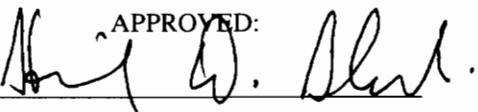


ENHANCED LOWER BOUNDS AND AN ALGORITHM FOR A WATER DISTRIBUTION
NETWORK DESIGN MODEL

by

Rajiv Totlani

Thesis submitted to the faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
Master of Science
in
Industrial and Systems Engineering

APPROVED:


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July 24, 1996

Blacksburg, Virginia

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(ABSTRACT)

The design of water distribution systems has received a great deal of attention in the last three decades because of its importance to industrial growth and its crucial role in society for community health, fire-fighting capability, and quality of life. The cost of installing a water distribution system is typically in the tens of millions of dollars. These systems also account for the largest costs in the municipal maintenance budgets. Furthermore, existing systems are being burdened by increasing urban development and water use. All these factors cause the pipe sizing decisions to be a critical task in designing a cost effective water distribution system that is capable of handling the demand and satisfying the minimum pressure head and hydraulic redundancy requirements.

A number of research efforts have focused on the least cost pipe sizing decision, each of them generating improved solutions for several standard test problems from literature, but so far, very little work has been done to test the quality of these solutions. In this thesis, two lower bounding schemes are proposed to evaluate the quality of these solutions. These lower bounding schemes make use of the special concave-convex nature of the nonlinear frictional loss terms. We show that the first is a dual to *Eiger et al.'s* [1994] bounding procedure while the second method produces far tighter lower bounds with comparable ease. Results on applying these lower bounding schemes to some standard test problems from literature are presented.

The second lower bounding scheme is then embedded in a branch-and-bound procedure along with an upper bounding scheme by suitably restricting the flows at each node of the search tree. By branching successively, we attempt to narrow the gap from optimality to generate near optimal solutions to the least cost pipe sizing problem. This results in a comprehensive reduced cost network design that satisfies all pressure and flow requirements for realistically sized problems. The proposed method is applied to standard test problems from the literature. It is hoped that this method will provide a useful tool for city engineers to design a cost effective water distribution system that meets specified hydraulic requirements.

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1. INTRODUCTION

1.1 Motivation

Because of its importance to the needs of society and industrial growth, considerable emphasis has recently been given to the condition of the nation's infrastructure. To support US industrial competitiveness in the world, large capital expenditures will be needed to bring several critical systems to higher levels of serviceability. One of the most vital services to industrial growth is an adequate water supply system, without which industry cannot survive. The quality of water distribution systems (WDS) also plays a crucial role in society because of its strong contribution to community health, fire-fighting capability, and quality of life. A survey by *Clark et al.* [1982] of previous water supply projects concluded that the distribution facilities in water supply systems account for the largest cost item in future maintenance budgets.

Many of the existing pipe networks in the older urbanized areas are old and deteriorating. Deterioration in many areas has translated into a high proportion of water leakage. This results in not only a loss of valuable resource, but it also raises concerns of unsafe drinking water caused by possible contamination through cracked pipes. For example, *Smith* [1993] reports that some districts in UK have as much as 40% leakage from their water distribution systems. Some other figures reported by *Mays* [1989] are : 17% in Boston, 15% in St. Louis, 15% in Cleveland, 14% in Pittsburgh, and 14% in Tulsa. The reliability of existing aging systems is continually decreasing. In 1973, in Houston, Texas, there were 5149 breaks in the 3998 miles of water mains, which amounts to 1290 breaks per year per 1000 miles of mains. In New Orleans, there was an average of 680 breaks per year per 1000 miles of mains in the 1519 miles of mains during the period 1969-1978. *O'Day et al.* [1987] point out the main problems caused by the deterioration of a water distribution system as

- water quality problems related to tuberculation and internal pipe corrosion;
- low pressure and high head loss problems due to tuberculation;
- distribution and transmission leakage due to pipe movement or soil movement;
- main breaks due to deterioration from internal and/or external corrosion;
- over deflection or excess joint pull, resulting in cracks and breaks;
- inoperable valves and hydrants.

Meanwhile, new sectors are being continually added to existing networks in many areas that are undergoing rapid urbanization, such as in the southwestern United States, in order to accommodate growing communities and new industries. The water distribution network design problem is further complicated by

issues such as network reliability, hydraulic redundancy, and the need to accommodate a range of expected emergency conditions such as broken pipes, fire demands, pump failure, control valve failure and power outages.

The costs associated with the overhaul and expansion of these systems are prohibitive and can easily escalate into tens of millions of dollars. This has spurred the search for cost-effective methods to build new networks, or replace old and deteriorating sections of existing networks with new configurations having enhanced capacity, that integrate network reliability and redundancy issues, network expansion and pipe sizing decisions, and multi-period economic analysis, all within a holistic framework. A paper by *Sherali and Smith* [1993] discusses such an approach.

The objective of a WDS is to find a least cost feasible network. Ideally a feasible network is one that satisfies hydraulic requirements (flow and pressure requirements) under all likely demand and failure scenarios, and thus incorporates a high level of reliability and redundancy. The most general problem is to modify and/or expand the design of an existing network by prescribing sufficiently high energy heads and large enough pipe diameters of suitable smoothness to supply the varied anticipated demand for water at required pressure levels, even while experiencing pipe breakage in the network. If the network is designed with low energy heads and rough pipes, then the flow/pressure requirements will not be met during demand peaks or various pipe failure scenarios. On the other hand, if the energy sources and pipes are overdesigned, or if there are too many redundant paths, then increased costs may lead to a very expensive design. As traditional methods for designing WDS are heuristic in nature and incorporate high levels of redundancy, they tend to be overly conservative, and therefore expensive. Over the last two decades, a number of models have been developed to address this issue, and several methods have been developed to solve the nonlinear, nonconvex and nonsmooth optimization subproblem for designing a WDS. All these methods generate a local optimum, or at best, a series of improving local optima that have yielded better and better approximate solutions for various standard test problems. However, these methods eschew determination of an adequate lower bound that might assist in evaluating the quality of the local optimum. Two notable exceptions in this regard are the recent papers by *Eiger et al.* [1994] and *Sherali and Smith* [1995] in which lower bounding procedures are developed and are embedded within globally convergent branch and bound algorithms. These methods have been applied to solve some standard test problems. In particular, *Sherali and Smith* solve certain problems to optimality for the first time, assuming no a priori or unproven bounds on optimal flow values. The work presented in this thesis uses a different approach to obtain good lower bounds by successively relaxing the nonlinear constraints via polyhedral outer

approximations in the network optimization problem and solving the corresponding linear programming problem. These bounds are shown to theoretically and empirically dominate *Eiger et al.*'s lower bounds.

1.2 Proposed Approach

The network optimization problem is nonlinear and nonconvex primarily due to constraints resulting from frictional head losses in the pipes. This frictional head loss is a function of the flow rate in the distribution links of the network and the length and diameter of the segments in a link. In our approach, we bound the flow values in each link between some initial heuristically determined minimum and maximum values (see *Sherali and Smith [1995]* for a theoretical derivation of appropriate bounds). The network optimization problem is then reformulated as a linear lower bounding problem which can be solved to obtain a lower bound. Two lower bounding schemes are used. The lower bounding linear programming problem developed in the first scheme is obtained by relaxing certain constraints and using polyhedral approximations to the nonlinearly constrained region in the network optimization problem. The lower bounding scheme used by *Eiger et al. [1994]* will be shown to be the dual of this first lower bounding scheme. Our derivation of this lower bounding scheme, being more direct, is simpler to enhance. This leads to our second lower bounding scheme that incorporates certain additional constraints in order to tighten the relaxation. Solving the lower bounding linear programming problem also yields a solution that can be perturbed to derive good quality feasible solutions to the original nonlinear problem. The procedure is embedded in a branch-and-bound scheme in an attempt to successively narrow the gap from optimality, by partitioning on the flow values generated for each link in the previous step. By branching successively in this fashion, the procedure converges to a near global optimal solution.

1.3 Organization of this Thesis

The remainder of this thesis is organized as follows. Chapter 2 discusses some of the previous approaches to solve the single-stage network optimization problem. Chapter 3 deals with the development of the network optimization model. In Chapter 4 we develop two linear programming lower bounding problems LB1 and LB2 for the nonlinear design problem. In Chapter 5, we prove that the lower bounding scheme of *Eiger et al. [1994]* is the dual of our LB1 formulation. The branch-and-bound procedure used is discussed in Chapter 6. Chapter 7 presents the results of applying the methodology developed to some standard test problems from literature. Finally, in Chapter 8, we analyze the computational performance of the approach and present some recommendations for future research.

2. LITERATURE REVIEW

The purpose of a water distribution system is to supply users with the amount of water demanded and to supply it with adequate pressure. Presented with the problem of sizing the components of a system which must operate under a set of loading conditions, a number of approaches have been suggested for the optimal or minimum cost design of water distribution systems. This chapter presents a review of previous approaches to this design problem.

Networks can be considered in two categories, branched or looped, with the latter being more difficult to solve. In a looped network more than one pipe may supply flow to a node. *Swamee and Khanna* [1984] showed that the optimal design for a single demand pattern is a branched network. However, looped networks are better suited to address the hydraulic redundancy issue which is an important factor in evaluating network performance. Although a considerable amount of research has addressed the design of looped water distribution systems, the determination of an optimal solution for a looped system is particularly difficult because the requirement for redundancy (closed loops) is vague. A truly minimum cost system for a single demand pattern is clearly devoid of redundancy, since the required flow can be carried at a lower cost by a single large pipe than by several small pipes in parallel. Thus, it is necessary to devise a constraint which, in one way or another, reflects the redundancy requirements. The mere application of a minimum diameter constraint is frequently unsatisfactory because it results in a design consisting of a branched structure with loops closed by pipes of minimum diameter. A review of papers for the looped problem follows. The optimization techniques for these networks can be classified as stochastic optimization methods and deterministic optimization methods. The current review will focus more on the deterministic optimization methods.

2.1 Deterministic Optimization Methods

Early examples of the nonlinear programming approaches described in literature are variations of the Newton-Raphson method (e.g., *Shamir* [1964]; *Pitchai* [1966]; *Shamir and Howard* [1968]). *Jacoby* [1968] used the cost of pipes and pumping as the objective, and incorporated the continuity equations in the objective using a penalty term. The procedure used an augmented Lagrangian penalty function. No guidance is, however, given on how to change the penalty weights. *Watanatada* [1973] and *Shamir* [1974] presented algorithms using the penalty function method where the initial cost of pumps and multiple loads could be considered in the formulation. *Watanatada* [1973] used a variable metric method to solve the nonlinear program. *Shamir* [1974] applied the reduced gradient method with a Newton-Raphson solver.

One of the most significant decomposition methods to solve the WDS problem was proposed by *Alperovits and Shamir* [1977] who developed the successive linear programming gradient (LPG) method. Here hydraulic loops are formulated for each basic loop in the network and for each path from a source to a demand node. The problem is then projected onto the space of the flow variables. For each (initially assumed) flow distribution, the other decision variables are optimized by linear programming. (Note that fixing the flows linearizes the head loss equations.) A postoptimality analysis of the linear program provides the information necessary to compute the gradient of the total cost with respect to changes in the flow distribution. The gradient is used to change the flows and the procedure is repeated until convergence to a local optimum is obtained. An initial feasible flow is assumed. Although the method is applied to a test problem that has a single source node, the authors also discuss an extension of their method to complex systems with multiple loading, pumps, valves and reservoirs.

Bhave [1978] formulated the single stage network optimization problem as a nonlinear problem with continuous pipe diameters for each pipe link. He solved the problem by identifying primary links (using critical path concepts) and determining the flow in the primary and secondary links at each iteration, assuming the nodal heads to be fixed. The heads are adjusted between iterations based on a cost-head-loss ratio criterion. The procedure is terminated when there is only a very small improvement in the total cost or when the changes in nodal heads are very small. After optimization, the pipes having nonstandard diameters are replaced with two segments of the next smaller and the next larger diameters such that the combined head loss remains the same across the link. This method was applied to the Bandung, Indonesia water supply system that has one source, 58 nodes and 90 links reducing the cost after only four iterations to 10% below the best currently available methods.

Collins et al., [1978] used standard techniques such as the Frank-Wolfe and Convex Simplex methods to solve the optimization problem, while taking advantage of the network structure to rapidly accelerate the solution process. They assumed that the energy levels of the sources are fixed and that flow directions are known *a priori*. Convergence was determined to be erratic, and strongly dependent on the initial solution. However, they were able to approximately solve a large problem based on the Dallas, Texas network (having 452 nodes, including 21 reservoirs, 516 links and 21 pumps) to within 1.3% of the best known solution.

Quindry et al. [1979] showed that *Alperovits and Shamir* [1977] had missed certain terms in their gradient expression. They modified the LPG method to incorporate these terms and demonstrated an improvement in solving the aforementioned small test problem using the same starting solution. However, the inclusion of these terms rendered the procedure ineffective for large-scale problems.

Quindry et al. [1981] then published an algorithm in which the pressure was considered as fixed in the subproblem and updated by a hierarchical scheme. The subproblem was based on the work of *Lai and Schaake* [1969] and determined the optimal continuous diameter for a link. A gradient term was derived from the node equation and directed the change in the pressure distribution for the next iteration. Because of the form of the subproblem only pipes could be sized. Also, due to the nature of the algorithm, the analysis of multiple loads, although theoretically possible is difficult to perform in practice.

A two-level approach was also used by *Rowell and Barnes* [1982] for determining the layout and design of a distribution system. After an initial assumption concerning the optimum hydraulic gradient which allows the pipe diameters to be selected independent of the pump heads and tank elevations, a nonlinear problem results which is only a function of the network flows. This new nonlinear problem is a minimum cost network flow problem with a separable objective function. Thus it can be solved for the optimum core tree using a piecewise approximation of the objective by network flow techniques. Given the layout of a spanning tree a flow covering integer subproblem which considers other loading conditions is solved for the required redundant links. *Goulter and Morgan* [1983] question the validity of the optimum hydraulic gradient assumption to looped networks and suggest a feedback mechanism in the optimization to ensure its correctness.

Gessler [1982] presented a model based on an enumeration scheme to determine a solution. Given a set of possible pipe sizes, the algorithm performs cost and hydraulic consistency tests on all potential combinations to determine a series of pareto optimal solutions. Logic is incorporated in the model to perform size and cost tests which eliminate some combinations without computing the pressure distribution. The enumeration scheme is not an optimization model and has no guarantee of optimality.

Bhave [1983a] developed a two-stage linear programming approach for optimizing multiple source looped WDS. In the first stage, he heuristically constructs a distribution graph of primary links using shortest path routes from source nodes to demand nodes, in order to determine the set of links to be included in the tree. These links are sized by solving a linear transportation problem, assuming the availability of continuous

pipe diameters. In the second stage, some flows are assigned to the non primary links and the flows in the primary links are thereby determined. Once all the flows are determined, the resulting LP problem is solved, this time allowing only commercially available diameter segments. In a second paper, *Bhave* [1983b], the method is applied to a small test problem having two sources, eight demand nodes and 13 links.

A further advancement of the decomposition approach is given by *Mahjoub* [1983], who avoids the LP approximation by solving alternatively convex and concave programs as follows: first, link flows are fixed, and the resulting convex program (i.e. convex minimization over a convex set) is solved for link head losses, using the classical gradient projection method (see *Rosen* [1960]). Then the obtained link head losses are fixed, and the resulting concave program (i.e. concave minimization over a convex set) is heuristically solved for link flows. The whole procedure is repeated until no improvement can be obtained.

Lansey and Mays [1985] simplified the solution of the nonlinear single-stage model of the LPG method by incorporating a network simulator within the optimization model, and applying general reduced gradient and optimal control theory concepts. *Gessler* [1985], *Loubser and Gessler* [1993] used an enumeration approach that examines all possible combinations of decision options, testing each for feasibility and cost. The obvious combinatorial explosion of this approach is somewhat mitigated by grouping together similar classes of pipes and by pruning infeasible or inferior combinations that might be evident from the candidate list of solutions.

Morgan and Goulter [1985] developed an algorithm to analyze networks under multiple loads and determine the optimum layout and design of looped water distribution systems. The optimization problem is formulated as a linear program using heuristics to define the critical nodes under different loading conditions with their corresponding path equations acting as active constraints in the linear program. The linear program considers the change of length of one pipe size for another in a given link thus requiring constraints in the model to insure the sum of lengths of pipe equal the total distance between nodes. A second heuristic was developed to remove unneeded links using a weighting procedure. The overall algorithm works as follows; first an initial solution is analyzed by a Hardy Cross solver; then weights are generated for each link; unnecessary links are removed using heuristics; next given the flow distributions and the weights the linear program model determines the least cost pipe diameters to achieve the pressure requirements. Finally the new pipe sizes are returned to the network solver to compute a new flow distribution and the iterative procedure continues until no change in pipe diameters is made by the linear program. A heuristic has also been added to determine the best layout.

Fujiwara et al. [1987] and *Kessler and Shamir* [1989] proposed alternative derivations of the linear programming based gradient expressions along with other algorithmic enhancements to improve the computational efficiency of the LPG approach. *Fujiwara et al.* [1987] derived a complete expression for the gradient term assuming a single source network. They managed to significantly improve the computational efficiency of the LPG algorithm by incorporating a quasi-Newton direction method along with suitable line search techniques to determine step sizes. *Kessler and Shamir* [1989] used a graph theoretic formulation, adding insight to the problem and reformulating the problem in matrix notation.

Fujiwara and Khang [1990] proposed a two-phase decomposition approach which, unlike previous approaches, is not content with determining a single local optimum alone, but generates a sequence of improving local solutions. The first phase of the method uses the LPG approach to produce a local optimum. In the second phase, the link head losses of this local optimum solution are fixed and the resulting concave program is solved for the link flows and the pumping heads; these then serve to restart the first phase to obtain an improved local optimum solution.

One of the first methods which explicitly addressed redundancy issues in the design problem was proposed by *Loganathan, Sherali and Shah* [1990]. The authors proposed a two-phase design and optimization procedure for constructing a water distribution network having a built-in degree of reliability. The first phase is comprised of an algorithm which iteratively constructs a tree pipe network. Starting with a shortest path tree, the procedure employs a linear programming subproblem to systematically modify this tree by adding and deleting one link at a time, with the aim of reducing the cost of the network while satisfying the flow continuity, energy balance and pressure head requirement constraints. The second phase of the algorithm is concerned with the issue of reliability. In this phase, the tree network constructed by the first phase is augmented through the addition of links so that there are at least two arc-disjoint paths from each source to every demand node it serves. The augmentation is performed through the use of a set covering problem which recommends the links to be added, and by using a Hardy-Cross solver for redesigning the perturbed network to ensure feasibility, while attempting to minimize the overall design cost.

A decomposition/projection algorithm was proposed by *Sherali and Smith* [1993], where the problem is projected onto the space of the network design variables (pipe lengths of various diameters and source elevation heads), and an auxiliary convex cost network flow subproblem of the type analyzed by *Collins et al.* [1978] is used to guide the variations in design variables. The authors also propose a global optimization

procedure based on a modification of the Reformulation-Linearization technique developed by *Sherali and Tuncbilek* [1991]. A good quality starting solution, from the results of the decomposition strategy, is used to provide an initial upper bound in the global optimization strategy.

Several other refinements have been proposed in the design of water distribution system. All the above procedures employ heuristics for the single-stage network optimization problem that, at best, might converge to locally optimal solutions. A first global optimization approach was proposed by *Eiger et al.* [1994]. The procedure proposed by them enforces hydraulic consistency requirements via an enumeration of all possible basic loops and source-to-demand node paths in the network, as opposed to a link-wise formulation of these constraints. A branch-and-bound algorithm is proposed based on partitioning the hyperrectangle restricting the flows into several subrectangles. At each node of the branch-and-bound tree, a subgradient-based heuristic is applied to determine an upper bound via the nonsmooth, nonconvex, projection of the problem onto the space of the flow variables. A different relaxed, duality-based, linear programming formulation is used to compute lower bounds.

Sherali and Smith [1995] present another global optimization approach to a water distribution network design problem. The algorithm proposed by the authors uses a Reformulation-Linearization Technique (RLT) to construct a tight linear programming relaxation for the given problem in order to compute tight lower bounds. The procedure is embedded in a branch-and-bound scheme. Convergence to an optimal solution is induced by coordinating this process with an appropriate partitioning scheme. The authors also propose several algorithmic refinements and enhancements that need to be made in order to make the algorithm practical and effective for large-scale problems.

2.2 Stochastic Optimization Methods

Simulation optimization procedures like simulated annealing have recently become very popular. *Loganathan et al.* [1995] use an outer flow search-inner optimization procedure to identify improved local minima. Each pipe link in the network is subjected to the outer search scheme that chooses alternative flow configurations to find an optimal flow division among pipes. An inner linear program is used for the design of least cost pipe diameters. Two global search schemes are used to permit a local-optimum-seeking method to migrate among various local minima. Simulated annealing iteratively improves the objective function by finding successively better points, and, to escape out of a local minimum by an occasional acceptance of a worse point.

Another technique that has gained popularity as an optimization heuristic is the use of genetic algorithms. A genetic algorithm is a member of a class of search algorithms based on artificial evolution [Holland, 1975]. Genetic algorithms simulate mechanisms of population genetics and natural rules of survival in pursuit of the ideas of adaptation. *Dandy et al.* [1996] use the genetic algorithm formulation for the pipe network optimization of a water distribution system. The authors propose an improved genetic algorithm that uses an adjacency or creeping mutation operator. Previous research using genetic algorithms on pipe network optimization by *Murphy and Simpson* [1992], *Dandy et al.* [1993], *Simpson et al.* [1993], *Murphy et al.* [1993] and *Simpson et al.* [1994] used only bitwise mutation operators. The genetic algorithm proposed by *Dandy et al.* uses variable power scaling of the fitness function. Gray codes rather than binary codes are used to represent the set of decision variables which make up the pipe network design.

Most of the work to date has focused on developing optimization procedures for the least cost pipe sizing in a water distribution network. Some of the authors discuss extension of their methods to include other components such as pumps, tanks and valves. The techniques employed range from traditional nonlinear Hardy-Cross solver and Newton-Raphson methods to more complicated hierarchical decomposition methods based on linear programming approximations. The use of stochastic optimization techniques like simulated annealing and genetic algorithms has also been employed with considerable success. All these methods report improved solutions for some standard test problems, but at best generate only local optimum without any method to test the quality of solutions obtained. *Eiger et al.* [1994] and *Sherali and Smith* [1995] have developed global optimization procedures based on flow partitioning and branch-and-bound techniques. They also generate lower bounds to test the quality of their solutions. It is the purpose of this thesis to develop a more effective lower bounding scheme to test the quality of solutions. The next chapter in this thesis deals with the formulation of the network optimization problem (NOP) which is central to the development of our lower bounding scheme. ◀

3. MODEL FORMULATION

3.1 Network Optimization Problem Formulation

The network optimization model discussed in this chapter for a water distribution system is similar to the formulation of *Sherali and Smith* [1993].

For a distribution network, consider a set of reservoirs or supply nodes and a set of consumption or demand nodes. Let these be identified by the index set $N=\{1,2,\dots,n\}$. Let the node set be partitioned into a set of source nodes denoted by $S\subset N$ and a set of demand nodes denoted by $D\subset N$ such that $N=S\cup D$. Associated with each node is a quantity b_i , the net water supply rate or demand rate corresponding to node i in the index set N . We will assume that $b_i>0$ for $i\in S$ and $b_i<0$ for $i\in D$. To ensure feasibility, we assume that the total supply rate equals the total demand rate, i.e., the network is balanced. If the total supply exceeds total demand, then it can be converted to a balanced network problem by creating an additional dummy node having demand $b_d = -\sum_{i\in N} b_i$ and connecting this to all the supply nodes at zero cost.

We will define *sections* of pipes to be short (20"-30") lengths of pipe that are used to physically construct a pipeline. A *segment* is defined to be a length of pipe of constant properties of diameter, roughness and annualized cost, perhaps composed of many sections. *Links* are defined as a collection of segments between two nodes, the lengths of which add up to the required length of the pipeline between the nodes.

For each pipe connecting node pairs i and j , where $i,j \in N$, $i < j$, we create a (notationally) directed arc $(i,j) \in A$. For each $(i,j) \in A$, let L_{ij} denote the pipe length corresponding to an existing connection in the network between the nodes i and j . We will assume that each such link is constructed from segments of lengths of standard available diameters, chosen from the set $\{d_k, k=1,\dots,K\}$. Associated with each diameter is the cost per unit length c_k for a pipe of diameter d_k . Henceforth, the column vector of allowable diameters will be referred to as d , and the column vector of the cost per unit length for the allowable diameters will be referred to as c .

Associated with each link connecting node pairs (i,j) is the decision variable q_{ij} representing the flow rate (m^3/hr) which may be non-negative or negative thus permitting flow in either direction. A positive flow value means that flow is along the conventional direction of the arc.

Our next set of decision variables relate to the length of each segment of different diameters chosen from the set of available diameters to construct each link of the network. Let x_{ijk} denote the length of segment of diameter k in the link L_{ij} and let x_{ij} be the vector having components $(x_{ijk}, k=1, \dots, K)$. Now, let us consider the energy heads at the various nodes $i \in N$ in the network. For each node $i \in S$ let E_i denote the ground elevation (m) of node $i \in N$ and let H_i denote the established head above E_i . For the source nodes $i \in S$, let F_i denote the maximum available energy (fixed) head. We will assume that there is an additional opportunity to raise the head at each source by an amount H_{Si} at an annualized cost $c_{Si} > 0$ per unit energy head, as suggested by *Rowell and Barnes* [1982]. Furthermore, for each node $i \in D$, suppose that there is the requirement that the head $(H_i + E_i)$ at this node at flow equilibrium lies in the interval $[H_{iL}, H_{iU}]$, where $H_{iL} < H_{iU}$.

The pressure loss (or head loss) in a pipe due to friction, given by $[(H_i + E_i) - (H_j + E_j)]$ for a link (i, j) depends on the pipe characteristics such as diameter, roughness, length and the water flow rate through the pipe. The frictional head loss in a segment of pipe under smooth flow conditions is approximately described by the empirical Hazen-William equation as follows (*Alperovits and Shamir* [1977] and *Walski* [1984]), where the sign depends on the direction of flow. That is, for a link (i, j) the frictional loss is taken to be positive if computed along the direction of flow and negative if opposite to it.

$$\Phi(q, C_{HW}, d, x) = (1.52) 10^4 \text{ sign}(q) |q/C_{HW}| 1.852 d^{-4.87} x \quad (3.1a)$$

where

Φ = pressure head loss assuming smooth flow conditions in a given pipe segment

q = water flow rate in the pipe (meter³/hr)

C_{HW} = Hazen Williams coefficient based on roughness and diameter

d = pipe diameter (in inches)

x = pipe length (in meters).

For our model, the head loss in a pipe that has several potential segments of varying diameter and roughness is computed as follows

$$\Phi_{ij}(q_{ij}, x_{ij}) = \sum_{k=1}^K \Phi(q_{ij}, C_{HW(ijk)}, d_k, x_{ijk}). \quad (3.1b)$$

For any connected network, basic loops can be obtained by extracting a spanning tree structure from the network and connecting the remaining arcs to the tree one by one to get back the original network. The cycle formed by connecting back an arc to the tree is a basic loop. The notation B_s will be used to denote

the s th basic loop in the network. The chain from node m to node n will be denoted by C_{mn} . The basic loop set is a minimal set of loops such that all other loops can be represented as a linear combination of them. Also, the basic loop set is not necessarily unique, since several underlying spanning tree subgraphs typically exist.

The flow values q_{ij} associated with each link are assumed to lie between some heuristically determined minimum and maximum values $q_{\min ij}$ and $q_{\max ij}$ that may appropriately be of either sign. We define the hyperrectangle restricting the flows Ω as

$$\Omega = \{q: q_{\min} \leq q \leq q_{\max}\} \quad (3.1c)$$

where the notation q_{\min} and q_{\max} with the subscripts dropped denotes the vectors of lower and upper bounds.

3.2 Network Optimization Problem NOP

Based on the above model and using the notation explained, the network optimization problem NOP, restricted on Ω can now be formulated as follows.

NOP (Ω): Minimize

$$\sum_{(i,j) \in A} \sum_{k=1}^K c_k x_{ijk} + \sum_{i \in S} c_{si} H_{si} \quad (3.2a)$$

subject to

$$\Phi_{ij}(q_{ij}, x_{ij}) = (H_i + E_i) - (H_j + E_j) \quad \forall (i,j) \in A \quad (3.2b)$$

$$\sum_{k=1}^K x_{ijk} = L_{ij} \quad \forall (i,j) \in A \quad (3.2c)$$

$$\sum_{(i,k) \in A} q_{ik} - \sum_{(j,i) \in A} q_{ji} = b_i \quad \forall i \in D \quad (3.2d)$$

$$\sum_{(i,k) \in A} q_{ik} - \sum_{(j,i) \in A} q_{ji} + q_{id} = b_i \quad \forall i \in S \quad (3.2e)$$

$$- \sum_{(i,d) \in A} q_{id} = b_d \quad \text{for the dummy node } d \quad (3.2f)$$

$$q_{\min ij} \leq q_{ij} \leq q_{\max ij} \quad \forall (i,j) \in A \quad (3.2g)$$

$$H_i + E_i \leq F_i + H_{si} \quad \forall i \in S \quad (3.2h)$$

$$H_{iL} \leq H_i + E_i \leq H_{iU} \quad \forall i \in D \quad (3.2i)$$

$$H_{si} \geq 0 \quad \forall i \in S \quad (3.2j)$$

$$x_{ijk} \geq 0 \quad \forall (i,j) \in A \text{ and } k=1, \dots, K. \quad (3.2k)$$

The objective function, Equation (3.2a), denotes the total cost of the pipes and the cost of the additional head generated at each source node.

Constraints (3.2b) are the conservation of energy equations. Along with Constraints (3.2i) they ensure that the hydraulic energy loss over each chain in the network is less than some required value so that the minimum head requirements (H_{iL}) are met for each demand node. Conservation of energy also requires that the total energy loss in each basic loop should be zero. For each basic loop in the network, this requirement takes the following form

$$\sum_{(i,j) \in B_s} \Phi_{ij}(q_{ij}, x_{ij}) = 0 \quad \forall B_s. \quad (3.2l)$$

It should be noted that Constraints (3.2b) implicitly enforce that the hydraulic energy loss in each basic loop is zero. This is demonstrated in the formulation of the example problem in Section 3.3

The link length constraints are represented by Equation (3.2c).

Equation (3.2d) enforces conservation of flow at each demand node.

Equation (3.2e) enforces conservation of flow at each source node; the excess of supply over demand is the consumption by the dummy demand node d .

Equation (3.2f) enforces conservation of flow for the dummy demand node.

Equation (3.2g) bounds the flow values in a link between some heuristically determined minimum and maximum values.

Constraints (3.2h) represents the maximum variable head at each source node. Equality is enforced if q_{jd} is required to be zero.

Constraints (3.2i) restrict the minimum and maximum head at each demand node.

Equations (3.2j) and (3.2k) are the nonnegativity constraints for additional head at a source node and the link lengths.

The problem is obviously nonlinear due to the frictional head loss constraints. Our principal set of decision variables are the lengths x_{ijk} of different segments in each link $(i,j) \in A$ and the additional head H_{si} to be developed at each source node $i \in S$. The resulting heads H_i at each node $i \in N$ (above E_i the elevation of the node) and the flows q_{ij} in the links $(i,j) \in A$ are also problem dependent variables.

3.3 NOP Example

As an illustration, the network optimization problem is derived for the network shown in Figure (3.3a). The demand nodes in the network are supplied by gravity from a single source node. In addition, $H_{si} = 0$. Tables (3.3a) and (3.3b) summarize the data for this example problem. For simplicity, we assume that the Hazen William's coefficient $C_{HW(ijk)}$ is the same for all diameters. In general, this may not be the case.

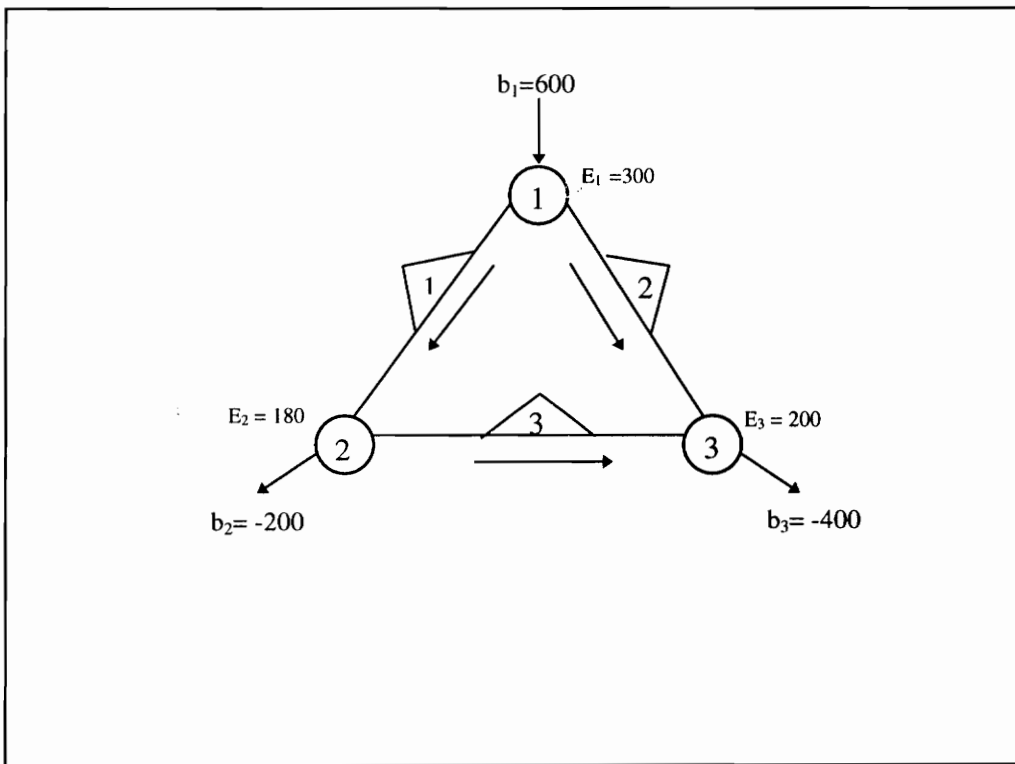


Figure (3.3a) Network Configuration for example problem.

Table (3.3a) Arc data for example problem.

Link Number	Arcs (i,j)	Length (m)	C_{HW}
1	(1,2)	500	100
2	(1,3)	700	100
3	(2,3)	800	100

Table (3.3b) Node data for example problem.

Node	Elevation (m)	Minimum Pressure required (m)	Supply or Demand (m^3/hr)
1	300	-	600
2	180	20	-200
3	200	40	-400

Set of commercially available diameters d in inches = {6,10,15}.

Corresponding cost per unit length of available diameters c in \$ = {20,40,80}.

The maximum flow in a link (i,j) is determined as $q_{ij} \leq b_s + b_i$ where b_s is the sum of all the other supply nodes in the example network. To determine the minimum flow, for all links (i,j) where $i \in S$ and $j \in D$, we set $q_{ij} = 0$. For all other links (i,j), we reverse the direction of arcs in the network and use the above mentioned bound for determining the maximum flow. This gives the maximum flow for the arcs with the direction reversed and hence the minimum flow in the initially assumed direction of the arcs. Based on this computation, the minimum and maximum flow values determined for each link are as in Table (3.3c). *Sherali and Smith* [1995], discuss techniques for deriving tighter bounds on the flows.

Table (3.3c) Minimum and maximum flow in each link

Link Number	qmin	qmax
1	0	600
2	0	600
3	-200	400

Based on the data, we generate the following values which to used as coefficients for the variables in the NOP example formulation.

$$d^{-4.87} = \{16.23, 1.345, 0.1873\} 10^{-5} \quad (3.3a)$$

$$(1.52) 10^4 d^{-4.87} (C_{HW})^{-1.852} = \{48.77, 4.042, 0.5628\} 10^{-5}. \quad (3.3b)$$

3.4 Optimization Problem for the Example Network

The optimization problem for the example network can now be formulated as

Minimize

$$20x_{121} + 40x_{122} + 80x_{123} + 20x_{131} + 40x_{132} + 80x_{133} + 20x_{231} + 40x_{232} + 80x_{233} \quad (3.4a)$$

subject to

$$\text{sign}(q_{12}) \left| q_{12} \right|^{1.852} (48.77x_{121} + 4.042x_{122} + 0.5628x_{123}) 10^{-5} + H_2 = 120 \quad (3.4b)$$

$$\text{sign}(q_{13}) \left| q_{13} \right|^{1.852} (48.77x_{131} + 4.042x_{132} + 0.5628x_{133}) 10^{-5} + H_3 = 100 \quad (3.4c)$$

$$\text{sign}(q_{23}) \left| q_{23} \right|^{1.852} (48.77x_{231} + 4.042x_{232} + 0.5628x_{233}) 10^{-5} - H_2 + H_3 = -20 \quad (3.4d)$$

$$x_{121} + x_{122} + x_{123} = 500 \quad (3.4e)$$

$$x_{131} + x_{132} + x_{133} = 700 \quad (3.4f)$$

$$x_{231} + x_{232} + x_{233} = 800 \quad (3.4g)$$

$$q_{12} + q_{13} = 600 \quad (3.4h)$$

$$q_{23} - q_{12} = -200 \quad (3.4i)$$

$$-q_{13} - q_{23} = -400 \quad (3.4j)$$

$$0 \leq q_{12} \leq 600 \quad (3.4k)$$

$$0 \leq q_{13} \leq 600 \quad (3.4l)$$

$$-200 \leq q_{23} \leq 400 \quad (3.4m)$$

$$20 \leq H_2 \leq 120 \quad (3.4n)$$

$$40 \leq H_3 \leq 100 \quad (3.4o)$$

$$x_{121}, x_{122}, x_{123}, x_{131}, x_{132}, x_{133}, x_{231}, x_{232}, x_{233} \geq 0. \quad (3.4p)$$

Equation (3.4a) denotes the total cost of pipe segments of different diameters in each link.

Constraints (3.4b), (3.4c) and (3.4d) are the conservation of energy equations. Along with Equations (3.4n) and (3.4o), they ensure that the minimum head requirements are met for each demand node. There is one basic loop in the problem. The conservation of energy equation for the basic loop gives

$$\begin{aligned} &\text{sign}(q_{12}) \left| q_{12} \right|^{1.852} (48.77x_{121} + 4.042x_{122} + 0.5628x_{123}) 10^{-5} - \\ &\text{sign}(q_{13}) \left| q_{13} \right|^{1.852} (48.77x_{131} + 4.042x_{132} + 0.5628x_{133}) 10^{-5} + \\ &\text{sign}(q_{23}) \left| q_{23} \right|^{1.852} (48.77x_{231} + 4.042x_{232} + 0.5628x_{233}) 10^{-5} = 0 \end{aligned} \quad (3.4q)$$

It is easily seen that this equation can be obtained by adding Constraints (3.4b) and (3.4d) and subtracting Constraint (3.4c) from the sum. Hence as noted earlier, the basic loop energy equations are implied by the energy conservation constraints.

Equations (3.4e), (3.4f) and (3.4g) are the link length constraints.

Conservation of flow is enforced by Equations (3.4h), (3.4i) and (3.4j)

Equations (3.4k), (3.4l) and (3.4m) bound the flow in the links between heuristically determined minimum and maximum values.

Equations (3.4n) and (3.4o) are bounds on the head developed at each demand node. They ensure that the minimum head requirements are met and that the maximum available head at each node is less than the maximum allowable head. If the maximum allowable head is not specified, then the largest of the maximum head developed at each source node can be used.

The nonnegativity constraints are represented by Constraints (3.4p).

As noted earlier, the above formulation is nonlinear due to the presence of the frictional head loss terms in Equations (3.4b), (3.4c) and (3.4d). The next chapter in this thesis discusses two methods to develop suitable relaxations for the frictional loss terms to develop a linear programming lower bounding problem formulation.

4. DERIVATION OF LOWER BOUNDING PROBLEMS

4.1 Structure of the Problem NOP

The problem NOP has a linear objective function. The frictional head loss terms in the Constraints (3.2b) cause the problem to become nonlinear and nonconvex. All other constraints are linear. In our first formulation of the lower bounding problem, we project the problem NOP onto a set of newly defined variables and delete the flow conservation constraints which now become nonlinear in the projected space of these new decision variables. The above procedure takes advantage of the monotone nature of the frictional head loss equation.

4.2 Rewriting the Frictional Head Loss Terms

Consider Equation (3.1b) which appears in Constraints (3.2b). Recall that the frictional losses are positive in the direction of flow and negative opposed to it, and using Equation (3.1b), we can rewrite them as follows

$$\Phi_{ij}(q_{ij}, x_{ij}) = \sum_{k=1}^K \text{sign}(q_{ij}) |q_{ij}|^{1.852} (1.52 \cdot 10^4 (C_{HW}(ijk))^{-1.852} d_k^{-4.87}) x_{ijk} \quad (4.2a)$$

where $|q_{ij}|$ denotes the absolute value of the flow q_{ij} . Denoting new variables

$$v_{ij}(q_{ij}) = \text{sign}(q_{ij}) |q_{ij}|^{1.852} \quad (4.2b)$$

$$\text{and } \alpha_{ijk} = (1.52 \cdot 10^4 (C_{HW}(ijk))^{-1.852} d_k^{-4.87}) \quad (4.2c)$$

we can rewrite Equation (4.2a) as

$$\Phi_{ij}(q_{ij}, x_{ij}) = \sum_{k=1}^K v_{ij}(q_{ij}) \alpha_{ijk} x_{ijk}. \quad (4.2d)$$

The functional relation for $v_{ij}(q_{ij})$ as given by Equation (4.2b), has a convex-concave structure as shown in Figure (4.2a). For $q_{ij} < 0$, $v_{ij}(q_{ij})$ is a concave function of q_{ij} , and for $q_{ij} > 0$, $v_{ij}(q_{ij})$ is a convex function of q_{ij} . By the monotonicity of $v_{ij}(q_{ij})$ we can represent its value for any q_{ij} as the convex combination

$$v_{ij}(q_{ij}) = \lambda_{ij} v_{ij}(q_{\min ij}) + (1 - \lambda_{ij}) v_{ij}(q_{\max ij}) \quad \text{for } \forall (i,j) \in A \quad (4.2e)$$

where $0 \leq \lambda_{ij} \leq 1$.

For convenience, we will refer to the following expressions according to the notation specified below

$$v_{\min ij} = v_{ij}(q_{\min ij}) \quad (4.2f)$$

$$\text{and } v_{\max ij} = v_{ij}(q_{\max ij}). \quad (4.2g)$$

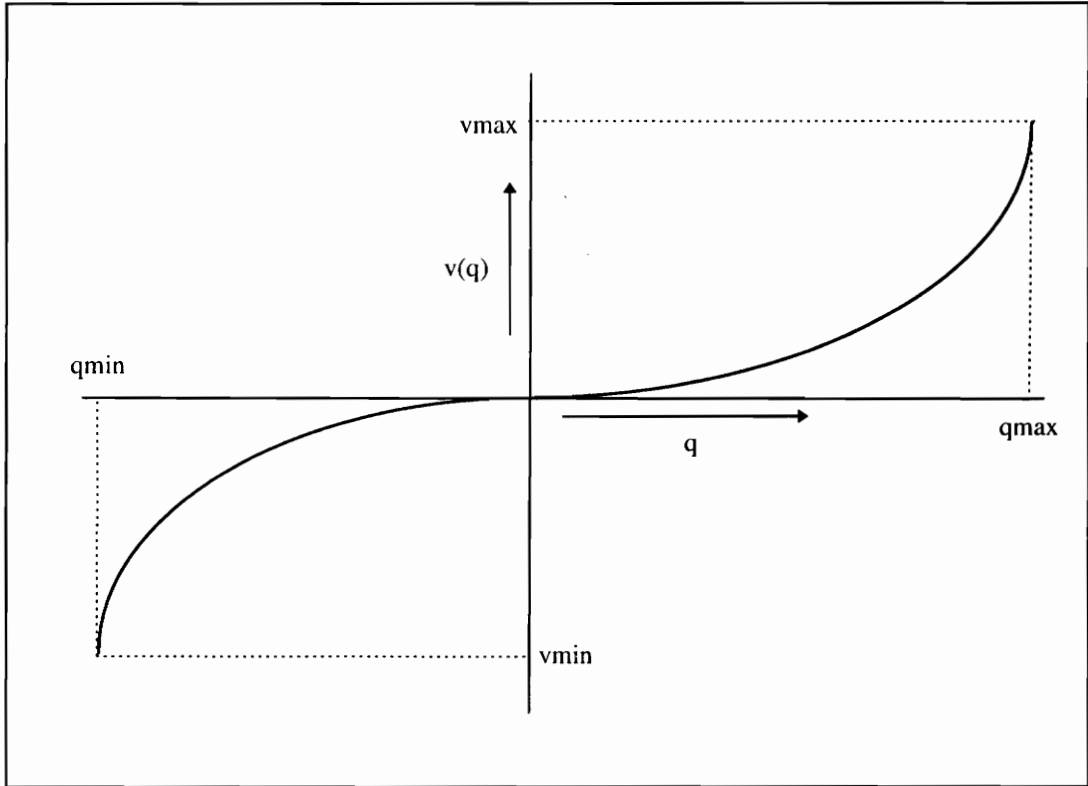


Figure (4.2a) Plot of $v_{ij}(q_{ij})$

Using expression (4.2e) in Equation (4.2d), we get,

$$\Phi_{ij}(q_{ij}, x_{ij}) = \sum_{k=1}^K (\lambda_{ij} v_{min_{ij}} + (1-\lambda_{ij}) v_{max_{ij}}) \alpha_{ijk} x_{ijk}. \quad (4.2h)$$

Rearranging terms,

$$\Phi_{ij}(q_{ij}, x_{ij}) = \sum_{k=1}^K v_{min_{ij}} \alpha_{ijk} (\lambda_{ij} x_{ijk}) + \sum_{k=1}^K v_{max_{ij}} \alpha_{ijk} (1-\lambda_{ij}) x_{ijk}. \quad (4.2i)$$

We now define our new decision variables as

$$x^1_{ijk} = \lambda_{ij} x_{ijk} \quad (4.2j)$$

$$\text{and } x^2_{ijk} = (1-\lambda_{ij}) x_{ijk} \quad (4.2k)$$

so that,

$$x_{ijk} = x^1_{ijk} + x^2_{ijk}. \quad (4.2l)$$

Also, note that

$$\sum_{k=1}^K x^1_{ijk} = \lambda_{ij} \sum_{k=1}^K x_{ijk} = \lambda_{ij} L_{ij} \quad (4.2m)$$

$$\text{and } \sum_{k=1}^K x^2_{ijk} = (1-\lambda_{ij})L_{ij}. \quad (4.2n)$$

Equation (4.2i) can now be rewritten in terms of the new decision variables as

$$\Phi_{ij}(q_{ij}, x_{ij}) = \sum_{k=1}^K v \min_{ij} \alpha_{ijk} x^1_{ijk} + \sum_{k=1}^K v \max_{ij} \alpha_{ijk} x^2_{ijk}. \quad (4.2o)$$

By writing Equation (4.2e) in terms of the new decision variables x^1 and x^2 and variables λ_{ij} , we have linearized the energy conservation constraints, but at the expense of introducing nonlinearity in the flow conservation equations (via the inverse relationship), which were linear in the NOP formulation.

4.3 Lower Bounding Problem LB1

Rewriting the frictional head loss as described above allows us to reformulate the network optimization problem NOP in terms of the new decision variables x^1_{ijk} and x^2_{ijk} . Deleting the transformed nonlinear flow conservation constraints we obtain the following linear lower bounding problem restricted on Ω .

LB1(Ω) : Minimize

$$\sum_{(i,j) \in A} \sum_{k=1}^K c_k (x^1_{ijk} + x^2_{ijk}) + \sum_{i \in S} c_{si} H_{si} \quad (4.3a)$$

subject to

$$\sum_{k=1}^K v \min_{ij} \alpha_{ijk} x^1_{ijk} + \sum_{k=1}^K v \max_{ij} \alpha_{ijk} x^2_{ijk} = (H_i + E_i) - (H_j + E_j) \quad \forall (i,j) \in A \quad (4.3b)$$

$$\sum_{k=1}^K (x^1_{ijk} + x^2_{ijk}) = L_{ij} \quad \forall (i,j) \in A \quad (4.3c)$$

$$H_i + E_i \leq F_i + H_{si} \quad \forall i \in S \quad (4.3d)$$

$$H_{iL} \leq H_i + E_i \leq H_{iU} \quad \forall i \in D \quad (4.3e)$$

$$H_{si} \geq 0 \quad \forall i \in S \quad (4.3f)$$

$$x^1_{ijk}, x^2_{ijk} \geq 0 \quad \forall (i,j) \in A \text{ and } k=1, \dots, K. \quad (4.3g)$$

Equation (4.3a) is the total construction cost of the network. Constraints (4.3b) are the energy conservation constraints. Constraints (4.3c) are the link length constraints. Constraints (4.3d), (4.3e) are the minimum and maximum head constraints.

In Chapter 5 we prove that the lower bounding problem of *Eiger et al.* [1994] is the dual of our LB1 formulation. The above approach being more direct is easier to enhance as shown in the sequel.

4.4 LB1 Formulation Example

The LB1 formulation is applied to the example problem of Section 3.3. From Equation (4.2c), we obtain the α_{ijk} values as

$$\alpha_{121} = \alpha_{131} = \alpha_{231} = (4.878) 10^{-4} \quad (4.4a)$$

$$\alpha_{122} = \alpha_{132} = \alpha_{232} = (4.054) 10^{-5} \quad (4.4b)$$

$$\alpha_{123} = \alpha_{133} = \alpha_{233} = (5.627) 10^{-6} \quad (4.4c)$$

From Table (3.3c), using Equation (4.2b), Table (4.4a) is obtained .

Table (4.4a) Values vmin and vmax for each link.

Link Number	vmin	vmax
1	0	139680.608
2	0	139680.608
3	-18260.285	65919.682

The problem LB1 for the example network can now be formulated as follows :

Minimize

$$20x^1_{121} + 40x^1_{122} + 80x^1_{123} + 20x^1_{131} + 40x^1_{132} + 80x^1_{133} + 20x^1_{231} + 40x^1_{232} + 80x^1_{233} + 20x^2_{121} + 40x^2_{122} + 80x^2_{123} + 20x^2_{131} + 40x^2_{132} + 80x^2_{133} + 20x^2_{231} + 40x^2_{232} + 80x^2_{233} \quad (4.4d)$$

subject to

$$68.136x^2_{121} + 5.663x^2_{122} + 0.786x^2_{123} + H_2 = 120 \quad (4.4e)$$

$$68.136x^2_{131} + 5.663x^2_{132} + 0.786x^2_{133} + H_3 = 100 \quad (4.4f)$$

$$-8.907x^1_{231} - 0.740x^1_{232} - 0.103x^1_{233} + 26.724x^2_{231} + 2.672x^2_{232} + 0.371x^2_{233} - H_2 + H_3 = -20 \quad (4.4g)$$

$$x^1_{121} + x^1_{122} + x^1_{123} + x^2_{121} + x^2_{122} + x^2_{123} = 500 \quad (4.4h)$$

$$x^1_{131} + x^1_{132} + x^1_{133} + x^2_{131} + x^2_{132} + x^2_{133} = 700 \quad (4.4i)$$

$$x^1_{231} + x^1_{232} + x^1_{233} + x^2_{231} + x^2_{232} + x^2_{233} = 800 \quad (4.4j)$$

$$20 \leq H_2 \leq 120 \quad (4.4k)$$

$$40 \leq H_3 \leq 100 \quad (4.4l)$$

$$\begin{aligned} &x^1_{121}, x^1_{122}, x^1_{123}, x^1_{131}, x^1_{132}, x^1_{133}, x^1_{231}, x^1_{232}, x^1_{233}, x^2_{121}, x^2_{122}, x^2_{123}, \\ &x^2_{131}, x^2_{132}, x^2_{133}, x^2_{231}, x^2_{232}, x^2_{233} \geq 0. \end{aligned} \quad (4.4m)$$

Constraints (4.4e), (4.4f) and (4.4g) are the energy conservation constraints for flow in links 1,2 and 3, respectively. Constraints (4.4h), (4.4i) and (4.4j) are the link length constraints. Constraints (4.4k) and (4.4l) establish minimum and maximum pressure heads developed at each demand node.

4.5 Enhancement of Lower Bounds

We can write the flow q_{ij} in any link (i,j) in terms of v_{ij} from Equation (4.2b) as,

$$q_{ij}(v_{ij}) = \text{sign}(v_{ij}) |v_{ij}|^{1/1.852} \quad (4.5a)$$

where v_{ij} is given by Equation (4.2e). This enables us to represent the flow q_{ij} in any link as an explicit function of λ_{ij} which we will refer to as $q_{ij}(\lambda)$. Since the relations derived subsequently hold for each link, the subscripts defining the link will be dropped for convenience.

$$q(\lambda) = \text{sign}(\lambda v_{\min} + (1-\lambda) v_{\max}) |\lambda v_{\min} + (1-\lambda) v_{\max}|^{1/1.852}. \quad (4.5b)$$

The LBI formulation avoids nonlinearity by simply deleting the flow conservation constraints. Our intention now is to obtain a suitable relaxation of Equation (4.5b) and reintroduce the flow variables to obtain an enhanced lower bounding problem. A plot of Equation (4.5b) when $q_{\max_{ij}} > 0$ and $q_{\min_{ij}} < 0$ is shown in Figure (4.5a).

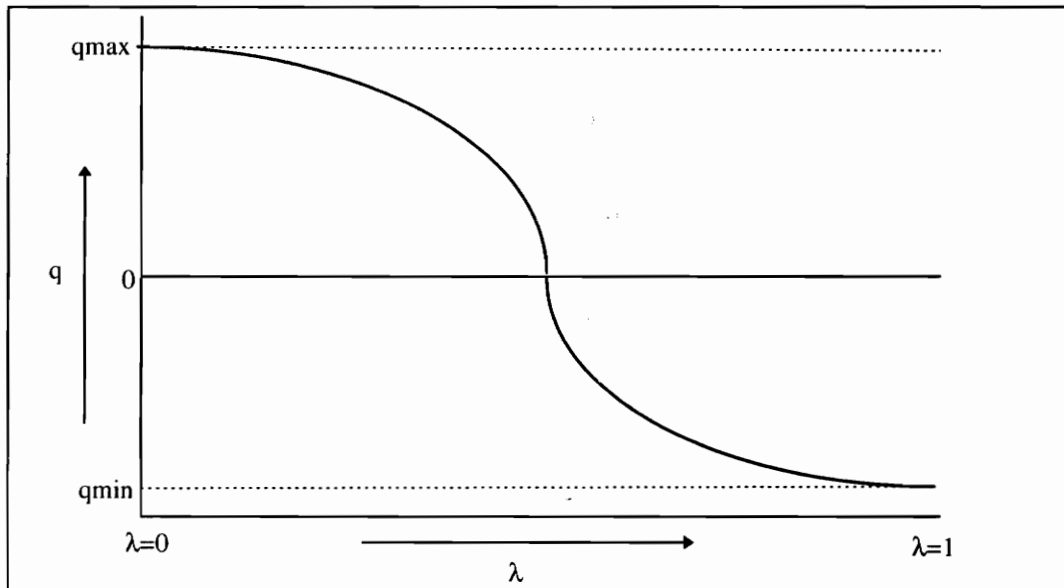


Figure (4.5a) Plot of $q(\lambda)$.

Our relaxation of Equation (4.5b) is based on exploiting the concave-convex nature of the function. In general, we can have the following three cases for the minimum and maximum flow in each link.

- (i) $q_{\max} > 0$ and $q_{\min} < 0$
- (ii) $q_{\max} > 0$ and $q_{\min} \geq 0$
- (iii) $q_{\max} \leq 0$ and $q_{\min} < 0$.

Case(i)

The function is concave-convex as shown in Figure (4.5a) for $0 \leq \lambda \leq 1$. By examining q as a function of λ over its positive and negative ranges separately, it can be readily verified that the derivative or slope of $q(\lambda)$ denoted by $q'(\lambda)$, is given by

$$q'(\lambda) = \frac{(v_{\min} - v_{\max})}{1.852} |\lambda v_{\min} + (1 - \lambda)v_{\max}|^{\frac{-0.852}{1.852}} \quad (4.5c)$$

As a relaxation for $q(\lambda)$ for this case, we can obtain a band width as shown in Figure (4.5b). To obtain the bandwidth, we need to determine the points $\bar{\lambda}$ and $\hat{\lambda}$ shown in Figure (4.5b). These points can be obtained as solutions of Equations (4.5d) and (4.5e), respectively, given below.

$$q_{\min} - q(\bar{\lambda}) - (1 - \bar{\lambda}) q'(\bar{\lambda}) = 0 \quad (4.5d)$$

$$\text{and } q_{\max} - q(\hat{\lambda}) + \hat{\lambda} q'(\hat{\lambda}) = 0. \quad (4.5e)$$

Equations (4.5d) and (4.5e) can be solved by performing a bisection search. Solving Equations (4.5d) and (4.4e), and noting that the following inequality holds,

$$0 < \bar{\lambda}, \hat{\lambda} < 1 \quad (4.5f)$$

we can construct the following relaxation for flow in a link

$$q(\hat{\lambda}) + (\lambda - \hat{\lambda}) q'(\hat{\lambda}) \leq q \leq q(\bar{\lambda}) + (\lambda - \bar{\lambda}) q'(\bar{\lambda}) \quad (4.5g)$$

and where

$$0 \leq \lambda \leq 1. \quad (4.5h)$$

Figure (4.5b) depicts the relaxations represented by Equation(4.5g).

The relaxation can be further tightened by noting that for $0 \leq \lambda \leq \bar{\lambda}$, the function $q(\lambda)$ lies below the tangent plane at λ . $\lambda = 0$ is always one such point. For $\hat{\lambda} \leq \lambda \leq 1$, the function $q(\lambda)$ lies above the tangent plane at λ . $\lambda = 1$ is one such point. This gives us the following constraints for the flow in each such link.

$$q \leq q_{\max} + \lambda q'(0). \quad (4.5i)$$

$$q \geq q_{\min} + (\lambda - 1) q'(\lambda). \quad (4.5j)$$

Figure (4.5c) depicts the above two constraints and the band estimate.

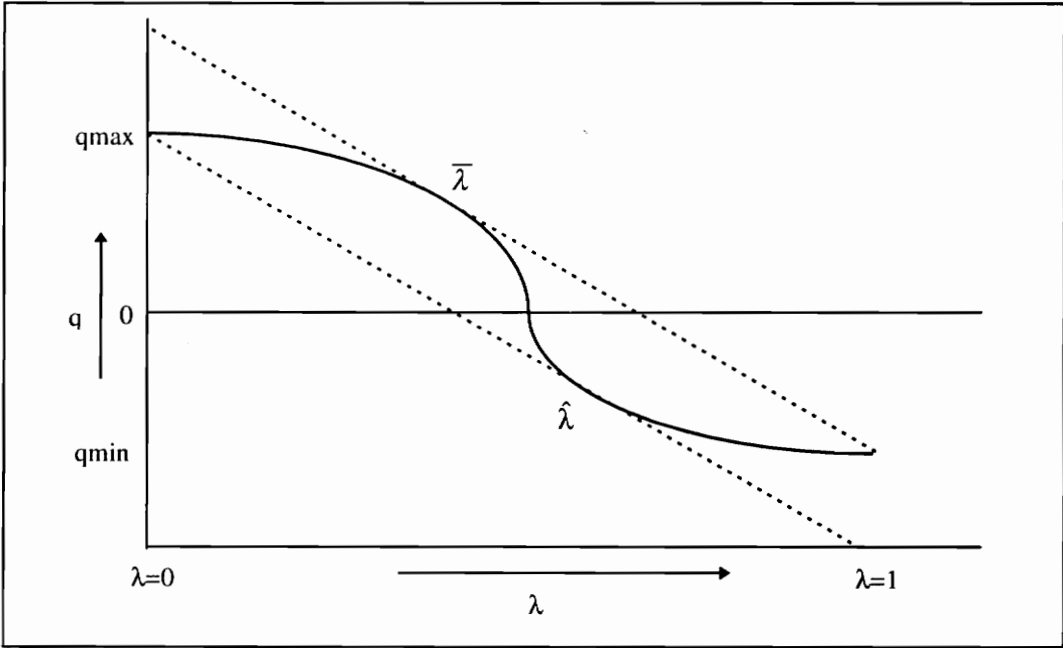


Figure (4.5b) Band relaxation of $q(\lambda)$ for case (i).

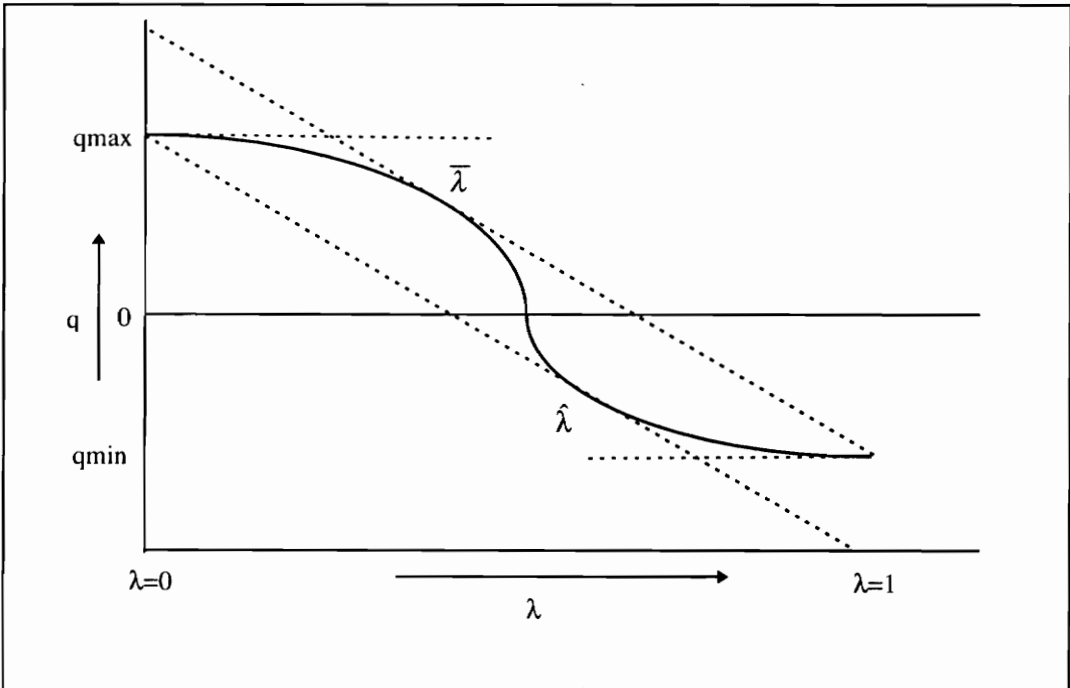


Figure (4.5c) Relaxation of $q(\lambda)$ for case (i).

Case(ii)

The direction of flow in the link is determined and agrees with the designated flow direction. The function is concave as shown in Figure (4.5d) for $0 \leq \lambda \leq 1$. Utilizing the fact that a concave function lies above the chord defined by its minimum and maximum values, yields the following constraint :

$$q \geq \lambda q_{\min} + (1-\lambda) q_{\max}. \tag{4.5k}$$

Also, a concave function lies below the tangent plane at any point. At the point $\lambda = 1$, the slope of tangent plane is infinite causing computational difficulties and so this point should be avoided. We take three such points as $\lambda = 0, 0.5$ and 0.9 . This gives us the constraints defined by the following equations :

$$q \leq q(\tilde{\lambda}) + (\lambda - \tilde{\lambda}) q'(\tilde{\lambda}) \quad \text{for } \tilde{\lambda} = 0, 0.5, 0.9. \tag{4.5l}$$

Figure (4.5d) depicts the above constraints.

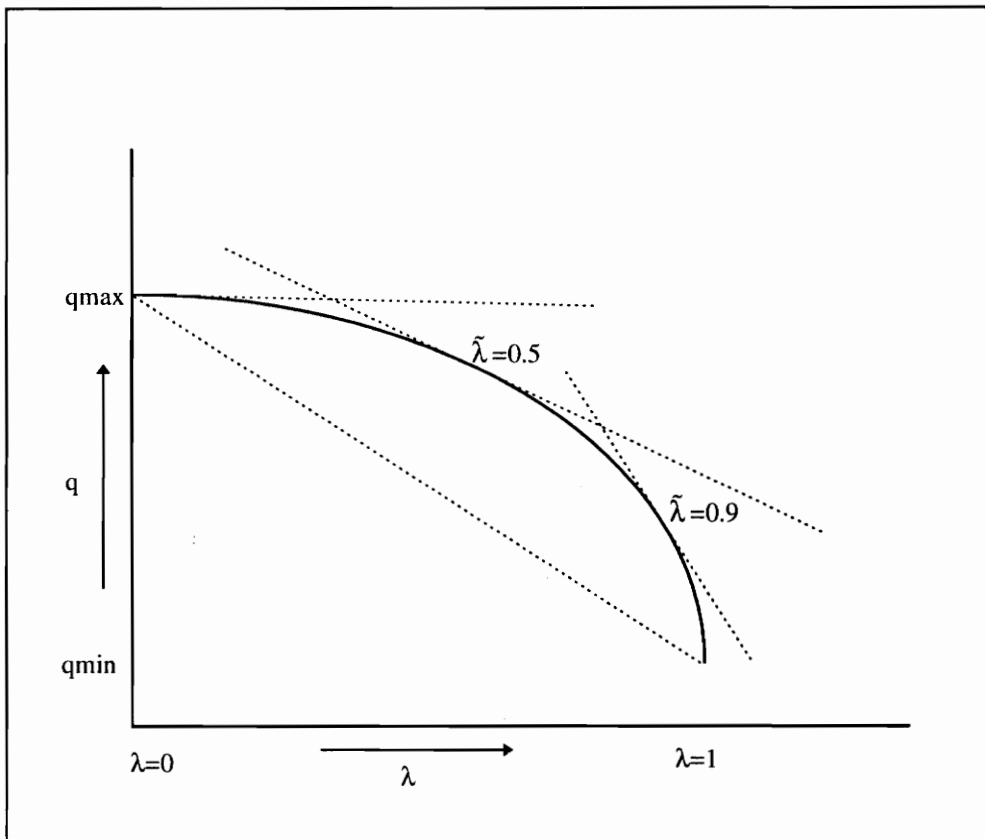


Figure (4.5d) Relaxation of $q(\lambda)$ for case (ii).

Case (iii)

The direction of flow in the link is determined and is opposite to the initially designated flow direction. Considering the case when q_{max} is nonpositive, similar to case (ii), we can bound the flow values exploiting the convex nature of the function to get the following bounding expressions

$$q \leq \lambda q_{min} + (1-\lambda) q_{max}. \quad (4.5m)$$

$$q \geq q(\tilde{\lambda}) + (\lambda - \tilde{\lambda}) q'(\tilde{\lambda}) \quad \text{for } \tilde{\lambda} = 0.1, 0.5, 1. \quad (4.5n)$$

4.6 Enhanced Lower Bounding Problem LB2

We now formulate the enhanced lower bounding problem in the x^1, x^2, λ and q variable space.

LB2(Ω) : Minimize

$$\sum_{(i,j) \in A} \sum_{k=1}^K c_k (x_{ijk}^1 + x_{ijk}^2) + \sum_{i \in S} c_{Si} H_{Si} \quad (4.6a)$$

subject to

$$\sum_{k=1}^K v \min_{ij} \alpha_{ijk} x_{ijk}^1 + \sum_{k=1}^K v \max_{ij} \alpha_{ijk} x_{ijk}^2 = (H_i + E_i) - (H_j + E_j) \quad \forall (i,j) \in A \quad (4.6b)$$

$$\sum_{k=1}^K (x_{ijk}^1 + x_{ijk}^2) = L_{ij} \quad \forall (i,j) \in A \quad (4.6c)$$

$$\sum_{k=1}^K x_{ijk}^1 = \lambda_{ij} L_{ij} \quad \forall (i,j) \in A \quad (4.6d)$$

$$\sum_{(i,k) \in A} q_{ik} - \sum_{(j,i) \in A} q_{ji} = b_i \quad \forall i \in D \quad (4.6e)$$

$$\sum_{(i,k) \in A} q_{ik} - \sum_{(j,i) \in A} q_{ji} + q_{id} = b_i \quad \forall i \in S \quad (4.6f)$$

$$- \sum_{(i,d) \in A} q_{id} = b_d \quad \text{for node } d \quad (4.6g)$$

$$q_{ij} \geq q_{ij}(\hat{\lambda}_{ij}) + (\lambda_{ij} - \hat{\lambda}_{ij}) q'_{ij}(\hat{\lambda}_{ij}) \quad \forall (i,j) \ni q_{min_{ij}} < 0, q_{max_{ij}} > 0 \quad (4.6h)$$

$$q_{ij} \leq q_{ij}(\bar{\lambda}_{ij}) + (\lambda_{ij} - \bar{\lambda}_{ij}) q'_{ij}(\bar{\lambda}_{ij}) \quad \forall (i,j) \ni q_{min_{ij}} < 0, q_{max_{ij}} > 0 \quad (4.6i)$$

$$q_{ij} \leq q_{max_{ij}} + \lambda_{ij} q'_{ij}(0) \quad \forall (i,j) \ni q_{min_{ij}} < 0, q_{max_{ij}} > 0 \quad (4.6j)$$

$$q_{ij} \geq q_{min_{ij}} + (\lambda_{ij} - 1) q'_{ij}(1) \quad \forall (i,j) \ni q_{min_{ij}} < 0, q_{max_{ij}} > 0 \quad (4.6k)$$

$$q_{ij} \geq \lambda_{ij} q_{min_{ij}} + (1 - \lambda_{ij}) q_{max_{ij}} \quad \forall (i,j) \ni q_{min_{ij}} \geq 0 \quad (4.6l)$$

$$q_{ij} \leq q_{ij}(\tilde{\lambda}_{ij}) + (\lambda_{ij} - \tilde{\lambda}_{ij}) q'_{ij}(\tilde{\lambda}_{ij}) \quad \forall (i,j) \ni q_{min_{ij}} \geq 0, \tilde{\lambda}_{ij} = 0, 0.5, 0.9 \quad (4.6m)$$

$$q_{ij} \leq \lambda_{ij} q_{min_{ij}} + (1 - \lambda_{ij}) q_{max_{ij}} \quad \forall (i,j) \ni q_{max_{ij}} \leq 0 \quad (4.6n)$$

$$q_{ij} \geq q_{ij}(\tilde{\lambda}_{ij}) + (\lambda_{ij} - \tilde{\lambda}_{ij}) q'_{ij}(\tilde{\lambda}_{ij}) \quad \forall (i,j) \ni q_{max_{ij}} \leq 0, \tilde{\lambda}_{ij} = 0.1, 0.5, 1 \quad (4.6o)$$

$$H_i + E_i \leq F_i + H_{Si} \quad \forall i \in S \quad (4.6p)$$

$$H_{iL} \leq H_i + E_i \leq H_{iu} \quad \forall i \in D \quad (4.6q)$$

$$H_{si} \geq 0 \quad \forall i \in S \quad (4.6r)$$

$$q_{\min ij} \leq q_{ij} \leq q_{\max ij} \quad \forall (i,j) \in A \quad (4.6s)$$

$$x^1_{ijk}, x^2_{ijk} \geq 0 \quad \forall (i,j) \in A \text{ and } k=1, \dots, K \quad (4.6t)$$

$$0 \leq \lambda_{ij} \leq 1 \quad \forall (i,j) \in A. \quad (4.6u)$$

4.7 LB2 Formulation Example

The formulation LB2 is constructed for the example problem of Section 3.3. For the arc (1,3), the values of $\bar{\lambda}_{23}$ and $\hat{\lambda}_{23}$ are computed by bisection search from Equations (4.5d) and (4.5e) as 0.744 and 0.925, respectively. Table (4.7a) summarizes the numerical values of the coefficients of some of the constraints. Note that since arcs (1,2) and (2,3) have the same minimum and maximum flow, this helps to reduce computation of the coefficients of the constraints.

Table (4.7a) Constraint coefficients.

Arc (i,j)	Nature of coefficient	Computed value
(1,2) and (1,3)	$q_{ij}(0)$	600
	$q_{ij}(0.5)$	412.657
	$q_{ij}(0.9)$	173.059
	$q'_{ij}(0)$	-323.974
	$q'_{ij}(0.5)$	-445.653
	$q'_{ij}(0.9)$	-934.444
(2,3)	$q_{ij}(0.724)$	79.602
	$q_{ij}(0.925)$	-159.303
	$q'_{ij}(0.724)$	-1091.386
	$q'_{ij}(0.925)$	-604.327
	$q'_{ij}(0)$	-275.812
	$q'_{ij}(1)$	-497.840

The problem LB2 for the example network can now be formulated as

Minimize

$$20x^1_{121} + 40x^1_{122} + 80x^1_{123} + 20x^1_{131} + 40x^1_{132} + 80x^1_{133} + 20x^1_{231} + 40x^1_{232} + 80x^1_{233} + 20x^2_{121} + 40x^2_{122} + 80x^2_{123} + 20x^2_{131} + 40x^2_{132} + 80x^2_{133} + 20x^2_{231} + 40x^2_{232} + 80x^2_{233} \quad (4.7a)$$

subject to

$$68.136x^2_{121} + 5.663x^2_{122} + 0.786x^2_{123} + H_2 = 120 \quad (4.7b)$$

$$68.136x^2_{131} + 5.663x^2_{132} + 0.786x^2_{133} + H_3 = 100 \quad (4.7c)$$

$$-8.907x^1_{231} - 0.740x^1_{232} - 0.103x^1_{233} + 26.724x^2_{231} + 2.672x^2_{232} + 0.371x^2_{233} - H_2 + H_3 = -20 \quad (4.7d)$$

$$x^1_{121} + x^1_{122} + x^1_{123} + x^2_{121} + x^2_{122} + x^2_{123} = 500 \quad (4.7e)$$

$$x^1_{131} + x^1_{132} + x^1_{133} + x^2_{131} + x^2_{132} + x^2_{133} = 700 \quad (4.7f)$$

$$x^1_{231} + x^1_{232} + x^1_{233} + x^2_{231} + x^2_{232} + x^2_{233} = 800 \quad (4.7g)$$

$$q_{12} + q_{13} = 600 \quad (4.7h)$$

$$q_{23} - q_{12} = -200 \quad (4.7i)$$

$$-q_{13} - q_{23} = -400 \quad (4.7j)$$

$$q_{23} + 604.327\lambda_{23} \geq 399.699 \quad (4.7k)$$

$$q_{23} + 1091.386\lambda_{23} \leq 869.765 \quad (4.7l)$$

$$q_{23} + 275.812\lambda_{23} \leq 400 \quad (4.7m)$$

$$q_{23} + 497.840\lambda_{23} \geq 297.84 \quad (4.7n)$$

$$q_{12} + 600\lambda_{12} \geq 600 \quad (4.7o)$$

$$q_{12} + 323.974\lambda_{12} \leq 600 \quad (4.7p)$$

$$q_{12} + 445.653\lambda_{12} \leq 635.484 \quad (4.7q)$$

$$q_{12} + 934.444\lambda_{12} \leq 1014.059 \quad (4.7r)$$

$$q_{13} + 600\lambda_{13} \geq 600 \quad (4.7s)$$

$$q_{13} + 323.974\lambda_{13} \leq 600 \quad (4.7t)$$

$$q_{13} + 445.653\lambda_{13} \leq 635.484 \quad (4.7u)$$

$$q_{13} + 934.444\lambda_{13} \leq 1014.059 \quad (4.7v)$$

$$0 \leq q_{12} \leq 600 \quad (4.7w)$$

$$0 \leq q_{13} \leq 600 \quad (4.7x)$$

$$-200 \leq q_{23} \leq 400 \quad (4.7y)$$

$$20 \leq H_2 \leq 120 \quad (4.7z)$$

$$40 \leq H_3 \leq 100 \quad (4.7aa)$$

$$x^1_{121}, x^1_{122}, x^1_{123}, x^1_{131}, x^1_{132}, x^1_{133}, x^1_{231}, x^1_{232}, x^1_{233}, x^2_{121}, x^2_{122}, x^2_{123}, x^2_{131}, x^2_{132}, x^2_{133}, x^2_{231}, x^2_{232}, x^2_{233} \geq 0 \quad (4.7ab)$$

$$0 \leq \lambda_{12}, \lambda_{13}, \lambda_{23} \leq 1. \quad (4.7ac)$$

Constraints (4.7k) and (4.7l) represent the lower and upper bands for flow in link (2,3). Constraints (4.7m) and (4.7n) tighten this relaxation by means of additional supports. Constraint (4.7o) is generated using Equation (4.5k). Constraints (4.7p), (4.7q) and (4.7r) are derived from Equation (4.5l).

5. PROBLEM LB1 AS THE DUAL OF EIGER ET AL.'S [1994] LOWER BOUNDING PROBLEM

This Chapter proves that the lower bounding problem of *Eiger et al.*[1994] is the dual of our LB1 formulation. We will first briefly review the problem formulation of *Eiger et al.* and how it slightly differs from the formulation of *Sherali and Smith* [1995] used in this thesis.

5.1 Eiger et al.'s Problem Formulation

Eiger et al. use the formulation of *Kessler and Shamir* [1989] which, concisely, is as follows:

P : Minimize

$$\sum_{(i,j) \in A} \sum_{k=1}^K c_k x_{ijk} \quad (5.1a)$$

subject to

$$\sum_{(i,j) \in C_m} \Phi_{ij}(q_{ij}, x_{ij}) \leq (H_m + E_m) - (H_{nL} + E_n) \quad \forall m \in S \text{ and } \forall n \in D \quad (5.1b)$$

$$\sum_{(i,j) \in B_s} \Phi_{ij}(q_{ij}, x_{ij}) = 0 \quad \forall s \in B \quad (5.1c)$$

$$\sum_{k=1}^K x_{ijk} = L_{ij} \quad \forall (i,j) \in A \quad (5.1d)$$

$$x_{ijk} \geq 0 \quad \forall (i,j) \in A \text{ and } k=1, \dots, K \quad (5.1e)$$

$$q_{ij} \in Q \quad \forall (i,j) \in A \quad (5.1f)$$

where,

Q is the set of all flow vectors which maintain continuity of flow at all network nodes. Constraints (5.1b) are the minimum head constraints. Constraints (5.1c) are the basic loop equations. Constraints (5.1d) are the link length constraints.

Note that *Eiger et al.* do not use the head generated at each node as a decision variable. Note that H_{nL} is playing that role in Constraint (5.1b). As noted in Section 3.4, the *Sherali and Smith* formulation implicitly enforces energy loss among all loops in the network to be zero. In the formulation used by *Eiger et al.*, Constraints (5.1c) are required to explicitly enforce conservation of energy along all basic loops in the network. The specific nature of the energy conservation constraints is the same and uses the Hazen-Williams equation, Equation (3.1a). The set Q is the set of all flow conserving vectors and ensures continuity at each node. By letting b be the vector of demands such that its i th component b_i is the demand

for node i and R be the directed incidence matrix of links on nodes according to some initially chosen direction for each link such that,

$R_{ij} = 1$ if link j is directed away from node i ,

$R_{ij} = -1$ if link j is directed towards node i ,

and $R_{ij} = 0$ if link j is not adjacent to node i ,

Eiger et al. define the unbounded set Q as

$$Q = \{q: Rq = b\}. \quad (5.1g)$$

and the bounded set Q^* as the set Q restricted by the hyperrectangle Ω as defined by Expression (3.1c) as

$$Q^* = \{q: Rq = b, q \in \Omega\} \quad (5.1h)$$

5.2 Example Problem Using Eiger et al.'s Formulation

Using *Eiger et al.*'s formulation, the optimization problem for the example network can be formulated as

Minimize

$$20x_{121} + 40x_{122} + 80x_{123} + 20x_{131} + 40x_{132} + 80x_{133} + 20x_{231} + 40x_{232} + 80x_{233} \quad (5.2a)$$

subject to

$$\text{sign}(q_{12}) \left| q_{12} \right|^{1.852} (48.77x_{121} + 4.042x_{122} + 0.5628x_{123}) 10^{-5} \leq 100 \quad (5.2b)$$

$$\text{sign}(q_{13}) \left| q_{13} \right|^{1.852} (48.77x_{131} + 4.042x_{132} + 0.5628x_{133}) 10^{-5} \leq 60 \quad (5.2c)$$

$$\begin{aligned} &\text{sign}(q_{12}) \left| q_{12} \right|^{1.852} (48.77x_{121} + 4.042x_{122} + 0.5628x_{123}) 10^{-5} - \\ &\text{sign}(q_{13}) \left| q_{13} \right|^{1.852} (48.77x_{131} + 4.042x_{132} + 0.5628x_{133}) 10^{-5} + \\ &\text{sign}(q_{23}) \left| q_{23} \right|^{1.852} (48.77x_{231} + 4.042x_{232} + 0.5628x_{233}) 10^{-5} = 0 \end{aligned} \quad (5.2d)$$

$$x_{121} + x_{122} + x_{123} = 500 \quad (5.2e)$$

$$x_{131} + x_{132} + x_{133} = 700 \quad (5.2f)$$

$$x_{231} + x_{232} + x_{233} = 800 \quad (5.2g)$$

$$-q_{12} - q_{13} = -600 \quad (5.2h)$$

$$-q_{23} + q_{12} = 200 \quad (5.2i)$$

$$q_{13} + q_{23} = 400 \quad (5.2j)$$

$$q_{12}, q_{13}, q_{23} \geq 0 \quad (5.2k)$$

$$x_{121}, x_{122}, x_{123}, x_{131}, x_{132}, x_{133}, x_{231}, x_{232}, x_{233} \geq 0. \quad (5.2l)$$

5.3 The Lower Bounding Problem

Deleting the flow conservation constraints in problem P, we can rewrite it in matrix notation as

P1 : Minimize

$$c^T x \quad (5.3a)$$

subject to

$$A_1(q)x \leq F \quad (5.3b)$$

$$A_2(q)x = 0 \quad (5.3c)$$

$$A_3x = L \quad (5.3d)$$

$$x \geq 0. \quad (5.3e)$$

Constraints (5.3b) represent Constraints (5.1b) and in problem P. Constraints (5.3c) represent Constraints (5.1c). Constraints (5.3d) represent Constraints (5.1d). The flow conservation constraints, Constraints (5.1f) have been deleted in P1. In Constraints (5.3b), F is the vector of maximum head losses permissible in the chain from a source node to a demand node. For a source node i supplying at a pressure H_i , and a demand node j, the corresponding component in F is given by

$$F_{ij} = (E_i + H_i) - H_n L. \quad (5.3f)$$

By standard duality theory, associating dual variable vectors y_1 , y_2 , and y_3 with Constraints (5.3b), (5.3c) and (5.3d), *Eiger et al.* give the dual of problem P1 as

D1 : Maximize

$$-F^T y_1 - 0^T y_2 - L^T y_3 \quad (5.3g)$$

subject to

$$A_1(q)^T y_1 + A_2(q)^T y_2 + A_3^T y_3 + c \geq 0 \quad (5.3h)$$

$$y_1 \geq 0 \quad (5.3i)$$

$$y_2, y_3 \text{ unrestricted.} \quad (5.3j)$$

Problem D1 is a semi-infinite linear problem with a linear objective function and infinitely many linear constraints. *Eiger et al.*'s lower bounding problem is a more restricted, finite version of the problem D1. By monotonicity of $A_1(q)^T y$ and $A_2(q)^T y$ (as discussed in Section 4.2), it would be sufficient to rewrite Constraint(5.3h) at the minimum and maximum flows instead of the infinite set of flow values. Based on this, the lower bounding problem of *Eiger et al.* is derived as the following linear program

D2 : Maximize

$$-F^T y_1 - 0^T y_2 - L^T y_3 \quad (5.3k)$$

subject to

$$A_1(q_{\min})^T y_1 + A_2(q_{\min})^T y_2 + A_3^T y_3 + c \geq 0 \quad (5.3l)$$

$$A_1(q_{\max})^T y_1 + A_2(q_{\max})^T y_2 + A_3^T y_3 + c \geq 0 \quad (5.3m)$$

$$y_1 \geq 0 \quad (5.3n)$$

$$y_2, y_3 \text{ unrestricted.} \quad (5.3o)$$

5.4 Dual of the Lower Bounding Problem D2

Associating dual variables x^1 and x^2 with the Constraints (5.3l) and (5.3m) respectively, we obtain the dual of the problem D2 as

Minimize : P2

$$c^T(x^1 + x^2) \quad (5.4a)$$

subject to

$$A_1(q_{\min})x^1 + A_1(q_{\max})x^2 \leq F \quad (5.4b)$$

$$A_2(q_{\min})x^1 + A_2(q_{\max})x^2 = 0 \quad (5.4c)$$

$$A_3(x^1 + x^2) = L \quad (5.4d)$$

$$x^1, x^2 \geq 0. \quad (5.4e)$$

A little consideration shows that problem P2 is equivalent to the lower bounding problem LB1 of Section 4.3 using fixed head at sources, and no upper bounds on the heads at each demand node. This proves that the lower bounding problem D2 of *Eiger et al.* is in essence the dual of our LB1 formulation of Section 4.3.

The success of a global optimization scheme using a branch and bound procedure depends on the sharpness of the upper and lower bounds. A tighter lower bound given by LB2 in Section 4.6 allows us to better evaluate the quality of solutions found in the branch-and-bound procedure, which is discussed in more detail in the next Chapter.

6. Global Search

In this Chapter, we propose a branch-and-bound algorithm for globally solving the pipe network design problem within any specified tolerance $\epsilon > 0$. Branch-and-bound is one of the most commonly used techniques in global optimization. Branching refers to the successive partitioning of the feasible domain, and bounding refers to the computation of lower and upper bounds for the global optimum. Bounding helps to identify inferior subregions which can be deleted from the search. Each branch-and-bound node will principally differ in the specification of the hyperrectangle Ω . The hyperrectangle associated with node t of the branch-and-bound tree will be denoted by $\Omega^t = \{q: q_{\min}^t \leq q \leq q_{\max}^t\}$. The individual intervals bounding the flow in an arc (i,j) will be denoted by $[q_{\min_{ij}}, q_{\max_{ij}}]$. In our implementation of the branch-and-bound procedure, we will partition the hyperrectangle defined by the initial bounds (Ω^0) on the flow variables into smaller and smaller hyperrectangles. For a hyperrectangle Ω , the solution to $LB2(\Omega)$ gives an excellent lower bound for the node subproblem $NOP(\Omega)$. The next section describes an effective method to compute an upper bound for $NOP(\Omega)$.

6.1 Upper Bounds

A feasible solution to the $LB2(\Omega)$ problem gives a flow conserving vector in Ω . For a flow conserving vector, an inner linear program can be used for the design of least cost pipe diameters. Given a flow conserving vector $q \in \Omega$, the inner problem is as follows:

ILP(Ω) : Minimize

$$\sum_{(i,j) \in A} \sum_{k=1}^K c_k x_{ijk} + \sum_{i \in S} c_{si} H_{si} \quad (6.1a)$$

subject to

$$\sum \Phi_{ij}(q_{ij}, x_{ij}) = (H_i + E_i) - (H_j + E_j) \quad \forall (i,j) \in A \quad (6.1b)$$

$$\sum_{k=1}^K x_{ijk} = L_{ij} \quad \forall (i,j) \in A \quad (6.1c)$$

$$H_i + E_i \leq F_i + H_{si} \quad \forall i \in S \quad (6.1d)$$

$$H_{iL} \leq H_i + E_i \leq H_{iU} \quad \forall i \in D \quad (6.1e)$$

$$H_{si} \geq 0 \quad \forall i \in S \quad (6.1f)$$

$$x_{ijk} \geq 0. \quad \forall (i,j) \in A \text{ and } k=1, \dots, K. \quad (6.1g)$$

If the solution to $LB2(\Omega)$ is infeasible, the corresponding node can be fathomed. If the solution is feasible, a feasible solution to $ILP(\Omega)$ provides an upper bound to the problem $NOP(\Omega)$. If this upper bound is better than the best upper bound (BUB), we update the best upper bound to this value. The node can be fathomed if the lower bound, $LB2(\Omega) \leq \epsilon$, or else, we partition the hyperrectangle associated with this node into two sub-hyperrectangles based on the branching variable selection strategies discussed in the next section.

6.2 Branching Variable Selection Strategy

Two branching variable selection strategies are proposed. The choice of a branching variable dictated by Strategy 1 attempts to reduce the gap from optimality. The branching variable choice dictated by Strategy 2 tries to obtain the best possible improvement on the lower bound.

6.2.1 Branching Variable Selection Strategy 1

The choice of a branching variable should be such that it reduces the gap from optimality for the nonlinear Constraints (3.2b) and their relaxed version, Constraints (4.6b).

Let $(q, x^1, x^2, H, \lambda)$ be the optimal solution to $LB2(\Omega^t)$. Define

$$\Delta H_{ij} = (H_i + E_i) - (H_j + E_j) \quad \forall (i,j) \in A \quad (6.2.1a)$$

$$\Delta \hat{H}_{ij} = \text{sign}(q_{ij}) |q_{ij}|^{1.852} \sum_{k=1}^K \alpha_{ijk} x_{ijk} \quad \forall (i,j) \in A \quad (6.2.1b)$$

where x_{ijk} is given by Equation (4.2l).

The solution to $LB2(\Omega^t)$ is feasible to $NOP(\Omega^t)$ if $\Delta H_{ij} = \Delta \hat{H}_{ij}$. For this to be true, the following two equations should hold :

$$x^1_{ijk} = \lambda_{ij} x_{ijk} \quad \forall (i,j) \in A \text{ and } k=1, \dots, K \quad (6.2.1c)$$

$$v_{ij}(q_{ij}) = \lambda_{ij} v_{\min ij} + (1 - \lambda_{ij}) v_{\max ij} \quad \forall (i,j) \in A \quad (6.2.1d)$$

where $v_{ij}(q_{ij})$ is computed according to Equation (4.2b).

Proposition 1. If for an arc, $q_{\min ij} = q_{\max ij}$, then $\Delta H_{ij} = \Delta \hat{H}_{ij}$.

Proof: If $q_{\min ij} = q_{\max ij}$, then, $v_{\min ij} = v_{\max ij}$ and the head loss in a link given by Equation (4.2o) reduces to

$$\Delta H_{ij} = v_{\min ij} \sum_{k=1}^K \alpha_{ijk} (x^1_{ijk} + x^2_{ijk}) = \text{sign}(q_{ij}) |q_{ij}|^{1.852} \sum_{k=1}^K \alpha_{ijk} x_{ijk} = \Delta \hat{H}_{ij}.$$

This completes the proof.

Proposition 2. If $\lambda_{ij} = 0$ or $\lambda_{ij} = 1$, then, $\Delta H_{ij} = \Delta \hat{H}_{ij}$.

Proof: If $\lambda_{ij} = 0$, $q_{ij} = q_{\max_{ij}}$ and noting that $x^1_{ijk} = 0 \forall k$, Equation (4.2o) which gives the head loss in a link is equivalent to

$$\Delta H_{ij} = v_{\max_{ij}} \sum_{k=1}^K \alpha_{ijk} x^2_{ijk} = \text{sign}(q_{ij}) |q_{ij}|^{1.852} \sum_{k=1}^K \alpha_{ijk} x_{ijk} = \Delta \hat{H}_{ij}.$$

If $\lambda_{ij} = 1$, $q_{ij} = q_{\min_{ij}}$ and noting that $x^2_{ijk} = 0 \forall k$,

$$\Delta H_{ij} = v_{\min_{ij}} \sum_{k=1}^K \alpha_{ijk} x^1_{ijk} = \text{sign}(q_{ij}) |q_{ij}|^{1.852} \sum_{k=1}^K \alpha_{ijk} x_{ijk} = \Delta \hat{H}_{ij}.$$

This completes the proof.

It should be noted that if Equation (6.2.1d) holds, and $0 < \lambda_{ij} < 1$, then $\Delta H_{ij} \neq \Delta \hat{H}_{ij}$ unless Equation (6.2.1c) holds as well.

Our branching strategy 1 is based on the above two propositions. For each $(i,j) \in A$, we compute the following discrepancy in the critical nonlinear Constraints (4.6b).

$$\delta_{ij} = |\Delta H_{ij} - \Delta \hat{H}_{ij}| \quad \forall (i,j) \in A. \quad (6.2.1e)$$

Note that if $\delta_{pq} = 0$, then we have achieved a feasible optimum solution to the node subproblem and hence the corresponding node can be fathomed.

Let $\Delta = \text{argmax} \{ \delta_{ij} : (i,j) \in A \}$ and I be the set of arcs obtained as $I = \{(r,s) : \delta_{rs} \geq 0.9\Delta\}$.

The branching variable choice is then given by

$$(p,q) = \text{argmax} \{ q_{\max_{rs}} - q_{\min_{rs}} : (r,s) \in I \} \quad (6.2.1f)$$

If $\delta_{pq} > 0$, we partition the interval $[q_{\min_{pq}}, q_{\max_{pq}}]$ into the two intervals as follows:

$$(i) \text{ If } q_{\max_{pq}} > 0 \text{ and } q_{\min_{pq}} < 0, \text{ then the intervals are } [q_{\min_{pq}}, 0] \text{ and } [0, q_{\max_{pq}}] \quad (6.2.1g)$$

$$(ii) \text{ Else, the intervals are } [q_{\min_{pq}}, (q_{\min_{pq}} + q_{\max_{pq}})/2] \text{ and } [(q_{\min_{pq}} + q_{\max_{pq}})/2, q_{\max_{pq}}]. \quad (6.2.1h)$$

6.2.1 Branching Variable Selection Strategy 2

The lower bounds obtained by the LB2 problem are very sensitive to the difference in the minimum and maximum flows for the node subproblems. Our second strategy is to obtain the best possible improvement on the global lower bound initially, by choosing the branching variable (p,q) where

$$(p,q) = \text{argmax} \{ q_{\max_{ij}} - q_{\min_{ij}} : (i,j) \in A \}. \quad (6.2.1a)$$

The intervals for the node subproblems are as given by Equations (6.2.1g) and (6.2.1h). After a few iterations if no significant improvement in the global lower bound is obtained in successive iterations i.e. if for iterations k and $k+1$, and corresponding global lower bounds GLB_k and GLB_{k+1} if

$$GLB_{k+1} - GLB_k < 0.8GLB_{k+1} \quad (6.2.1b)$$

we switch to branching variable selection strategy 1. This method allows us to initially focus on improving the quality of the global lower bound, and later on attempt to reduce the gap from nonlinearity.

6.3 Branch-and-Bound Algorithm

The branch-and-bound algorithm proposed is in the spirit of *Sherali and Tuncbilek* [1992] where general polynomial programs are addressed. At any stage r of the branch-and-bound algorithm, we have a set of nonfathomed denoted as T_r . We select an active node t^* in T_r that has the least lower bound, breaking ties arbitrarily. Solving the corresponding $LB2(\Omega^{t^*})$ and $ILP(\Omega^{t^*})$ problems yields lower and upper bounds for the node. If the nodelevel is equal to the maximum nodelevel, or if the lower bound is greater than the best known upper bound (BUB) or some prescribed percentage of the best known upper bound, we fathom the node. If the computation time is greater than the maximum permissible computation time (CT), we stop. If neither of the above holds, we partition the hyperrectangle associated with this node into two sub-hyperrectangles based on a branching variable selected according to the strategy discussed in Section 6.2.

Step 0. Initialization Step

Set a numerical value for the tolerance ϵ and maximum permissible computation time, CT. Set the stage counter $r=1$, nodelevel $l=0$, $T_1=\{1\}$ and Clock=0. Heuristically determine the initial flow bounds. Denote the initial hyperrectangle defined by the flow bounds as $\Omega^{r,l} = \Omega^{1,1} \equiv \Omega^0$. Set $t^* = 1$. Solve the node zero problem $LB2(\Omega^{1,1})$. If infeasible, then stop; the given problem is infeasible. Otherwise, find an optimal solution and initialize the global lower bound, GLB to this optimal value. Determine the branching variable index (p,q) . If $\delta_{pq} = 0$, then stop as this solution solves the NOP. If $\delta_{pq} > 0$, solve the corresponding $ILP(\Omega^{1,1})$. If infeasible, initialize the BUB to $+\text{inf}$. If feasible, initialize the BUB to its optimal solution. If $LB_{1,1} \geq BUB(1-\epsilon)$, then stop with the optimal solution to $ILP(\Omega^{r,t^*})$ as the solution to the NOP.

Step 1. Partitioning Step

Having the active node (r,t^*) to be partitioned, and given the choice (p,q) of the branching variable, partition this node into two sub-nodes associated with the two hyperrectangles Ω^{r,t_1} and Ω^{r,t_2} that are identical to Ω^{r,t^*} except that the two respective interval restrictions on q_{pq} are given by Expression (6.2.1g) or (6.2.1h), as appropriate. Set $\text{nodelevel}(t_1) = \text{nodelevel}(t_2) = \text{nodelevel}(t^*) + 1$. Update $T_{r+1} = (T_r - \{t^*\}) \cup \{t_1, t_2\}$. Solve $LB2(\Omega^{r,t_1})$. If this problem is infeasible, then fathom the corresponding node. Otherwise, find an optimal solution and denote its objective value by LB_{r,t_1} . Using this optimal solution, determine the corresponding branching variable index (p,q) . If $\delta_{pq} = 0$, then fathom the corresponding node as this

solution solves the node subproblem $\text{NOP}(\Omega^r, t_1)$. If $\delta_{pq} > 0$, solve $\text{ILP}(\Omega^r, t_1)$ to obtain an upper bound UB_{r,t_1} for $\text{NOP}(\Omega^r, t_1)$. If $\text{UB}_{r,t_1} < \text{BUB}$, update $\text{BUB} \leftarrow \text{UB}_{r,t_1}$. Repeat the bounding procedure by replacing t_1 with t_2 and solving $\text{LB2}(\Omega^r, t_2)$. From the list of nodes T_{r+1} , identify all nodes $p \in T_{r+1}$ such that their corresponding lower bounds, $\text{LB}_2 \geq \text{BUB}(1-\epsilon)$ and update $T_{r+1} = T_{r+1} - \{p\}$. Set $r=r+1$. If $\text{Clock} > \text{CT}$, stop with BUB as the best found solution.

Step 2. Node Selection Step

If $T_r = \emptyset$ then stop; the incumbent solution is optimal (within the ϵ -tolerance). Else, select an active node (r, t^*) where $t^* \in \text{argmin}\{\text{LB}_{r,t} : t \in T_r\}$ is associated with the least lower bound over the active nodes at stage r . Return to Step 1.

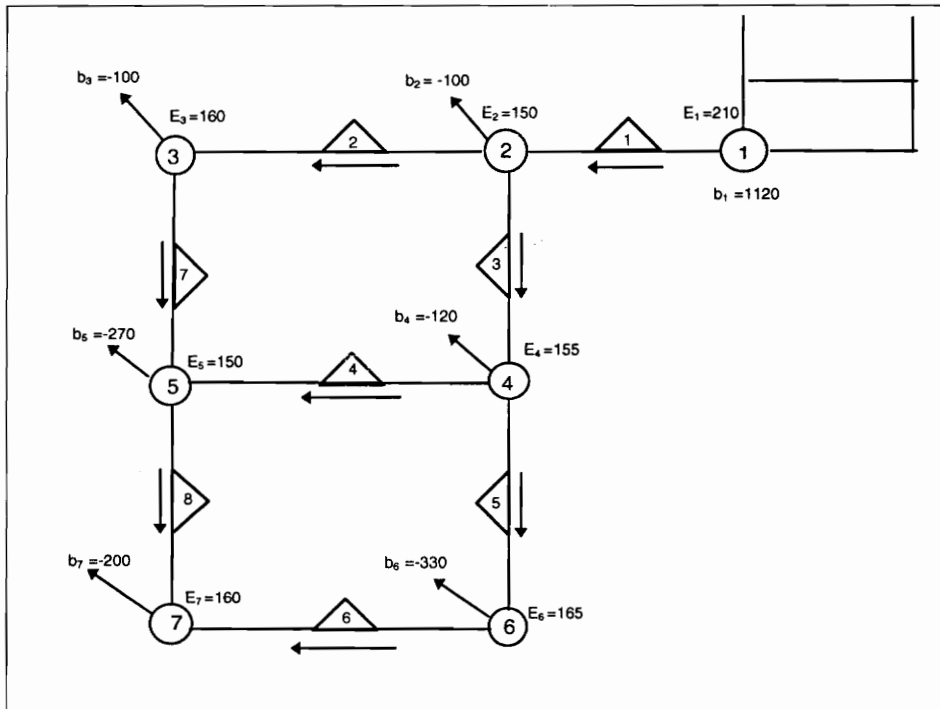
The next Chapter in this thesis presents some results of applying this algorithm to some standard test problems from literature.

7. COMPUTATIONAL EXPERIENCE ON SOME STANDARD TEST PROBLEMS

The branch-and-bound procedure presented in Chapter 6 was applied to two different standard water distribution network problems using the lower bounding scheme LB2 discussed in Chapter 4. The algorithms were implemented on a SUN SPARC 10 UNIX workstation, using the CPLEX callable library to solve the linear programming problems. The computer code was written in C++. The CPLEX code provided by the CPLEX Corporation is written in C. The LB2 lower bounding scheme was used in the algorithm. The algorithm was implemented for both the branching variable selection strategies discussed in Chapter 6 for target gaps of 10%, 5%, 1% and 0.5% between the LB2 global lower bound and the global optimum. At each node enumerated by the algorithm, the LB1 lower bound was also computed in order to determine the global LB1 lower bound for the sake of comparison.

7.1 The Two Loop Network

The two loop network is a single source test problem originally presented by *Alperovits and Shamir* [1977].



Figure(7.1a) Two Loop Network

The test problem is repeated in Section 7.1.1 for the sake of convenience. Two cases of pipe diameter restrictions were tested for different flow bounds. In Section 7.1.2, we consider the set of candidate pipe diameters restricted for each link, from a set of commercially available diameters, as the ones given in the original paper. A number of other researchers like *Fujiwara and Khang* [1990], *Eiger et al.* [1994] and *Loganathan et al.* [1995] permit the complete set of commercially available standard pipe diameters to be the set of candidate diameters for all links. Section 7.1.3 presents results for this case.

7.1.1 Test Problem Data

Set of commercially available diameters $d = \{1, 2, 3, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24\}$.

Cost per unit length = $\{2, 5, 8, 11, 16, 23, 32, 50, 60, 90, 130, 170, 300, 550\}$.

Table (7.1.1a) Arc Data.

Link Index	Arcs	Length (m)	C_{HW}
1	(1,2)	1000	130
2	(2,3)	1000	130
3	(2,4)	1000	130
4	(4,5)	1000	130
5	(4,6)	1000	130
6	(6,7)	1000	130
7	(3,5)	1000	130
8	(5,7)	1000	130

Table (7.1.1b) Node Data.

Node	Elevation (m)	Minimum Pressure required	Supply or Demand
1	210	0	1120
2	150	30	-100
3	160	30	-100
4	155	30	-120
5	150	30	-270
6	165	30	-330
7	160	30	-200

Table (7.1.1c) Diameter Restrictions on the Links from the Commercially Available Set.

Link Index	Arcs	Restricted Diameters (in.)
1	(1,2)	12, 14, 16, 18, 20
2	(2,3)	6, 8, 10, 12, 14
3	(2,4)	10, 12, 14, 16, 18
4	(4,5)	3, 4, 6, 8
5	(4,6)	10, 12, 14, 16, 18
6	(6,7)	8, 10, 12, 14, 16
7	(3,5)	6, 8, 10, 12, 14
8	(5,7)	6, 8, 10, 12, 14

7.1.2 Two Loop Network Results For The Restricted Diameter Case

The derived lower bounds and the best objective function value obtained by the branch-and-bound procedure are presented for two different instances of flow bounds. The first instance uses heuristically determined minimum and maximum flow values. The heuristic used for determining the flow bounds in Table (7.1.2b) is as follows:

- 1) Assume the starting minimum and maximum flows for links 2 and 3 to be (0,1020) and (0,1020).
- 2) Examine all permutations of flow directions such that node 4 is an end node(i.e. its forward star is a null set) and identify the minimum and maximum flows in each link over all the permutations. Repeat this procedure with nodes 5, 6 and 7 as end nodes instead of node 4.
- 3) The minimum and maximum flow in a link is obtained by taking the minimum and maximum of all the flow values obtained for that link in Step 2.

Table (7.1.2b) Initial Minimum and Maximum Flow.

Link Index	qmin (m ³ /hr)	qmax (m ³ /hr)
1	1120	1120
2	0	1020
3	0	1020
4	-650	900
5	-120	900
6	-450	570
7	-100	920
8	-370	650

Table (7.1.2c) gives the results for the global lower bounds, LB1 and LB2, the global upper bounds and the computational effort involved for different target gaps for the branch-and-bound algorithm for the first branching variable selection strategy. Table (7.1.2d) gives the results for the second branching variable selection strategy. The count for the number of linear programs solved as reported in Table (7.1.2c) and Table (7.1.2d) and subsequent similar tables does not include the linear programs solved to compute the LB1 lower bound. Table (7.1.2e) gives the least cost pipe sizings and the optimum flows for the best solution obtained for a target gap of 0.5% in Table (7.1.2c).

Table (7.1.2c) Results for Branching Variable Selection Strategy 1.

Target Gap	Global LB1	Global LB2	Global Optimum	Nodes Enumerated	Linear Programs Solved	Computation Time (seconds)
10%	\$333,702	\$407,555	\$441,603	127	213	17
5%	\$388,309	\$421,393	\$438,916	179	307	28
1%	\$404,535	\$433,593	\$437,426	345	606	53
0.5%	\$421,946	\$434,845	\$436,931	443	782	69

Table (7.1.2d) Results for Branching Variable Selection Strategy 2.

Target Gap	Global LB1	Global LB2	Global Optimum	Nodes Enumerated	Linear Programs Solved	Computation Time (seconds)
10%	\$333,092	\$404,650	\$444,524	63	116	10
5%	\$352,453	\$423,598	\$439,400	121	214	18
1%	\$405,036	\$434,125	\$437,013	223	401	35
0.5%	\$421,496	\$435,044	\$436,915	265	474	42

It can be noted from the results that the second branching variable selection strategy outperforms the first. The best previous known solution for this case was by *Alperovits* and *Shamir* and has an objective function value of \$479,525. *Sherali* and *Smith* [1995] report a solution having an objective function value of \$425,821 by relaxing the pipe diameter restrictions to permit inclusion of 4" pipe segments in link 8.

Table (7.1.2e) Branch-and-Bound Results.

Pipe Section	Sections having Length (m) of Diameter (in)		Flow (m ³ /hr)	Head Loss (m)	Heads and Bounds			
	Node (i)	H _{iL} (m)			H _i + E _i (m)	H _{iU} (m)		
1	1000.00	18"	1120.00	6.75	1	n/a	210.00	n/a
2	1000.00	14"	448.80	4.22	2	180.00	203.25	210.00
3	225.00	14"	571.20	4.15	3	190.00	199.03	210.00
4	775.00	16"	9.40	5.84	4	185.00	199.10	210.00
	978.00	3"						
5	22.00	4"	441.80	4.10	5	180.00	193.26	210.00
	1000.00	14"						
6	1000.00	8"	111.80	4.91	6	195.00	195.00	210.00
7	22.30	10"	348.80	5.78	7	190.00	190.09	210.00
	977.60	12"						
8	1000.00	8"	88.20	3.16				

To examine the sensitivity of the algorithm on the initial bounds imposed on the flow variables, a second, much more restricted set of minimum and maximum flow values was used. The set of flow bounds for this instance is from *Sherali and Smith*, Report #HDS95-6 [1995].

Table (7.1.2f) Second Set of Minimum and Maximum Flows.

Link Index	qmin (m ³ /hr)	qmax (m ³ /hr)
1	1120	1120
2	250	350
3	650	750
4	0	100
5	500	600
6	200	300
7	-100	250
8	-100	100

The comparative results for different target gaps and branching variable selection strategies are presented in Table (7.1.2g) and Table (7.1.2h).

Table (7.1.2g) Results for Branching Variable Selection Strategy 1.

Target Gap	Global LB1	Global LB2	Global Optimum	Nodes Enumerated	Linear Programs Solved	Computation Time (secs)
10%	\$395,721	\$423,889	\$449,651	1	2	0.5
5%	\$400,237	\$425,164	\$443,449	3	6	1
1%	\$408,947	\$439,418	\$442,979	37	65	6
0.5%	\$413,348	\$440,986	\$442,763	63	115	11

Table (7.1.2h) Results for Branching Variable Selection Strategy 2.

Target Gap	Global LB1	Global LB2	Global Optimum	Nodes Enumerated	Linear Programs Solved	Computation Time (seconds)
10%	\$395,721	\$423,890	\$445,691	1	2	0.5
5%	\$402,677	\$433,599	\$443,811	7	12	1
1%	\$404,328	\$439,083	\$442,959	17	32	3
0.5%	\$422,750	\$440,818	\$442,801	41	80	7

Tightening the bounds on the flow greatly improves the quality of lower bounds obtained, reduces the number of nodes enumerated and consequently, greatly reduces the computation time. At the same time, it may be noted that the quality of solutions is slightly compromised. The least cost solution of \$442,763 is about 1.3% higher than the solution of \$436,931 obtained for the flow bounds as in Table (7.1.2b). Hydraulic redundancy often dictates tight bounds on the flow variables. This also demonstrates that for a given layout, hydraulic redundancy requirements can often be effectively implemented at a small extra cost.

7.1.3 Two Loop Network Results For the Unrestricted Diameter Case

Most of the researchers have done away with the separate diameter restrictions for each link from the set of commercially available diameters. To test the effectiveness of the algorithm, and compare its performance

with the existing ones, we apply the algorithm for this case with the initial, heuristic flow bounds as in Table (7.2.1b).

Table (7.1.3a) and Table (7.1.3b) give the results for different target gaps and branching variable selection strategies.

Table (7.1.3a) Results for Branching Variable Selection Strategy 1.

Target Gap	Global LB1	Global LB2	Global Optimum	Nodes Enumerated	Linear Programs Solved	Computation Time (seconds)
10%	\$302,857	\$382,914	\$419,491	93	170	21
5%	\$339,251	\$394,663	\$414,251	123	225	28
1%	\$368,309	\$399,943	\$403,708	375	682	88
0.5%	\$376,169	\$401,965	\$403,390	437	788	102

Table (7.1.3b) Results for Branching Variable Selection Strategy 2.

Target Gap	Global LB1	Global LB2	Global Optimum	Nodes Enumerated	Linear Programs Solved	Computation Time (seconds)
10%	\$281,439	\$387,986	\$423,560	121	228	28
5%	\$319,642	\$397,137	\$409,440	183	346	44
1%	\$376,244	\$401,960	\$405,561	385	718	95
0.5%	\$401,118	\$403,214	\$405,378	473	867	114

Fujiwara et al. [1990] report a solution with an objective function value of \$415,271. *Loganathan et al.* [1990] report a solution of \$412,931. *Eiger et al.* [1994] report a solution with an objective value of \$402,352 and a global lower bound of \$400,703 for a target gap of 0.5%. However, their results permit small violations in the flow conservation constraints. It may be noted that the best global lower bound of \$403,214 obtained by the algorithm is greater than *Eiger et al.*'s near feasible solution of \$402,352. The best known optimal solution for this case was by *Loganathan et al.* [1995] for a network with a total cost of \$403,657. Our best solution of \$403,390 is slightly better than this previously best known solution . It may

also be noted that the optimum flow values presented for our best solution in Table (7.1.3c) are very nearly the same as the optimum flow values presented by *Loganathan et al.* [1995].

Table (7.1.3c) Branch-and-Bound Results.

Pipe Section	Sections having Length (m) of Diameter (in)		Flow (m ³ /hr)	Head Loss (m)	Heads and Bounds			
	Node (i)	H _{iL} (m)			H _i + E _i (m)	H _{iU} (m)		
1	1000.00	18"	1120.00	6.75	1	n/a	210.00	n/a
2	795.41	10"	368.33	13.25	2	180.00	203.25	210.00
	204.59	12"						
3	1000.00	16"	651.67	4.39	3	190.00	190.00	210.00
4	999.50	1"	0.97	18.86	4	185.00	198.86	210.00
	8.50	2"						
5	310.35	14"	530.70	3.86	5	180.00	180.00	210.00
	689.65	16"						
6	11.14	8"	208.70	5.00	6	195.00	195.00	210.00
	988.86	10"						
7	98.48	8"	268.33	10.00	7	190.00	190.00	210.00
	901.52	10"						
8	1000.00	1"	-0.70	10.00				

7.2 The Hanoi Network

The Two Loop network problem is the most popular test problem among researchers who want to test the efficacy of their algorithms. However, since it is a very small size problem, and real world networks tend to be of a much more larger size, we apply the algorithm to a large water distribution known in literature as the Hanoi network. The network consists of three basic loops, thirty two nodes and thirty four links. All the nodes are at the same elevation. All the pipes are specified to have the same Hazen-William's coefficient. The data presented is as it appears in *Fujiwara and Khang* [1990]. In the network layout shown in Figure(7.2a), the direction of some of the arcs has been reversed from as it appears in the previous papers, to be consistent with our definition of an arc in a network.

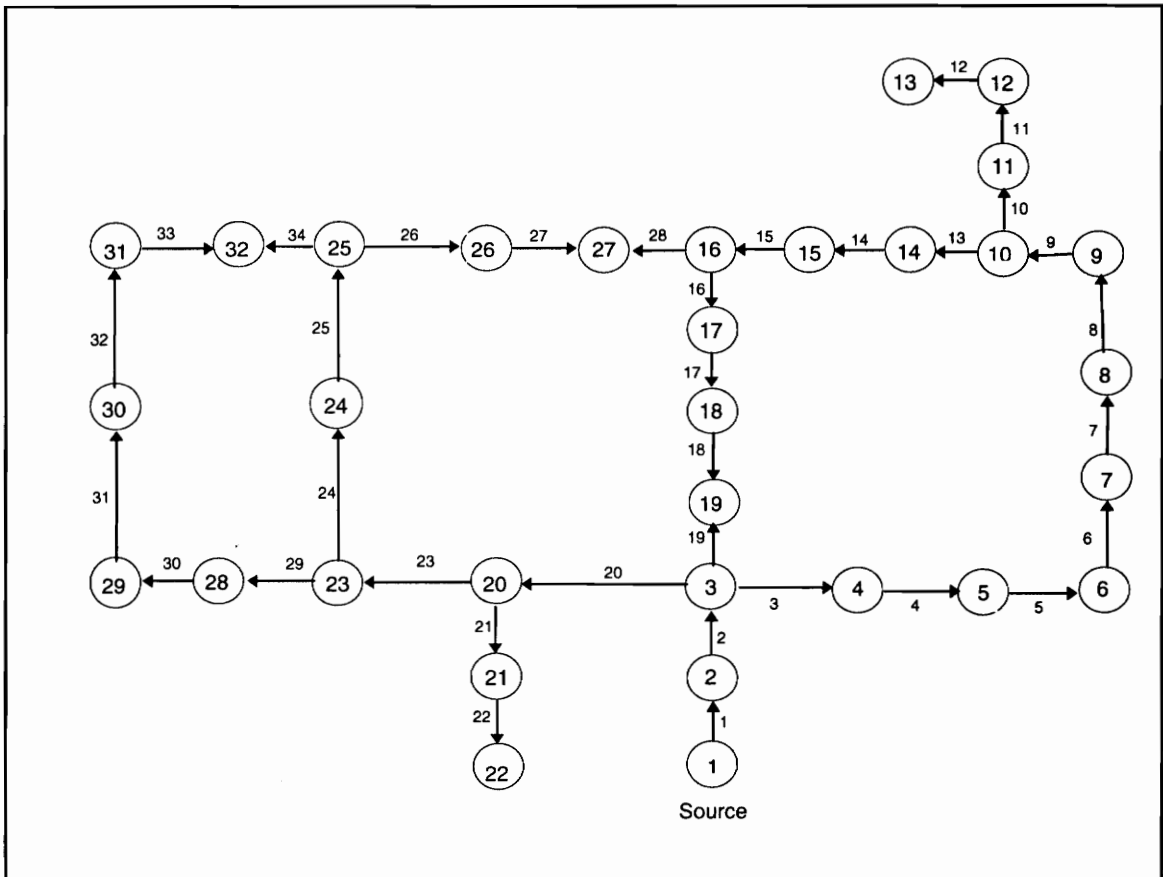
7.2.1 Test Problem Data

Set of commercially available diameters, $d = \{12, 16, 20, 24, 30, 40\}$.

For a section of diameter d_k , cost per unit length $c_k = 1.1d_k^{1.5}$.

Based on this relation the cost per unit length for different diameters is computed as

$c = \{45.73, 70.40, 98.39, 129.33, 180.74, 278.28\}$.



Figure(7.2a) Hanoi Network

Table (7.2.1a) Arc Data.

Link Index	Arcs	Link Length (m)	C_{HW}
1	(1,2)	100	130
2	(2,3)	1350	130
3	(3,4)	900	130

Table(7.1.2a) Arc Data (contd.)

Link Index	Arcs	Link Length (m)	C _{HW}
4	(4,5)	1150	130
5	(5,6)	1450	130
6	(6,7)	450	130
7	(7,8)	850	130
8	(8,9)	850	130
9	(9,10)	800	130
10	(10,11)	950	130
11	(11,12)	1200	130
12	(12,13)	3500	130
13	(10,14)	800	130
14	(14,15)	500	130
15	(15,16)	550	130
16	(16,17)	2730	130
17	(17,18)	1750	130
18	(18,19)	800	130
19	(3,19)	400	130
20	(3,20)	2200	130
21	(20,21)	1500	130
22	(21,22)	500	130
23	(20,23)	2650	130
24	(23,24)	1230	130
25	(24,25)	1300	130
26	(25,26)	850	130
27	(26,27)	300	130
28	(16,27)	750	130
29	(23,28)	1500	130
30	(28,29)	2000	130
31	(29,30)	1600	130
32	(30,31)	150	130
33	(31,32)	860	130
34	(25,32)	950	130

Table (7.2.1b) Node Data.

Node	Minimum Pressure required	Supply or Demand
1	100	19940
2	30	-890
3	30	-850
4	30	-130
5	30	-725
6	30	-1005
7	30	-1350
8	30	-550
9	30	-525
10	30	-525
11	30	-500
12	30	-560
13	30	-940
14	30	-615
15	30	-280
16	30	-310
17	30	-865
18	30	-1345
19	30	-60
20	30	-1275
21	30	-930
22	30	-485
23	30	-1045
24	30	-820
25	30	-170
26	30	-900
27	30	-370
28	30	-290
29	30	-360
30	30	-360
31	30	-105
32	30	-805

7.2.2 Hanoi Network Results

Table (7.2.2a) gives the heuristically determined initial bounds on the flow variables. The heuristic used for determining the set of minimum flows is as follows:

- 1) For the network layout, extract the shortest path spanning tree SP. Let L be the list of all leaf nodes in SP.
- 2) Select a node $p \in L$ with the smallest index; in the chain from p to the root node, delete three nodes from SP, including node p.
- 3) Set $L=L-\{p\}$.
- 4) If $L=\emptyset$, stop. Else, go to step 2.
- 5) For the tree SP, compute the flow in each link and designate this as the set of minimum flow in the network.

The heuristic used for determining the set of maximum flows is as follows:

- 1) For the network layout, extract the shortest path spanning tree SP. Let L be the list of all leaf nodes in SP.
- 2) Select a node $p \in L$ with the smallest index; select three nearest nodes to p in the layout, not in SP and such that their addition to SP does not create a cycle.
- 3) Add these nodes to SP, at the corresponding leaf node p, as new nodes with demands as in the original layout. Add an extra supply equivalent to the demand created by the addition of these nodes to SP, to node p.
- 4) Set $L=L-\{p\}$.
- 5) If $L=\emptyset$, stop. Else, go to step 2.
- 6) For the tree SP, compute the flow in each link and designate this as the set of maximum flow in the network.

The above heuristics are based on the fact that for this network, there is a single supply node, the Hazen William's coefficient is the same for all the pipe diameters and the minimum pressure required at each node is also the same.

Table (7.2.2a) Initial Minimum and Maximum Flow.

Link Index	qmin (m ³ /hr)	qmax (m ³ /hr)
1	19940	19940
2	19050	19050
3	4885	9455
4	4755	9325
5	4030	8600
6	3025	7595
7	1675	6245

Table (7.2.2a) Initial Minimum and Maximum Flow (contd.).

Link Index	qmin (m ³ /hr)	qmax (m ³ /hr)
8	1125	5695
9	600	5170
10	2000	2000
11	1500	1500
12	940	940
13	-1925	2645
14	-2540	2030
15	-2820	1750
16	-3070	0
17	-3935	-865
18	-5280	-2210
19	2270	5340
20	6475	8255
21	1415	1415
22	485	485
23	3785	5565
24	1265	4230
25	445	3410
26	-3240	4400
27	-4140	3500
28	-3130	4510
29	290	1920
30	0	1630
31	-360	1270
32	-720	910
33	-825	805
34	0	1630

Table (7.2.2b) and Table (7.2.2c) give the comparative results for the branch-and-bound algorithm for different target gaps. Table (7.2.2c) gives the least cost pipe sizing and the optimal set of flows in each link.

Table (7.2.2b) Results for Branching Variable Selection Strategy 1.

Target Gap	Global LB1	Global LB2	Global Optimum	Nodes Enumerated	Linear Programs Solved	Computation Time (seconds)
10%	\$5,122,559	\$5,587,179	\$6,187,475	21	31	21
5%	\$5,437,810	\$5,830,734	\$6,122,612	345	518	308
1%	\$5,673,626	\$6,003,585	\$6,062,713	1407	2113	1235
0.5%	\$5,862,912	\$6,029,076	\$6,059,841	1927	2877	1671

Table (7.2.2c) Results for Branching Variable Selection Strategy 2.

Target Gap	Global LB1	Global LB2	Global Optimum	Nodes Enumerated	Linear Programs Solved	Computation Time (seconds)
10%	\$5,153,214	\$5,657,781	\$6,181,796	15	30	18
5%	\$5,200,906	\$5,846,452	\$6,137,858	107	160	99
1%	\$5,880,833	\$6,003,184	\$6,062,494	867	1301	791
0.5%	\$5,900,752	\$6,029,554	\$6,058,976	1773	2687	1604

Eiger et al. [1994] report a solution with an objective function value of \$6,026,660 with a gap of 0.5% between the global lower bound and the optimum. However, their solution permits small violations in the flow conservation constraints. By defining a narrow box of $(q-100, q+100)$ around the optimum flow values q of their solution, we obtained the lower bound for that subspace as 6,260,197 and the optimal solution as 6,060,197. It may be noted that our global lower bound is higher than their near optimal solution.

Table (7.2.2c) Branch-and-Bound Results.

Pipe Section	Sections having Length (m) of Diameter (in)		Flow (m^3/hr)	Head Loss (m)	Heads and Bounds			
					Node (i)	H_{iL} (m)	$H_i + E_i$ (m)	H_{iU} (m)
1	100.00	40"	19940.00	2.86	1	n/a	100	n/a
2	1350.00	40"	19050.00	35.48	2	30.00	97.14	100.00
3	900.00	40"	7956.61	4.70	3	30.00	61.66	100.00
4	1150.00	40"	7826.61	5.82	4	30.00	56.97	100.00

Table (7.2.2c) Branch-and-Bound Results (contd.).

Pipe Section	Sections having Length (m) of Diameter (in)	Flow (m ³ /hr)	Head Loss (m)	Heads and Bounds			
				Node (i)	H _{iL} (m)	H _i + E _i (m)	H _{iU} (m)
5	1450.00 40"	7101.61	6.13	5	30.00	51.14	100.00
6	450.00 40"	6096.61	1.43	6	30.00	45.01	100.00
7	850.00 40"	4746.61	1.70	7	30.00	45.58	100.00
8	850.00 40"	4196.61	1.36	8	30.00	41.88	100.00
9	85.83 30"	3671.61	1.32	9	30.00	40.52	100.00
	714.17 40"						
10	950.00 30"	2000	1.56	10	30.00	39.19	100.00
11	1200.00 24"	1500	3.43	11	30.00	37.63	100.00
12	3500.00 24"	940	4.21	12	30.00	34.21	100.00
13	256.67 16"	1146.61	5.51	13	30.00	30.00	100.00
	543.33 20"						
14	500.00 16"	531.61	1.52	14	30.00	33.67	100.00
15	550.00 12"	251.61	1.70	15	30.00	32.16	100.00
16	2730.00 12"	-161.65	3.68	16	30.00	30.45	100.00
17	1750.00 16"	-1026.65	17.84	17	30.00	34.14	100.00
18	162.81 20"	-2371.65	6.89	18	30.00	51.98	100.00
	637.19 24"						
19	400.00 24"	2431.65	2.80	19	30.00	58.87	100.00
20	2200.00 40"	7811.74	11.09	20	30.00	50.57	100.00
21	496.03 16"	1415	15.42	21	30.00	35.16	100.00
	1003.97 20"						
22	500.00 12"	485	5.16	22	30.00	30.00	100.00
23	2650.00 40"	5121.75	6.11	23	30.00	44.46	100.00
24	1230.00 30"	3488.75	5.66	24	30.00	38.81	100.00
25	1130.00 30"	2668.75	3.64	25	30.00	35.17	100.00
26	53.98 16"	1166.74	4.16	26	30.00	31.01	100.00
	796.02 20"						
27	300.00 12"	266.74	1.01	27	30.00	30.00	100.00

Table (7.2.2c) Branch-and-Bound Results (contd.).

Pipe Section	Sections having Length (m) of Diameter (in)	Flow (m ³ /hr)	Head Loss (m)	Heads and Bounds			
				Node (i)	H _{iL} (m)	H _i + E _i (m)	H _{iU} (m)
28	750.00 12"	103.26	0.45	28	30.00	38.29	100.00
29	68.83 12"	587.99	6.18	29	30.00	30.00	100.00
	1431.17 16"						
30	2000.00 12"	297.99	8.29	30	30.00	30.38	100.00
31	1600.00 12"	-62.00	0.38	31	30.00	30.68	100.00
32	150.00 16"	-422.00	0.30	32	30.00	32.98	100.00
33	726.90 16"	-527.00	2.30				
	133.10 20"						
34	950.00 24"	1332.00	2.18				

The computation time can be reduced by tightening the initial flow bounds. The next chapter discusses the significance of the results obtained and recommendations for future research.

8. CONCLUSION AND RECOMMENDATIONS FOR FUTURE RESEARCH

8.1 Summary and Significance of Results

For problems where functional evaluations at discrete points alone do not guarantee a global optimum in finite time, and for problems where necessary and sufficient optimality conditions to verify that a solution is indeed the global optimum do not exist, we must resort to computing lower and upper bounds over ranges of values of a function. Computation of sharp lower and upper bounds is crucial for the success of any global optimization scheme. In this thesis, we proposed an improved method to compute global lower bounds for the pipe network optimization problem. The new lower bounding scheme takes advantage of the monotonicity of the nonlinear constraints by developing suitable linear relaxations. It yields a flow conserving solution which allows us to compute upper bounds for the node subproblems. The improved lower and upper bounds were used in a global branch-and-bound optimization scheme on some standard test problems from the literature.

The proposed global optimization algorithm does very well on small sized problems, generating solutions as close as 1% to the global lower bound. The algorithm was found to be very sensitive to the initial bounds on the flow variables. The success of its application to larger problems (25 links or more) depends on being able to develop very tight initial bounds on the flow variables. The procedure can also benefit by computing sharper upper bounds. It was also noted that for a large number of node subproblems, the flow conserving solutions obtained by the lower bounding scheme were infeasible to the upper bounding heuristic. The algorithm can benefit by using a gradient based upper bounding scheme as proposed by *Alperovits and Shamir* [1977] which allows the upper bound to migrate to a local optimum. Not only will this yield better quality solutions, but it will also significantly reduce the size of the search space by fathoming the nodes which are infeasible to the gradient based upper bounding scheme.

8.2 Conclusions and Recommendations for Future Research

All the global optimization strategies (including the one proposed in this thesis) for the pipe network design problem in the literature exhibit great sensitivity to the initial bounds on the flow variables. Tight initial bounds reduce the size of the search space to manageable proportions for these algorithms. Our recommendation for future research lies in developing effective methods for generating tight initial bounds.

Another related suggestion for future research is to devise procedures for generating better upper bounds, given a solution to the lower bounding problem. The algorithm can benefit by computing sharper upper bounds for the reasons mentioned above. A local optimization scheme is recommended to obtain better

quality solutions and to fathom the infeasible node subproblems. Alternative branching strategies that lead to convergence should be examined and computationally tested.

Finally, we could investigate and computationally test the usage of a number of supporting hyperplanes (between ten to fifteen), in the lower bounding scheme, once the direction of flow in a link is determined. These supporting hyperplanes would serve to tighten the relaxation for the corresponding concave or convex constraint. However, it could significantly increase the size of the problem, but is expected to yield much tighter lower bounds. Alongwith a suitable upper bounding scheme, it is expected that larger problems can be solved within a reasonable effort.

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