Chapter 2

Conservation Equations

2.1 Introduction

The objective of the two-phase flow model is the determination of unknown velocities and temperatures for each phase as well as the pressure and vapor volume fraction (void fraction). Corresponding to these unknowns, a set of partial differential equations can be written which represent the conservation of mass, momentum, and energy for each phase. The equations are coupled through source terms expressing the interfacial transfer of these conserved properties. The complete set of conservation equations (including source terms) is then cast in non-dimensional form and transformed to a special coordinate system which fits the geometry of the flow and facilitates discretization. Appropriate boundary conditions are specified to complete the mathematical formulation. The purpose of this chapter is to describe these operations, particularly the conservation equations and source terms.

2.1.1 Two-Phase Flow Formulation Methods

In general, two formulation methods are possible in describing two-phase flows. These formulations actually represent fundamental ways of thinking about such flows and hence provide additional insight into the method chosen here. The first is the particle-source-in-cell method where the dispersed phase is treated from a Lagrangian point of view in which the individual particles are tracked. The continuous phase is seen from an Eulerian point of view with the effect of the dispersed phase entering through

source terms in the conservation equations. This method, also known as the Eulerian-Lagrangian formulation, is physically intuitive but is not computationally practical for other than very dilute dispersed phases. The second method is the two-fluid model presented here in which each phase is seen from an Eulerian point of view. This method regards the phases as "interpenetrating continua" (Spalding [34]), much like two mixed gases. The degree to which this description is accurate depends on how dispersed and dense each phase is, similar to the applicability of the continuum flow assumption in single-phase flows (Crowe [35]). Although not as accurate as the first method, the two-fluid model is the only alternative for practical computations when the dispersed phase is dense. It is therefore the method chosen here.

Rigorous derivations of the two-fluid model have been made by several authors (i.e., Ishii [25]). The usual procedure is to write the local instantaneous conservation equations for each phase separately. Conservation laws are also written for the infinitesimally thin boundary layer between the two phases. Several authors refer to these equations for the interfaces as "jump conditions". The instantaneous equations, while physically valid, are intractable computationally because the phases and their interfaces are constantly changing with position and time. This condition is analogous in single-phase flow to using the local instantaneous Navier-Stokes equations to solve a turbulent flow problem. The solution, as with turbulent flow, is to average the equations and corresponding interfacial conditions. Several such techniques such as time or volume averaging are treated in the literature. The resulting solutions yield approximate average values of the desired flow variables and interfacial effects. If the averaging is done rigorously this is the most accurate technique and leads to a computationally tractable set of equations.

Derivation of the conservation laws for two-phase flow by this method is highly mathematical and to some extent can be viewed as a separate subject within two-phase flow. The equations presented here were not derived with this level of mathematical rigor.

16

Instead it was simply assumed that the phases were dispersed enough over a control volume that assuming average properties would be accurate. Thus the method used here assumed average values from the beginning.

This assumption leads to a derivation process similar to that used in single-phase flow. Using a simple rectangular control volume of infinitesimal size (White, p.199 [36]), fluxes and diffusion terms at the boundaries can then be written taking into account the fact that each control surface is only partially occupied by a given phase. This fact is handled by multiplying each term by the void fraction of the phase in question. When interfacial source terms are added, the result is a set of two-fluid conservation equations. While this method produces equations which are not exactly the same as obtained by rigorous averaging, the terms in which they differ are negligible or not applicable to the flow considered here.

2.2 Two-Phase Conservation Equations

The two-fluid model equations presented below rest on several basic assumptions:
1. The flow is parabolic.
2. Steady-state.
3. Gravitational effects are negligible.
4. The fluid is Newtonian.

2.2.1 Conservation of Mass

The conservation of mass for two-phase flow in axisymmetric coordinates is:

$$\frac{\partial}{\partial x}(r\rho\alpha u) + \frac{\partial}{\partial r}(r\rho\alpha v) = r\Gamma \tag{2.1}$$

17

where α and Γ denote the volume fraction (of whichever phase is under consideration) and interfacial mass transfer (evaporation/condensation), respectively. Note that this equation is similar to its single-phase counterpart except for these two terms. As noted earlier, the volume (or area) fraction accounts for the fact that each cell is occupied by two phases, each occupying a certain fraction of the cell. The volume fraction, like velocity or temperature, is one of the basic unknowns of the problem which needs to be solved for.

The source term on the right hand side in this equation represents, of course, the mass lost or gained by the phase due to evaporation or condensation. It is strongly coupled to the interfacial heat transfer, to be discussed later.

2.2.2 Conservation of Momentum

The derivation of the conservation of momentum uses the first of the basic assumptions made earlier, that the flow is parabolic. This condition is possible when convection is so predominant in one direction that downstream effects cannot penetrate upstream by molecular diffusion. The mathematical approximations necessary for parabolic flow are made in the same manner as for the single-phase case. The cross-stream velocity is assumed negligible compared to the streamwise velocity and the cross-stream gradients are assumed much greater than the streamwise gradients. Also, by a scaling argument, pressure can be shown to depend on only the streamwise direction, making only the streamwise momentum equation significant. Thus even the effect of pressure is only felt in one direction. Parabolic flow yields a marching solution with significant savings in computer time and memory requirements.

Two problems exist with the assumption of parabolic flow, however. The first is that in the motive nozzle, where the cross-sectional area is variable, the cross-stream pressure gradients begin to be significant. This effect depends on the variability of the area. It has been shown, however, that this effect is tolerably small by comparing a single phase parabolic model to a single-phase semi-parabolic model in a converging duct similar to the motive nozzle. The second, and more serious problem, is that parabolic flow models cannot account for flow separation. When this project was started, it was not anticipated that analysis of a separated flow would be needed because no ejector would be designed with flow characteristics so highly detrimental to its performance. However, one of the experimental ejectors that is being used to validate the results of this model has indeed shown separation in the mixing section. This apparently occurs more readily in two-phase ejectors because the motive flow, with the higher density liquid, has much greater momentum than the suction flow. The separation problem is discussed at length later. Despite this problem, up to this point no attempt has been made to correct the parabolic approximation by using, for instance, an elliptic model.

The conservation of axial momentum in axisymmetric coordinates is:

$$\underbrace{\frac{\partial}{\partial x}\left(r\rho\alpha u^2\right) + \frac{\partial}{\partial r}(r\rho\alpha uv)}_{\text{convection}} = \underbrace{-\alpha\frac{d(rP_x)}{dx}}_{\text{pressure}} + \underbrace{\frac{\partial}{\partial r}\left(r\alpha\mu\frac{\partial u}{\partial r}\right)}_{\text{shear stress}} + \underbrace{rF_x}_{\text{interfacial drag}} + \underbrace{rM_x} \qquad (2.2)$$

Several points about this equation can be made. The two terms on the left hand side representing convection of momentum are the same as for single-phase flow except for the volume fraction. On the right hand side the forces arising due to viscous stresses are again the same as for a single-phase Newtonian fluid with the addition of the volume fraction. The term that represents the change in stress in the axial direction has been neglected according to the parabolic approximation. The first interfacial source term (F_x) represents the drag force exerted by the other phase on the one under consideration. The second

source term (M_x) represents the momentum gained or lost due to interfacial mass transfer. Chapter 3 contains a more detailed explanation of all the interfacial source terms and Appendix B has a step-by-step derivation of the basic equations.

Two additional points should be made about the pressure gradient term. First, since the flow is parabolic, the pressure is only a function of axial position and hence the total rather than the partial derivative is used. Second, the volume fraction is taken outside the derivative (whereas it is inside in all the other terms) because this term actually represents the following addition:

$$\frac{d}{dx}(P\alpha) - P\frac{d}{dx}(\alpha) = \alpha\frac{dP}{dx} \tag{2.3}$$

The first term represents the change in pressure forces acting over the control volume, accounting for the fact that volume fraction is also changing. The second term represents the fact that within the control volume a pressure force can exist even if the pressure itself is constant, because the area over which the pressure acts is changing if the volume fraction is changing. Thus a pressure force exists internally in the control volume, not just at the boundaries.

Another way of looking at the pressure term is to regard the first term above as the pressure forces acting on the control volume from outside (as in single-phase flow). The second term represents the pressure force exerted internally by one phase on another, so it is really an interfacial pressure force (see Fig. 2.1). These terms must be added to get the correct representation of pressure forces on each phase.
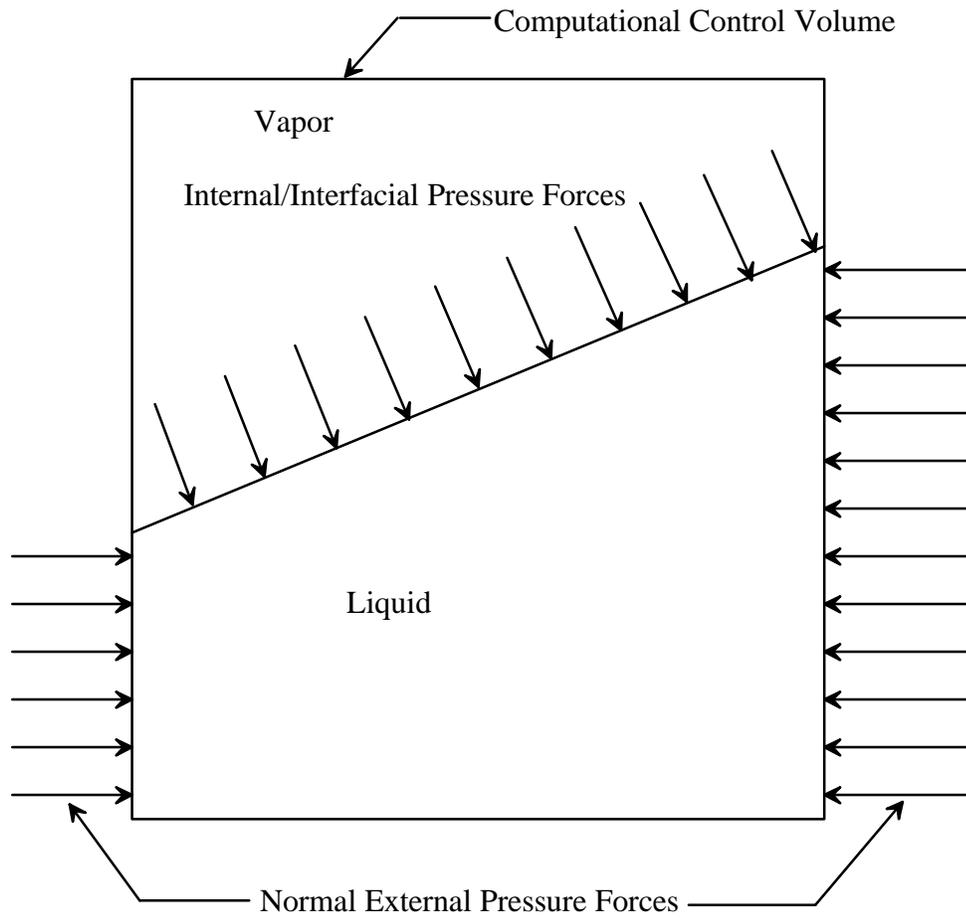
Computational Control Volume

Vapor

Internal/Interfacial Pressure Forces

Liquid

Normal External Pressure Forces

Figure 2.1 - Illustration of Interfacial Pressure Force due to Void Fraction Variation

In similar fashion, the radial momentum can be written:

$$\underbrace{\frac{\partial}{\partial r}\left(r\rho\alpha v^2\right) + \frac{\partial}{\partial x}(r\rho\alpha uv)}_{\text{convection}} = \underbrace{-r\alpha\frac{dP_r}{dr}}_{\text{pressure}} + \underbrace{\left(2r\alpha\mu\frac{\partial v}{\partial r}\right) + \frac{\partial}{\partial x}\left(r\alpha\mu\frac{\partial u}{\partial r}\right) - \frac{2\mu\alpha v}{r}}_{\text{shear forces}}$$

$$+ \quad \underbrace{rF_r}_{\text{interfacial drag}} \quad + \quad \underbrace{rM_r}_{\text{int. mass transfer}} \qquad (2.4)$$

The reader may question why the radial momentum equation is of interest at all in a parabolic flow. The answer is that while the radial momentum is indeed small compared to the axial momentum, this equation still needs to be solved because an additional unknown, the volume fraction, is present in two-phase flow. In order to obtain the radial distribution of volume fraction, the radial momentum equation must be solved. This issue will be elaborated upon in Chapter 4.

One aspect of the two-fluid model which poses some difficulty is the representation of the shear stress terms for the discontinuous phase. In the above equations it is assumed that the discontinuous phase can be represented using the Stokes deformation law, in the same manner as the continuous phase, except for the additional void fraction term. This assumption is only accurate to the extent that the discontinuous phase is dispersed (ie, how well it approximates a continuum). Thus, a well dispersed phase would consist of small particles (bubbles or droplets) which, by bumping into each other, create shear stress, just as shear stress is created by colliding molecules in single phase fluids. Since it is expected that the flow through the ejector is highly turbulent, the assumption that the discountinuous phase is highly dispersed seems appropriate. However, this aspect of modelling should be kept in mind should the flow regime be found to change.

22

2.2.3 Conservation of Energy

The conservation of energy in axisymmetric coordinates is:

$$\frac{\partial}{\partial r}(r\alpha\rho vT) + \frac{\partial}{\partial x}(r\alpha\rho uT) = \frac{1}{c_p}\frac{\partial}{\partial r}\left(rk\alpha\frac{\partial T}{\partial r}\right) + \frac{\beta T}{c_p}\left(ru\alpha\frac{dP_x}{dx}\right)$$
$$+ \frac{r}{c_p}(E_{ht} + E_{mt} + E_{ke} + E_{wt}) \qquad (2.5)$$

This equation is derived by first writing the First Law of Thermodynamics for an open system. Next, terms representing work done by shear stresses are cancelled out by substituting the mechanical energy equation (dot product of velocity and momentum) into this First Law expression. The result is the simplified and more relevant thermal energy equation. Also, for simplicity, convective terms which include enthalpy are converted to temperature using the thermodynamic property relationship (Bejan, p.12 [37])

$$dh = c_p dT + \frac{1}{\rho}(1 - \beta T)dP \qquad (2.6)$$

where β is the coefficient of thermal expansion of the fluid. Appendix B contains more details on the derivation of Eqn. 2.5.

The left hand side of the thermal energy equation represents the convection terms. The first term on the right hand side is the conduction heat transfer occurring within a particular phase. The next term is a compressibility effect introduced from Eqn. 2.6. The next two terms are thermal energy transferred due to mass transfer and the interfacial heat transfer. The last two terms are the kinetic energy transferred due to mass transfer and the work done by the interfacial drag force. These two terms are actually viscous dissipation since their mechanical energy components cancelled out with the substitution of the mechanical energy equation.

It is expected that the terms for the compressibility effect and the interfacial source terms representing viscous dissipation are small. However this is not known a priori with any certainty. It is prudent to wait for the results of the computation to make this determination conclusively and is the reason why the terms are presented here.

2.3 Nondimensionalization

In preparation for discretization the set of partial differential equations and source terms is nondimensionalized and transformed to a body fitted coordinate system. Nondimensionalization is done in a conventional manner and yields a set of equations which can be characterized by Reynold's number, Prandtl number, and Eckert number. It should be noted that this step is not required for the solution of the flow problem but makes the problem set-up and subsequent analysis more convenient. The following set of non-dimensional variables is used along with the appropriate derivative transformations.

$$x^* = \frac{x}{r_o} \qquad r^* = \frac{r}{r_o} \qquad u^* = \frac{u}{u_o} \qquad v^* = \frac{v}{v_o} \qquad P^* = \frac{P - P_o}{\rho_o u_o^2} \qquad T^* = \frac{T}{T_o}$$

$$\rho^* = \frac{\rho}{\rho_o} \qquad \mu^* = \frac{\mu}{\mu_o} \qquad c_p^* = \frac{c_p}{c_{p,o}} \qquad k^* = \frac{k}{k_o} \qquad \beta^* = \frac{\beta}{1/T_o}$$

$$\Gamma^* = \Gamma \frac{r_o}{\rho_o u_o} \qquad F^* = F \frac{r_o}{\rho_o u_o^2} \qquad M^* = M \frac{r_o}{\rho_o u_o^2} \qquad E^* = E \frac{r_o}{c_p^o T_o \rho_o u_o}$$

$$Re = \frac{r_o \rho_o u_o}{\mu_o} \qquad Pr = \frac{\mu_o c_{p,o}}{k_o} \qquad Ec = \frac{u_o^2}{T_o c_{p,o}} \qquad \frac{\partial}{\partial x} = \frac{1}{r_o} \frac{\partial}{\partial x^*} \qquad \frac{\partial}{\partial r} = \frac{1}{r_o} \frac{\partial}{\partial r^*} \qquad (2.7)$$

Here, values with the o subscript represent constants at some location in the flow, generally liquid values at the inlet to the mixing section. Thus r_o is the radius of the mixing section at the inlet, u_o is the velocity of the liquid at the motive nozzle outlet, and

so on. This transformation leads to the following set of non-dimensional conservation equations:

Continuity:

$$\frac{\partial}{\partial x^*}(r^*\rho^*\alpha u^*) + \frac{\partial}{\partial r^*}(r^*\rho^*\alpha v^*) = r^*\Gamma^* \tag{2.8}$$

Axial Momentum:

$$\frac{\partial}{\partial x^*}\left(r^*\rho^*\alpha u^{*2}\right) + \frac{\partial}{\partial r^*}(r^*\rho^*\alpha u^*v^*) = -\alpha\frac{d(r^*P_x^*)}{dx^*} + \frac{\partial}{\partial r^*}\left(\frac{r^*\alpha\mu^*}{Re}\frac{\partial u^*}{\partial r^*}\right)$$
$$+ r^*F_x^* + r^*M_x^* \tag{2.9}$$

Radial Momentum:

$$\frac{\partial}{\partial r^*}\left(r^*\rho^*\alpha v^{*2}\right) + \frac{\partial}{\partial x^*}(r^*\rho^*\alpha u^*v^*) = -r^*\alpha\frac{dP_r^*}{dr^*}$$
$$+ \frac{\partial}{\partial r^*}\left(\frac{2r^*\alpha\mu^*}{Re}\frac{\partial v^*}{\partial r^*}\right) + \frac{\partial}{\partial x^*}\left(\frac{r^*\alpha\mu^*}{Re}\frac{\partial u^*}{\partial r^*}\right) - \frac{2\mu^*\alpha v^*}{Rer^{*2}}r^* + r^*F_r^* + r^*M_r^* \tag{2.10}$$

Energy:

$$\frac{\partial}{\partial r^*}(r^*\alpha\rho^*v^*T^*) + \frac{\partial}{\partial x^*}(r^*\alpha\rho^*u^*T^*) = \frac{1}{c_p^*}\frac{\partial}{\partial r^*}\left(\frac{r^*k^*\alpha}{RePr}\frac{\partial T^*}{\partial r^*}\right)$$
$$+ \frac{Ec\beta^*T^*r^*u^*\alpha}{c_p^*}\frac{dP_x^*}{dx^*} + \frac{r^*}{c_p^*}\left[E_{ht}^* + E_{mt}^* + E_{ke}^* + E_{wt}^*\right] \tag{2.11}$$

The expressions for the interfacial source terms (described in Chapter 3) are also nondimensionalized and lead to equations involving the new non-dimensional variables multiplied by functions of *Re*, *Pr*, and *Ec*.

25

2.4 Boundary Conditions

The remaining pieces of information are the boundary conditions for the flow. Since this is a parabolic problem, it is required to have values of the unknowns at the inlet to the domain and at the top and bottom boundaries. No outlet boundary condition is required. Two types of boundaries (plus inlet conditions, which are known) are necessary in this problem, an adiabatic wall and a line of symmetry. Accordingly, for the wall boundary condition:

$$u_{wall} = 0 \qquad v_{wall} = 0 \qquad \left(\frac{\partial T}{\partial r}\right)_{wall} = 0 \tag{2.12}$$

Along the line of symmetry the following conditions hold:

$$\left(\frac{\partial u}{\partial r}\right)_{symm} = 0 \qquad v_{symm} = 0 \qquad \left(\frac{\partial T}{\partial r}\right)_{symm} = 0 \tag{2.13}$$

No new boundary conditions are required for the volume fraction. Since this unknown is solved for by the continuity equation, the no-flow condition at the wall and axis of symmetry is all that is required for its solution.

For the ejector geometry, the top boundary always corresponds to a wall. The line of symmetry is the bottom boundary for the motive nozzle, mixing section, and diffuser (Fig. 4.1). For the suction nozzle, however, the bottom boundary is also a wall.