THE ELASTIC CONSTANTS AND WAVE VELOCITIES FOR AN AXIALLY SYMMETRIC MEDIUM

by

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The notations used in this thesis are summarized below:

Symbol	Definition	Units
F	Unit of force	
L	Unit of length	
T	Unit of time	
W	Strain energy function	FL/L^3
T_{ij} or T'_{ij}	Components of stress	F/L^2
e_{ij} or e'_{ij}	Components of strain	L/L
C_{ijkl}	General elastic constants	F/L^2
b_{ij}	Independent elastic constants	F/L^2
$\frac{\partial W}{\partial e_{ij}}$	Partial derivative of the strain energy function with respect to the components of strain	F/L^2
x_i or x'_i	A set of rectangular cartesian coordinates	L
l_i or m_i	A set of direction cosines for a vector in the x_i coordinate system	L/L
a_{ij}	Direction cosines of x'_i relative to x_i	L/L
E_3	T_{33}/e_{33}, Young's modulus in the x_3 direction	F/L^2
E_1	$T_{11}/e_{11} = T_{22}/e_{22}$	F/L^2
G_1	$T_{12}/2e_{12}$, shear modulus	F/L^2

Symbol	Definition	Units
G_3	$T_{23}/2e_{23} = T_{13}/2e_{13}$	F/L^2
σ_1	$- e_{11}/e_{33}$, Poisson's ratio	L/L
σ_2	$- e_{22}/e_{11}$	L/L
σ_3	$- e_{33}/e_{11}$	L/L
X_i	Body forces	F/L^3
u_i	Displacement of a particle in the x_i direction	L
ρ	Mass density per unit volume	FT^2/L^4
$u_{i,j}$	Denotes differentiation of the displacement with respect to x_j	
$T_{ij,j}$	Denotes differentiation of the components of stress T_{ij} with respect to the x_j	
V_i	Velocities of propagation	L/T
E'_{33}	T'_{33}/e'_{33}	F/L^2
G'_{12}	$T'_{12}/2e'_{12}$	F/L^2
G'_{13}	$T'_{13}/2e'_{13}$	F/L^2
G'_{23}	$T'_{23}/2e'_{23}$	F/L^2

IV. INTRODUCTION

The theory of aves is one of the most universal fields of mechanics. Wave mechanics deals with solids, liquids, gases, electricity, electronics, and heat flow. One of the major fields in wave mechanics of solids is seismology.

Seismology treats of those solids composing the earth. Such solids are definitely not isotropic and it is the purpose of this thesis to advance the theory for wave propagation in non-isotropic solids. Some of the earth's crust, being of a stratified nature, may be considered as axially symmetric media, and that is why that type of media has been chosen for investigation for this thesis.

In a review of the literature, it was found that the theory of elastic waves in isotropic solids is well established and is used extensively. Sometimes the simple theory is used to approximate the theory for non-isotropic media because the exact theory is not known and the results are satisfactory or because it is much easier to use.

Very little was found in the field for non-isotropic media, so it seems worthwhile here to investigate some of the physical properties for a general axially symmetric medium.

Thus, the purpose of this thesis is to investigate the properties and wave velocities for an axially symmetric medium.

V. SYNOPSIS

The investigation consists of four parts. In the first part, the physical properties of the medium are defined. Then the stress-strain relations for the case under consideration are obtained from the general case. This is done by imposing the condition of symmetry on the strain energy function.

Next the measurable constants are found in terms of the natural constants. This is done by applying simple extensions and shears to the material. After the measurable constants are determined in terms of the natural constants, then the relationship is inverted and the natural constants are found in terms of the measurable constants. Some elastic constants are then determined for an arbitrary direction, as it is not likely that the stresses will always be imposed along an axis of symmetry.

Following this major part, the equations of motion for a vibrating medium are determined in terms of the natural constants by substituting the stress-strain relations. This is very straightforward, but must be done.

The final step is made by finding the velocities of propagation of the waves by using the equations of motion. This is done by assuming a solution and substituting into the equations of motion. From these equations, a cubic equation defining the three principal velocities arises. The solution of this cubic equation is the culmination of this investigation.

VI. INVESTIGATION

A. The Stress-Strain Relations For an Axially Symmetric Medium

Axial symmetry can easily be defined with the aid of a coordinate system. Using a set of rectangular cartesian coordinates, x_i, and letting x_3 be the axis of symmetry, then the $x_1 - x_2$ plane is a plane of symmetry. (See Figure 1).

Definition:

An axially symmetric medium is one such that an arbitrary rotation of the x_i axes about x_3 produces no change in the stress-strain relations, implying isotropy in the $x_1 - x_2$ plane and all parallel planes.

This definition will be used to derive the particular stress-strain relations desired.

From the theory of elasticity,

$$T_{ij} = \frac{\partial W}{\partial e_{ij}} \tag{A-1}$$

where $2W = C_{ijkl}\ e_{ij}\ e_{kl}$. (A-2)

Due to symmetry in the stress and strain components and in the constants themselves, the eighty-one elastic constants, C_{ijkl}, may be reduced to twenty-one which cover all cases. These twenty-one will be included in b_{ij} where i and j range over the values $(i,j = 1,2,\ldots,6.)$. Using the twenty-one constants, leaves the strain energy function in the form,

$$\begin{aligned} 2W = {} & b_{11}\ e_{11}^2 + 2b_{12}\ e_{11}\ e_{22} + 2b_{13}\ e_{11}\ e_{33} + 2b_{14}\ e_{11}\ e_{23} \\ & + 2b_{15}\ e_{11}\ e_{31} + 2b_{16}\ e_{11}\ e_{12} \\ & + b_{22}\ e_{22}^2 + 2b_{23}\ e_{22}\ e_{33} + 2b_{24}\ e_{22}\ e_{23} + 2b_{25}\ e_{22}\ e_{31} \\ & + 2b_{26}\ e_{22}\ e_{12} \end{aligned}$$

$$+ b_{33} e_{33}^2 + 2b_{34} e_{33} e_{23} + 2b_{35} e_{33} e_{31} + 2b_{36} e_{33} e_{12}$$
$$+ b_{44} e_{23}^2 + 2b_{45} e_{23} e_{31} + 2b_{46} e_{23} e_{12}$$
$$+ b_{55} e_{31}^2 + 2b_{56} e_{31} e_{12}$$
$$+ b_{66} e_{12}^2 \qquad \text{(A-3)}$$

This is the general strain-energy function from which the stress-strain relations will arise. The number of elastic constants in (A-3) may be reduced by imposing the definition of axial symmetry. The first step then is to rotate the x_1 and x_2 axes 180 degrees about x_3. The following relations between the old and new strain components will then exist:

$$e_{11} = {}'e_{11} \qquad e_{22} = {}'e_{22} \qquad e_{33} = {}'e_{33}$$
$$e_{12} = {}'e_{12} \qquad e_{13} = -\,{}'e_{13} \qquad e_{23} = -\,{}'e_{33} \qquad \text{(A-4)}$$

W must be the same function of e_{ij} and $'e_{ij}$. Therefore, to find the form of the function, $'e_{ij}$ must be substituted into W. The result is shown below:

$$2W = b_{11}\,{}'e_{11}^2 + 2b_{12}\,{}'e_{11}\,{}'e_{22} + 2b_{13}\,{}'e_{11}\,{}'e_{33} - 2b_{14}\,{}'e_{11}\,{}'e_{23}$$
$$- 2b_{15}\,{}'e_{11}\,{}'e_{31} + 2b_{16}\,{}'e_{11}\,{}'e_{12}$$
$$+ b_{22}\,{}'e_{22}^2 + 2b_{23}\,{}'e_{22}\,{}'e_{33} - 2b_{24}\,{}'e_{22}\,{}'e_{23} - 2b_{25}\,{}'e_{22}\,{}'e_{31}$$
$$+ 2b_{26}\,{}'e_{22}\,{}'e_{12}$$
$$+ b_{33}\,{}'e_{33}^2 - 2b_{34}\,{}'e_{33}\,{}'e_{23} - 2b_{35}\,{}'e_{33}\,{}'e_{31} + b_{36}\,{}'e_{33}\,{}'e_{12}$$
$$+ b_{44}\,{}'e_{23}^2 + 2b_{45}\,{}'e_{23}\,{}'e_{31} - 2b_{46}\,{}'e_{23}\,{}'e_{12}$$
$$+ b_{55}\,{}'e_{31}^2 - 2b_{56}\,{}'e_{31}\,{}'e_{12}$$
$$+ b_{66}\,{}'e_{12}^2 \qquad \text{(A-5)}$$

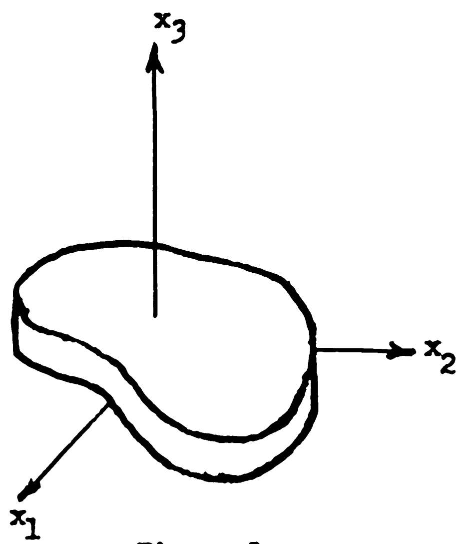

Figure 1

x_3 - axis of symmetry

$x_1 - x_2$ - plane of symmetry

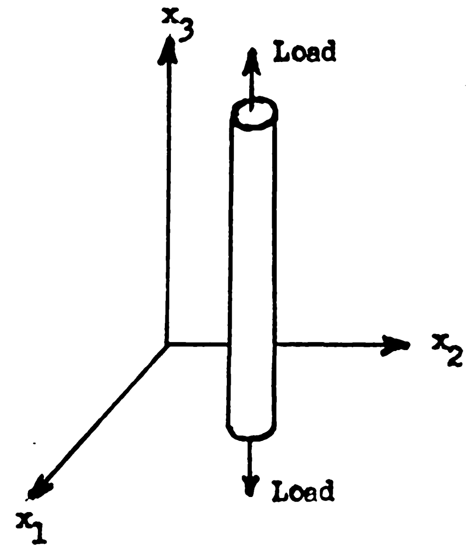

Figure 2

Load in x_2 direction

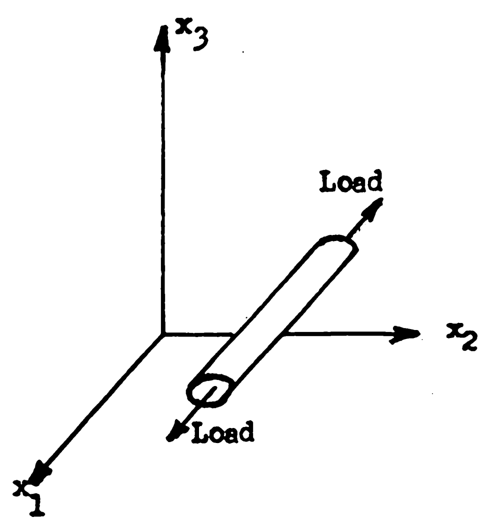

Figure 3

Load in x_1 direction

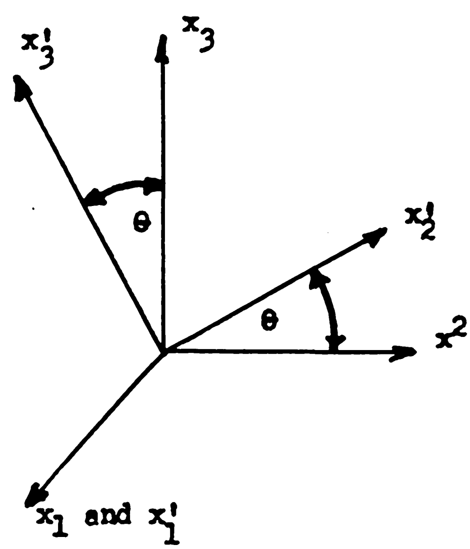

Figure 4

Arbitrary axes,

x_3' - direction of propagation

Since b_{14} cannot $= - b_{14}$ except that $b_{14} = 0$, b_{14} must $= 0$ and similarly for

$$b_{14} = b_{15} = b_{24} = b_{25} = b_{34} = b_{35} = b_{46} = b_{56} = 0 \qquad \text{(A-6)}$$

The number of constants may be reduced further by rotating the x_1 and x_2 axes 90 degrees about x_3. The relations then become:

$$e_{11} = {}'e_{22} \qquad e_{22} = {}'e_{11} \qquad e_{33} = {}'e_{33}$$

$$e_{12} = -{}'e_{12} \qquad e_{23} = {}'e_{13} \qquad e_{13} = -{}'e_{23} \qquad \text{(A-7)}$$

Substituting (A-7) into (A-5) discloses that:

$$b_{11} = b_{22}; \quad b_{13} = b_{23}; \quad b_{16} = -b_{26}$$

$$b_{36} = 0 \quad ; \quad b_{44} = b_{55}; \quad b_{45} = 0 \qquad \text{(A-8)}$$

Continuing, the next rotation is of 45 degrees and the relations between the components are:

$$2e_{11} = {}'e_{11} + 2'e_{12} + {}'e_{22}$$

$$2e_{22} = {}'e_{11} - 2'e_{12} + {}'e_{22}$$

$$e_{33} = {}'e_{33}$$

$$2e_{12} = -{}'e_{11} + {}'e_{22}$$

$$\sqrt{2}\, e_{13} = {}'e_{13} + {}'e_{23}$$

$$\sqrt{2}\, e_{23} = -{}'e_{13} + {}'e_{23} \qquad \text{(A-9)}$$

When (A-8) and (A-9) are substituted into (A-5), then it is found that $b_{16} = 0$; $b_{66} = 2(b_{11} - b_{12})$. (A-10)

Using (A-6), (A-8) and (A-10) in (A-3), gives the strain energy function in its final form with only five independent constants left. Now:

$$2W = b_{11}(e_{11}^2 + 2e_{12}^2 + e_{22}^2) + 2b_{12}(e_{11}e_{22} - e_{12}^2)$$
$$+ 2b_{13}e_{33}(e_{11} + e_{22}) + b_{33}e_{33}^2 + b_{44}(e_{13}^2 + e_{23}^2). \quad \text{(A-11)}$$

To prove that this is the final form, it is only required that the x_1 and x_2 axes be rotated through an arbitrary angle about the x_3 axis and thus show that there is no further reduction in the number of independent constants. For this rotation, the direction cosines become

$$a_{11} = a_{22} = \cos\theta \,;\ a_{12} = -a_{21} = \sin\theta \,;\ a_{33} = 1 \,;$$
$$a_{13} = a_{31} = a_{23} = a_{32} = 0$$

The relations between the components are found by using

$$e_{ij} = e'_{mm}\, a_{mi}\, a_{nj} \quad \text{(A-12)}$$

For this arbitrary rotation, the relations between the components are

$$e_{11} = {}'e_{11}\cos^2\theta + 2{}'e_{12}\sin\theta\, 0\cos\theta + {}'e_{22}\sin^2 0$$
$$e_{22} = {}'e_{11}\sin^2\theta - 2{}'e_{12}\sin\theta\cos\theta + {}'e_{22}\cos^2\theta$$
$$e_{33} = {}'e_{33}$$
$$e_{12} = -{}'e_{11}\sin\theta\cos\theta + {}'e_{12}(\cos^2\theta - \sin^2\theta) + {}'e_{22}\sin\theta\cos\theta$$
$$e_{13} = {}'e_{13}\cos\theta + {}'e_{23}\sin\theta$$
$$e_{23} = -{}'e_{13}\sin\theta + {}'e_{23}\cos\theta \quad \text{(A-13)}$$

Substituting (A-13) into (A-11) and expanding, it follows that

$$2W = 2b_{12}\left\{ {}'e_{11}\,{}'e_{22}(\cos^4\theta + \sin^4\theta) + 2\,{}'e_{11}\,{}'e_{22}\sin^2\theta\cos^2\theta \right\}$$
$$+ b_{11}\left\{ {}'e_{11}^2\cos^4\theta + {}'e_{11}^2\sin^4\theta + {}'e_{22}^2\sin^4\theta \right.$$
$$\left. + {}'e_{22}^2\cos^4\theta + 2{}'e_{11}^2\sin^2\theta\, \mathrm{dos}^2\theta + 2{}'e_{22}\sin^2\theta\cos^2\theta \right\}$$

$$+2b_{11}\{4\ 'e_{12}^2\ \cos^2\theta\sin^2\theta + 'e_{12}^2\ (\cos^2\theta - \sin^2\theta)^2\}$$

$$-\ 2b_{12}\{4\ 'e_{12}^2\cos^2\theta\sin^2\theta + 'e_{12}^2\ (\cos^2\theta - \sin^2\theta)^2\}$$

$$+\ b_{11}\{4\ 'e_{11}\ 'e_{12}\cos^3\theta\sin\theta + 2\ 'e_{11}\ 'e_{22}\cos^2\theta\sin^2\theta$$

$$+\ 4\ 'e_{22}\ 'e_{12}\cos\theta\sin^3\theta - 4'e_{22}\ 'e_{12}\cos^3\theta\sin\theta$$

$$+\ 2'e_{22}\ 'e_{11}\sin^2\theta\cos^2\theta - 4\ 'e_{12}\ 'e_{11}\cos\theta\sin^3\theta$$

$$-\ 4\ 'e_{11}\ 'e_{22}\sin^2\theta\cos^2\theta + 4\ 'e_{11}\ 'e_{12}\sin^3\theta\cos\theta$$

$$-\ 4\ 'e_{11}\ 'e_{12}\sin\theta\cos^3\theta + 4\ 'e_{22}\ 'e_{12}\sin\theta\cos^3\theta$$

$$-\ 4\ 'e_{22}\ 'e_{12}\sin^3\theta\cos\theta\}$$

$$+\ b_{12}\{2\ 'e_{11}^2\sin^2\theta\cos^2\theta + 2\ 'e_{22}^2\sin^2\theta\cos^2\theta$$

$$+\ 4\ 'e_{11}\ 'e_{12}\cos\theta\sin^3\theta - 4\ 'e_{11}\ 'e_{12}\cos^3\theta\sin\theta$$

$$+\ 4\ 'e_{22}\ 'e_{12}\sin\theta\cos^3\theta - 4\ 'e_{22}\ 'e_{12}\sin^3\theta\cos\theta$$

$$-\ 2\ 'e_{11}^2\sin^2\theta\cos^2\theta - 2\ 'e_{22}^2\ \sin^2\theta\cos^2\theta$$

$$-\ 4\ 'e_{11}\ 'e_{12}\sin^3\theta\cos\theta + 4\ 'e_{11}\ 'e_{12}\sin\theta\cos^3\theta$$

$$-\ 4\ 'e_{22}\ 'e_{12}\sin\theta\cos^3\theta + 4\ 'e_{22}\ 'e_{12}\sin^3\theta\cos\theta\}$$

$$+\ 2b_{13}\ 'e_{33}\ ('e_{11} + 'e_{22}) + b_{33}\ 'e_{33}^2 + b_{44}\ ('e_{13}^2 + 'e_{23}^2). \quad \text{(A-14)}$$

which reduces to (A-11).

B. The Measurable Elastic Constants In Terms of The Natural Constants

The first step here is to find the stress-strain relations for the axially symmetric medium. This is done by using (A-11) in (A-1) and it turns out that:

$$T_{11} = b_{11}\, e_{11} + b_{12}\, e_{22} + b_{13}\, e_{33},$$
$$T_{22} = b_{12}\, e_{11} + b_{11}\, e_{22} + b_{13}\, e_{33},$$
$$T_{33} = b_{13}\, e_{11} + b_{13}\, e_{22} + b_{33}\, e_{33},$$
$$T_{12} = 2(b_{11} - b_{12})\, e_{12},$$
$$T_{23} = b_{44}\, e_{23}, \text{ and}$$
$$T_{13} = b_{44}\, e_{13} \qquad \text{(B-1)}$$

The next step is to use these stress-strain relations to find the measurable constants in terms of the natural constants. These measurable elastic constants are the Young's moduli and Poisson's ratios. Finding the Young's modulus for the x_3 direction is done by taking the x_3 axis to be the axis of symmetry and applying a simple tensile load in the x_3 direction. (See Figure 2). Let all the other stresses be zero. That is

$$T_{33} = T_{33} \;;\; T_{ij} = 0,\; i \neq j \neq 3 \qquad \text{(B-2)}$$

Imposing condition (B-2) on (B-1) gives

$$b_{11}\, e_{11} + b_{12}\, e_{22} + b_{13}\, e_{33} = 0$$
$$b_{12}\, e_{11} + b_{11}\, e_{22} + b_{13}\, e_{33} = 0$$
$$b_{13}\, e_{11} + b_{13}\, e_{22} + b_{33}\, e_{33} = T_{33} \qquad \text{(B-3)}$$

Solving (B-3) for $E_3 = T_{33}/e_{33}$ gives

$$E_3 = b_{33}\,(b_{11} + b_{12}) - 2\, b_{13}^2 \;/(b_{11} + b_{12}). \qquad \text{(B-4)}$$

And solving (B-3)

$$\sigma_1 = -\frac{e_{11}}{e_{33}} = b_{13}/(b_{11} + b_{12}) \qquad \text{(B-5)}$$

Similarly, applying a simple tension test along the x_1 axis (see Figure 3) gives

$$b_{11}\,e_{11} + b_{12}\,e_{22} + b_{13}\,e_{33} = T_{11},$$
$$b_{12}\,e_{11} + b_{11}\,e_{22} + b_{13}\,e_{33} = 0, \text{ and}$$
$$b_{13}\,e_{11} + b_{13}\,e_{22} + b_{33}\,e_{33} = 0. \qquad \text{(B-6)}$$

Solving (B-6)

$$E_1 = T_{11}/e_{11} = \frac{(b_{11} - b_{12})}{(b_{13}^2 - b_{11}\,b_{33})}(2\,b_{13}^2 - b_{11}\,b_{33} - b_{12}\,b_{33})$$

$$E_1 = \frac{(b_{12}^2 - b_{11}^2)}{(b_{13}^2 - b_{11}\,b_{33})}\;E_3 \qquad \text{(B-7)}$$

$$\sigma_2 = -\,e_{22}/e_{11} = \frac{b_{12}\,b_{33} - b_{13}^2}{b_{11}\,b_{33} - b_{13}^2} \qquad \text{(B-8)}$$

$$\sigma_3 = -\,e_{33}/e_{11} = \frac{b_{13}(b_{12} - b_{11})}{(b_{13}^2 - b_{11}\,b_{33})} \qquad \text{(B-9)}$$

Using the stress strain relations for the shear components of stress gives rise to the following:

$$T_{12} = 2(b_{11} - b_{12})\,e_{12}$$

$$\frac{T_{12}}{2e_{12}} = (b_{11} - b_{12}) = G_1 \qquad \text{(B-10)}$$

$$T_{23} = b_{44}\,e_{23}$$

$$\frac{T_{23}}{2e_{23}} = \frac{b_{44}}{2} = G_3 \qquad \text{(B-11)}$$

Thus the measurable constants are given in terms of the natural constants. These constants are collected below for convenience.

Table 1 - The Measurable Elastic Constants

$$E_1 = \frac{(b_{12}^2 - b_{11}^2)}{(b_{13}^2 - b_{11}\, b_{33})}\, E_3 \qquad \text{(B-7)}$$

$$E_3 = \frac{(b_{11}\, b_{33} + b_{12}\, b_{33} - 2b_{13}^2)}{(b_{11} + b_{12})} \qquad \text{(B-4)}$$

$$G_1 = b_{11} - b_{12} \qquad \text{(B-10)}$$

$$G_3 = \frac{b_{44}}{2} \qquad \text{(B-11)}$$

$$\sigma_1 = \frac{b_{13}}{b_{11} + b_{12}} \qquad \text{(B-5)}$$

$$\sigma_2 = \frac{b_{12}\, b_{33} - b_{13}^2}{b_{11}\, b_{33} - b_{13}^2} \qquad \text{(B-8)}$$

$$\sigma_3 = \frac{b_{13}\, (b_{12} - b_{11})}{b_{13}^2 - b_{11}\, b_{33}} \qquad \text{(B-9)}$$

(B-12), (B-13), and (B-14), which are some interesting relations, may be noted here. They are obtained in the following manner. (B-5) and (B-9) are substituted into (B-7), getting

$$E_3 = \frac{\sigma_1}{\sigma_3}\, E_1 \qquad \text{(B-12)}$$

Now, dividing (B-7) by (B-10) gives

$$E_1 = G_1\, (1 + \sigma_2) \qquad \text{(B-13)}$$

$$E_3 = \frac{\sigma_1}{\sigma_3}\, (1 + \sigma_2)\, G_1 \qquad \text{(B-14)}$$

To give the stress-strain relations a full meaning, the natural constants are needed in terms of the measurable constants. Therefore, the existing relations must be inverted.

Some of the steps will be shown to add clarity and continuity. Adding (B-10) and (B-5) to remove b_{12} gives

$$2b_{11} = \frac{b_{13}}{\sigma_1} + G_1 \tag{B-15}$$

When (B-10), (B-5), (B-15), and (B-12) are substituted into (B-7)

$$\frac{b_{33}}{2}\left(\frac{b_{13}}{\sigma_1} + G_1\right) - b_{13}^2 = \frac{G_1\, b_{13}}{\sigma_3} \tag{B-16}$$

is the result.

$$b_{13} = \frac{\sigma_1\, G_1\, (1 + \sigma_2)}{(1 - \sigma_2 - 2\sigma_1\, \sigma_3)} \tag{B-17}$$

is the result upon substituting (B-14), (B-5), and (B-16) into (B-4).

By substituting (B-17) into (B-15), it is found that

$$b_{11} = \frac{G_1\, (1 - \sigma_1\, \sigma_3)}{1 - \sigma_2 - \sigma_1\, \sigma_3} \tag{B-18}$$

Using this result, (B-18), in (B-10) gives

$$b_{12} = \frac{G_1\, (\sigma_2 + \sigma_1\, \sigma_3)}{1 - \sigma_2 - \sigma_1\, \sigma_3} \tag{B-19}$$

Finally substitutihg (B-17) into (B-16) gives

$$b_{33} = \frac{\sigma_1\, G_1\, (1 + \sigma_2)\, (1 - \sigma_2)}{\sigma_3\, (1 - \sigma_2 - 2\sigma_1\, \sigma_3)} \tag{B-20}$$

For convenience, the natural constants are tabulated here:

Table 2 - The Natural Elastic Constants

$b_{11} = \dfrac{G_1 (1 - \sigma_1 \sigma_3)}{1 - \sigma_2 - 2 \sigma_1 \sigma_3}$	(B-18)
$b_{12} = \dfrac{G_1 (\sigma_2 + \sigma_1 \sigma_3)}{1 - \sigma_2 - 2 \sigma_1 \sigma_3}$	(B-19)
$b_{13} = \dfrac{\sigma_1 G_1 (1 + \sigma_2)}{1 - \sigma_2 - 2\sigma_1 \sigma_3}$	(B-17)
$b_{33} = \dfrac{\sigma_1 G_1 (1 + \sigma_2) (1 - \sigma_2)}{\sigma_3 (1 - \sigma_2 - 2 \sigma_1 \sigma_3)}$	(B-20)
$b_{44} = 2 G_3$	(B-21)

C. The Measurable Elastic Constants For An Arbitrary Direction.

The measurable elastic constants given above can only be used when the stresses are in the direction of the three axes. Therefore, the various Young's moduli and shearing moduli will be found for the case when the direction is arbitrary.

Finding these constants for an arbitrary direction will require the stress and strain components for an arbitrary direction. The easiest way to get these components is to take a new set of rectangular cartesian coordinates x_i' such that the x'_2 and x'_3 are rotated about x_1' through some arbitrary angle. (See Figure 4). The direction numbers then for this case are $a_{11} = 1$; $a_{22} = a_{33} = l_2$, and $a_{12} = a_{13} = a_{21} = a_{31} = 0$, and $a_{23} = - a_{32} = l_3$. There will be no loss in generality by letting x_1 and x_1' coincide due to the nature of the medium.

To obtain the relationships, the following formulas are required:

$$T_{mn} = T'_{ij}\, a_{im}\, a_{jn}, \tag{C-1}$$

$$T_{mn} = C_{mnkl}\, e_{kl}, \text{ and} \tag{C-2}$$

$$e_{ij} = e_{mn}\, a_{im}\, a_{jn} \tag{C-3}$$

First, a shearing modulus relating T'_{12} and e'_{12} will be found. Let T'_{12} have a value and all other $T'_{ij} = 0$. Using (C-1), the stress components are

$$T_{12} = T'_{12}\, l_2 \text{ and}$$
$$T_{13} = T'_{12}\, l_3. \tag{C-4}$$

Using (B-1) and (B-21), the following relations arise:

$$T'_{12}\, l_2 = 2\, G_1\, e_{12}$$

$$T'_{12}\, l_3 = 2\, G_3\, e_{13} \tag{C-5}$$

From (C-3) comes the relationship

$$e'_{12} = e_{12}\, l_2 + e_{13}\, l_3. \tag{C-6}$$

Substitution of (C-5) into (C-6), gives

$$\frac{1}{G'_{12}} = \frac{2e'_{12}}{T'_{12}} = \frac{G_2\, l_2^2 + G_1\, l_3^2}{G_1\, G_3} \tag{C-7}$$

Continuing in this direction, let T'_{13} have a value and other $T'_{ij} = 0$. Similarly:

$$\frac{1}{G'_{13}} = \frac{2e'_{13}}{T'_{13}} = \frac{G_3\, l_3^2 + G_1\, l_2^2}{G_1\, G_3} \tag{C-8}$$

Let T'_{23} have a value and other $T'_{ij} = 0$, then from (C-1) and (B-1),

$$T_{11} = 0 \qquad = b_{11} e_{11} + b_{12} e_{22} + b_{13} e_{33}$$

$$T_{22} = -2 T'_{23} l_2 l_3 = b_{12} e_{11} + b_{11} e_{22} + b_{13} e_{33}$$

$$T_{33} = 2 T'_{23} l_2 l_3 = b_{13} e_{11} + b_{13} e_{22} + b_{33} e_{33}$$

$$T_{23} = T'_{23} (l_2^2 - l_3^2) = 2 a_3 e_{23}$$

$$e'_{32} = - e_{22} l_2 l_3 - e_{23} l_3^2 + e_{23} l_2^2 + e_{33} l_2 l_3 \qquad \text{(C-9)}$$

let $A = \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{11} & b_{13} \\ b_{13} & b_{13} & b_{33} \end{vmatrix}$

$B = \begin{vmatrix} b_{11} & 0 & b_{13} \\ b_{12} & -2T'_{23} l_2 l_3 & b_{13} \\ b_{13} & 2T'_{23} l_2 l_3 & b_{33} \end{vmatrix}$

$C = \begin{vmatrix} b_{11} & b_{12} & 0 \\ b_{12} & b_{11} & -2T'_{23} l_2 l_3 \\ b_{13} & b_{13} & 2T'_{23} l_2 l_3 \end{vmatrix}$

Let it be noted here that the first three equations of (C-9) are simultaneous equations in e_{11}, e_{22}, and e_{33}. Then "A" is simply the determinant of the coefficients of the e_{11}, e_{22}, and e_{33} in these equations.

"B" and "C" are formed from "A" by replacing the second and third columns respectively with the constant terms. The solutions to the equations are obtained by Cramer's Rule.

Then $e_{22} = B/A$; $e_{33} = C/A$.

Substituting for e_{22} and e_{23} and e_{23} in (C-9), gives

$$\frac{1}{G'_{23}} = \frac{2e'_{23}}{T'_{23}} = \frac{(l_2^2 - l_3^2)^2}{G_3} \; \frac{+4\, l_2^2\, l_3^2}{A} \begin{vmatrix} 4b_{11} & 2b_{12} + 2b_{13} \\ 2b_{12} + 2b_{13} & b_{11} + 2b_{13} + b_{33} \end{vmatrix} \qquad \text{(C-10)}$$

Finally, let T'_{33} have a value and all other $T_{ij} = 0$.

$$T_{11} = 0$$
$$T_{22} = T'_{33}\, l_3^2$$
$$T_{33} = T'_{33}\, l_2^2$$
$$T_{23} = - T'_{33}\, l_2\, l_3$$
$$e'_{33} = e_{22}\, l_3^2 + e_{33}\, l_2^2$$

Performing operations similar to those above and remembering the definition of "A"

$$\frac{1}{E'_{33}} = \frac{e'_{33}}{T'_{33}} = \frac{l_3^4}{A} (b_{11}\, b_{33} - b_{13}^2) + \frac{l_2^4}{A} G_1\, (b_{11} + b_{12})$$
$$- \frac{2\, l_2^2\, l_3^2}{A} G_1\, b_{13} \qquad \text{(C-11)}$$

Collecting these elastic constants for arbitrary directions gives:

Table 3. The Measurable Elastic Constants

For An Arbitrary Direction.

$$\frac{1}{E'_{33}} = \frac{l_3^4}{A}\left(b_{11}\, b_{33} - b_{13}^2\right) + \frac{l_2^4}{A}\, G_1 \left(b_{11} + b_{12}\right)$$

$$- 2\, \frac{l_2^2\, l_3^2}{A}\, G_1\, b_{13} \qquad \text{(C-11)}$$

$$\frac{1}{G'_{12}} = \frac{G_3\, l_2^2 + G_1\, l_3^2}{G_1\, G_3} \qquad \text{(C-7)}$$

$$\frac{1}{G'_{13}} = \frac{G_3\, l_3^2 + G_1\, l_2^2}{G_1\, G_3} \qquad \text{(C-8)}$$

$$\frac{1}{G'_{23}} = \frac{(l_2^2 - l_3^2)^2}{G_3} + \frac{4\, l_2^2\, l_3^2}{A} \begin{vmatrix} 4b_{11} & 2b_{12} + 2b_{13} \\ 2b_{12} + 2b_{13} & b_{11} + 2b_{13} + b_{33} \end{vmatrix} \qquad \text{(C-10)}$$

D. The Velocities of Propagation

Getting the equations of motion for the disturbed medium is the first step towards getting the velocities of propagation. The equations of motion come from the equilibrium equations of elasticity

$$T_{ij,j} + x_i = 0 \qquad \text{(D-1)}$$

with the inertia force added. To include the inertia force, simply add $-\rho \frac{\partial^2 u_i}{\partial t^2}$, the mass times the acceleration of the particle. Neglecting the body forces, x_i, gives

$$T_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad \text{(D-2)}$$

This is a mixed equation of stresses and displacements. This equation for this investigation tells more when it is expressed exclusively in terms of displacement, so it will be expressed that way.

By definition,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{D-3}$$

Substituting (D-3) into (B-1) and this into (D-2), gives

$$\rho \frac{\partial^2 u_1}{\partial t^2} = b_{11} \frac{\partial^2 u_1}{\partial {x_1}^2} + b_{12} \frac{\partial^2 u_2}{\partial x_2 \partial x_1} + b_{13} \frac{\partial^2 u_3}{\partial x_3 \partial x_1}$$

$$+ (b_{11} - b_{12}) \frac{\partial^2 u_1}{\partial {x_2}^2} + (b_{11} - b_{12}) \frac{\partial^2 u_2}{\partial x_1 \partial x_2}$$

$$+ \frac{b_{44}}{2} \frac{\partial^2 u_1}{\partial {x_3}^2} + \frac{b_{44}}{2} \frac{\partial^2 u_3}{\partial x_1 \partial x_3} \tag{D-4}$$

$$\rho \frac{\partial^2 u_2}{\partial t^2} = (b_{11} - b_{12}) \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + (b_{11} - b_{12}) \frac{\partial^2 u_2}{\partial {x_1}^2}$$

$$+ b_{12} \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + b_{11} \frac{\partial^2 u_2}{\partial {x_2}^2} + b_{13} \frac{\partial^2 u_3}{\partial x_2 \partial x_3}$$

$$+ \frac{b_{44}}{2} \frac{\partial^2 u_2}{\partial {x_3}^2} + \frac{b_{44}}{2} \frac{\partial^2 u_3}{\partial x_2 \partial x_3} \tag{D-5}$$

$$\rho \frac{\partial^2 u_3}{\partial t^2} = \frac{b_{44}}{2} \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \frac{b_{44}}{2} \frac{\partial^2 u_3}{\partial {x_1}^2}$$

$$+ \frac{b_{44}}{2} \frac{\partial^2 u_2}{\partial x_2 \partial x_3} + \frac{b_{44}}{2} \frac{\partial^2 u_3}{\partial {x_2}^2}$$

$$+ b_{13} \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + b_{13} \frac{\partial^2 u_2}{\partial x_2 \partial x_3} + b_{33} \frac{\partial^2 u_3}{\partial x_3^2} \qquad \text{(D-6)}$$

Now that the equations of motion are in terms of displacement only, they can be used in the investigation of some of the phenomena of wave propagation. The first case that will be studied is that of a plane wave whose front is parallel to the $x_1 - x_2$ plane; that is, $u_i = u_i(x_3 - Vt)$ where V is the velocity of propagation of the wave. Substituting into (D-4), (D-5), and (D-6), gives

$$2\rho \frac{\partial^2 u_1}{\partial t^2} = b_{44} \frac{\partial^2 u_1}{\partial x_3^2}$$

$$2\rho \frac{\partial^2 u_2}{\partial t^2} = b_{44} \frac{\partial^2 u_2}{\partial x_3^2}$$

$$\rho \frac{\partial^2 u_3}{\partial t^2} = b_{33} \frac{\partial^2 u_3}{\partial x_3^2} \qquad \text{(D-7)}$$

Now differentiate with respect to the corrdinates, that is $\frac{\partial u_i}{\partial x_i}$.

$$\rho \frac{\partial^2 u_{3,3}}{\partial t^2} = b_{33} \frac{\partial^2 u_{3,3}}{\partial x_3^2} \qquad \text{(D-8)}$$

This defines the velocity of dilatation in the x_3 direction,

$$\rho V^2 = b_{33} \qquad \text{(D-9)}$$

Before finding the velocities for the rotary wave, note here that the components of rotation are given by

$$2\omega_1 = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right),$$

$$2\omega_2 = \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right),$$

$$2\omega_3 = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right). \qquad \text{(D-10)}$$

Performing the above indicated operations, gives

$$2\rho \quad \frac{\partial^2 u_{2,3}}{\partial t^2} = b_{44} \frac{\partial^2 u_{2,3}}{\partial x_3^2}$$

$$2\rho \quad \frac{\partial^2 u_{1,3}}{\partial t^2} = b_{44} \frac{\partial^2 u_{1,3}}{\partial x_3^2} \quad . \qquad \text{(D-11)}$$

Then

$$\rho V^2 = G_3 \text{ for } \omega_1$$

$$\rho V^2 = G_3 \text{ for } \omega_2 \qquad \text{(D-12)}$$

when the wave is initiated in the x_3 direction.

When the case for a plane wave perpendicular to the x_1 axis is investigated, it is found by a similar analysis that the following relations hold. The velocity of the dilatory wave is defined by

$$\rho V^2 = b_{11}. \qquad \text{(D-13)}$$

Taking the curl for this set of equations of motion, gives

$$\rho V^2 = G_1 \text{ for } \omega_3$$

$$\rho V^2 = G_3 \text{ for } \omega_2 \qquad \text{(D-14)}$$

when the wave is initiated in the x_1 direction.

The final case to be investigated is that for a wave initially plane, propagated in an arbitrary direction. This may be done if the wave considered is so far from its point of inception that its curvature is very small. It is assumed that the wave form is a trigonometric function of the form $u_1 = m_1 A \cos \frac{2\pi}{\lambda}(x_2 l_2 + x_3 l_3 - Vt)$ where m_1 is the unit vector in the direction of displacement. A is the amplitude of the wave and λ is the wave length. Due to the axial symmetry, any arbitrary direction is defined by the angle between the direction of propagation and the $x_1 - x_2$ plane. (See Figure 4). That is, the direction is a function of one angle only and therefore $l_2^2 + l_3^2 = 1$. This shows there is no loss in generality by assuming this form.

Substituting into (D-3), (D-4) and (D-5) and clearing

$$m_1 \rho V^2 = (b_{11} - b_{12})\, l_2^2\, m_1 + \frac{b_{44}}{2}\, l_3^2\, m_1 \tag{D-15}$$

$$m_2 \rho V^2 = b_{11}\, l_2^2\, m_2 + b_{13}\, l_2\, l_3\, m_3 + \frac{b_{44}}{2}\, l_3^2\, m_2 + \frac{b_{44}}{2}\, l_2\, l_3\, m_3 \tag{D-16}$$

$$m_3 \rho V^2 = \frac{b_{44}}{2}\, l_2\, l_3\, m_2 + \frac{b_{44}}{2}\, l_2^2\, m_3 + b_{13}\, l_2\, l_3\, m_2 + b_{33}\, l_3^2\, m_3 \tag{D-17}$$

Now for m_i to have a value, other than zero, the determinant of their coefficients must vanish.

$$\begin{vmatrix} G_1 l_2^2 + G_3 l_3^2 - \rho V^2 & 0 & 0 \\ 0 & b_{11} l_2^2 + G_3 l_3^2 - \rho V^2 & (b_{13} + G_3)\, l_2\, l_3 \\ 0 & (b_{13} + G_3)\, l_2\, l_3 & G_3 l_2^2 + b_{33} l_3^2 - \rho V^2 \end{vmatrix} = 0 \tag{D-18}$$

Therefore $\rho V_1^2 = G_1 l_2^2 + G_3 l_3^2$ and

$$(b_{11} l_2^2 + G_3 l_3^2 - \rho V^2)\,(G_3 l_2^2 + b_{33} l_3^2 - \rho V^2) - (b_{13} + G_3)^2\, l_2^2\, l_3^2 = 0$$

which is quadratic in ρV^2. Using the quadratic formula, gives

$$2\rho\, V^2 = b_{11}\, l_2^2 + G_3\, l_3^2 + G_3\, l_2^2 + b_{33}\, l_3^2$$

$$\pm\sqrt{(b_{11}\, l_2^2 + G_3\, l_3^2 - G_3\, l_2^2 - b_{11}\, l_3^2)^2 + 4(b_{13} + G_3)^2\, l_2^2\, l_3^2} \quad \text{(D-19)}$$

To check this with the cases for known direction of propagation, let

$$l_3 = 1, \quad l_2 = 0.$$

$$\rho\, V_1^2 = G_3 \text{ for the rotary wave}$$

$$\rho\, V_2^2 = G_3 \text{ for the rotary wave}$$

$$\rho\, V_3^2 = b_{33} \text{ for the dilatory wave.} \quad \text{(D-20)}$$

Now let $l_3 = 0, \quad l_2 = 1.$

$$\rho\, V_1^2 = G_1 \text{ for the rotary wave}$$

$$\rho\, V_2^2 = b_{11} \text{ for the dilatory wave}$$

$$\rho\, V_3^2 = G_3 \text{ for the rotary wave.} \quad \text{(D-21)}$$

Now that the velocities of propagation have been found, next the relations that must exist between the m_i will be found. Substituting the value for $\rho\, V_1^2$ into (D-15), (D-16), and (D-17) gives

$$0\, m_1 = 0$$

$$(b_{11} - G_1)\, l_2^2\, m_2 + (b_{13} + G_3)\, l_2\, l_3\, m_3 = 0$$

$$(b_{13} + G_3)\, l_2\, l_3\, m_2 + (G_3\, l_2^2 - G_3\, l_3^2 + b_{33}\, l_3^2 - G_1\, l_2^2) m_3 = 0 \quad \text{(D-22)}$$

For m_2 and m_3 to have a value other than zero, the determinant of their coefficients must vanish. Putting everything in terms of b_{ij} and checking this condition, shows

$$2b_{12}\, l_2^2\, (b_{44}\, l_2^2 - b_{44}\, l_3^2 - 2b_{11}\, l_2^2 + 2b_{12}\, l_2^2 + 2b_{33}\, l_3^2)$$
$$- (2b_{13} + b_{44})^2\, l_2^2\, l_3^2 \neq 0$$

as all these quantities are independent and arbitrary. Due to $m_i\, m_i = 1$, then $^{(1)}m_1 = 1$ as $^{(1)}m_2 = {}^{(1)}m_3 = 0$.

Working in a similar fashion and substituting for ρV_2^2 and ρV_3^2 respectively, it is found that $^{(2)}m_1 = 0$

$$^{(2)}m_2 : {}^{(2)}m_3 = \frac{G_3\, l_2^2 + b_{33}\, l_3^2 - b_{11}\, l_2^2 - G_3\, l_3^2}{2(b_{13} + G_3)l_2\, l_3}$$

$$- \frac{\sqrt{(b_{11}\, l_2^2 + G_3\, l_3^2 - G_3\, l_2^2 - b_{33}\, l_3^2)^2 + 4(b_{13} + G_3)^2\, l_2^2\, l_3^2}}{2(b_{13} + G_{23})\, l_2\, l_3} \qquad \text{(D-23)}$$

$$^{(3)}m_1 = 0$$

$$^{(3)}m_2 : {}^{(3)}m_3 = \frac{G_3\, l_2^2 + b_{33}\, l_3^2 - b_{11}\, l_2^2 - G_3\, l_3^2}{2(b_{13} + G_3)\, l_2\, l_3}$$

$$+ \frac{\sqrt{(b_{11}\, l_2^2 + G_3\, l_3^2 - G_3\, l_2^2 - b_{33}\, l_3^2)^2 + 4(b_{13} + G_3)^2\, l_2^2\, l_3^2}}{2(b_{13} + G_3)\, l_2\, l_3}$$

$^{(1)}m_1$, $^{(2)}m_1$, and $^{(3)}m_1$ are the unit vectors corresponding, respectively to V_1, V_2, and V_3.

Now for convenience, these velocities are tabulated below:

Table 4. Velocities for $l_3 = 1$

ρV_1^2	$=$	G_3	- rotary wave	
ρV_2^2	$=$	G_3	- rotary wave	
ρV_3^2	$=$	b_{33}	- dilatory wave	(D-20)

Table 5. Velocities for $l_2 = 1$

$\rho V_1^2 = G_1$	-	rotary wave	
$\rho V_2^2 = b_{11}$	-	dilatory wave	
$\rho V_3^2 = G_3$	-	rotary wave	(D-21)

Table 6. Velocities for arbitrary directions

$$\rho V_1^2 = G_1 l_2^2 + G_3 l_3^2 \qquad \text{(D-19)}$$

$$2\rho V_2^2 = b_{11} l_2^2 + G_3 l_3^2 + G_3 l_2^2 + b_{33} l_3^2 + \sqrt{(b_{11} l_2^2 + G_3 l_3^2 - G_3 l_2^2 - b_{33} l_3^2)^2 + 4(b_{13} + G_3)^2 l_2^2 l_3^2}$$

$$2\rho V_3^2 = b_{11} l_2^2 + G_3 l_3^2 + G_3 l_2^2 + b_{33} l_3^2 - \sqrt{(b_{11} l_2^2 + G_3 l_3^2 - G_3 l_2^2 - b_{33} l_3^2)^2 + 4(b_{13} + G_3)^2 l_2^2 l_3^2}$$

Table 7. Relations between the $^{(j)}m_i$ for the various velocities

$$^{(1)}m_1 : {}^{(1)}m_2 : {}^{(1)}m_3 = 1 : 0 : 0$$

$$^{(2)}m_1 : {}^{(2)}m_2 : {}^{(2)}m_3 = 0 : \left\{ G_3 l_2^2 + b_{33} l_3^2 - b_{11} l_2^2 - G_3 l_3^2 - \sqrt{(b_{11} l_2^2 + G_3 l_3^2 - G_3 l_2^2 - b_{33} l_3^2)^2 + 4(b_{13} + G_3)^2 l_2^2 l_3^2} \right\} : \left\{ 2(b_{13} + G_3) l_2 l_3 \right\}$$

$$^{(3)}m_1 : {}^{(3)}m_2 : {}^{(3)}m_3 = 0 : \left\{ G_3 l_2^2 + b_{33} l_3^2 - b_{11} l_2^2 - G_3 l_3^2 + \sqrt{(b_{11} l_2^2 + G_3 l_3^2 - G_3 l_2^2 - b_{33} l_3^2)^2 + 4(b_{13} + G_3)^2 l_2^2 l_3^2} \right\} : \left\{ 2(b_{13} + G_3) l_2 l_3 \right\}$$

VII. DISCUSSION OF RESULTS AND PRACTICAL APPLICATIONS

The results presented here give the exact theory for an axially symmetric medium. Due to the form of the solution, containing only l_2 and l_3, it will be a little more difficult to work with than if it contained l_1, l_2, and l_3. The solution is just as correct as if the most general case using all three direction cosines for the axially symmetric case were used. This form of the solution then will require more understanding on the part of anyone who uses it because the problem must be changed to fit the form used here.

Each topic is fairly complete in itself and is of some use that way. The first topic gives an insight into some of the properties of non-isotropic media for those interested.

The second topic is of interest to the elastician who is concerned only with stress and strain of bodies statically loaded. As stated before, some engineering materials are axially symmetric. The material presented here will be of help in obtaining complete solutions to the multitude of unsolved problems of axially symmetric elasticity.

The velocities of propagation in terms of the elastic constants and the elastic constants in terms of the velocities will be most useful to the seismologist. If he knows the properties of the earth, he can locate the origin of terrestial disturbances by the differences in velocities. The seismologist by measuring the velocities of propagation from a known source can measure the properties of inaccessible media. Likewise, the materials technician in the laboratory can investigate the properties of materials with this theory.

In addition, the work done here will be a help in extending the theory of reflection of waves from straight and curvilinear boundaries for an axially symmetric medium.

VIII. CONCLUSIONS

From the author's experience here, it seems that the consideration of a more general medium would become intolerably complicated. A solution was sought for the case here containing all three direction cosines, but was never obtained. Of course in the investigation, it was discovered that using the two direction cosines would suffice. The only apparent solution to the cubic equation involving all three direction cosines seemed to be by the cubic formula. This would seem to indicate that for any medium not axially symmetric, the only results to be obtained from the cubic equation would be by the formula. This thought practically excludes any further work in that direction.

As an aid in the investigation, the author made a numerical analysis, which presented no difficulty even when all three direction cosines were used. It is concluded here then that any work done on a more general medium should be done on a numerical basis or with some simplifying assumptions.

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