THE GLOBAL STABILITY OF A RIGID SOLID SUPPORTED BY ELASTIC COLUMNS

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Abstract—A new stability index, the global critical load, is advocated. This index is useful for flexible structures prone to large disturbances such as earthquakes. A symmetric rigid body supported by flexible legs is studied in detail. The nonlinear equilibrium equations are solved and the results show that global stability depends heavily on the height of the mass center and the distance between the legs.

1. INTRODUCTION

Consider a slender elastic column subjected to increasing compression and loads. The column would remain straight until a certain critical load is reached, where the column begins to bifurcate from the straight state. This bifurcation critical load has long been used as an index for elastic stability (see e.g., [Timoshenko and Gere (1961)]), i.e., if the force is below the bifurcation load the column would return to its undeformed state after removal of any added *small* perturbations. We emphasize the word *small* here since the bifurcation load is obtained through a linearization about the trivial state.

At this stage we define the global critical load F_{gc} as that load below which the column remains stable, or returns to its original state, after the removal of *any* applied disturbances, however large. It so happens that for the column buckling problem, the bifurcation load is also F_{gc}, since the bifurcation curve is monotonic, single-valued with a global minimum at the bifurcation point.

However, there are cases where F_{gc} is smaller than the bifurcation load. An example is the snap buckling of shallow arches under uniform pressure [Timoshenko and Gere (1961)]. The arch is stable only for small disturbance pressures. Application and subsequent removal of large disturbance pressures may cause snap through to another state. Recently Thacker *et al.* (1993) studied the large deformations of a rigid platform supported by two flexible elastic columns. It was found that the global buckling load in most cases is smaller than the bifurcation load. The results are quite significant for flexible structures that are subjected to large disturbances such as earthquakes.

The present note extends the results of Thacker *et al.* (1993) to the important case of a symmetric rigid body supported by elastic columns. Specifically, we wish to investigate the effect of the height of the rigid body on the global critical load.

2. FORMULATION and METHOD

Suppose a solid has two vertical orthogonal planes of symmetry and is supported by four symmetrically placed columns. It is obvious the structure is least stable in the plane of the closer spaced columns. Figure 1(a) shows the undeformed end view of this plane. Let the supporting columns be of length L, slender, perfectly elastic, with negligible weight. The ends of the columns are fixed to the base and to the rigid solid. The distance between the ends of the two columns is βL. The total weight of the solid is W acting at the center of mass which in the undeformed state is $\alpha L + L$ from the base. We normalize all lengths by L and all forces by EI/L^2. Figure 1(b) shows the deformed state while Fig. 1(c) shows the forces and moments acting on each column. Here $F_1 + F_2 = WL^2/2EI \equiv F$ and G is the horizontal reactive force.

The elastica equations for large deformations of slender columns are (Frisch-Fay, 1962)

$$\frac{d^2\theta_1}{ds^2} + F_1 \sin\theta_1 - G\cos\theta_1 = 0, \tag{1}$$

$$\frac{d^2\theta_2}{ds^2} + F_2 \sin\theta_2 + G\cos\theta_2 = 0, \tag{2}$$

where θ_i is the inclination of the ith column with the vertical and s is the arc length. The kinematic relations are

$$\frac{dx_i}{ds} = \cos\theta_i, \qquad \frac{dy_i}{ds} = \sin\theta_i, \qquad i = 1, 2. \tag{3}$$

The boundary conditions are

$$\theta_1(0) = \theta_2(0) = x_1(0) = x_2(0) = y_1(0) = 0, \tag{4}$$

$$y_2(0) = \beta, \tag{5}$$

$$\theta_1(1) = \theta_2(1), \tag{6}$$

$$\beta\cos\theta_1(1) = y_2(1) - y_1(1), \tag{7}$$

$$\beta\sin\theta_1(1) = x_1(1) - x_2(1), \tag{8}$$

$$\frac{d\theta_1}{ds}(0) + \frac{d\theta_2}{ds}(0) + F_2\beta = F\left[\frac{y_1(1) + y_2(1)}{2} + \alpha\sin\theta_1(1)\right]. \tag{9}$$

The last equation is from an overall moment balance on the whole structure about the origin. There are 10 equations and 10 unknowns. The set of equations reduces to those of the flat platform [Thacker *et al.* 1993] when $\alpha = 0$.

Although the elastica equations (1, 2) have closed form first integrals, the unknowns θ_i are implicit in the resulting elliptic integrals, making the boundary conditions impossible to apply. One can linearize about small θ_i and find that the linear bifurcation load is $2\pi^2 EI/L^2$ due to side sway [Chan and Liu, (1987)]. However, since the global critical load results from large deformation phenomena, the nonlinear equations must be integrated numerically. The techniques used are similar to those used in Thacker *et al.* (1993). Starting from a known solution with $\alpha = 0$, HOMPACK routines [Watson *et al.* (1987)] were used to find a family of nontrivial solutions for $\alpha = 0$ to $\alpha = 5$ with fixed F. One of these solutions was then used as a starting point to track the solution curve as F varies. The solution curves are not monotone in F, and therefore the more general homotopy algorithms in HOMPACK, rather than simple continuation, are necessary. A similar process was used to find the solution curves for different β values.

3. RESULTS AND DISCUSSION

We define displacement δ as the vertical (downward) translation of the center of mass after deformation,

$$\delta = \alpha + 1 - \left[\frac{y_1(1) + y_2(1)}{2} + \alpha \cos \theta_1(1) \right]. \tag{10}$$

Figure 2a shows the force-displacement curves for width $\beta = 1$ and different constant values of center of mass height α. When $\alpha = 0$ the results reduce to those of the thin platform studied by Thacker *et al.* (1993). For small deformations, all curves bifurcate from $2\pi^2$, which is the linear stability criterion for side sway independent of both α and β. As δ is increased, the corresponding load F first increases, then decreases, then increases again. Thus for a given load, there may be up to three equilibrium solutions. The increase in height α greatly magnifies this nonlinear behavior and thus drastically decreases F_{gc}, the maximum load below which the system would always spring back to the undeformed state. The effect of α is diminished for larger β as shown in Fig. 2b. For fixed α, the effect of β is to change the character of the curves. Figures 3a and 3b show that when β is large not only does the global critical load become the bifurcation load $2\pi^2$, but the nonuniqueness may also disappear.

The computed global critical load is shown in Fig. 4 for various β and α. The value of F_{gc} rises with β but is limited by the bifurcation load $2\pi^2$ which is independent of β. We see that increasing the height α drastically decreases F_{gc}. Figure 4 should be quite useful in the design of solid bodies supported by elastic columns.

The current example shows that, in certain cases, the linear stability index or bifurcation load is entirely erroneous in predicting the safe load under finite disturbances. The importance of using the nonlinear stability index F_{gc} for flexible structures cannot be overemphasized.

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FIGURE CAPTIONS

Fig. 1. a) Configuration before deformation.

Fig. 1. b) Configuration after deformation.

Fig. 1. c) Forces and moments on each column.

Fig. 2. a) Displacement δ as a function of F, $\beta = 1$.

Fig. 2. b) Displacement δ as a function of F, $\beta = 2$.

Fig. 3. a) Displacement δ as a function of F, $\alpha = 1$.

Fig. 3. b) Displacement δ as a function of F, $\alpha = 5$.

Fig. 4. The global critical load.