

Chapter 5

Measuring Describing Functions and Nonlinear Model Identification

In this chapter we discuss the experimental setup and procedures of measuring the describing functions of laminar premixed flame dynamics. The data should enable the prediction of the limit cycle amplitude and frequency. Moreover, an empirical nonlinear flame model is identified from the measured describing function data.

It is probably beyond the state of the art to obtain a complete, global, nonlinear dynamic model of the heat-release dynamics of a combustion system. Therefore, a practical, first-cut solution is to build a nonlinear model that captures the describing function of the nonlinear system. A describing function describes the response of a nonlinear system to a sinusoidal input at the fundamental frequency of the input. The magnitude of the describing function is the ratio of magnitude of the output at the fundamental frequency to the magnitude of the input sinusoid. The phase of the describing function is the phase angle between the input and output at the fundamental frequency. With a sinusoidal velocity perturbation to the flame, we can determine the describing function by measuring the fundamental frequency amplitude and phase of the output heat release rate signal, which we assume is proportional to the integrated OH* emission from the flame. The describing function, which is complex, is a function of input velocity amplitude and frequency.

The describing function technique is often used to analyze nonlinear feedback systems to predict limit cycle frequencies and amplitudes. The thermoacoustic

interactions in the tube combustor can be described by the feedback interconnection of a linear acoustics model and the nonlinear heat release dynamics. The linear acoustics model can be obtained from a first-principles analysis. In this chapter, we focus on the techniques and procedures of the experimental design and measurement of the flame describing functions. A flat flame burner is used to capture the nonlinear flame dynamics from the input velocity perturbations to the output dynamic heat release rate.

Furthermore, a system identification technique, which is based on the principle of harmonic balancing, is applied to identify a dynamic model with a cubic nonlinearity that fits the measured describing function. The stability of the nonlinear dynamic model is studied and digital simulation is provided for verification. This low-order nonlinear flame model can be used in a dynamic model of the thermoacoustic instability that is useful for efficient control system simulation.

5.1 Experimental setup and data measurement

Figure 5.1 shows the physical feedback loop that always exists between the system acoustics and the flame for any combustion system. The experimental flat-flame burner, depicted in Figure 5.2, was designed such that there is no acoustic resonance within the bandwidth of 20-400Hz, which is the frequency range of interest. By measuring the input acoustic velocity u and heat release rate perturbation q , this experiment isolates the flame dynamics block. A side-mounted speaker is excited as the source of the acoustic velocity perturbation and premixed methane and air, with equivalence ratio ϕ , is injected as shown, with a total flow rate of Q_{tot} . Two pressure transducers are placed just below the flame holder to measure the acoustic velocity u and a PMT sensor is placed above the flame to measure the OH* signal, which is assumed to be proportional to the heat release rate q .

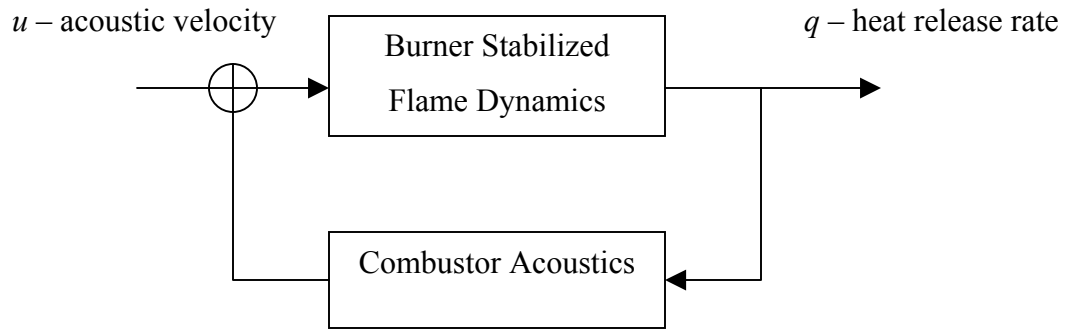


Figure 5.1 Block Diagram of the Experimental Flat Flame Burner

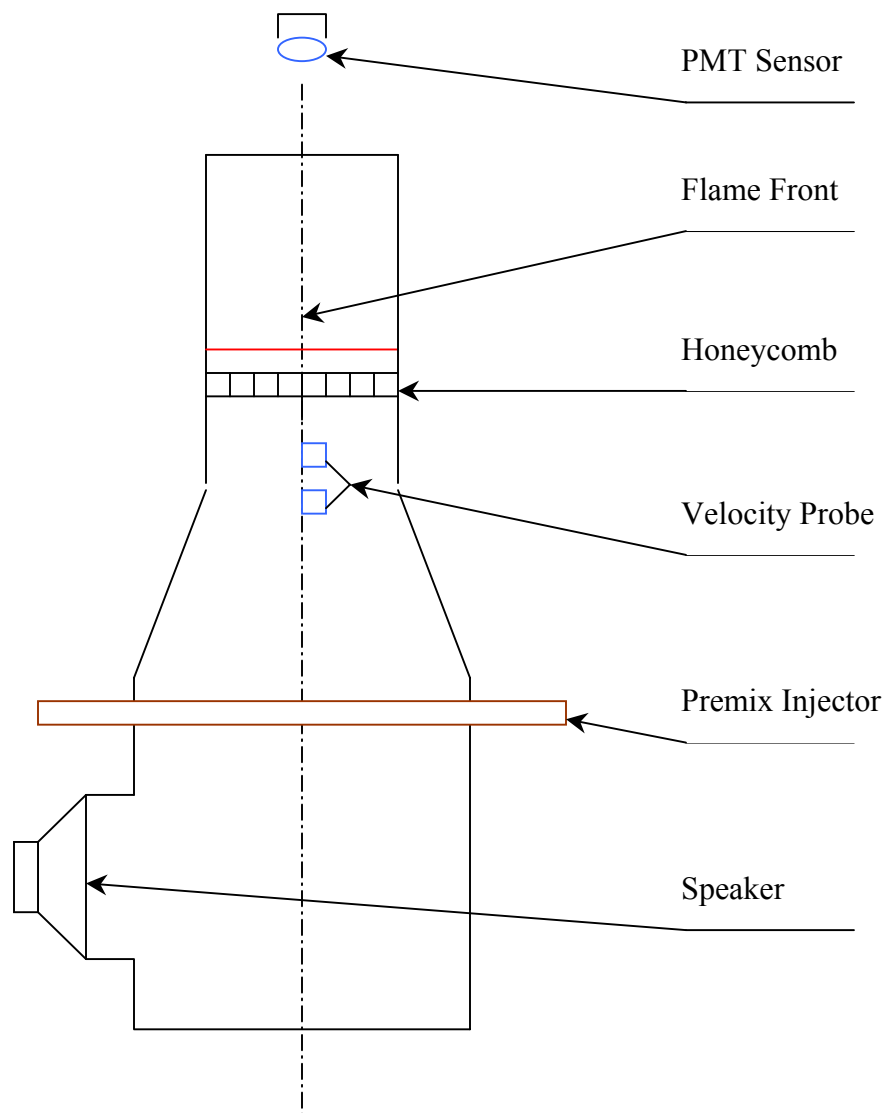


Figure 5.2 Schematic of the Experimental Flat Flame Burner

The burner consists of a plenum, a bell reducer, a ceramic honeycomb flame stabilizer and a quartz combustion chamber. The plenum has a diameter of 100mm, and the bell reducer interfaces to the flame holder with diameter of 65mm. The thickness of flame holder is 17mm, and the length of the quartz chimney is about 150mm. The length of the chimney is chosen to ensure there is no acoustic resonance within the frequency band 20-400Hz, ensuring a stable system so that the measurement can be conducted. Figure 5.3 shows a picture of the experimental setup and pictures of flames.

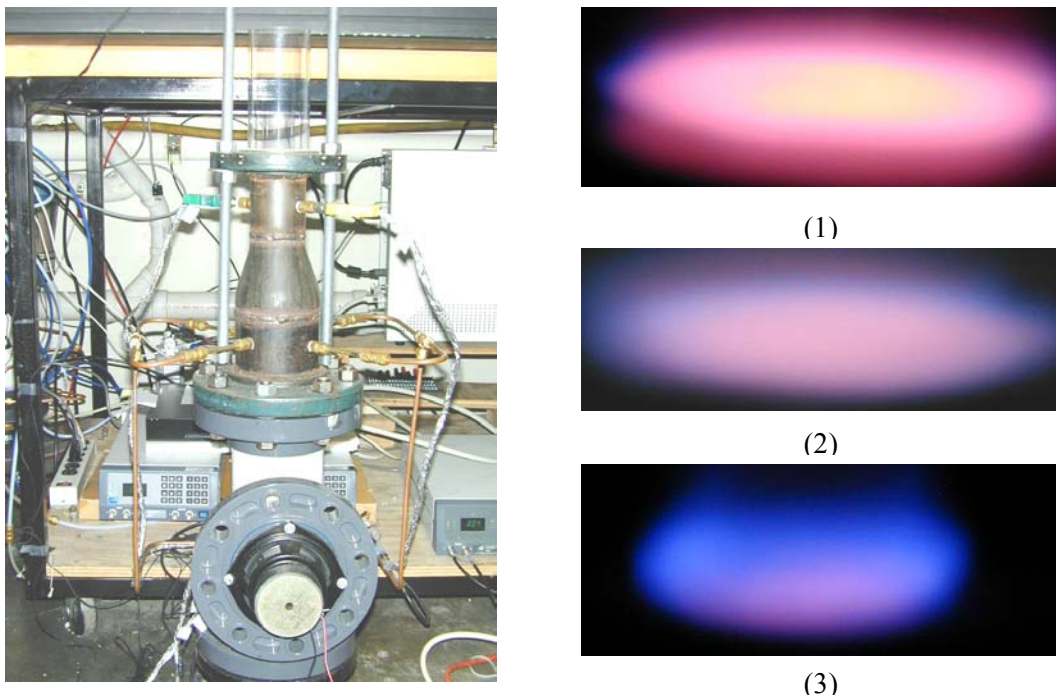


Figure 5.3 Picture of the experimental setup and flame samples

Note that the sample flame pictures are taken at various flow conditions: (1) flow rate 130cc/s, $\phi = 0.45$, without external perturbations; (2) flow rate 180cc/s, $\phi = 0.6$, without external excitations; (3) flow rate 180cc/s, $\phi = 0.6$, with speaker voltage at 10 volts. As we can see, with external velocity perturbation, the flame area shrinks, and the flame itself becomes more volatile.

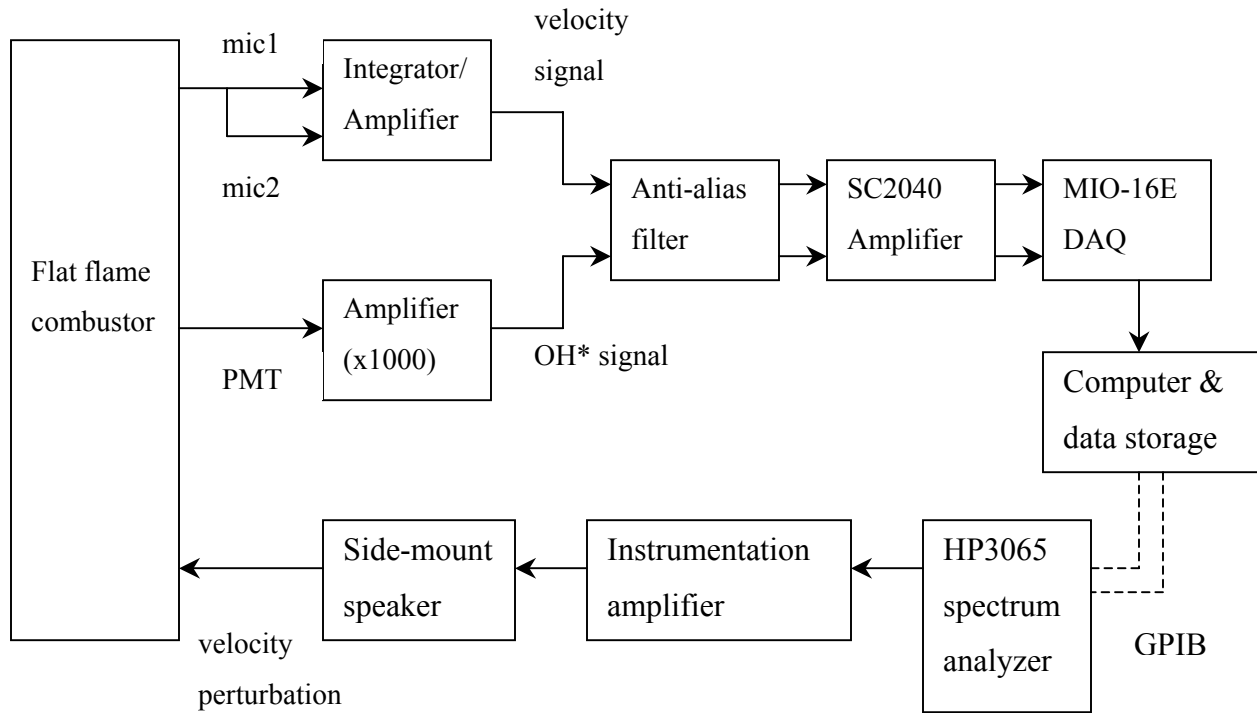


Figure 5.4 Instrument diagram of flame experiment setup

The instrumental layout of the flame dynamics measurement system is shown in Figure 5.4. The velocity perturbation signal u' is generated using the two-microphone technique. The OH* signal is captured by a PMT sensor through an optical system, and it is considered proportional to heat release rate changes q' . Both channels are filtered by an anti-alias filter before entering the SC2040 8-channel strain gage amplifier. Notice that the filters with 0dB magnitudes add the same phase delay to both channels, but do not affect the computed describing function in which the phase difference between both channels is cancelled out. The data acquisition card then captures samples of the data at a rate of 8912 samples per second, and for each specific speaker source setting, a data series is stored for a time span of 60 seconds. The DAQ is simultaneous S/H. Further spectrum analysis to extract the fundamental component of input and output is performed with Matlab[®], and the describing function is then obtained.

On the input side, acoustic velocity is computed from two pressure signals using the 1-D Euler's equation [Kha01].

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p \quad (5.1)$$

Integration gives,

$$u(x) = -\frac{1}{\rho_0} \int \frac{\partial p(x, t)}{\partial x} dt \quad (5.2)$$

Using the difference between the pressure sensed by the two microphones to approximate the spatial derivative at the flame location x_f , and denoting the distance between the two microphones as Δx yields,

$$u(x_f, t) = -\frac{1}{\rho_0} \int \frac{(p_2 - p_1)}{\Delta x} dt \quad (5.3)$$

Thus, by sending the difference of the pressure signals through an integrator, a signal proportional to the velocity is obtained. The detailed calibration procedure is given in Appendix A. At frequency 180 Hz, the instrumental gain from measured voltage to actual velocity is about 26.9 cm/(s-V).

Compared to the linear flame dynamics study [Kha01] that used the same experimental rig, the difference is that the amplitude of the induced velocity has to be increased to the point that nonlinear effects are observed. Raising the voltage applied to the speaker increases the source velocity. For the experimental determination of the describing function, the frequency of the speaker voltage was held constant at various values within a frequency range of 160-200Hz, and the amplitude of the velocity was swept from small to large by setting the speaker voltage at different levels.

Our Rijke tube experiment exhibits a limit cycle with a frequency near 180Hz. Considering the frequency shift with equivalence ratio ϕ and total flow Q_{tot} , the describing function should be measured over a bandwidth of 160-200Hz. The speaker voltage is driven as high as possible without blowing off the flame. Ideally this experiment should generate acoustic waves with amplitudes comparable to those occurring in the unstable tube combustor.

5.2 Review of Describing Function

The describing function was initially introduced by R. Kochenburger as a linearization technique to address the existence of limit cycles [Kol92][Ath75]. For a linear system, the transfer function is an important description of the system in the frequency domain. At a given frequency, the gain and phase are constants, independent of the system input. However, a nonlinear system's response can be much more complicated. Jump phenomena, frequency shifting, and self-excited oscillators can occur in nonlinear differential equations [Nay93]. So, to generate a complete description of the frequency response of a nonlinear system is not trivial. The describing function provides a quasilinear representation for a nonlinear system subjected to a sinusoid input. It is widely used in the field of nonlinear feedback control.

For a sinusoidal input $u = A \cos(\omega t)$, the output at the fundamental frequency can be written as $F(A, \omega) \cos[\omega t + \phi(A, \omega)]$. The describing function has a magnitude (gain) of $\frac{F(A, \omega)}{A}$ and phase given by $\angle \phi(A, \omega)$. If the describing function phase is zero, then it is referred to as the “equivalent gain”.

Due to nonlinear effects, a pure sinusoidal input at frequency ω produces an output that contains not only a fundamental component at frequency ω , but also harmonics. Fourier series theory is applicable to extract the fundamental component. The output time sequence $f(t)$, over an interval $[t_0, t_0 + 2\pi/\omega]$, can be represented by a Fourier series as,

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} [A_n \cos(n\omega t) + B_n \sin(n\omega t)], \quad (5.4)$$

where its Fourier coefficients are,

$$\begin{aligned} A_n &= \frac{\omega}{\pi} \int_{t_0}^{t_0 + \frac{2\pi}{\omega}} f(t) \cos(n\omega t) dt \\ B_n &= \frac{\omega}{\pi} \int_{t_0}^{t_0 + \frac{2\pi}{\omega}} f(t) \sin(n\omega t) dt, \quad n = 0, 1, 2, 3, \dots \end{aligned} \quad (5.5)$$

The fundamental output component is

$$\tilde{f}(t) = A_1 \cos(\omega t) + B_1 \sin(\omega t) = F \cos(\omega t + \phi), \quad (5.6)$$

and the describing function is represented as,

$$N(A, \omega) = \frac{F}{A} \angle \phi \quad (5.7)$$

An example of a describing function is shown below for a saturation nonlinearity, $y = \text{sat}(x)$. For input amplitudes A larger than unity, the output $y(t)$ is forced to one. The sinusoidal response of the nonlinear function is depicted in Figure 5.5. Direct computation using (2) gives

$$N(A, \omega) = \begin{cases} 1 & \text{if } A \leq 1 \\ \frac{2}{\pi} \left[\sin^{-1}\left(\frac{1}{A}\right) + \frac{1}{A} \sqrt{1 - \left(\frac{1}{A}\right)^2} \right] & \text{if } A > 1 \end{cases} \quad (5.8)$$

In this case, the describing function $N(A, \omega)$ is an equivalent gain, as the phase is zero everywhere. Figure 5.6 plots the describing function of the saturation nonlinearity.

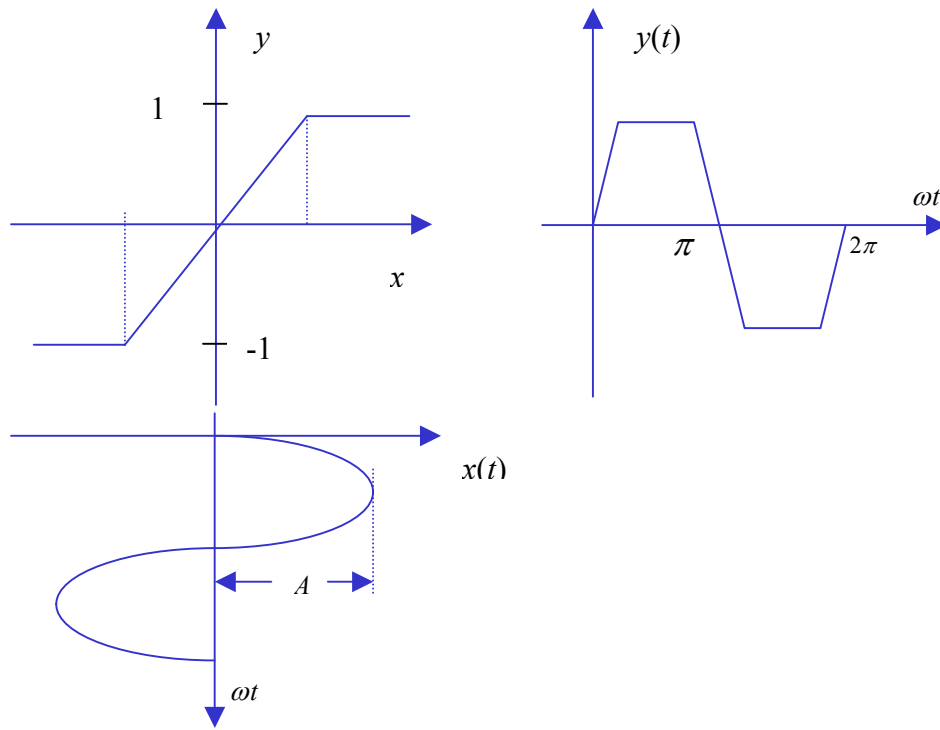


Figure 5.5 Response of a saturation unit to a sinusoid

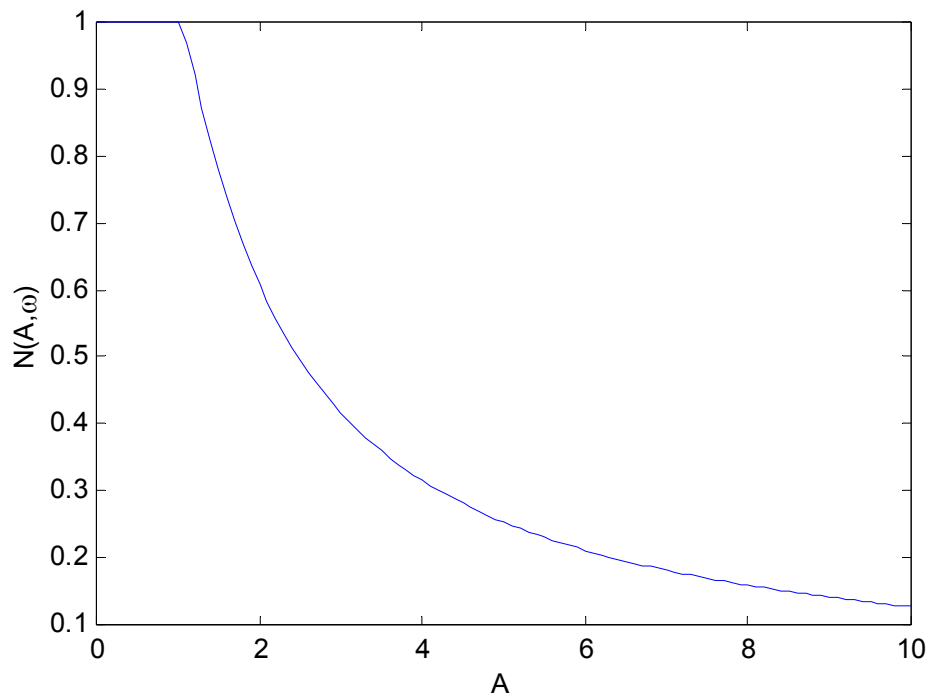


Figure 5.6 Describing function of a saturation unit

5.3 Measured Describing Functions

To measure the describing function, a sinusoid voltage source is applied to the speaker to create the velocity perturbation. Applying various speaker voltage levels at each frequency, the describing function is then computed from the fundamental components of the collected u' and q' data. Note that this describing function of the flame dynamics $N(A,\omega)$ is a function of velocity amplitude A and frequency ω .

The output heat release rate response, sensed by the OH* signal, is measured as a function of velocity amplitude A at each frequency ω . Due to the speaker dynamics and the effect of the acoustic dynamics of the test rig itself, the input velocity amplitude is not exactly proportional to the applied speaker voltage. Figure 5.7 shows the input perturbation velocity amplitude at different speaker voltages under the experimental condition: total flow rate $Q_{tot} = 180$ cc/s and methane-air mixture equivalence ratio $\phi = 0.6$. When the speaker voltage is above 10 volts, increasing the voltage does not result in much change in the velocity. This potentially could be a constraint of this experiment for studying large pressure oscillations. In the following sections, we use the measured voltage value from the velocity probe as the amplitude. 1volt is equivalent to acoustic velocity amplitude of 26.91 cm/s.

As we discussed earlier, the Rijke tube limit cycle frequency is near 180Hz. The describing function is measured within a bandwidth of 160-200Hz and speaker voltage level is sweeping from small to large. The describing function technique requires a sinusoidal input as excitation and measurement of the output component at the fundamental frequency. Even if a pure sinusoidal voltage source were applied to the speaker, the input acoustic velocity would not be a pure sinusoidal signal due to speaker and heat-release nonlinearities. Similarly, the output heat release rate measured from the OH* signal contains various harmonics as well because of the input velocity and the nonlinear characteristics of the heat release rate. A spectrum analyzer can be used to extract the fundamental component and compute the describing function.

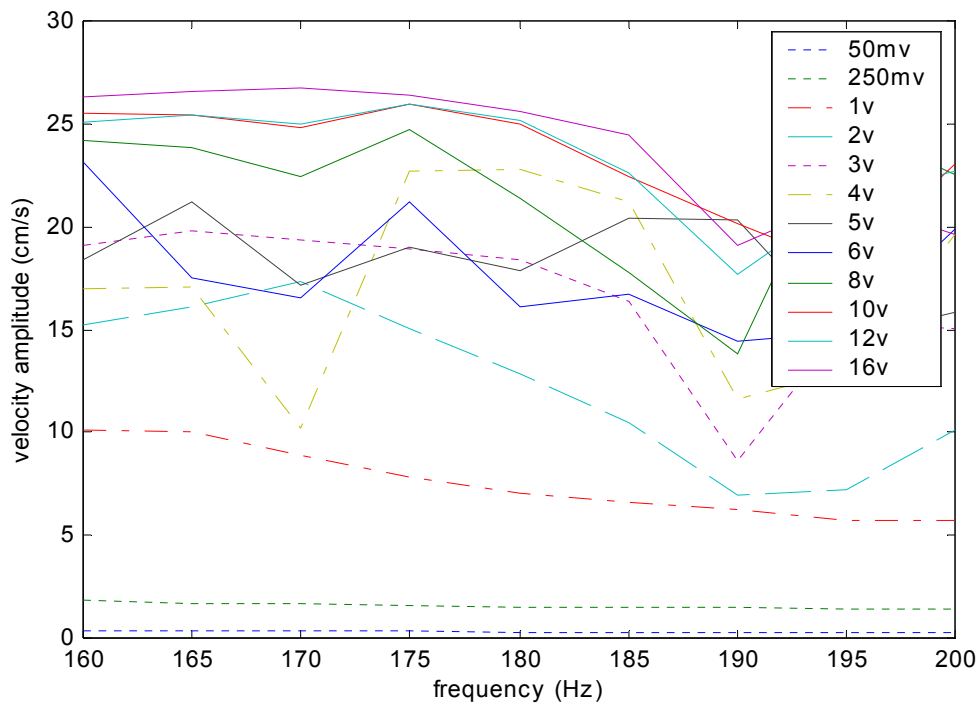


Figure 5.7 Input velocity amplitudes at various speaker voltages

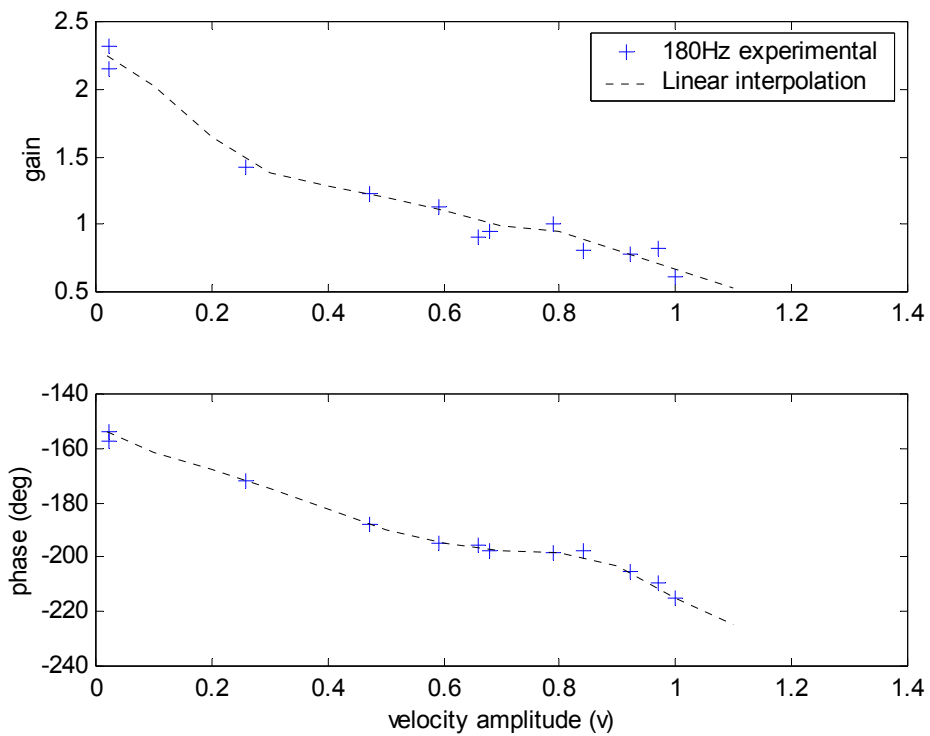


Figure 5.8 An example of experimental describing function at 180 Hz

Figure 5.8 shows a set of experimental describing function data at 180Hz with various velocity amplitudes. ($Q_{tot} = 180 \text{ cc/s}$, $\phi = 0.6$) Linear interpolation is used between the points. As the velocity amplitude increases from 0.02 to unity, the nonlinear effect becomes more pronounced, with the output gain dropping from 2.4 to 0.6, and the phase decreasing from -150° to about -220° .

Figure 5.9 shows the gain and phase response of the heat release rate at 175Hz, 180Hz and 185Hz ($Q_{tot} = 180 \text{ cc/s}$, $\phi = 0.6$). When the input velocity amplitude is near 0.02, the flame response is very close to the linear flame dynamics. Comparing these 3 sets of data, the difference in the gains is subtle, while the phase is sensitive to the small differences in frequency when the input velocity is large.

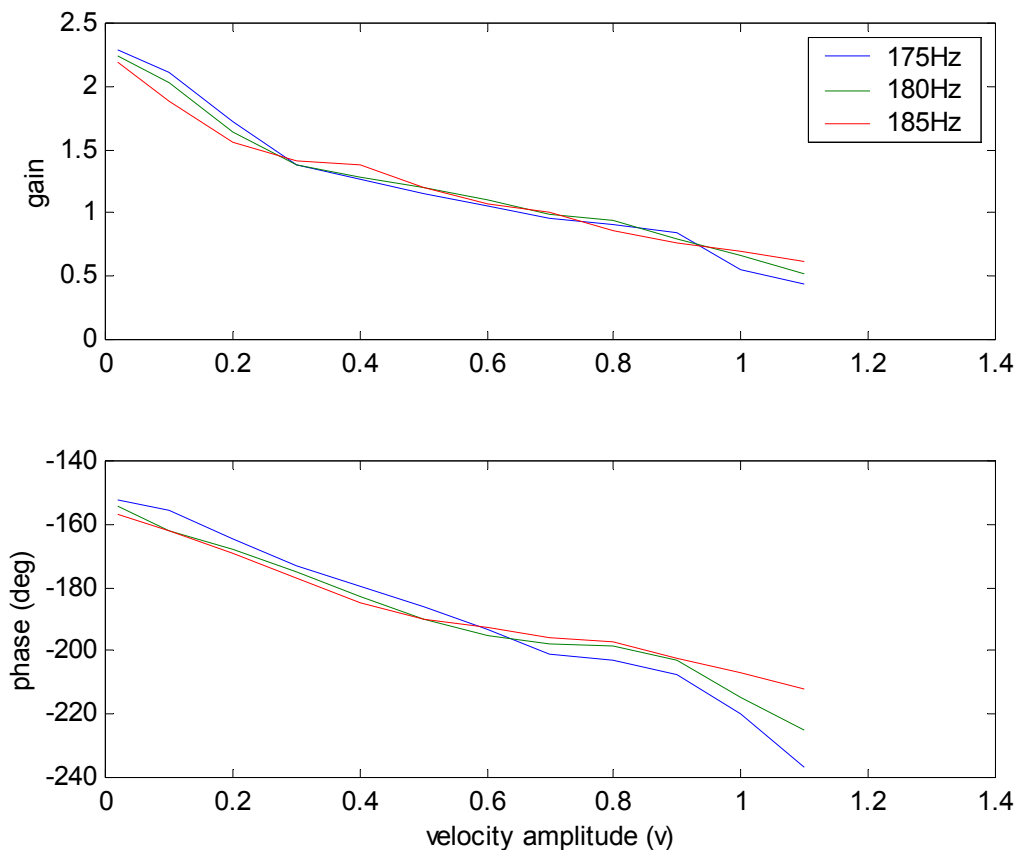


Figure 5.9 Experimental describing functions at frequency 175, 180 and 185 Hz

5.4 Identification of an Empirical Nonlinear Flame Model

In this section, we compute a low order nonlinear dynamic flame model that reproduces the describing function data that we measured from experiments. The linear transfer function of the flame can be easily extracted from small amplitude measurements as a function of frequency, as shown in [Kha01]. The goal in this section is to add nonlinear terms that capture the nonlinear amplitude effects embodied in the describing function data. Only odd nonlinear terms affect the fundamental frequency response data, and hence a number of cubic nonlinear terms are used to approximate the experimental DF data. Stability of the resulting nonlinear model is studied, and numerical simulation is used to validate the dynamic model.

5.4.1 Need for a Dynamic Nonlinear Model

The describing function technique is a first cut method to model complicated nonlinear systems, and it is commonly used to predict limit cycles. However, a describing function is not a dynamic model. A dynamic model has several advantages: (1) a dynamic model is easy to use for simulation; (2) it can be used to estimate transient response, which is useful when studying the stability region of the nonlinear model; (3) It can be used as a tool for control parameter selection and performance evaluation.

For the nonlinear flame dynamics, the form of the nonlinearity is not clear to us at this time. From the describing function, we can see that the phase changes while increasing the amplitude of the input. This implies that the nonlinearity is not just a static nonlinearity that only exist at the input or output. Therefore, it is not proper to model the system as a simple Wiener or Hammerstein system. Thus, a dynamic model is needed to capture the observed nonlinearity.

5.4.2 From Describing Function to Dynamic Model

A dynamic model is going to be identified to fit the describing functions that we measured from the flame dynamics experiments. However, the describing function only contains nonlinear dynamic response for a sinusoidal input. If an input is not a sinusoid, the output response is not predicted by the describing functions. Meanwhile, for the flame model, the nonlinear describing function $N(A, \omega)$ is both a function of input amplitude A and a function of input frequency ω . Due to a finite number of experiments, the describing functions that we measured cover only a small subset of this two-dimensional space. Therefore, fitting describing function using a dynamic model is a simple approach that does not model the system completely.

An n th order nonlinear dynamic model can be written as,

$$y^{(n)} = f(y^{(n-1)}, \dots, \dot{y}, y, u^{(n)}, \dots, \dot{u}, u) \quad (5.9)$$

The right hand side is a function of $2n$ variables. As we see, our experimentally measured describing function data only concerns a very small subset of the domain of this function. Therefore, the transformation from describing functions to dynamic model is an extensional process. The model's range of validity is enlarged while assuming the linear dynamics still match with the actual experimental system.

This eventually leads to the problems that we encountered during our study of model stability and validation. Even though the dynamic model fits the describing function, it does not guarantee the stability of the model during the simulation run. For example, the describing function that we capture is within the frequency range of 160-200Hz and amplitude range from 0 to 1v. Given a sinusoidal input u to the dynamic model, the transient response could drive the output y out of the amplitude range, and the system could easily go unstable. Therefore, a low order model is desirable and a cascade form of subsystem modeling is employed in the following section to incorporate the describing function data.

5.4.3 Identification with Cubic Nonlinearity

The linear part of the model can be obtained from the frequency response function measured with small velocity perturbations. An inverse frequency response method is then applied to fit an n th order linear flame model. The dynamic nonlinear model is constructed by appending nonlinear terms to the n th order input-output differential equations. These nonlinear coefficients are determined based on the principle of harmonic balancing. The linear part of the flame model is 4th order, and can be treated as a cascade form of two 2nd order units with resonance frequencies at 31Hz and 156Hz given by

$$G_1(s) = \frac{1.029(s + 600.4)(s + 50.27)}{(s^2 + 174.4s + 37830)}$$

$$G_2(s) = \frac{0.2596(s^2 - 3961s + 8486000)}{(s^2 + 996.6s + 965700)}$$

Since the nonlinear data that we measured is near 180Hz, it is reasonable to separate the system into a linear 2nd order module with frequency response $P_1(j\omega)$ resonant at 30.9 Hz, cascaded with a 2nd order nonlinear module $P_2(A, \omega)$ with linear resonance frequency at 156.4 Hz . This form should retain all of the linear dynamics and incorporate relevant nonlinearity in the high frequency range. The describing function for the P_2 system must satisfy

$$P_1(j\omega)P_2(A, \omega) = N(A, \omega) \quad (5.10)$$

and can be computed from the available describing function data.

With describing function data, the procedure to identify a 2nd order nonlinear model is straightforward based on principle of harmonic balancing. With a desired form of input output differential equation, apply harmonic balancing to both sides of the equation for each frequency component, and use the least squares method to solve the

resulting equations for the unknown coefficients. In this case, only the fundamental frequency components are considered. All linear coefficients (a_1, a_0, b_2, b_1, b_0) are assumed known from the small-perturbation linear identification of the system, as in $G_2(s)$, and the only goal is to pick proper nonlinear terms and identify parameters. The system equation is now,

$$\ddot{q} = -a_1\dot{q} - a_0q + b_2\ddot{u} + b_1\dot{u} + b_0u + (\text{nonlinear_terms}) \quad (5.11)$$

There are 125 possible cubic terms, of which the following were selected for the nonlinear identification.

$$q^2u, \quad qu^2, \quad u^3, \quad \dot{q}u^2, \quad \dot{q}qu, \quad \dot{q}q^2, \quad \dot{u}u^2, \quad \dot{u}qu, \quad \dot{u}q^2, \quad \ddot{u}qu, \quad \ddot{u}u^2$$

Two principles guided the choice of nonlinear terms. The first was to avoid using the q^3 term, because it introduces 2 more fixed points. Usually these fixed points are saddle points, which limits the stability region of the model. The second was to give priority to terms containing the input u , \dot{u} , or \ddot{u} term, since the inputs are known, and there are less nonlinear terms in the feedback loop. Using the least squares method to obtain a fit to the experimental data resulted in the coefficients. The coefficient of each term is listed below in Table 5.1. Figure 5.10 shows an example of the fit at 180Hz. The model matches well with all of the experimental data.

\dot{q}	q	\ddot{u}	\dot{u}	u	
-9.9655e2	-9.6569e4	2.5960e-1	-1.0283e2	2.203e4	
q^2u	qu^2	u^3	$\dot{q}u^2$	$\dot{q}qu$	$\dot{q}q^2$
1.5233e7	2.6939e7	-3.9366e6	-3.3148e3	-1.4863e4	-2.1490e4
$\dot{u}u^2$	$\dot{u}qu$	$\dot{u}q^2$	$\ddot{u}q^2$	$\ddot{u}qu$	$\ddot{u}u^2$
-1.8648e3	-5.0001e3	-1.2995e4	1.2926e1	1.9670e1	-1.6839

Table 5.1 Coefficients of the identified dynamic model

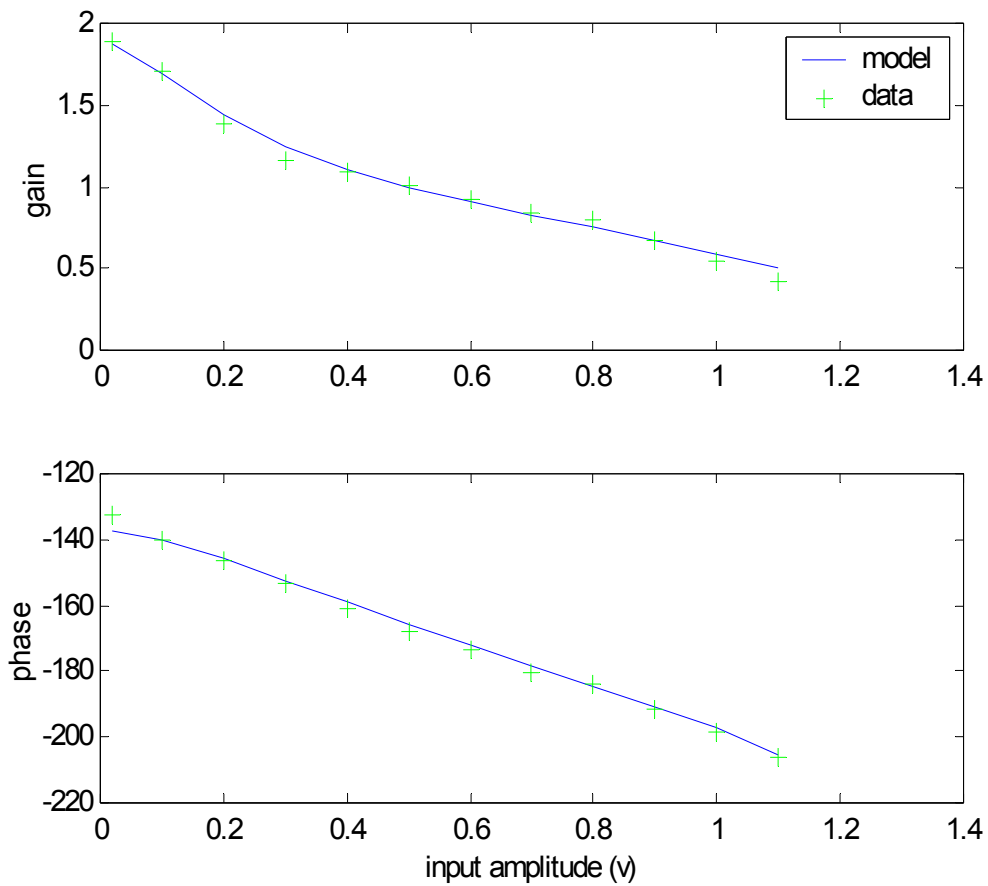


Figure 5.10 Comparison of dynamic model to experimental data at 180Hz

5.4.4 Validation of the Dynamic Model

To validate the dynamic model, numerical simulation was performed to reproduce the data in the time-domain. Simulation is the simplest way to verify stability of the model over the relevant range and spectral analysis enables us to extract the fundamental frequency component, which is always dominant as expected, to compute the describing function data and compare it with the original nonlinear data. Figure 5.11 shows a time domain simulation with input velocity amplitude 1 at 180Hz. A phase-plane plot is presented to show nonlinear response of the model.

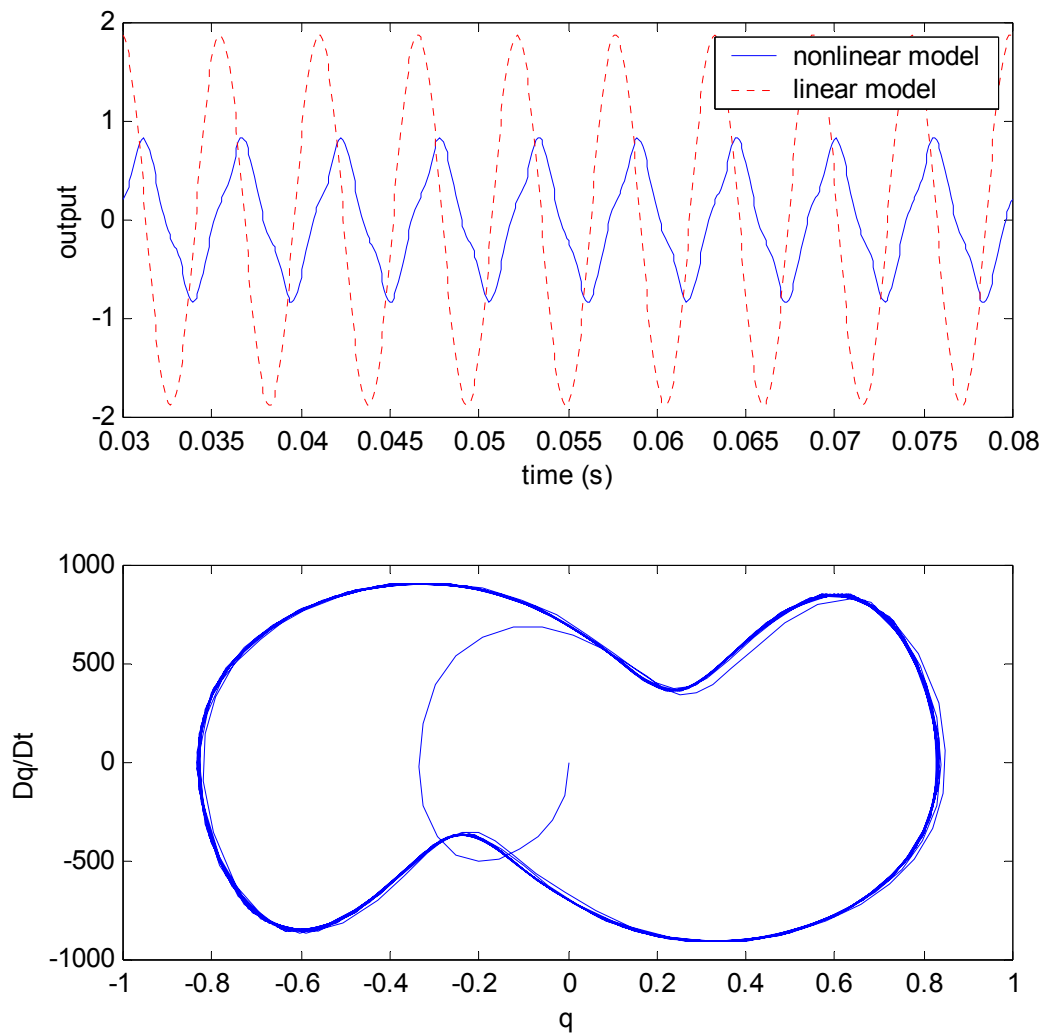


Figure 5.11 Time-domain simulation of model with input ($A = 1$, $f = 180\text{Hz}$)

Compared to the nonlinear frequency response data listed in Figure 5.10, time domain simulation shows the right gain and phase values. In summary, the identified dynamic nonlinear model accurately reproduces the desired describing function data. This model will be useful for stability prediction, due to its accurate linear characteristics. In addition, since the response of the thermoacoustic system to control action will be dominantly sinusoidal with a frequency in the neighborhood of the instability frequency, the model should be useful for control-oriented simulations. There is no expectation that non-sinusoidal transients in the nonlinear region will be handled accurately by this model.