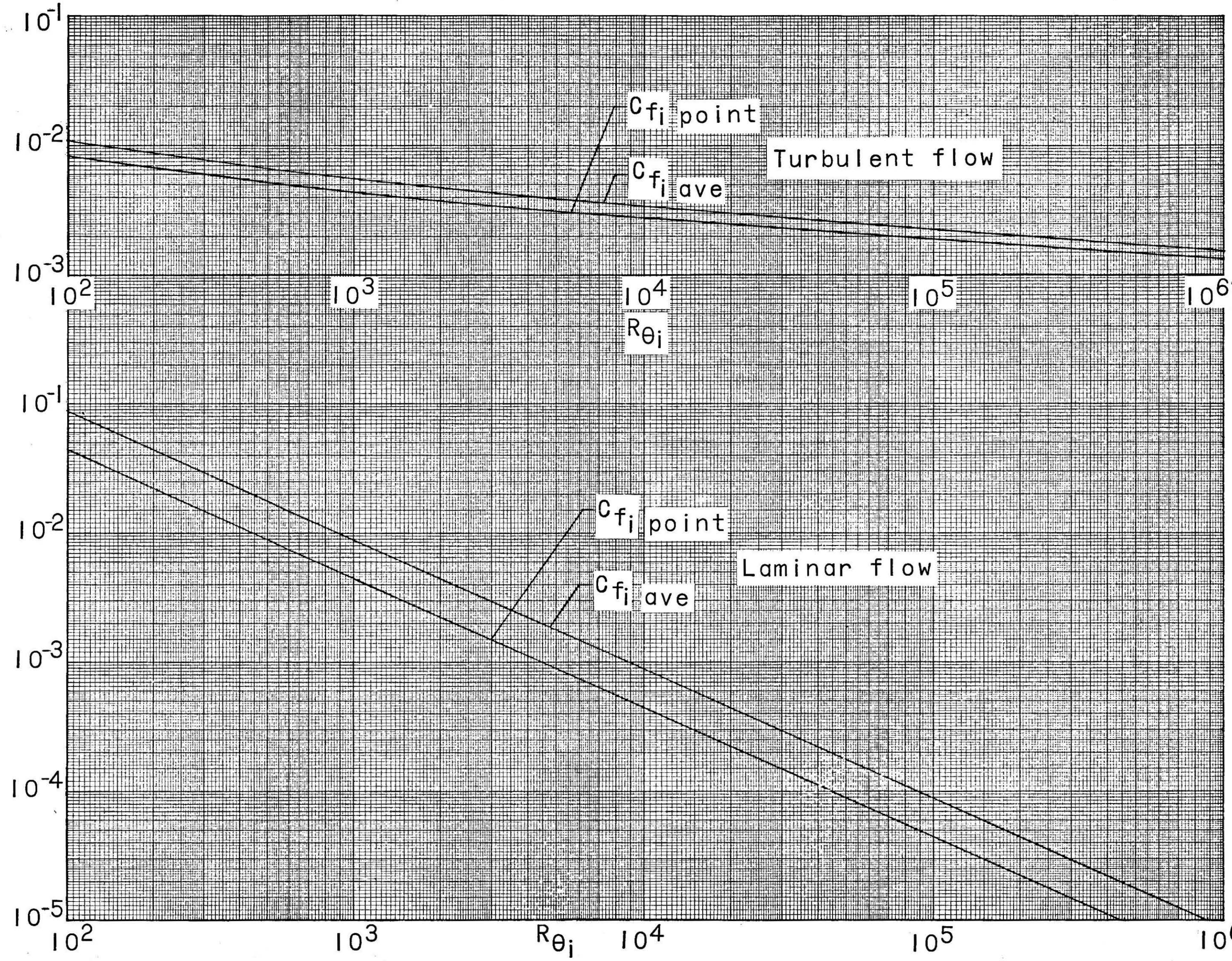


Figure 11.- Incompressible laminar and turbulent average and point friction factor curves.

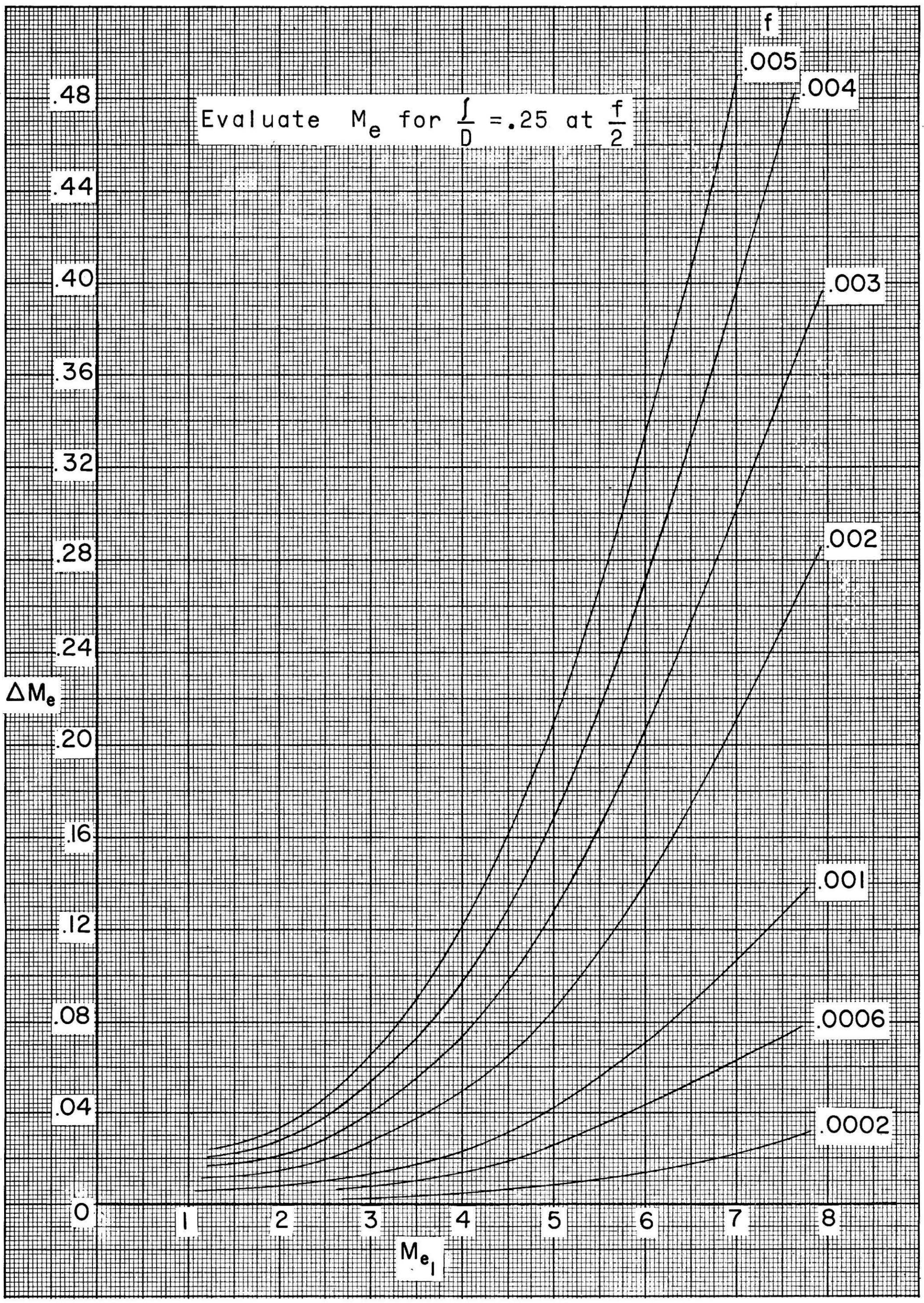


Figure 12.- Effective Mach number loss for $\frac{l}{D} = 0.5$ at constant friction factor f.

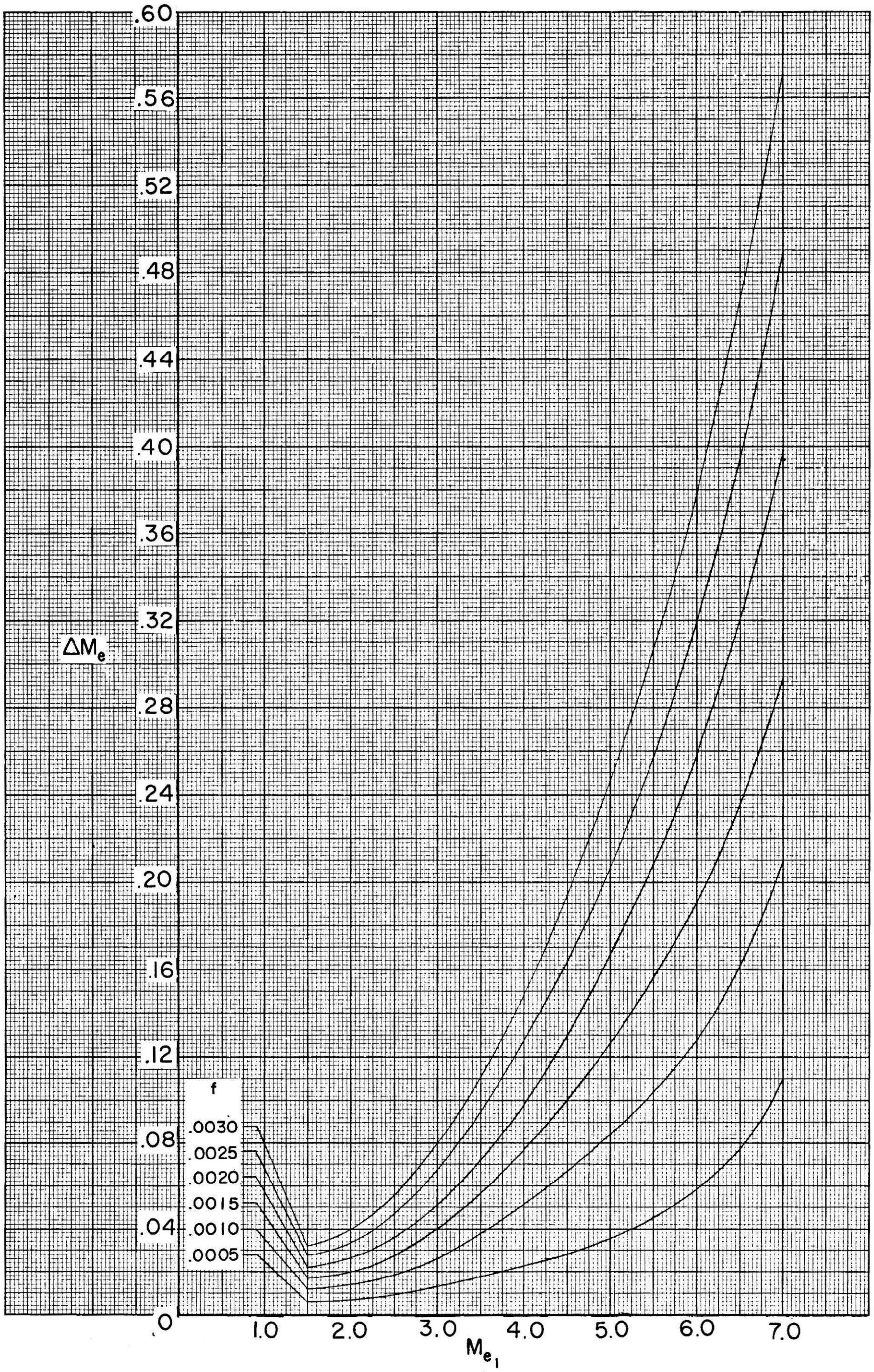


Figure 13.- Effective Mach number loss for $\frac{l}{D} = 1.0$ and a constant friction factor f.

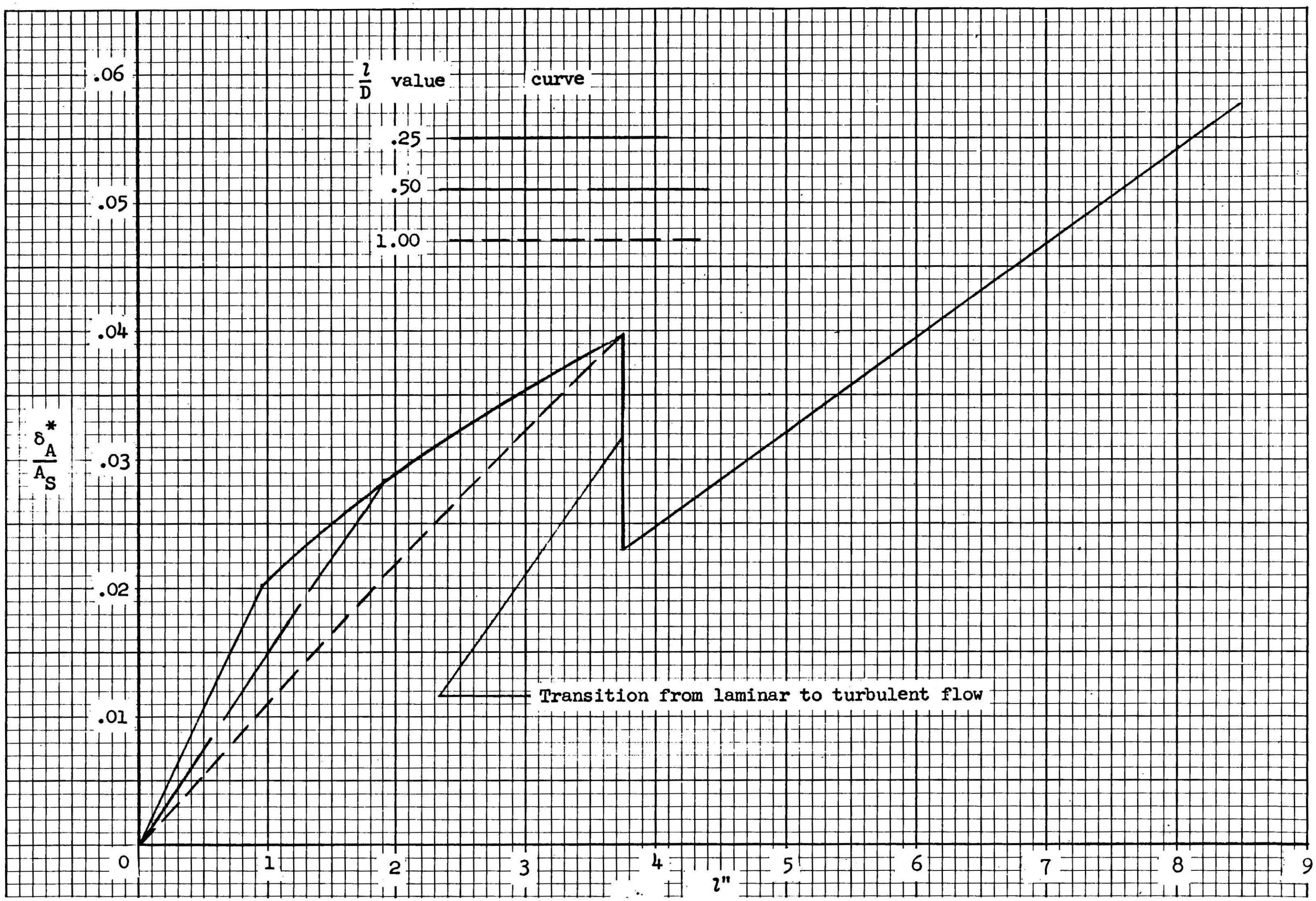


Figure 14.- Model II curves of $\frac{\delta_A^*}{A_s}$ versus l (in.) for $\frac{l}{D}$ = 0.25, 0.50, and 1.0.

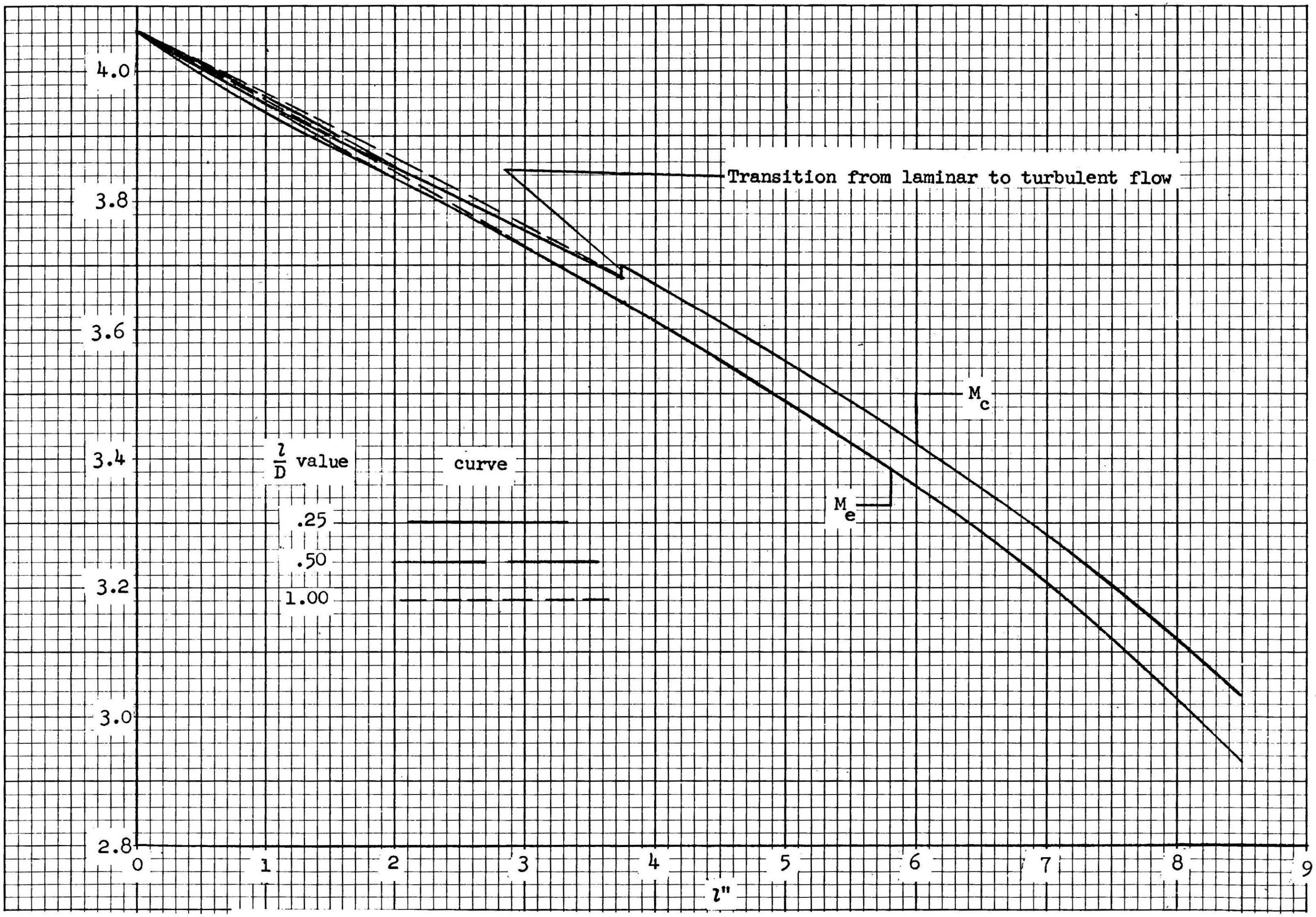


Figure 15.- Model II curves of M_e and M_c versus l (in.) for $\frac{l}{D}$ = 0.25, 0.50, and 1.0.

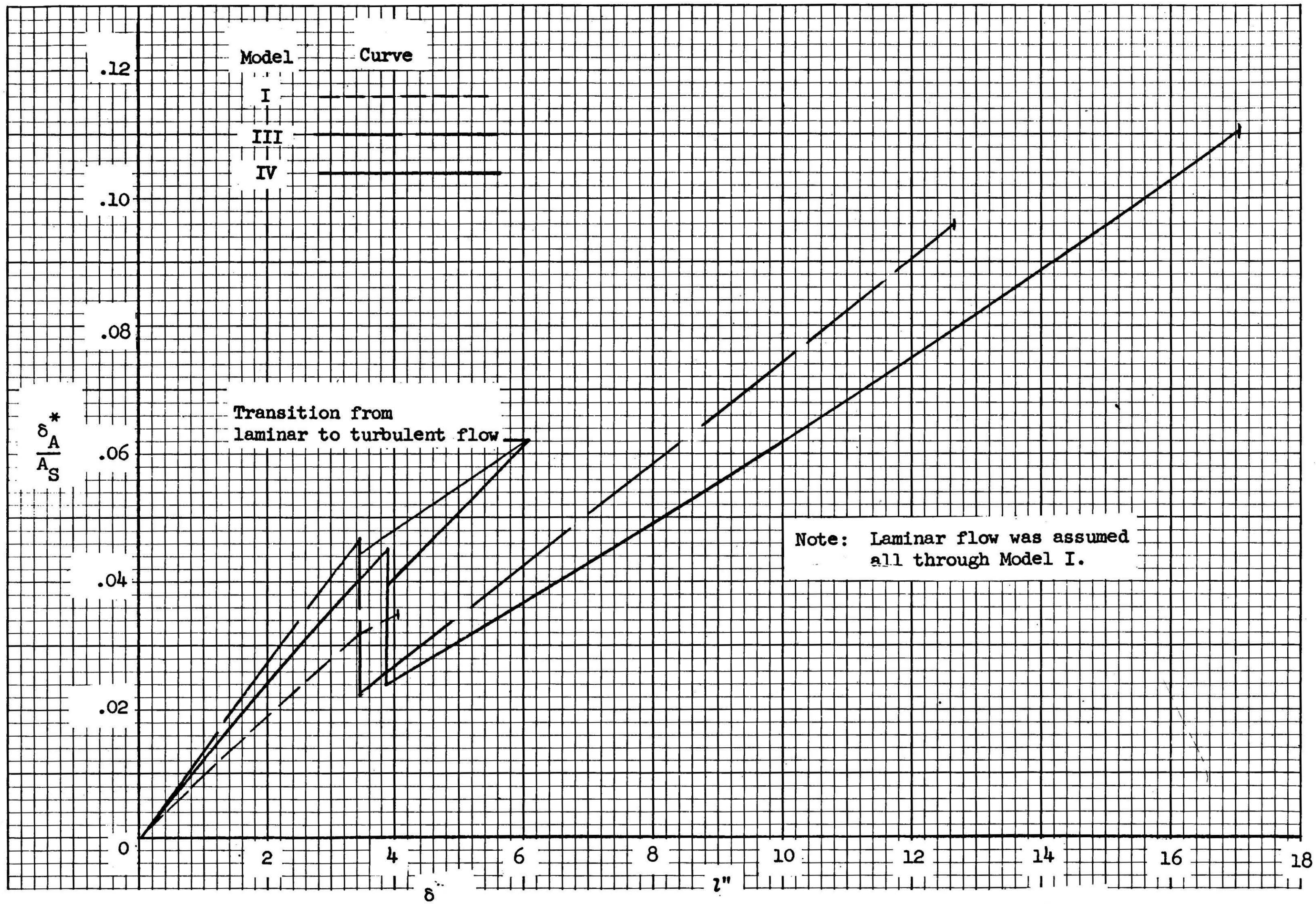


Figure 16.- Curves of $\frac{\delta_A^*}{A_S}$ versus l (in.) for models I, III, and IV for $\frac{l}{D} = 1.0$.

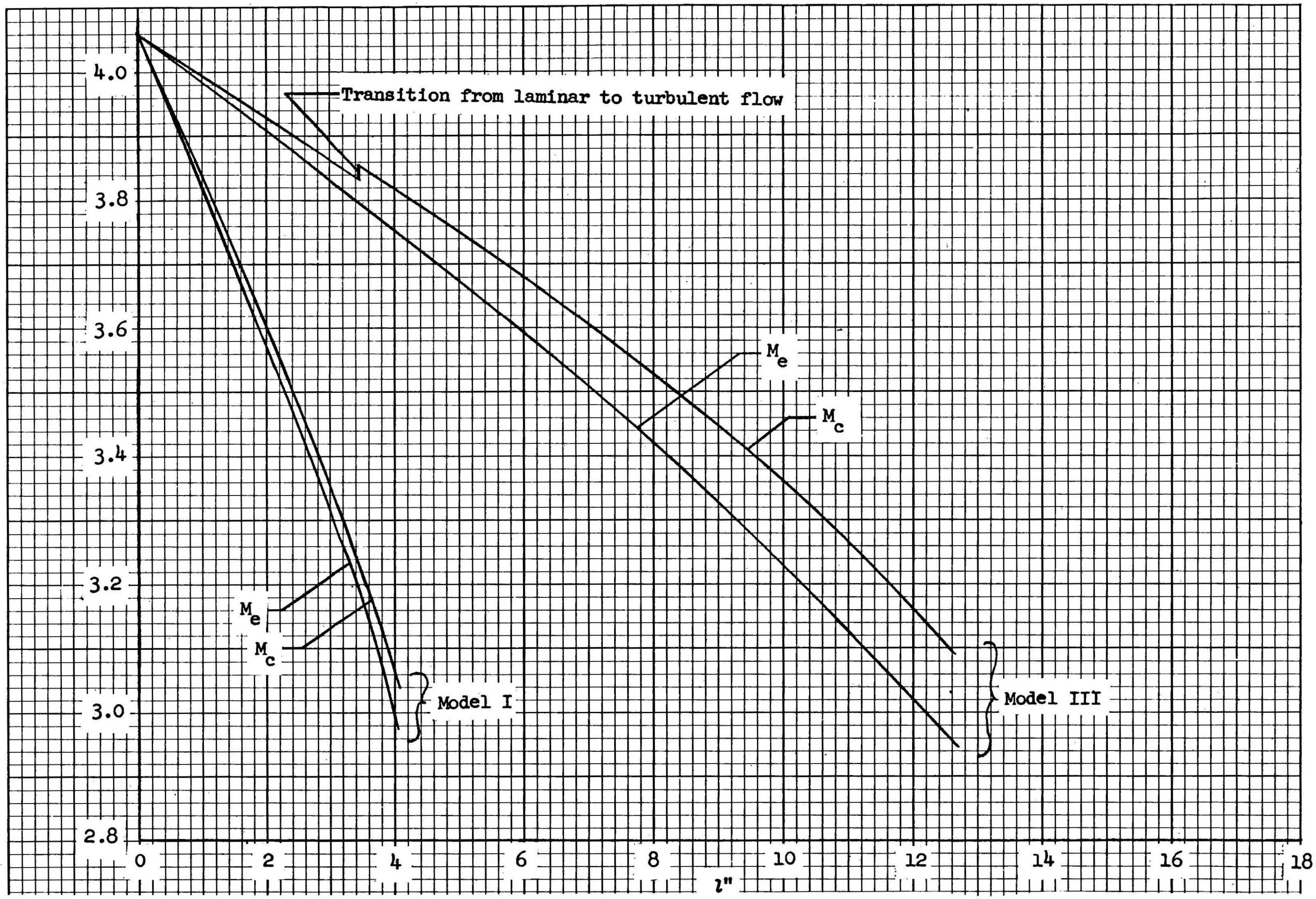
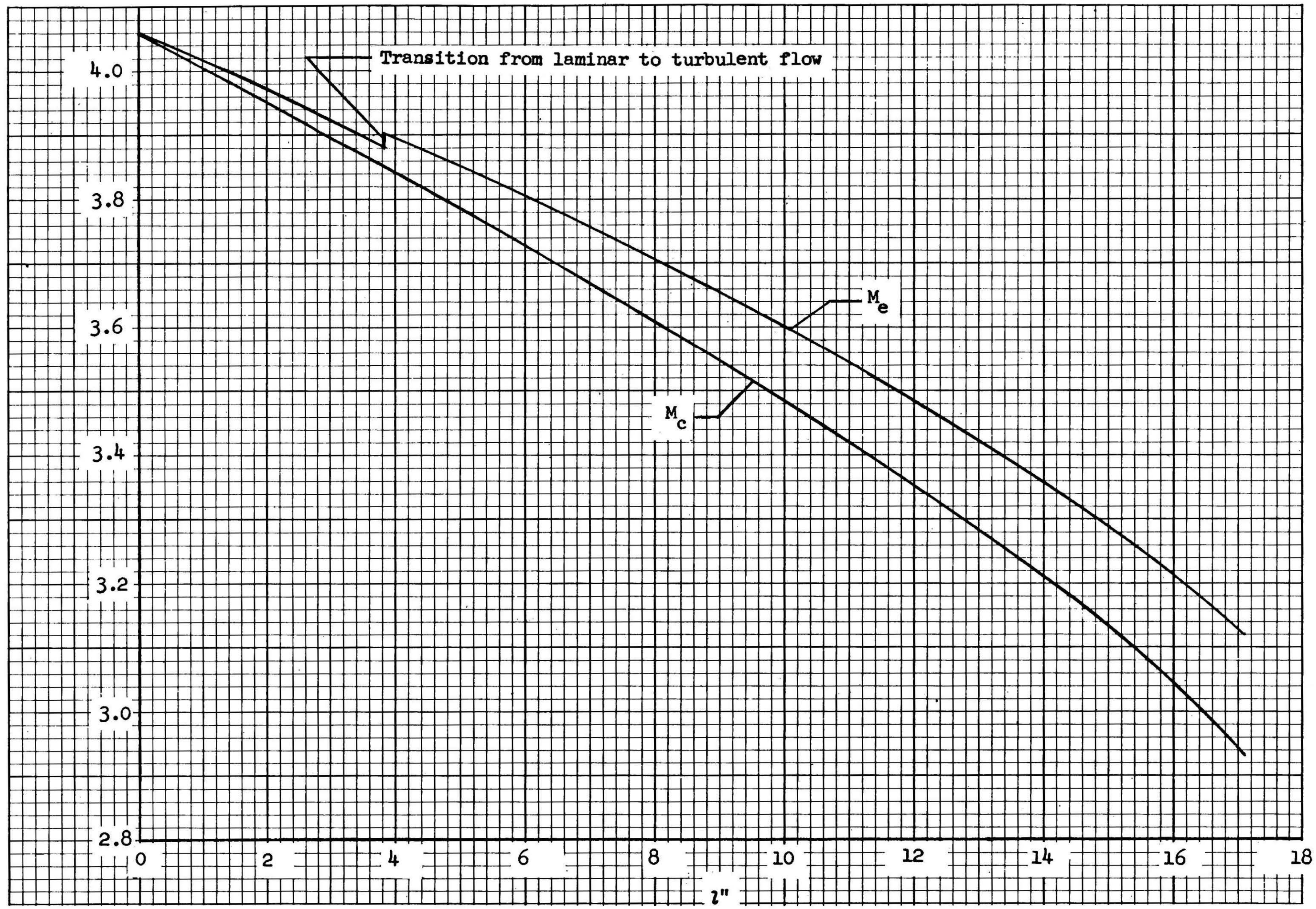


Figure 17(a).- Curves of M_c and M_e versus l (in.) for models I and III.



(b) Curves of M_c and M_e versus l (in.) for model IV and $\frac{l}{D} = 1.0$.

Figure 17.- Concluded.

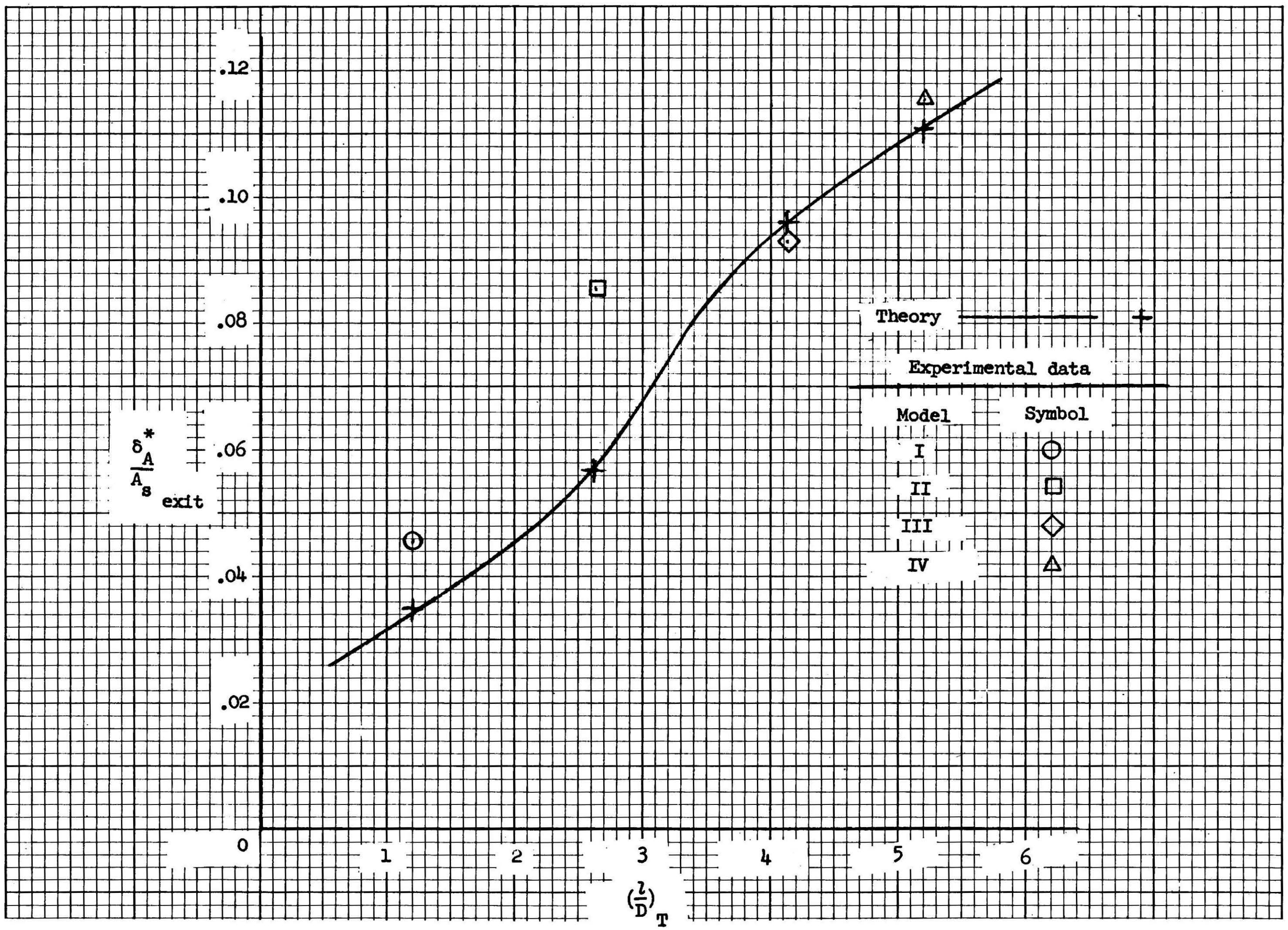


Figure 18.- Plot of experimental and theoretical model exit values of $\frac{\delta_A^*}{A_s}$ versus model total over-all $\left(\frac{l}{D}\right)_T$.

APPLICATION TO SUPERSONIC DIFFUSERS OF A

ONE-DIMENSIONAL FLUID FLOW EQUATION

OF THE PFAFFIAN TYPE

By S. Z. Pinckney

ABSTRACT

The equation investigated in the present paper is

$$\frac{(1 - M_e^2)}{M\left(1 + \frac{\gamma - 1}{2} M_e^2\right)} dM_e + \frac{dA}{A} - 2\gamma M_e^2 f d\left(\frac{l}{D}\right) = 0$$

which is the friction-loss equation for one-dimensional duct flow. An investigation into the complete solution of an equation of the form,

$$P_a dx + Q_a dy + R_a dz = 0$$

which is the form of the equation considered, revealed two methods of solution. The first method of solution applies when the equation of condition is satisfied and the second method of solution applies when the equation of condition is not satisfied. The equation of condition is

$$P_a\left(\frac{\partial Q_a}{\partial z} - \frac{\partial R_a}{\partial y}\right) + Q_a\left(\frac{\partial R_a}{\partial x} - \frac{\partial P_a}{\partial z}\right) + R_a\left(\frac{\partial P_a}{\partial y} - \frac{\partial Q_a}{\partial x}\right) = 0$$

An equation for which the equation of condition is not satisfied is called a Pfaffian equation. The equation investigated in the present

paper is of the Pfaffian type. Upon applying the method of Pfaff to the equation, the solution was found to be extremely complicated and awkward.

In order to simplify the solution the term $\frac{dA}{A}$ was dropped and the resulting equation integrated assuming the friction factor, f, constant through the integration. In order to apply the resulting constant area friction loss equation to supersonic inlets, having varying areas, the assumption was made that the friction losses through a variable area supersonic inlet could be approximated by the friction losses through a series of constant area elements. The complicated numerical method used to apply the constant area friction loss equation to a variable area supersonic inlet is developed in detail. Examples were calculated to determine the effect of element size on the calculated results; element sizes $\frac{l}{D} = 0.25$, 0.50, and 1.0 were considered. For the examples considered, the resulting conclusion was that the choice of element size does not affect the calculated effective Mach number, M_e, change due to friction through a two-dimensional supersonic inlet except near the leading edge of the inlet. After further calculations using element sizes equal to the total $\left(\frac{l}{D}\right)$ of the inlets considered, it was found that this conclusion has to be restricted in that the element size has to be taken small as far as the total $\left(\frac{l}{D}\right)$ of the inlet. A comparison with experimental data taken for the inlet models calculated show the calculations to correspond well with experimental results when shock strengths are small.