AN ALGORITHM FOR A TWO-PHASE

STRATEGY FOR PREVENTIVE MAINTENANCE

by

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I. INTRODUCTION

Preventive maintenance is of paramount importance in many systems. A well designed, well engineered, thoroughly tested system should never fail in operation. But experience speaks otherwise. The failure of a system at any time is inopportune. But at certain times it is definitely catastrophic. The failure of the system affects not only the engineer and the manufacturer; quite often the user of the system bears the heaviest consequences. The failure of a system should be avoided. Preventive maintenance is planned maintenance. Its purpose is to minimize the probability of system failure.

Preventive maintenance does not eliminate all system failures. It will become clear later on that with any realistic preventive maintenance there will always be some probability that the equipment will fail prior to its scheduled inspection time. Therefore, the main concern is when preventive maintenance should be scheduled in order to reduce the number of failures. This involves consideration of the cost of downtime and maintenance services.

A good maintenance policy contributes a great deal in reducing the costs incurred due to random system failure. The government has found that in some instances the cost of maintenance for unreliable equipment is in excess of ten times the original cost and that a good maintenance program reduces this expense considerably (1).

In industry the scheduling of the frequency of preventive maintenance has been primarily based on experience or convenience. The schedules commonly involve:

1. Inspection on a daily or weekly basis.
2. Cleaning, adjusting and repairing on a biweekly or monthly basis.
3. Overhauling semi-annually or annually.

If preventive maintenance is to be economically advantageous, then it must be scheduled at a time such that the probability of system failure is low. The cost of system failure is exhorbitantly high.

Quite a few models have been developed for preventive maintenance. Initially, preventive maintenance policies were developed without consideration of the deterioration of the system. Some papers published recently have rectified this situation. But the models developed have been essentially of one unit system. The purpose of this research is to progress one more step--to consider deterioration as well as multiple failures. A system has q identical components and it fails when k $(k < q)$ components fail. This research aims at developing an algorithm for a two-phase strategy of scheduling maintenance. When the system is in a transient state, there will be frequent inspection. But when the system has reached steady state, there will be less frequent inspection until the system fails. The criterion for scheduling maintenance would be the critical number of units. Maintenance will be scheduled if the number of component failures is equal to or greater than the critical number of units.

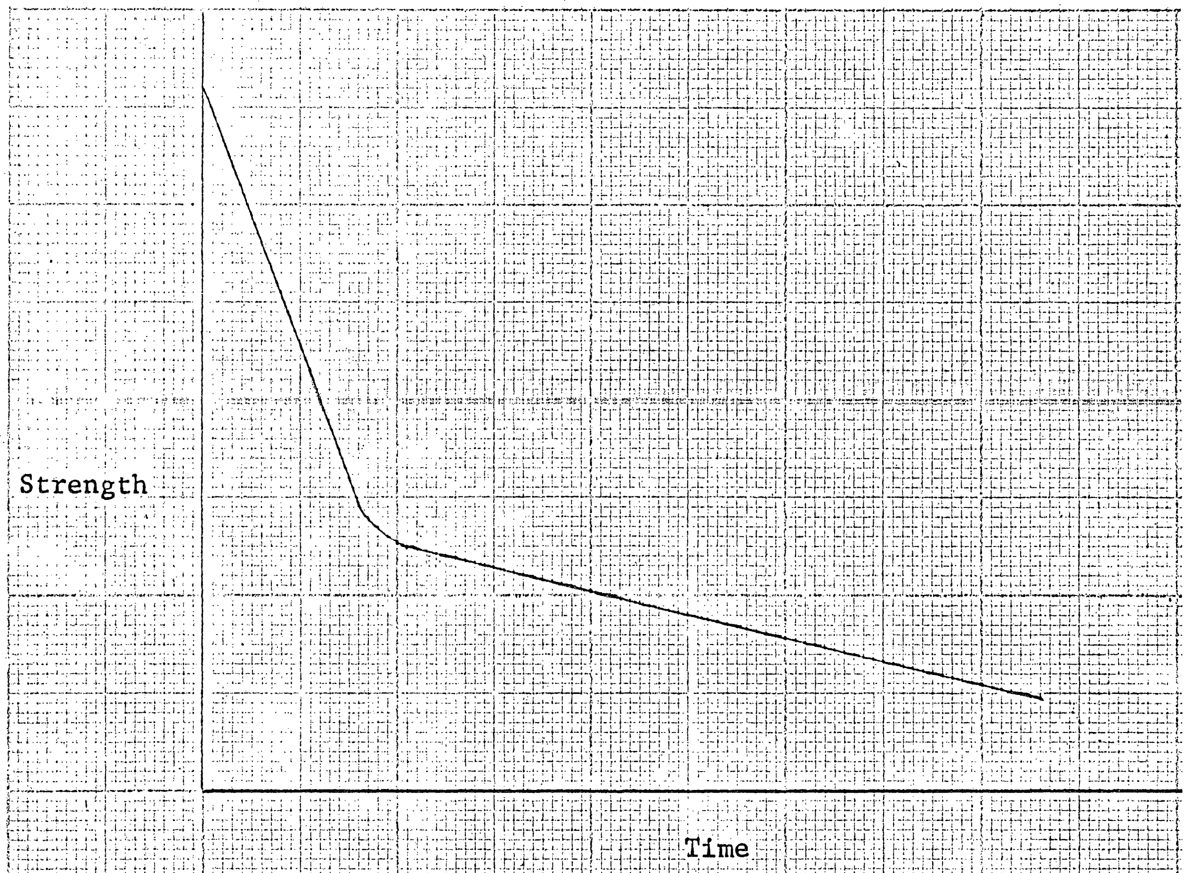

Fig. 1. A typical deterioration curve.

Procedure

The component deteriorates with the passage of time. It is not possible to consider every point on the deterioration curve (Fig. 1). Therefore, the life of the component is divided into a sufficient number of states to describe the deterioration curve. This chain of states is represented by S_1, S_2 ... S_h. S_1, the initial state, represents the new component. S_n, the final state, represents the component failure. A component subjected to normal deterioration passes through this sequence of states. However, if too much deterioration occurs, failure can occur from any state of the component. From one time period to another it can make only one of the following three transitions:

1. Remain in the same state - The component does not deteriorate sufficiently to change its state.
2. Change to the next succeeding state - The component deteriorates sufficiently to change to the next succeeding state.
3. Fail completely - The component becomes inoperable.

Probabilities are associated with each of the above three events. These are called as transition probabilities or deterioration probabilities since a transition of the component takes place from one state to another. It is convenient to arrange the various transition probabilities in a form of a square matrix, called a stochastic matrix or transition matrix.

$$P = \begin{bmatrix} P_{11} & P_{12} & - & - & - & - & P_{1n} \\ - & P_{22} & P_{23} & - & - & - & P_{2n} \\ - & - & P_{33} & P_{34} & - & - & P_{3n} \\ - & - & - & - & - & - & - \\ P_{n1} & - & - & - & - & - & P_{nn} \end{bmatrix}$$

Deterioration in one time period is not sufficient for the component to change through two states. In any one time period the component can exist in any one of the S_1, S_2 ... S_n states. This system can be represented by a Markov Process.

Therefore, failure is an extreme event. The failure probabilities are calculated from the distributions of the strength and the stress. These distributions can be represented asymptotically by the double exponential distributions. A component can fail because it has low strength or it is subjected to high stress or both. The strength of a component will be greater than the load applied to it for only a finite time. The strength is generally decreasing due to deterioration. A component which is subjected to a series of load values L_1, L_2 ... L_n there will be a largest value. If failure is to be avoided, the strength of the component must be greater than this largest load at the instant of occurrence of this event.

Historical

The mathematical theory of reliability has grown out of the demands of modern technology and particularly out of the experiences in World War II with complex military systems. One of the first areas of reliability to be approached with any mathematical sophistication was the area of machine maintenance (Khintechine 1932, and C. Palm 1947). The techniques used to solve these problems grew out of the successful experiences of A. K. Erlang, C. Palm, and others in solving telephone trucking problems.

In the late 1930's the subject of fatigue life in materials and the related subject of extreme value theory were studied by W. Weibull 1939, Gumbel 1939, and Epstein 1948, among others. Gumbel's book (8) 1958 supplies data to illustrate the use of each of the extreme value (asymptotic) distributions to represent lifetimes.

Richard Barlow and Larry Hunter (2) consider two types of preventive maintenance policies. Policy I--Preventive maintenance is done after 'to' hours of continuing operation without failure. If system fails before 'to' hours, perform maintenance at the time of failure. Policy II--Preventive maintenance is done on the system after it has been operating 't' hours regardless of the number of intervening failures. After each failure only minimal repair is done. Alan J. Truelove (13) discusses the system availability and preventive maintenance. He uses Policy I of Richard Barlow and Larry Hunter for preventive maintenance.

C. Derman and J. Sacks (6) discuss the problem of choosing an optimal replacement rule for deteriorating equipment. They assumed that the amount of deterioration is observable.

Richard Barlow and Larry Hunter (3) consider a one unit system. The system can exist in two states--the 'on' state and the 'off' state. They have developed the mathematical analysis relevant for the reliability of this system.

Morton Klein (11) and Cyrus Derman (5) consider the deterioration of the system as a Markov Chain. They have used linear programming formulation.

In 1965, Barlow, Proschan, and Hunter (1) published a text on reliability. In this text they have applied mathematical formulation to preventive maintenance. They have discussed replacement policies, renewal theory and the application of several distributions.

A complete exposition of the theory of Markov Processes is in a text by Howard (10).

P. M. Ghare and D. J. Guarino (7) developed a model for scheduling preventive maintenance. They considered the deterioration of the system. Their system had twelve presses.

II. THE MATHEMATICAL MODEL

List of Symbols

a = measure of concentration around the mode of the double exponential distribution

b_n = instantaneous strength

b_o = original strength

c = constant of the deterioration function

d = inspection period

k = number of components for system failure

m = mode of the double exponential distribution

q = number of components in the system

p = probability of failure

p_{ij} = probability of changing from state i to state j

P = transition matrix

h = number of states

$S_1, S_2 \ldots S_h$ = states of the component

t_n = time taken to reduce the original strength b_o to strength b_n

T = return period

u = critical number of units

V = system status vector

γ = Euler's constant = 0.57722

Failure is an Extreme Event

Z. W. Birnbaum (4) describes the missile reliability as follows:

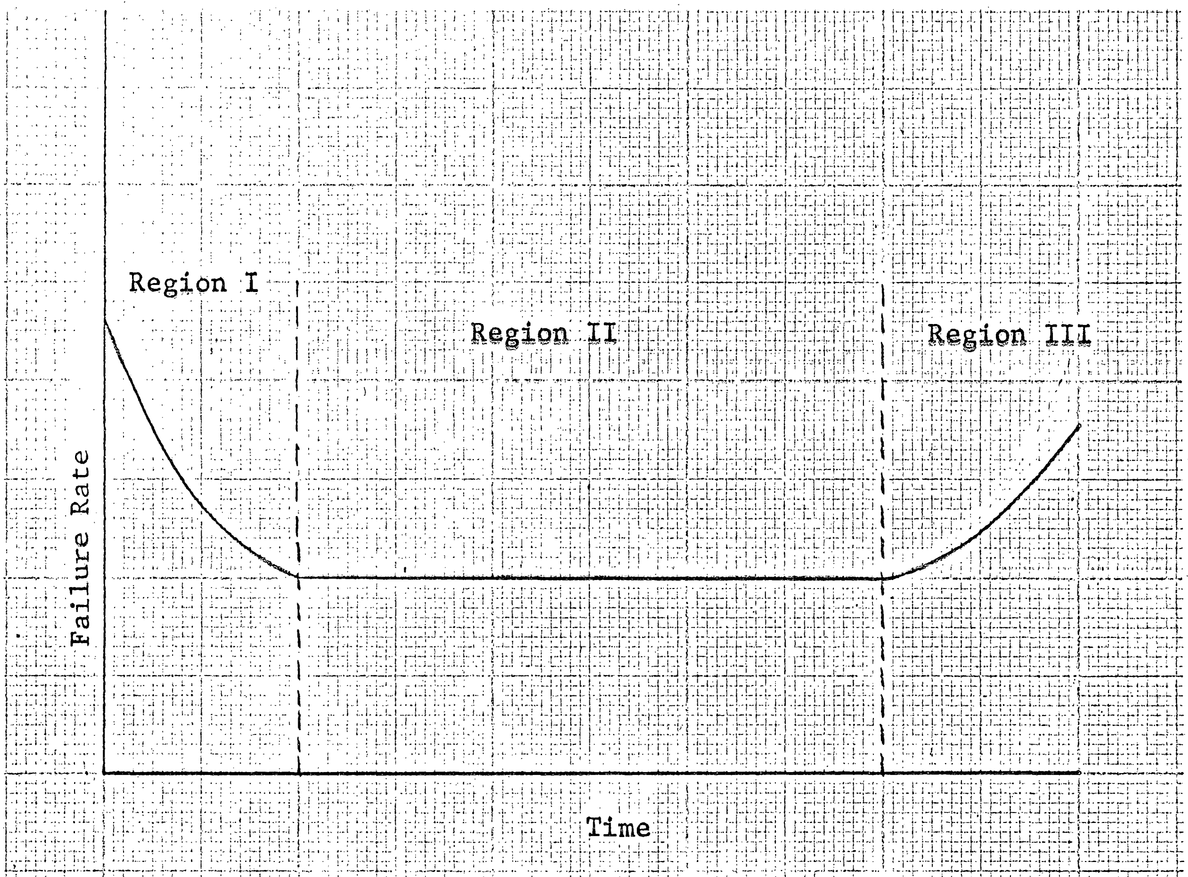

Fig. 2. Failure is a function of time.
"A Bathtub Function"

> If structural components of a mechanism are mass produced, the strength at failure Y of each single component may be considered a random variable. The component is installed in an assembly and exposed to a stress which reaches its maximum value X, again a random variable. If $Y < X$, then the component will fail in use.

A component can fail becuase it has low strength or it is subjected to high stress or both. From this it is evident that failure is an extreme event. Haviland (9) has shown that extreme values from a large class of probability distributions can be represented asymptotically by the double exponential distribution. The cumulative distribution is given by $F(X) = e^{-e^{-a(x - m)}}$. This is called double exponential because it has an exponential of an exponential.

The strength of a component is not a constant with respect to time. It decreases with respect to time. This is called deterioration. Haviland (9) has shown that in the case of the flow of materials between the object and the environment the instantaneous strength at time t is given by

$$\log_e b_n = \log_e b_o - ct$$

where b_n = instantaneous strength

b_o = original strength.

c = a constant

t = time

Stress is not a function of time. Therefore, a time will come when the strength will be less than the stress. When that occurs, failure results. Consequently, failure is a function of time. More specifically, it is a "bathtub" function (Fig. 2).

This curve (Fig. 2) can be divided into three distinct regions:

1. The region of early failures ("infant mortality"). The failure rate is initially high because the weak components are weeded out.
2. The region of chance or random failures. The failure rate is practically constant.
3. The region of wear-out failures. The failure rate increases slowly as the components reach the end of their useful life.

It is assumed that in a given time period one of the following events can take place for an operating component:

1. Failure - The component fails.
2. Deterioration - The unit changes state.
3. Nothing - It remains in the same state.
4. Improvement due to preventive maintenance.

The component that had failed in the previous time period can remain in either the failure state or become operable due to corrective maintenance.

As mentioned in Chapter I, the life span of the component can be divided into a number of states. When the component is new, it is in state S_1. When it has failed it is in state S_h. It can exist in any one of the states from state S_1 to state S_h. Between states S_1 and S_h it is in a process of deterioration. When maintenance is done on the components, corrective maintenance is done on the components that have failed. These components move from state S_h to S_1. Preventive

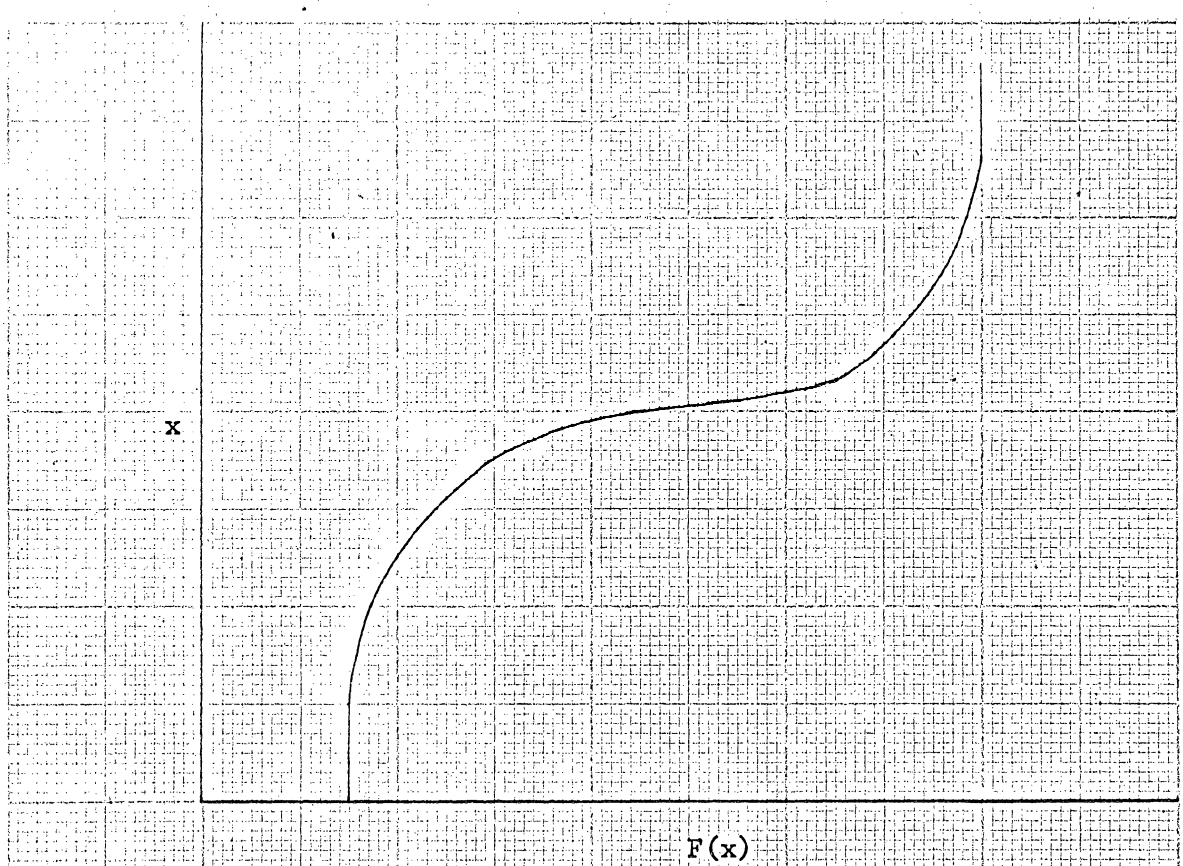

Fig. 3. Extreme values on normal graph.

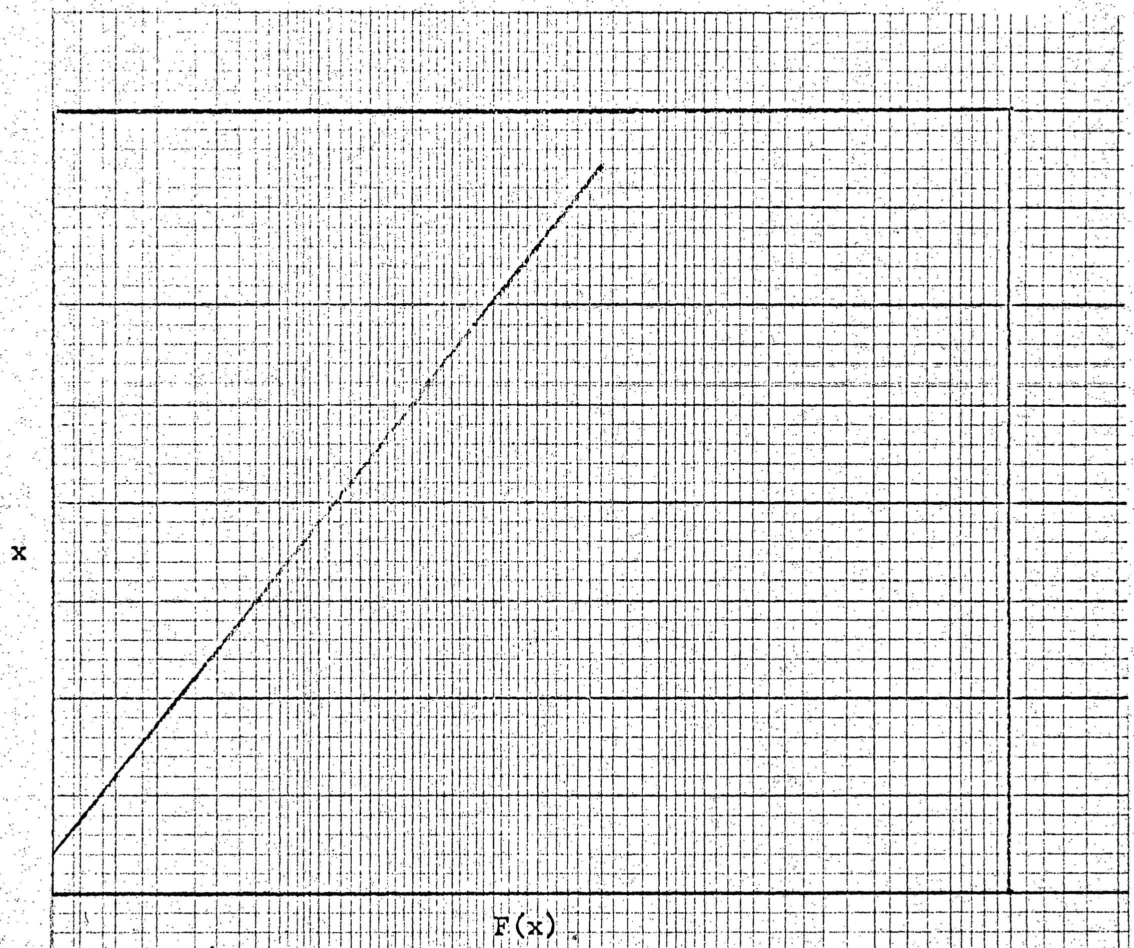

Fig. 4. Extreme values on extreme value graph.

maintenance is done on the functioning components. Therefore, it is assumed that the state of the component improves by a certain number of states.

Computation of Transition Probabilities

The probability of failure is obtained from the cumulative distribution of the load and the cumulative distribution of the strength of the component. As stated earlier, it has been experimentally found that if a group of similar components are tested for strength, the density function is given by

$$f(x_s) = a_s \, e^{-a_s(x_s - m_s)} e^{-e^{-a_s(x_s - m_s)}} \qquad (2.1)$$

where a_s = measure of concentration around the mode

x_s = random variable

m_s = mode of the distribution

The cumulative distribution is given by

$$F(x_s) = e^{-e^{-a_s(x_s - m_s)}} \qquad (2.2)$$

The E(x), a, and m are derived in Appendix I. The cumulative distribution is plotted in Fig. 3. This is a difficult curve to work with. A paper has been designed such that the cumulative distribution of the double exponential distribution is represented by means of a straight line on this paper (Fig. 4). The strength of the component deteriorates with the passage of time. This causes a reduction in the mode m_s which is given by

$$m_s = E(x_s) - \frac{\gamma}{a_s} \text{ (see Appendix I)} \qquad (2.3)$$

γ = Euler's constant

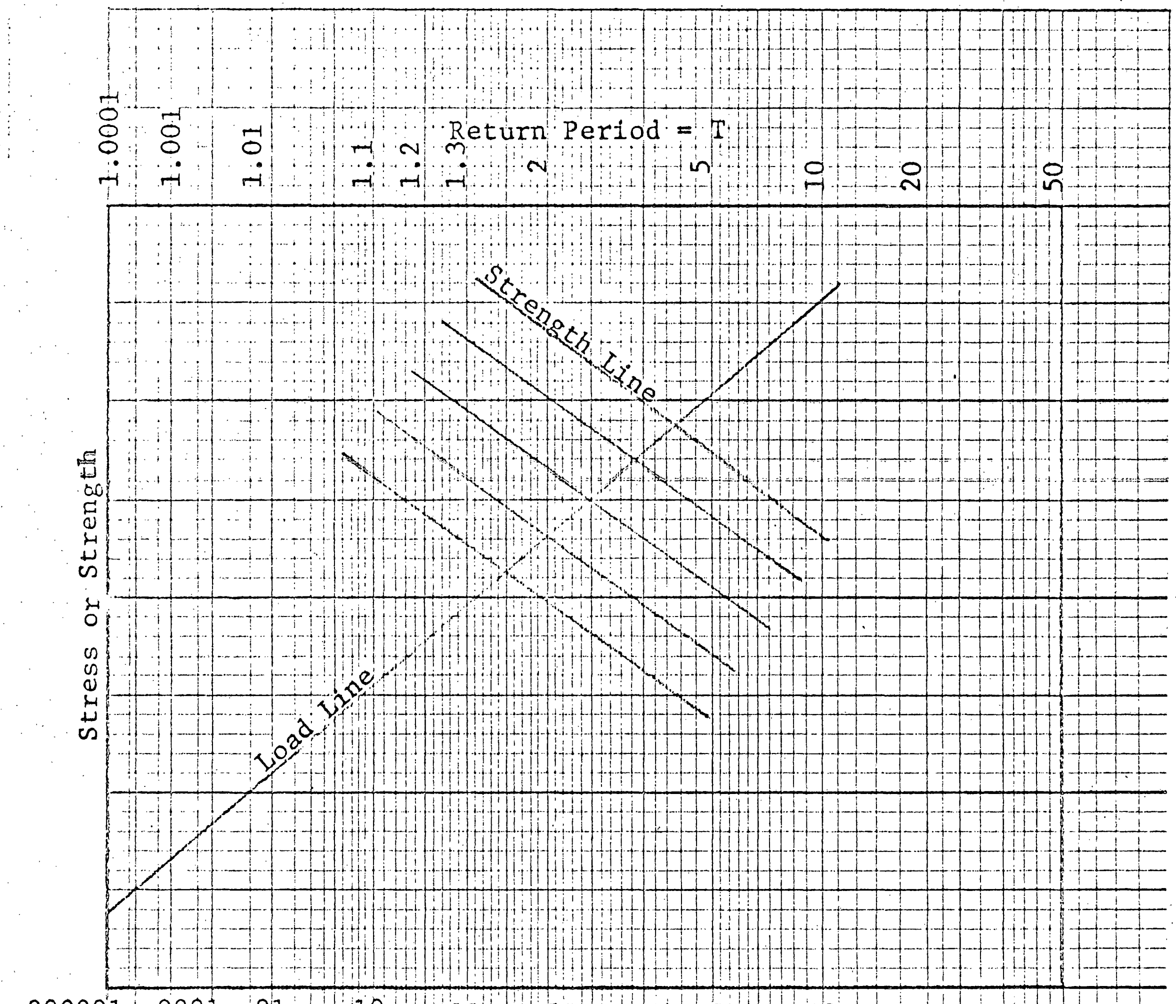

Fig. 5. Probability of failure and return function.

For decreasing values of the mode parallel straight lines are obtained.

The cumulative distribution of the extreme value of the load is given by

$$F(x_L) = e^{-e^{-a_L(x_L - m_L)}} \tag{2.4}$$

A straight line is obtained if the load and $F(x_L)$ is plotted on the extreme value paper.

From the intersection of the load line and the strength line the probability of failure p is obtained with a certain return period (Fig. 5). The return period is the mean time to failure. The return period T is given by

$$T = \frac{1}{1 - p} \tag{2.5}$$

The time to failure is assumed to have an exponential distribution. From this assumption the probability of failure during one time period is calculated as

$$\text{Prob[Failure]} = \int_o^1 \frac{1}{T} e^{-\frac{x}{T}} dx$$

$$= 1 - e^{-1/T} \tag{2.6}$$

The probability of deterioration is obtained from the classical equation of deterioration [p. 24 of (9)]. The validity of this equation has been experimentally determined.

$$t_n = \frac{\log_e b_o - \log_e b_n}{c}$$

$$t_{n+1} = \frac{\log_e b_o - \log_e b_{n+1}}{c}$$

$$t = t_{n+1} - t_n = \frac{\log_e \dfrac{b_n}{b_{n+1}}}{c}$$

Probability of deterioration is given by

$$\text{Prob[Deterioration]} = \frac{1}{t}\,(1 - \text{probability of failure}) \qquad (2.7)$$

The Probability of Remaining in the Same State

The probabilities of the transition matrix are conditional probabilities. The sum of the three probabilities is equal to one because these three describe the probabilities of all the possible events. Hence, the probability of remaining in the same state is given by

$$\text{Prob[Remain in the same state]} =$$

$$1 - \text{Prob[Failure]} - \text{Prob[Deterioration]} \qquad (2.8)$$

Markov Process

A Markov Process is embedded in the maintenance process. The status of the system at any given time can be represented by the status vector V. Any element v_i of the status vector V represents the probability that a component exists in state i at that time. P is the transition matrix with each element expressing the probability of the component making a transition from one specific state to another.

$$P = \begin{array}{cc} & \begin{array}{ccccccccc} \text{To } 1 & 2 & & & & & & & n \end{array} \\ \begin{array}{r} \text{From } 1 \\ 2 \\ 3 \\ \\ \\ \\ \end{array} & \begin{bmatrix} P_{11} & P_{12} & - & - & - & - & - & - & P_{1n} \\ - & P_{22} & P_{23} & - & - & - & - & - & P_{2n} \\ - & - & P_{33} & P_{34} & - & - & - & - & P_{3n} \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ P_{n1} & - & - & - & - & - & - & - & P_{nn} \end{bmatrix} \end{array}$$

The size of the matrix is determined by the number of states in which the component can exist. The transition over one time period can be expressed as

$$V_1 = P V_o$$

If the process starts with the system in status V_o, after transitions through r time periods the status of the system would become V_r

$$V_r = P^r V_o$$

The number of failures after r time periods is given by the last element of V_r multiplied by the total number of components. If the number of failures is equal to or greater than the critical number of failures then maintenance will be performed. The cost of doing maintenance is equal to the cost of corrective maintenance x number of components that have failed + cost of preventive maintenance x operable components + cost of inspection. Initially, when the system is in the transient state, there will be more frequency inspections. But when the system has reached steady state, there will be less frequent inspection. This is called the two-phase strategy.

The next chapter describes an algorithm for calculating the status vector and the cost equation.

III. THE ALGORITHM

Problem

A system has q components. It fails when k components fail. The objective is to determine the two-phase strategy for scheduling maintenance and the critical number of units.

Assumptions

1. If a decision is made to do maintenance, corrective maintenance is done on the components that have failed and preventive maintenance is done on the functioning components. Due to preventive maintenance the life of the component increases by five states.
2. Maintenance cost includes the cost of preventive maintenance, cost of corrective maintenance, cost of inspection and cost of risk if the system were to fail.
3. Components can fail due to 'm' types of failures. But when the component fails, it is assumed that the component has failed due to only one type of failure. In this model only one type of failure is considered. This can be easily expanded to 'm' types of failures.
4. System status vector is assumed such that all the components are in state S_1; i.e., they are all new.
5. It is assumed that the life of the component can be divided into forty states.

Input data required for this model is the minimum strength values of the components and the maximum load values to which the components are subjected. These observations can be arranged in the form of a frequency histogram. From this frequency histogram the parameters 'a' and 'm' of the double exponential distribution can be calculated. The procedure for obtaining the maximum likelihood estimate is shown in Appendix II. As no real world data was available for the purpose of illustration, simulated data was generated using a random number generator. In actual application, however, the frequency histogram of minimum strength or maximum load values would have to be used. In the same fashion the constant of the deterioration equation was assumed to be known. In practice these would have to be estimated from the observations of the deterioration and the strength of the component.

Data Collection and Analysis

1. Collect the strength data of the components or make observations of the strengths of the components.
2. Estimate the parameters 'a' (measure of concentration around the mode) and 'm' (mode) of the double-exponential distribution for the strength.
3. Collect the load data of the components or make observations of the loading condition of the components.
4. Estimate the parameters 'a' and 'm' of the double-exponential distribution for the load.

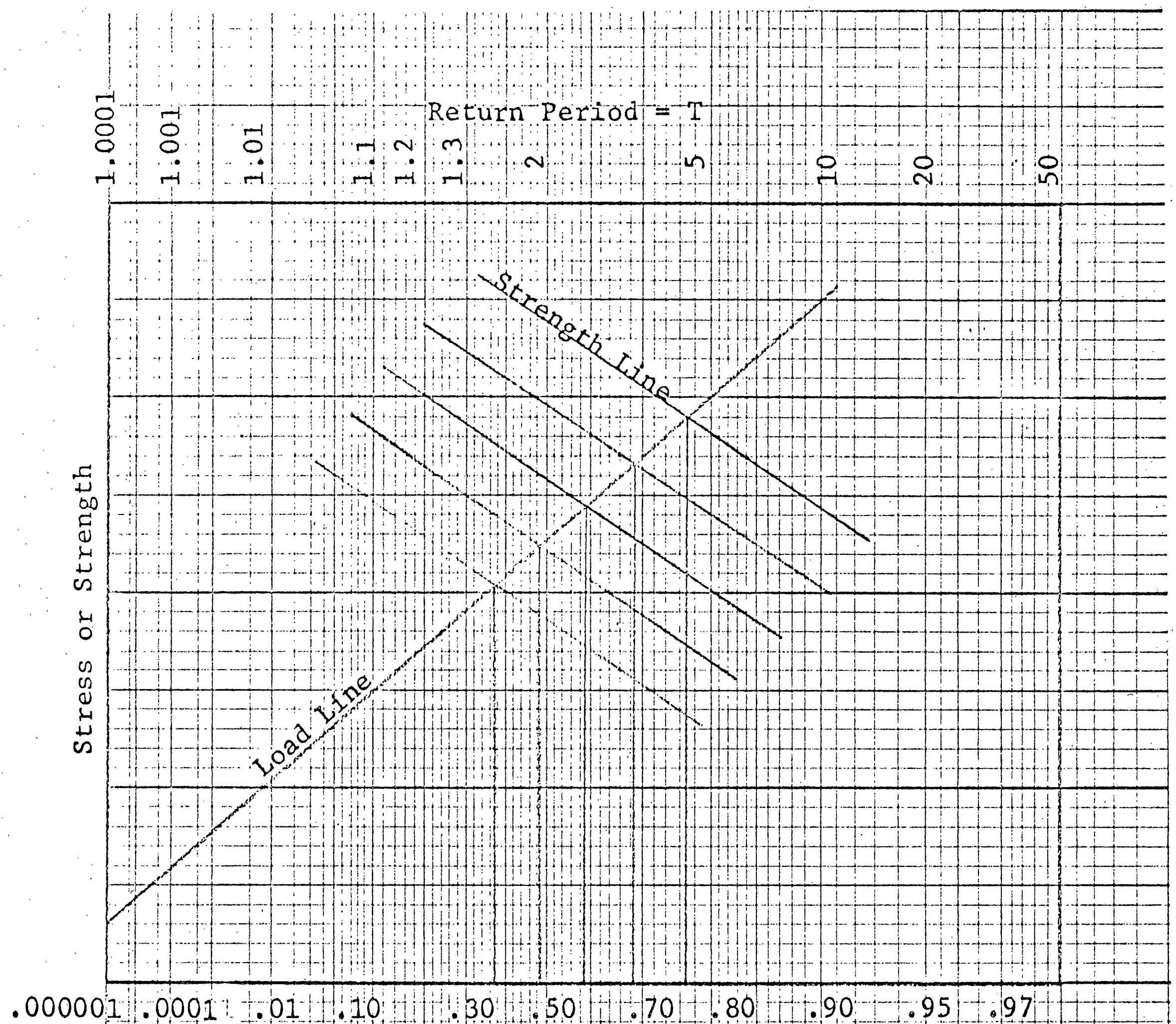

Fig. 6. Probability of failures for each mode.

Formulation of the Model

5. Plot a graph of load (x_L) versus $F(x_L)$ on the extreme value paper-called load line.

6. Determine the number of states. There should be sufficient number of states to describe the deterioration function. The strengths of the component in two consecutive states should differ by a constant value (θ).

   b_o = the strength of the component in state S_1

   $b_1 = b_o - \frac{b_o}{\theta}$ the strength of the component in state S_2

   $b_k = b_o - \frac{kb_o}{\theta}$ the strength of the component in state S_{k+1}

7. Obtain the mode for every mean instantaneous strength.

   $m = E(x) - \frac{\gamma}{a}$

8. Plot strength (x_s) versus $1 - F(x_s)$ on the extreme value paper-called strength line.

9. Obtain strength lines for each and every mode (Fig. 6).

10. Obtain the point of intersection of the strength line with the load line. There will be as many points of intersection as there are modes. Each point of intersection corresponds to one state.

11. Obtain the return function for each point of intersection

    $T = \frac{1}{1 - F(x)}$.

12. Obtain the probability of failure during one time period for each state.

13. Obtain the probability of deterioration which is equal to $\frac{1}{t}$ [1 - Prob(Failure)].

14. Obtain the probability of remaining in the same state.

15. Form the transition matrix (P).

16. Estimate the system status vector (V_o).

17. Compute $V_r = P^r V_o$ until all the components fail.

18. Obtain the probability of failure for the system for every time period.

19. Estimate the various costs - The cost of preventive maintenance, cost of corrective maintenance, and the cost of inspection.

Optimization

Total system cost function of two decision variables inspection period (d) and the critical number of units (u). Optimization consists of minimum total system cost subject to the constraint that there be system failure with acceptable probability of failure. Since d and u are integer values the total cost function is described only on the discrete grid points of the two dimensional (d, u) space. The optimal could be obtained by complete enumeration; however, it would almost always be prohibitively costly. Therefore, a two-dimensional search technique has to be adopted. A suggested method would be to use a Simplical Method by E. P. Box and Wilson or a Multivariable Dichotomous Elimination by Wilde. In this thesis only a few acceptable policies have been illustrated in order to conserve computer time.

20. Repeat step 17 but this time perform inspection and do maintenance if necessary. Use the probabilities obtained

in step 18 to find the probability of system failure before the next inspection. Assume

$$\frac{\text{units required for system failure}}{\text{days required for system failure}}$$

as the mean of the poisson distribution. Compute the maintenance cost. Maintenance cost = number of units failed x cost of corrective maintenance + remaining units x cost of preventive maintenance + inspection cost.

21. Repeat step 20 for each combination of d, and u required by the chosen search technique until a global minimum is obtained.

IV. THE PROGRAM FOR THE MODEL

The major portion of the calculations required in the determination of the optimum schedule involves matrix multiplication. Because of the accuracy required in the mathematical manipulation, a computer program was used to perform the optimization of the model. The program is presented in Appendix III. The computer program for the model contains three parts:

A. Generate data and estimate the parameters of the double-exponential distribution.

B. Obtain the failure probabilities of the system per time period.

C. Obtain the two phase strategy for scheduling maintenance and the critical number of units.

A. Estimation of the Parameters

This performs steps 1 to 4 of the algorithm. A random number generator is used to generate numbers between 0 and 1. Using the property of the cumulative distribution that $0 \leq F(x) \leq 1$ the random variable x is generated from $F(x) = e^{-e^{-a(x - m)}}$. The random variable x is used to generate the estimators $\hat{m} = \frac{1}{a} \log_e \frac{N}{\Sigma e^{-axi}}$ and $\hat{a} = \frac{N}{\Sigma xi - N\Sigma e^{-axi}}$.

B. Failure Probabilities

This performs steps 5 to 18 of the algorithm. This program has three parts:

1. Main program - This program obtains the probability of failure for every state per time period. The original strength of the component is assumed as 19 units while the maximum load is as 4 units, giving a factor of safety of 4.7. The life of the component is divided into 40 states. The probability of failure for each state is obtained by using Eqs. 2.2, 2.3, 2.4, 2.5, 2.6.
2. Subroutine GOHOME - This program performs the necessary computations to obtain the failure probabilities. Total number of components is assumed as 60. The system fails when 19 components fail. The cycle is of 300 days duration. The probability of deterioration is obtained by Eq. 2.7. The probability of remaining in the same state is obtained by Eq. 2.8. It forms the transition matrix and assumes the system state vector. It obtains the expected number of failures per time period.
3. Subroutine MATPRO - This program performs matrix multiplication. This subroutine performs the actual matrix multiplication of the transition matrix and the system state vector V.

C. Optimum Schedule

This performs steps 19 to 21 of the algorithm. This part of the program has 4 sections:

1. Main program - It conducts a search for the combination of the period of inspection and the critical number of units. Computes the total cost for the optimal combination.

2. Subroutine FINSUB - This is the main program of B(1).
3. Subroutine GOHOME - This program is essentially the same as B(2) but with the following additions: The probability of failure of the system per time period as obtained in B. It obtains the probability of the system failure before the next inspection. The program also performs the calculation for obtaining the total cost of maintenance for every combination of the inspection, period and the critical number of units. Performs the maintenance work, i.e., changes the system state vector when maintenance is done.
4. Subroutine MATPRO - Same program as in B(3).

Numerical Values Used in the Program

Number of components in the system = 60

System fails if 8 components fail

Number of days in the cycle = 300

Number of states of the component = 40

Load Data

$$F(x) = e^{-e^{-a_L(x_L - m_L)}}$$

$a_L = 0.5$

$m_L = 4.0$

Strength Data

$$F(x) = e^{-e^{-a_s(x_s - m_s)}}$$

$a_s = 0.5$

$$\log_e b_n = \log_e b_o - ct$$

So = 19

C = .002

Cost Data

Cost of doing preventive maintenance = \$75/component

Cost of doing corrective maintenance = \$100/component

Cost of inspection = \$200

Cost of system failure = infinite

<u>Results from the Model</u>

From the results of the model the following good policies were obtained:

Policy I - Inspect every 5 days and the critical number of units is 4. The first maintenance will be performed on the 70th day. After that maintenance will be performed on every 55th day. The probability of system failure will be .004 and the total cost of maintenance and inspection will be \$26,800. The two-phase inspection policy would be to inspect every 5 days till the 70th day. Then perform inspection on every 55th day till the system fails. When the system fails revert back to inspection every 5 days.

Policy II - Inspect every 4 days and the critical number of units is 5. The first maintenance will be performed on the 76th day. After that maintenance will be performed on every 56th day. The probability of system failure will be .019 and the total cost of maintenance and inspection will be $23,100. The two-phase inspection policy would be to inspect every 4 days till the 76th day. Then perform inspection on every 56th day till the system fails. When the system fails revert back to inspection every 4 days.

Policy III - Inspect every 3 days and the critical number of units is 6. The first maintenance will be performed on the 81st day. After that maintenance will be performed on every 57th day. The probability of system failure will be .08 and the total cost of maintenance and inspection will be $24,800. The two-phase inspection policy would be to inspect every 3 days till the 81st day. Then perform inspection on every 57th day till the system fails. When the system fails revert back to inspection every 3 days.

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APPENDIX I

Mean of a Double Exponential Distribution

$$F(x) = e^{-e^{-a(x - m)}}$$

$$f(x) = \frac{\partial F(x)}{\partial x} = ae^{-e^{-a(x - m)}} e^{-a(x - m)}$$

$$f(x) = ae^{-a(x - m)} e^{-e^{-a(x - m)}}$$

To obtain the characteristic function

$$d_x(t) = a \int_{-\infty}^{\infty} e^{jtx} e^{-a(x - m)} e^{-e^{-a(x - m)}} dx$$

let

$$z = -a(x - m)$$

$$dz = -adx$$

$$x = \frac{am - z}{a}$$

when

$$x = \infty \qquad z = -\infty$$

$$x = -\infty \qquad z = \infty$$

$$d_z(t) = -\int_{\infty}^{-\infty} e^{jt\left(\frac{am - z}{a}\right)} e^z e^{-e^z} dz$$

$$= e^{jtm} \int_{-\infty}^{\infty} e^{-\frac{jt}{a}z} e^z e^{-e^z} dz$$

$$= e^{jtm} \int_{-\infty}^{\infty} e^{z\left(1 - \frac{jt}{a}\right)} e^{-e^z} dz$$

But

$$\int_{-\infty}^{\infty} e^{-e^x} e^{ux} dx = \Gamma(u)$$

$$d_x(t) = e^{jmt} \Gamma\left(1 - \frac{jt}{a}\right)$$

To find the moment generating function

$$dx(-jt) = M_x(t) = e^{jmt(-j)}\ (1 - \frac{jt}{a}x - j)$$

$$M_x(t) = e^{jmt(-j)} \int_o^\infty x^{-t/a}\ e^{-x}\ dx$$

$$= e^{mt} \int_o^\infty x^{-t/a}\ e^{-x}\ dx$$

To obtain the mean $\left.\frac{\partial M_x(t)}{\partial t}\right|_{t=0}$

$$\frac{\partial M_x(t)}{\partial t} = me^{mt} \int_o^\infty x^{-t/a}\ e^{-x}\ dx - \frac{e^{mt}}{a} \int_o^\infty x^{-t/a}\ e^{-x} \log x\ dx$$

Putting $t = 0$ we obtain

$$= m \int_o^\infty e^{-x}\ dx - \frac{1}{a} \int_o^\infty e^{-x} \log x\ dx$$

$$= m + \frac{\gamma}{a}$$

Since $$\int_o^\infty e^{-x} \log x\ dx = -\gamma$$

$$E(x) = m + \frac{\gamma}{a}$$

APPENDIX II

Estimation of Parameters 'a' and 'm' of the Double Exponential Distribution

Likelihood function is given by

$$\pi_i f(x_i, m, a) = \pi_i a e^{-a(x_i - m)} e^{-e^{-a(x_i - m)}}$$

Taking logarithms we get

$$L(\theta) = \log\{\pi_i a e^{-a(x_i - m)} e^{-e^{-a(x_i - m)}}\}$$

$$= \sum_{i=1}^{N} \{\log a - a(x_i - m) - e^{-a(x_i - m)}\}$$

differentiating w.r.t. 'm' and setting the equation to zero

$$\frac{\partial L(m)}{\partial m} = \sum_{i=1}^{N} (0 + a - a e^{-a(x_i - m)}) = 0$$

$$Na = a \Sigma e^{-a(x_i - m)}$$

$$N = e^{am} \Sigma e^{-ax_i}$$

$$e^{am} = \frac{N}{\Sigma e^{-ax_i}}$$

$$am = \log_e \frac{N}{\Sigma e^{-axi}}$$

$$\hat{m} = \frac{1}{a} \log_e \frac{N}{\Sigma e^{-axi}}$$

To obtain the estimator of 'a'

Differentiating the logarithm equation w.r.t. 'a' and setting the equation to zero

$$\frac{\partial L(a)}{\partial a} = \sum_{i=1}^{N} \{\frac{1}{a} - (x_i - m) + (x_i - m)e^{-a(x_i - m)}\} = 0$$

$$= \{\frac{N}{a} - \sum_{i=1}^{N} (x_i - m) + \sum_{i=1}^{N} (x_i - m)e^{-a(x_i - m)}\} = 0$$

$$= \frac{N}{a} - \sum_{i=1}^{N} (x_i - m) + e^{am} \sum_{i=1}^{N} (x_i - m)e^{-axi} = 0$$

But
$$e^{am} = \frac{N}{\Sigma e^{-axi}}$$

$$\frac{N}{a} - \sum_{i=1}^{N} (x_i - m) + e^{am}\, \Sigma(x_i - m)e^{-axi} = 0$$

$$= \frac{N}{a} - \sum_{i=1}^{N} (x_i - m) + \frac{N}{\sum_{i=1}^{N} e^{-axi}} \sum_{i=1}^{N} (x_i - m)e^{-axi} = 0$$

$$= \frac{N}{a} - \sum_{i=1}^{N} (x_i - m) + \frac{N \sum_{i=1}^{N} e^{-axi}xi}{\sum_{i=1}^{N} e^{-axi}} - Nm = 0$$

$$= \frac{N}{a} - \sum_{i=1}^{N} x_i + Nm + \frac{N \sum_{i=1}^{N} e^{-axi}xi}{\sum_{i=1}^{N} e^{-axi}} - Nm = 0$$

$$= \frac{N}{a} - \sum_{i=1}^{N} x_i + \frac{N \sum_{i=1}^{N} e^{-axi}xi}{\sum_{i=1}^{N} e^{-axi}} = 0$$

$$\hat{a} = \frac{N}{\sum_{i=1}^{N} xi - \dfrac{N \sum_{i=1}^{N} e^{-axi} xi}{\sum_{i=1}^{N} e^{-axi}}}$$

APPENDIX III

The Computer Program

The program for the algorithm described in Chapter III was written in Fortran IV language for an IBM 360. The variables used in the computer program are as follows:

AS	Measure of concentration around the mode for strength distribution
AL	Measure of concentration around the mode for load distribution
DETER	Constant c of the deterioration function
ENF	Expected number of failures
IP	Inspection period
NDAYS	Total number of time periods
ONENIN	Number of components for system failure
OPNM	Critical number of units
PRISK	Probability of failure for a component at each state
PROB	Probability of failure for the system per time period
PSF	Probability of failure for the system before next inspection
S	Instantaneous strength
TMAT	Transition matrix
TN	Total number of components in the system
TCM	Total cost of maintenance and inspection

VT	System status vector
XML	Mode for the load distribution
XMS	Mode for the strength distribution

THE PROGRAM FOR ESTIMATION OF THE PARAMETERS

```
      DIMENSION X(200),Y(200),Z(3),FX(3,200)
      THETA=4.0
      ASSUME=0.5
      TN=50.0
      WRITE(6,16)THETA,TN
 16   FORMAT(1H0,F10.3,2X,F10.3)
      SUMX=0.0
      EX=0.0
      E=0.0
      A=.01
      IX=13575
      DO10I=1,50
      CALL RANDU(IX,IY,YFL)
      Y(I)=YFL+0.01
      IF(Y(I) .GE. 0.99)Y(I)=0.99
      WRITE(6,51)Y(I)
 51   FORMAT(1H0,F8.4)
      BAC=ALOG(1.0/Y(I))
      BAD=ALOG(BAC)
      X(I)=THETA-BAD/ASSUME
 12   SUMX=SUMX+X(I)
      WRITE(6,17)SUMX,X(I),Y(I),BAC,BAD
 17   FORMAT(1H0,5(F10.3,4X))
 10   CONTINUE
 21   DO20I=1,50
      EX=EX+EXP(-A*X(I))
      E=E+X(I)*(EXP(-A*X(I)))
 20   CONTINUE
      EP=TN*E/EX
      ET=SUMX-EP
      A1=TN/ET
      WRITE(6,15)A1,ET,EP
 15   FORMAT(1H0,5HA1 = ,F6.3,4X,2(F10.3,4X))
      SUB=A1-A
      IF(ABS(SUB) .LE. .001)GOTO31
      A=(A+A1)/2.0
      EX=0.0
      E=0.0
      GOTO21
 31   THETES=(1/A)*(ALOG(TN/EX))
      WRITE(6,100) A,THETES
 100  FORMAT(1H0,4HA = ,F6.3,4X,20HESTIMATE OF THETA = ,F10.2)
      STOP
      END
```

```
      SUBROUTINE RANDU(IX,IY,YFL)
      IY=IX*65539
      IF(IY)5,6,6
5     IY=IY+2147483647+1
6     YFL=IY
      YFL=YFL*.4656613E-9
      IX=IY
      RETURN
      END
```

THE PROGRAM FOR FAILURE PROBABILITIES PER TIME PERIOD

```
      IP=4
      OPNM=5.0
      WRITE(6,1000)OPNM,IP
 1000 FORMAT(1H0,27HCRITICAL NUMBER OF UNITS = ,F5.0,20HINSPECTION PERIO
     1D = ,I5)
      CALL FINSUB(IP,OPNM)
 20   CONTINUE
 10   CONTINUE
      STOP
      END
      SUBROUTINE FINSUB(IP,OPNM)
      DIMENSION Z(2),FX(2,100),PRISK(60)
      COMMON NSTATE,SO,C,YL,TSJ
      XML=4.0
      SO=19.0
      MC=0
      GAMMA=0.57722
      AS=0.5
      AL=0.5
      C=0.008
      SL=ALOG(SO)
      Q=4.0
      EN=3.0
      YL=Q/EN
      P=.01
      XL=0.0
      TSJ=0.02*SO
      NSTATE=0
      S=SO
      DO40I=1,50
      NSTATE=NSTATE+1
      T=1
      S=S-TSJ
      XMS=S-GAMMA/AS
      Z(1)=XMS-1.0
      Z(2)=XMS-3.0
      DO50J=1,2
      XL=XL+.15
      AE=EXP(-AS*(Z(J)-XMS))
      FX(J,T)=1.0-EXP(-AE)
      AEX=EXP(-AL*(XL-XML))
      FL=EXP(-AEX)
      IF(J .EQ. 2)GOTO51
      EF=-ALOG(ALOG(1.0/FL))
 50   CONTINUE
 51   GF=-ALOG(ALOG(1.0/FL)) -EF
      AF=-ALOG(ALOG(1.0/FX(1,T)))
      BF=-ALOG(ALOG(1.0/FX(2,T)))-AF
      FF=0.15
      XLM=XL-0.15
```

```
      PXY=(2*AF)/BF+XLM-Z(1)-(FF*EF)/GF
      PXU=2.0/BF-FF/GF
      XK2=PXY/PXU
      PI=EXP(-EXP(-XK2))
17    MC=MC+1
      PRISK(MC)=1.0-EXP(PI-1.0)
      IF(MC .EQ. 40)GOTO13
      GOTO40
13    DO14IM=1,40
      PRISK(IM)=PRISK(IM)/PRISK(40)
      WRITE(6,100) PRISK(IM)
100   FORMAT(1H0,6HPRISK = ,F10.6)
      IF(IM .EQ.40)GOTO12
14    CONTINUE
40    CONTINUE
12    CALL GOHOME(IP,OPNM,PRISK)
      RETURN
      END
      SUBROUTINE GOHOME(IP,OPNM,PF)
      DIMENSION TMAT(60,60),PF(60),VT(60),R(60),AT(60,60),PROB(61)
      COMMON NSTATE,SO,C,YL,TSJ
      TINSP=200.0
      KDAYS=9
      TP=75.0
      TC=100.0
      GNENIN=8
      FIX=0.5
      NDAYS=300
      TN=60.0
      DETER=C*EXP(-YL)
      S=SO
      SCA=ALOG(S)
      DO 60I=1,60
      DO 60J=1,60
      TMAT(I,J)=0.0
60    CONTINUE
      DO70I=1,NSTATE
      J=I+1
      TMAT(I,NSTATE)=PF(I)
      IF(I .EQ. NSTATE)TMAT(NSTATE,1)=1.0-PF(I)
      IF(J .GT. NSTATE)GOTO72
      S=S-TSJ
      SCB=ALOG(S)
      SCC=DETER/(SCA-SCB)
      TMAT(I,J)=SCC*(1.0-PF(I))
      SCA=SCB
12    IF(J .EQ. NSTATE)TMAT(I,J)=PF(I)
      TMAT(I,I)=1.0-TMAT(I,J)-PF(I)
      IF(TMAT(I,I) .LE. 0.0)TMAT(I,I)=0.0
13    IF(J .EQ. NSTATE)TMAT(I,I)=1.0-PF(I)
```

```
12    WRITE(6,400)I,J,TMAT(I,1),TMAT(I,J),TMAT(I,NSTATE)
400   FORMAT(1H0,2HI = ,I4,2X,4HJ = ,I4,2X,3(F10.6,2X))
30    CONTINUE
      DO15I=1,NSTATE
      DO15J=1,NSTATE
      AT(J,I)=TMAT(I,J)
15    CONTINUE
1223  VT(1)=1.0
      DO40I=2,NSTATE
      VT(I)=0.0
40    CONTINUE
      TCM=0.0
      DO 90 I=1,NDAYS
      CALL MATPRD(AT,VT,W)
      DO10J=1,NSTATE
      VT(J)=W(J)
10    CONTINUE
      EMF=TMP*VT(NSTATE)
      IF(EMF .LT. FIX)EMF=0.0
      IF(EMF .LT. FIX)GOTO67
      DO921 INT=1,40
      TMI=INT
      IF(EMF .LT. FIX*TMI)GOTO922
      GOTO921
922   EMF=TMI
      GOTO67
67    WRITE(6,399)EMF
399   FORMAT(1H0,7HFARID = ,F6.2)
90    CONTINUE
      RETURN
      END
      SUBROUTINE MATPRD (A,B,C)
      DIMENSION A(60,60),B(60),C(60)
      COMMON NSTATE
      DO10I=1,NSTATE
      C(I)=0.0
      DO20K=1,NSTATE
      C(I)=C(I)+A(I,K)*B(K)
 20   CONTINUE
10    CONTINUE
      RETURN
      END
```

THE PROGRAM FOR OPTIMUM SCHEDULE

```
      IP=4
      UPNN=5.0
      WRITE(6,1000)UPNN,IP
 1000 FORMAT(1H0,27HCRITICAL NUMBER OF UNITS = ,F5.0,20HINSPECTION PERIO
     1D = ,I5)
      CALL FINSUB(IP,UPNN)
 20   CONTINUE
 10   CONTINUE
      STOP
      END
      SUBROUTINE FINSUB(IP,UPNN)
      DIMENSION Z(2),FX(2,150),FNIEX(60)
      COMMON NSTATE,SO,C,YL,TSJ
      XML=4.0
      SO=19.0
      NC=N
      GAMMA=0.57722
      AS=0.3
      AL=0.3
      C=0.008
      SL=ALOG(SO)
      B=4.0
      XN=4.0
      YL=C/XN
      P=.01
      XL=0.0
      TSJ=0.02*SO
      NSTATE=0
      S=SO
      DO 40 I=1,30
      NSTATE=NSTATE+1
      J=1
      S=S-TSJ
      XMS=S-GAMMA/AS
      Z(1)=XMS-1.0
      Z(2)=XMS-3.0
      DO 50 J=1,2
      XL=XL+.15
      AE=EXP(-AS*(Z(J)-XMS))
      FX(J,I)=1.0-EXP(-AE)
      ALX=EXP(-AL*(XL-XML))
      PL=EXP(-ALX)
      IF(J .EQ. 2)GOTO51
      XP=-ALOG(ALOG(1.0/PL))
 50   CONTINUE
 51   BP=-ALOG(ALOG(1.0/PL)) -XP
      AF=-ALOG(ALOG(1.0/FX(1,I)))
      BF=-ALOG(ALOG(1.0/FX(2,I)))-AF
      PF=0.15
      XLN=XL-0.15
```

```
      PXY=(2*AP)/BP+XLM-Z(1)-(PP*BP)/BP
      PXU=2.0/BP-PP/BP
      XK2=PXY/PXU
      P1=EXP(-EXP(-XK2))
17    MC=MC+1
      PRISK(MC)=1.0-EXP(P1-1.0)
      IF(MC .EQ. 40)GOTO13
      GOTO40
13    DO14IM=1,40
      PRISK(IM)=PRISK(IM)/PRISK(40)
      WRITE(6,100) PRISK(IM)
100   FORMAT(1H0,6HPRISK = ,F10.6)
      IF(IM .EQ.40)GOTO12
14    CONTINUE
40    CONTINUE
12    CALL GOHORE(IP,OPNM,PRISK)
      RETURN
      END
      SUBROUTINE GOHORE(IP,OPNM,PF)
      DIMENSION TMAT(60,60),PF(60),VT(60),R(60),AT(60,60),PROB(61)
      COMMON NSTATE,SD,C,YL,TSJ
      PROB(1)=0.0
      PROB(2)=1.0
      PROB(3)=2.0
      PROB(4)=3.0
      PROB(5)=4.0
      PROB(6)=5.0
      PROB(7)=6.0
      PROB(8)=7.0
      PROB(9)=8.0
      PROB(10)=9.0
      PROB(11)=10.0
      PROB(12)=11.0
      PROB(13)=12.0
      PROB(14)=13.0
      PROB(15)=14.0
      PROB(16)=15.0
      PROB(17)=16.0
      PROB(18)=17.0
      PROB(19)=18.0
      PROB(20)=19.0
      PROB(21)=20.0
      PROB(22)=21.0
      PROB(23)=22.0
      PROB(24)=23.0
      PROB(25)=24.0
      PROB(26)=25.0
      PROB(27)=26.0
      PROB(28)=27.0
      PROB(29)=28.0
```

```
      PROB(30)=29.0
      PROB(31)=30.0
      PROB(32)=31.0
      PROB(33)=32.0
      PROB(34)=33.0
      PROB(35)=34.0
      PROB(36)=35.0
      PROB(37)=36.0
      PROB(38)=37.0
      PROB(39)=38.0
      PROB(40)=39.0
      PROB(41)=40.0
      PROB(42)=41.0
      PROB(43)=42.0
      PROB(44)=43.0
      PROB(45)=44.0
      PROB(46)=45.0
      PROB(47)=46.0
      PROB(48)=47.0
      PROB(49)=48.0
      PROB(50)=49.0
      PROB(51)=50.0
      PROB(52)=51.0
      PROB(53)=52.0
      PROB(54)=53.0
      PROB(55)=54.0
      PROB(56)=55.0
      PROB(57)=56.0
      PROB(58)=57.0
      PROB(59)=58.0
      PROB(60)=59.0
      PROB(61)=60.0
      TINSP=200.0
      KDAYS=9
      TP=75.0
      TC=100.0
      ONENIN=8
      FIX=0.5
      NDAYS=300
      TN=60.0
      DETER=C*EXP(-YL)
      S=SO
      SCA=ALOG(S)
      DO 60I=1,60
      DO 60J=1,60
      TMAT(I,J)=0.0
60    CONTINUE
      DO70I=1,NSTATE
      J=I+1
      TMAT(I,NSTATE)=PF(I)
```

```
      IF(I .EQ. NSTATE)TMAT(NSTATE,I)=1.0-PF(I)
      IF(J .GT. NSTATE)GOTO72
      S=S-TSJ
      SCB=ALOG(S)
      SCC=DETER/(SCA-SCB)
      TMAT(I,J)=SCC*(1.0-PF(I))
      SCA=SCB
12    IF(J .EQ. NSTATE)TMAT(I,J)=PF(I)
      TMAT(I,I)=1.0-TMAT(I,J)-PF(I)
      IF(TMAT(I,I) .LE. 0.0)TMAT(I,I)=0.0
13    IF(J .EQ. NSTATE)TMAT(I,I)=1.0-PF(I)
72    WRITE(6,400)I,J,TMAT(I,I),TMAT(I,J),TMAT(I,NSTATE)
400   FORMAT(1H0,4HI = ,I4,2X,4HJ = ,I4,2X,3(F10.6,2X))
70    CONTINUE
      DO15I=1,NSTATE
      DO15J=1,NSTATE
      AT(J,I)=TMAT(I,J)
15    CONTINUE
1229 VT(1)=1.0
      DO80I=2,NSTATE
      VT(I)=0.0
80    CONTINUE
      TCM=0.0
      DO 90 I=1,NDAYS
      CALL MATPRO(AT,VT,R)
      DO10J=1,NSTATE
      VT(J)=R(J)
10    CONTINUE
      ENF=TN*VT(NSTATE)
      IF(ENF .LT. FIX)ENF=0.0
      IF(ENF .LT. FIX)GOTO67
      DO921 IMT=1,60
      TMI=IMT
      IF(ENF .LT. FIX+TMI)GOTO922
      GOTO921
922   ENF=TMI
      GOTO67
921   CONTINUE
67    DO 900 KJ=IP,NDAYS,IP
      IF(I .EQ. KJ .AND. ENF .GE. OPNM)GOTO91
900   CONTINUE
      GOTO90
91    DO 1222 MK=1,61
      IF(PROB(MK) .GE. ENF)GOTO1223
1222 CONTINUE
1223 JM=MK
      WRITE(6,1112)ENF,I,IP
1112 FORMAT(1H0,11HFAILURES = ,F6.3,4X,7HDAYS = ,I5,17HINSPECTION EVERY
     1 ,I5,4HDAYS)
1224 LTHETA=KDAYS-JM
```

```
      NF=ONENIN+ENF
      IF(LTHETA .LE. 0)GOTO1024
      PMEAN=NF/LTHETA
      GOTO1025
1024  PMEAN=170.0
1025  SUM=0.0
      TNM=1.0
      NFO=NF+1
      DO 1225 KN=1,NFO
      TON=KN-1
      TNM=TON*TNM
      POM=PMEAN**TON
      IF(TNM .EQ. 0)POM=1.0
      IF(TNM .EQ. 0)TNM=1.0
      SUM=SUM+(POM*(EXP(-PMEAN)))/TNM
1225  CONTINUE
      PSF=1.0-SUM
      TCM=TCM+(TN-PROB(JM))*TP+PROB(JM)*TC+TINSP
      WRITE(6,600)TCM,PSF,OPNM
600   FORMAT(1H0,6HTCM = ,F10.3,4X,6HPSF = ,F6.3,4X,20HCRITICAL NO UNITS
     1 = ,F6.3)
      IF(ENF .GE. 19)GOTO2021
      VT(1)=VT(1)+VT(2)+VT(3)+VT(4)+VT(5)+VT(NSTATE)
      VT(NSTATE)=0.0
      DO1021 IKJM=2,34
      VT(IKJM)=VT(IKJM+5)
1021  CONTINUE
      GOTO90
2021  VT(1)=1.0
      DO 2022 IVT=2,NSTATE
      VT(IVT)=0.0
2022  CONTINUE
90    CONTINUE
      RETURN
      END
      SUBROUTINE MATPRO (A,B,C)
      DIMENSION A(60,60),B(60),C(60)
      COMMON NSTATE
      DO10I=1,NSTATE
      C(I)=0.0
      DO20K=1,NSTATE
      C(I)=C(I)+A(I,K)*B(K)
 20   CONTINUE
10    CONTINUE
      RETURN
      END
```

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AN ALGORITHM FOR A TWO-PHASE

STRATEGY FOR PREVENTIVE MAINTENANCE

Deepak Panjabi

Abstract

The primary object of this research was to develop a two-phase strategy for preventive maintenance and the critical number of units. Maintenance is done if the number of failures is equal to or greater than the critical number of units.

The system under consideration had q components. The system failed when k ($k < q$) components or more failed. This system when subjected to preventive maintenance can be described by a Markov Process. The transition probabilities of the Markov Process were obtained from the distributions of the strength and stress of the components. The underlining distributions were assumed to be double exponential. Various combinations of the inspection period and the critical number of units were used to obtain the global minimum. The criteria were that the system should not fail and minimum cost.