ONE-TO-ONE CORRESPONDENCE BETWEEN MAXIMAL SETS OF ANTISYMMETRY AND MAXIMAL PROJECTIONS OF ANTISYMMETRY

by

Jiann-Shiun Huang

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

R.F. Olin, Chairman

R.A. McCoy

J.A.Arnold

J.F. Rossi

P.E. Haskell

June, 1991

Blacksburg, Virginia

ONE-TO-ONE CORRESPONDENCE BETWEEN MAXIMAL SETS OF ANTISYMMETRY AND MAXIMAL PROJECTIONS OF ANTISYMMETRY

by

Jiann-Shiun Huang

Committee Chairman:Robert F. Olin

Mathematics

(ABSTRACT)

Let $\mathbf{X}$ be a compact Hausdorff space and $\mathbf{A}$ a uniform algebra on $\mathbf{X}$. Let π be an isometric unital representation that maps $\mathbf{A}$ into bounded linear operators on a Hilbert space. This research investigated that there is a one-to-one correspondence between the collection of maximal sets of antisymmetry for $\mathbf{A}$ and that of maximal projections of antisymmetry for $\pi(\mathbf{A})$ under the extension of π if π satisfies a certain regularity property.

ACKNOWLEDGEMENT

I would like to express my sincere thanks to Professor R. F. Olin for his encouragement to study mathematics when I was a graduate student in another discipline at Virginia Tech. I also want to thank him for his introduction to function algebras and his persistent patience in advising on this thesis.

Finally I wish to thank Virginia Tech and National Science Foundation for their financial support during my graduate career.

TABLE OF CONTENTS

ABSTRACT ii

ACKNOWLEDGEMENT iii

INTRODUCTION 1

CHAPTER 1 5

GENERALIZED PEAK SETS

CHAPTER 2 15

EXTENSION OF CONTRACTIVE UNITAL REPRESENTATIONS

CHAPTER 3 28

ONE-TO-ONE CORRESPONDENCE BETWEEN
MAXIMAL SETS OF ANTISYMMETRY AND
MAXIMAL PROJECTIONS OF ANTISYMMETRY

REFERENCES 44

VITAE 46

INTRODUCTION

1. Notation and Definitions

Let $\mathbf{X}$ be a compact Hausdorff space and $\mathbf{C(X)}$ the algebra of all complex-valued continuous functions on $\mathbf{X}$. For $\mathbf{f} \in \mathbf{C(X)}$, we set $\|\mathbf{f}\| \equiv \max\{|\mathbf{f(x)}| : \mathbf{x} \in \mathbf{X}\}$ and call $\|\mathbf{f}\|$ the uniform norm of $\mathbf{f}$. A norm closed subset of $\mathbf{C(X)}$ will be referred to as uniformly closed.

We shall call $\mathbf{A}$ a uniform algebra on $\mathbf{X}$ if $\mathbf{A}$ is a uniformly closed subalgebra of $\mathbf{C(X)}$ which contains the constants and separates the points of $\mathbf{X}$.

Let $\mathbf{A}$ be a nonempty subset of $\mathbf{C(X)}$. A set $\mathbf{K} \subseteq \mathbf{X}$ is a set of antisymmetry for $\mathbf{A}$ if $\mathbf{K} \neq \emptyset$ and every $\mathbf{f} \in \mathbf{A}$ which is real-valued on $\mathbf{K}$ is constant on $\mathbf{K}$. Since the union of sets of antisymmetry for $\mathbf{A}$ that contain a common point is a set of antisymmetry for $\mathbf{A}$, every set of antisymmetry for $\mathbf{A}$ is contained in a maximal set of antisymmetry for $\mathbf{A}$. The collection $\mathcal{K}_{\mathbf{A}}$ of maximal sets of antisymmetry for $\mathbf{A}$ forms a pairwise disjoint, closed covering of $\mathbf{X}$.

Let $\mathcal{H}$ be a Hilbert space over the complex numbers $\mathbb{C}$ with the inner product $\langle\ ,\ \rangle$, and define

$$\|\mathbf{y}\| \equiv \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}$$

for any $\mathbf{y} \in \mathcal{H}$. Let $\mathbf{B}(\mathcal{H})$ stand for the algebra of all bounded linear operators on $\mathcal{H}$. For $\mathbf{T} \in \mathbf{B}(\mathcal{H})$, we set $\|\mathbf{T}\| \equiv \sup\{\|\mathbf{T}\,\mathbf{y}\| : \mathbf{y} \in \mathcal{H}, \text{ and } \|\mathbf{y}\| = 1\}$ and $\mathbf{T}^*$ denotes the adjoint of $\mathbf{T}$. If $\mathbf{T} = \mathbf{T}^*$ then $\mathbf{T}$ is called self-adjoint. If $\mathbf{P} \in \mathbf{B}(\mathcal{H})$ satisfies $\mathbf{P} = \mathbf{P}^2 = \mathbf{P}^*$, then $\mathbf{P}$ is called a projection. Denote the identity map on $\mathcal{H}$ by $\mathbf{I}$. Denote the zero map on $\mathcal{H}$ by $\mathbf{O}$.

Let $\mathcal{U}$ be a nonempty subset of $\mathbf{B}(\mathcal{H})$. A projection $\mathbf{P}$ in $\mathbf{B}(\mathcal{H})$ is called a projection of antisymmetry for $\mathcal{U}$ if

(a) $\mathbf{P} \neq \mathbf{O}$,

(b) $\mathbf{P} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{P}$ for all $\mathbf{T} \in \mathcal{U}$, and

(c) if $\mathbf{T} \cdot \mathbf{P}$ is self-adjoint for some $\mathbf{T} \in \mathcal{U}$ then $\mathbf{T} \cdot \mathbf{P} = \lambda \cdot \mathbf{P}$ for some real number λ.

In a natural way we define a maximal projection of antisymmetry for $\mathcal{U}$:

A projection $\mathbf{P}$ in $\mathbf{B}(\mathcal{H})$ is called a maximal projection of antisymmetry for $\mathcal{U}$ if

(a) $\mathbf{P}$ is a projection of antisymmetry for $\mathcal{U}$, and

(b) if $\mathbf{Q}$ is a projection of antisymmetry for $\mathcal{U}$ such that $\mathbf{Q} \geq \mathbf{P}$ then $\mathbf{P} = \mathbf{Q}$.

Applying Zorn's lemma one can show the range of every projection of antisymmetry for $\mathcal{U}$ is contained in that of a maximal projection of antisymmetry for $\mathcal{U}$.

Let $\mathbf{A}$ be a uniform algebra of $\mathbf{C}(\mathbf{X})$. A mapping $\pi : \mathbf{A} \to \mathbf{B}(\mathcal{H})$ is called a unital representation of $\mathbf{A}$ into $\mathbf{B}(\mathcal{H})$ if

(a) $\pi(1) = \mathbf{I}$,

(b) $\pi(\mathbf{f} + \mathbf{g}) = \pi(\mathbf{f}) + \pi(\mathbf{g})$,

(c) $\pi(\lambda \cdot \mathbf{f}) = \lambda \cdot \pi(\mathbf{f})$, and

(d) $\pi(\mathbf{f} \cdot \mathbf{g}) = \pi(\mathbf{f}) \cdot \pi(\mathbf{g})$

for all $\mathbf{f}, \mathbf{g} \in \mathbf{A}$, and $\lambda \in \mathbb{C}$

2. Historical Background

We begin with a history on sets of antisymmetry.

Let $\mathbf{A}$ be a closed algebra on a compact Hausdorff space $\mathbf{X}$. Denote the set of all regular complex measures on the Borel subsets of $\mathbf{X}$ by $\mathbf{M}(\mathbf{X})$. Let $\mathbf{b}(\mathbf{A}^{\perp}) \equiv \{\, \mu \in \mathbf{M}(\mathbf{X}) \mid \int_{\mathbf{X}} \mathbf{f}\, d\mu = 0$ for all $\mathbf{f} \in \mathbf{A}$ and the total variation of μ is at most 1 $\}$. Then

by Alaoglu's theorem and the fact that a closed subset of a compact set is compact, $\mathbf{b}(\mathbf{A}^{\perp})$ is compact in the weak topology induced by $\mathbf{C}(\mathbf{X})$ under integration.

Suppose $\mathbf{A} \neq \mathbf{C}(\mathbf{X})$, then $\mathbf{b}(\mathbf{A}^{\perp}) \neq \{0\}$. Since $\mathbf{b}(\mathbf{A}^{\perp})$ is also convex, by Krein-Milman theorem, $\mathbf{b}(\mathbf{A}^{\perp})$ has a nonzero extreme point, say μ_0. In [5], deBranges proved the Stone-Weierstrass theorem using the concept below:
If $\mathbf{f} \in \mathbf{A}$ and $\mathbf{f}$ is real-valued on the support of μ_0, then $\mathbf{f}$ is constant on the support of μ_0.

In [9], Glicksberg gave a formal definition of sets of antisymmetry and proved some important properties of maximal sets of antisymmetry.

Next we have some history on representations and projections of antisymmetry.

Let $\mathbf{T}$ be a bounded linear operator on $\mathcal{H}$ with $\mathbf{X}$ as a spectral set (in the sense von Neumann meant). Then there is a natural contractive unital representation, say π, from the uniform closure of the rational functions with poles off $\mathbf{X}$ in $\mathbf{C}(\mathbf{X})$, denoted by $\mathbf{R}(\mathbf{X})$, into $\mathbf{B}(\mathcal{H})$. Suppose that $\mathbb{C}\backslash\mathbf{X}$ is connected. Sarason [19] saw how this representation determines a relationship between reducing subspaces for $\mathbf{T}$ and the components of the interior of the spectrum of $\mathbf{T}$. Lautzenheiser [12] extended the result by showing that nontrivial Gleason parts for $\mathbf{R}(\mathbf{X})$ yielded reducing subspaces for $\mathbf{T}$. The work of ([15], Melnikov) established that these subspaces are nontrivial.

Mlak [16] further replaced π by any contractive unital representation, mapping a uniform algebra, say $\mathbf{A}$, of $\mathbf{C}(\mathbf{X})$ into $\mathbf{B}(\mathcal{H})$. Mlak developed a process that associates to each generalized peak set for $\mathbf{A}$ a projection that commutes with the range of π. The projection may be trivial.

Suppose $\mathbf{K}$ is a maximal set of antisymmetry for $\mathbf{A}$. In ([9], Glicksberg) it is shown that $\mathbf{K}$ is also a generalized peak set for $\mathbf{A}$. Hence it is natural to ask if the projection generated by the Mlak's process from $\mathbf{K}$, denoted by $\mathbf{P}_{\mathbf{K}}$, is a projection of antisymmetry for $\pi(\mathbf{A})$, or even a maximal projection of antisymmetry for $\pi(\mathbf{A})$.

Furthermore, if the answer to the latter part is affirmative, we want to investigate whether there is a one-to-one correspondence between maximal sets of antisymmetry and maximal projections of antisymmetry.

In ([22], Szymanski) a maximal projection of antisymmetry was introduced. Szymanski also proved the following result:

Let $\mathbf{N}$ be a normal operator on a Hilbert space $\mathcal{H}$ with spectrum $\mathbf{X}$. Let $\mathbf{P}(\mathbf{X})$ be the uniform closure of all polynomials in $\mathbf{C}(\mathbf{X})$. Denote the collection of maximal sets of antisymmetry for $\mathbf{P}(\mathbf{X})$ by $\mathcal{K}_{\mathbf{P}(\mathbf{X})}$. Define $\pi : \mathbf{P}(\mathbf{X}) \to \mathbf{B}(\mathcal{H})$ by $\pi(\mathbf{p}) = \mathbf{p}(\mathbf{N})$ for any $\mathbf{p}$ in $\mathbf{P}(\mathbf{X})$. Suppose for every $\mathbf{K} \in \mathcal{K}_{\mathbf{P}(\mathbf{X})}$ with $\mathbf{P}_{\mathbf{K}} \neq \mathbf{O}$, then $\mathbf{K}$ = the spectrum of $(\mathbf{N} \cdot \mathbf{P}_{\mathbf{K}})$. Then Szymanski showed that there is a one-to-one correspondence between the collection of maximal sets of antisymmetry $\mathbf{K}$ for $\mathbf{P}(\mathbf{X})$ that satisty $\mathbf{P}_{\mathbf{K}} \neq \mathbf{O}$ and the collection of all maximal projections of antisymmetry for $\pi(\mathbf{P}(\mathbf{X}))$.

3. Objectives

We will prove that a contractive unital representation $\pi : \mathbf{A} \to \mathbf{B}(\mathcal{H})$ can be extended so that the characteristic function of a maximal set of antisymmetry for $\mathbf{A}$ is in the domain of the extension. If the representation is an isometry and a certain regularity property is satisfied for each maximal set of antisymmetry for $\mathbf{A}$, then the extension will map the characteristic function of a maximal set of antisymmetry for $\mathbf{A}$ to a maximal projection of antisymmetry for $\pi(\mathbf{A})$. We show also that every maximal projection of antisymmetry for $\pi(\mathbf{A})$ is the image of the characteristic function of a unique maximal set of antisymmetry for $\mathbf{A}$ under the extension of π. That is, there is a one-to-one correspondence between the collection of maximal sets of antisymmetry for $\mathbf{A}$ and that of maximal projections of antisymmetry for $\pi(\mathbf{A})$.

CHAPTER 1

GENERALIZED PEAK SETS

In ([9], Glicksberg) it is shown that a maximal set of antisymmetry for a uniform algebra $\mathbf{A}$ is a generalized peak set for $\mathbf{A}$. In this chapter we will define generalized peak sets and prove some related properties of generalized peak sets we will use later.

We will continue our notation in Introduction. Let $\mathbf{X}$ denote a compact Hausdorff space, and $\mathbf{A}$ a uniform algebra on $\mathbf{X}$.

First we define peak sets.

Definition (Peak Sets). *A set* $\mathbf{F} \subseteq \mathbf{X}$ *is a peak set for* $\mathbf{A}$ *if* $\mathbf{F} \neq \emptyset$ *and there is a function* $\mathbf{f} \in \mathbf{A}$ *such that* $\mathbf{f} = 1$ *on* $\mathbf{F}$ *and* $|\mathbf{f}| < 1$ *on* $\mathbf{X} \backslash \mathbf{F}$ *(the complement of* $\mathbf{F}$ *in* $\mathbf{X}$*). Any such function* $\mathbf{f}$ *is said to peak on* $\mathbf{F}$.

Notes:

(a) Clearly $\mathbf{F} = \{\, \mathbf{x} \in \mathbf{X} \mid \mathbf{f}(\mathbf{x}) = 1 \,\}$ is a closed set.

(b) Since $\mathbf{F}$ can also be expressed as the intersection of the countable open sets $\{\, \mathbf{x} \in \mathbf{X} : |\mathbf{f}(\mathbf{x})| > 1 - \frac{1}{n} \,\}_{n=1}^{\infty}$, a peak set is a $\mathbf{G}_\delta$-set.

Next we give some examples about peak sets.

1.1 Examples.

(a) For any uniform algebra $\mathbf{A}$ on $\mathbf{X}$, the constant function $\mathbf{1}$ peaks on $\mathbf{X}$. Hence $\mathbf{X}$ is always a peak set for any uniform algebra on $\mathbf{X}$.

(b) Denote the unit circle in the complex plane, $\{\, \mathbf{c} \in \mathbb{C} : |\mathbf{c}| = 1 \,\}$, by Γ. Let $\mathbf{A}$ be the disk algebra, i.e., the uniform closure of all polynomials in $\mathbf{C}(\Gamma)$. For any

point $\mathbf{c}$ in Γ, the polynomial $\mathbf{f}(\mathbf{x}) = \bar{\mathbf{c}} \cdot (\mathbf{x} + \mathbf{c})/2$ is a function peaking on $\{\mathbf{c}\}$. Therefore, every point in the unit circle is a peak set for the disk algebra.

(c) Let $\mathbf{K}_1, \mathbf{K}_2$ be two disjoint compact subsets in the complex plane $\mathbb{C}$. Define $\mathbf{X} \equiv \mathbf{K}_1 \cup \mathbf{K}_2$. Denote the uniform closure of the rational functions with poles off $\mathbf{X}$ in $\mathbf{C}(\mathbf{X})$ by $\mathbf{R}(\mathbf{X})$. Then both $\mathbf{K}_1$ and $\mathbf{K}_2$ are peak sets for $\mathbf{R}(\mathbf{X})$. This follows from the fact that $\mathbb{C}$ is a normal space and application of Runge's theorem.

By the following two lemmas, a finite or countable intersection of peak sets for a uniform algebra $\mathbf{A}$ is still a peak set for $\mathbf{A}$.

1.2 Lemma. *Let $\mathbf{A}$ be a uniform algebra on $\mathbf{X}$. Then a finite intersection of peak sets for $\mathbf{A}$ is a peak set for $\mathbf{A}$.*

PROOF: Let $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \ldots, \mathbf{F}_n$ be peak sets for $\mathbf{A}$ with $\mathbf{f}_j$ peaks on $\mathbf{F}_j$. Set

$$\mathbf{f} \equiv \mathbf{f}_1 \cdot \mathbf{f}_2 \cdot \mathbf{f}_3 \cdots \mathbf{f}_n.$$

Then $\mathbf{f} \in \mathbf{A}$ and $\mathbf{f}$ peaks on $\bigcap\{\mathbf{F}_j \mid j = 1, 2, 3, \ldots, n\}$. Therefore, $\bigcap\{\mathbf{F}_j \mid j = 1, 2, 3, \ldots, n\}$ is a peak set for $\mathbf{A}$. ■■

1.3 Lemma. *Let $\mathbf{A}$ be a uniform algebra on $\mathbf{X}$. Then a countable intersection of peak sets for $\mathbf{A}$ is a peak set for $\mathbf{A}$.*

PROOF: Let $\{\mathbf{F}_j \mid j = 1, 2, 3, \ldots\}$ be a sequence of peak sets for $\mathbf{A}$ with $\mathbf{f}_j$ peaks on $\mathbf{F}_j$. Set

$$\mathbf{f} \equiv \sum_{j=1}^{\infty} \frac{\mathbf{f}_j}{2^j}.$$

Then $\mathbf{f} \in \mathbf{A}$ and $\mathbf{f}$ peaks on $\bigcap\{\mathbf{F}_j \mid j = 1, 2, 3, \ldots\}$. Therefore, $\bigcap\{\mathbf{F}_j \mid j = 1, 2, 3, \ldots\}$ is a peak set for $\mathbf{A}$. ■■

Definition (Generalized Peak Sets). *A generalized peak set is an intersection of peak sets.*

Notes:

(a) A peak set for a uniform algebra $\mathbf{A}$ is a generalized peak set for $\mathbf{A}$.

(b) An intersection of closed sets is a closed set, so a generalized peak set is a closed set.

(c) For a compact metrizable space $\mathbf{X}$, the collection of peak sets for a uniform algebra $\mathbf{A}$ on $\mathbf{X}$ coincides with that of generalized peak sets for $\mathbf{A}$.

PROOF: A compact metrizable space is second countable. For a second countable space, every open subspace is second countable, and hence Lindelöf. Considering the complement, then the intersection of a collection of closed sets can be expressed as the intersection of a finite or countable subcollection of closed sets. Therefore, a generalized peak set in a compact metrizable space can be written as the intersection of a finite or countable peak sets. Applying Lemma 1.2 or 1.3, the claim is proved. ■■

The following example shows that a generalized peak set for a uniform algebra $\mathbf{A}$ is not necessarily a peak set for $\mathbf{A}$.

1.4 Example. Let Ω be an uncountable well-ordered set with a largest element ω_1. The set Ω has the property that if $\alpha \in \Omega$ with $\alpha < \omega_1$, then $\{\beta \in \Omega \mid \beta \leq \alpha\}$ is countable.

Take as a subbase of the topology on Ω all sets of the form $\{\beta \in \Omega \mid \beta > \alpha\}$ and $\{\beta \in \Omega \mid \beta < \alpha\}$, for all $\alpha \in \Omega$. By 17.2 (c) of ([24], Willard), Ω is a compact Hausdorff space.

Define $\mathbf{A} \equiv \{\, \mathbf{f} \in \mathbf{C}(\Omega) \mid$ there exists a constant $\mathbf{c} \in \mathbb{C}$, and $\alpha < \omega_1$ such that $\mathbf{f}(\beta) = \mathbf{c}$ for all $\beta \geq \alpha \,\}$. Then one can check that $\mathbf{A}$ is a uniform algebra on Ω.

Let α be an element in Ω with $\alpha < \omega_1$. Since $\chi_{[\alpha,\omega_1]}$, the characteristic function of $[\alpha, \omega_1]$, belonging to $\mathbf{A}$ peaks on $[\alpha, \omega_1]$, the closed interval $[\alpha, \omega_1]$ is a peak set for $\mathbf{A}$. Hence $\{\omega_1\} = \bigcap\{\, [\alpha, \omega_1] \mid \alpha < \omega_1 \,\}$ is a generalized peak set for $\mathbf{A}$. Clearly, $\{\omega_1\}$ is not a peak set for $\mathbf{A}$ by the definition of $\mathbf{A}$.

To prove the main theorems of this chapter we need the following technical lemma.

1.5 Lemma ([7], Gamelin). *Let* $\mathbf{B}$ *be a closed subspace of* $\mathbf{C}(\mathbf{X})$. *Let* $\mathbf{G} \subseteq \mathbf{X}$ *be closed and* $\mathbf{f} \in \mathbf{C}(\mathbf{G})$ *satisfies*

$$\|\mathbf{f}\|_{\mathbf{G}} \equiv \max \{\, |\mathbf{f}(\mathbf{x})| : \mathbf{x} \in \mathbf{G} \,\} \leq 1.$$

Suppose there exist two constants $\mathbf{M}$ *and* $\mathbf{a}$ *with* $0 < \mathbf{a} < 1 < \mathbf{M}$ *such that for each open set* $\mathbf{V} \supseteq \mathbf{G}$, *there exists* $\mathbf{g} \in \mathbf{B}$ *such that (i)* $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbf{G}$, *(ii)* $\|\mathbf{g}\| \leq \mathbf{M}$, *and (iii)* $|\mathbf{g}| \leq \mathbf{a}$ *on* $\mathbf{X} \backslash \mathbf{V}$. *Then there exists* $\mathbf{h} \in \mathbf{B}$ *satisfying* $\|\mathbf{h}\| \leq 1$ *and* $\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbf{G}$. *Furthermore,* $\mathbf{h}$ *can be chosen so that* $|\mathbf{h}| < 1$ *on any prescribed* $\mathbf{F}_\sigma$*-subset of* $\mathbf{X} \backslash \mathbf{G}$.

Note: An $\mathbf{F}_\sigma$-set is a countable union of closed sets.

PROOF: Fix a number $\mathbf{s}$ such that

$$1 > \mathbf{s} > \frac{\mathbf{M} - 1}{\mathbf{M} - \mathbf{a}}.$$

Let $\{\, \delta_n \mid n = 1, 2, 3, \ldots \}$ be a decreasing sequence with δ_n converges to 0.

Let $\epsilon_1 = \delta_1$, and for $n \geq 2$, let

$$\epsilon_n = \min \{\, \delta_n, \frac{\mathbf{s}^k[\mathbf{s}(\mathbf{M} - \mathbf{a}) - (\mathbf{M} - 1)]}{1 - \mathbf{s}^k} \mid k = 2, 3, 4, \ldots, n \,\}.$$

One can verify $\{\epsilon_n : n = 1, 2, 3, \dots\}$ is a decreasing sequence with ϵ_n converges to 0.

Now suppose $\mathbf{E}$ is an $\mathbf{F}_\sigma$-subset of $\mathbf{X}\backslash\mathbf{G}$, say $\mathbf{E} = \bigcup\{\mathbf{E}_n \mid n = 1, 2, 3, \dots\}$, where each $\mathbf{E}_n$ is a closed set.

We construct a sequence $\{\mathbf{h}_n \mid n = 1, 2, 3, \dots\} \subseteq \mathbf{B}$ as follows.

Since $\mathbf{X}$ is open and $\mathbf{X} \supseteq \mathbf{G}$, by hypothesis, there exists $\mathbf{h}_1 \in \mathbf{B}$ such that $\mathbf{h}_1(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{G}$, and $\|\mathbf{h}_1\| \leq \mathbf{M}$.

Suppose $\mathbf{h}_1, \mathbf{h}_2, \dots, \mathbf{h}_n$ have been chosen such that for each $j = 1, 2, \dots, n$, $\mathbf{h}_j(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{G}$ and $\|\mathbf{h}_j\| \leq \mathbf{M}$. Let

$$\begin{aligned}\mathbf{W}_n &\equiv \{\mathbf{x} \in \mathbf{X} : \max_{1 \leq j \leq n} |\mathbf{h}_j(\mathbf{x})| \geq 1 + \epsilon_n\} \\ &= \{\mathbf{x} \in \mathbf{X} : |\mathbf{h}_j(\mathbf{x})| \geq 1 + \epsilon_n \text{ for some } j \text{ with } 1 \leq j \leq n\}.\end{aligned}$$

Clearly, $\mathbf{W}_n$ is a closed set, and $\mathbf{W}_n \subseteq \mathbf{W}_{n+1}$ for all $n = 1, 2, 3, \dots$. Since $\|\mathbf{f}\|_{\mathbf{G}} \leq 1$, and for all $\mathbf{x} \in \mathbf{G}$, $\mathbf{h}_j(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for each $j = 1, 2, 3 \dots$, we see that $\mathbf{W}_n \cap \mathbf{G} = \emptyset$ for all $n = 1, 2, 3, \dots$.

From $\mathbf{E}_n \subseteq \mathbf{E} \subseteq \mathbf{X}\backslash\mathbf{G}$ and $\mathbf{W}_n \cap \mathbf{G} = \emptyset$, the open set $\mathbf{X}\backslash(\mathbf{W}_n \cup \mathbf{E}_n) \supseteq \mathbf{G}$. By hypothesis, there exists $\mathbf{h}_{n+1} \in \mathbf{B}$ such that $\mathbf{h}_{n+1}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{G}$, $\|\mathbf{h}_{n+1}\| \leq \mathbf{M}$, and $|\mathbf{h}_{n+1}| \leq \mathbf{a}$ on $\mathbf{W}_n \cup \mathbf{E}_n$.

Set

$$\mathbf{h} \equiv (1 - \mathbf{s}) \sum_{j=1}^{\infty} \mathbf{s}^{j-1} \mathbf{h}_j.$$

Since the $\mathbf{h}_j$'s are uniformly bounded by $\mathbf{M}$, the series converges uniformly on $\mathbf{X}$. Hence $\mathbf{h} \in \mathbf{B}$.

For any $\mathbf{x} \in \mathbf{G}$,

$$\begin{aligned}
\mathbf{h}(\mathbf{x}) &= (1-\mathbf{s})\sum_{j=1}^{\infty} \mathbf{s}^{j-1}\mathbf{h}_j(\mathbf{x}) \\
&= (1-\mathbf{s})\sum_{j=1}^{\infty} \mathbf{s}^{j-1}\mathbf{f}(\mathbf{x}) \\
&= (1-\mathbf{s})\frac{\mathbf{f}(\mathbf{x})}{1-\mathbf{s}} \\
&= \mathbf{f}(\mathbf{x}).
\end{aligned}$$

To complete the proof we need to show $|\mathbf{h}| < 1$ on $\mathbf{E}$ and $|\mathbf{h}| \leq 1$ on $\mathbf{X}\backslash\mathbf{G}$.

It suffices to prove:

(1) If $\mathbf{x} \in \mathbf{X}\backslash(\bigcup\{\mathbf{W}_n \mid n = 1,2,3,\ldots\})$, then $|\mathbf{h}(\mathbf{x})| \leq 1$;

(2) If $\mathbf{x} \in \mathbf{E}\bigcap[\mathbf{X}\backslash(\bigcup\{\mathbf{W}_n \mid n = 1,2,3,\ldots\})]$, then $|\mathbf{h}(\mathbf{x})| < 1$; and

(3) If $\mathbf{x} \in \bigcup\{\mathbf{W}_n \mid n = 1,2,3,\ldots\}$, then $|\mathbf{h}(\mathbf{x})| < 1$.

To prove (1), let $\mathbf{x} \in \mathbf{X}\backslash(\bigcup\{\mathbf{W}_n \mid n = 1,2,3,\ldots\})$. Then for any $j = 1,2,3,\ldots$, the value $|\mathbf{h}_j(\mathbf{x})| < 1 + \epsilon_n$ for all $n \geq j$. The condition ϵ_n converges to 0 implies $|\mathbf{h}_j(\mathbf{x})| \leq 1$, for $j = 1,2,3,\ldots$. Therefore,

$$|\mathbf{h}(\mathbf{x})| = |(1-\mathbf{s})\sum_{j=1}^{\infty} \mathbf{s}^{j-1}\mathbf{h}_j(\mathbf{x})| \leq (1-\mathbf{s})\sum_{j=1}^{\infty} \mathbf{s}^{j-1} = 1.$$

To prove (2), let $\mathbf{x} \in \mathbf{E}\bigcap[\mathbf{X}\backslash(\bigcup\{\mathbf{W}_n \mid n = 1,2,3,\ldots\})]$. Clearly, $\mathbf{x} \in \mathbf{X}\backslash(\bigcup\{\mathbf{W}_n \mid n = 1,2,3,\ldots\})$. From the result of (1), $|\mathbf{h}_j(\mathbf{x})| \leq 1$ for $j = 1,2,3,\ldots$. Suppose $\mathbf{x} \in \mathbf{E}_k$ for some positive integer k. Then $|\mathbf{h}_{k+1}(\mathbf{x})| \leq \mathbf{a} < 1$. Therefore, $|\mathbf{h}(\mathbf{x})| < 1$ by the definition of $\mathbf{h}$.

To prove (3), let $\mathbf{x} \in \bigcup\{\mathbf{W}_n \mid n = 1,2,3,\ldots\}$. Since $\mathbf{W}_1 \subseteq \mathbf{W}_2 \subseteq \cdots \subseteq \mathbf{W}_n \subseteq \mathbf{W}_{n+1} \subseteq \cdots$, either ($\mathbf{x} \in \mathbf{W}_1$) or (there exists an $m \geq 2$ such that $\mathbf{x} \in \mathbf{W}_m$ and $\mathbf{x} \notin \mathbf{W}_j$ for all $j < m$).

If $\mathbf{x} \in \mathbf{W}_1$, then $|\mathbf{h}_j(\mathbf{x})| \leq \mathbf{a}$ for all $j > 1$, and

$$\begin{aligned}
|\mathbf{h}(\mathbf{x})| &\leq (1-\mathbf{s})\sum_{j=1}^{\infty} \mathbf{s}^{j-1}|\mathbf{h}_j(\mathbf{x})| \\
&= (1-\mathbf{s})[|\mathbf{h}_1(\mathbf{x})| + \sum_{j=2}^{\infty} \mathbf{s}^{j-1}|\mathbf{h}_j(\mathbf{x})|] \\
&\leq (1-\mathbf{s})[\mathbf{M} + \sum_{j=2}^{\infty} \mathbf{s}^{j-1}\mathbf{a}] \\
&= (1-\mathbf{s})[\mathbf{M} + \frac{\mathbf{as}}{1-\mathbf{s}}] \\
&= (1-\mathbf{s})\mathbf{M} + \mathbf{as} \\
&= \mathbf{M} - \mathbf{s}(\mathbf{M}-\mathbf{a}) \\
&< \mathbf{M} - \frac{\mathbf{M}-1}{\mathbf{M}-\mathbf{a}}(\mathbf{M}-\mathbf{a}) \\
&= 1.
\end{aligned}$$

That is, $|\mathbf{h}(\mathbf{x})| < 1$ if $\mathbf{x} \in \mathbf{W}_1$.

Suppose $\mathbf{x} \in \mathbf{W}_m$ for some $m \geq 2$, and $\mathbf{x} \notin \mathbf{W}_j$, for all $j < m$. In particular, $\mathbf{x} \notin \mathbf{W}_{m-1}$ and hence $|\mathbf{h}_j(\mathbf{x})| < 1 + \epsilon_{m-1}$ for all $j = 1, 2, \ldots, m-1$. Therefore,

$$\begin{aligned}
&|\mathbf{h}(\mathbf{x})| \leq (1-\mathbf{s})\sum_{j=1}^{\infty} \mathbf{s}^{j-1}|\mathbf{h}_j(\mathbf{x})| \\
&= (1-\mathbf{s})[\sum_{j=1}^{m-1} \mathbf{s}^{j-1}|\mathbf{h}_j(\mathbf{x})| + \mathbf{s}^{m-1}|\mathbf{h}_m(\mathbf{x})| + \sum_{j=m+1}^{\infty} \mathbf{s}^{j-1}|\mathbf{h}_j(\mathbf{x})|] \\
&< (1-\mathbf{s})[\sum_{j=1}^{m-1} \mathbf{s}^{j-1}(1+\epsilon_{m-1}) + \mathbf{s}^{m-1}\mathbf{M} + \sum_{j=m+1}^{\infty} \mathbf{s}^{j-1}\mathbf{a}] \\
&= (1-\mathbf{s})[\frac{1-\mathbf{s}^{m-1}}{1-\mathbf{s}}(1+\epsilon_{m-1}) + \mathbf{s}^{m-1}\mathbf{M} + \frac{\mathbf{as}^m}{1-\mathbf{s}}] \\
&= (1-\mathbf{s}^{m-1})(1+\epsilon_{m-1}) + (1-\mathbf{s})\mathbf{s}^{m-1}\mathbf{M} + \mathbf{as}^m \\
&= 1 + \epsilon_{m-1}(1-\mathbf{s}^{m-1}) - \mathbf{s}^{m-1}[1-(1-\mathbf{s})\mathbf{M} - \mathbf{as}] \\
&\leq 1 + \frac{\mathbf{s}^{m-1}[\mathbf{s}(\mathbf{M}-\mathbf{a}) - (\mathbf{M}-1)]}{1-\mathbf{s}^{m-1}}(1-\mathbf{s}^{m-1}) - \mathbf{s}^{m-1}[\mathbf{s}(\mathbf{M}-\mathbf{a}) - (\mathbf{M}-1)] \\
&= 1.
\end{aligned}$$

Therefore, $|\mathbf{h}(\mathbf{x})| < 1$ for all $\mathbf{x} \in \bigcup\{\, \mathbf{W}_n \mid n = 1, 2, 3, \ldots \}$.

This completes the proof of the lemma. ■■

Applying the previous lemmas, we can prove the following theorem.

1.6 Theorem ([7], Gamelin). *Let* $\mathbf{A}$ *be a uniform algebra on* $\mathbf{X}$, *and* $\mathbf{G}$ *a generalized peak set for* $\mathbf{A}$. *Let* $\mathbf{f} \in \mathbf{C}(\mathbf{G})$ *with* $\mathbf{f}(\mathbf{x}) = \mathbf{q}(\mathbf{x})$ *for some* $\mathbf{q} \in \mathbf{A}$ *and all* $\mathbf{x} \in \mathbf{G}$. *Then there exists* $\mathbf{h} \in \mathbf{A}$ *such that* $\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbf{G}$ *and* $\|\mathbf{h}\|_{\mathbf{X}} = \|\mathbf{f}\|_{\mathbf{G}}$.

PROOF: Without loss of generality, we assume $\|\mathbf{f}\|_{\mathbf{G}} = 1$.

Let $\mathbf{V}$ be an open set that contains $\mathbf{G}$. We claim there exists a peak set for $\mathbf{A}$, say $\mathbf{F}$, such that $\mathbf{G} \subseteq \mathbf{F} \subseteq \mathbf{W} \equiv \mathbf{V} \cap \{\, \mathbf{x} \in \mathbf{X} : |\mathbf{q}(\mathbf{x})| < 2 \,\}$.

PROOF OF CLAIM: As an intersection of two open sets, $\mathbf{W}$ is open. Since

$$\begin{aligned}
\mathbf{G} &= \{\, \mathbf{x} \in \mathbf{G} : |\mathbf{f}(\mathbf{x})| \leq 1 \,\} \\
&= \{\, \mathbf{x} \in \mathbf{G} : |\mathbf{q}(\mathbf{x})| \leq 1 \,\} \\
&\subseteq \{\, \mathbf{x} \in \mathbf{X} : |\mathbf{q}(\mathbf{x})| \leq 1 \,\} \\
&\subseteq \{\, \mathbf{x} \in \mathbf{X} : |\mathbf{q}(\mathbf{x})| < 2 \,\},
\end{aligned}$$

we have $\mathbf{W} \supseteq \mathbf{G}$.

The set $\mathbf{G}$ is a generalized peak set for $\mathbf{A}$, by definition,

$$\mathbf{G} = \bigcap\{\, \mathbf{F}_\alpha \mid \alpha \in \Lambda \,\},$$

where Λ is an index set, and $\mathbf{F}_\alpha$ is a peak set for $\mathbf{A}$ for each $\alpha \in \Lambda$.

By taking the complement of $\mathbf{W} \supseteq \mathbf{G}$, we have

$$\begin{aligned}\mathbf{X}\backslash\mathbf{W} &\subseteq \mathbf{X}\backslash\mathbf{G} \\ &= \mathbf{X}\backslash(\bigcap\{\,\mathbf{F}_\alpha \mid \alpha \in \Lambda\,\}) \\ &= \bigcup\{\,\mathbf{X}\backslash\mathbf{F}_\alpha \mid \alpha \in \Lambda\,\}.\end{aligned}$$

Since $\mathbf{X}\backslash\mathbf{W}$ is compact, there exist $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \Lambda$ such that

$$\begin{aligned}\mathbf{X}\backslash\mathbf{W} &\subseteq \bigcup\{\,\mathbf{X}\backslash\mathbf{F}_\alpha \mid \alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\,\} \\ &= \mathbf{X}\backslash(\bigcap\{\,\mathbf{F}_\alpha \mid \alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\,\}).\end{aligned}$$

So $\mathbf{W} \supseteq \bigcap\{\,\mathbf{F}_\alpha \mid \alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\,\}$. Let $\mathbf{F} \equiv \bigcap\{\,\mathbf{F}_\alpha \mid \alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\,\}$. By Lemma 1.2, $\mathbf{F}$ is a peak set for $\mathbf{A}$ and $\mathbf{G} \subseteq \mathbf{F} \subseteq \mathbf{W}$. The claim is proved. ■

Let $\mathbf{p} \in \mathbf{A}$ peak on $\mathbf{F}$. Then as n approaches ∞, we see that $\mathbf{p}^n$ converges to 0 uniformly on

$$\mathbf{X}\backslash(\mathbf{V} \cap \{\,\mathbf{x} \in \mathbf{X} : |\mathbf{q}(\mathbf{x})| < 2\,\}) = (\mathbf{X}\backslash\mathbf{V}) \cup \{\,\mathbf{x} \in \mathbf{X} : |\mathbf{q}(\mathbf{x})| \geq 2\,\}.$$

For sufficiently large n, the function $\mathbf{g} \equiv \mathbf{p}^n \cdot \mathbf{q} \in \mathbf{A}$ satisfies (i)$\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{G}$, (ii)$\|\mathbf{g}\|_{\mathbf{X}} \leq 2$, and (iii)$|\mathbf{g}| \leq \frac{1}{2}$ on $\mathbf{X}\backslash\mathbf{V}$. By Lemma 1.5, with $\mathbf{M} = 2$ and $\mathbf{a} = \frac{1}{2}$, the desired function $\mathbf{h} \in \mathbf{A}$ exists, and this wraps up the proof of the theorem. ■■

The following theorem shows the restriction of a uniform algebra to a generalized peak set is still a uniform algebra on that set.

1.7 Theorem. *Let* $\mathbf{A}$ *be a uniform algebra on* $\mathbf{X}$ *and* $\mathbf{G}$ *a generalized peak set for* $\mathbf{A}$. *Define* $\mathbf{i_G} : \mathbf{G} \to \mathbf{X}$ *by* $\mathbf{i_G}(\mathbf{x}) = \mathbf{x}$ *for all* $\mathbf{x} \in \mathbf{G}$. *Then* $\mathbf{A_G} \equiv \{\,\mathbf{f} \circ \mathbf{i_G} \mid \mathbf{f} \in \mathbf{A}\,\}$, *the restriction of* $\mathbf{A}$ *to* $\mathbf{G}$, *is a uniform algebra on* $\mathbf{G}$.

PROOF: The restriction of an algebra to a subset of $\mathbf{X}$ is an algebra on that set. The restriction of a constant function to a subset of $\mathbf{X}$ is a constant function on that set.

An algebra that separates points of a set separates points of a subset. Therefore, its restriction to that subset separates points. Hence we now proceed to prove that $\mathbf{A_G}$ is uniformly closed to complete the claim.

To this end, define $\psi : \mathbf{A} \to \mathbf{A_G}$ by $\psi(\mathbf{f}) = \mathbf{f} \circ \mathbf{i_G}$ for all $\mathbf{f} \in \mathbf{A}$. Then ψ is linear, onto with kernel space, denoted by $\mathbf{ker}\ \psi$, the set of functions in $\mathbf{A}$ that vanish on $\mathbf{G}$.

Define $\mathbf{A}/\mathbf{ker}\ \psi \equiv \{\, \mathbf{f}+\mathbf{ker}\ \psi \mid \mathbf{f} \in \mathbf{A} \,\}$ with the norm $\|\mathbf{f}+\mathbf{ker}\ \psi\| \equiv \inf \{\, \|\mathbf{f}+\mathbf{g}\| : \mathbf{g} \in \mathbf{ker}\ \psi \,\}$ for any $\mathbf{f} \in \mathbf{A}$. Since $\mathbf{A}$ is a complete normed space, by III.4.2 of ([2], Conway), $\mathbf{A}/\mathbf{ker}\ \psi$ is a complete normed space.

The mapping $\Psi : \mathbf{A}/\mathbf{ker}\ \psi \to \mathbf{A_G}$ defined by $\Psi(\mathbf{f} + \mathbf{ker}\ \psi) \equiv \mathbf{f} \circ \mathbf{i_G}$ is one-to-one, onto, and linear. Furthermore, by Theorem 1.6, Ψ is an isometry. Therefore, $\mathbf{A_G}$ is a complete normed space, and hence uniformly closed. ■■

CHAPTER 2

EXTENSION OF CONTRACTIVE UNITAL REPRESENTATIONS

Let $\mathbf{A}$ be a uniform algebra on $\mathbf{X}$. A unital representation $\pi : \mathbf{A} \to \mathbf{B}(\mathcal{H})$ is said to be contractive if $\|\pi(\mathbf{f})\| \leq \|\mathbf{f}\|$ for all $\mathbf{f} \in \mathbf{A}$. Denote the collection of generalized peak sets for $\mathbf{A}$ by $(\mathcal{GP})_{\mathbf{A}}$. Then in this chapter we want to extend the domain of π to include $\chi_{\mathbf{G}}$, the characteristic function of $\mathbf{G}$, for each $\mathbf{G} \in (\mathcal{GP})_{\mathbf{A}}$. We will see that the range of $\chi_{\mathbf{G}}$ under the extension of π is a projection that commutes with the range of π. Therefore, if the projection is not $\mathbf{O}$ or $\mathbf{I}$, then the range of the projection is a non-trivial reducing subspace for the range of π. The main result of this chapter, Theorem 2.7, is due to ([16], Mlak).

We first need some definitions for our following examples.

Definition (Spectral Sets). *If $\mathbf{T}$ is a bounded linear operator on a Hilbert space $\mathcal{H}$, then a compact subset $\mathbf{X}$ of the complex numbers $\mathbb{C}$ is called a spectral set for $\mathbf{T}$ if it contains the spectrum of $\mathbf{T}$ and satisfies*

$$\|\mathbf{r}(\mathbf{T})\| \leq \sup \{ |\mathbf{r}(\mathbf{x})| : \mathbf{x} \in \mathbf{X} \}$$

for all rational functions $\mathbf{r}$ with poles off $\mathbf{X}$.

(For definition of $\mathbf{r}(\mathbf{T})$, see VII.4 of ([2], Conway).)

Definition (von Neumann Operators). *A bounded linear operator $\mathbf{T}$ on a Hilbert space $\mathcal{H}$ is a von Neumann operator if the spectrum of $\mathbf{T}$ is a spectral set for $\mathbf{T}$.*

Here we present some examples of contractive unital representations.

2.1 Examples

(a) Let $\mathbf{X}$ be a compact Hausdorff space and let μ be a positive Borel measure on $\mathbf{X}$. Denote the Hilbert space of square integrable functions on $(\mathbf{X}, \mu)$ by $\mathbf{L}^2(\mu)$. Define $\pi : \mathbf{C}(\mathbf{X}) \to \mathbf{B}(\mathbf{L}^2(\mu))$ by $\pi(\mathbf{f}) = \mathbf{M_f}$, where $\mathbf{M_f}(\mathbf{y}) = \mathbf{f} \cdot \mathbf{y}$ for any $\mathbf{y} \in \mathbf{L}^2(\mu)$. One can check that π is a contractive unital representation.

(b) Let $\mathbf{N}$ be a normal operator on a Hilbert space $\mathcal{H}$ and let $\mathbf{X}$ be the spectrum of $\mathbf{N}$. By the spectral theorem for normal operators, for every $\mathbf{f}$ in $\mathbf{C}(\mathbf{X})$ there exists an operator $\mathbf{f}(\mathbf{N})$ on $\mathcal{H}$. Define $\pi : \mathbf{C}(\mathbf{X}) \to \mathbf{B}(\mathcal{H})$ by $\pi(\mathbf{f}) = \mathbf{f}(\mathbf{N})$. Then one can use the spectral theorem to check that π is a contractive unital representation. (Actually, π is an isometry.)

(c) Let $\mathbf{T}$ be a von Neumann operator on a Hilbert space $\mathcal{H}$. (For a brief discussion of von Neumann operators see ([4], Conway).) Let $\mathbf{X}$ be the spectrum of $\mathbf{T}$. Denote the uniform closure of all rational functions with poles off $\mathbf{X}$ in $\mathbf{C}(\mathbf{X})$ by $\mathbf{R}(\mathbf{X})$. Define $\pi : \mathbf{R}(\mathbf{X}) \to \mathbf{B}(\mathcal{H})$ by $\pi(\mathbf{f}) = \mathbf{f}(\mathbf{T})$. By invoking the definition of von Neumann operators one can verify that π is a contractive unital representation.

(d) Let $\mathbf{X}$ in $\mathbb{C}$ be a spectral set for an operator $\mathbf{T}$ on a Hilbert space $\mathcal{H}$. (See ([4], Conway) for reference on spectral sets.) Let $\mathbf{R}(\mathbf{X})$ and π be defined as in (c) above. Then π is a contractive unital representation.

Denote the collection of peak sets for $\mathbf{A}$ by $\mathcal{P}_{\mathbf{A}}$. Next we extend the domain of π to include $\chi_{\mathbf{F}}$, for any $\mathbf{F} \in \mathcal{P}_{\mathbf{A}}$. Before that we need several lemmas.

2.2 Lemma. *Let $\mathbf{A}$ be a uniform algebra on a compact Hausdorff space $\mathbf{X}$, and π a contractive unital representation that maps $\mathbf{A}$ into $\mathbf{B}(\mathcal{H})$. Then for any $\mathbf{y}, \mathbf{z} \in \mathcal{H}$*

there exists a regular complex Borel measure $\mu_{\mathbf{y},\mathbf{z}}$ *on* $\mathbf{X}$ *such that*

$$\langle \pi(\mathbf{f})\,\mathbf{y}, \mathbf{z}\rangle = \int_{\mathbf{X}} \mathbf{f}\, d\mu_{\mathbf{y},\mathbf{z}} \quad \textit{for all } \mathbf{f} \in \mathbf{A}, \tag{1}$$

and

$$\|\mu_{\mathbf{y},\mathbf{z}}\| \leq \|\mathbf{y}\| \cdot \|\mathbf{z}\|. \tag{2}$$

(Note: $\|\mu_{\mathbf{y},\mathbf{z}}\|$ denotes the total variation of $\mu_{\mathbf{y},\mathbf{z}}$.)

PROOF: Fix $\mathbf{y}, \mathbf{z} \in \mathcal{H}$ and define the map $\phi : \mathbf{A} \to \mathbb{C}$ by

$$\phi(\mathbf{f}) = \langle \pi(\mathbf{f})\,\mathbf{y}, \mathbf{z}\rangle.$$

By linearity of inner product one can see that ϕ is a linear functional. Also

$$|\phi(\mathbf{f})| = |\langle \pi(\mathbf{f})\,\mathbf{y}, \mathbf{z}\rangle| \leq \|\pi(\mathbf{f})\,\mathbf{y}\| \cdot \|\mathbf{z}\| \leq \|\pi(\mathbf{f})\| \cdot \|\mathbf{y}\| \cdot \|\mathbf{z}\| \leq \|\mathbf{f}\| \cdot \|\mathbf{y}\| \cdot \|\mathbf{z}\|$$

implies that $\|\phi\|$ is bounded by $\|\mathbf{y}\| \cdot \|\mathbf{z}\|$.

By the Hahn-Banach theorem there exists a bounded linear functional, say Φ, defined on $\mathbf{C}(\mathbf{X})$ such that $\Phi(\mathbf{f}) = \phi(\mathbf{f})$ for all $\mathbf{f} \in \mathbf{A}$, and $\|\Phi\| = \|\phi\|$. From Riesz representation theorem there exists a complex regular Borel measure on $\mathbf{X}$, say $\mu_{\mathbf{y},\mathbf{z}}$, with $\|\mu_{\mathbf{y},\mathbf{z}}\| = \|\Phi\| \leq \|\mathbf{y}\| \cdot \|\mathbf{z}\|$, and

$$\Phi(\mathbf{g}) = \int_{\mathbf{X}} \mathbf{g}\, d\mu_{\mathbf{y},\mathbf{z}} \quad \text{for all } \mathbf{g} \in \mathbf{C}(\mathbf{X}).$$

In particular, for $\mathbf{f} \in \mathbf{A} \subseteq \mathbf{C}(\mathbf{X})$

$$\langle \pi(\mathbf{f})\,\mathbf{y}, \mathbf{z}\rangle = \phi(\mathbf{f}) = \Phi(\mathbf{f}) = \int_{\mathbf{X}} \mathbf{f}\, d\mu_{\mathbf{y},\mathbf{z}}.$$

The proof is complete. ■■

Definition(Elementary Measures of π). *Any arbitrary regular complex Borel measure that satisfies (1) and (2) in Lemma 2.2 is called an elementary measure for* $(\mathbf{y}, \mathbf{z}) \in \mathcal{H} \times \mathcal{H}$ *of the contractive unital representation* π.

2.3 Lemma. *Let* $\mathbf{A}, \mathbf{X}$, *and* π *be as in Lemma 2.2. If* $\mu_{\mathbf{y},\mathbf{z}}$, *and* $\mu'_{\mathbf{y},\mathbf{z}}$ *are elementary measures for* $(\mathbf{y}, \mathbf{z}) \in \mathcal{H} \times \mathcal{H}$ *of* π, *then for any* $\mathbf{F} \in \mathcal{P}_{\mathbf{A}}$

$$\int_{\mathbf{X}} \chi_{\mathbf{F}} \, d\mu_{\mathbf{y},\mathbf{z}} = \int_{\mathbf{X}} \chi_{\mathbf{F}} \, d\mu'_{\mathbf{y},\mathbf{z}},$$

where $\chi_{\mathbf{F}}$ *denotes the characteristic function of* $\mathbf{F}$.

PROOF: Let $\mathbf{F} \in \mathcal{P}_{\mathbf{A}}$, and let $\mathbf{p} \in \mathbf{A}$ peak on $\mathbf{F}$.

Since $\mathbf{p}^n$ converges to $\chi_{\mathbf{F}}$ pointwise, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
\int_{\mathbf{X}} \chi_{\mathbf{F}} \, d\mu_{\mathbf{y},\mathbf{z}} &= \lim_{n\to\infty} \int_{\mathbf{X}} \mathbf{p}^n \, d\mu_{\mathbf{y},\mathbf{z}} \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{p}^n)\, \mathbf{y}, \mathbf{z} \rangle \\
&= \lim_{n\to\infty} \int_{\mathbf{X}} \mathbf{p}^n \, d\mu'_{\mathbf{y},\mathbf{z}} \\
&= \int_{\mathbf{X}} \chi_{\mathbf{F}} \, d\mu'_{\mathbf{y},\mathbf{z}}.
\end{aligned}$$

Thus the lemma is proved. ■■

With Lemma 2.3 we can now extend π to include $\{ \chi_{\mathbf{F}} \mid \mathbf{F} \in \mathcal{P}_{\mathbf{A}} \}$ in its domain. That is, for any $\mathbf{F} \in \mathcal{P}_{\mathbf{A}}$ we define $\pi(\chi_{\mathbf{F}})$ to be $\mathbf{P}_{\mathbf{F}}$ in the following theorem.

2.4 Theorem ([16], Mlak). *Let* $\mathbf{A}, \mathbf{X}$, *and* π *be as in Lemma 2.2. For any* $\mathbf{F} \in \mathcal{P}_{\mathbf{A}}$ *there exists a unique projection* $\mathbf{P}_{\mathbf{F}} \in \mathbf{B}(\mathcal{H})$ *such that*

$$\langle \mathbf{P}_{\mathbf{F}}\, \mathbf{y}, \mathbf{z} \rangle = \int_{\mathbf{X}} \chi_{\mathbf{F}} \, d\mu_{\mathbf{y},\mathbf{z}},$$

and $\mathbf{P}_{\mathbf{F}} \cdot \pi(\mathbf{f}) = \pi(\mathbf{f}) \cdot \mathbf{P}_{\mathbf{F}}$ *for all* $\mathbf{f} \in \mathbf{A}$.

PROOF: For any fixed $\mathbf{F} \in \mathcal{P}_{\mathbf{A}}$, define $\mathbf{S}_{\mathbf{F}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by :

$$\mathbf{S}_{\mathbf{F}}(\mathbf{y}, \mathbf{z}) = \int_{\mathbf{X}} \chi_{\mathbf{F}} \, d\mu_{\mathbf{y},\mathbf{z}},$$

where $\mathbf{y}, \mathbf{z} \in \mathcal{H}$, and $\mu_{\mathbf{y},\mathbf{z}}$ is an elementary measure for $(\mathbf{y}, \mathbf{z})$ of π.

By Lemma 2.3, $\mathbf{S_F}$ is well-defined.

Let $\mathbf{p} \in \mathbf{A}$ peak on $\mathbf{F}$, we have

$$\begin{aligned}\mathbf{S_F}(\mathbf{y}, \mathbf{z}) &= \int_{\mathbf{X}} \chi_{\mathbf{F}}\, d\mu_{\mathbf{y},\mathbf{z}} \\ &= \lim_{n\to\infty} \int_{\mathbf{X}} \mathbf{p}^n\, d\mu_{\mathbf{y},\mathbf{z}} \\ &= \lim_{n\to\infty} \langle \pi(\mathbf{p}^n)\, \mathbf{y}, \mathbf{z}\rangle. \end{aligned} \tag{3}$$

From the above relation one can verify $\mathbf{S_F}$ is a sesquilinear functional.

Also

$$\begin{aligned}|\mathbf{S_F}(\mathbf{y}, \mathbf{z})| &= |\int_{\mathbf{X}} \chi_{\mathbf{F}}\, d\mu_{\mathbf{y},\mathbf{z}}| \\ &\leq \|\mu_{\mathbf{y},\mathbf{z}}\| \\ &\leq \|\mathbf{y}\| \cdot \|\mathbf{z}\|.\end{aligned}$$

Hence $\mathbf{S_F}$ is a bounded sesquilinear functional on $\mathcal{H} \times \mathcal{H}$. By Theorem 1 of § 22 in ([10], Halmos) there exists a unique operator, say $\mathbf{P_F}$, on $\mathcal{H}$ such that

$$\mathbf{S_F}(\mathbf{y}, \mathbf{z}) = \langle \mathbf{P_F}\, \mathbf{y}, \mathbf{z}\rangle \quad \text{for all } \mathbf{y}, \mathbf{z} \in \mathcal{H}.$$

For any $\mathbf{y} \in \mathcal{H}$,

$$\begin{aligned}\|\mathbf{y}\|^2 = \langle \mathbf{y}, \mathbf{y}\rangle &= \langle \pi(1)\, \mathbf{y}, \mathbf{y}\rangle \\ &= \int_{\mathbf{X}} 1\, d\mu_{\mathbf{y},\mathbf{y}} \\ &= \mu_{\mathbf{y},\mathbf{y}}(\mathbf{X}) \\ &\leq \|\mu_{\mathbf{y},\mathbf{y}}\| \\ &\leq \|\mathbf{y}\| \cdot \|\mathbf{y}\| \\ &= \|\mathbf{y}\|^2.\end{aligned}$$

This implies $\mu_{\mathbf{y},\mathbf{y}}$ is a positive measure.

For any $\mathbf{y} \in \mathcal{H}$,

$$\langle \mathbf{P_F}\, \mathbf{y}, \mathbf{y} \rangle = \mathbf{S_F}(\mathbf{y}, \mathbf{y}) = \int_{\mathbf{X}} \chi_{\mathbf{F}}\, d\mu_{\mathbf{y},\mathbf{y}} = \mu_{\mathbf{y},\mathbf{y}}(\mathbf{F}) \geq 0.$$

Therefore, $\mathbf{P_F}$ is a positive operator on $\mathcal{H}$.

Next we show $\mathbf{P_F}$ commutes with $\pi(\mathbf{f})$ for any $\mathbf{f} \in \mathbf{A}$.

For any $\mathbf{y}, \mathbf{z} \in \mathcal{H}$, and any $\mathbf{f} \in \mathbf{A}$,

$$\begin{aligned}
\langle \mathbf{P_F} \cdot \pi(\mathbf{f})\, \mathbf{y}, \mathbf{z} \rangle &= \lim_{n\to\infty} \langle \pi(\mathbf{p}^n) \cdot \pi(\mathbf{f})\, \mathbf{y}, \mathbf{z} \rangle && \text{(Use (3).)} \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{p}^n \cdot \mathbf{f})\, \mathbf{y}, \mathbf{z} \rangle \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{f} \cdot \mathbf{p}^n)\, \mathbf{y}, \mathbf{z} \rangle \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{f}) \cdot \pi(\mathbf{p}^n)\, \mathbf{y}, \mathbf{z} \rangle \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{p}^n)\, \mathbf{y}, \pi(\mathbf{f})^*\, \mathbf{z} \rangle \\
&= \langle \mathbf{P_F}\, \mathbf{y}, \pi(\mathbf{f})^*\, \mathbf{z} \rangle && \text{(Use (3).)} \\
&= \langle \pi(\mathbf{f}) \cdot \mathbf{P_F}\, \mathbf{y}, \mathbf{z} \rangle.
\end{aligned}$$

Therefore, $\mathbf{P_F} \cdot \pi(\mathbf{f}) = \pi(\mathbf{f}) \cdot \mathbf{P_F}$ for all $\mathbf{f} \in \mathbf{A}$.

To complete the proof we need to show $\mathbf{P_F}^2 = \mathbf{P_F}$.

For any fixed positive integer k, and any $\mathbf{y}, \mathbf{z} \in \mathcal{H}$,

$$\begin{aligned}
\langle \mathbf{P_F}\, \mathbf{y}, \mathbf{z} \rangle &= \lim_{n\to\infty} \langle \pi(\mathbf{p}^n)\, \mathbf{y}, \mathbf{z} \rangle && \text{(Use (3).)} \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{p}^{n+k})\, \mathbf{y}, \mathbf{z} \rangle \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{p}^n) \cdot \pi(\mathbf{p}^k)\, \mathbf{y}, \mathbf{z} \rangle \\
&= \langle \mathbf{P_F} \cdot \pi(\mathbf{p}^k)\, \mathbf{y}, \mathbf{z} \rangle && \text{(Use (3).)} \\
&= \langle \pi(\mathbf{p}^k)\, \mathbf{y}, \mathbf{P_F}\, \mathbf{z} \rangle.
\end{aligned}$$

Since this is true for any k,

$$\begin{aligned}\langle \mathbf{P_F}\,\mathbf{y}, \mathbf{z}\rangle &= \lim_{k\to\infty} \langle \pi(\mathbf{p}^k)\,\mathbf{y}, \mathbf{P_F}\,\mathbf{z}\rangle \\ &= \langle \mathbf{P_F}\,\mathbf{y}, \mathbf{P_F}\,\mathbf{z}\rangle && \text{(Use (3).)} \\ &= \langle \mathbf{P_F}^2\,\mathbf{y}, \mathbf{z}\rangle.\end{aligned}$$

This proves $\mathbf{P_F} = \mathbf{P_F}^2$, and the theorem is proved completely. ■■

To further extend the domain of π to include $\{\,\chi_{\mathbf{G}} \mid \mathbf{G} \in (\mathcal{GP})_{\mathbf{A}}\,\}$ we need some more preparations.

2.5 Lemma. *Let* $\mathbf{G} \in (\mathcal{GP})_{\mathbf{A}}$ *and* μ *a regular complex Borel measure on* $\mathbf{X}$. *Then there is a sequence* $\{\,\mathbf{F}_n \mid n = 1, 2, 3, \ldots\} \subseteq \mathcal{P}_{\mathbf{A}}$ *such that* $\mathbf{G} \subseteq \mathbf{F}_n$ *for each* $n = 1, 2, 3, \ldots$, *and*

$$\lim_{n\to\infty} |\mu|(\mathbf{F}_n \backslash \mathbf{G}) = 0.$$

Note: $|\mu|(\mathbf{F}_n \backslash \mathbf{G})$ denotes the total variation of $\mu(\mathbf{F}_n \backslash \mathbf{G})$.

Proof: Clearly $\mathbf{G} = \bigcap\{\,\mathbf{F}_\alpha \mid \mathbf{G} \subseteq \mathbf{F}_\alpha \text{ and } \mathbf{F}_\alpha \in \mathcal{P}_{\mathbf{A}}\,\}$.

The set $\mathbf{X}\backslash\mathbf{G}$ is open, and $|\mu|$ is regular, so there is a sequence $\{\,\mathbf{K}_n \mid n = 1, 2, 3, \ldots\}$ of compact sets such that $\mathbf{K}_n \subseteq \mathbf{X}\backslash\mathbf{G}$ and

$$\lim_{n\to\infty} |\mu|(\mathbf{K}_n) = |\mu|(\mathbf{X}\backslash\mathbf{G}).$$

We claim for any $\mathbf{K}_n$ there is some $\mathbf{F}_n$ with $\mathbf{G} \subseteq \mathbf{F}_n \in \mathcal{P}_{\mathbf{A}}$ such that $\mathbf{K}_n \cap \mathbf{F}_n = \emptyset$.

Proof of claim: Suppose not. Then for some fixed n, $\{\,\mathbf{K}_n \cap \mathbf{F}_\alpha \mid \mathbf{G} \subseteq \mathbf{F}_\alpha$ and $\mathbf{F}_\alpha \in \mathcal{P}_{\mathbf{A}}\,\}$ is a collection of nonempty closed sets with finite intersection property.

By compactness of $\mathbf{X}$, the set $\bigcap\{\,\mathbf{K}_n \cap \mathbf{F}_\alpha \mid \mathbf{G} \subseteq \mathbf{F}_\alpha,\ \text{and}\ \mathbf{F}_\alpha \in \mathcal{P}_\mathbf{A}\,\} \neq \emptyset$. But

$$\begin{aligned}
\emptyset &= \mathbf{K}_n \cap \mathbf{G} \\
&= \mathbf{K}_n \cap (\bigcap\{\,\mathbf{F}_\alpha \mid \mathbf{G} \subseteq \mathbf{F}_\alpha \text{ and } \mathbf{F}_\alpha \in \mathcal{P}_\mathbf{A}\,\}) \\
&= \bigcap\{\,\mathbf{K}_n \cap \mathbf{F}_\alpha \mid \mathbf{G} \subseteq \mathbf{F}_\alpha \text{ and } \mathbf{F}_\alpha \in \mathcal{P}_\mathbf{A}\,\} \\
&\neq \emptyset.
\end{aligned}$$

A contradiction arrives. Thus the claim is proved. ■

Continuing the proof of the lemma, we have

$$\begin{aligned}
0 \leq |\mu|(\mathbf{F}_n \backslash \mathbf{G}) &= |\mu|(\mathbf{F}_n) - |\mu|(\mathbf{G}) \\
&\leq |\mu|(\mathbf{X} \backslash \mathbf{K}_n) - |\mu|(\mathbf{G}) \\
&= |\mu|[(\mathbf{X} \backslash \mathbf{K}_n) \backslash \mathbf{G}] \\
&= |\mu|[(\mathbf{X} \backslash \mathbf{G}) \backslash \mathbf{K}_n] \\
&= |\mu|(\mathbf{X} \backslash \mathbf{G}) - |\mu|(\mathbf{K}_n).
\end{aligned}$$

Since $|\mu|(\mathbf{K}_n)$ converges to $|\mu|(\mathbf{X} \backslash \mathbf{G})$ as n approaches ∞, $|\mu|(\mathbf{F}_n \backslash \mathbf{G})$ converges to 0 as n approaches ∞. Therefore, the lemma is proved. ■■

The key to the proof of the next theorem is the following:

2.6 Lemma. *Let* $\mathbf{G} \in (\mathcal{GP})_\mathbf{A}$ *and* μ *a regular complex Borel measure on* $\mathbf{X}$ *such that*

$$\int_\mathbf{X} \mathbf{f}\, d\mu = 0 \quad \textit{for all } \mathbf{f} \in \mathbf{A}.$$

Then

$$\int_\mathbf{X} \mathbf{f} \cdot \chi_\mathbf{G}\, d\mu = 0 \quad \textit{for all } \mathbf{f} \in \mathbf{A}.$$

PROOF: By Lemma 2.5, there exists a sequence $\{ \mathbf{F}_n \mid n = 1, 2, 3 \ldots \} \subseteq \mathcal{P}_{\mathbf{A}}$ such that $\mathbf{G} \subseteq \mathbf{F}_n$ for each $n = 1, 2, 3, \ldots$, and

$$\lim_{n \to \infty} |\mu|(\mathbf{F}_n \backslash \mathbf{G}) = 0.$$

Let $\mathbf{p}_n \in \mathbf{A}$ peak on $\mathbf{F}_n$ for each $n = 1, 2, 3, \ldots$. For any positive integer n, and any $\mathbf{f} \in \mathbf{A}$,

$$\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{F}_n} \, d\mu = \lim_{m \to \infty} \int_{\mathbf{X}} \mathbf{f} \cdot \mathbf{p}_n{}^m \, d\mu = 0. \tag{4}$$

For any $\mathbf{f} \in \mathbf{A}$,

$$\begin{aligned} |\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{G}} \, d\mu| &= |\int_{\mathbf{X}} \mathbf{f} \cdot (\chi_{\mathbf{F}_n} - \chi_{\mathbf{F}_n \backslash \mathbf{G}}) \, d\mu| \\ &\leq |\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{F}_n} \, d\mu| + |\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{(\mathbf{F}_n \backslash \mathbf{G})} \, d\mu| \\ &= |\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{(\mathbf{F}_n \backslash \mathbf{G})} \, d\mu| \qquad \text{(Use (4).)} \\ &\leq \|\mathbf{f}\| \cdot |\mu|(\mathbf{F}_n \backslash \mathbf{G}). \end{aligned}$$

As n approaches ∞, $|\mu|(\mathbf{F}_n \backslash \mathbf{G})$ converges to 0. This proves $\int_{\mathbf{X}} \mathbf{f} \chi_{\mathbf{G}} \, d\mu = 0$ for all $\mathbf{f} \in \mathbf{A}$. ■■

Now we are ready to extend the domain of π to include $\{ \chi_{\mathbf{G}} \mid \mathbf{G} \in (\mathcal{GP})_{\mathbf{A}} \}$. We define, for any $\mathbf{G}$ in $(\mathcal{GP})_{\mathbf{A}}$, the image $\pi(\chi_{\mathbf{G}})$ to be $\mathbf{P}_{\mathbf{G}}$ in the following theorem.

2.7 Theorem ([16], Mlak). *Let $\mathbf{A}, \mathbf{X}$, and π be as in Lemma 2.2. For any $\mathbf{G} \in (\mathcal{GP})_{\mathbf{A}}$ there exists a unique projection $\mathbf{P}_{\mathbf{G}} \in \mathbf{B}(\mathcal{H})$ such that*

$$\langle \mathbf{P}_{\mathbf{G}} \, \mathbf{y}, \mathbf{z} \rangle = \int_{\mathbf{X}} \chi_{\mathbf{G}} \, d\mu_{\mathbf{y},\mathbf{z}},$$

and $\mathbf{P}_{\mathbf{G}} \cdot \pi(\mathbf{f}) = \pi(\mathbf{f}) \cdot \mathbf{P}_{\mathbf{G}}$ for all $\mathbf{f} \in \mathbf{A}$.

PROOF: For any fixed $\mathbf{G} \in (\mathcal{GP})_{\mathbf{A}}$, define $\mathbf{S}_{\mathbf{G}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ by

$$\mathbf{S}_{\mathbf{G}}(\mathbf{y}, \mathbf{z}) = \int_{\mathbf{X}} \chi_{\mathbf{G}} \, d\mu_{\mathbf{y},\mathbf{z}},$$

where $\mathbf{y}, \mathbf{z} \in \mathcal{H}$, and $\mu_{\mathbf{y},\mathbf{z}}$ is an elementary measure for $(\mathbf{y}, \mathbf{z})$ of π.

First we check that $\mathbf{S_G}$ is well-defined.

Let $\mu_{\mathbf{y},\mathbf{z}}$ and $\mu'_{\mathbf{y},\mathbf{z}}$ be two elementary measures for $(\mathbf{y}, \mathbf{z})$ of π. Then $\mu_{\mathbf{y},\mathbf{z}} - \mu'_{\mathbf{y},\mathbf{z}}$ is a regular complex Borel measure on $\mathbf{X}$ such that

$$\int_{\mathbf{X}} \mathbf{f}\, d(\mu_{\mathbf{y},\mathbf{z}} - \mu'_{\mathbf{y},\mathbf{z}}) = 0 \quad \text{for all } \mathbf{f} \in \mathbf{A}.$$

From Lemma 2.6, it follows that

$$\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{G}}\, d(\mu_{\mathbf{y},\mathbf{z}} - \mu'_{\mathbf{y},\mathbf{z}}) = 0 \quad \text{for all } \mathbf{f} \in \mathbf{A}.$$

Take $\mathbf{f} = 1$, then

$$\int_{\mathbf{X}} \chi_{\mathbf{G}}\, d\mu_{\mathbf{y},\mathbf{z}} = \int_{\mathbf{X}} \chi_{\mathbf{G}}\, d\mu'_{\mathbf{y},\mathbf{z}}.$$

Consequently, $\mathbf{S_G}$ is well-defined.

For $\mathbf{y}_1, \mathbf{y}_2, \mathbf{z} \in \mathcal{H}$, and any $\mathbf{f} \in \mathbf{A}$,

$$\begin{aligned}
\int_{\mathbf{X}} \mathbf{f}\, d\mu_{\mathbf{y}_1+\mathbf{y}_2,\mathbf{z}} &= \langle \pi(\mathbf{f})(\mathbf{y}_1 + \mathbf{y}_2), \mathbf{z} \rangle \\
&= \langle \pi(\mathbf{f})\, \mathbf{y}_1, \mathbf{z} \rangle + \langle \pi(\mathbf{f})\, \mathbf{y}_2, \mathbf{z} \rangle \\
&= \int_{\mathbf{X}} \mathbf{f}\, d\mu_{\mathbf{y}_1,\mathbf{z}} + \int_{\mathbf{X}} \mathbf{f}\, d\mu_{\mathbf{y}_2,\mathbf{z}}.
\end{aligned}$$

This proves that for any $\mathbf{f} \in \mathbf{A}$,

$$\int_{\mathbf{X}} \mathbf{f}\, d(\mu_{\mathbf{y}_1+\mathbf{y}_2,\mathbf{z}} - \mu_{\mathbf{y}_1,\mathbf{z}} - \mu_{\mathbf{y}_2,\mathbf{z}}) = 0.$$

Using Lemma 2.6 again, we have

$$\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{G}}\, d(\mu_{\mathbf{y}_1+\mathbf{y}_2,\mathbf{z}} - \mu_{\mathbf{y}_1,\mathbf{z}} - \mu_{\mathbf{y}_2,\mathbf{z}}) = 0,$$

or, equivalently, we have for all $\mathbf{f}$ in $\mathbf{A}$ that

$$\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{G}}\, d\mu_{\mathbf{y}_1+\mathbf{y}_2,\mathbf{z}} = \int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{G}}\, d\mu_{\mathbf{y}_1,\mathbf{z}} + \int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{G}}\, d\mu_{\mathbf{y}_2,\mathbf{z}}.$$

If $\mathbf{f} = 1$, we see that

$$\mathbf{S_G}((\mathbf{y}_1 + \mathbf{y}_2), \mathbf{z}) = \mathbf{S_G}(\mathbf{y}_1, \mathbf{z}) + \mathbf{S_G}(\mathbf{y}_2, \mathbf{z}).$$

In a similar way we can prove $\mathbf{S_G}$ is a sesquilinear functional. Also

$$|\mathbf{S_G}(\mathbf{y}, \mathbf{z})| = |\int_{\mathbf{X}} \chi_{\mathbf{G}}\, d\mu_{\mathbf{y},\mathbf{z}}| \leq \|\mu_{\mathbf{y},\mathbf{z}}\| \leq \|\mathbf{y}\| \cdot \|\mathbf{z}\|.$$

Hence, $\mathbf{S_G}$ is a bounded sesquilinear functional. There exists a unique operator, say $\mathbf{P_G}$, on $\mathcal{H}$ such that $\mathbf{S_G}(\mathbf{y}, \mathbf{z}) = \langle \mathbf{P_G}\, \mathbf{y}, \mathbf{z} \rangle$ for all $\mathbf{y}, \mathbf{z} \in \mathcal{H}$.

Also for any $\mathbf{y} \in \mathcal{H}$,

$$\langle \mathbf{P_G}\, \mathbf{y}, \mathbf{y} \rangle = \int_{\mathbf{X}} \chi_{\mathbf{G}}\, d\mu_{\mathbf{y},\mathbf{y}} = \mu_{\mathbf{y},\mathbf{y}}(\mathbf{G}) \geq 0,$$

since $\mu_{\mathbf{y},\mathbf{y}}$ is a positive measure as proved in Theorem 2.4. Therefore, $\mathbf{P_G}$ is a positive operator on $\mathcal{H}$.

Next we show $\mathbf{P_G}$ commutes with $\pi(\mathbf{f})$ for any $\mathbf{f} \in \mathbf{A}$.

For any $\mathbf{y}, \mathbf{z} \in \mathcal{H}$, and any $\mathbf{f} \in \mathbf{A}$, by Lemma 2.5, there exists a sequence $\{\, \mathbf{F}_n \mid \mathbf{G} \subseteq \mathbf{F}_n,$ where $n = 1, 2, 3, \ldots \} \subseteq \mathcal{P}_{\mathbf{A}}$ such that $|\mu_{\pi(\mathbf{f})\mathbf{y},\mathbf{z}}|(\mathbf{F}_n \backslash \mathbf{G})$ approaches 0, and $|\mu_{\mathbf{y},\pi(\mathbf{f})^*\mathbf{z}}|(\mathbf{F}_n \backslash \mathbf{G})$ approaches 0 as n tends to ∞.

Then for any $\mathbf{f} \in \mathbf{A}$, and any $\mathbf{y}, \mathbf{z} \in \mathcal{H}$,

$$\begin{aligned}
\langle \mathbf{P_G} \cdot \pi(\mathbf{f})\,\mathbf{y}, \mathbf{z}\rangle &= \int_{\mathbf{X}} \chi_{\mathbf{G}}\, d\mu_{\pi(\mathbf{f})\,\mathbf{y},\mathbf{z}} \\
&= \lim_{n\to\infty} \int_{\mathbf{X}} \chi_{\mathbf{F}_n}\, d\mu_{\pi(\mathbf{f})\,\mathbf{y},\mathbf{z}} \\
&= \lim_{n\to\infty} \langle \mathbf{P_{F_n}} \cdot \pi(\mathbf{f})\,\mathbf{y}, \mathbf{z}\rangle \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{f}) \cdot \mathbf{P_{F_n}}\,\mathbf{y}, \mathbf{z}\rangle \qquad \text{(Use Theorem 2.4.)} \\
&= \lim_{n\to\infty} \langle \mathbf{P_{F_n}}\,\mathbf{y}, \pi(\mathbf{f})^*\,\mathbf{z}\rangle \\
&= \lim_{n\to\infty} \int_{\mathbf{X}} \chi_{\mathbf{F}_n}\, d\mu_{\mathbf{y},\pi(\mathbf{f})^*\,\mathbf{z}} \qquad \text{(Use Theorem 2.4)} \\
&= \int_{\mathbf{X}} \chi_{\mathbf{G}}\, d\mu_{\mathbf{y},\pi(\mathbf{f})^*\,\mathbf{z}} \\
&= \langle \mathbf{P_G}\,\mathbf{y}, \pi(\mathbf{f})^*\,\mathbf{z}\rangle \\
&= \langle \pi(\mathbf{f}) \cdot \mathbf{P_G}\,\mathbf{y}, \mathbf{z}\rangle.
\end{aligned}$$

Thus $\mathbf{P_G} \cdot \pi(\mathbf{f}) = \pi(\mathbf{f}) \cdot \mathbf{P_G}$ for any $\mathbf{f} \in \mathbf{A}$.

It remains to show ${\mathbf{P_G}}^2 = \mathbf{P_G}$.

To this end, fix $\mathbf{y} \in \mathcal{H}$. We can find, by Lemma 2.5, a sequence $\{\mathbf{F}_n \mid \mathbf{G} \subseteq \mathbf{F}_n$, where $n = 1, 2, 3, \ldots\} \subseteq \mathcal{P}_{\mathbf{A}}$ such that $|\mu_{\mathbf{y},\mathbf{y}}|(\mathbf{F}_n \backslash \mathbf{G})$ converges to 0, as n approaches ∞. Then

$$\begin{aligned}
0 &\le \langle (\mathbf{P_{F_n}} - \mathbf{P_G})\,\mathbf{y}, (\mathbf{P_{F_n}} - \mathbf{P_G})\,\mathbf{y}\rangle \\
&\le \langle (\mathbf{P_{F_n}} - \mathbf{P_G})\,\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}} \langle (\mathbf{P_{F_n}} - \mathbf{P_G})(\mathbf{P_{F_n}} - \mathbf{P_G})\,\mathbf{y}, (\mathbf{P_{F_n}} - \mathbf{P_G})\,\mathbf{y}\rangle^{\frac{1}{2}} \\
&\le [|\mu_{\mathbf{y},\mathbf{y}}|(\mathbf{F}_n \backslash \mathbf{G}) \cdot \|\mathbf{P_{F_n}} - \mathbf{P_G}\|^3 \cdot \|\mathbf{y}\|^2]^{\frac{1}{2}},
\end{aligned}$$

where the second inequality follows from Schwartz inequality. This proves that as n approaches ∞,

$$\mathbf{P_{F_n}}\mathbf{y} \to \mathbf{P_G}\mathbf{y}. \tag{5}$$

Therefore,

$$
\begin{aligned}
\langle \mathbf{P_G}\, \mathbf{y}, \mathbf{y} \rangle &= \int_{\mathbf{X}} \chi_{\mathbf{G}}\, d\mu_{\mathbf{y},\mathbf{y}} \\
&= \lim_{n\to\infty} \int_{\mathbf{X}} \chi_{\mathbf{F}_n}\, d\mu_{\mathbf{y},\mathbf{y}} \\
&= \lim_{n\to\infty} \langle \mathbf{P_{F_n}}\, \mathbf{y}, \mathbf{y} \rangle \\
&= \lim_{n\to\infty} \langle \mathbf{P_{F_n}}^2\, \mathbf{y}, \mathbf{y} \rangle && \text{(Use Theorem 2.4.)} \\
&= \lim_{n\to\infty} \langle \mathbf{P_{F_n}}\, \mathbf{y}, \mathbf{P_{F_n}}\, \mathbf{y} \rangle \\
&= \langle \mathbf{P_G}\, \mathbf{y}, \mathbf{P_G}\, \mathbf{y} \rangle && \text{(Use (5).)} \\
&= \langle \mathbf{P_G}^2\, \mathbf{y}, \mathbf{y} \rangle.
\end{aligned}
$$

This implies $\mathbf{P_G} = \mathbf{P_G}^2$, and the proof is complete. ■■

CHAPTER 3

ONE-TO-ONE CORRESPONDENCE BETWEEN MAXIMAL SETS OF ANTISYMMETRY AND MAXIMAL PROJECTIONS OF ANTISYMMETRY

Let $\mathbf{A}$ be a uniform algebra on $\mathbf{X}$, and π a contractive unital representation mapping $\mathbf{A}$ into $\mathbf{B}(\mathcal{H})$. Later in this chapter we assume further that (i)π is an isometry and (ii)for each maximal set of antisymmetry for $\mathbf{A}$, say $\mathbf{K}$, and any $\epsilon > 0$, there is an open set, say $\mathbf{V}$, with $\mathbf{V} \supseteq \mathbf{K}$ such that

$$|\mu_{\mathbf{y},\mathbf{z}}|(\mathbf{V}\backslash\mathbf{K}) < \epsilon \text{ for all } \mathbf{y}, \mathbf{z} \in \mathcal{H} \text{ with } \|\mathbf{y}\| = \|\mathbf{z}\| = 1,$$

where $\mu_{\mathbf{y},\mathbf{z}}$ is an elementary measure for $(\mathbf{y}, \mathbf{z})$.

Under these conditions, in Corollary 3.9 we show that there is a one-to-one correspondence between the collection of maximal sets of antisymmetry for $\mathbf{A}$ and that of maximal projections of antisymmetry for $\pi(\mathbf{A})$.

In Chapter 2 we proved for each $\mathbf{G} \in (\mathcal{GP})_{\mathbf{A}}$ there exists a projection $\mathbf{P_G} \in \mathbf{B}(\mathcal{H})$ such that $\mathbf{P_G} \cdot \pi(\mathbf{f}) = \pi(\mathbf{f}) \cdot \mathbf{P_G}$ for any $\mathbf{f} \in \mathbf{A}$, and

$$\langle \mathbf{P_G}\, \mathbf{y}, \mathbf{z} \rangle = \int_{\mathbf{X}} \chi_{\mathbf{G}}\, d\mu_{\mathbf{y},\mathbf{z}},$$

where $\mu_{\mathbf{y},\mathbf{z}}$ is any elementary measure for $(\mathbf{y}, \mathbf{z})$ of π.

In actuality, we also have an integral form for the expression $\langle \pi(\mathbf{f}) \cdot \mathbf{P_G}\, \mathbf{y}, \mathbf{z} \rangle$, for any $\mathbf{f} \in \mathbf{A}$, $\mathbf{y}, \mathbf{z} \in \mathcal{H}$, and $\mathbf{G} \in (\mathcal{GP})_{\mathbf{A}}$. First we obtain the desired form for those $\mathbf{F} \in \mathcal{P}_{\mathbf{A}}$.

3.1 Lemma. *Let* $\mathbf{F}$ *be a peak set for* $\mathbf{A}$. *Then for any* $\mathbf{f} \in \mathbf{A}$, *and* $\mathbf{y}, \mathbf{z} \in \mathcal{H}$, *we have*

$$\langle \pi(\mathbf{f}) \cdot \mathbf{P_F}\, \mathbf{y}, \mathbf{z} \rangle = \int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{F}}\, d\mu_{\mathbf{y},\mathbf{z}},$$

where $\mu_{\mathbf{y},\mathbf{z}}$ *is any elementary measure for* $(\mathbf{y},\mathbf{z})$ *of* π.

PROOF: Let $\mathbf{p} \in \mathbf{A}$ peak on $\mathbf{F}$, we have

$$\begin{aligned}
\langle \pi(\mathbf{f}) \cdot \mathbf{P}_{\mathbf{F}}\, \mathbf{y}, \mathbf{z} \rangle &= \langle \mathbf{P}_{\mathbf{F}} \cdot \pi(\mathbf{f})\, \mathbf{y}, \mathbf{z} \rangle \qquad \text{(Use Theorem 2.4.)} \\
&= \lim_{n\to\infty} \int_{\mathbf{X}} \mathbf{p}^n \, d\mu_{\pi(\mathbf{f})\,\mathbf{y},\mathbf{z}} \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{p}^n) \cdot \pi(\mathbf{f})\, \mathbf{y}, \mathbf{z} \rangle \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{p}^n \cdot \mathbf{f})\, \mathbf{y}, \mathbf{z} \rangle \\
&= \lim_{n\to\infty} \int_{\mathbf{X}} \mathbf{p}^n \cdot \mathbf{f} \, d\mu_{\mathbf{y},\mathbf{z}} \\
&= \int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{F}} \, d\mu_{\mathbf{y},\mathbf{z}}. \quad \blacksquare\blacksquare
\end{aligned}$$

Next we prove the equality in Lemma 3.1 holds true for any $\mathbf{G} \in (\mathcal{GP})_{\mathbf{A}}$.

3.2 Lemma. *Let* $\mathbf{G}$ *be a generalized peak set for* $\mathbf{A}$. *For any* $\mathbf{y}, \mathbf{z} \in \mathcal{H}$, *and* $\mathbf{f} \in \mathbf{A}$, *we have*

$$\langle \pi(\mathbf{f}) \cdot \mathbf{P}_{\mathbf{G}}\, \mathbf{y}, \mathbf{z} \rangle = \int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{G}} \, d\mu_{\mathbf{y},\mathbf{z}},$$

where $\mu_{\mathbf{y},\mathbf{z}}$ *is any elementary measure for* $(\mathbf{y},\mathbf{z})$ *of* π.

PROOF: Fix $\mathbf{y}, \mathbf{z} \in \mathcal{H}$, any elementary measure $\mu_{\mathbf{y},\mathbf{z}}$ for $(\mathbf{y},\mathbf{z})$ of π, and $\mathbf{f} \in \mathbf{A}$.

By Lemma 2.5, there exists a sequence $\{\, \mathbf{F}_n \mid n = 1, 2, 3, \ldots \} \subseteq \mathcal{P}_{\mathbf{A}}$ such that

$$\lim_{n\to\infty} |\mu_{\mathbf{y},\mathbf{z}}|(\mathbf{F}_n \backslash \mathbf{G}) = 0,$$

and

$$\lim_{n\to\infty} |\mu_{\pi(\mathbf{f})\,\mathbf{y},\mathbf{z}}|(\mathbf{F}_n \backslash \mathbf{G}) = 0.$$

Then

$$\begin{aligned}
\langle \pi(\mathbf{f}) \cdot \mathbf{P}_{\mathbf{G}}\, \mathbf{y}, \mathbf{z} \rangle &= \langle \mathbf{P}_{\mathbf{G}} \cdot \pi(\mathbf{f})\, \mathbf{y}, \mathbf{z} \rangle && \text{(Use Theorem 2.7.)} \\
&= \int_{\mathbf{X}} \chi_{\mathbf{G}}\, d\mu_{\pi(\mathbf{f})\,\mathbf{y},\mathbf{z}} \\
&= \lim_{n\to\infty} \int_{\mathbf{X}} \chi_{\mathbf{F}_n}\, d\mu_{\pi(\mathbf{f})\,\mathbf{y},\mathbf{z}} \\
&= \lim_{n\to\infty} \langle \mathbf{P}_{\mathbf{F}_n} \cdot \pi(\mathbf{f})\, \mathbf{y}, \mathbf{z} \rangle && \text{(Use Theorem 2.4.)} \\
&= \lim_{n\to\infty} \langle \pi(\mathbf{f}) \cdot \mathbf{P}_{\mathbf{F}_n}\, \mathbf{y}, \mathbf{z} \rangle && \text{(Use Theorem 2.4.)} \\
&= \lim_{n\to\infty} \int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{F}_n}\, d\mu_{\mathbf{y},\mathbf{z}} && \text{(Use Lemma 3.1.)} \\
&= \int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{G}}\, d\mu_{\mathbf{y},\mathbf{z}}. \quad \blacksquare\blacksquare
\end{aligned}$$

3.3 Corollary. *Let* $\mathbf{G}_1, \mathbf{G}_2$ *be two generalized peak sets for* $\mathbf{A}$. *For any* $\mathbf{y}, \mathbf{z} \in \mathcal{H}$, *we have*

$$\langle \mathbf{P}_{\mathbf{G}_1} \cdot \mathbf{P}_{\mathbf{G}_2}\, \mathbf{y}, \mathbf{z} \rangle = \int_{\mathbf{X}} \chi_{\mathbf{G}_1 \cap \mathbf{G}_2}\, d\mu_{\mathbf{y},\mathbf{z}},$$

where $\mu_{\mathbf{y},\mathbf{z}}$ *is any elementary measure for* $(\mathbf{y}, \mathbf{z})$ *of* π.

PROOF: Fix $\mathbf{y}, \mathbf{z} \in \mathcal{H}$.

Using Lemma 2.5, there exists a sequence $\{\, \mathbf{F}_n \mid n = 1, 2, 3, \ldots \} \subseteq \mathcal{P}_{\mathbf{A}}$ with $\mathbf{G}_1 \subseteq \mathbf{F}_n$ for each $n = 1, 2, 3, \ldots$, such that

$$\lim_{n\to\infty} |\mu_{\mathbf{y},\mathbf{z}}|(\mathbf{F}_n \backslash \mathbf{G}_1) = 0,$$

and

$$\lim_{n\to\infty} |\mu_{\mathbf{P}_{\mathbf{G}_2}\,\mathbf{y},\mathbf{z}}|(\mathbf{F}_n \backslash \mathbf{G}_1) = 0.$$

Also let $\mathbf{p}_n$ peak on $\mathbf{F}_n$ for each $n = 1, 2, 3, \ldots$. Then

$$\begin{aligned}
\langle \mathbf{P}_{\mathbf{G}_1} \cdot \mathbf{P}_{\mathbf{G}_2} \mathbf{y}, \mathbf{z} \rangle &= \int_{\mathbf{X}} \chi_{\mathbf{G}_1} \, d\mu_{\mathbf{P}_{\mathbf{G}_2} \mathbf{y}, \mathbf{z}} && \text{(Use Theorem 2.7.)} \\
&= \lim_{n \to \infty} \int_{\mathbf{X}} \chi_{\mathbf{F}_n} \, d\mu_{\mathbf{P}_{\mathbf{G}_2} \mathbf{y}, \mathbf{z}} \\
&= \lim_{n \to \infty} \lim_{m \to \infty} \int_{\mathbf{X}} \mathbf{p}_n{}^m \, d\mu_{\mathbf{P}_{\mathbf{G}_2} \mathbf{y}, \mathbf{z}} \\
&= \lim_{n \to \infty} \lim_{m \to \infty} \langle \pi(\mathbf{p}_n{}^m) \cdot \mathbf{P}_{\mathbf{G}_2} \mathbf{y}, \mathbf{z} \rangle \\
&= \lim_{n \to \infty} \lim_{m \to \infty} \int_{\mathbf{X}} \mathbf{p}_n{}^m \cdot \chi_{\mathbf{G}_2} \, d\mu_{\mathbf{y}, \mathbf{z}} && \text{(Use Lemma 3.2.)} \\
&= \lim_{n \to \infty} \int_{\mathbf{X}} \chi_{\mathbf{F}_n} \cdot \chi_{\mathbf{G}_2} \, d\mu_{\mathbf{y}, \mathbf{z}} \\
&= \lim_{n \to \infty} \int_{\mathbf{X}} \chi_{\mathbf{F}_n \cap \mathbf{G}_2} \, d\mu_{\mathbf{y}, \mathbf{z}} \\
&= \int_{\mathbf{X}} \chi_{\mathbf{G}_1 \cap \mathbf{G}_2} \, d\mu_{\mathbf{y}, \mathbf{z}}. \quad \blacksquare\blacksquare
\end{aligned}$$

Remarks:

(a) $\mathbf{P}_{\mathbf{G}_1 \cap \mathbf{G}_2} = \mathbf{P}_{\mathbf{G}_1} \cdot \mathbf{P}_{\mathbf{G}_2}$.

(b) If $\mathbf{G}_1 \cap \mathbf{G}_2 = \emptyset$, then $\mathbf{P}_{\mathbf{G}_1} \cdot \mathbf{P}_{\mathbf{G}_2} = \mathbf{O}$.

In the following we further assume that π is an isometry, i.e., $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ for all $\mathbf{f} \in \mathbf{A}$.

3.4 Lemma. *Let π be a unital representation that maps $\mathbf{A}$ into $\mathbf{B}(\mathcal{H})$. Also assume $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ for all $\mathbf{f} \in \mathbf{A}$. If $\mathbf{f}_0 \in \mathbf{A}$, and $\pi(\mathbf{f}_0)$ is self-adjoint, then $\mathbf{f}_0$ is real-valued on $\mathbf{X}$.*

Proof: Suppose on the contrary $\mathbf{f}_0$ is not real-valued on $\mathbf{X}$. Then there exists $\mathbf{x}_0 \in \mathbf{X}$ such that $\mathbf{f}_0(\mathbf{x}_0) = \mathbf{a} + i\mathbf{b}$ with $\mathbf{a}, \mathbf{b}$ real numbers and $\mathbf{b} \neq 0$.

Define $\phi : \pi(\mathbf{A}) \to \mathbb{C}$ by

$$\phi(\pi(\mathbf{f})) = \mathbf{f}(\mathbf{x}_0) \quad \text{for } \mathbf{f} \in \mathbf{A}.$$

We check ϕ is well-defined.

Suppose $\pi(\mathbf{f}) = \pi(\mathbf{g})$ for some $\mathbf{f}, \mathbf{g} \in \mathbf{A}$. Then

$$0 = \|\pi(\mathbf{f}) - \pi(\mathbf{g})\| = \|\pi(\mathbf{f} - \mathbf{g})\| = \|\mathbf{f} - \mathbf{g}\|.$$

This implies $(\mathbf{f} - \mathbf{g})(\mathbf{x}_0) = 0$, or $\mathbf{f}(\mathbf{x}_0) = \mathbf{g}(\mathbf{x}_0)$.

One can check ϕ is a nonzero multiplicative linear functional defined on $\pi(\mathbf{A})$. By Theorem VII.8.6 of ([2], Conway), the spectrum of $\pi(\mathbf{f}_0)$ is:
$\{\, \psi(\pi(\mathbf{f}_0)) \mid \text{where } \psi \text{ is any nonzero multiplicative linear functional on } \pi(\mathbf{A}) \,\}$.

Since $\pi(\mathbf{f}_0)$ is self-adjoint, the spectrum of $\pi(\mathbf{f}_0)$ is a subset of real numbers. But $\phi(\pi(\mathbf{f}_0)) = \mathbf{f}_0(\mathbf{x}_0)$ is not a real number. A contradiction arrives. Thus $\mathbf{f}_0$ must be real-valued on $\mathbf{X}$. ■■

Recall that the collection of maximal sets of antisymmetry for $\mathbf{A}$ is denoted by $\mathcal{K}_{\mathbf{A}}$. In ([9], Glicksberg), it is shown $\mathcal{K}_{\mathbf{A}} \subseteq (\mathcal{GP})_{\mathbf{A}}$.

3.5 Corollary. *Let π be a unital representation that maps $\mathbf{A}$ into $\mathbf{B}(\mathcal{H})$. Also assume that $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ for all $\mathbf{f} \in \mathbf{A}$, and $\mathcal{K}_{\mathbf{A}} = \{\, \mathbf{X} \,\}$. Then the only self-adjoint operator in $\pi(\mathbf{A})$ is $\lambda \cdot \mathbf{I}$ for some real number λ.*

Proof: Suppose $\pi(\mathbf{f})$ is self-adjoint for some $\mathbf{f} \in \mathbf{A}$. We see from Lemma 3.4 that $\mathbf{f}$ is real-valued on $\mathbf{X}$. Since $\mathbf{X}$ is the maximal set of antisymmetry for $\mathbf{A}$, we have $\mathbf{f} = \lambda \cdot 1$ for some real number λ. Thus, $\pi(\mathbf{f}) = \pi(\lambda \cdot 1) = \lambda \cdot \pi(1) = \lambda \cdot \mathbf{I}$. ■■

Remark: The corollary says $\mathbf{I}$ is a projection of antisymmetry, and hence a maximal projection of antisymmetry, for $\pi(\mathbf{A})$ in the case $\mathbf{X}$ is the only maximal set of

antisymmetry for $\mathbf{A}$.

3.6 Theorem. *Let π be a unital representation mapping a uniform algebra $\mathbf{A} \subseteq \mathbf{C}(\mathbf{X})$ into $\mathbf{B}(\mathcal{H})$ with $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ for all $\mathbf{f} \in \mathbf{A}$. Let $\mathbf{K}$ be a maximal set of antisymmetry for $\mathbf{A}$ such that for any $\epsilon > 0$ there exists an open set $\mathbf{V}$ with $\mathbf{K} \subseteq \mathbf{V} \subseteq \mathbf{X}$ such that*

$$|\mu_{\mathbf{y},\mathbf{z}}|(\mathbf{V}\backslash\mathbf{K}) < \epsilon \text{ for all } \mathbf{y}, \mathbf{z} \in \mathcal{H} \text{ with } \|\mathbf{y}\| = \|\mathbf{z}\| = 1,$$

where $\mu_{\mathbf{y},\mathbf{z}}$ is an elementary measure for $(\mathbf{y}, \mathbf{z})$. Then $\mathbf{P_K}$ is a projection of antisymmetry for $\pi(\mathbf{A})$.

PROOF: Let $\mathbf{i_K}$ denote the mapping from $\mathbf{K}$ to $\mathbf{X}$ such that $\mathbf{i_K}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbf{K}$.

Define $\mathbf{A_K} \equiv \{\, \mathbf{f} \circ \mathbf{i_K} \mid \mathbf{f} \in \mathbf{A} \,\}$. Then, by Theorem 1.7, $\mathbf{A_K}$ is a uniform algebra on $\mathbf{K}$ with $\mathbf{K}$ as the only maximal set of antisymmetry for $\mathbf{A_K}$.

Define $\pi_{\mathbf{K}} : \mathbf{A_K} \to \mathbf{B}(\mathbf{P_K}\mathcal{H})$ by $\pi_{\mathbf{K}}(\mathbf{f} \circ \mathbf{i_K}) \equiv \pi(\mathbf{f}) \cdot \mathbf{P_K}$.

We need to check $\pi_{\mathbf{K}}$ is well-defined.

Suppose $\mathbf{f} \circ \mathbf{i_K} = \mathbf{g} \circ \mathbf{i_K}$ for some $\mathbf{f}, \mathbf{g} \in \mathbf{A}$. We want to show $\pi(\mathbf{f}) \cdot \mathbf{P_K} = \pi(\mathbf{g}) \cdot \mathbf{P_K}$. To this end, for any $\mathbf{y}, \mathbf{z} \in \mathcal{H}$,

$$\begin{aligned} \langle \pi(\mathbf{f}) \cdot \mathbf{P_K}\, \mathbf{y}, \mathbf{z} \rangle &= \int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{K}}\, d\mu_{\mathbf{y},\mathbf{z}} \qquad \text{(Use Lemma 3.2.)} \\ &= \int_{\mathbf{X}} \mathbf{g} \cdot \chi_{\mathbf{K}}\, d\mu_{\mathbf{y},\mathbf{z}} \\ &= \langle \pi(\mathbf{g}) \cdot \mathbf{P_K}\, \mathbf{y}, \mathbf{z} \rangle. \end{aligned}$$

Thus $\pi(\mathbf{f}) \cdot \mathbf{P_K} = \pi(\mathbf{g}) \cdot \mathbf{P_K}$.

Using the property that $\mathbf{P_K}$ commutes with $\pi(\mathbf{f})$ for all $\mathbf{f} \in \mathbf{A}$, one can verify $\pi_{\mathbf{K}}$ is a unital representation that maps $\mathbf{A_K}$ into $\mathbf{B}(\mathbf{P_K}\mathcal{H})$.

To apply Corollary 3.5 we need to show $\pi_{\mathbf{K}}$ is an isometry, i.e., $\|\pi_{\mathbf{K}}(\mathbf{f} \circ \mathbf{i_K})\| = \|\mathbf{f} \circ \mathbf{i_K}\|$, or $\|\pi(\mathbf{f}) \cdot \mathbf{P_K}\| = \|\mathbf{f}\|_{\mathbf{K}}$.

Suppose this is so, we claim $\mathbf{P_K}$ is a projection of antisymmetry for $\pi(\mathbf{A})$. We see that

$$\|\mathbf{P_K}\| = \|\pi(1) \cdot \mathbf{P_K}\| = \|1\|_{\mathbf{K}} = 1,$$

hence $\mathbf{P_K} \neq \mathbf{O}$. By Theorem 2.7, $\pi(\mathbf{f}) \cdot \mathbf{P_K} = \mathbf{P_K} \cdot \pi(\mathbf{f})$ for any $\mathbf{f} \in \mathbf{A}$. Suppose $\pi(\mathbf{f}) \cdot \mathbf{P_K}$ is self-adjoint. By Corollary 3.5 we have $\pi(\mathbf{f}) \cdot \mathbf{P_K} = \lambda \cdot \mathbf{P_K}$ for some real number λ.

To show $\|\pi(\mathbf{f}) \cdot \mathbf{P_K}\| = \|\mathbf{f}\|_{\mathbf{K}}$, we first claim that $\|\pi(\mathbf{f}) \cdot \mathbf{P_K}\| \leq \|\mathbf{f}\|_{\mathbf{K}}$.

Let $\mathbf{y}, \mathbf{z} \in \mathcal{H}$ with $\|\mathbf{y}\| = \|\mathbf{z}\| = 1$, and $\mathbf{f} \in \mathbf{A}$. Then

$$\begin{aligned} |\langle \pi(\mathbf{f}) \cdot \mathbf{P_K}\, \mathbf{y}, \mathbf{z} \rangle| &= |\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{K}}\, d\mu_{\mathbf{y},\mathbf{z}}| \qquad \text{(Use Lemma 3.2.)} \\ &\leq \|\mathbf{f}\|_{\mathbf{K}} \cdot \|\mu_{\mathbf{y},\mathbf{z}}\| \\ &\leq \|\mathbf{f}\|_{\mathbf{K}} \cdot \|\mathbf{y}\| \cdot \|\mathbf{z}\| \\ &= \|\mathbf{f}\|_{\mathbf{K}}. \end{aligned}$$

Thus $\|\pi(\mathbf{f}) \cdot \mathbf{P_K}\| \leq \|\mathbf{f}\|_{\mathbf{K}}$.

To prove the other direction of inequality, fix any $\mathbf{f} \in \mathbf{A}$.

Since $\mathbf{K}$ is a maximal set of antisymmetry for $\mathbf{A}$, by ([9], Glicksberg), $\mathbf{K}$ is a generalized peak set for $\mathbf{A}$. By Theorem 1.6, there exists $\mathbf{h} \in \mathbf{A}$ such that $\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{K}$ and $\|\mathbf{h}\|_{\mathbf{X}} = \|\mathbf{f}\|_{\mathbf{K}}$.

Given any $\epsilon > 0$, by hypothesis, there exists an open set $\mathbf{V}$ with $\mathbf{V} \supseteq \mathbf{K}$ such that $|\mu_{\mathbf{y},\mathbf{z}}|(\mathbf{V} \backslash \mathbf{K}) < \epsilon$ for any $\mathbf{y}, \mathbf{z} \in \mathcal{H}$ with $\|\mathbf{y}\| = \|\mathbf{z}\| = 1$.

Write $\mathbf{K} = \bigcap\{\, \mathbf{F}_\alpha \mid \mathbf{K} \subseteq \mathbf{F}_\alpha \in \mathcal{P}_{\mathbf{A}} \,\} \subseteq \mathbf{V}$. By a standard argument about compactness and the property that a finite intersection of peak sets for $\mathbf{A}$ is a peak set for $\mathbf{A}$, one can show there exists a peak set, say $\mathbf{F}$, for $\mathbf{A}$ such that $\mathbf{K} \subseteq \mathbf{F} \subseteq \mathbf{V}$.

Let $\mathbf{g}$ peak on $\mathbf{F}$. By raising $\mathbf{g}$ to a sufficiently large power we may assume $|\mathbf{g}(\mathbf{x})| < \epsilon$ for any $\mathbf{x} \in \mathbf{X} \backslash \mathbf{V}$.

Since π is an isometry, we can find $\mathbf{y}_0, \mathbf{z}_0 \in \mathcal{H}$ with $\|\mathbf{y}_0\| = \|\mathbf{z}_0\| = 1$ such that

$$\begin{aligned} |\int_{\mathbf{X}} \mathbf{g} \cdot \mathbf{h} \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| &= |\langle \pi(\mathbf{g} \cdot \mathbf{h}) \, \mathbf{y}_0, \mathbf{z}_0 \rangle| \\ &\geq \|\mathbf{g} \cdot \mathbf{h}\|_{\mathbf{X}} - \epsilon \\ &= \|\mathbf{h}\|_{\mathbf{X}} - \epsilon \\ &= \|\mathbf{f}\|_{\mathbf{K}} - \epsilon. \end{aligned} \tag{1}$$

Thus

$$\begin{aligned} &|\langle \pi(\mathbf{f}) \cdot \mathbf{P}_{\mathbf{K}} \, \mathbf{y}_0, \mathbf{z}_0 \rangle| = |\int_{\mathbf{X}} \mathbf{f} \cdot \chi_{\mathbf{K}} \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| \qquad \text{(Use Lemma 3.2.)} \\ &= |\int_{\mathbf{X}} \mathbf{g} \cdot \mathbf{h} \cdot \chi_{\mathbf{K}} \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| \\ &\geq |\int_{\mathbf{X}} \mathbf{g} \cdot \mathbf{h} \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| - |\int_{\mathbf{X}} (\mathbf{g} \cdot \mathbf{h} - \mathbf{g} \cdot \mathbf{h} \cdot \chi_{\mathbf{K}}) \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| \\ &= |\int_{\mathbf{X}} \mathbf{g} \cdot \mathbf{h} \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| - |\int_{\mathbf{X}} (\mathbf{g} \cdot \mathbf{h} - \mathbf{f} \cdot \chi_{\mathbf{K}}) \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| \\ &\geq |\int_{\mathbf{X}} \mathbf{g} \cdot \mathbf{h} \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| - |\int_{\mathbf{V}} (\mathbf{g} \cdot \mathbf{h} - \mathbf{f} \cdot \chi_{\mathbf{K}}) \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| - |\int_{\mathbf{X}\backslash\mathbf{V}} (\mathbf{g} \cdot \mathbf{h} - \mathbf{f} \cdot \chi_{\mathbf{K}}) \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| \\ &= |\int_{\mathbf{X}} \mathbf{g} \cdot \mathbf{h} \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| - |\int_{\mathbf{V}\backslash\mathbf{K}} \mathbf{g} \cdot \mathbf{h} \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| - |\int_{\mathbf{X}\backslash\mathbf{V}} \mathbf{g} \cdot \mathbf{h} \, d\mu_{\mathbf{y}_0,\mathbf{z}_0}| \\ &\geq (\|\mathbf{f}\|_{\mathbf{K}} - \epsilon) - \|\mathbf{g} \cdot \mathbf{h}\|_{\mathbf{X}} \cdot |\mu_{\mathbf{y}_0,\mathbf{z}_0}|(\mathbf{V}\backslash\mathbf{K}) - \|\mathbf{g} \cdot \mathbf{h}\|_{\mathbf{X}\backslash\mathbf{V}} \cdot |\mu_{\mathbf{y}_0,\mathbf{z}_0}|(\mathbf{X}\backslash\mathbf{V}) \\ &\geq (\|\mathbf{f}\|_{\mathbf{K}} - \epsilon) - \|\mathbf{f}\|_{\mathbf{K}} \cdot \epsilon - \epsilon \cdot \|\mathbf{h}\|_{\mathbf{X}} \cdot \|\mu_{\mathbf{y}_0,\mathbf{z}_0}\| \qquad \text{(Use (1).)} \\ &\geq (\|\mathbf{f}\|_{\mathbf{K}} - \epsilon) - \|\mathbf{f}\|_{\mathbf{K}} \cdot \epsilon - \epsilon \cdot \|\mathbf{f}\|_{\mathbf{K}} \\ &= \|\mathbf{f}\|_{\mathbf{K}}(1 - 2\epsilon) - \epsilon. \end{aligned}$$

Since ϵ was arbitrary, $\|\pi(\mathbf{f}) \cdot \mathbf{P}_{\mathbf{K}}\| \geq \|\mathbf{f}\|_{\mathbf{K}}$ for all $\mathbf{f} \in \mathbf{A}$. Hence $\pi_{\mathbf{K}}$ is an isometry.

■■

Remark: Suppose $\mathbf{G} \in (\mathcal{GP})_{\mathbf{A}}$ satisfies the regularity property in Theorem 3.6, i.e., for any $\epsilon > 0$ there exists an open set $\mathbf{V}$ with $\mathbf{G} \subseteq \mathbf{V} \subseteq \mathbf{X}$ such that

$$|\mu_{\mathbf{y},\mathbf{z}}|(\mathbf{V}\backslash\mathbf{G}) < \epsilon \text{ for all } \mathbf{y}, \mathbf{z} \in \mathcal{H} \text{ with } \|\mathbf{y}\| = \|\mathbf{z}\| = 1,$$

where $\mu_{\mathbf{y},\mathbf{z}}$ is an elementary measure for $(\mathbf{y}, \mathbf{z})$.

In the proof of Theorem 3.6 we actually showed that $\|\pi(\mathbf{f}) \cdot \mathbf{P_G}\| = \|\mathbf{f}\|_{\mathbf{G}}$ for all $\mathbf{f} \in \mathbf{A}$. In particular, $\mathbf{P_G} \neq \mathbf{O}$.

The following example taken from ([22], Szymanski) shows the regularity hypothesis in Theorem 3.6 is indispensable.

3.7 Example. Let

(a) $\Gamma \equiv \{\, \mathbf{c} \in \mathbb{C} : |\mathbf{c}| = 1 \,\}$, i.e., Γ denotes the unit circle,

(b) $\{\, \mathbf{c}_n \mid n = 1, 2, 3, \ldots \}$ be a bounded sequence of complex numbers with $|\mathbf{c}_n| > 1$ for each n and such that Γ is the cluster points of $\{\, \mathbf{c}_n \mid n = 1, 2, 3, \ldots \}$,

(c) $\mathbf{K}$ denote the real number interval $[0, \frac{1}{2}]$, and

(d) $\mathbf{X} \equiv \Gamma \cup \{\, \mathbf{c}_n \mid n = 1, 2, 3, \ldots \} \cup \mathbf{K}$.

Let $\mathbf{P}(\mathbf{X}) \equiv$ the uniform closure of all polynomials in $\mathbf{C}(\mathbf{X})$. Denote the point mass at $\mathbf{c}_n$ and the linear Lebesgue measure restricted to $\mathbf{K}$ by δ_n and $m_{\mathbf{K}}$, respectively. Define $\mu \equiv m_{\mathbf{K}} + \sum_{n=1}^{\infty} 2^{-n}\delta_n$. It is clear μ is a positive, finite Borel measure on $\mathbf{X}$, and $\mu(\Gamma) = 0$.

Denote the collection of square integrable functions on the measure μ by $\mathbf{L}^2(\mu)$. Define $\pi : \mathbf{P}(\mathbf{X}) \to \mathbf{B}(\mathbf{L}^2(\mu))$ by $\pi(\mathbf{f}) = \mathbf{M_f}$ for any $\mathbf{f} \in \mathbf{P}(\mathbf{X})$, where $\mathbf{M_f}(\mathbf{y}) = \mathbf{f} \cdot \mathbf{y}$ for any $\mathbf{y} \in \mathbf{L}^2(\mu)$.

One can check :

(i) The mapping π is a unital representation with $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ for all $\mathbf{f} \in \mathbf{P}(\mathbf{X})$;

(ii) For any $\mathbf{y}, \mathbf{z} \in \mathbf{L}^2(\mu)$, the measure $\mathbf{y} \cdot \bar{\mathbf{z}}\, d\mu$ is an elementary measure for $(\mathbf{y}, \mathbf{z})$;

(iii) The set $\Gamma \cup \mathbf{K}$ is a maximal set of antisymmetry for $\mathbf{P}(\mathbf{X})$; and

(iv) For any open set $\mathbf{V}$ with $\mathbf{X} \supseteq \mathbf{V} \supseteq (\Gamma \cup \mathbf{K})$, there exists a $\mathbf{c}_n \in \mathbf{V} \backslash (\Gamma \cup \mathbf{K})$. Let

$\mathbf{y}_n \equiv \sqrt{2^n} \cdot \chi_{\{\mathbf{c}_n\}}$. Then $\mathbf{y}_n \in \mathbf{L}^2(\mu)$, and $\|\mathbf{y}_n\| = 1$, and yet $|\mu_{\mathbf{y}_n,\mathbf{y}_n}|[\mathbf{V}\backslash(\Gamma \cup \mathbf{K})] = 1$. Hence the regularity hypothesis in Theorem 3.6 is not satisfied.

In the following we show $\mathbf{P}_{\Gamma\cup\mathbf{K}}$ is not a projection of antisymmetry for $\pi(\mathbf{P}(\mathbf{X}))$.

PROOF: The function $\mathbf{i} : \mathbf{X} \to \mathbf{X}$ defined by $\mathbf{i}(\mathbf{x}) = \mathbf{x}$ is in $\mathbf{P}(\mathbf{X})$. For any $\mathbf{y} \in \mathbf{L}^2(\mu)$,

$$\begin{aligned}
\langle \pi(\mathbf{i}) \cdot \mathbf{P}_{\Gamma\cup\mathbf{K}}\, \mathbf{y}, \mathbf{y} \rangle &= \int_{\mathbf{X}} \mathbf{i} \cdot \chi_{\Gamma\cup\mathbf{K}}\, d\mu_{\mathbf{y},\mathbf{y}} \\
&= \int_{\mathbf{X}} \mathbf{i} \cdot \chi_{\Gamma\cup\mathbf{K}} \cdot \mathbf{y} \cdot \bar{\mathbf{y}}\, d\mu \\
&= \int_{\Gamma\cup\mathbf{K}} \mathbf{i} \cdot \mathbf{y} \cdot \bar{\mathbf{y}}\, d\mu \\
&= \int_{\mathbf{K}} \mathbf{i} \cdot \mathbf{y} \cdot \bar{\mathbf{y}}\, d\mu.
\end{aligned}$$

Since $\mathbf{i}$ is real-valued on $\mathbf{K}$ and $\mathbf{y} \cdot \bar{\mathbf{y}} \geq 0$, the operator $\pi(\mathbf{i}) \cdot \mathbf{P}_{\Gamma\cup\mathbf{K}}$ is self-adjoint.

If $\mathbf{P}_{\Gamma\cup\mathbf{K}}$ is a projection of antisymmetry for $\pi(\mathbf{P}(\mathbf{X}))$, then $\pi(\mathbf{i}) \cdot \mathbf{P}_{\Gamma\cup\mathbf{K}} = \lambda \cdot \mathbf{P}_{\Gamma\cup\mathbf{K}}$ for some real number λ. This is absurd, because for any real number λ, we can find $\mathbf{y} \in \mathbf{P}_{\Gamma\cup\mathbf{K}}(\mathbf{L}^2(\mu))$ with $\|\mathbf{y}\| = 1$ such that $\int_{\mathbf{K}} \mathbf{i} \cdot \mathbf{y} \cdot \bar{\mathbf{y}}\, d\mu \neq \lambda$. ■■

3.8 Theorem. *Let* $\mathbf{A}$ *be a uniform algebra on* $\mathbf{X}$ *and* π *a unital representation mapping* $\mathbf{A}$ *into* $\mathbf{B}(\mathcal{H})$ *with* $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ *for all* $\mathbf{f} \in \mathbf{A}$. *Suppose also for any* $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$, *and* $\epsilon > 0$ *there exists an open set* $\mathbf{V}$ *with* $\mathbf{K} \subseteq \mathbf{V} \subseteq \mathbf{X}$ *such that*

$$|\mu_{\mathbf{y},\mathbf{z}}|(\mathbf{V}\backslash\mathbf{K}) < \epsilon \text{ for all } \mathbf{y}, \mathbf{z} \in \mathcal{H} \text{ with } \|\mathbf{y}\| = \|\mathbf{z}\| = 1,$$

where $\mu_{\mathbf{y},\mathbf{z}}$ *is an elementary measure for* $(\mathbf{y}, \mathbf{z})$.

If $\mathbf{P}$ *is a maximal projection of antisymmetry for* $\pi(\mathbf{A})$, *then* $\mathbf{P} = \mathbf{P}_{\mathbf{K}}$ *for some* $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$.

PROOF: Let $\mathbf{Z} \equiv \bigcup\{$ support of $\mu_{\mathbf{P}\,\mathbf{y},\mathbf{P}\,\mathbf{y}} \mid \mathbf{y} \in \mathcal{H}$ and $\mu_{\mathbf{P}\,\mathbf{y},\mathbf{P}\,\mathbf{y}}$ is any elementary measure for $(\mathbf{P}\,\mathbf{y}, \mathbf{P}\,\mathbf{y})\}$.

We first claim $\mathbf{Z}$ is a set of antisymmetry for $\mathbf{A}$.

PROOF OF CLAIM: Let $\mathbf{f} \in \mathbf{A}$ be real-valued on $\mathbf{Z}$. Then for any $\mathbf{y} \in \mathcal{H}$,

$$\langle \pi(\mathbf{f}) \cdot \mathbf{P}\,\mathbf{y}, \mathbf{y} \rangle = \langle \pi(\mathbf{f}) \cdot \mathbf{P}\,\mathbf{y}, \mathbf{P}\,\mathbf{y} \rangle = \int_{\mathbf{X}} \mathbf{f}\, d\mu_{\mathbf{P}\,\mathbf{y},\mathbf{P}\,\mathbf{y}}.$$

Since $\mathbf{f}$ is real-valued on support of $\mu_{\mathbf{P}\,\mathbf{y},\mathbf{P}\,\mathbf{y}}$, and $\mu_{\mathbf{P}\,\mathbf{y},\mathbf{P}\,\mathbf{y}} \geq 0$, we conclude that $\langle \pi(\mathbf{f}) \cdot \mathbf{P}\,\mathbf{y}, \mathbf{y} \rangle$ is real-valued for all $\mathbf{y} \in \mathcal{H}$. Therefore, $\pi(\mathbf{f}) \cdot \mathbf{P}$ is self-adjoint.

By the definition of projection of antisymmetry, we have $\pi(\mathbf{f}) \cdot \mathbf{P} = \lambda \cdot \mathbf{P}$ for some real number λ. Thus $\mathbf{O} = [\pi(\mathbf{f}) - \lambda] \cdot \mathbf{P} = \pi(\mathbf{f} - \lambda) \cdot \mathbf{P}$. Since $\pi(\mathbf{f} - \lambda) \cdot \mathbf{P} = \mathbf{P} \cdot \pi(\mathbf{f} - \lambda)$, we see

$$\mathbf{P} \cdot \pi(\mathbf{f} - \lambda) \cdot \mathbf{P} = \pi(\mathbf{f} - \lambda) \cdot \mathbf{P}^2 = \pi(\mathbf{f} - \lambda) \cdot \mathbf{P}. \tag{2}$$

Thus

$$\begin{aligned}
\mathbf{O} &= [\pi(\mathbf{f} - \lambda) \cdot \mathbf{P}] \cdot [\pi(\mathbf{f} - \lambda) \cdot \mathbf{P}] \\
&= \pi(\mathbf{f} - \lambda) \cdot [\mathbf{P} \cdot \pi(\mathbf{f} - \lambda) \cdot \mathbf{P}] \\
&= \pi(\mathbf{f} - \lambda) \cdot [\pi(\mathbf{f} - \lambda) \cdot \mathbf{P}] && \text{(Use (2).)} \\
&= \pi[(\mathbf{f} - \lambda)^2] \cdot \mathbf{P}.
\end{aligned}$$

Therefore, for any $\mathbf{y} \in \mathcal{H}$,

$$0 = \langle \pi[(\mathbf{f} - \lambda)^2] \cdot \mathbf{P}\,\mathbf{y}, \mathbf{y} \rangle = \langle \pi[(\mathbf{f} - \lambda)^2] \cdot \mathbf{P}\,\mathbf{y}, \mathbf{P}\,\mathbf{y} \rangle = \int_{\mathbf{X}} (\mathbf{f} - \lambda)^2\, d\mu_{\mathbf{P}\,\mathbf{y},\mathbf{P}\,\mathbf{y}}.$$

Since $(\mathbf{f} - \lambda)^2 \geq 0$, and $\mu_{\mathbf{P}\,\mathbf{y},\mathbf{P}\,\mathbf{y}} \geq 0$, this implies $\mathbf{f} - \lambda = 0$ on support of $\mu_{\mathbf{P}\,\mathbf{y},\mathbf{P}\,\mathbf{y}}$. Hence $\mathbf{f} = \lambda$ on support of $\mu_{\mathbf{P}\,\mathbf{y},\mathbf{P}\,\mathbf{y}}$, for any $\mathbf{y} \in \mathcal{H}$. Thus $\mathbf{Z}$ is a set of antisymmetry for $\mathbf{A}$. ■

We return to the proof of the theorem. Let $\mathbf{K}$ be the maximal set of antisymmetry

for $\mathbf{A}$ that contains $\mathbf{Z}$. Then for any $\mathbf{y} \in \mathbf{P}\mathcal{H}$, we have

$$\begin{aligned}
\langle \mathbf{P_K}\,\mathbf{y}, \mathbf{y}\rangle &= \int_{\mathbf{X}} \chi_{\mathbf{K}}\, d\mu_{\mathbf{y},\mathbf{y}} \\
&= \int_{\mathbf{X}} \chi_{\{\text{ support of } \mu_{\mathbf{y},\mathbf{y}}\ \}}\, d\mu_{\mathbf{y},\mathbf{y}} \\
&= \|\mu_{\mathbf{y},\mathbf{y}}\| \\
&= \langle \mathbf{y}, \mathbf{y}\rangle \\
&= \langle \mathbf{P}\,\mathbf{y}, \mathbf{y}\rangle.
\end{aligned}$$

This proves $\mathbf{P_K} \geq \mathbf{P}$.

We also know $\mathbf{P_K}$ is a projection of antisymmetry for $\pi(\mathbf{A})$ from Theorem 3.6. Thus $\mathbf{P_K} = \mathbf{P}$, by the fact $\mathbf{P}$ is a maximal projection of antisymmetry for $\pi(\mathbf{A})$. ■■

3.9 Corollary. *Let* $\mathbf{A}$ *be a uniform algebra on* $\mathbf{X}$ *and* π *a unital representation mapping* $\mathbf{A}$ *into* $\mathbf{B}(\mathcal{H})$ *with* $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ *for all* $\mathbf{f} \in \mathbf{A}$. *Suppose also for any* $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$, *and* $\epsilon > 0$ *there exists an open set* $\mathbf{V}$ *with* $\mathbf{K} \subseteq \mathbf{V} \subseteq \mathbf{X}$ *such that*

$$|\mu_{\mathbf{y},\mathbf{z}}|(\mathbf{V}\backslash\mathbf{K}) < \epsilon \text{ for all } \mathbf{y}, \mathbf{z} \in \mathcal{H} \text{ with } \|\mathbf{y}\| = \|\mathbf{z}\| = 1,$$

where $\mu_{\mathbf{y},\mathbf{z}}$ *is an elementary measure for* $(\mathbf{y}, \mathbf{z})$. *Then there is a one-to-one correspondence between* $\mathcal{K}_{\mathbf{A}}$ *and the collection of maximal projections of antisymmetry for* $\pi(\mathbf{A})$.

PROOF: Define $\phi : \mathcal{K}_{\mathbf{A}} \to \mathbf{B}(\mathcal{H})$ by $\phi(\mathbf{K}) = \mathbf{P_K}$ for all $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$.

Let $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$. Using Theorem 3.6, we see that $\mathbf{P_K}$ is a projection of antisymmetry for $\pi(\mathbf{A})$. Hence $\mathbf{P_K} \leq \mathbf{P}$, where $\mathbf{P}$ is a maximal projection of antisymmetry for $\pi(\mathbf{A})$. By Theorem 3.8, we have $\mathbf{P} = \mathbf{P_{K_0}}$ for some $\mathbf{K_0} \in \mathcal{K}_{\mathbf{A}}$.

Since $\mathcal{K}_{\mathbf{A}}$ forms a pairwise disjoint partition of $\mathbf{X}$, either $\mathbf{K} \cap \mathbf{K_0} = \emptyset$, or $\mathbf{K} = \mathbf{K_0}$. If $\mathbf{K} \cap \mathbf{K_0} = \emptyset$, then, by Remark (b) after the proof of Corollary 3.3, $\mathbf{P_K} = \mathbf{P_K} \cdot \mathbf{P_{K_0}} =$

$\mathbf{O}$. A contradiction to the fact $\mathbf{P_K} \neq \mathbf{O}$. Thus $\mathbf{K} = \mathbf{K_0}$. Therefore, $\phi(\mathbf{K}) = \mathbf{P_K}$ is a maximal projection of antisymmetry for $\pi(\mathbf{A})$ for all $\mathbf{K} \in \mathcal{K}_\mathbf{A}$. Hence ϕ maps $\mathcal{K}_\mathbf{A}$ into the collection of maximal projections of antisymmetry for $\pi(\mathbf{A})$.

By Theorem 3.8, every maximal projection of antisymmetry for $\pi(\mathbf{A})$ is $\mathbf{P_K}$ for some $\mathbf{K} \in \mathcal{K}_\mathbf{A}$. Thus, ϕ maps $\mathcal{K}_\mathbf{A}$ onto the collection of maximal projections of antisymmetry for $\pi(\mathbf{A})$.

To complete the proof it remains to show ϕ is one-to-one. If $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{K}_\mathbf{A}$ and $\mathbf{K}_1 \neq \mathbf{K}_2$, then $\mathbf{K}_1 \cap \mathbf{K}_2 = \emptyset$. Hence, $\mathbf{P_{K_1}} \cdot \mathbf{P_{K_2}} = \mathbf{O}$. Since $\mathbf{P_{K_1}} \neq \mathbf{O}$ and $\mathbf{P_{K_2}} \neq \mathbf{O}$, $\mathbf{P_{K_1}} \neq \mathbf{P_{K_2}}$. This proves ϕ is one-to-one. ■■

The following corollary strengthens the result of Theorem 3.6. Note in the following we do not assume that every maximal set of antisymmetry for $\mathbf{A}$ satisfies the regularity property.

3.10 Corollary. *Let π be a unital representation mapping a uniform algebra $\mathbf{A} \subseteq \mathbf{C}(\mathbf{X})$ into $\mathbf{B}(\mathcal{H})$ with $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ for all $\mathbf{f} \in \mathbf{A}$. Let $\mathbf{K}$ be a maximal set of antisymmetry for $\mathbf{A}$ such that for any $\epsilon > 0$ there exists an open set $\mathbf{V}$ with $\mathbf{K} \subseteq \mathbf{V} \subseteq \mathbf{X}$ such that*

$$|\mu_{\mathbf{y},\mathbf{z}}|(\mathbf{V} \backslash \mathbf{K}) < \epsilon \text{ for all } \mathbf{y}, \mathbf{z} \in \mathcal{H} \text{ with } \|\mathbf{y}\| = \|\mathbf{z}\| = 1,$$

where $\mu_{\mathbf{y},\mathbf{z}}$ is an elementary measure for $(\mathbf{y}, \mathbf{z})$. Then $\mathbf{P_K}$ is a maximal projection of antisymmetry for $\pi(\mathbf{A})$.

PROOF: By Theorem 3.6, the projection $\mathbf{P_K}$ is a projection of antisymmetry for $\pi(\mathbf{A})$. Hence $\mathbf{P_K} \leq \mathbf{P}$, where $\mathbf{P}$ is a maximal projection of antisymmetry for $\pi(\mathbf{A})$.

By following the proof of Theorem 3.8, we see that there is $\mathbf{K_0} \in \mathcal{K}_\mathbf{A}$ such that $\mathbf{P} \leq \mathbf{P_{K_0}}$. Thus $\mathbf{P_K} \leq \mathbf{P_{K_0}}$. Since $\mathcal{K}_\mathbf{A}$ forms a pairwise disjoint partition of $\mathbf{X}$,

$\mathbf{K} \cap \mathbf{K}_0 = \emptyset$, or $\mathbf{K} = \mathbf{K}_0$.

If $\mathbf{K} \cap \mathbf{K}_0 = \emptyset$, then, by Remark (b) after the proof of Corollary 3.3, $\mathbf{O} = \mathbf{P}_{\mathbf{K}} \cdot \mathbf{P}_{\mathbf{K}_0} = \mathbf{P}_{\mathbf{K}}$. A contradiction to the fact that $\mathbf{P}_{\mathbf{K}}$ is a projection of antisymmetry for $\pi(\mathbf{A})$.

Thus $\mathbf{K} = \mathbf{K}_0$, and therefore, $\mathbf{P}_{\mathbf{K}} = \mathbf{P}_{\mathbf{K}_0} = \mathbf{P}$. This completes the proof that $\mathbf{P}_{\mathbf{K}}$ is a maximal projection of antisymmetry for $\pi(\mathbf{A})$. ■■

In the next example we show that the regularity hypothesis is not necessary in Corollary 3.9.

3.11 Example. Let $\Gamma, \{\, \mathbf{c}_n \mid n = 1, 2, 3, \ldots \}$, and δ_n be as defined in Example 3.7. Let $\mathbf{X} \equiv \Gamma \cup \{\, \mathbf{c}_n \mid n = 1, 2, 3, \ldots \}$.

Denote the uniform closure of all polynomials in $\mathbf{C}(\mathbf{X})$ by $\mathbf{P}(\mathbf{X})$. For each $n = 1, 2, 3, \ldots$, the singleton $\{\mathbf{c}_n\} \in \mathcal{K}_{\mathbf{P}(\mathbf{X})}$ by Runge's theorem. To claim that $\mathcal{K}_{\mathbf{P}(\mathbf{X})} = \{\, \{\mathbf{c}_n\} \mid n = 1, 2, 3, \ldots \} \cup \{\Gamma\}$, it remains to show that Γ is a set of antisymmetry for $\mathbf{P}(\mathbf{X})$.

To this end, let $\mathbf{p} \in \mathbf{P}(\mathbf{X})$ with $\mathbf{p}$ real-valued on Γ. The function $\mathbf{p}$ restricted to Γ is in the uniform closure of all polynomials in $\mathbf{C}(\Gamma)$, or the disk algebra. That is, $\mathbf{p}$ restricted to Γ is in the disk algebra and real-valued. This implies that for all $\mathbf{x} \in \Gamma$, $\mathbf{p}(\mathbf{x}) = \lambda$ for some real number λ, by the fact that Γ is the maximal set of antisymmetry for the disk algebra.

Denote the normalized Lebesgue measure on Γ by m. Let $\mu \equiv m + \sum_{n=1}^{\infty} 2^{-n} \delta_n$. Then μ is a positive, finite Borel measure on $\mathbf{X}$. As in Example 3.7, regularity hypothesis is not satisfied by Γ. Define $\pi : \mathbf{C}(\mathbf{X}) \to \mathbf{B}(\mathbf{L}^2(\mu))$ as in Example 3.7.

For each $n = 1, 2, 3, \ldots$, the singleton $\{\mathbf{c}_n\}$ satisfies the regularity hypothesis. By Corollary 3.10, the projection $\mathbf{P}_{\{\mathbf{c}_n\}}$ is a maximal projection of antisymmetry for

$\pi(\mathbf{P}(\mathbf{X}))$.

Next we show $\mathbf{P}_\Gamma$ is also a maximal projection of antisymmetry for $\pi(\mathbf{P}(\mathbf{X}))$. For any $\mathbf{f} \in \mathbf{P}(\mathbf{X})$,

$$\begin{aligned}\|\pi(\mathbf{f}) \cdot \mathbf{P}_\Gamma\| &= \sup \{ |\langle \pi(\mathbf{f})\, \mathbf{y}, \mathbf{z}\rangle| : \mathbf{y}, \mathbf{z} \in \mathbf{L}^2(m), \text{ and } \|\mathbf{y}\| = \|\mathbf{z}\| = 1. \} \\ &= \sup \{ |\int_\Gamma \mathbf{f} \cdot \mathbf{y} \cdot \bar{\mathbf{z}}\, dm| : \mathbf{y}, \mathbf{z} \in \mathbf{L}^2(m), \text{ and } \|\mathbf{y}\| = \|\mathbf{z}\| = 1. \} \\ &= \|\mathbf{f}\|_\Gamma.\end{aligned}$$

This implies the mapping $\pi_\Gamma : \mathbf{P}(\mathbf{X})_\Gamma \to \mathbf{B}(\mathbf{L}^2(m))$ defined by $\pi_\Gamma(\mathbf{f} \circ \mathbf{i}_\Gamma) = \pi(\mathbf{f}) \cdot \mathbf{P}_\Gamma$ is an isometry. We conclude that $\mathbf{P}_\Gamma$ is a projection of antisymmetry for $\pi(\mathbf{P}(\mathbf{X}))$ by following the proof of Theorem 3.6.

There exists some maximal projection of antisymmetry for $\pi(\mathbf{P}(\mathbf{X}))$, say $\mathbf{P}$, such that $\mathbf{P}_\Gamma \leq \mathbf{P}$. Following the proof of Theorem 3.8 we can find $\mathbf{K} \in \mathcal{K}_{\mathbf{P}(\mathbf{X})}$ such that $\mathbf{P} \leq \mathbf{P}_\mathbf{K}$. But $\mathbf{P}_\Gamma \leq \mathbf{P} \leq \mathbf{P}_\mathbf{K}$, thus $\Gamma = \mathbf{K}$. For otherwise $\mathbf{P}_\Gamma = \mathbf{P}_\Gamma \cdot \mathbf{P}_\mathbf{K} = \mathbf{O}$, a contradiction to the fact $\mathbf{P}_\Gamma \neq \mathbf{O}$.

Hence π is a unital representation with $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ for all $\mathbf{f} \in \mathbf{P}(\mathbf{X})$. There exists a one-to-one correspondence between $\mathcal{K}_{\mathbf{P}(\mathbf{X})}$ and the collection of all maximal projections of antisymmetry for $\pi(\mathbf{P}(\mathbf{X}))$. But $\Gamma \in \mathcal{K}_{\mathbf{P}(\mathbf{X})}$ does not satisfy the regularity hypothesis.

Suppose $\mathbf{A}$ is a uniform algebra on $\mathbf{X}$ and $\mathbf{f} \in \mathbf{C}(\mathbf{X})$ with $\mathbf{f} \circ \mathbf{i}_\mathbf{K} \in \mathbf{A}_\mathbf{K}$, where $\mathbf{i}_\mathbf{K}$ and $\mathbf{A}_\mathbf{K}$ as defined in the proof of Theorem 3.6, for each $\mathbf{K} \in \mathcal{K}_\mathbf{A}$. Then, in ([9], Glicksberg), it is shown that $\mathbf{f} \in \mathbf{A}$. We have a similar result on $\pi(\mathbf{A})$.

3.12 Theorem. *Let* $\mathbf{A}$ *be a uniform algebra on* $\mathbf{X}$ *and* π *a unital representation mapping* $\mathbf{A}$ *into* $\mathbf{B}(\mathcal{H})$ *with* $\|\pi(\mathbf{f})\| = \|\mathbf{f}\|$ *for all* $\mathbf{f} \in \mathbf{A}$. *Suppose also* $\mathbf{K}$ *is both open and closed for each* $\mathbf{K} \in \mathcal{K}_\mathbf{A}$. *If* $\mathbf{T} \in \mathbf{B}(\mathcal{H})$ *with* $\mathbf{T} \cdot \mathbf{P}_\mathbf{K} \in \pi(\mathbf{A}) \cdot \mathbf{P}_\mathbf{K}$ *for each*

$\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$, *then* $\mathbf{T} = \pi(\mathbf{f})$ *for some* $\mathbf{f} \in \mathbf{A}$.

Notes:

(a) Since each $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$ is open, the hypotheses of Theorem 3.6 are satisfied.

(b) Since $\mathbf{X}$ is compact, each $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$ is open is equivalent to $\mathcal{K}_{\mathbf{A}}$ is a finite collection.

PROOF: Suppose $\mathbf{T} \cdot \mathbf{P_K} = \pi(\mathbf{f_K}) \cdot \mathbf{P_K}$ for some $\mathbf{f_K} \in \mathbf{A}$. Define

$$\mathbf{f} \equiv \sum_{\mathbf{K} \in \mathcal{K}_{\mathbf{A}}} \mathbf{f_K} \cdot \chi_{\mathbf{K}}.$$

The function $\mathbf{f}$ is well-defined since the sets in $\mathcal{K}_{\mathbf{A}}$ are pairwise disjoint.

Furthermore, $\mathbf{f}$ is continuous since each $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$ is an open set, and for any open set $\mathbf{V} \subseteq \mathbb{C}$,

$$\mathbf{f}^{-1}(\mathbf{V}) = \bigcup \{ \mathbf{K} \cap \mathbf{f_K}^{-1}(\mathbf{V}) \mid \mathbf{K} \in \mathcal{K}_{\mathbf{A}} \}.$$

By the fact mentioned before the statement of the theorem, $\mathbf{f} \in \mathbf{A}$ since $\mathbf{f} \circ \mathbf{i_K} = \mathbf{f_K} \circ \mathbf{i_K}$ for all $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$.

Also for each $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$, the representation $\pi_{\mathbf{K}} : \mathbf{A_K} \to \mathbf{B}(\mathbf{P_K}\mathcal{H})$ defined by $\pi_{\mathbf{K}}(\mathbf{g} \circ \mathbf{i_K}) = \pi(\mathbf{g}) \cdot \mathbf{P_K}$ for each $\mathbf{g} \in \mathbf{A}$ is well-defined as shown in the proof of Theorem 3.6.

For any $\mathbf{K} \in \mathcal{K}_{\mathbf{A}}$, $\mathbf{f} \circ \mathbf{i_K} = \mathbf{f_K} \circ \mathbf{i_K}$, so we have $\pi_{\mathbf{K}}(\mathbf{f} \circ \mathbf{i_K}) = \pi_{\mathbf{K}}(\mathbf{f_K} \circ \mathbf{i_K})$, or $\pi(\mathbf{f}) \cdot \mathbf{P_K} = \pi(\mathbf{f_K}) \cdot \mathbf{P_K}$.

Hence,

$$\begin{aligned}
\mathbf{T} = \sum_{\mathbf{K} \in \mathcal{K}_{\mathbf{A}}} \mathbf{T} \cdot \mathbf{P_K} &= \sum_{\mathbf{K} \in \mathcal{K}_{\mathbf{A}}} \pi(\mathbf{f_K}) \cdot \mathbf{P_K} \\
&= \sum_{\mathbf{K} \in \mathcal{K}_{\mathbf{A}}} \pi(\mathbf{f}) \cdot \mathbf{P_K} \\
&= \pi(\mathbf{f}) \cdot \sum_{\mathbf{K} \in \mathcal{K}_{\mathbf{A}}} \mathbf{P_K} \\
&= \pi(\mathbf{f}),
\end{aligned}$$

and the proof is complete. ■■

REFERENCES

[1] Sterling K. Berberian, *Notes on Spectral Theory*, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1966.

[2] John B. Conway, *A Course in Functional Analysis*, Springer-Verlag New York Inc., New York, New York, 1985.

[3] –, *Functions of One Complex Variable*, Springer-Verlag New York Inc., New York, New York, 1978.

[4] –, *Subnormal Operators*, Pitman Books Limited, London, 1981.

[5] Louis deBranges, The Stone-Weierstrass theorem, *Proc. Amer. Math. Soc.*, **10** (1959), p. 822–824.

[6] Ronald G. Douglas and Vern I. Paulsen, *Hilbert modules over function algebra*, Longman Scientific & Technical, Harlow, Essex, England, 1989.

[7] Theodore W. Gamelin, *Rational Approximation Theory*, course notes.

[8] –, *Uniform Algebras*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1969.

[9] Irving Glicksberg, Measures orthogonal to algebras and sets of antisymmetry, *Trans. Amer. Math. Soc.*, **105** (1962) p. 415–435.

[10] Paul R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*, Chelsea, New York, 1951.

[11] Edwin Hewitt and Karl Stromberg, *Real and Abstract Analysis*, Springer-Verlag Berlin Heidelberg, 1965.

[12] Roger G. Lautzenheiser, Spectral sets, reducing subspaces, and function algebras, Ph.D. thesis, Indiana University, 1973.

[13] –, Spectral theory for subnormal operators, *Trans. Amer. Math. Soc.* **255**, p. 301–314.

[14] Gerald M. Leibowitz, *Lectures on Complex Function Algebras*, Scott, Foresman and Company, Glenview, Illinois, 1970.

[15] M. S. Melnikov, On the Gleason parts of the algebra R(X), (Russian) *Mat. Sb.* 101 (134), No.2 (10), 1976, p. 293–300.

[16] W. Mlak, Decomposition of operator-valued representations of function algebras, *Studia Math.* **36** (1970), p. 111–123.

[17] Vern I. Paulsen, *Completely bounded maps and dilations*, Longman Scientific & Technical, Harlow, Essex, England, 1986.

[18] Walter Rudin, *Real and Complex Analysis*, McGraw-Hill, Inc., New York, 1966.

[19] Donald Sarason, On spectral sets having connected complement, *Acta Sci. Math.* (Szeged), **26** (1965), p. 289–299.

[20] W. F. Stinespring, Positive functions on C^*-algebras, *Proc. Amer. Math. Soc.* **6** (1955), p. 211–216.

[21] Waclaw Szymanski, Antisymmetric operator algebras, I, *Annales Pol. Math.* **37** (1980), p. 263–274.

[22] –, Antisymmetric operator algebras, II, *Annales Pol. Math.* **37** (1980), p. 299–311.

[23] Angus E. Taylor, *General Theory of Functions and Integration*, Dover Publications, Inc., Mineola, New York, 1985.

[24] Stephen Willard, *General Topology*, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1970.

VITAE

Jiann-Shiun Huang was born April 19, 1954 in Taipei, Taiwan, Republic of China. In June, 1976 he received a Bachelor of Science degree in Industrial Engineering from Tunghai University, Taichung, Taiwan. He was employed as a manufacturing supervisor, a process engineer, and a computer programmer in Taoyuan, Taiwan from 1977 through 1983.

In August, 1983 the author entered the Graduate School of Virginia Polytechnic Institute and State University. He received a Master of Science degree in Statistics in December, 1984. In June, 1985 he began to pursue a Doctor of Philosophy degree in Mathematics, and finished the requirement of his Ph.D. degree in June, 1991.

Jiann-Shiun Huang

Jiann-Shiun Huang