Inverse scattering in 1D nonhomogeneous media and recovery of the wave speed

Tuncay Aktosun, Martin Klaus, and Cornelis van der Mee

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Tuncay Aktosun
Department of Mathematics, Southern Methodist University, Dallas, Texas 75275

Martin Klaus
Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

Cornelis van der Mee
Department of Physics and Astronomy, Free University, Amsterdam, The Netherlands

The inverse scattering problem for the 1-D Schrödinger equation $d^2\psi/dx^2 + k^2\psi = k^2P(x)\psi + Q(x)\psi$ is studied. This equation is equivalent to the 1-D wave equation with speed $1/\sqrt{1 - P(x)}$ in a nonhomogeneous medium where $Q(x)$ acts as a restoring force. When $Q(x)$ is integrable with a finite first moment, $P(x) < 1$ and bounded below and satisfies two integrability conditions, $P(x)$ is recovered uniquely when the scattering data and $Q(x)$ are known. Some explicitly solved examples are provided.

I. INTRODUCTION

Consider the one-dimensional Schrödinger equation

$$\psi''(k,x) + k^2\psi(k,x) = k^2P(x)\psi(k,x) + Q(x)\psi(k,x), \qquad (1.1)$$

where $x \in \mathbb{R}$ is the space coordinate, $k^2 \in \mathbb{R}$ is energy, $Q(x)$ is a potential, and $k^2P(x)$ is the potential proportional to energy. Note that throughout the paper we use the prime to denote the derivative with respect to x. The physical solutions ψ_l from the left and ψ_r from the right of (1.1) satisfy the boundary conditions

$$\psi_l(k,x) = \begin{cases} T(k)e^{ikx} + o(1), & x \to \infty, \\ e^{ikx} + L(k)e^{-ikx} + o(1), & x \to -\infty, \end{cases}$$

and

$$\psi_r(k,x) = \begin{cases} e^{-ikx} + R(k)e^{ikx} + o(1), & x \to \infty, \\ T(k)e^{-ikx} + o(1), & x \to -\infty, \end{cases}$$

where T is the transmission coefficient, and L and R are the reflection coefficients from the left and from the right, respectively. The scattering matrix $S(k)$ is defined as

$$S(k) = \begin{bmatrix} T(k) & R(k) \\ L(k) & T(k) \end{bmatrix}.$$

The Faddeev solutions from the right and from the left are given by

$$m_l(k,x) = [1/T(k)]e^{-ikx}\psi_l(k,x)$$

$$m_r(k,x) = [1/T(k)]e^{ikx}\psi_r(k,x), \qquad (1.2)$$

and they satisfy the boundary conditions

$$m_l(k,x) = 1 + o(1),$$

$$m_l'(k,x) = o(1), \quad x \to \infty,$$

and

$$m_r(k,x) = 1 + o(1),$$

$$m_r'(k,x) = o(1), \quad x \to -\infty.$$

Letting $H(x) = \sqrt{1 - P(x)}$ and

$$Y_l(k,x) = \frac{e^{ik\int_0^x dz\, H(z)}}{\sqrt{H(x)}},$$

$$Y_r(k,x) = \frac{e^{-ik\int_0^x dz\, H(z)}}{\sqrt{H(x)}},$$

we can write[1] the physical solutions of (1.1) as

$$\psi_l(k,x) = T(k)e^{ik\int_0^\infty[1-H]}Y_l(k,x)Z_l(k,x),$$

$$\psi_r(k,x) = T(k)e^{ik\int_{-\infty}^0[1-H]}Y_r(k,x)Z_r(k,x),$$

and from (1.2) we obtain

$$m_l(k,x) = [1/\sqrt{H(x)}]e^{ik\int_x^\infty[1-H]}Z_l(k,x), \qquad (1.3)$$

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$$m_r(k,x) = [1/\sqrt{H(x)}]e^{ik\int_{-\infty}^{x}[1-H]}Z_r(k,x). \qquad (1.4)$$

The transformation $u(t,x) = e^{ikt}\psi(k,x)$ from the frequency k domain into the time t domain changes (1.1) into the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2} = Q(x)u, \qquad (1.5)$$

where $c(x) = 1/\sqrt{1-P(x)}$ is the wave speed and $Q(x)$ is the restoring force. The equation in (1.5) describes the propagation of waves (e.g., sound, electromagnetic, or elastic waves) in nondispersive media where the wave speed and the restoring force depend on position. The direct scattering problem for (1.1) consists of finding the scattering matrix $S(k)$ when $P(x)$ and $Q(x)$ are known, and it has been studied in Ref. 1. One inverse problem for (1.1) is to recover $Q(x)$ when $S(k)$, $P(x)$, and the bound state energies and the normalization constants are known, and this has been studied also in Ref. 1. Another inverse problem is to recover $P(x)$ when $Q(x)$, $S(k)$, and the bound state information are known, and this inverse problem will be studied elsewhere. In this paper we study a version of the second inverse problem, namely we recover $P(x)$ when we know $Q(x)$, the bound state information, one of the reflection coefficients, and the delay time caused by the nonhomogeneity when the signal travels from an arbitrary point to either of $\pm\infty$. Such a delay time can be specified by giving either A_+ or A_-, where

$$A_\pm = \pm \int_0^{\pm\infty} dz[1-H(z)]. \qquad (1.6)$$

The second inverse scattering problem for (1.1) is important because this problem is equivalent to the determination of the wave speed $c(x)$ when the scattering data and the restoring force are known, and this has many important applications in acoustic imaging, nondestructive evaluation, and various fields of geophysics such as seismology.

As in Ref. 1 let us define

$$G(x) = -\frac{H''(x)}{2H(x)^2} + \frac{3}{4}\frac{H'(x)^2}{H(x)^3} - \frac{Q(x)}{H(x)}. \qquad (1.7)$$

All the results given in this paper hold for real potentials satisfying the conditions $P,Q,G \in L_1^1(\mathbf{R})$, $P(x) < 1$ and is bounded below, where $L_1^1(\mathbf{R})$ is the class of Lebesgue integrable functions having a finite first moment.

When $P(x) \equiv 0$ in (1.1), we obtain the Schrödinger equation

$$d^2\psi^{[0]}(k,x)/dx^2 + k^2\psi^{[0]}(k,x) = Q(x)\psi^{[0]}(k,x). \qquad (1.8)$$

Let us use $m_l^{[0]}(k,x)$ and $m_r^{[0]}(k,x)$ to denote the Faddeev solutions of (1.8); i.e., let

$$m_l^{[0]}(k,x) = [1/T_0(k)]e^{-ikx}\psi_l^{[0]}(k,x)$$

and

$$m_r^{[0]}(k,x) = [1/T_0(k)]e^{ikx}\psi_r^{[0]}(k,x),$$

where the scattering matrix for (1.8) is denoted by

$$S_0(k) = \begin{bmatrix} T_0(k) & R_0(k) \\ L_0(k) & T_0(k) \end{bmatrix}.$$

The scattering and inverse scattering problems for (1.8) are well understood[2-5] for $Q \in L_1^1(\mathbf{R})$. When we solve the inverse scattering problem, we will exploit the fact that for $k=0$ the Faddeev solutions of (1.1) and (1.8) satisfy

$$m_l(0,x) = m_l^{[0]}(0,x), \qquad (1.9)$$

$$m_r(0,x) = m_r^{[0]}(0,x). \qquad (1.10)$$

The equality in (1.9) holds because each function there satisfies

$$\frac{d^2\eta}{dx^2} = Q(x)\eta \qquad (1.11)$$

with the boundary conditions $\eta(\infty) = 1$ and $\eta'(\infty) = 0$. Similarly, each function in (1.10) satisfies (1.11) with the boundary conditions $\eta(-\infty) = 1$ and $\eta'(-\infty) = 0$, and thus (1.10) holds.

This paper is organized as follows. In Sec. II we formulate the key Riemann–Hilbert problem arising in the inverse scattering problem, in Sec. III we give the solution of the inverse scattering problem, and in Sec. IV we provide some explicit examples to illustrate the inversion method.

II. RIEMANN–HILBERT PROBLEM

Since k appears as k^2 in (1.1), $\psi_l(-k,x)$ and $\psi_r(-k,x)$ are also solutions of (1.1) whenever $\psi_l(k,x)$ and $\psi_r(k,x)$ are the physical solutions. The solution vectors

$$\psi(\pm k,x) = \begin{bmatrix} \psi_l(\pm k,x) \\ \psi_r(\pm k,x) \end{bmatrix}$$

are related to each other[1] as

$$\psi(-k,x) = S(-k)^t q\psi(k,x), \quad k\in\mathbf{R}, \qquad (2.1)$$

where

$$q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the superscript t denotes the matrix transpose. Letting

$$Z(k,x) = \begin{bmatrix} Z_l(k,x) \\ Z_r(k,x) \end{bmatrix},$$

where $Z_l(k,x)$ and $Z_r(k,x)$ are the functions defined in (1.3) and (1.4), respectively, we can write (2.1) as

$$Z(-k,x) = \Lambda(k,x)qZ(k,x), \quad k \in \mathbf{R}, \tag{2.2}$$

where we have defined

$$\Lambda(k,x)$$

$$= \begin{bmatrix} T(k)e^{iAk} & -R(k)e^{2ikx + \int_x^\infty [1-H]} \\ -L(k)e^{-2ikx + \int_{-\infty}^x [1-H]} & T(k)e^{iAk} \end{bmatrix}$$

where

$$A = \int_{-\infty}^{\infty} dz [1 - H(z)]. \tag{2.3}$$

Note that $A_+ + A_- = A$, as seen from (1.6). When the scattering matrix $S(k)$ is given, A can be obtained from $T(k)$ because[1] $e^{iAk}T(k) = 1 + O(1/k)$ as $k \to \pm \infty$. In terms of $y = \int_0^x H$, the so-called travel-time coordinate,[6] we can write the matrix $\Lambda(k,x)$ as

$$\Lambda(k,x)$$

$$= \begin{bmatrix} T(k)e^{iAk} & -R(k)e^{2iky + 2ikA_+} \\ -L(k)e^{-2iky + 2ikA_-} & T(k)e^{iAk} \end{bmatrix}. \tag{2.4}$$

Let

$$\hat{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The following is known.[1] The vector $Z(k,x)$ is continuous in $k \in \overline{\mathbf{C}^+}$, has an analytic extension in k to $\mathbf{C}^+$ for each x, and $Z(k,x) - \hat{1} = O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Similarly, $Z(-k,x)$ is continuous in $k \in \overline{\mathbf{C}^-}$, has an analytic extension in k to $\mathbf{C}^-$ for each x, and $Z(-k,x) - \hat{1} = O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^-}$. Furthermore, $Z(k,x)$ is Hölder continuous.[7] Hence, when the matrix $\Lambda(k,x)$ is known, solving (2.2) for $Z(k,x)$ constitutes a Riemann–Hilbert problem. There are various methods to solve this Riemann–Hilbert problem and a solution by the Marchenko method is given in Ref. 1. When there are bound

states, the normalization constant for each bound state must be specified in addition to the matrix $\Lambda(k,x)$.

Defining $\tau(k) = T(k)e^{ikA}$, $\rho(k) = R(k)e^{2ikA_+}$, and $\ell(k) = L(k)e^{2ikA_-}$, we can write (2.4) in the form

$$\Lambda(k,x) = \begin{bmatrix} \tau(k) & -\rho(k)e^{2iky} \\ -\ell(k)e^{-2iky} & \tau(k) \end{bmatrix}, \quad k \in \mathbf{R}, \tag{2.5}$$

and thus from (2.2), we obtain

$$Z_l(k,x) = [1/\tau(k)][Z_r(-k,x) + \ell(k)Z_r(k,x)e^{-2iky}], \tag{2.6}$$

$$Z_r(k,x) = [1/\tau(k)][Z_l(-k,x) + \rho(k)Z_l(k,x)e^{2iky}], \tag{2.7}$$

which enables us to express $Z_l(k,x)$ in $Z_r(k,x)$ and conversely. Define

$$B(x,z) = \begin{bmatrix} B_l(x,z) \\ B_r(x,z) \end{bmatrix} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikz}[Z(k,x) - \hat{1}].$$

Since $Z(k,x)$ has an analytic extension[1] in k to $\mathbf{C}^+$ for each x and $Z(k,x) - \hat{1} = O(1/k)$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$, it follows that $B(x,z) = 0$ for $z < 0$ so that

$$Z(k,x) = \begin{bmatrix} Z_l(k,x) \\ Z_r(k,x) \end{bmatrix} = \hat{1} + \int_0^\infty dz\, e^{ikz}B(x,z). \tag{2.8}$$

Then, by taking the Fourier transform of (2.2) after writing it in the form

$$Z(-k,x) - \hat{1} = [\Lambda(k,x) - \mathbf{I}]qZ(k,x) + q[Z(k,x) - \hat{1}],$$

we obtain the identity

$$B(x,z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} [\Lambda(k,x) - \mathbf{I}]qZ(k,x)e^{ikz}, \tag{2.9}$$

where $\mathbf{I}$ is the unit matrix. If there are $\mathcal{N}$ bound states with energies $-\beta_1^2, \ldots, -\beta_{\mathcal{N}}^2$, the function $\tau(k)$ is meromorphic in $\mathbf{C}^+$ with simple poles at $k = i\beta_1, \ldots, i\beta_{\mathcal{N}}$ with $\beta_j > 0$ for $j = 1, \ldots, \mathcal{N}$; then for the unique solvability of (2.2), one needs to specify the normalization constants

$$\alpha_j = e^{2\beta_j y}Z_r(i\beta_j,x)/Z_l(i\beta_j,x), \quad j = 1, \ldots, \mathcal{N}. \tag{2.10}$$

Once these normalization constants are specified, (2.2) can be solved uniquely.[1]

In the special case, when $\rho(k)$ has a meromorphic extension to $\mathbf{C}^+$ with finitely many simple poles at k

$= i\kappa_1,...,i\kappa_{N_R}$ with residues $\rho_1,...,\rho_{N_R}$, respectively, by calculus of residues we get from (2.9) after using (2.10)

$$B_l(x,z) = i \sum_{j=1}^{\mathcal{N}} e^{-\beta_j(z+2y)} \tau_j \alpha_j Z_l(i\beta_j,x)$$

$$- i \sum_{j=1}^{N_R} e^{-\kappa_j(z+2y)} \rho_j Z_l(i\kappa_j,x), \quad y > 0,$$

so that

$$Z_l(k,x) = 1 - \sum_{j=1}^{\mathcal{N}} e^{-2\beta_j y} \tau_j \alpha_j \frac{Z_l(i\beta_j,x)}{k+i\beta_j}$$

$$+ \sum_{j=1}^{N_R} e^{-2\kappa_j y} \rho_j \frac{Z_l(i\kappa_j,x)}{k+i\kappa_j}, \quad y > 0, \quad (2.11)$$

where τ_j denote the residues of $\tau(k)$ at $k = i\beta_j$. Analogously, in the special case when $\ell(k)$ has a meromorphic extension to $\mathbf{C}^+$ with finitely many simple poles at $k = i\lambda_1,...,i\lambda_{N_L}$ with residues $\ell_1,...,\ell_{N_L}$, respectively, we get from (2.9) after using (2.10)

$$B_r(x,z) = i \sum_{j=1}^{\mathcal{N}} e^{-\beta_j(z-2y)} \frac{\tau_j}{\alpha_j} Z_r(i\beta_j,x)$$

$$- i \sum_{j=1}^{N_L} e^{-\lambda_j(z-2y)} \ell_j Z_r(i\lambda_j,x), \quad y < 0,$$

so that

$$Z_r(k,x) = 1 - \sum_{j=1}^{\mathcal{N}} e^{2\beta_j y} \frac{\tau_j}{\alpha_j} \frac{Z_r(i\beta_j,x)}{k+i\beta_j}$$

$$+ \sum_{j=1}^{N_L} e^{2\lambda_j y} \ell_j \frac{Z_r(i\lambda_j,x)}{k+i\lambda_j}, \quad y < 0. \quad (2.12)$$

The unknowns $Z_l(i\kappa_j,x)$ and $Z_r(i\lambda_j,x)$ are easily found by solving two systems of linear equations that are obtained from (2.12) at $k = i\beta_1,...,i\beta_{\mathcal{N}}, i\kappa_1,...,i\kappa_{N_R}$ and from (2.12) at $k = i\beta_1,...,i\beta_{\mathcal{N}}, i\lambda_1,...,i\lambda_{N_L}$, respectively. From (2.6) and (2.7) one then finds $Z_l(k,x)$ for $y < 0$ and $Z_r(k,x)$ for $y > 0$, respectively.

III. RECOVERY OF THE POTENTIAL

Note that if $P(x)$ vanishes on either half line, $A_\pm$ can be obtained from $S(k)$. This is because, as seen from (2.3) and (1.6), if one of $A_\pm$ vanishes, the other must be equal to A, and A is known when $S(k)$ is given. In particular, when the nonhomogeneity $P(x)$ does not extend to both of $\pm \infty$, the information about $A_\pm$ is contained in the scattering matrix.

Once the Riemann–Hilbert problem (2.2) is solved,[1] from (1.3) and (1.4), we have

$$Z_l(0,x) = \sqrt{H(x)} m_l(0,x), \quad (3.1)$$

$$Z_r(0,x) = \sqrt{H(x)} m_r(0,x). \quad (3.2)$$

Using (1.9) and (1.10), from (3.1) and (3.2) we then obtain

$$Z_l(0,x) = \sqrt{H(x)} m_l^{[0]}(0,x), \quad (3.3)$$

$$Z_r(0,x) = \sqrt{H(x)} m_r^{[0]}(0,x), \quad (3.4)$$

where $m_l^{[0]}(k,x)$ and $m_r^{[0]}(k,x)$ are the Faddeev solutions of (1.8). Note that as seen from (2.3), x enters $\Lambda(k,x)$ only in the form $y = \int_0^x H$, and as a result the solution $Z(k,x)$ of (2.2) contains x only through y. Thus, both $Z_l(0,x)$ and $Z_r(0,x)$ are functions of y only. Furthermore, given $Q(x)$, we can obtain $m_l^{[0]}(0,x)$ and $m_r^{[0]}(0,x)$ and these two are functions of x only. Therefore, using $dy/dx = H(x)$, we see that (3.3) and (3.4) are first-order separable ordinary differential equations, and thus we can write them in the separated form as

$$\frac{dy}{Z_l(0,x)^2} = \frac{dx}{m_l^{[0]}(0,x)^2}, \quad (3.5)$$

$$\frac{dy}{Z_r(0,x)^2} = \frac{dx}{m_r^{[0]}(0,x)^2}. \quad (3.6)$$

Using the initial condition $y=0$ when $x=0$, we can obtain y in terms of x by integrating either (3.5) or (3.6). Once y is obtained in terms of x, replacing it in $Z_l(0,x)$ or in $Z_r(0,x)$ by its equivalent in terms of x, using

$$H(x) = \frac{Z_l(0,x)^2}{m_l^{[0]}(0,x)^2} = \frac{Z_r(0,x)^2}{m_r^{[0]}(0,x)^2},$$

we obtain $H(x)$ in terms of x only. Then the potential $P(x)$ can be obtained by using $P(x) = 1 - H(x)^2$.

In the absence of bound states, in order to obtain the potential $P(x)$, it is sufficient to know either $R(k)e^{2ikA_+}$ or $L(k)e^{2ikA_-}$; this is because the former quantity gives us $Z_l(k,x)$ and the latter gives us $Z_r(k,x)$ by the Marchenko procedure;[1] in order to solve the inverse problem studied here, it is sufficient to know either $Z_l(0,x)$ or $Z_r(0,x)$.

In the special case $Q(x) \equiv 0$, we have

$$m_l^{[0]}(0,x) = m_r^{[0]}(0,x) = 1,$$

and hence, from (3.5) and (3.6) we obtain

$$x = \int_0^y \frac{d\xi}{Z_l(0,\xi)^2} = \int_0^y \frac{d\xi}{Z_r(0,\xi)^2} . \qquad (3.7)$$

The result in (3.7) was obtained before by using a different argument.[3] Note that when $Q(x) = 0$, (1.1) cannot have any bound state solutions.[1,3]

IV. EXAMPLES

In this section we present some examples to illustrate the method described in Secs. II and III.

As a first example, consider the scattering matrix

$$S(k) = \begin{bmatrix} \tau(k)e^{-ikA} & \rho(k)e^{-2ikA_+} \\ \ell(k)e^{-2ikA_-} & \tau(k)e^{-ikA} \end{bmatrix}, \qquad (4.1)$$

where we have

$$\tau(k) = (k+i)/(k+2i) \qquad (4.2)$$

$$\rho(k) = \sqrt{3}\, i/(k+2i) \qquad (4.3)$$

$$\ell(k) = \frac{\sqrt{3}\, i}{k+2i} \frac{k+i}{k-i} . \qquad (4.4)$$

The Riemann–Hilbert problem (2.2) can be solved explicitly using (2.11) and (2.6) for $y > 0$ and (2.12) and (2.7) for $y < 0$. We obtain

$$Z_l(k,x) = 1, \quad x \geqslant 0$$

$$Z_r(k,x) = \frac{1}{\tau(k)} + \frac{\rho(k)}{\tau(k)} e^{2iky}, \quad x \geqslant 0$$

$$Z_l(k,x) = \frac{1}{\tau(k)} Z_r(-k,x)$$

$$+ \frac{\ell(k)}{\tau(k)} Z_r(k,x)e^{-2iky}, \quad x \leqslant 0$$

$$Z_r(k,x) = 1 + \frac{i}{k+i} \frac{2}{\sqrt{3}\, e^{-2y} - 1}, \quad x \leqslant 0,$$

where $\tau(k)$, $\rho(k)$, and $\ell(k)$ are the quantities given in (4.2), (4.3), and (4.4), respectively. Hence, $Z_r(0,x) = 2 + \sqrt{3}$ for $x \geqslant 0$ and $Z_r(0,x) = (\sqrt{3} + e^{2y})/(\sqrt{3} - e^{2y})$ for $x \leqslant 0$.

Let us use the potential

$$Q(x) = \sqrt{3}\delta(x) - \theta(x) \frac{2\sqrt{3}\, e^x}{[1 + \sqrt{3}\, e^x]^2},$$

where $\delta(x)$ is the delta function and $\theta(x)$ is the Heaviside function. The scattering matrix corresponding to this $Q(x)$, with $P(x) = 0$ is given by

$$S_0(k) = \begin{bmatrix} \dfrac{k+i/2}{k+i} & -\dfrac{\sqrt{3}\, i/2}{k+i} \dfrac{k+i/2}{k-i/2} \\ -\dfrac{\sqrt{3}i/2}{k+i} & \dfrac{k+i/2}{k+i} \end{bmatrix} .$$

The Faddeev function with this potential is given by

$$m_r^{[0]}(k,x) = 1, \quad x < 0,$$

$$m_r^{[0]}(k,x) = \frac{k+i}{k+i/2} \left[1 + \frac{i}{k-i/2} \frac{1}{1+\sqrt{3}\, e^x} \right]$$

$$- \frac{\sqrt{3}\, i/2}{k-i/2} e^{2ikx} \left[1 - \frac{i}{k+i/2} \frac{1}{1+\sqrt{3}\, e^x} \right],$$

$$x \geqslant 0.$$

Thus, we have

$$m_r^{[0]}(0,x) = 1, \quad x \leqslant 0,$$

$$m_r^{[0]}(0,x) = (2+\sqrt{3}) \frac{\sqrt{3}\, e^x - 1}{\sqrt{3}\, e^x + 1}, \quad x \geqslant 0.$$

In this example, (3.6) becomes for $x \geqslant 0$

$$\frac{dy}{(2+\sqrt{3})^2} = \left(\frac{\sqrt{3}\, e^x + 1}{\sqrt{3}\, e^x - 1} \right)^2 \frac{dx}{(2+\sqrt{3})^2} . \qquad (4.5)$$

Integrating (4.5) with the initial condition $y = 0$ when $x = 0$, we obtain for $x \geqslant 0$

$$y = x + 2\frac{\sqrt{3}+1}{\sqrt{3}-1} - 2\frac{\sqrt{3}\, e^x + 1}{\sqrt{3}\, e^x - 1},$$

and

$$H(x) = \left(\frac{\sqrt{3}\, e^x + 1}{\sqrt{3}\, e^x - 1} \right)^2, \quad x \geqslant 0.$$

Note that $A_+ = -2(\sqrt{3} + 1)$.

On the other hand, for $x \leqslant 0$, (3.6) becomes

$$dx = \left(\frac{\sqrt{3} - e^{2y}}{\sqrt{3} + e^{2y}} \right)^2 dy. \qquad (4.6)$$

After using the initial condition $y = 0$ when $x = 0$, we can solve (4.6) for $x \leqslant 0$ and obtain

$$x = y + \sqrt{3} - 3 + \frac{2\sqrt{3}}{\sqrt{3} + e^{2y}} .$$

Note that $A_- = 1 - \sqrt{3}$ and thus $A = -1 - 3\sqrt{3}$. In the last equation y can be computed numerically in terms of
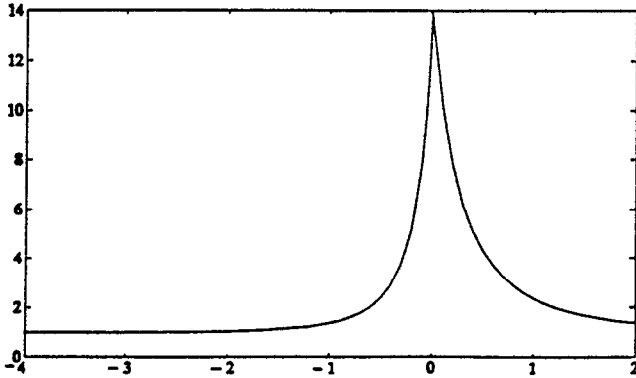
FIG. 1. Plot of $H(x)$ for the first example.

x to any desired accuracy. In Fig. 1 we have the plot of $H(x)$. In this example $H(x)$ is continuous everywhere and

$$H(0) = \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2.$$

Note that $P(x) = 1 - H(x)^2$.

As a second example, let us use

$$Q(x) = \frac{-8\sqrt{3}\, e^{2x}}{(\sqrt{3}+e^{2x})^2},$$

which corresponds to the reflectionless scattering matrix

$$S_0(k) = [(k+i)/(k-i)]\mathbf{I}.$$

This corresponds to the Faddeev solutions

$$m_l^{[0]}(k,x) = \frac{1}{k+i}\left[k+i\frac{e^{2x}-\sqrt{3}}{e^{2x}+\sqrt{3}}\right]$$

$$m_r^{[0]}(k,x) = \frac{1}{k+i}\left[k-i\frac{e^{2x}-\sqrt{3}}{e^{2x}+\sqrt{3}}\right]$$

with one bound state at $k=i$ where the normalization constant is chosen in such a way that $m_r^{[0]}(i,x)/m_l^{[0]}(i,x) = \sqrt{3}\, e^{-2x}$. Consider the scattering matrix $S(k)$ in (4.1), with

$$\tau(k) = (k+i\epsilon)/(k-i\epsilon) \quad \text{and} \quad \rho(k) = \ell(k) = 0$$

with one bound state at $k=i\epsilon$, where ϵ is a positive constant. Solving (2.2) using (2.11) and (2.12), we obtain

$$Z_l(k,x) = \frac{1}{k+i\epsilon}\left[k+i\epsilon\frac{e^{2\epsilon y}-\alpha}{e^{2\epsilon y}+\alpha}\right] \qquad (4.7)$$

$$Z_r(k,x) = \frac{1}{k+i\epsilon}\left[k-i\epsilon\frac{e^{2\epsilon y}-\alpha}{e^{2\epsilon y}+\alpha}\right], \qquad (4.8)$$

where α is the normalization constant such that $Z_r(i\epsilon,x)/Z_l(i\epsilon,x) = \alpha e^{-2\epsilon y}$. We see from (4.7) and (4.8) that α cannot be negative; otherwise, the corresponding bound state solutions of (1.1) would not be square integrable. Thus, we have

$$m_l^{[0]}(0,x) = \frac{e^{2x}-\sqrt{3}}{e^{2x}+\sqrt{3}} = \tanh\left(x-\frac{1}{4}\ln 3\right)$$

$$Z_l(0,x) = \frac{e^{2\epsilon y}-\alpha}{e^{2\epsilon y}+\alpha} = \tanh\left(\epsilon y-\frac{1}{2}\ln\alpha\right).$$

Hence, (3.5) becomes

$$dy\,\coth^2(\epsilon y-\tfrac{1}{2}\ln\alpha) = dx\,\coth^2(x-\tfrac{1}{4}\ln 3), \qquad (4.9)$$

and thus

$$\frac{dy}{dx} = H(x) = \left(\frac{e^{2\epsilon y}-\alpha}{e^{2\epsilon y}+\alpha}\right)^2\left(\frac{e^{2x}+\sqrt{3}}{e^{2x}-\sqrt{3}}\right)^2. \qquad (4.10)$$

The general solution of (4.9) is given by

$$y - \frac{1}{\epsilon}\coth\left(\epsilon y-\frac{1}{2}\ln\alpha\right)-\frac{1}{2\epsilon}\ln\alpha$$

$$= x - \coth\left(x-\frac{1}{4}\ln 3\right)-\frac{1}{4}\ln 3 + c. \qquad (4.11)$$

Note that $x_0 = \frac{1}{4}\ln 3$ is a singular point for the differential equation (4.9). In order to have $H(x) > 0$ and bounded above, we see that the numerator in (4.10) can vanish only when the denominator vanishes also. Thus, we must assume $\alpha > 1$. A further analysis of (4.9) shows that all solutions as well as their derivatives have finite limits as $x \to x_0$ from either side, and we find that

$$\lim_{x \to x_0} y(x) = (1/2\epsilon)\ln\alpha,$$

$$\lim_{x \to x_0} y'(x) = \lim_{x \to x_0} H(x) = 1/\epsilon^2, \qquad (4.12)$$

$$\lim_{x \to x_0} y''(x) = \lim_{x \to x_0} H'(x) = 4c/\epsilon^2.$$

Note that the continuity of $y(x)$ and $y'(x)$ at x_0 holds independently of the value of the constant c in (4.11). The condition $y(0) = 0$ fixes the value of c as
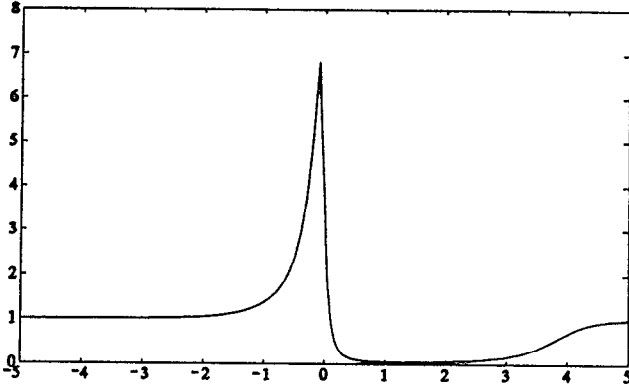
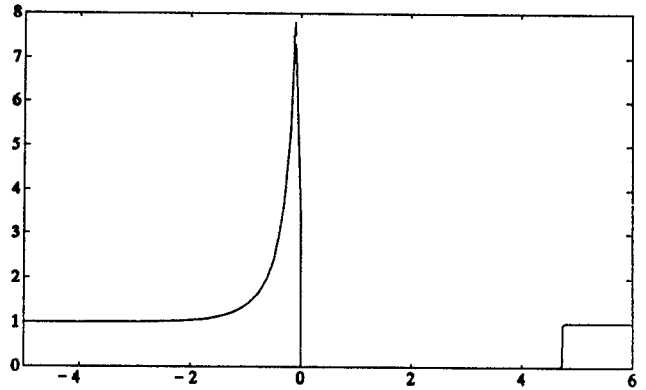FIG. 2. Plot of $H(x)$ of the second example with $\alpha = 3$, $\epsilon = 2$.

FIG. 3. Plot of $H(x)$ of the second example with $\alpha = 3$, $\epsilon = 500$.

$$c = \frac{1}{\epsilon}\coth\left(\frac{1}{2}\ln\alpha\right) - \frac{1}{2}\ln\alpha - \coth\left(\frac{1}{4}\ln 3\right) + \frac{1}{4}\ln 3$$

with regard to the interval $(-\infty, x_0)$. Furthermore, $y''(x)$ needs to be continuous at x_0 because $\psi'(k,x)$ in (1.1) contains y'', and hence in order for (1.1) to have a solution at x_0, we need to have y'' continuous at x_0. Therefore, the constant c must have the same value on (x_0, ∞) as on $(-\infty, x_0)$. The resulting equation for $y(x)$ then becomes

$$y - \frac{1}{\epsilon}\coth\left(\epsilon y - \frac{1}{2}\ln\alpha\right) - \frac{1}{\epsilon}\coth\left(\frac{1}{2}\ln\alpha\right)$$
$$= x - \coth\left(x - \frac{1}{4}\ln 3\right) - \coth\left(\frac{1}{4}\ln 3\right) \qquad (4.13)$$

for $x < x_0$ as well as for $x > x_0$. It also follows that $y(x)$ and consequently $H(x)$ are analytic functions of x near x_0. In Fig. 2 we have the plot of $H(x)$ for $\alpha = 3$ and $\epsilon = 2$.

From (4.12) and (4.13) it is seen that for fixed x, as $\epsilon \to \infty$, we have

$$y(x) = (\ln\alpha)/2\epsilon + O(\epsilon^{-2}), \quad 0 < x < x_1,$$

and

$$\lim_{\epsilon \to \infty} y(x) = x + \coth(\tfrac{1}{4}\ln 3 - x) - \coth(\tfrac{1}{4}\ln 3),$$

$$x \in (-\infty, 0) \cup (x_1, \infty),$$

where $x_1 = 4.7323...$ is the positive root of the equation

$$x + \coth(\tfrac{1}{4}\ln 3 - x) - \coth(\tfrac{1}{4}\ln 3) = 0,$$

which is obtained from (4.13) in the limit $\epsilon \to \infty$. Consequently,

$$\lim_{\epsilon \to \infty} H(x) = 0, \quad 0 < x < x_1,$$

and

$$\lim_{\epsilon \to \infty} H(x) = \coth^2(\tfrac{1}{4}\ln 3 - x),$$

$$x \in (-\infty, 0) \cup (x_1, \infty).$$

In Fig. 3 we have $H(x)$ for $\alpha = 3$ and $\epsilon = 500$, which illustrates the above facts for large ϵ.

At fixed x, we obtain

$$\lim_{\epsilon \to 0} y(x) = \tanh^2(\tfrac{1}{2}\ln\alpha)[x + \coth(\tfrac{1}{4}\ln 3 - x)$$

$$- \coth(\tfrac{1}{4}\ln 3)], \quad x < x_0,$$

and

$$y(x) \sim (\tfrac{1}{2}\ln\alpha + \gamma)/\epsilon, \quad x > x_0, \qquad (4.14)$$

where $\tfrac{1}{2}\ln\alpha + \gamma$ is equal to the positive root ϵy of the equation

$$\epsilon y - \coth(\epsilon y - \tfrac{1}{2}\ln\alpha) - \coth(\tfrac{1}{2}\ln\alpha) = 0,$$

which is obtained from (4.13) in the limit as $\epsilon \to 0$. We also have

$$\lim_{\epsilon \to 0} H(x) = \tanh^2(\tfrac{1}{2}\ln\alpha)\coth^2(\tfrac{1}{4}\ln 3 - x), \quad x < x_0,$$

$$\lim_{\epsilon \to 0} H(x) = \coth^2(\tfrac{1}{4}\ln 3 - x)/\coth^2\gamma, \quad x > x_0.$$

To illustrate the above results with small ϵ, in Fig. 4 we give $H(x)$ for $\alpha = 3$ and $\epsilon = 0.1$. In this case the value of γ in (4.14) is given by $\gamma = 2.46524...$.
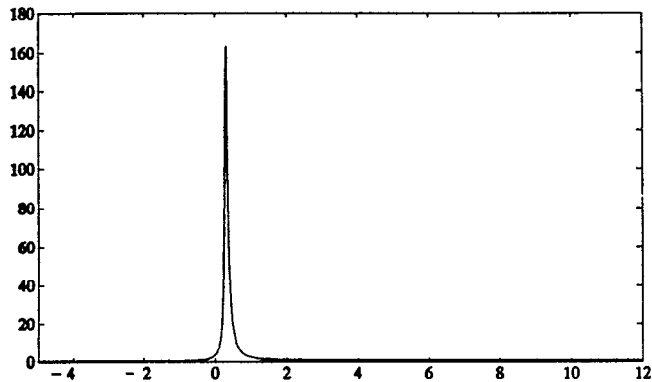
FIG. 4. Plot of $H(x)$ of the second example with $\alpha = 3$, $\epsilon = 0.1$.
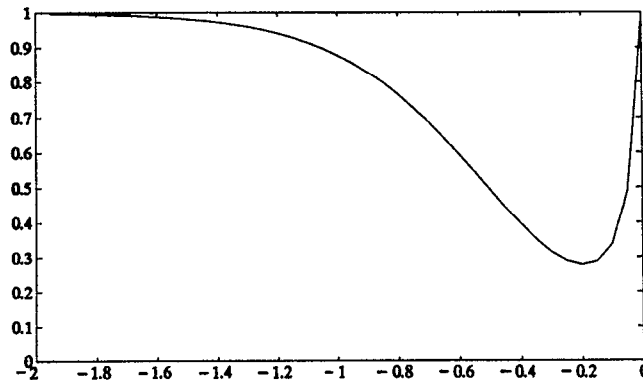

FIG. 5. Plot of $H(x)$ of the third example.

As a third example, let $Q(x) \equiv 0$ and consider the scattering matrix $S(k)$ given in (4.1) where

$$\tau(k) = \frac{(k+2i)(k+3i)}{(k+i)(k+6i)}, \qquad (4.15)$$

$$\rho(k) = \frac{\sqrt{24} ik}{(k+i)(k+6i)}, \qquad (4.16)$$

$$\ell(k) = \frac{\sqrt{24} ik}{(k+i)(k+6i)} \frac{k+2i}{k-2i} \frac{k+3i}{k-3i}. \qquad (4.17)$$

The solution of the Riemann–Hilbert problem (2.2) using (2.11) and (2.12) gives us

$$Z_l(k,x) = 1, \quad x \geqslant 0$$

$$Z_r(k,x) = \frac{1}{\tau(k)} + \frac{\rho(k)}{\tau(k)} e^{2iky}, \quad x \geqslant 0,$$

where $\tau(k)$ and $\rho(k)$ are the quantities given in (4.15) and (4.16), respectively. Thus, $Z_l(0,x) = Z_r(0,x) = 1$ for $x \geqslant 0$ and $H(x) = 1$ for $x \geqslant 0$. In terms of $y = \int_0^x dz\, H(z)$, when $x \leqslant 0$, we obtain

$$Z_r(k,x) = 1 - \frac{10i/3}{k+2i} \frac{\sqrt{6} e^{4y} + e^{10y}}{\Delta(y)}$$

$$+ \frac{5i}{k+3i} \frac{\sqrt{6} e^{6y} + e^{10y}}{\Delta(y)},$$

where

$$\Delta(y) = 1 + \frac{5}{\sqrt{6}} e^{4y} - \frac{5}{\sqrt{6}} e^{6y} - \frac{1}{6} e^{10y}.$$

Thus, for $x \leqslant 0$ we have

$$Z_r(0,x) = 1 + \frac{10}{\sqrt{6}} \frac{e^{6y} - e^{4y}}{\Delta(y)},$$

and hence from (3.7), we obtain for $y \leqslant 0$

$$x = \int_0^y dz \left[\frac{1 + (5/\sqrt{6}) e^{4z} - (5/\sqrt{6}) e^{6z} - \frac{1}{6} e^{10z}}{1 - (5/\sqrt{6}) e^{4z} + (5/\sqrt{6}) e^{6z} - \frac{1}{6} e^{10z}} \right]^2.$$

The last integral can be computed numerically to any desired accuracy. In this example $H(x) = Z_r(0,x)^2$ is continuous everywhere, and $H'(x)$ is continuous everywhere except at $x=0$; to be precise, we have $H'(0+) = 0$ and $H'(0-) = 8\sqrt{6}$. Again we can recover $P(x)$ using $P(x) = 1 - H(x)^2$. In Fig. 5 we have the graph of $H(x)$ for this example.

ACKNOWLEDGMENTS

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