Two Level Weight Optimization of Composite Laminates using Integer Programming

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Optimization of composite laminates requires the satisfaction of constraints where the design ply thicknesses and orientations can only take discrete values prescribed by the manufacturers. Heuristics such as particle swarm or genetic algorithms are inefficient in such cases because they provide sub-optimal solutions when the number of design variables is large. They also are computationally expensive in handling the combinatorial nature of the problem. In contrast, with the help of binary decision variables, Mixed Integer Programming (MIP) can be adopted to optimize such laminates efficiently. This paper presents an approach to reformulate lamination parameters and failure constraints as functions of binary decision variables. The buckling load maximization for a simply-supported laminated plate is initially demonstrated using integer linear programming. Next, the laminate weight is minimized by varying the number of plies for a given external bi-axial compressive load and subjected to buckling and material failure constraints. A variation of laminate weight minimization is demonstrated by fixing the number of plies and assuming discrete changes in ply thicknesses. This is achieved using a sequential two-level optimization (STLO) for laminates having uniform ply thickness. Finally, a scalability study is performed to evaluate the performance of MIP for different problem sizes. It is demonstrated that all three formulations with integer programming achieve significant performance gain and robustness over standard heuristic solvers.

I. Nomenclature

a	=	laminate length (in.)
b	=	laminate width (in.)
h	=	total thickness of the laminate (in.)
m_{def}	=	maximum number of half sine waves in x-direction

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n_{def} = maximum number of half sine waves in y-direction

N = number of plies for half symmetric laminates

N_{xx} = X component of load acting on the laminated panel edge (lb/in.)

N_{yy} = Y component of load acting on the laminated panel edge (lb/in.)

p = size of the prescribed fiber orientations set

q = size of the prescribed ply thickness set

T = ply thickness set

U = material invariants

V = lamination parameters

x = x-axis corresponding to laminate length

y = y-axis corresponding to the laminate width

z = z-axis corresponding to the laminate thickness

$\mathbf{x}$ = vector of decision variables associated with fiber orientations

$\mathbf{y}$ = vector of decision variables associated with ply thicknesses

ε_{al} = allowable strains

Θ = fiber orientations set

λ = load scaling parameter

λ_{cr} = critical scaling parameter

λ_0 = design buckling load factor

λ_b = buckling load factor

II. Introduction

Optimization of composite laminates is a non-convex optimization problem that involves finding values of fiber orientation and thickness of multiple plies. The goal is to minimize the total weight of the laminate by satisfying various stability and strength constraints. These constraints can be linear or non-linear functions of lamination parameters which themselves are non-linear functions of design parameters. Over the last two decades, many studies have aimed to develop efficient optimization algorithms that can take into account such constraints and the discrete nature of design variables. Evolutionary algorithms have been widely used to find a globally optimal design for composite laminate configuration. Le Riche and Haftka [1] used a genetic algorithm (GA) in laminate optimization and showed that its efficiency in obtaining a good quality solution increases with an increase in the number of design variables but decreases with an increase in the number of nonlinear constraints. In that study, to increase the number of design variables, the total number of plies are increased while keeping the number of available ply orientations to be same. An improved version with combination of fixed and variable penalty function was explored in [2]. The particle swarm algorithm (PSO) used in the stacking sequence optimization for maximum fundamental frequency by Bargh and Sadr [3] was more efficient than the GA. Since multiple combinations of fiber orientations and ply thicknesses can give same-valued lamination parameter [4], most optimization algorithms struggle to obtain a converged solution when the design space is large.

Other studies developed sophisticated PSO algorithms such as vector evaluated particle swarm (VEPSO) [5] and quantum values particle swarm (QPSO) [6] for multi-objective requirements. A discrete PSO algorithm is presented by Chang et al. [7]. Simulated annealing is another reliable technique used in the literature for optimization of composite laminates [8–10]. These studies show the use of standard or modified meta-heuristic approaches which are best suited for solving problems having continuous design space. Generally, in these formulations, continuous variables are rounded to the nearest integer to get discrete values of design variables. However, because of the stochastic characteristics of such meta-heuristics, they have difficulty in converging to a global optimum or near-optimal solutions, in a reasonable time. The parameters governing these algorithms such as the population size, crossover and mutation probability in GA or swarm size and particle velocity scaling in PSO are problem dependent and need adjustment over multiple runs. Usually, these methods are coupled with non-heuristic optimization algorithms, such as gradient based algorithms to improve the convergence efficiency.

Many studies have demonstrated the advantages of using multi-level formulations that use different types of optimization algorithms at different optimization levels. The key concept in multi-level optimization is to decompose the problem into multiple sub optimization problems based on convexity, nature of design variables and the types of constraints. These sub optimization problems are solved in a nested or sequential manner using appropriate algorithms to reduce the complexity of the main problem and thus reducing the computational cost of optimization. A bi-level decomposition technique was demonstrated by Schmit and Mehrinfar [11]. In this formulation, the upper level is used to minimize the structure weight subjected to strength and buckling constraints whereas the lower level is used to optimize the geometry of composite panels. A two-level wing-box optimization is presented by Liu et al. [12], where the problem is decomposed into wing and panel levels. The stacking sequence of each panel is optimized for buckling load in the panel level using a genetic algorithm while the ply thickness is optimized at wing level using GENESIS software.

A two-level optimization scheme by dividing the size and shape variables is shown in [13–15]. In these studies, the variables representing the geometry or shape of the structure are optimized first by fixing the sizing design variables such as panel thickness and stiffener cross sections. The newly obtained shape variables are then fixed and the design is optimized for sizing variables. This is repeated to minimize panel weight. Zhao and Kapania [16] demonstrated a bi-level approach to minimize composite laminate weight by using PSO in the upper level and a gradient based optimizer (GBO) in the lower level. In the foregoing study, the GBO is used to optimize design variables such as ply thicknesses which are assumed to be continuous and the PSO is used to get the optimal ply orientations which have discrete design space subjected to buckling and material failure constraints. The formulation significantly reduces the computation cost in comparison to the standard PSO or GA, but it is noticed that the PSO still reduces the algorithm's efficiency when there are large number of plies considered for the composite laminates.

In the field of composite laminate design, Herencia et al. [17] presented a two step approach where the optimal lamination parameters are found in step 1 using GBO and the stacking sequence is retrieved using GA. A similar approach is demonstrated by Peeters et al. [18] for a combined topology and lamination parameter optimization. In this method, the lamination parameters can be optimized relatively easily using any global or local optimization algorithm due to smaller and continuous design space. The challenge lies in retrieving the stacking sequence for a predetermined number of plies which can match the values of optimum lamination parameter values as closely as possible. A beam search technique is used by Fedon et al. [19] to efficiently retrieve fiber orientations for a large number of plies.

An alternative to tackle laminate design problem is the use of mixed integer programs (MIPs) where discrete design variables such as ply thickness and orientations can be represented in terms of binary decision variables in linear or quadratic programs formulations. A few decades ago, MIP in optimization of composite laminates was deemed intractable due to large computational expenses and inability to effectively handle constraint nonlinearities. However, advancements in computer technology and development of advance (branch-and-cut) algorithms for solving MIPs [20], have made it worthwhile to reevaluate the option.

Broadly speaking, these exact algorithms implicitly enumerate the solution space by generating a binary tree where at each node, the linear/continuous programming relaxation of another MIP, defined over a subset of the original solution space, is solved. As a result, these algorithms have higher chances of quickly arriving at a global optimum. Haftka et al. [21] used mixed integer linear programming (MILP) for optimization of a composite panel for determining the most appropriate laminate stacking sequence. In their work, a MILP solver is used to maximize the buckling load and minimize the panel weight by varying the number of plies that have a fixed thickness value. Minimizing the panel weight subjected to buckling loads with the use of decision variables for distinct ply thicknesses is a complex problem to solve using MIP. This is due to the cubic nature of lamination parameters with respect to thickness design variables. This problem is the one being solved in this paper.

A large-scale truss design optimization was demonstrated by Shahabsafa et. al [22] which uses mixed integer neighborhood search. In this method, the subproblems are only solved to get an improved solution and the next subproblem is defined in the neighborhood of this solution. A bi-level integer programming technique has been used by Zein [23] for the optimization of blended composite structures. The efficiency of MIP or Mixed Integer Non-linear Programs (MINLP) in solving linear or quadratic problems is high compared to meta-heuristic solvers. But in the case of minimum weight optimization of a structure subjected to both the material failure and buckling constraints, there is a coupling between binary variables for ply orientations and thickness leading to higher order terms. Therefore, the problem must be decomposed into separate sub problems each of which can either be optimized using MIP or a combination of GBO and MIP.

Three different formulations are presented for optimization of rectangular composite laminates under bi-axial compressive loads subjected to buckling and material failure constraints. The first formulation is used to optimize ply fiber orientations and maximize the buckling load for a given set of ply thicknesses. This formulation consists of buckling constraints where the lamination parameter variations are linear with respect to fiber orientation decision variables. It is generalized to work with any possible orientation values chosen from a finite set. The second formulation presents an approach to minimize weight by minimizing the number of plies for a fixed ply thickness. To handle non-linear constraints where the lamination parameters are cubic and higher order with respect to design variables, a two level approach is chosen to separate stacking sequence and thickness design variables. A sequential two-level methodology is adopted in the third formulation, where the stacking sequence and the ply thicknesses are optimized in separate loops in every iteration. As opposed to varying the number of plies, the ply thicknesses are optimized along with the stacking sequence for a fixed number of plies.

The following paper is organized as follows: Section III presents static and buckling analysis of a composite laminate plate under in-plane loads. The formulation of constraint expressions using binary design variables is demonstrated in section IV. Section V provides the relevant theories for the three approaches for optimization with uniform ply thicknesses. The results for these studies are presented in section VI followed by conclusions in section VII.

III. Laminate Analysis

A symmetric composite laminate plate under study is subjected to static bi-axial compressive loads λN_{xx} and λN_{yy} in x and y directions, respectively, as shown in Fig. 1, where λ is a scaling parameter. The panel has a fixed number of plies N with orientations, $\theta_1, \theta_2, \cdots,$ and θ_N, and ply thicknesses, $t_1, t_2, \cdots,$ and t_N. The ply thicknesses are treated as fixed parameters in the optimization to maximize buckling load, whereas they are considered as design variables when the objective is to minimize panel weight. The panel is assumed to have a fixed and uniform geometry and material properties. We also assume that the loads are constant over the edges on which they are defined.

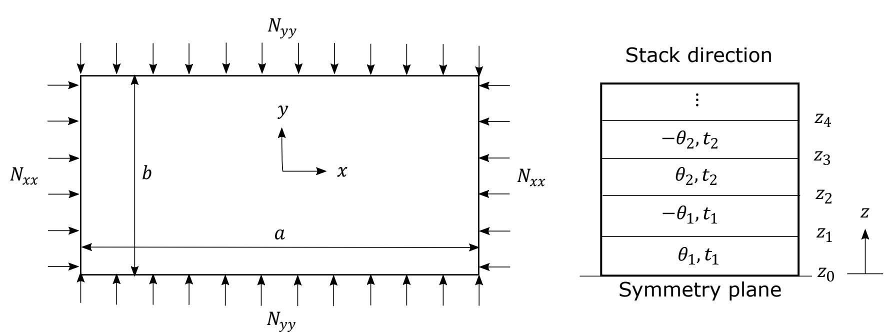


Fig. 1 Rectangular composite panel (a = 20 in., b = 5 in.) subjected to bi-axial loading.

For a symmetric and balanced laminate, the axial and flexural stiffness coefficients are defined in terms of material invariants, U, and lamination parameters, V [24],

$$
\begin{aligned}
A_{11} &= U_1 V_{0A} + U_2 V_{1A} + U_3 V_{3A} & D_{11} &= U_1 V_{0D} + U_2 V_{1D} + U_3 V_{3D} \\
A_{22} &= U_1 V_{0A} - U_2 V_{1A} + U_3 V_{3A} & D_{22} &= U_1 V_{0D} - U_2 V_{1D} + U_3 V_{3D} \\
A_{12} &= U_4 V_{0A} - U_3 V_{3A} & D_{12} &= U_4 V_{0D} - U_3 V_{3D} \\
A_{66} &= U_5 V_{0A} - U_3 V_{3A} & D_{66} &= U_5 V_{0D} - U_3 V_{3D}
\end{aligned}
\tag{1}
$$

where the material invariants $U_i, i \in \{1, \ldots, 5\}$ are given in terms of (known) local ply stiffnesses $Q_{jk}, j, k \in \{1, \ldots, 6\}$ as:

$$
\begin{aligned}
U_1 &= \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}) \\
U_2 &= \frac{1}{2}(Q_{11} - Q_{22}) \\
U_3 &= \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}) \\
U_4 &= \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}) \\
U_5 &= \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66})
\end{aligned}
\tag{2}
$$

and V_0, V_1, and V_3 are dependent on ply orientations and thicknesses as follows:

$$
\begin{aligned}
V_{0A} &= \int_{-h/2}^{h/2} dz & V_{0D} &= \int_{-h/2}^{h/2} z^2 dz \\
V_{1A} &= \int_{-h/2}^{h/2} \cos 2\theta dz & V_{1D} &= \int_{-h/2}^{h/2} z^2 \cos 2\theta dz \\
V_{3A} &= \int_{-h/2}^{h/2} \cos 4\theta dz & V_{3D} &= \int_{-h/2}^{h/2} z^2 \cos 4\theta dz
\end{aligned}
\tag{3}
$$

Here, h is the total thickness of the laminate and z is the distance of a ply from the plane of symmetry. Writing the extensional and bending stiffnesses in terms of material invariants and lamination parameters helps in separating the terms that are dependent on design variables (ply fiber orientations and ply thicknesses). In turn, it helps us introduce binary decision variables comfortably in the lamination parameters expressions as shown in the next section. Panel failure occurs if either the maximum load exceeds the critical buckling load or the local strains induced in any of the plies exceed the allowable limit. The buckling load factor λ_b for applied bi-axial compressive loads is given as in [25],

$$
\lambda(m, n) = \pi^2 \frac{D_{11}(m/a)^4 + 2(D_{12} + 2D_{66})(m/a)^2(n/b)^2 + D_{22}(n/b)^4}{(m/a)^2 N_{xx} + (n/b)^2 N_{yy}}
\tag{4}
$$

where m and n are integer numbers that show the number of half waves of buckling mode shapes in x and y directions, respectively. The laminate instability occurs when a critical load scaling parameter, λ_{cr} exceeds the minimum value of $\lambda_b(m, n)$ for all m and n. The material strength is characterised by using a maximum strain theory. The ply strains are a function of axial stiffnesses A_{11}, A_{22}, A_{12}, and A_{66}. For bi-axial loads and symmetric and balanced laminate, mid-plane shear strains, γ_{xy} go to zero under the action of normal strains. Thus, for a unit value of λ we can write the longitudinal strain, ε_{xx}, and transverse strain, ε_{yy}, as a function of A_{11}, A_{22}, and A_{12},

$$\begin{aligned} \varepsilon_{xx} &= \frac{N_{xx}A_{22} - N_{yy}A_{12}}{A_{11}A_{22} - A_{12}^2} \\ \varepsilon_{yy} &= \frac{N_{yy}A_{11} - N_{xx}A_{12}}{A_{11}A_{22} - A_{12}^2} \end{aligned} \tag{5}$$

In the material coordinate system, we can obtain these strains using coordinate framework transformation. Thus, for ply $k \in \{1, \ldots, N\}$, the longitudinal strain ε_1^k , transverse strain ε_2^k and shear strain ε_{12}^k are defined as follows

$$\begin{aligned} \varepsilon_1^k &= \cos^2\theta_k\varepsilon_{xx} + \sin^2\theta_k\varepsilon_{yy} \\ \varepsilon_2^k &= \sin^2\theta_k\varepsilon_{xx} + \cos^2\theta_k\varepsilon_{yy} \\ \varepsilon_{12}^k &= \sin 2\theta_k(\varepsilon_{yy} - \varepsilon_{xx}) \end{aligned} \tag{6}$$

The material is failed when the normal and/or shear strains in k^{th} ply are larger than the corresponding failure strains.

IV. Constraint Formulation

Composite laminate failure may happen either due to instability or material failure. Even though most panel optimization studies [1, 5, 8–10] only use stability constraints as it is usually the dominant failure mode, in certain design configurations, the panels may fail due to material failure before the critical buckling load is reached. Thus both, buckling and strain constraints are used to govern the failure in this study. Binary decision variables are used to choose ply orientation and thickness from a predefined set in a MINLP formulation where the buckling and strain constraints are defined in terms of the binary variables. To use a multi-level optimization technique, the formulation of constraints with respect to two classes of binary variables namely, stacking sequence variables and ply thickness variables is shown next.

A. Buckling Constraints w.r.t Stacking Sequence Design Variables

The buckling load factor for bi-axial load case is given in Eq. (4). Suppose, a composite laminate is made up of N plies with symmetry (total plies = $2N$) where each ply can take a value from a given finite set of orientations $\Theta = \{\theta_1, \theta_2, \ldots, \theta_p\}$. For a single ply, with p available orientations to choose from, there are p binary variables associated with it. Thus, for N plies, the total number of binary design variables for the fiber path orientations are Np. We write the vector **x** of binary design variables as

$$\mathbf{x} = \left[x_{11}, x_{21} \cdots x_{p1}, x_{12}, x_{22} \cdots x_{p2} \cdots x_{lk} \cdots x_{1N}, x_{2N} \cdots x_{pN}\right]^T \tag{7}$$

where, x_{lk} is the binary variable associated with ply k and orientation $\theta_l \in \Theta$, i.e., $x_{lk} = 1$ if the ply k has orientation θ_l and 0 otherwise. Note that for each ply k, $\sum_{l=1}^{p} x_{lk} = 1$ ensures that only one orientation is assigned to the ply. As a result, we can express the integral for V_{0D}, V_{1D} and V_{3D} from Eq. (3) in terms of x_{lk} as

$$\begin{aligned}
V_{0D} &= \frac{2}{3} \sum_{k=1}^{N} \left[(z_k^3 - z_{k-1}^3) \sum_{l=1}^{p} x_{lk} \right] \\
V_{1D} &= \frac{2}{3} \sum_{k=1}^{N} \left[(z_k^3 - z_{k-1}^3) \sum_{l=1}^{p} x_{lk} \cos 2\theta_l \right] \\
V_{3D} &= \frac{2}{3} \sum_{k=1}^{N} \left[(z_k^3 - z_{k-1}^3) \sum_{l=1}^{p} x_{lk} \cos 4\theta_l \right]
\end{aligned} \tag{8}$$

where z_k is the thickness coordinate for the top surface of the $k-$th ply and z_{k-1} is the bottom surface thickness coordinate of the $k-$th ply. Note that since ply thickness is fixed, the buckling expressions in Eq. (8) are linear functions of **x**. We can simplify the above calculation by separating the design variable and writing the lamination parameters and flexural stiffnesses in vector form i.e., using each term that is inside the summation as a vector element. The stiffnesses D_{11}, D_{22}, D_{12}, and D_{66} can be represented as

$$D_{11} = \mathbf{d_{(11)}}^T\mathbf{x}; \quad D_{22} = \mathbf{d_{(22)}}^T\mathbf{x}; \quad D_{12} = \mathbf{d_{(12)}}^T\mathbf{x}; \quad D_{66} = \mathbf{d_{(66)}}^T\mathbf{x} \tag{9}$$

where $\mathbf{d_{(ij)}}, i, j \in \{(1,1), (2,2), (1,2), (6,6)\}$, are vectors of size Np whose elements $d_{l(ij)}^k$ represent the flexural stiffness D_{ij} of ply k having an orientation θ_l. The task of the optimizer is to choose values from the above defined vectors for flexural stiffnesses such that the summation of chosen values will lead to optimal flexural stiffnesses of the laminate.

B. Strain Constraints w.r.t Stacking Sequence Design Variables

In symmetric laminates, strains are a function of extensional stiffness A_{ij}, which needs to be represented using binary variables **x**. By using similar approach as that used for obtaining Eq. (9), we can write the extensional stiffness as a function of binary design variables as:

$$A_{11} = \mathbf{a_{(11)}}^T\mathbf{x}; \;\; A_{22} = \mathbf{a_{(22)}}^T\mathbf{x}; \;\; A_{12} = \mathbf{a_{(12)}}^T\mathbf{x}. \tag{10}$$

The element $a^k_{l(ij)}$ of vector $\mathbf{a_{(ij)}}$ represent the axial stiffness A_{ij} of ply k having an orientation θ_l where $(i, j) \in \{(1, 1), (2, 2), (1, 2)\}$. We can rewrite the global strains in Eq. (5) as:

$$\begin{aligned} \varepsilon_{xx} &= \frac{[N_{xx}\mathbf{a_{(22)}}^T - N_{yy}\mathbf{a_{(12)}}^T]\mathbf{x}}{\mathbf{x}^T[\mathbf{a_{(11)}}\mathbf{a_{(22)}}^T - \mathbf{a_{(12)}}\mathbf{a_{(12)}}^T]\mathbf{x}} \\ \varepsilon_{yy} &= \frac{[N_{yy}\mathbf{a_{(11)}}^T - N_{xx}\mathbf{a_{(12)}}^T]\mathbf{x}}{\mathbf{x}^T[\mathbf{a_{(11)}}\mathbf{a_{(22)}}^T - \mathbf{a_{(12)}}\mathbf{a_{(12)}}^T]\mathbf{x}} \end{aligned} \tag{11}$$

To transform the strains from Eq. (11) into material coordinate system, we define three column vectors $\mathbf{c}$, $\mathbf{s}$ and $\mathbf{s_2}$ to represent the values for $\cos^2\theta$, $\sin^2\theta$ and $\sin 2\theta$, respectively; i.e., $\mathbf{c} = [\cos^2\theta_1, \cos^2\theta_2, \ldots, \cos^2\theta_p]$, $\mathbf{s} = [\sin^2\theta_1, \sin^2\theta_2, \ldots, \sin^2\theta_p]$ and $\mathbf{s_2} = [\sin 2\theta_1, \sin 2\theta_2, \ldots, \sin 2\theta_p]$. To get the strain in ply k, these vectors are used along with a transformation matrix H_k such that only design variables associated with ply k are multiplied with the respective p orientations. If $j_s = (k - 1)p + 1$ is the starting index of the design variables associated with ply k, then the ending index $j_e = kp$. The transformation matrix H_k is a sparse matrix having dimensions $(Np \times p)$ with elements in rows and columns ranging from j_s to j_e forming an identity matrix. We can write ply strains from Eq. (6) as a function of stacking sequence binary variables as:

$$\begin{aligned} \varepsilon^k_1 &= \frac{\mathbf{x}^T\{H_k\mathbf{c}[N_{xx}\mathbf{a_{(22)}}^T - N_{yy}\mathbf{a_{(12)}}^T] + H_k\mathbf{s}[N_{yy}\mathbf{a_{(11)}}^T - N_{xx}\mathbf{a_{(12)}}^T]\}\mathbf{x}}{\mathbf{x}^T[\mathbf{a_{(11)}}\mathbf{a_{(22)}}^T - \mathbf{a_{(12)}}\mathbf{a_{(12)}}^T]\mathbf{x}} \\ \varepsilon^k_2 &= \frac{\mathbf{x}^T\{H_k\mathbf{s}[N_{xx}\mathbf{a_{(22)}}^T - N_{yy}\mathbf{a_{(12)}}^T] + H_k\mathbf{c}[N_{yy}\mathbf{a_{(11)}}^T - N_{xx}\mathbf{a_{(12)}}^T]\}\mathbf{x}}{\mathbf{x}^T[\mathbf{a_{(11)}}\mathbf{a_{(22)}}^T - \mathbf{a_{(12)}}\mathbf{a_{(12)}}^T]\mathbf{x}} \\ \varepsilon^k_{12} &= \frac{\mathbf{x}^T H_k\mathbf{s_2}\{N_{yy}\mathbf{a_{(11)}}^T + (N_{yy} - N_{xx})\mathbf{a_{(12)}}^T - N_{xx}\mathbf{a_{(22)}}^T\}\mathbf{x}}{\mathbf{x}^T[\mathbf{a_{(11)}}\mathbf{a_{(22)}}^T - \mathbf{a_{(12)}}\mathbf{a_{(12)}}^T]\mathbf{x}} \end{aligned} \tag{12}$$

Equations (12) have quadratic expressions in the form of $\mathbf{x}^T Q\mathbf{x}$, in both the numerator and the denominator. It can be equated with the allowable strain values and rearranged to get a quadratic expression with respect to design variables $\mathbf{x}$. This holds true if N_{xx} and N_{yy} are constant. If Eq. (12) is to be used in buckling load maximization problem, the variable for buckling load factor λ must be multiplied to the R.H.S. making it a cubic mixed integer programming problem. We avoid this problem by including strain constraints only at the thickness optimization level.

C. Buckling Constraints w.r.t Ply Thickness Design Variables

Using the buckling constraint in the minimum weight design requires inclusion of thickness variables. This leads to a 4^{th} order polynomial constraint that can be handled by efficient linearization of constraints and use of multi-level optimization techniques. The order of the buckling constraint w.r.t design variables can be reduced by using two optimization levels. In one level, ply thicknesses are fixed and the laminate is optimized for stacking sequence leading to a linear and quadratic expressions of constraints w.r.t $\mathbf{x}$. The second level is used for finding optimal thickness values by fixing ply orientations resulting in a cubic expression. This is because the bending stiffness matrix, $\boldsymbol{D}$, is a cubic function of ply thickness (see Eqs. (1) and (3)). Let $\mathbf{y}$ be the vector of binary design variables associated with a finite set of thickness values $\mathrm{T} = \{t_1, t_2, \cdots, t_q\}$ such that

$$\mathbf{y} = [y_{11}, y_{21} \cdots y_{q1}, y_{12}, y_{22} \cdots y_{q2} \cdots y_{lk} \cdots y_{1N}, y_{2N} \cdots y_{qN}]^T \tag{13}$$

where, y_{lk} is the binary variable associated with ply k and thickness $t_l \in \mathrm{T}$. The lamination parameters V_{0D}, V_{1D} and V_{3D} are cubic functions of ply thickness coordinates z_k. For a fixed ply orientation $\hat{\theta}_k \in \Theta$, Eq. (8) for a symmetric layup can be rewritten as:

$$\begin{aligned}
V_{0D} &= \frac{2}{3}\sum_{k=1}^{N}(z_k^3 - z_{k-1}^3) = \frac{2}{3}z_N^3 \\
V_{1D} &= \frac{2}{3}\sum_{k=1}^{N}(z_k^3 - z_{k-1}^3)\cos 2\hat{\theta}_k \\
&= \frac{2}{3}\left[z_N^3 \cos 2\hat{\theta}_N + z_{N-1}^3(\cos 2\hat{\theta}_{N-1} - \cos 2\hat{\theta}_N) + \cdots + z_1^3(\cos 2\hat{\theta}_1 - \cos 2\hat{\theta}_2)\right] \\
V_{3D} &= \frac{2}{3}\sum_{k=1}^{N}(z_k^3 - z_{k-1}^3)\cos 4\hat{\theta}_k \\
&= \frac{2}{3}\left[z_N^3 \cos 4\hat{\theta}_N + z_{N-1}^3(\cos 4\hat{\theta}_{N-1} - \cos 4\hat{\theta}_N) + \cdots + z_1^3(\cos 4\hat{\theta}_1 - \cos 4\hat{\theta}_2)\right]
\end{aligned} \tag{14}$$

Special Case with all Plies having Same Thickness. If we assume that all plies have same thickness, then the number of binary variables in $\mathbf{y}$ reduces from Nq to q. Specifically, $\mathbf{y} := [y_1, y_2, \ldots, y_q]^T \in \{0, 1\}^q$ where $y_l = 1$ when all plies have thickness $t_l \in T$ and 0, otherwise. Let $\mathbf{t} := [t_1, t_2, \ldots, t_q]^T$. Then, the z-distance of each ply in terms of ply thicknesses are

$$\begin{aligned}
z_0 &= 0 \\
z_k &= z_{k-1} + \sum_{l=1}^{q} t_l y_l = k\mathbf{t}^T\mathbf{y}, \quad \text{for } k = 1, \ldots, N, \\
\therefore\ z_k^3 &= k^3\left(\mathbf{t}^T\mathbf{y}\right)^3
\end{aligned} \tag{15}$$

It is important to note that $\sum_{l=1}^{q} y_l = 1$ because all the plies have the same thickness and they can have only one value from T. As a consequence, $\Pi_{l\in S}\, y_i = 0$ for all $S \subseteq \{1, \ldots, q\}$ where $|S| \geq 2$. This implies that any product of y variables with at least different indices is equal to 0. Therefore, $\left(\mathbf{t}^T\mathbf{y}\right)^3 = \sum_{l=1}^{q} t_l^3 y_l^3 = \sum_{l=1}^{q} t_l^3 y_l$. The last equality is because $y_l \in \{0, 1\}$. This gives:

$$z_k^3 = k^3 \sum_{l=1}^{q} t_l^3 y_l = k^3 \left(\mathbf{t}^{\circ 3}\right)^T \mathbf{y} \tag{16}$$

where, the notation '$\circ$' represents Hadamard or element-wise operation. In this special case, the cubic expression of z - distances in terms of ply thicknesses (16) is linear w.r.t $\mathbf{y}$. Thus, the lamination parameters in Eq. (14) become linearly dependent on ply thickness variables as

$$\begin{aligned}
V_{0D} &= \frac{2}{3}\sum_{k=1}^{N}(z_k^3 - z_{k-1}^3) = \frac{2}{3}N^3\left(\mathbf{t}^{\circ 3}\right)^T \mathbf{y} \\
V_{1D} &= \frac{2}{3}\sum_{k=1}^{N}\left\{\left[k^3 - (k-1)^3\right]\cos 2\hat{\theta}_k\right\}\left(\mathbf{t}^{\circ 3}\right)^T \mathbf{y} \\
V_{3D} &= \frac{2}{3}\sum_{k=1}^{N}\left\{\left[k^3 - (k-1)^3\right]\cos 4\hat{\theta}_k\right\}\left(\mathbf{t}^{\circ 3}\right)^T \mathbf{y}
\end{aligned} \tag{17}$$

This is similar to calculating flexural stiffnesses D_{ij} and buckling load factor λ for a given set of thicknesses and selecting the thickness that gives the optimal buckling load factor.

D. Strain Constraints w.r.t Ply Thickness Design Variables

Strains in ply k with fixed ply orientations $\hat{\theta}_k$ and variable thickness can be written as

$$\begin{aligned}
\varepsilon_1^k &= \frac{\{\cos^2\hat{\theta}_k[N_{xx}\mathbf{a_{(22)}}^T - N_{yy}\mathbf{a_{(12)}}^T] + \sin^2\hat{\theta}_k[N_{yy}\mathbf{a_{(11)}}^T - N_{xx}\mathbf{a_{(12)}}^T]\}\mathbf{y}}{\mathbf{y}^T[\mathbf{a_{(11)}}\mathbf{a_{(22)}}^T - \mathbf{a_{(12)}}\mathbf{a_{(12)}}^T]\mathbf{y}} \\
\varepsilon_2^k &= \frac{\{\sin^2\hat{\theta}_k[N_{xx}\mathbf{a_{(22)}}^T - N_{yy}\mathbf{a_{(12)}}^T] + \cos^2\hat{\theta}_k[N_{yy}\mathbf{a_{(11)}}^T - N_{xx}\mathbf{a_{(12)}}^T]\}\mathbf{y}}{\mathbf{y}^T[\mathbf{a_{(11)}}\mathbf{a_{(22)}}^T - \mathbf{a_{(12)}}\mathbf{a_{(12)}}^T]\mathbf{y}} \\
\varepsilon_{12}^k &= \frac{\sin 2\hat{\theta}_k\{N_{yy}\mathbf{a_{(11)}}^T + (N_{yy} - N_{xx})\mathbf{a_{(12)}}^T - N_{xx}\mathbf{a_{(22)}}^T\}\mathbf{y}}{\mathbf{y}^T[\mathbf{a_{(11)}}\mathbf{a_{(22)}}^T - \mathbf{a_{(12)}}\mathbf{a_{(12)}}^T]\mathbf{y}}
\end{aligned} \tag{18}$$

Rearranging Eq. (18) will result in quadratic expressions in $\mathbf{y}$ in the form: $\mathbf{y}^T Q\mathbf{y} - B\mathbf{y} = 0$.

Special Case. For plies with uniform thickness, the strain expression can be linearized similar to the buckling expressions. By fixing the orientation values in vectors $\mathbf{c}$, $\mathbf{s}$, and $\mathbf{s2}$ and using element-wise operations ($\circ$), the system of linear equations for $\varepsilon_\mathbf{1}, \varepsilon_\mathbf{2}$ and $\varepsilon_\mathbf{12}$ can be obtained.

$$\begin{aligned}
\varepsilon_\mathbf{1} &= \frac{\{\mathbf{c}[N_{xx}\mathbf{a_{(22)}}^T - N_{yy}\mathbf{a_{(12)}}^T] + \mathbf{s}[N_{yy}\mathbf{a_{(11)}}^T - N_{xx}\mathbf{a_{(12)}}^T]\}\mathbf{y}}{[\mathbf{a_{(11)}} \circ \mathbf{a_{(22)}} - \mathbf{a_{(12)}} \circ \mathbf{a_{(12)}}]^T\mathbf{y}} \\
\varepsilon_\mathbf{2} &= \frac{\{\mathbf{s}[N_{xx}\mathbf{a_{(22)}}^T - N_{yy}\mathbf{a_{(12)}}^T] + \mathbf{c}[N_{yy}\mathbf{a_{(11)}}^T - N_{xx}\mathbf{a_{(12)}}^T]\}\mathbf{y}}{[\mathbf{a_{(11)}} \circ \mathbf{a_{(22)}} - \mathbf{a_{(12)}} \circ \mathbf{a_{(12)}}]^T\mathbf{y}} \\
\varepsilon_\mathbf{12} &= \frac{\mathbf{s2}\{N_{yy}\mathbf{a_{(11)}}^T + (N_{yy} - N_{xx})\mathbf{a_{(12)}}^T - N_{xx}\mathbf{a_{(22)}}^T\}\mathbf{y}}{[\mathbf{a_{(11)}} \circ \mathbf{a_{(22)}} - \mathbf{a_{(12)}} \circ \mathbf{a_{(12)}}]^T\mathbf{y}}
\end{aligned} \tag{19}$$

V. Optimization Problem Formulation

Optimization of composite laminates can either be for maximizing the failure load for a fixed ply thickness or minimizing the weight for a fixed failure load threshold. The latter problem has additional design variables for thicknesses. This section demonstrates three ways for solving laminate optimization problem using MILP/MINLP.

A. Buckling Load Maximization - Single Level Optimization 1 (SLO1)

Maximization of the failure load having linear buckling constraints can be formulated as a MILP, as shown in [21] and given in section IV. The objective is to maximize the minimum buckling load obtained from different half sine wave configurations. A maximum of five half sine waves are chosen in both the x and y directions to capture the minimum value of buckling load factor ($m_{def} = n_{def} = 5$). This is required since the buckling mode number with minimum load factor value is not fixed for composite laminates. First constraint ensures that minimum λ of all possible buckling load factors is maximized. Second constraint ensures that only one orientation is assigned to each ply. The third constraint enforces ply contiguity to a maximum of 4 plies of same orientation while the fourth constraint enforces balance by ensuring that every alternate ply of a particular fiber orientation is followed by a ply of the same negative fiber orientation. This constraint requires that the number of symmetric plies be even. The formulation is a MILP optimization with λ_{cr} as a continuous design variable and x_{lk} as a binary variable for $k = 1, \dots, N$ and $l = 1, \dots, p$.

$$
\begin{array}{lll}
\textbf{maximize} & \lambda_{cr} & \\
\textbf{w.r.t} & \mathbf{x} & \\
\textbf{s.t.} & \lambda_{cr} \leq \lambda_b(\mathbf{x}, m, n) & m = 1, \dots, m_{def};\;\; n = 1, \dots, n_{def} \\
& \sum_{l=1}^{p} x_{lk} = 1 & k = 1, \dots, N \\
& \sum_{i=k}^{k+3} x_{li} \leq 4 & k = 1, \dots, N-4;\;\; l = 1, \dots, p \\
& x_{l_1,k} - x_{l_2,k+1} = 0 & \forall\;\; l_1, l_2 \in \{1, \dots, p\} \text{ with } \theta_{l_1} = -\theta_{l_2};\, k = 1, 3, \dots, N-1 \\
& \mathbf{x} \in \{0, 1\}^{pN} & \\
& \lambda_{cr} \geq 0 &
\end{array}
$$

B. Weight Minimization - Single Level Optimization 2 (SLO2)

For the problem with a focus on weight minimization, the total number of plies with a fixed ply thickness is used as the minimization function. The buckling load factor formulated with Eq. (8) and failure strains from Eq. (12) are used as linear and quadratic constraints. The idea is to assume an upper limit for the total number of plies and set the values of corresponding binary variables to 0 or 1 if a ply is occupied or unoccupied, respectively. This maximum limit on the number of plies defines the total number of binary design variables that are required for the optimization.

Here, the laminate weight is given by $W = \sum_{k=1}^{N} \sum_{l=1}^{p} x_{lk}$ assuming ply thickness and material properties are constant across the laminate. Parameter λ_0 is the design buckling load factor of the laminate. Likewise, parameters $\varepsilon_1^{t,c}$, $\varepsilon_2^{t,c}$ and

$$
\begin{array}{lll}
\textbf{minimize} & W(\mathbf{x}) = \sum_{k=1}^{N} \sum_{l=1}^{p} x_{lk} & \\
\textbf{w.r.t} & \mathbf{x} & \\
\textbf{s.t.} & \lambda_0 \leq \lambda_b(\mathbf{x}, m, n) & m = 1, \ldots, m_{def}; \;\; n = 1, \ldots, n_{def} \\
& \sum_{l=1}^{p} x_{lk} \leq 1 & k = 1, \ldots, N \\
& \sum_{i=k}^{k+3} x_{li} \leq 4 & k = 1, \ldots, N-4; \;\; l = 1, \ldots, p \\
& x_{l_1,k} - x_{l_2,k+1} = 0 & \forall \;\; l_1, l_2 \in \{1, \ldots, p\} \text{ with } \theta_{l_1} = -\theta_{l_2}; k = 1, 3, \ldots, N-1 \\
& \sum_{l=1}^{p} x_{lk+1} - \sum_{l=1}^{p} x_{lk} \leq 0 & k = 1, \ldots, N-1 \\
& \varepsilon_1^k \leq \varepsilon_1^{t,c} & \\
& \varepsilon_2^k \leq \varepsilon_2^{t,c} & k = 1, \ldots, N \\
& \varepsilon_{12}^k \leq \varepsilon_{12}^s & \\
& \mathbf{x} \in \{0, 1\}^{pN} &
\end{array}
$$

ε_{12}^s are allowable longitudinal, transverse and shear strains in tension and compression, respectively evaluated at the buckling load. The second constraint from SLO1 is changed to an inequality in SLO2 to allow the laminate to have empty plies. The third and fourth constraints are same as that from SLO1 whereas the fifth constraint ensures that empty plies if any lie on the outside. The problem can be solved as an integer linear program (ILP) with only buckling constraints or a quadratically constrained linear program (QCLP) with the inclusion of strain constraints.

C. Weight Minimization - Sequential Two-Level Optimization (STLO)

Another way to minimize the weight of a laminate is to change the ply thickness instead of number of plies. The addition of thickness design variables renders both buckling and strain constraints to be 4^{th} order polynomials in terms of binary design variables (orientation and thickness). The optimization problem can be split into two different optimization problems which can be solved in a sequential manner. It allows for faster convergence and enables the use of MILP in both sub problems. The STLO process is shown in Figure 2.

$$
\begin{array}{ll}
\textbf{minimize} & W = Nab(\mathbf{t}^T\mathbf{y}) \text{ (Level 2)} \\
\textbf{w.r.t} & \mathbf{y} \\
\textbf{s.t.} & \lambda_0 \leq \lambda_b(\mathbf{y}, m, n) \\
& \varepsilon_1^k \leq \varepsilon_1^{t,c} \\
& \varepsilon_2^k \leq \varepsilon_2^{t,c} \\
& \varepsilon_{12}^k \leq \varepsilon_{12}^s \\
& \mathbf{y} \in \{0, 1\}^q
\end{array}
\qquad
\begin{array}{lll}
\textbf{maximize} & \lambda_{cr} \text{ (Level 1)} & \\
\textbf{w.r.t} & \mathbf{x} & \\
\textbf{s.t.} & \lambda_{cr} \leq \lambda_b(\mathbf{x}, m, n) & m = 1, \ldots, m_{def}; n = 1, \ldots, n_{def} \\
& \sum_{l=1}^{p} x_{lk} = 1 & k = 1, \ldots, N \\
& \sum_{i=k}^{k+3} x_{li} \leq 4 & k = 1, \ldots, N-4; \;\; l = 1, \ldots, p \\
& x_{l_1,k} - x_{l_2,k+1} = 0 & \forall \;\; l_1, l_2 \in \{1, \ldots, p\} \text{ with } \theta_{l_1} = -\theta_{l_2} \\
& \lambda_{cr} \geq 0 & \\
& \mathbf{x} \in \{0, 1\}^{pN} &
\end{array}
$$

Both optimization levels are solved sequentially and independently, but they share only the optimal ply orientation and thickness information. Level 1 of the optimizer is used to maximize the buckling load for given initial ply thickness

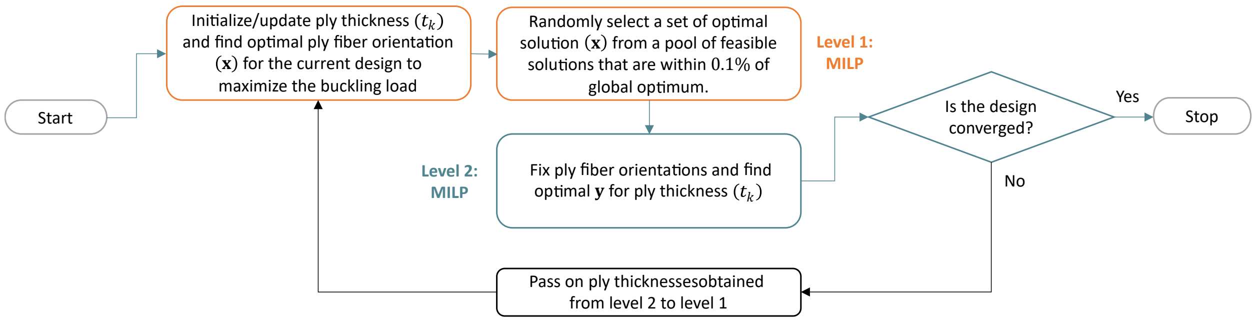


Fig. 2 Sequential two level optimization process.

values. The ply fiber orientations obtained from level 1 are passed on to level 2 as fixed values and the weight is minimized by finding new values of ply thicknesses. Thickness values obtained from level 2 are passed back to level 1 for stacking sequence optimization.

In level 1, MIP is solved for a global optimum for fiber orientations for fixed ply thicknesses whereas at level 2, it is solved for a global optimum for ply thickness for fixed fiber orientations. Even if MILP gives global optimum solution for maximum buckling load and minimum thickness separately in each level, when two levels are paired together, a global optimum solution is not guaranteed for the whole problem. The STLO can oscillate between two sub-optimal values. Thus, to improve the chances of finding a global optimum for the whole problem, the information about multiple local optimum solutions for ply stacking sequence optimization (level 1) is used. It is possible that a sub-optimal stacking sequence can yield better ply thickness values and in turn yields a better solution overall. The solution pool containing values for local optima (buckling load) is used to randomly select a solution which is within 0.1% of the global optimum. The stacking sequence corresponding to this local solution is passed on to level 2 for weight minimization. At the end of each cycle, the best design is retained by comparing it to the previous best. This also mitigates the possibility of oscillation between two sub optimal solutions.

VI. Optimization Results

A. Optimization for Maximum Buckling Load Design using SLO1

To demonstrate the capability of MILP for different size of problems, the results from SLO1 are compared to those from using a standard PSO. Initially, the SLO1 is used for optimization of laminate configurations having total number of plies to be $2N = 48$ and 144. Each laminate is optimized for a stacking sequence with ply orientations chosen from the following two sets.

$$\begin{aligned} &\text{Set 1: } \Theta \in \{0^\circ, 45^\circ, 90^\circ\} \\ &\text{Set 2: } \Theta \in \{0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ\} \end{aligned} \tag{20}$$

In the first set, θ can take values in the interval of 45° whereas in the second set, a 15° interval is used to increase the design space and complexity. The total laminate thickness is kept constant at 0.24 in. This results in fixed ply thickness of 0.005 in. (=0.24/48) and 0.00167 in. (=0.24/144), respectively, for the two configurations studied here. With advancements in manufacturing technology, it is now possible to make laminates with very thin plies (ply thickness as small as 7.0E-4 in. [26]) that has advantages in terms of material failure over conventional thick plies. The second configuration depicts this by reducing the ply thickness but keeping the overall laminate thickness same. The buckling load factor is computed for $\frac{N_{yy}}{N_{xx}} = 0.5$ where $N_{xx} = 1$ lb/in.. Material properties used in this study are given in Table 1. The optimization is done using a commercially available software, *Gurobi* [27] which uses branch and cut algorithm. All default parameters except '*PoolSearchMode*' are used to setup the solver. The '*PoolSearchMode*' parameter is set to 1 to find multiple nearby optimum values. A machine with an Intel Xeon W2265 (12 cores, 24 threads) CPU and 128 GB RAM is used for all studies. The solver automatically uses parallelization on all cores while performing node search in the branch and bound technique. The optimum buckling load factor values and solver times (given in parenthesis) are compared in Table 2. The optimization is run twice in each case to check its robustness. The optimal stacking sequences obtained using SLO1 are given in Table 3.

Table 1 Material properties for T300/5208

E_1(msi)	E_2(msi)	G_{12}(msi)	ν_{12}
18.5	1.89	0.93	0.3

Table 2 Maximum buckling load factor comparison using SLO1.

Cases	λ_{cr} (run time in seconds)			
	PSO		SLO1 with MILP	
	run #1	run #2	run #1	run #2
(a) N = 24, Set 1	9999.12 (6.43)	9999.15 (6.29)	9999.15 (0.12)	9999.15 (0.13)
(b) N = 24, Set 2	10136.14 (180)	10140.26 (180)	10686.17 (0.08)	10686.17 (0.08)
(c) N = 72, Set 1	10059.65 (18.65)	10059.82 (17.96)	10059.87 (0.08)	10059.87 (0.08)
(d) N = 72, Set 2	Ø(180)	Ø(180)	10750.44 (0.1)	10750.44 (0.1)

Note: Ø denotes an infeasible solution resulting from non-convergence of results in 100 PSO iterations.

From the four cases, i.e., $N \in \{24, 72\}$ and $\Theta \in$ {Set 1, Set 2}, MILP is much faster than PSO with a solver time of less than 1 second in comparison to 6.36 seconds (on average) for the PSO for case (a) and upwards of 180 seconds for case (b). The optimal values of PSO and MIP are very close to each other and agree well with those available in the literature [16]. In cases where the number of available orientations are high (cases (b) and (d)), the PSO struggles to obtain a converged solution before the limit of 100 iterations is reached with 1000 particles (100,000 function evaluations). This can be attributed to the large increase in combinatorial space in case (b) and case (d) with

Table 3 Optimal stacking sequence obtained using SLO1 (same sequence is obtained using both runs).

Case	Optimal stacking sequence
(a)	$[\pm45/90_4/(\pm45)_3/90_2/\pm45/90_2/\pm45/90_4]_S$
(b)	$[(\pm60)_5/(\pm75)_2/(\pm60)_4/\pm75]_S$
(c)	$[(\pm45)_2/90_4/\pm45/90_2/(\pm45)_2/90_4/\pm45/90_4/(\pm45)_4/90_2/\pm45/90_2/(\pm45)_3/90_2/$ $\pm45/90_2/(\pm45)_3/90_2/(\pm45)_5/90_2]_S$
(d)	$[(\pm60)_2/\pm75/(\pm60)_5/\pm75/(\pm60)_{18}/\pm75/(\pm60)_3/\pm75/(\pm60)_4]_S$

$7^{24} = 1.9 \times 10^{20}$ and $7^{72} = 7 \times 10^{60}$ possible orientation combinations respectively. Increase in the combinatorial space significantly increases the total number of local minima. Case (c) has 2.25×10^{35} possible combinations which is more than case (b) but it is able to converge to an optimum.

One reason for the PSO to not converge in the specified time is that, even though the number of possible combinations in case (b) are less than that in case (c), the number of optimal combinations are higher as there are more choices for the optimizer. Another reason might be because of rounding off of continuous design variables to their nearest integer values. The higher the number of orientation values needed to round off, higher are the chances for the PSO to drift away from the optimum. Thus, despite having less number of design variables in case (b), case (c) is able to converge much faster to the optimal solution. Increasing the number of design variables by increasing the number of plies and orientations renders the PSO inefficient. It cannot yield a feasible solution in reasonable time frame which is taken to be 180s. The time required for MILP to get a solution is orders of magnitude lower than a standard PSO when the problem has a linear objective function and constraints. Therefore, the extremely low computational cost of MILP can be leveraged in a multi-level formulation.

B. Optimization for Minimum Weight Design using SLO2

In SLO2, the value of critical buckling load found from buckling load maximization study is used as the design buckling load factor λ_0 on the panel with $N_{xx} = 1$ lb/in. and $\frac{N_{yy}}{N_{xx}} = 0.5$. The allowable strains are kept constant across layups at $\varepsilon_{11} = 0.008$, $\varepsilon_{22} = 0.029$ and $\varepsilon_{12} = 0.015$ with an additional safety factor of 1.5. For easier process validation, four cases are chosen for SLO2 that correspond to the four SLO1 cases, with $N_{max} \in \{50, 100\}$ and $\Theta \in \{\text{Set1}, \text{Set2}\}$ where N_{max} is the maximum number of plies the laminate can have in the optimization. The number of binary variables used of each case is given by $N_{max} \times p$. Since, the maximum load for the cases with ply thickness of 0.005in. and 0.00167in. from Table 2 is used as the limit load, the optimization should return the same number of plies as that used for the corresponding SLO1 cases to obtain the minimum weight design with buckling load constraint. Additionally, the optimization is performed with buckling only constraints and with both, buckling and strain constraints to assess the performance of the QCLP solver. In the absence of bending loads, the strain constraints are computed for each fiber orientation instead of each ply to reduce the total number of constraints in the model. The number of quadratic constraints in such problems is equal to $6p$ (accounting for both, negative and positive strains).

From Table 4 and 5, SLO2 with ILP obtains a minimum number of plies N_{opt} = 24 for case (a) and case (b), and N_{opt} = 72 for case (c) and case (d). This is easily validated from SLO1 case inputs. Setting N_{max} = 50 leads to 150 binary variables for case (a) and 350 binary variables for case (b) whereas N_{max} = 100 leads to 300 and 700 binary variables for case (c) and case (d) respectively. Similar to that observed in SLO1, PSO takes significantly longer to obtain a converged solution for all cases and especially, in case (d) where the combinatorial space is large. With the addition of quadratic strain constraints, the optimum number of plies is higher in all the cases compared to buckling only optimization as seen from Table 6 with the laminate stacking sequence provided in Table 7. As ply contiguity and adjacency constraints are used, even a small violation of failure constraints leads to an addition of a ply pair to maintain a balanced configuration leading to designs with only even number of plies. Additionally, it is observed that there is a significant increase in solution times with the QCLP compared to ILP as the number of design variables is increased. But the actual wall clock time is much less indicating that the method is efficient in solving quadratically constrained laminate optimization problems.

Table 4 Minimum weight optimization with buckling constraints using SLO2.

Cases	λ_0	Ply thickness (in.)	N_{opt} (run time in seconds)			
			PSO		SLO2 with ILP	
			run #1	run #2	run #1	run #2
(a) N_{max} = 50, Set 1	9999.15	0.005	24 (12.87)	24 (11.56)	24 (0.07)	24 (0.07)
(b) N_{max} = 50, Set 2	10686.17	0.005	32 (180)	31 (180)	24 (0.1)	24 (0.1)
(c) N_{max} = 100, Set 1	10059.87	0.00167	92 (180)	93 (180)	72 (0.08)	72 (0.08)
(d) N_{max} = 100, Set 2	10750.44	0.00167	Ø(180)	Ø(180)	72 (0.12)	72 (0.12)

Note: Ø denotes an infeasible solution resulting from non-convergence of results in the specified time frame of 180s or 100 PSO iterations.

Table 5 Optimal stacking sequence obtained using SLO2 with buckling constraints (same sequence is obtained using both runs).

Case	Optimal stacking sequence
(a)	$[90_2/(\pm45)_3/90_2/\pm45/90_4/\pm45/90_2/\pm45/90_2/\pm45]_S$
(b)	$[(\pm60)_5/(\pm75)_4/(\pm60)_4/\pm75]_S$
(c)	$[90_4/\pm45/90_4/\pm45/90_4/(\pm45)_7/90_2/(\pm45)_{13}/90_2/(\pm45)_6]_S$
(d)	$[(\pm60)_5/\pm75/(\pm60)_6/\pm75/(\pm60)_2/\pm75/(\pm60)_{12}/\pm75/(\pm60)_3/\pm75/(\pm60)_3]_S$

C. Optimization for Minimum Weight Design using Sequential Two-Level Approach (STLO)

In this approach, the STLO is adopted by using MILP at both levels enabling the use of discrete design space for ply thicknesses. Linear expressions of lamination parameters from Eq. (8) and Eq. (17) are used in level 1 and level 2

Table 6 Minimum weight optimization with buckling and strain constraints using SLO2.

Cases	λ_0	Ply thickness (in.)	N_{opt} (run time in seconds) PSO		SLO2 with QCLP	
			run #1	run #2	run #1	run #2
(a) N_{max} = 50, Set 1	9999.15	0.005	24 (66.45)	24 (68.81)	26 (0.23)	26 (0.23)
(b) N_{max} = 50, Set 2	10686.17	0.005	Ø(180)	Ø(180)	26 (1.63)	26 (1.64)
(c) N_{max} = 100, Set 1	10059.87	0.00167	Ø(180)	Ø(180)	74 (1.24)	74 (1.25)
(d) N_{max} = 100, Set 2	10750.44	0.00167	Ø(180)	Ø(180)	74 (5.2)	74 (5.18)

Note: Ø denotes an infeasible solution resulting from non-convergence of results in the specified time frame of 180s or 100 PSO iterations.

Table 7 Optimal stacking sequence obtained using SLO2 with buckling and strain constraints (same sequence is obtained using both runs).

Case	Optimal stacking sequence
(a)	$[0_2/90_2/\pm45/0_2/90_2/\pm45/90_4/\pm45/90_2/0_2/(\pm45)_2]_S$
(b)	$[0_2/(\pm60)_3/\pm75/(\pm60)_3/(\pm45)_2/(\pm15)_2/0_2]_S$
(c)	$[0_2/90_4/\pm45/90_2/\pm45/90_2/(\pm45)_8/90_4/\pm45/90_4/(\pm45)_8/0_2/\pm45/0_4/(\pm45)_2/0_2/\pm45/0_2]_S$
(d)	$[(\pm60)_3/\pm75/\pm60/(\pm75)_2/\pm60/\pm75/\pm60/\pm30/\pm60/\pm30/(\pm60)_2/\pm75/\pm45/(\pm30)_3/$ $90_2/\pm15/0_2/90_2/(\pm45)_2/\pm30/\pm60/90_2/(\pm15)_2/\pm30/0_2/\pm45/\pm15/0_2/\pm15]_S$

respectively to build buckling constraints. The effect of including strain constraints only at level 2 on the optimization results is also demonstrated. Cases (a) and (d) from SLO1 optimization study are used for the STLO study.

An initial ply thickness needs to be specified to run the optimization from level 1. Ply thickness values are chosen from a set T = {0.001in., 0.00105in., 0.0011in., . . . , 0.008in.} for case (a) and T = {0.0008in., 0.00085in., 0.0009in., . . . , 0.003in.} for case (d). The design is optimized such that the buckling load factor is higher than $\lambda_0 = 9999.15$ and $\lambda_0 = 10,750.44$ with $N_{xx} = 1$ lb/in. and $\frac{N_{yy}}{N_{xx}} = 0.5$. for the two cases respectively. The results in comparison with those from PSO for discrete thicknesses are given in Table 8. The failure load is chosen based on values of maximum buckling load obtained from the single level optimization (SLO1). As the thickness values used in SLO1 for cases (a) and (d) are 0.005 in. and 0.00167 in. respectively, the optimizer must output these values as global optimum ply thickness. To check the impact of initial design point (ply thickness) on the solution, optimization is done using different values of initial ply thicknesses.

Results in Table 8 show optimal solution for the case (a) and Table 9 shows optimal values for the case (d). It is observed that STLO obtains optimum values for all initial design points considered. These values can be verified as global optimum from the results of SLO1 and SLO2. The STLO converges in 4 iterations with a run time of less than 2 seconds in most cases. This is much faster and robust than the PSO. As seen in the results for previous two formulations, the PSO struggles with higher number of discrete design variables (Table 9).

Even though the results for $(N, \Theta$ set$) = (72, 2)$ converge for 'buckling constraint only' optimization, addition of strain constraints increases the ply thickness from 0.0017 in. to over 0.002 in. in all cases. The stacking sequence is

Table 8 Optimal result comparison for case (a) using STLO.

Constraints included	Parameters	PSO runs		STLO Initial thickness (in.)			
		#1	#2	0.001	0.003	0.005	0.007
Buckling	Volume (in.^{3})	24	24	24	24	24	24
	ply thickness (in.)	0.005	0.005	0.005	0.005	0.005	0.005
	run time (s)	9.165	9.54	1.017	1.32	1.2	1.084
Buckling and Strain	Volume (in.^{3})	24.24	24.24	24.24	24.24	24.24	24.24
	ply thickness (in.)	0.00505	0.00505	0.00505	0.00505	0.00505	0.00505
	run time (s)	15.58	13.745	1.78	1.32	1.26	1.104

Table 9 Optimal result comparison for case (d) using STLO.

Constraints included	Parameters	PSO runs		STLO Initial thickness (in.)			
		#1	#2	0.001	0.003	0.005	0.007
Buckling	Volume (in.^{3})	Ø	Ø	24.48	24.48	24.48	24.48
	ply thickness (in.)	Ø	Ø	0.0017	0.0017	0.0017	0.0017
	run time (s)	180 (max)	180 (max)	2.13	2.04	1.96	1.97
Buckling and Strain	Volume (in.^{3})	Ø	Ø	28.8	28.8	28.8	29.52
	ply thickness (in.)	Ø	Ø	0.002	0.002	0.002	0.00205
	run time (s)	180 (max)	180 (max)	3.06	2.92	3.18	2.59

Note: Ø denotes an infeasible solution resulting from non-convergence of results in a predefined number of PSO iterations.

Table 10 Comparison between results obtained from SLO2 and STLO.

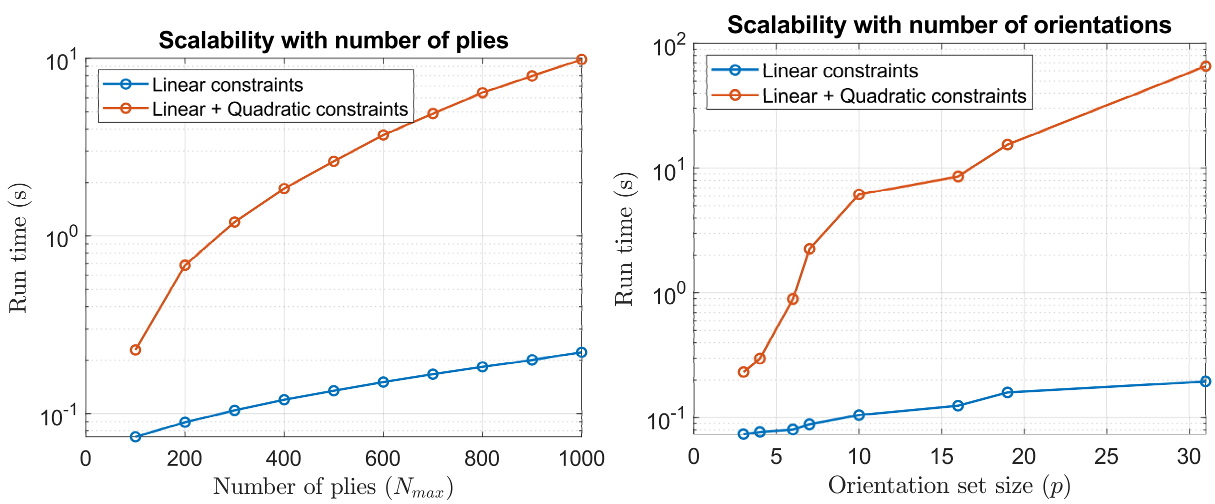
Constraints	Case	Volume (in.^{3})		Average run time (s)	
		SLO2	STLO	SLO2	STLO
Buckling	(a)	24	24	0.07	1.15
	(d)	24.72	24.48	1.24	2.025
Buckling and Strain	(a)	26	24.24	0.23	1.37
	(d)	24.716	28.8	5.19	2.93

optimized in level 1 for buckling constraints only, and the optimal configuration to prevent buckling may or may not be ideal to prevent material failure. Using a fixed stacking sequence during level 2 optimization which is optimized for buckling load only can result in an increase in ply thickness to satisfy strain constraints. A way to mitigate this and to further improve weight reduction is to use non-uniform ply thicknesses to change the thickness of only those plies in which strain constraints are violated. In such a scenario, the buckling constraints at level 2 can no longer be simplified directly; they require a reformulation for their linearization. It will also enable the use of hybrid laminates having plies with different materials that have different thicknesses. The application of reformulation linearization techniques (RLT) in laminate optimization will be investigated in a separate study.

A comparison between SLO2 and STLO results is also given in Table 10. SLO2 and STLO give two different approaches to weight minimization depending on the optimization problems. Both formulations result in the same optimal volume when only buckling constraints are included. Any small discrepancy is due to ply thickness sets T used in STLO that do not include the ply thickness value used in SLO2. It forces the optimizer in STLO to choose a higher thickness value from set T. This is mainly observed in case (d) results which include strain constraints. SLO2 is much faster than STLO for linear constraints whereas the use of quadratic constraints for strains leads to increased computational costs with larger design spaces. STLO has the advantage of having linear constraints for strains which reduces the computational costs in such scenarios.

D. Scalability with Problem Size

To check the optimization scalability, SLO2 is chosen since it has linear as well as quadratic constraints. The number of design variables depend on the maximum number of plies N_{max} and the size of orientation set p. The number of linear constraints depend on the number of buckling modes to be included $(m \times n)$, N_{max} and p. The number of quadratic constraints depend only on p. In this study, the number of buckling modes are fixed at 25 $(m = n = 5)$, N_{max} and p are varied and the solver time is plotted against these variations.



(a) (b)

Fig. 3 Scalability study; (a) Varying number of plies N_{max} and a fixed number of possible orientations $p = 3$, (b) Varying number of possible orientations p and a fixed number of plies $N_{max} = 100$.

Figure 3 shows the optimizer efficiency in handling problems with different number of plies and orientation values. Increasing the number of plies by keeping p constant increases the number of linear constraints associated with ply restrictions. It does not increase the number of quadratic constraints associated with strains since only in-plane loads are applied. Although this is the case, since the size of quadratic coefficient matrix Q increases with the number of design variables, an increased run time is observed with more plies. Increasing the number of orientations by keeping N_{max} constant increases both type of constraints. Moreover, the size as well as number of quadratic constraints are increased in such cases. Hence, there is a steeper increase in run time when quadratic constraints are involved. From the two scalability studies, we can state that the optimization problem can be solved within a minute for a reasonably large number of plies and orientations.

VII. Conclusion

This paper studied optimization of composite laminate for a minimum weight design using Mixed Integer Programming (MIP) available in software, *Gurobi*. The studied composite laminate is subjected to buckling and strain constraints which are non-linear with respect to the binary design variables. In this study, we expanded the capabilities of MIP solvers to handle non-linear quadratic and cubic constraints with respect to fiber orientation and ply thickness design parameters by using different optimization formulations. The MIP formulations use binary design variables to represent ply orientations. Three different optimization formulations of increasing complexity using MIP were studied, with each building on the previous one for buckling load maximization and weight minimization of composite laminates. The first formulation, Single Level Optimization 1 (SLO1), generalizes the Mixed Integer Linear program (MILP) formulation demonstrated, earlier by Haftka et al. [21], for any given set of ply fiber orientations. The second formulation, Single Level Optimization 2 (SLO2) is an optimization approach for laminate weight minimization by minimizing the number of plies using Integer Linear Programming (ILP). With the addition of strain constraints, the optimization is solved as a Quadratically Constrained Linear Program (QCLP). A Sequential Two Level Optimization (STLO) is next employed to facilitate the use of MILP by minimizing the total laminate weight by minimizing ply thickness, given a discrete set of ply thicknesses.

The results for buckling load maximization show the capability of MIP in solving stacking sequence problems having linear constraints with a high efficiency, even for large number of design variables as compared to a Particle Swarm Optimizer (PSO). Similar results are observed when the ILP is incorporated in SLO2. It is seen that the PSO struggles to obtain a feasible solution when the design space of combinatorial problem is large, MIP handles it more efficiently. The PSO is rendered inefficient as it is inherently built for continuous design spaces. Using SLO2 and STLO significantly improves the efficiency of the weight minimization process over PSO.

Based on the design requirements and manufacturing feasibility, either SLO2 or STLO can be used for laminate design. SLO2 is easier to implement as compared to STLO as SLO2 optimizes all design variables in one optimization pass. SLO2 is more efficient for small to medium sized optimization models. Since SLO2 is entirely governed by the MIP solver, the problem is guaranteed to be solved to optimality with convex design space. STLO is found to be more efficient for large design spaces but the optimality must be determined through user defined convergence criteria. The method is fast even for orientation sets with a large size as shown in the scalability study. The current STLO requires the number of plies to be predetermined but it can be easily modified to add the capability to vary number of plies. Laminate optimization with a large number of plies, as those observed in thin-ply composites, can significantly benefit from this approach.

A limitation of MIP solvers is that they require explicit expressions of constraints and the objective function. The formulations demonstrated in this study work well with the problem of a rectangular symmetric balanced laminate subjected to bi-axial compressive loads. In practical application, there is no explicit expression for the buckling load and strains as presented in this work. The semi-analytical approach, such as Ritz method or the Finite Element Method are used for analyzing complex structures. To this end, sequential linear mixed integer programming can be used to solve the complex laminate configuration with the methods devised in this work. Another direct application of this work can be for stacking sequence retrieval from optimized lamination parameters and will be explored next.

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