GLOBAL EXISTENCE IN L^1 FOR THE SQUARE-WELL KINETIC EQUATION

by

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(ABSTRACT)

An attractive square-well is incorporated into the Enskog equation, in order to model the kinetic theory of a moderately dense gas with intermolecular potential. The existence of solutions to the Cauchy problem in L^1, global in time and for arbitrary initial data, is proved.

A simple derivation of the square-well kinetic equation is given. Lewis's method is used, which starts from the Liouville equation of statistical mechanics. Then various symmetries of the collisional integrals are established. An H-theorem for entropy, mass, and momentum conservation is obtained, as well as an energy estimate, and key gain-loss estimates.

Approximate equations for the square-well kinetic equation are constructed that preserve symmetries of the collisional integral. Existence of nonnegative solutions of the approximate equations and weak compactness are obtained. The velocity averaging lemma of Golse is then a principal tool in demonstrating the convergence of the approximate solutions to a solution of the renormalized square well kinetic equation. The existence of weak solution of the initial value problem for the square-well kinetic equation is thus proved.

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Section V Approximate solution

Section VI Some results due to weak limit

§I. Introduction

Although the Boltzmann equation has been a mainstay of kinetic theory for more than a century, it is properly the description of a rarefied gas, yielding only the transport coefficients of an ideal gas. Enskog, in 1921, first attempted to model more correctly moderately dense gases, taking into account the dimensions of the gas molecules. The Enskog equation, in various modified and revised versions (to correct hydrodynamics), has been remarkably successful in describing the one-particle distribution function for real gases.

A drawback to Enskog theory is that the Enskog equation, unlike the Boltzmann equation. includes no intermolecular potential, describing purely the elastic collision of hard spheres. In recent years G. Stell, H. van Beijeren and others in the physics community [1-4] have proposed a variety of *kinetic reference theories* which would better describe gases in the moderately dense regime, by incorporating long range interactions into the Enskog description. The most direct fashion to accomplish this is to include a binary intermolecular potential at the Liousville level and follow the Enskog derivation, thus obtaining an Enskog-like equation or system of equations. Several such models have been proposed in the literature[2-8], but to date no results were available on the Cauchy problem for such models.

Herein we obtain the first such results for an Enskog equation with an intermolecular potential. We have chosen to study the solution of the initial value problem for the square-well kinetic equation with an attractive square well potential added to the hard core repulsion.

The square well potential,

$$\phi(x) = \begin{cases} \infty, & 0 < x < a \\ -\varepsilon, & a \le x < Ra \\ 0, & x \ge Ra, \end{cases} \tag{1.1}$$

must be introduced at the Liousville level. The result is an Enskog equation with four types of collisions [1] (see figure 1.1): collisions at the hard core; collisions entering the square well; collisions leaving the square well; and bound state collisions.

We shall see in the next section that we are led thus to consider the Cauchy problem in $((0,\infty)\times R^3\times R^3)$:

$$
\begin{aligned}
&\left[\frac{\partial}{\partial t}+\vec{v}_1\cdot\nabla_r\right]f(\vec{r}_1,\vec{v}_1,t)=\\
&=a^2\int_{R^3}d\vec{v}_2\int_{S^2}d\vec{\sigma}\vec{\sigma}\cdot\vec{p}\theta(\vec{\sigma}\cdot\vec{p})[g(\vec{r}_1,\vec{r}_1+a\vec{\sigma}|n)f(\vec{r}_1,\vec{v}_1',t)f(r_1+a\vec{\sigma},\vec{v}_2',t)\\
&\quad-g(\vec{r}_1,\vec{r}_1-a\vec{\sigma}|n)f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1-a\vec{\sigma},\vec{v}_2,t)]\\
&\quad+R^2a^2\int_{R^3}d\vec{v}_2\int_{S^2}d\vec{\sigma}\vec{\sigma}\cdot\vec{p}\theta(\vec{\sigma}\cdot\vec{p})[g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n)f(\vec{r}_1,\vec{v}_1'',t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2'',t)]\\
&\quad-g(\vec{r}_1,\vec{r}_1-Ra\vec{\sigma}|n)f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1-Ra\vec{\sigma},\vec{v}_2,t)]\\
&\quad+R^2a^2\int_{R^3}d\vec{v}_2\int_{S^2}d\vec{\sigma}\vec{\sigma}\cdot\vec{p}\theta(\vec{\sigma}\cdot\vec{p}-\sqrt{4\epsilon})[g(\vec{r}_1,\vec{r}_1-Ra\vec{\sigma}|n)f(\vec{r}_1,\vec{v}_1''',t)f(\vec{r}_1-Ra\vec{\sigma},\vec{v}_2''',t)\\
&\quad-g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n)f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2,t)]\\
&\quad+R^2a^2\int_{R^3}d\vec{v}_2\int_{S^2}d\vec{\sigma}\vec{\sigma}\cdot\vec{p}\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon}-\vec{\sigma}\cdot\vec{p})\\
&\qquad\qquad\times[g(\vec{r}_1,\vec{r}_1-Ra\vec{\sigma}|n)f(\vec{r}_1,\vec{v}_1',t)f(\vec{r}_1-Ra\vec{\sigma},\vec{v}_2',t)\\
&\qquad\qquad\quad-g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n)f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2,t)]\\
&=E(f,f)=C_E+C_2+C_3+C_4
\end{aligned}
\tag{1.2}
$$

with initial condition

$$
f(\vec{r}_1,\vec{v}_1,0)=f_0(\vec{r}_1,\vec{v}_1), \tag{1.3}
$$

where $\vec{p}=\vec{v}_2-\vec{v}_1$, $\theta(z)=1$ for $z>0$. $\theta(z)=0$ for $z\le 0$, C_E is the first term on the right side of (1.2) and C_i, $(i=2,3,4)$ are second, third and forth terms, respectively. Here,

$$\begin{aligned}
\vec{v}_1' &= \vec{v}_1 + \vec{\sigma}\vec{\sigma}\cdot\vec{p} \\
\vec{v}_1'' &= \vec{v}_1 + \frac{1}{2}\vec{\sigma}\{\vec{\sigma}\cdot\vec{p} - [(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}}\} \\
\vec{v}_1''' &= \vec{v}_1 + \frac{1}{2}\vec{\sigma}\{\vec{\sigma}\cdot\vec{p} - [(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{\frac{1}{2}}\} \\
\vec{v}_2' &= \vec{v}_2 - \vec{\sigma}\vec{\sigma}\cdot\vec{p} \\
\vec{v}_2'' &= \vec{v}_2 - \frac{1}{2}\vec{\sigma}\{\vec{\sigma}\cdot\vec{p} - [(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}}\} \\
\vec{v}_2''' &= \vec{v}_2 - \frac{1}{2}\vec{\sigma}\{\vec{\sigma}\cdot\vec{p} - [(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{\frac{1}{2}}\},
\end{aligned} \tag{1.4}$$

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3), |\vec{\sigma}| = 1$ and $n(\vec{r}, t, f) = \int_{R^3} f(\vec{r}, \vec{v}, t)d\vec{v}$. We assume $g(\vec{r}_1, \vec{r}_2) \equiv g(\vec{r}_1, \vec{r}_2|n) = g(n(\vec{r}_1, t, f), n(\vec{r}_1, t, f)) \geq 0$ for $0 \leq f \in L^1_+(R^3 \times R^3)$ and $g(\mu, \nu)$, $\mu, \nu \in R$ is uniformly bounded, symmetic under the interchange of μ and ν, continuous about μ and ν. We also suppose that the initial value $f_0 \geq 0$ satisfies the bound

$$\iint\limits_{R^3 \times S^2} (1 + |\vec{r}_1|^2 + |\vec{v}_1|^2 + |\log f_0|) f_0(\vec{r}_1, \vec{v}_1) d\vec{r}_1 d\vec{v}_1 \leq c_0 < \infty \tag{1.5}$$

This corresponds to bounded total mass, energy and entropy, and vanishing behaviour as $\vec{r}_1 \to \infty$.

In recent years, the study of the initial value problem for Boltzmann and Enskog equations has made great advancement. For the case of local in time solutions, M. Lachowicz obtained the existence of solutons of the initial value problem for the Enskog equation[9]. In whole space R^3, G. Toscani and N. Bellomo showed global existence in the case of near vacuum for the Enskog-Boltzmann equation[10]. Under the assumption of the initial value going to zero fast enough at infinity and the mean path being sufficiently large, J. Polewczak[11] proved the existence of a classical solution of the Enskog equation in all R^3, which is global in time. For the space inhomogeneous Enskog equation, and collision factor $g = 1$, C. Cercignani[12] proved existence of a global solution with the initial data in a weighted L^1. In [13-20], L.

Arkeryd, C. Cercignani, J. Polewczak, M.J. Esteban, B. Perthame, N. Bellomo, A. Palczewski, G. Toscani and Y. Shizuta, under various assumptions about small initial data, local in time and spatial homogeneity, obtained existence of solutions of initial value problems for Boltzmann and Enskog equations.

For arbitrary initial data in L^1 for the Boltzmann equation, R.J. DiPerna and P.L. Lions[21] successfully employed a compactness lemma[22] in the proof of global existence for the Cauchy problem. For the Enskog equation J. Polewczak [23] extended their argument in solving the initial value problem for a special case of g. For collisional factor $g \equiv 1$, L. Arkeryd and C. Cercignani[24] added a full scattering term with parameter $\delta \to 0$ to the truncated collisional operator, thereby obtaining global existence.

In Section 2 we employ Lewis's method[25] to derive the square-well kinetic equation. Our aim is only laid on the derivation of the equation. In Section 3 we will give symmetries, bounds and estimates related to the Enskog–Square well collision operator. In the following section we will state precisely the notions of solution for the initial value problem. Section 5 will be concerned with solutions of approximating equations. Finally, in the last two sections we provide an existence theorem for the initial value problem.

§II. Derivation of the square-well kinetic equation

In order to study the square-well kinetic equation, we start from its statistical mechanics derivation. We first derive Boltzmann and Enskog equations. Then the square-well kinetic equation can be obtained as a generalization.

Let us first look at a derivation of the Boltzmann equation. We know that a system of N classical identical interacting particles of mass m confined to a box of volume V can be described by the Hamiltonian

$$H_N = \sum_{i=1}^{N} \{ \frac{\vec{p}_i^{\,2}}{2m} + U^{Box}(\vec{r}_i) \} + \sum_{i<j}^{N} \phi_{ij}(|\vec{r}_i - \vec{r}_j|). \tag{2.1}$$

$\phi_{ij}(|\vec{r}_i - \vec{r}_j|)$ indicates that the particles interact through central forces, and U^{Box} enforces that particles are confined to the box. For the description of classical many body dynamics, we also want to know what is the specification of the number of member systems in the element of phase space $\vec{r}_i \rightarrow \vec{r}_i + d\vec{r}_i, \vec{p}_i \rightarrow \vec{p}_i + d\vec{p}_i, i = 1, 2, ...$. We introduce a density in phase space $D_N(x_1, ..., x_N, t), (x_i = (\vec{r}_i, \vec{p}_i))$, such that $D_N(x_1, ..., x_N, t) dx_1 ... dx_N$ is the probability of finding system in the state: particle i in $x_i \rightarrow x_i + d\vec{x}_i, i = 1, ..., N$, at time t. D_N is symmetric in the variables $(x_1, ..., x_N)$, normalized by the condition $\int D_N dx_1 ... dx_N = 1$, and is the solution of an initial value problem for the Liousville equation

$$\frac{\partial}{\partial t} D_N(x_1, ..., x_N, t) = \{ H_N, D_N(x_1, ..., x_N) \} \tag{2.2}$$

The Liousville equation describes the time evolution of D_N, which relates the theory to macroscopic phenomena, where

$$\{ H_N, D_N \} = \sum_{i=1}^{N} (\frac{\partial H_N}{\partial \vec{r}_i} \cdot \frac{\partial D_N}{\partial \vec{p}_i} - \frac{\partial H_N}{\partial \vec{p}_i} \cdot \frac{\partial D_N}{\partial \vec{r}_i}). \tag{2.3}$$

The solution of the Liousville equation is equivalent to the solution of an n-body problem, but where n is very large, the initial conditions are, in general, unknown, and therefore the solution of (2.2) is not practical.

In order to circumvent these difficulties, one introduces the N s-particle distribution functions [25]

$$F_s^N(x_1, ..., x_s, t) = V^s \int dx_{s+1} ... dx_N D_N(x_1, ..., x_N, t)$$

$$s = 0, 1, 2, ... \tag{2.4}$$

which satisfies $\int dx_1 ... dx_s F_s^N = V^s$, a normalization condition, because $\int d^N x D_N(x^N, t) = 1$. We note $F_0^N = 1$.

From (2.2), using the symmetry of D_N and taking the thermodynamic limit $N \to \infty, V \to \infty, N/V = n = constant$, one obtains the BBGKY hierarchy[25]

$$\frac{\partial}{\partial t} F_s(x^s, t) = \{H_s, F_s(x^s, t)\} + n \int dx_{s+1} \sum_{i=1}^{s} \frac{\partial}{\partial \vec{r}_i} \phi_{is+1} \cdot \frac{\partial}{\partial \vec{p}_i} F_{s+1}(x^{s+1}, t), \tag{2.5}$$

where $H_s = \sum_{i=1}^{s} \frac{\vec{p}_i}{2m} + \sum_{i<j}^{s} \phi_{ij}$, $\phi_{ij} = \phi(|\vec{r}_i - \vec{r}_j|)$ and $F_s = \lim_{N\to\infty} F_s^N$.

We will utilize the integral form of (2.5)[25]:

$$F_s(x^s, t + \tau) = \sum_{k=1}^{\infty} n^k \int dx_{s+1} ... dx_{s+k} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!(k-j)!} T_{-\tau}^{(j+s)} F_{k+s}(x^{k+s}, t), \tag{2.6}$$

where $T_{-\tau}^{(j+s)} F_{k+s} = F_{k+s}(S_{-\tau}^{(j+s)} x^{j+s}, S_{-\tau}^{(1)} x_{j+s+1}, ... S_{-\tau}^{(1)} x_{k+s}, t)$ and $S_{-\tau}^{(j)} = \exp \tau\{H_j, \}$, $S_t^{(n)}$ is the flow operator of the n-particle mechanical system, $\{x'_1, ..., x'_n\} = S_t^{(n)}\{x_1, ..., x_n\}$ means that if $\{x_1, ..., x_n\}$ is the state at $t = 0$, then $\{x'_1, ..., x'_n\}$ is the state at time t.

The BBGKY hierarchy connects the evolution of an s-particle distribution function, F_s, to the distribution function F_{s+1} of $s + 1$ particles. In order to solve F_1, distribution functions of all order and streaming operators $S_{-\tau}^{(j)}$ must be known. In general, they are difficult to determine. However many physical quantities of greatest interest can be expressed in terms of the first and second order distribution functions. In the following, we will see that a low/moderate density assumption can decouple the system and lead to the derivation of kinetic equations for the distribution function F_1 .

From (2.6), for $s = 1$, we get

$$
\begin{aligned}
&F_1(x_1, t+\tau) - T^{(1)}_{-\tau} F_1(x_1, t) \\
&= n \int dx_2 [T^{(2)}_{-\tau} F_2(x_1, x_2, t) - T^{(1)}_{-\tau} F_2(x_1, x_2, t)] \\
&+ n^2 \int dx_2 dx_3 [\frac{1}{2} T^{(3)}_{-\tau} F_3(x^3, t) - T^{(2)}_{-\tau} F_3(x^3, t) + \frac{1}{2} T^{(1)}_{-\tau} F_3(x^3, t)] + \ldots \quad .
\end{aligned}
\tag{2.7}
$$

We can make the replacement $x_1 \to S^{(1)}_\tau x_1 = (\vec{r}_1 + \tau \frac{\vec{p}_1}{m}, \vec{p}_1)$, let x_i be replaced by $S^{(1)}_\tau x_i$, while dx_i remain unchanged. Then (2.7) becomes

$$
\begin{aligned}
&F_1(S^{(1)}_\tau x_1, t+\tau) - F_1(x_1, t) \\
&= n \int dx_2 \{S^{(2)}_{-\tau} F_2(S^{(1)}_\tau x_1, S^{(1)}_\tau x_2, t) - F_2(x_1, x_2, t)\} \\
&+ \frac{1}{2} n^2 \int dx_2 dx_3 \{S^{(3)}_{-\tau} F_3(S^{(1)}_\tau x_1, S^{(1)}_\tau x_2, S^{(1)}_\tau x_3, t) \\
&\quad - 2F_3(S^{(2)}_{-\tau}[S^{(1)}_\tau x_1, S^{(1)}_\tau x_2], x_3, , t) + F_3(x^3, t)\} + \ldots \quad .
\end{aligned}
\tag{2.8}
$$

Also,

$$
\begin{aligned}
&F_1(S^{(1)}_\tau x_1, t+\tau) - F_1(x_1, t) \\
&= \int_0^\tau ds \frac{d}{ds} F_1(\vec{r}_1 + \frac{s}{m}\vec{p}_1, \vec{p}_1, t+s) \\
&= \frac{d}{dw} \int_0^\tau F_1(\vec{r}_1 + \frac{s+w}{m}\vec{p}_1, \vec{p}_1, t+w+s)|_{w=0} \\
&= \left[\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m} \cdot \nabla_1\right] \int_0^\tau ds F_1(S^{(1)}_s x_1, t+s) \\
&= \tau \left[\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m} \cdot \nabla_1\right] \tilde{F}_1(\vec{r}_1, \vec{p}_1, t),
\end{aligned}
\tag{2.9}
$$

where $\tilde{F}_1(\vec{r}_1, \vec{p}_1, t) = \frac{1}{\tau} \int_0^\tau ds F_1(S^{(1)}_s x_1, t+s)$.

We obtain from (2.8) thus

$$
\begin{aligned}
&[\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m} \cdot \nabla_1] \tilde{F}_1(\vec{r}_1, \vec{p}_1, t) = \\
&= \frac{n}{\tau} \int dx_2 \{S^{(2)}_{-\tau} F_2(S^{(1)}_\tau x_1, S^{(1)}_\tau x_2, t) - F_2(x^2, t)\}
\end{aligned}
$$

$$+\frac{1}{2}\frac{n^2}{\tau}\int dx_2 dx_3\{S_{-\tau}^{(3)}F_3(S_\tau^{(1)}x_1, S_\tau^{(1)}x_2, S_\tau^{(1)}x_3, t)$$
$$-2F_3(S_{-\tau}^{(2)}[S_\tau^{(1)}x_1, S_\tau^{(1)}x_2], x_3, t) + F_3(x^3, t)\} + \ldots \quad . \tag{2.10}$$

We can choose n or τ such that the ternary and higher-order terms will be very small, and $\tilde{F}_1(x_1, t) \approx F(x_1, t)$. We define the correlation function

$$g_2(x_1, x_2, t) = \frac{F_2(x_1, x_2, t)}{F_1(x_1, t)F_1(x_2, t)} \tag{2.11}$$

and introduce the notation $S_{-\tau}^{(2)}F_2(S_\tau^{(1)}x_1, S_\tau^{(1)}x_2, t) = F_2(x_1', x_2', t)$, where $(x_1', x_2') = S_{-\tau}^{(2)}(S_\tau^{(1)}x_1, S_\tau^{(1)}x_2)$ can be related to (x_1, x_2) from the knowledge of the potential.

Finally, after neglecting small corrections, we obtain

$$\tau[\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m}\cdot\nabla_{\vec{r}_1}]F_1(x_1, t)$$
$$= n\int dx_2[g_2(x_1', x_2', t)F_1(x_1', t)F_1(x_2', t) - g_2(x_1, x_2, t)F_1(x_1, t)F_1(x_2, t)]. \tag{2.12}$$

Looking at the above integral term, we see that for given $\vec{r}_1, \vec{p}_1, \tau$ and fixed $\vec{p}_2$, nonzero contributions for the above integral arise for those $\vec{r}_2$ such that $S_{-\tau}^{(2)}(S_\tau^{(1)}x_1, S_\tau^{(1)}x_2) \neq (x_1, x_2)$. Hence, we only need consider three cases: $S_\tau^{(1)}$ induces $(\vec{r}_1, \vec{r}_2)$ into, out of, or through the interaction region.

Consider, (see Figure 2.1) $\vec{R}_\tau = S_\tau^{(1)}\vec{r}_2 - S_\tau^{(1)}\vec{r}_1 = \vec{R}_0 + \frac{\tau}{m}\vec{p}_{21}$, where $\vec{P}_{21} = \vec{p}_2 - \vec{p}_1$, and (i) if $\vec{R}_0 \cdot \vec{P}_{21} \geq 0$, then $R_0 \leq R_\phi$, where R_ϕ is the range of the potential; and if (ii) $\vec{R}_0 \cdot \vec{P}_{21} < 0$, then $R_\phi \geq (R_0^2 - (\vec{R}_0 \cdot \frac{\vec{P}_{21}}{P_{21}})^2)^{1/2}$ and either $R_\tau \leq R_\phi$ or $\vec{R}_\tau \cdot \vec{P}_{21} > 0$ and $R_\tau > R_\phi$. After neglecting some small contributions, assuming $g_2(x_1, x_2) = 1$ and using

$$\int_B d\vec{r}_2 = \int dA \int_{-P_{21}\frac{\tau}{m}}^{0} dz, \tag{2.13}$$

we obtain the Boltzmann equation

$$[\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m}\cdot\nabla_{\vec{r}_1}]F(\vec{r}_1, \vec{p}_1, t) =$$

$$= n \int d\vec{p}_2 \int_0^{2\pi} d\theta \int_0^{R_\phi} dbb \frac{P_{21}}{m} \{F(\vec{r}_1, \vec{p}_1', t) F(\vec{r}_1, \vec{p}_2', t) - F(\vec{r}_1, \vec{p}_1, t) F(\vec{r}_1, \vec{p}_2, t)\}. \tag{2.14}$$

For the Enskog equation, the hard-core potential

$$\phi(r) = \begin{cases} \infty, & r < \sigma = R_\psi \\ 0, & r > \sigma \end{cases}$$

can be treated as the limit of the sequence $\phi_j(r) = (\frac{\sigma}{r})^j$, $j \rightarrow \infty$, hence, part A and C in Figure 2.1 have zero probability of occurrence. Nonzero collisional contributions arise from those $\vec{r}_2$ which, for fixed $\vec{r}_1, \vec{p}_1$ and $\vec{p}_2$, satisfy $S_\tau^{(2)} F_2(S_\tau^{(1)} x_1, S_\tau^{(1)} x_2, t) \neq F_2(x_1, x_2, t)$ in the limit $j \rightarrow \infty$, then $\tau \rightarrow 0$, (see Figure 2.2). As $j \rightarrow \infty, F_2 = 0$ on the set A, and $S_\tau^{(2)} F_2(S_\tau^{(1)} x_1, S_\tau^{(1)} x_2, t) = 0$ on the set B for $\tau \rightarrow 0$. So, for the hard sphere potential, in the limit $\tau \rightarrow 0^+$, the contributing regions become hemispherical skins near each interaction sphere surface, shown in Figure 2.3. The collision surface element is $\sigma^2 \frac{\tau}{m} |\vec{\sigma} \cdot \vec{P}_{21}| d\vec{\sigma} (\tau \rightarrow 0)$. We assume $F_2(x_1, x_2, t) = g_2(\vec{r}_1, \vec{r}_2, t) F_1(x_1, t) F_1(x_2, t)$.

From region A we get

$$n \int d\vec{p}_2 \int_{\vec{\sigma} \cdot \vec{P}_{21} > 0} d\vec{\sigma} \sigma^2 \frac{1}{m} |\vec{\sigma} \cdot \vec{P}_{21}| g_2(\vec{r}_1, \vec{r}_1 + \sigma\vec{\sigma}, t) F_1(\vec{r}_1, \vec{p}_1', t) F_1(\vec{r}_1 + \sigma\vec{\sigma}, \vec{p}_2', t).$$

From region B we get

$$-n \int d\vec{p}_2 \int_{\vec{\sigma}' \cdot \vec{P}_{21} < 0} d\vec{\sigma}' \sigma^2 \frac{1}{m} |\vec{\sigma}' \cdot \vec{P}_{21}| g_2(\vec{r}_1, \vec{r}_1 + \sigma\vec{\sigma}', t) F_1(\vec{r}_1, \vec{p}_1, t) F_1(\vec{r}_1 + \sigma\vec{\sigma}, \vec{p}_2', t).$$

Changing $\vec{\sigma}' \rightarrow -\vec{\sigma}$, we get

$$[\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m} \cdot \nabla_1] F_1(\vec{r}_1, \vec{p}_1, t)$$
$$= n \int d\vec{p}_2 \int_{\vec{\sigma} \cdot \vec{P}_{21} > 0} d\vec{\sigma} \sigma^2 \vec{\sigma} \cdot \frac{\vec{P}_{21}}{m} \{g_2(\vec{r}_1, \vec{r}_1 + \sigma\vec{\sigma}, t) F_1(\vec{r}_1, \vec{p}_1', t) F_1(\vec{r}_1 + \sigma\vec{\sigma}, \vec{p}_2', t)$$

$$- g_2(\vec{r}_1, \vec{r}_1 - \sigma\vec{\sigma}, t) F_1(\vec{r}_1, \vec{p}_1, t) F_1(\vec{r}_1 - \sigma\vec{\sigma}, \vec{p}_2 . t)\}. \tag{2.15}$$

Here, $\vec{p}_1' = \vec{p}_1 + \vec{\sigma}(\vec{\sigma} \cdot \vec{P}_{21})$ and $\vec{p}_2' = \vec{p}_2 - \vec{\sigma}(\vec{\sigma} \cdot \vec{P}_{21})$.

In the following, we consider the square well potential. We only need check four regions(see Figure 2.4):

Region A:

$$d\vec{r}_2 = R^2 a^2 \frac{\tau}{m} |\vec{P}_{21} \cdot \vec{\sigma}| d\vec{\sigma},$$

$$(x_1, x_2) = (\vec{r}_1, \vec{p}_1, \vec{r}_1 + Ra^- \vec{\sigma}, \vec{p}_2),$$

$$(x_1', x_2') = (\vec{r}_1, \vec{p}_1^A, \vec{r}_1 + Ra^- \vec{\sigma}, \vec{p}_2^A),$$

where $(\vec{p}_1^A, \vec{p}_2^A) = (\vec{p}_1 + \triangle\vec{p}^A, \vec{p}_1 - \triangle\vec{p}^A)$, with $\triangle\vec{p}^A = \frac{1}{2}\vec{\sigma}\{\vec{\sigma} \cdot \vec{P}_{21} - ((\vec{\sigma} \cdot \vec{P}_{21})^2 + 4\epsilon m)^{1/2}\}$.

Region B, C: same as for Enskog equation.

Region D:

for $\vec{P}_{21} \cdot \vec{\sigma}' < -(4\epsilon m)^{1/2}$,

$$d\vec{r}_2 = R^2 a^2 \frac{\tau}{m} |\vec{P}_{21} \cdot \vec{\sigma}'| d\vec{\sigma},$$

$$(x_1, x_2) = (\vec{r}_1, \vec{p}_1, \vec{r}_1 + Ra^+ \vec{\sigma}', \vec{p}_2),$$

$$(x_1', x_2') = (\vec{r}_1, \vec{p}_1^{e}, \vec{r}_1 + Ra^+ \vec{\sigma}, \vec{p}_2^{e}),$$

and for $-(4\epsilon m)^{1/2} < \vec{P}_{21} \cdot \vec{\sigma}' < 0$,

$$(x_1', x_2') = (\vec{r}_1, \vec{p}_1^{b}, \vec{r}_1 + Ra^+ \vec{\sigma}', \vec{p}_2^{b}),$$

where $(\vec{p}_1^{e}, \vec{p}_2^{e}) = (\vec{p}_1 + \triangle\vec{p}^{e}, \vec{p}_1 - \triangle\vec{p}^{e})$, with $\triangle\vec{p}^{e} = \frac{1}{2}\vec{\sigma}\{\vec{\sigma} \cdot \vec{P}_{21} - ((\vec{\sigma} \cdot \vec{P}_{21})^2 - 4\epsilon m)^{1/2}\}$. and $\vec{p}_1^{b} = \vec{p}_1 + \vec{\sigma}(\vec{\sigma} \cdot \vec{P}_{21})$, $\vec{p}_2^{b} = \vec{p}_2 - \vec{\sigma}(\vec{\sigma} \cdot \vec{P}_{21})$.

Changing $\vec{\sigma}' \rightarrow -\vec{\sigma}$, after neglecting some small corrections and rearranging terms, we get the square-well kinetic equation (1.2), which will be studied in this thesis.

We may consider the square-shoulder kinetic equation. The square shoulder potential is given by

$$\phi(x) = \begin{cases} \infty, & 0 < x < a \\ \epsilon, & a \leq x \leq Ra \\ 0, & x > Ra. \end{cases} \tag{2.16}$$

Similar to the case of the square well potential, we can derive the square-shoulder kinetic equation:

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \vec{v}_1 \nabla_{\vec{r}_1}\right] f(r_1, \vec{v}_1, t) = \\
&a^2 \int_{R^3} d\vec{v}_2 \int_{S^2} d\vec{\sigma}\vec{\sigma} \cdot \vec{p}\theta(\vec{\sigma} \cdot \vec{p})[g(\vec{r}_1, \vec{r}_1 + a^+\vec{\sigma}|n) f(\vec{r}_1, \vec{v}_1', t) f(r_1 + a\vec{\sigma}, \vec{v}_2', t) \\
&- g(\vec{r}_1, \vec{r}_1 - a^+\vec{\sigma}|n) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)] \\
&+ R^2 a^2 \int_{R^3} d\vec{v}_2 \int_{S^2} d\vec{\sigma}\vec{\sigma} \cdot \vec{p}\theta(\vec{\sigma} \cdot \vec{p})[g(\vec{r}_1, \vec{r}_1 - Ra^+\vec{\sigma}|n) f(\vec{r}_1, \vec{v}_1'', t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2'', t)] \\
&- g(\vec{r}_1, \vec{r}_1 + Ra^-\vec{\sigma}|n) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)] \\
&+ R^2 a^2 \int_{R^3} d\vec{v}_2 \int_{S^2} d\vec{\sigma}\vec{\sigma} \cdot \vec{p}\theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon}) \\
&\qquad \times [g(\vec{r}_1, \vec{r}_1 + Ra^-\vec{\sigma}|n) f(\vec{r}_1, \vec{v}_1''', t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2''', t) \\
&\qquad\quad - g(\vec{r}_1, \vec{r}_1 - Ra^+\vec{\sigma}|n) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)] \\
&+ R^2 a^2 \int_{R^3} d\vec{v}_2 \int_{S^2} d\vec{\sigma}\vec{\sigma} \cdot \vec{p}\theta(\vec{\sigma} \cdot \vec{p})\theta(\sqrt{4\epsilon} - \vec{\sigma} \cdot \vec{p}) \\
&\qquad \times [g(\vec{r}_1, \vec{r}_1 + Ra^-\vec{\sigma}|n) f(\vec{r}_1, \vec{v}_1', t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2', t) \\
&\qquad\quad - g(\vec{r}_1, \vec{r}_1 - Ra^+\vec{\sigma}|n) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)].
\end{aligned} \tag{2.17}$$

§III. Estimation of the collisional integral

(3.1). Some properties about the change of variables

In the following, we will study some properties about the changes of variables. which will be utilized in dealing with collisional integrals.

(*a*) $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1', \vec{v}_2')$

From (1.4), we get the Jacobian of the transform: $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1'. \vec{v}_2')$

$$\frac{\partial(\vec{v}_1', \vec{v}_2')}{\partial(\vec{v}_1, \vec{v}_2)} =$$

$$= \begin{bmatrix} 1-\sigma_1\sigma_1 & -\sigma_1\sigma_2 & -\sigma_1\sigma_3 & \sigma_1\sigma_1 & \sigma_1\sigma_2 & \sigma_1\sigma_3 \\ -\sigma_2\sigma_1 & 1-\sigma_2\sigma_2 & -\sigma_2\sigma_3 & \sigma_2\sigma_1 & \sigma_2\sigma_2 & \sigma_2\sigma_3 \\ -\sigma_3\sigma_1 & -\sigma_3\sigma_2 & 1-\sigma_3\sigma_3 & \sigma_3\sigma_1 & \sigma_3\sigma_2 & \sigma_3\sigma_3 \\ \sigma_1\sigma_1 & \sigma_1\sigma_2 & \sigma_1\sigma_3 & -\sigma_1\sigma_1 & -\sigma_1\sigma_2 & -\sigma_1\sigma_3 \\ \sigma_2\sigma_1 & \sigma_2\sigma_2 & \sigma_2\sigma_3 & -\sigma_2\sigma_1 & 1-\sigma_2\sigma_2 & -\sigma_2\sigma_3 \\ \sigma_3\sigma_1 & \sigma_3\sigma_2 & \sigma_3\sigma_3 & -\sigma_3\sigma_1 & -\sigma_3\sigma_2 & 1-\sigma_3\sigma_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \sigma_1\sigma_1 & \sigma_1\sigma_2 & \sigma_1\sigma_3 \\ 0 & 1 & 0 & \sigma_2\sigma_1 & \sigma_2\sigma_2 & \sigma_2\sigma_3 \\ 0 & 0 & 1 & \sigma_3\sigma_1 & \sigma_3\sigma_2 & \sigma_3\sigma_3 \\ 1 & 0 & 0 & 1-\sigma_1\sigma_1 & -\sigma_1\sigma_2 & -\sigma_1\sigma_3 \\ 0 & 1 & 0 & -\sigma_2\sigma_1 & 1-\sigma_2\sigma_2 & -\sigma_2\sigma_3 \\ 0 & 0 & 1 & -\sigma_3\sigma_1 & -\sigma_3\sigma_2 & 1-\sigma_3\sigma_3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2}-\sigma_1\sigma_1 & -\sigma_1\sigma_2 & -\sigma_1\sigma_3 \\ 0 & 0 & 0 & -\sigma_2\sigma_1 & \frac{1}{2}-\sigma_2\sigma_2 & -\sigma_2\sigma_3 \\ 0 & 0 & 0 & -\sigma_3\sigma_1 & -\sigma_3\sigma_2 & \frac{1}{2}-\sigma_3\sigma_3 \end{bmatrix}$$

$$= 8 \begin{bmatrix} \frac{1}{2}-\sigma_1\sigma_1 & -\sigma_1\sigma_2 & -\sigma_1\sigma_3 \\ -\sigma_2\sigma_1 & \frac{1}{2}-\sigma_2\sigma_2 & -\sigma_2\sigma_3 \\ -\sigma_3\sigma_1 & -\sigma_3\sigma_2 & \frac{1}{2}-\sigma_3\sigma_3 \end{bmatrix} = 8(-\frac{1}{8}) = -1. \tag{3.1}$$

Hence, $d\vec{v}_1 d\vec{v}_2 = d\vec{v}_1' d\vec{v}_2'$. For the transform $(\vec{v}_1, \vec{v}_1) \to (\vec{v}_1', \vec{v}_2')$, we also have

$$\vec{v}_1' = \vec{v}_1 + \vec{\sigma}\vec{\sigma} \cdot (\vec{v}_2 - \vec{v}_1)$$

$$\to \vec{v}_1' + \vec{\sigma}\vec{\sigma} \cdot (\vec{v}_2' - \vec{v}_1')$$

$$
\begin{aligned}
&= \vec{v}_1 + \vec{\sigma}\vec{\sigma}\cdot(\vec{v}_2-\vec{v}_1) + \vec{\sigma}\vec{\sigma}\cdot[\vec{v}_2-\vec{v}_1-2\vec{\sigma}\vec{\sigma}\cdot(\vec{v}_2-\vec{v}_1)] \\
&= \vec{v}_1 + \vec{\sigma}\vec{\sigma}\cdot(\vec{v}_2-\vec{v}_1) + \vec{\sigma}\vec{\sigma}\cdot(\vec{v}_2-\vec{v}_1) - 2\vec{\sigma}\vec{\sigma}\cdot(\vec{v}_2-\vec{v}_1) = \vec{v}_1. \qquad (3.2)
\end{aligned}
$$

Similarly, $\vec{v}_2' \to \vec{v}_2$.

(*b*) $(\vec{v}_1,\vec{v}_2) \to (\vec{v}_1'',\vec{v}_2'')$

We have

$$
\begin{aligned}
d\vec{v}_1'' &= d\vec{v}_1 + \frac{1}{2}\vec{\sigma}\{\vec{\sigma}\cdot d\vec{p} - \frac{1}{2}[(\vec{\sigma}\cdot\vec{p})^2+4\epsilon]^{-\frac{1}{2}}2(\vec{\sigma}\cdot\vec{p})\vec{\sigma}\cdot d\vec{p}\}, \\
d\vec{v}_2'' &= d\vec{v}_2 - \frac{1}{2}\vec{\sigma}\{\vec{\sigma}\cdot d\vec{p} - \frac{1}{2}[(\vec{\sigma}\cdot\vec{p})^2+4\epsilon]^{-\frac{1}{2}}2(\vec{\sigma}\cdot\vec{p})\vec{\sigma}\cdot d\vec{p}\}. \qquad (3.3)
\end{aligned}
$$

Letting $a_1 = 1-[(\vec{\sigma}\cdot\vec{p})^2+4\epsilon]^{-\frac{1}{2}}(\vec{\sigma}\cdot\vec{p})$, we get the Jacobian of the transform $(\vec{v}_1,\vec{v}_2) \to (\vec{v}_1'',\vec{v}_2'')$:

$$
\frac{\partial(\vec{v}_1'',\vec{v}_2'')}{\partial(\vec{v}_1,\vec{v}_2)} =
$$

$$
= \begin{bmatrix}
1-\frac{1}{2}a_1\sigma_1\sigma_1 & -\frac{1}{2}a_1\sigma_1\sigma_2 & -\frac{1}{2}a_1\sigma_1\sigma_3 & \frac{1}{2}a_1\sigma_1\sigma_1 & \frac{1}{2}a_1\sigma_1\sigma_2 & \frac{1}{2}a_1\sigma_1\sigma_3 \\
-\frac{1}{2}a_1\sigma_2\sigma_1 & 1-\frac{1}{2}a_1\sigma_2\sigma_2 & -\frac{1}{2}a_1\sigma_2\sigma_3 & \frac{1}{2}a_1\sigma_2\sigma_1 & \frac{1}{2}a_1\sigma_2\sigma_2 & \frac{1}{2}a_1\sigma_2\sigma_3 \\
-\frac{1}{2}a_1\sigma_3\sigma_1 & -\frac{1}{2}a_1\sigma_3\sigma_2 & 1-\frac{1}{2}a_1\sigma_3\sigma_3 & \frac{1}{2}a_1\sigma_3\sigma_1 & \frac{1}{2}a_1\sigma_3\sigma_2 & \frac{1}{2}a_1\sigma_3\sigma_3 \\
\frac{1}{2}a_1\sigma_1\sigma_1 & \frac{1}{2}a_1\sigma_1\sigma_2 & \frac{1}{2}a_1\sigma_1\sigma_3 & 1-\frac{1}{2}a_1\sigma_1\sigma_1 & -\frac{1}{2}a_1\sigma_1\sigma_2 & -\frac{1}{2}a_1\sigma_1\sigma_3 \\
\frac{1}{2}a_1\sigma_2\sigma_1 & \frac{1}{2}a_1\sigma_2\sigma_2 & \frac{1}{2}a_1\sigma_2\sigma_3 & -\frac{1}{2}a_1\sigma_2\sigma_1 & 1-\frac{1}{2}a_1\sigma_2\sigma_2 & -\frac{1}{2}a_1\sigma_2\sigma_3 \\
\frac{1}{2}a_1\sigma_3\sigma_1 & \frac{1}{2}a_1\sigma_3\sigma_2 & \frac{1}{2}a_1\sigma_3\sigma_3 & -\frac{1}{2}a_1\sigma_3\sigma_1 & -\frac{1}{2}a_1\sigma_3\sigma_2 & 1-\frac{1}{2}a_1\sigma_3\sigma_3
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & \frac{1}{2}a_1\sigma_1\sigma_1 & \frac{1}{2}a_1\sigma_1\sigma_2 & \frac{1}{2}a_1\sigma_1\sigma_3 \\
0 & 1 & 0 & \frac{1}{2}a_1\sigma_2\sigma_1 & \frac{1}{2}a_1\sigma_2\sigma_2 & \frac{1}{2}a_1\sigma_2\sigma_3 \\
0 & 0 & 1 & \frac{1}{2}a_1\sigma_3\sigma_1 & \frac{1}{2}a_1\sigma_3\sigma_2 & \frac{1}{2}a_1\sigma_3\sigma_3 \\
1 & 0 & 0 & 1-\frac{1}{2}a_1\sigma_1\sigma_1 & -\frac{1}{2}a_1\sigma_1\sigma_2 & -\frac{1}{2}a_1\sigma_1\sigma_3 \\
0 & 1 & 0 & -\frac{1}{2}a_1\sigma_2\sigma_1 & 1-\frac{1}{2}a_1\sigma_2\sigma_2 & -\frac{1}{2}a_1\sigma_2\sigma_3 \\
0 & 0 & 1 & -\frac{1}{2}a_1\sigma_3\sigma_1 & -\frac{1}{2}a_1\sigma_3\sigma_2 & -\frac{1}{2}a_1\sigma_3\sigma_3
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & \frac{1}{2}a_1\sigma_1\sigma_1 & \frac{1}{2}a_1\sigma_1\sigma_2 & \frac{1}{2}a_1\sigma_1\sigma_3 \\
0 & 1 & 0 & \frac{1}{2}a_1\sigma_2\sigma_1 & \frac{1}{2}a_1\sigma_2\sigma_2 & \frac{1}{2}a_1\sigma_2\sigma_3 \\
0 & 0 & 1 & \frac{1}{2}a_1\sigma_3\sigma_1 & \frac{1}{2}a_1\sigma_3\sigma_2 & \frac{1}{2}a_1\sigma_3\sigma_3 \\
0 & 0 & 0 & 1-a_1\sigma_1\sigma_1 & -a_1\sigma_1\sigma_2 & -a_1\sigma_1\sigma_3 \\
0 & 0 & 0 & -a_1\sigma_2\sigma_1 & 1-a_1\sigma_2\sigma_2 & -a_1\sigma_2\sigma_3 \\
0 & 0 & 0 & -a_1\sigma_3\sigma_1 & -a_1\sigma_3\sigma_2 & -a_1\sigma_3\sigma_3
\end{bmatrix}
$$

$$= 1 - a_1 = 1 - [1 - (\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon]^{-\frac{1}{2}}(\vec{\sigma} \cdot \vec{p}) = [(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon]^{-\frac{1}{2}}(\vec{\sigma} \cdot \vec{g}). \tag{3.4}$$

Therefore,

$$(\vec{\sigma} \cdot \vec{p}) d\vec{v}_1 d\vec{v}_2 = (\vec{\sigma} \cdot \vec{p}'') dv_1'' dv_2'', \tag{3.5}$$

because

$$\begin{aligned} \vec{p}'' &= \vec{v}_2'' - \vec{v}_1'' = \vec{p} - \vec{\sigma}\{(\vec{\sigma} \cdot \vec{p}) - [(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon]^{\frac{1}{2}}\} \\ \textit{and} \quad \vec{\sigma} \cdot \vec{p}'' &= \vec{\sigma} \cdot \vec{p} - (\vec{\sigma} \cdot \vec{p}) + [(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon]^{\frac{1}{2}} = [(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon]^{\frac{1}{2}}. \end{aligned} \tag{3.6}$$

Similar to part (a), we have for $\vec{\sigma} \cdot \vec{p} \geq 0$

$$\begin{aligned} \vec{v}_1''' &= \vec{v}_1 + \frac{1}{2}\vec{\sigma}\{(\vec{\sigma} \cdot \vec{p}) - [(\vec{\sigma} \cdot \vec{p})^2 - 4\epsilon]^{\frac{1}{2}}\} \\ &\rightarrow \vec{v}_1'' + \frac{1}{2}\vec{\sigma}\{(\vec{\sigma} \cdot \vec{p}'') - [(\vec{\sigma} \cdot \vec{p}'')^2 - 4\epsilon]^{\frac{1}{2}}\} \\ &= \vec{v}_1 + \frac{1}{2}\vec{\sigma}\{(\vec{\sigma} \cdot \vec{p}) - [(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon)^{\frac{1}{2}}\} + \frac{1}{2}\vec{\sigma}\{[(a \cdot \vec{p})^2 + 4\epsilon]^{\frac{1}{2}} - [(a \cdot \vec{p})^2 + 4\epsilon - 4\epsilon]^{\frac{1}{2}}\} \\ &= \vec{v}_1 + \frac{1}{2}\vec{\sigma}(\vec{\sigma} \cdot \vec{p}) - \frac{1}{2}\vec{\sigma}[(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon]^{\frac{1}{2}} + \frac{1}{2}\vec{\sigma}[(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon]^{\frac{1}{2}} - \frac{1}{2}\vec{\sigma}(\vec{\sigma} \cdot \vec{p}) = \vec{v}_1. \end{aligned} \tag{3.7}$$

$$\begin{aligned} \vec{v}_2''' &= \vec{v}_2 - \frac{1}{2}\vec{\sigma}\{(\vec{\sigma} \cdot \vec{p}) - [(\vec{\sigma} \cdot \vec{p})^2 - 4\epsilon)^{\frac{1}{2}}\} \\ &\rightarrow \vec{v}_2'' - \frac{1}{2}\vec{\sigma}\{(\vec{\sigma} \cdot \vec{g}'') - [(\vec{\sigma} \cdot \vec{p}'')^2 - 4\epsilon]^{\frac{1}{2}}\} \\ &= \vec{v}_2 - \frac{1}{2}\vec{\sigma}\{(\vec{\sigma} \cdot \vec{p}) - [(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon)^{\frac{1}{2}}\} - \frac{1}{2}\vec{\sigma}\{[(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon]^{\frac{1}{2}} - [(\vec{\sigma} \cdot \vec{p})^2 + 4\epsilon - 4\epsilon]^{\frac{1}{2}}\} \\ &= \vec{v}_2. \end{aligned} \tag{3.8}$$

(c) $(\vec{v}_1, \vec{v}_2) \rightarrow (\vec{v}_1''', \vec{v}_2''')$

We have

$$d\vec{v}_1''' = d\vec{v}_1 + \frac{1}{2}\vec{\sigma}\{\vec{\sigma} \cdot d\vec{p} - \frac{1}{2}[(\vec{\sigma} \cdot \vec{p})^2 - 4\epsilon]^{-\frac{1}{2}} 2(\vec{\sigma} \cdot \vec{p})\vec{\sigma} \cdot d\vec{p}\},$$

$$d\vec{v}_2''' = d\vec{v}_2 - \frac{1}{2}\vec{\sigma}\{\vec{\sigma}\cdot d\vec{p} - \frac{1}{2}[(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{-\frac{1}{2}}2(\vec{\sigma}\cdot\vec{p})\vec{\sigma}\cdot d\vec{p}\}. \tag{3.9}$$

Letting $a_2 = 1-[(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{-\frac{1}{2}}(\vec{\sigma}\cdot\vec{p})$, we get the Jacobian of the transform $(\vec{v}_1, \vec{v}_2) \rightarrow (\vec{v}_1''', \vec{v}_2''')$:

$$\frac{\partial(\vec{v}_1''', \vec{v}_2''')}{\partial(\vec{v}_1, \vec{v}_2)} = 1 - a_2 = 1 - \left[1 - [(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{-\frac{1}{2}}(\vec{\sigma}\cdot\vec{g})\right] = \left[(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon\right]^{-\frac{1}{2}}(\vec{\sigma}\cdot\vec{p}). \tag{3.10}$$

Hence,

$$(\vec{\sigma}\cdot\vec{p})d\vec{v}_1 d\vec{v}_2 = (\vec{\sigma}\cdot\vec{p}''')d\vec{v}_1''' dv_2''', \tag{3.11}$$

because

$$\vec{\sigma}\cdot\vec{p}''' = [(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{\frac{1}{2}}, \tag{3.12}$$

where $\vec{p}''' = \vec{v}_2''' - \vec{v}_1'''$. Similarly, we have, when $(\vec{v}_1, \vec{v}_2) \rightarrow (\vec{v}_1''', \vec{v}_2''')$, $\vec{v}_1''' \rightarrow \vec{v}_1$ and $\vec{v}_2''' \rightarrow \vec{v}_2$.

(3.2). Symmetries of the collisional integral

Let φ be a measurable function on $R^3 \times R^3 \times R^1_+$ and $0 \le f \in C_0(R^3 \times R^3 \times R^1_+)$. Using symmetry arguments and appropriate change of variables, we have

$$\begin{aligned}
&\iint\limits_{R^3\times R^3} \varphi(\vec{r}_1, \vec{v}_1, t)C_E(f, f)d\vec{r}_1 d\vec{v}_1 = \\
&= -a^2\int d\vec{r}_1 d\vec{v}_1 \int d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n(f)) \\
&\qquad \times f(\vec{r}_1, \vec{v}_1, t)f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)\varphi(\vec{r}_1, \vec{v}_1, t) \\
&\quad + a^2\int d\vec{r}_1 d\vec{v}_1 \int d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}|n(f)) \\
&\qquad \times f(\vec{r}_1, \vec{v}_1', t)f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2', t)\varphi(\vec{r}_1, \vec{v}_1', t).
\end{aligned} \tag{3.13}$$

The second term becomes

$$a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n(f))f(\vec{r}_1, \vec{v}_1, t)f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)\varphi(\vec{r}_1, \vec{v}_1', t)$$

by the changes of variable $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1^{\,\prime}, \vec{v}_2^{\,\prime})$ with $d\vec{v}_1 d\vec{v}_2 = d\vec{v}_1^{\,\prime} d\vec{v}_2^{\,\prime}$ and $\vec{\sigma}\cdot\vec{p} = -\vec{\sigma}\cdot\vec{p}^{\,\prime}$. So, changing $\vec{\sigma} \to -\vec{\sigma}$, we get

$$\iint\limits_{R^3 \times R^3} \varphi(\vec{r}_1, \vec{v}_1, t) C_E(f, f) d\vec{r}_1 d\vec{v}_1 =$$

$$= -a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma}\cdot\vec{p}) \theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma} | n(f))$$

$$\times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)[\varphi(\vec{r}_1, \vec{v}_1, t) - \varphi(\vec{r}_1, \vec{v}_1^{\,\prime}, t)]. \tag{3.14}$$

Under the changes $\vec{v}_1 \to \vec{v}_2, \vec{v}_2 \to \vec{v}_1, \vec{\sigma} \to -\vec{\sigma}$, we get

$$\iint\limits_{R^3 \times R^3} \varphi(\vec{r}_1, \vec{v}_1, t) C_E(f, f) d\vec{r}_1 d\vec{v}_1 =$$

$$= -a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma}\cdot\vec{p}) \theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 + a\vec{\sigma} | n(f))$$

$$\times f(\vec{r}_1, \vec{v}_2, t) f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_1, t)[\varphi(\vec{r}_1, \vec{v}_2, t) - \varphi(\vec{r}_1, \vec{v}_2^{\,\prime}, t)]. \tag{3.15}$$

By the transform $\vec{r}_1 \to \vec{r}_1 - a\vec{\sigma}$, the above term becomes

$$\iint\limits_{R^3 \times R^3} \varphi(\vec{r}_1, \vec{v}_1, t) C_E(f, f) d\vec{r}_1 d\vec{v}_1 =$$

$$= -a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma}\cdot\vec{p}) \theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1 - a\vec{\sigma}, \vec{r}_1 | n(f))$$

$$\times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)[\varphi(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t) - \varphi(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2^{\,\prime}, t)]. \tag{3.16}$$

Finally, combining (3.14),(3.16), and using the symmetry of g, we obtain

$$\iint\limits_{R^3 \times R^3} \varphi(\vec{r}_1, \vec{v}_1, t) C_E(f, f) d\vec{r}_1 d\vec{v}_1 =$$

$$= -\frac{1}{2} a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma}\cdot\vec{p}) \theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma} | n(f)) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)$$

$$\times [\varphi(\vec{r}_1, \vec{v}_1, t) - \varphi(\vec{r}_1, \vec{v}_1^{\,\prime}, t) + \varphi(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t) - \varphi(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2^{\,\prime}, t)].$$

(3.17)

In the same way,

$$\iint\limits_{R^3\times R^3} \varphi(\vec{r}_1,\vec{v}_1,t)C_4(f,f)d\vec{r}_1, d\vec{v}_1 =$$

$$= -\frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon}-\vec{\sigma}\cdot\vec{p})g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n)f(\vec{r}_1,\vec{v}_1,t)$$

$$\times f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2,t)[\varphi(\vec{r}_1,\vec{v}_1,t)+\varphi(\vec{r}_1+Ra\vec{\sigma},v_2,t)-\varphi(\vec{r}_1,\vec{v}_1',t)-\varphi(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2',t)]. \tag{3.18}$$

Next, we consider the terms $C_2(f,f)$ and $C_3(f,f)$.

$$\iint\limits_{R^3\times R^3} \varphi(\vec{r}_1,\vec{v}_1,t)C_2(f,f)d\vec{r}_1, d\vec{v}_1 =$$

$$-R^2a^2\int d\vec{r}_1 d\vec{v}_1 \int d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1,\vec{r}_1-Ra\vec{\sigma}|n(f))$$

$$\times f(\vec{r}_1,v_1,t)f(r_1-Ra\vec{\sigma},v_2,t)\varphi(\vec{r}_1,\vec{v}_1,t)$$

$$+R^2a^2\int d\vec{r}_1 dv_1 dv_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n(f))$$

$$\times f(\vec{r}_1,\vec{v}_1'',t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2'',t)\varphi(\vec{r}_1,\vec{v}_1,t). \tag{3.19}$$

By the changes of variables $(\vec{v}_1,\vec{v}_2)\to(\vec{v}_1'',\vec{v}_2'')$ with $\vec{\sigma}\cdot\vec{p}''d\vec{v}_1''d\vec{v}_2''=\vec{\sigma}\cdot\vec{p}d\vec{v}_1d\vec{v}_2$, we have, noting that $\theta(\vec{\sigma}\cdot\vec{p})\to\theta(\vec{\sigma}\cdot\vec{p}-\sqrt{4\epsilon})$,

$$R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n(f))$$

$$\times f(\vec{r}_1,\vec{v}_1'',t)f(\vec{r}_1+Ra\vec{v}_2'',t)\varphi(\vec{r}_1,\vec{v}_1,t)$$

$$=R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}-\sqrt{4\epsilon})g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n(f))$$

$$\times f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2,t)\varphi(\vec{r}_1,\vec{v}_1''',t). \tag{3.20}$$

For the term $C_3(f,f)$,

$$\iint\limits_{R^3\times R^3} \varphi(\vec{r}_1,\vec{v}_1,t)C_3(f,f)d\vec{r}_1 d\vec{v}_1$$

$$= -R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon})g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(f))f(\vec{r}_1, \vec{v}_1, t)$$
$$\times f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)\varphi(\vec{r}_1, \vec{v}_1, t)$$
$$+ R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon})g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(f))$$
$$\times f(\vec{r}_1, \vec{v}_1''', t)f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2''', t)\varphi(\vec{r}_1, \vec{v}_1, t). \tag{3.21}$$

By changing variables $(\vec{v}_1, \vec{v}_2) \rightarrow (\vec{v}_1''', \vec{v}_2''')$ with $\vec{\sigma} \cdot \vec{p}'' d\vec{v}_1''' d\vec{v}_2''' = (\vec{\sigma} \cdot \vec{p}) d\vec{v}_1 d\vec{v}_2$, noting that $\theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon}) \rightarrow \theta(\vec{\sigma} \cdot \vec{p})$, we get

$$R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon})g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(f))$$
$$\times f(\vec{r}_1, \vec{v}_1''', t)f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2''', t)\varphi(\vec{r}_1, \vec{v}_1, t)$$
$$= R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p})g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(f))$$
$$\times f(\vec{r}_1, \vec{v}_1, t)f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)\varphi(\vec{r}_1, \vec{v}_2'', t). \tag{3.22}$$

So,

$$\iint\limits_{R^3 \times S^2} (C_2 + C_3)\varphi(\vec{r}_1, \vec{v}_1, t) d\vec{r}_1 d\vec{v}_1$$
$$= -R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p})g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(f))$$
$$\times f(\vec{r}_1, v_1, t)f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)[\varphi(\vec{r}_1, \vec{v}_1, t) - \varphi(\vec{r}_1, \vec{v}_1'', t)]$$
$$-R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon})g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(f))$$
$$\times f(\vec{r}_1, \vec{v}_1, t)f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)[\varphi(\vec{r}_1, \vec{v}_1, t) - \varphi(\vec{r}_1, \vec{v}_1''', t)]. \tag{3.23}$$

By the change of variables $\vec{v}_1 \rightarrow \vec{v}_2, \vec{v}_2 \rightarrow \vec{v}_1$ and $\vec{\sigma} \rightarrow -\vec{\sigma}$, noting $\vec{v}_1'' \rightarrow \vec{v}_2'', \vec{v}_1''' \rightarrow \vec{v}_2'''$, we obtain

$$\iint\limits_{R^3 \times S^2} (C_2 + C_3)\varphi d\vec{r}_1 d\vec{v}_1 =$$
$$= -R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p})g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(f))$$

$$\times f(\vec r_1, \vec v_2, t) f(\vec r_1 + Ra\vec\sigma, \vec v_1, t)[\varphi(\vec r_1, \vec v_2, t) - \varphi(\vec r_1, \vec v_2'', t)]$$

$$-R^2a^2 \int d\vec r_1 d\vec v_1 d\vec v_2 \int d\vec\sigma(\vec\sigma\cdot\vec p)\theta(\vec\sigma\cdot\vec p - \sqrt{4\epsilon})g(\vec r_1, \vec r_1 - Ra\vec\sigma|n(f))$$

$$f(\vec r_1, \vec v_2, t) f(\vec r_1 - Ra\vec\sigma, \vec v_1, t)[\varphi(\vec r_1, \vec v_2, t) - \varphi(\vec r_1, \vec v_2''', t)]. \qquad (3.24)$$

By the change of variables $\vec r_1 \to \vec r_1 - Ra\vec\sigma$, $\vec r_1 \to \vec r_1 + Ra\vec\sigma$, respectively, in the above two terms,

$$\iint_{R^3\times S^2} (C_2 + C_3)\varphi(\vec r_1, \vec v_1, t) d\vec r_1 d\vec v_1 =$$

$$- R^2a^2 \int d\vec r_1 d\vec v_1 d\vec v_2 \int d\vec\sigma(\vec\sigma\cdot\vec p)\theta(\vec\sigma\cdot\vec p)g(\vec r_1, \vec r_1 - Ra\vec\sigma|n(f))$$
$$\times f(\vec r_1 - Ra\vec\sigma, \vec v_2, t) f(\vec r_1, \vec v_1, t)[\varphi(\vec r_1 - Ra\vec\sigma, \vec v_2, t) - \varphi(\vec r_1 - Ra\vec\sigma, \vec v_2'', t)]$$
$$- R^2a^2 \int d\vec r_1 d\vec v_1 d\vec v_2 \int d\vec\sigma(\vec\sigma\cdot\vec p)\theta(\vec\sigma\cdot\vec p - \sqrt{4\epsilon})g(\vec r_1, \vec r_1 + Ra\vec\sigma|n(f))$$
$$\times f(\vec r_1, \vec v_1, t) f(\vec r_1 + Ra\vec\sigma, \vec v_2, t)[\varphi(\vec r_1 + Ra\vec\sigma, \vec v_2, t) - \varphi(\vec r_1 + Ra\vec\sigma, \vec v_2''', t)].$$

$$(3.25)$$

Finally, combining (3.23)and (3.25), we obtain,

$$\iint_{R^3\times S^2} \varphi(\vec r_1, \vec v_1, t)(C_2 + C_3) d\vec r_1 d\vec v_1 =$$

$$- \frac{1}{2}R^2a^2 \int d\vec r_1 d\vec v_1 d\vec v_2 \int d\vec\sigma(\vec\sigma\cdot\vec p)\theta(\vec\sigma\cdot\vec p)g(\vec r_1, \vec r_1 - Ra\vec\sigma|n(f)) f(\vec r_1, \vec v_1, t) f(\vec r_1 - Ra\vec\sigma, \vec v_2, t)$$
$$\times [\varphi(\vec r_1, \vec v_1, t) + \varphi(\vec r_1 - Ra\vec\sigma, \vec v_2, t) - \varphi(\vec r_1, \vec v_1'', t) - \varphi(\vec r_1 - Ra\vec\sigma, \vec v_2'', t)]$$
$$- \frac{1}{2}R^2a^2 \int d\vec r_1 d\vec v_1 d\vec v_2 \int d\vec\sigma(\vec\sigma\cdot\vec p)\theta(\vec\sigma\cdot\vec p - \sqrt{4\epsilon})g(\vec r_1, \vec r_1 + Ra\vec\sigma|n(f)) f(\vec r_1, \vec v_1, t)$$
$$\times f(\vec r_1 + Ra\vec\sigma, \vec v_2, t)[\varphi(\vec r_1, \vec v_1, t) + \varphi(\vec r_1 + Ra\vec\sigma, \vec v_2, t) - \varphi(\vec r_1, \vec v_1''', t) - \varphi(\vec r_1 + Ra\vec\sigma, \vec v_2''', t)].$$

$$(3.26)$$

Combining (3.17), (3.18) and (3.26), we have

$$\iint\limits_{R^3 \times R^3} \varphi(\vec{r}_1, \vec{v}_1, t) E(f, f) d\vec{r}_1 d\vec{v}_1 =$$

$$
\begin{aligned}
&-\frac{1}{2}a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n(f)) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t) \\
&\times [\varphi(\vec{r}_1, \vec{v}_1, t) - \varphi(\vec{r}_1, \vec{v}_1', t) + \varphi(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t) - \varphi(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2', t)] \\
&-\frac{1}{2}R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon} - \vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n) f(\vec{r}_1, \vec{v}_1, t) \\
&\times f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)[\varphi(\vec{r}_1, \vec{v}_1, t) + \varphi(\vec{r}_1 + Ra\vec{\sigma}, v_2, t) - \varphi(\vec{r}_1, \vec{v}_1', t) - \varphi(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2', t)] \\
&-\frac{1}{2}R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(f)) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) \\
&\times [\varphi(\vec{r}_1, \vec{v}_1, t) + \varphi(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) - \varphi(\vec{r}_1, \vec{v}_1'', t) - \varphi(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2'', t)] \\
&-\frac{1}{2}R^2a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p} - \sqrt{4\epsilon}) g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(f)) f(\vec{r}_1, \vec{v}_1, t) \\
&\times f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)[\varphi(\vec{r}_1, \vec{v}_1, t) + \varphi(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t) - \varphi(\vec{r}_1, \vec{v}_1''', t) - \varphi(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2''', t)].
\end{aligned}
\tag{3.27}
$$

Equation (3.27) plays an important role in our proof of existence.

(3.3). H-Theorem

For the remainder of section III, we shall consider f a nonnegative solution of the kinetic equation (1.2), which may be written

$$\frac{d}{dt}f = \frac{\partial}{\partial t}f + \vec{v}_1 \cdot \nabla_1 f = E(f, f). \tag{3.28}$$

Formally, taking the derivative of $f \log f$, we have

$$
\begin{aligned}
\frac{d}{dt}(f log f) &= \log f \frac{d}{dt}f + \frac{d}{dt}f = (1 + \log f)\frac{d}{dt}f \\
&= E(f, f)(1 + \log f).
\end{aligned}
\tag{3.29}
$$

In (3.27), letting $\varphi = 1$, we can easily get

$$\iint\limits_{R^3 \times S^2} E(f, f) d\vec{r}_1 d\vec{v}_1 = 0. \tag{3.30}$$

Now, letting $\varphi = \log f$, consider the integral

$$\iint\limits_{R^3 \times S^2} E(f, f) \log f d\vec{r}_1 d\vec{v}_1. \tag{3.31}$$

First, we check the integral

$$\iint\limits_{R^3 \times S^2} (C_2 + C_3 + C_4) \log f d\vec{r}_1 d\vec{v}_1. \tag{3.32}$$

We get, from (3.18) and (3.26), setting $\varphi = \log f$,

$$\begin{aligned}
&\iint\limits_{R^3 \times R^3} (C_2 + C_3 + C_4) \log f d\vec{r}_1 d\vec{v}_1 = \\
&- \frac{1}{2} R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) \theta(\sqrt{4\epsilon} - (\vec{\sigma} \cdot \vec{p})) g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma} | n(f)) \\
&\quad \times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t) \log \frac{f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)}{f(\vec{r}_1, \vec{v}_1', t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2', t)} \\
&- \frac{1}{2} R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}) \\
&\quad \times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) \log \frac{f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)}{f(\vec{r}_1, \vec{v}_1'', t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2'', t)} \\
&- \frac{1}{2} R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta((\vec{\sigma} \cdot \vec{p}) - \sqrt{4\epsilon}) g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}) \\
&\quad \times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t) \log \frac{f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)}{f(\vec{r}_1, \vec{v}_1''', t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2''', t)}.
\end{aligned} \tag{3.33}$$

Applying the inequality $-x \log \frac{x}{y} \leq y - x$ to the integrands of the above integrals, we obtain the estimate

$$\iint\limits_{R^3 \times R^3} (C_2 + C_3 + C_4) \log f d\vec{r}_1 d\vec{v}_1 \leq$$

$$
\begin{aligned}
&\leq -\frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon}-(\vec{\sigma}\cdot\vec{p}))g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma})\\
&\qquad\times[f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2,t)-f(\vec{r}_1,\vec{v}_1',t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2',t)]\\
&-\frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1,\vec{r}_1-Ra\vec{\sigma}|n(f))\\
&\qquad\times[f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1-Ra\vec{\sigma},\vec{v}_2,t)-f(\vec{r}_1,v_1'',t)f(\vec{r}_1-Ra\vec{\sigma},\vec{v}_2'',t)]\\
&-\frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta((\vec{\sigma}\cdot\vec{p})-\sqrt{4\epsilon})g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma})\\
&\qquad\times[f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2,t)-f(\vec{r}_1,\vec{v}_1''',t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2''',t)].
\end{aligned}
\tag{3.34}
$$

We note, by the change of variables $(\vec{v}_1,\vec{v}_2)\to(\vec{v}_1',\vec{v}_2')$, $\vec{\sigma}\to-\vec{\sigma}$,

$$
\begin{aligned}
&\frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon}-(\vec{\sigma}\cdot\vec{p}))\\
&\quad\times g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma})f(\vec{r}_1,\vec{v}_1',t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2',t)\\
&=\frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon}-(\vec{\sigma}\cdot\vec{p}))\\
&\quad\times g(\vec{r}_1,\vec{r}_1-Ra\vec{\sigma}|n(f))f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1-Ra\vec{\sigma},\vec{v}_2,t).
\end{aligned}
\tag{3.35}
$$

In a similar way, by the change of variables, $(\vec{v}_1,\vec{v}_2)\to(\vec{v}_1'',\vec{v}_2'')$, we have

$$
\begin{aligned}
&\frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\\
&\quad\times g(\vec{r}_1,\vec{r}_1-Ra\vec{\sigma}|n(f))f(\vec{r}_1,\vec{v}_1''.t)f(\vec{r}_1-Ra\vec{\sigma},\vec{v}_2'',t)\\
&=\frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta((\vec{\sigma}\cdot\vec{p})-\sqrt{4\epsilon})\\
&\quad\times g(\vec{r}_1,\vec{r}_1-Ra\vec{\sigma}|n(f))f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1-Ra\vec{\sigma},\vec{v}_2,t),
\end{aligned}
\tag{3.36}
$$

and by the change of variables, $(\vec{v}_1,\vec{v}_2)\to(\vec{v}_1''',\vec{v}_2''')$,

$$
\begin{aligned}
&\frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta((\vec{\sigma}\cdot\vec{p})-\sqrt{4\epsilon})\\
&\quad\times g(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma})f(\vec{r}_1,\vec{v}_1''',t)f(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2''',t)
\end{aligned}
$$

$$= \frac{1}{2}R^2a^2 \int d\vec{r}_1 d\vec{v}_1 dv_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})$$

$$\times\, g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t). \tag{3.37}$$

Because $\theta(\vec{\sigma}\cdot\vec{p}) - \theta((\vec{\sigma}\cdot\vec{p}) - \sqrt{4\epsilon}) = \theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon} - (\vec{\sigma}\cdot\vec{p}))$, from (3.34)- (3.37), we find that

$$\iint\limits_{R^3\times R^3} (C_2 + C_3 + C_4)\log f d\vec{r}_1 d\vec{v}_1 \leq 0. \tag{3.38}$$

Let us compute $\iint\limits_{R^3\times R^3} C_E(f,f)\log f d\vec{r}_1 d\vec{v}_1$. We have, applying the inequality $-x\log\frac{x}{y} \leq y - x$,

$$\iint\limits_{R^3\times R^3} C_E(f,f)\log f d\vec{r}_1 d\vec{v}_1 =$$

$$= -\frac{1}{2}a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n(f)) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)$$

$$\times \log\frac{f(\vec{r}_1. \vec{v}_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)}{f(\vec{r}_1, \vec{v}'_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}'_2, t)}$$

$$\leq \frac{1}{2}a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n(f))$$

$$\times\, [f(\vec{r}_1, v'_1, t)) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}'_2, t) - f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)]$$

$$\leq \frac{1}{2}a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n(f)) f(\vec{r}_1, \vec{v}'_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}'_2, t)$$

$$= \frac{1}{2}a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}|n(f)) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + a\vec{\sigma}, v_2, t).$$

$$\equiv I(t, f) \tag{3.39}$$

The expression $I(t)$ was defined first by Jacek Polewczak[23] for the Enskog equation. In the above argument, we used the change $\vec{\sigma} \to -\vec{\sigma}$. So, we can define a Liapunov functional ("H-function") as follows:

$$H(t) = \iint\limits_{R^3\times R^3} f(\vec{r}_1, \vec{v}_1. t)\log f(\vec{r}_1, \vec{v}_1, t) d\vec{r}_1 d\vec{v}_1 - \int_0^t I(s, f) ds, \tag{3.40}$$

which satisfies

$$\frac{dH}{dt} \leq 0. \tag{3.41}$$

From the above discussion, we can also define

$$
\begin{aligned}
I_{sw}(t) = \; & \frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon}-(\vec{\sigma}\cdot\vec{p})) \\
& \times g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(f))f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1 - Ra\vec{\sigma},\vec{v}_2,t) \\
& + \frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta((\vec{\sigma}\cdot\vec{p})-\sqrt{4\epsilon}) \\
& \times g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(f))f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1 - Ra\vec{\sigma},\vec{v}_2,t) \\
& + \frac{1}{2}R^2a^2\int d\vec{r}_1 d\vec{v}_1 dv_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) \\
& \times g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1 + Ra\vec{\sigma}.\vec{v}_2,t) \\
& + \frac{1}{2}a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) \\
& \times g(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}|n(f))f(\vec{r}_1,\vec{v}_1,t)f(\vec{r}_1 + a\vec{\sigma},\vec{v}_2,t).
\end{aligned}
\tag{3.42}
$$

In this case, (3.41) with $I(t,f)$ replaced by $I_{sw}(t)$ still holds. When discussing the solutions of approximation equations, we will find that this is a natural choice.

(3.4). Conservation properties and bounded energy estimation

Letting $\varphi = 1, v$ in (3.27), we can get easily

$$\iint\limits_{R^3\times R^3} f(\vec{r}_1,\vec{v}_1,t)d\vec{r}_1 d\vec{v}_1 = \iint\limits_{R^3\times S^2} f_0(\vec{r}_1,\vec{v}_1)d\vec{r}_1 d\vec{v}_1 \qquad \text{mass conservation,} \tag{3.43}$$

$$\iint\limits_{R^3\times R^3} \vec{v}_1 f(\vec{r}_1,\vec{v}_1,t)d\vec{r}_1 d\vec{v}_1 = \iint\limits_{R^3\times S^2} \vec{v}_1 f_0(\vec{r}_1,\vec{v}_1)d\vec{r}_1 d\vec{v}_1 \qquad \text{momentum conservation,} \tag{3.44}$$

In [23] Polewczak introduced the boundedness assumption

$$\sup_{\tau,\sigma\geq 0} \tau g(\tau,\sigma) \leq M_g < \infty, \tag{3.45}$$

for some constant M_g, on the geometric factor g. This assumption allows for control of the energy and gain-loss estimations, which will be presented shortly. Since it is believed that for any real gas $g(\tau, \sigma) \to \infty$ as $\tau, \sigma \to n_{cp}$, the close packing density, (3.45) amounts to a cutoff condition, which would be taken, presumably, at densities higher than n_{cp}. Throughout, we will adopt the cutoff (3.45), where M_g is some constant.

By multiplying (3.28) by $\vec{v}_1^2$, formally integrating over $(\vec{r}_1, \vec{v}_1) \in R^3 \times R^3$, and using (3.27), (3.29) and (3.30) with $\varphi = \vec{v}_1^2$, noting

$$\begin{aligned} \vec{v}_1'^2 + \vec{v}_2'^2 &= \vec{v}_1^2 + \vec{v}_2^2, \\ \vec{v}_1''^2 + \vec{v}_2''^2 &= \vec{v}_1^2 + \vec{v}_2^2 + 2\epsilon, \\ \vec{v}_1'''^2 + \vec{v}_2'''^2 &= \vec{v}_1^2 + \vec{v}_2^2 - 2\epsilon. \end{aligned} \tag{3.46}$$

we have

$$\begin{aligned}
&\frac{d}{dt} \iint\limits_{R^3 \times R^3} \vec{v}_1^2 f(\vec{r}_1, \vec{v}_1, t) d\vec{r}_1 d\vec{v}_1 = \\
&= \iint\limits_{R^3 \times R^3} \vec{v}_1^2 E(f, f) d\vec{r}_1 d\vec{v}_1 \\
&= \epsilon R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) \\
&\qquad \times g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma} | n(f)) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) \\
&\quad - \epsilon R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta((\vec{\sigma} \cdot \vec{p}) - \sqrt{4\epsilon}) \\
&\qquad \times g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma} | n(f)) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t) \\
&\leq \epsilon R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) \\
&\qquad \times g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma} | n(f)) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) \\
&\leq \epsilon R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (|\vec{v}_1| + |\vec{v}_2|) \\
&\qquad \times g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}) f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) \\
&\leq \frac{1}{2} \epsilon R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (1 + \vec{v}_1^2 + 1 + \vec{v}_2^2)
\end{aligned}$$

$$
\begin{aligned}
&\times g(\vec{r}_1,\vec{r}_1 - Ra\vec{\sigma}) f(\vec{r}_1,\vec{v}_1,t) f(\vec{r}_1 - Ra\vec{\sigma}, v_2, t) \\
&= \epsilon R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} g(\vec{r}_1,\vec{r}_1 - Ra\vec{\sigma}|n(f)) f(\vec{r}_1,\vec{v}_1,t) f(\vec{r}_1 - Ra\vec{\sigma}, v_2, t) \\
&\quad + \frac{1}{2}\epsilon R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} \vec{v}_1^2 f(\vec{r}_1,\vec{v}_1,t) g(\vec{r}_1,\vec{r}_1 - Ra\vec{\sigma}|n(f)) f(\vec{r}_1 - Ra\vec{\sigma}, v_2, t) \\
&\quad + \frac{1}{2}\epsilon R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} \vec{v}_2^2 f(\vec{r}_1 - Ra\vec{\sigma}, v_2, t) g(\vec{r}_1,\vec{r}_1 - Ra\vec{\sigma}|n(f)) f(\vec{r}_1,\vec{v}_1,t) \\
&\leq K_1(M_g, f_0) + K_2(M_g) \iint\limits_{R^3 \times R^3} \vec{v}_1^2 f(\vec{r}_1,\vec{v}_1,t) d\vec{r}_1 d\vec{v}_1, \qquad (3.47)
\end{aligned}
$$

according to mass conservation and the inequality (3,45), where K_1 and K_2 are some constants.

Let $h(t) = \iint\limits_{R^3 \times R^3} \vec{v}_1^2 f(\vec{r}_1,\vec{v}_1,t) d\vec{r}_1 d\vec{v}_1$, so

$$
\frac{d}{dt} h(t) \leq K_1 + K_2 h(t),
$$

$$
\frac{d}{dt}\left[h(t)e^{-K_2 t}\right] \leq K_1 e^{-K_2 t},
$$

$$
h(t)e^{-K_2 t} - h(0) \leq K_1 \int_0^t e^{-K_2 s} ds = \frac{K_1}{K_2}(1 - e^{-K_2 t}),
$$

$$
h(t) \leq h(0) e^{K_2 t} + \frac{K_1}{K_2}(e^{K_2 t} - 1). \qquad (3.48)
$$

So, when $t \in [0, T]$,

$$
\begin{aligned}
&\iint\limits_{R^3 \times R^3} \vec{v}_1^2 f(\vec{r}_1,\vec{v}_1,t) d\vec{r}_1 d\vec{v}_1 \\
&\leq e^{K_2 T} \iint\limits_{R^3 \times R^3} \vec{v}_1^2 f_0(\vec{r}_1,\vec{v}_1) d\vec{r}_1 d\vec{v}_2 + \frac{K_1}{K_2}(e^{K_2 T} - 1) \\
&= C(T, f_0, M_g). \qquad (3.49)
\end{aligned}
$$

From (3.27), we have

$$
\frac{d}{dt} \iint\limits_{R^3 \times R^3} f(\vec{r}_1,\vec{v}_1,t) |\vec{r}_1 - \vec{v}_1 t|^2 d\vec{r}_1 d\vec{v}_1
$$

$$= -\frac{1}{2}a^2 \iiint d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma})$$

$$\times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)[2a(\vec{\sigma}\cdot\vec{p}]t$$

$$-\frac{1}{2}R^2 a^2 \iiint d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma})$$

$$\times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)\{-2\epsilon t^2 + Ra((\vec{\sigma}\cdot\vec{p}) - [(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}})t\}$$

$$-\frac{1}{2}R^2 a^2 \iiint d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p} - \sqrt{4\epsilon})g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})$$

$$\times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)\{2\epsilon t^2 - Ra((\vec{\sigma}\cdot\vec{p}) - [(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{\frac{1}{2}})t\}$$

$$-\frac{1}{2}R^2 a^2 \iiint d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon} - \vec{\sigma}\cdot\vec{p})g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})$$

$$\times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)\{-2Ra(\vec{\sigma}\cdot\vec{p})t\}.$$

Using (3.45), we get

$$\sup_{t\in[0.T]} \iint\limits_{R^3\times R^3} |\vec{r}_1 - \vec{v}_1 t|^2 f(\vec{r}_1, \vec{v}_1, t) d\vec{r}_1 d\vec{v}_1 \le C(T, f_0, M_g),$$

which results in

$$\sup_{t\in[0.T]} \iint\limits_{R^3\times R^3} |\vec{r}_1|^2 f(\vec{r}_1, \vec{v}_1, t) d\vec{r}_1 d\vec{v}_1 \le C(T, f_0, M_g). \tag{3.50}$$

Again, by use of the inequality (3.41), we obtain

$$\iint\limits_{R^3\times R^3} f(\vec{r}_1, \vec{v}_1, t)\log f(\vec{r}_1, \vec{v}_1, t) d\vec{r}_1 d\vec{v}_1 - \int_0^t I(\tau, f) d\tau$$

$$\le \iint\limits_{R^3\times R^3} f_0(\vec{r}_1, \vec{v}_1)|\log f_0(\vec{r}_1, \vec{v}_1)| d\vec{r}_1 d\vec{v}_1 \tag{3.51}$$

and

$$\iint\limits_{R^3\times R^3} f(\vec{r}_1, \vec{v}_1, t)\log f^+(\vec{r}_1, \vec{v}_1, t) d\vec{r}_1 d\vec{v}_1 \le$$

$$\le \iint\limits_{R^3\times R^3} f(\vec{r}_1, \vec{v}_1, t)\log f^-(\vec{r}_1, \vec{v}_1, t) d\vec{r}_1 d\vec{v}_1 + \int_0^t I(\tau, f) d\tau$$

$$+\iint\limits_{R^3\times R^3} f_0(\vec{r}_1,\vec{v}_1)|\log f_0(\vec{r}_1,\vec{v}_1)|d\vec{r}_1 d\vec{v}_1. \tag{3.52}$$

By an argument similar to the proof of (3.47), (3.48), we have, for $0 \le t < T$,

$$I(t,f) \le C(T, f_0, Mg). \tag{3.53}$$

For the first term of the right hand side in (3.52), because $-x\log x \le yx + \exp(-y-1)$ if $x, y > 0$, we get

$$\begin{aligned} &\iint_{R^3\times R^3} f\log f^- d\vec{r}_1 d\vec{v}_1 \\ &= \iint_{R^3\times R^3} (f(|\vec{r}_1|^2 + |\vec{v}_1|^2) + exp(-|\vec{r}_1|^2 - |\vec{v}_1|^2 - 1))d\vec{r}_1 d\vec{v}_1 \\ &\le C(T, f_0, M_g), \end{aligned} \tag{3.54}$$

for $t \in [0,T]$. Note, the last inequality is the result of (1.5), (3.49), and (3.50). Therefore, for $t \in [0,T]$,

$$\iint_{R^3\times R^3} f(\vec{r}_1,\vec{v}_1,t)\log f^+(\vec{r}_1,\vec{v}_1,t)d\vec{r}_1 d\vec{v}_1 \le C(T, f_0, M_g). \tag{3.55}$$

So, we obtain, for $t \in [0,T]$,

$$\iint\limits_{R^3\times R^3} f(\vec{r}_1,\vec{v}_1,t)|\log f(\vec{r}_1,\vec{v}_1,t)|d\vec{r}_1 d\vec{v}_1 \le C(T, f_0, M_g). \tag{3.56}$$

Together, these estimates gives the bound

$$\begin{aligned} &\sup_{t\in[0,T]} \iint\limits_{R^3\times R^3} f(\vec{r}_1,\vec{v}_1,t)(1 + |\vec{r}_1|^2 + |\vec{v}_1|^2 + |\log f(\vec{r}_1,\vec{v}_1,t)|)d\vec{r}_1 d\vec{v}_1 \\ &\le C(T, f_0, M_g) < \infty. \end{aligned} \tag{3.57}$$

(3.5). The gain–loss estimation

A key ingredient in applying a weak compactness argument to the kinetic equation is the so-called gain–loss estimation. For each $M > 1$, partitioning the space $R^3 \times S^2$ as $\{(\vec{v}_2, \vec{\sigma}) : \gamma \leq M\} \cup \{(\vec{v}_2, \vec{\sigma}) : \gamma > M\}$, for

$$\gamma = \frac{f(\vec{r}_1, \vec{v}_1{}', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2{}', t)}{f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2, t)},$$

we can estimate

$$\begin{aligned}
C_E^+(t,f) \leq & M a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) g(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}) \\
& \times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2, t) \\
& + \frac{1}{\log M} \alpha_E(f),
\end{aligned} \tag{3.58}$$

$$\begin{aligned}
C_2^+(t,f) \leq & M R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) (\vec{v}_1, \vec{v}_2) g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}) \\
& \times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t) \\
& + \frac{1}{\log M} \alpha_2(f),
\end{aligned} \tag{3.59}$$

$$\begin{aligned}
C_3^+(t,f) \leq & M R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon}) g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}) \\
& \times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) \\
& + \frac{1}{\log M} \alpha_3(f),
\end{aligned} \tag{3.60}$$

$$\begin{aligned}
C_4^+(t,f) \leq & M R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\sqrt{4\epsilon} - \vec{\sigma} \cdot \vec{p}) g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}) \\
& \times f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) \\
& + \frac{1}{\log M} \alpha_4(f),
\end{aligned} \tag{3.61}$$

where

$$\alpha_E(f) = a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) g(\vec{r}_1, \vec{r}_1 + a\vec{\sigma})$$

$$\times\, f(\vec{r}_1, \vec{v}_1', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2', t) \mid \log \frac{f(\vec{r}_1, \vec{v}_1', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2', t)}{f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2, t)} \mid$$

$$\alpha_2(f) = R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})$$

$$\times\, f(\vec{r}_1, \vec{v}_1'', t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2'', t) \mid \log \frac{f(\vec{r}_1, \vec{v}_1'', t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2'', t)}{f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)} \mid$$

$$\alpha_3(f) = R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon}) g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma})$$

$$\times\, f(\vec{r}_1, \vec{v}_1''', t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2''', t) \mid \log \frac{f(\vec{r}_1, \vec{v}_1''', t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2''', t)}{f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)} \mid$$

$$\alpha_4(f) = R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\sqrt{4\epsilon} - \vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma})$$

$$\times\, f(\vec{r}_1, \vec{v}_1', t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2', t) \mid \log \frac{f(\vec{r}_1, \vec{v}_1', t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2', t)}{f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)} \mid$$

Next, let $\beta_i(f)$ $(i = E, 2, 3, 4)$ be the integrand in $\alpha_i(f)$ with $|\log(\ldots)|$ replaced by $\log(\ldots)$. According to the discussion of the H-theorem in Section 3 and equality (3.27), we obtain, by changing variables $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1', \vec{v}_2'), \vec{\sigma} \to -\vec{\sigma}$,and $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1'', \vec{v}_2'')$ and $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1''', \vec{v}_2''')$, respectively, for any $T > 0$,

$$\iint\limits_{R^3 \times R^3} f \log f\, d\vec{r}_1\, d\vec{v}_1 - \iint\limits_{R^3 \times R^3} f^0 \log f^0\, d\vec{r}_1\, d\vec{v}_1 =$$

$$= -\frac{1}{2} \int_0^t \iiiint\!\!\int\limits_{R^3 \times R^3 \times R^3 \times S^2} [\beta_E(f) + \beta_2(f) + \beta_3(f) + \beta_4(f)]\, d\vec{v}_1\, d\vec{v}_2\, d\vec{\sigma}\, d\vec{r}_1\, ds. \tag{3.62}$$

Since $z(\log y - \log z) \leq y - z \leq y + z$, for $y, z > 0$,

$$\iiiint\!\!\int\limits_{[0,T] \times R^3 \times R^3 \times R^3 \times S^2} \max(-\beta_E(f), 0)\, d\vec{v}_1\, d\vec{v}_2\, d\vec{\sigma}\, d\vec{r}_1\, ds \leq$$

$$\leq \frac{1}{2} \iiiint\!\!\int\limits_{[0,T] \times R^3 \times R^3 \times R^3 \times S^2} a^2\, d\vec{v}_1\, d\vec{v}_2\, d\vec{\sigma}\, d\vec{r}_1\, ds (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) g(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}|n(f))$$

$$\times\, [f(\vec{r}_1, \vec{v}_1, t) f(\vec{r}_1 + a\vec{\sigma}. \vec{r}_2. t) + f(\vec{r}_1. \vec{v}_1'. t) f(\vec{r}_1 + a\vec{\sigma}. \vec{v}_2'. t)]. \tag{3.63}$$

Paralleling the method by which we obtained the inequality (3.49), we have

$$\underset{[0.T]\times R^3\times R^3\times R^3\times S^2}{\int\int\int\int\int} [-\beta_E(f)]^+ \, d\vec{v}_1\, d\vec{v}_2\, d\vec{\sigma}\, d\vec{r}_1\, ds \leq C(T. f_0, M_g). \tag{3.64}$$

Similarly, we can get, for $i = 2.3.4$

$$\underset{[0.T]\times R^3\times R^3\times R^3\times S^2}{\int\int\int\int\int} [-\beta_i(f)]^+ \, d\vec{v}_1\, d\vec{v}_2\, d\vec{\sigma}\, d\vec{r}_1\, ds \leq C(T. f_0, M_g). \tag{3.65}$$

The proof of (3.65) will be given in the following. We have

$$\underset{[0.T]\times R^3\times R^3}{\iiint} \max(-\beta_2(f), 0) d\vec{r}_1 d\vec{v}_1 \leq$$

$$\leq R^2 a^2 \underset{[0.T]\times R^3\times R^3}{\iiint} d\vec{r}_1 d\vec{v}_1 \underset{R^3\times S^2}{\iint} d\vec{v}_2 d\vec{\sigma} (\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(f))$$

$$\times\, [f(\vec{r}_1, \vec{v}_1'', t) f(\vec{r}_1 + Ra\vec{\sigma}. \vec{v}_2'', t) + f(\vec{r}_1, \vec{v}_1. t) f(\vec{r}_1 + Ra\vec{\sigma}. \vec{v}_2. t)]$$

$$\leq R^2 a^2 \underset{[0.T]\times R^3\times R^3\times R^3\times S^2}{\iint\iiint} dt d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 d\vec{\sigma} (\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) g(\vec{r}_1. \vec{r}_1 + Ra\vec{\sigma}|n(f))$$

$$\times\, f(\vec{r}_1, \vec{v}_1''. t) f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2'', t) + C(T, f_0. M_g).$$

Under the change of variable: $\vec{v}_1 \longrightarrow \vec{v}_1''', \vec{v}_2 \longrightarrow \vec{v}_2'''$, we get $\vec{v}_1'' \longrightarrow \vec{v}_1, \vec{v}_2'' \longrightarrow \vec{v}_2$, moreover,

$$\begin{aligned}
& d\vec{v}_1 d\vec{v}_2 (\vec{\sigma}\cdot\vec{p}) \qquad\qquad (\vec{\sigma}\cdot\vec{p} > 0) \\
& \longrightarrow ((\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon)^{\frac{1}{2}} (\vec{\sigma}\cdot\vec{p})^{-1} \vec{v}_1''' d\vec{v}_2''' (\vec{\sigma}\cdot\vec{g}''') \\
& = (\vec{\sigma}\cdot\vec{p}''') d\vec{v}_1''' d\vec{v}_2''' (\vec{\sigma}\cdot\vec{p})^{-1} (\vec{\sigma}\cdot\vec{p}''') \\
& = (\vec{\sigma}\cdot\vec{p}) d\vec{v}_1 d\vec{v}_2 (\vec{\sigma}\cdot\vec{p})^{-1} (\vec{\sigma}\cdot\vec{p}''') \\
& = (\vec{\sigma}\cdot\vec{p}''') d\vec{v}_1 d\vec{v}_2 \\
& = ((\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon)^{\frac{1}{2}} d\vec{v}_1 d\vec{v}_2
\end{aligned}$$

$$\le (\vec\sigma\cdot\vec p)d\vec v_1 d\vec v_2.$$

Hence.

$$\begin{aligned}
&\iiint\limits_{[0.T]\times R^3\times R^3} \max(-\beta_2(f),0)d\vec r_1 d\vec v_1 dt\\
&\le R^2a^2 \iint\iiint\limits_{[0,T]\times R^3\times R^3\times R^3\times S^2} dtd\vec r_1 d\vec v_1 d\vec v_2 d\vec\sigma(\vec\sigma\cdot\vec p)\theta(\vec\sigma\cdot\vec p)g(\vec r_1,\vec r_1+Ra\vec\sigma|n(f))\\
&\times f(\vec r_1,\vec v_1,t)f(\vec r_1+Ra\vec\sigma,\vec v_2.t)+C(T,f_0,M_g)\\
&\le C(T,f_0,M_g).
\end{aligned}$$

For β_3. we get

$$\begin{aligned}
&\iiint\limits_{[0,T]\times R^3\times R^3} \max(-\beta_3(f),0)d\vec r_1 d\vec v_1\\
&\le R^2a^2 \iiint\limits_{[0,T]\times R^3\times R^3} d\vec r_1 d\vec v_1 \iint\limits_{R^3\times S^2} d\vec v_2 d\vec\sigma(\vec\sigma\cdot\vec p)\theta(\vec\sigma\cdot\vec p-\sqrt{4\epsilon})g(\vec r_1,\vec r_1-Ra\vec\sigma|n(f))\\
&\times[f(\vec r_1,\vec v_1''',t)f(\vec r_1-Ra\vec\sigma,\vec v_2''',t)+f(\vec r_1,\vec v_1,t)f(\vec r_1-Ra\vec\sigma,\vec v_2,t)]\\
&\le R^2a^2 \iint\iiint\limits_{[0,T]\times R^3\times R^3\times R^3\times s^2} dtd\vec r_1 d\vec v_1 d\vec v_2 d\vec\sigma(\vec\sigma\cdot\vec p)\theta(\vec\sigma\cdot\vec p-\sqrt{4\epsilon})g(\vec r_1,\vec r_1-Ra\vec\sigma|n(f))\\
&\times f(\vec r_1,\vec v_1''',t)f(\vec r_1-Ra\vec\sigma,\vec v_2''',t)\quad+\quad C(T,f_0,M_g).
\end{aligned}$$

Under the change of variable: $\vec v_1 \longrightarrow \vec v_1'', \vec v_2 \longrightarrow \vec v_2''$, we get $\vec v_1''' \longrightarrow \vec v_1, \vec v_2''' \longrightarrow \vec v_2$, and

$$\begin{aligned}
&d\vec v_1 d\vec v_2(\vec\sigma\cdot\vec p) \qquad\qquad (\vec\sigma\cdot\vec p\ge\sqrt{4\epsilon})\\
&\longrightarrow ((\vec\sigma\cdot\vec p)^2+4\epsilon)^{\frac12}(\vec\sigma\cdot\vec p)^{-1}\vec v_1'' d\vec v_2'' \quad (\vec\sigma\cdot\vec g'')\\
&=\frac{(\vec\sigma\cdot\vec p)^2+4\epsilon)^{\frac12}}{(\vec\sigma\cdot\vec p)}d\vec v_1 d\vec v_2(\vec\sigma\cdot\vec p)\\
&=((\vec\sigma\cdot\vec p)^2+4\beta)^{\frac12}d\vec v_1 d\vec v_2\\
&\le((\vec\sigma\cdot\vec p)+\sqrt{4\epsilon})d\vec v_1 d\vec v_2.
\end{aligned}$$

Hence,

$$\iiint\limits_{[0,T]\times R^3\times R^3} \max(-\beta_3(f),0)d\vec{r}_1 d\vec{v}_1 dt$$
$$\leq R^2 a^2 \iint\iiint\limits_{[0,T]\times R^3\times R^3\times R^3\times S^2} dt d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 d\vec{\sigma}((\vec{\sigma}\cdot\vec{p}) + \sqrt{4\epsilon})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(f))$$
$$\times f(\vec{r}_1, \vec{v}_1, t)f(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) + C(T, f_0, M_g)$$
$$\leq C(T, f_0, M_g).$$

Similar to the case of $\beta_1(f)$, we have

$$\iiint\limits_{[0,T]\times R^3\times R^3} \max(-\beta_4(f),0)dt d\vec{r}_1 d\vec{v}_1 \leq C(T, f_0, M_g).$$

Using the boundedness of $\int f\log f$, we get,

$$\int\int\int\int\int\limits_{[0,T]\times R^3\times R^3\times R^3\times S^2} |-\beta_i(f)|\, d\vec{v}_1\, d\vec{v}_2\, d\vec{\sigma}\, d\vec{r}_1\, ds \leq C(T, f_0, M_g), \qquad i = E, 2, 3, 4 \quad . \tag{3.66}$$

At last, we can obtain

$$\int\int\int\limits_{[0,T]\times R^3\times R^3} ds\, d\vec{r}_1\, d\vec{v}_1 \alpha_i(f) \leq C(T, f_0, M_g), \qquad i = E, 2, 3, 4 \quad . \tag{3.67}$$

(3.67) is useful when we show weak compactness of collision integrals.

§IV. Equivalent Forms of Solution

Now, we define precisely some notions of solution of the kinetic equation[21].

(i) A nonnegative function $f \in L^1_{loc}(R^3 \times R^3 \times (0, T))$ is a mild solution of equation (1.2) if, for each $0 < T < \infty$, $C_i^{\pm}(t, f)(\vec{r}, \vec{v}, \cdot)$ $(i = E, 2, 3.4) \in L^1(0, T)$ a.e. in $(\vec{r}, \vec{v}) \in R^3 \times R^3$ and satisfies

$$f^{\#}(\vec{r}, \vec{v}. t) - f^{\#}(r, v, s) = \int_s^t E(f, f)^{\#}(\vec{r}, \vec{v}, s)\, ds, \qquad 0 < s < t \leq T \tag{4.1}$$

where

$$f^{\#}(\vec{r}, \vec{v}, t) = f(\vec{r}, \vec{r} + \vec{v}t, t) \tag{4.2}$$

(ii) A nonnegative $f \in L^1_{loc}(R^3 \times R^3 \times (0, T))$ is a renormalized solution of the equation (1.2) if

$$\frac{1}{1+f} C_i^{\pm}(f) \in L^1_{loc}(R^3 \times R^3 \times (0, T)), \qquad i = E, 2, 3, 4 \tag{4.3}$$

and

$$\frac{\partial}{\partial t} \log(1 + f) + \vec{v} \cdot \frac{\partial}{\partial x} \log(1 + f) = \frac{1}{1+f} E(f, f) \tag{4.4}$$

in $\mathcal{D}'(R^3 \times R^3 \times (0, T))$.

(iii) For $i = E, 2.3, 4$, let $F_i^{\#}(\vec{r}, \vec{v}, t) = \int_0^t L_i^{+}(f)^{\#}(\vec{r}, \vec{v}, s)\, ds$, where L_i^{+} is defined by $C_i^{-}(f) = f L_i^{+}(f)$. If $L_i^{+}(f) \in L^1_{loc}(R^3 \times R^3 \times (0, T))$ for any $T > 0$, then f satisfies the exponential multiplier form (1.2) if

$$\begin{aligned} f^{\#}(\vec{r}, \vec{v}, t) - f^{\#}(\vec{r}, \vec{v}, s) \exp\left\{ -\sum_{i=E}^{4} (F_i^{\#}(t) - F_i^{\#}(s)) \right\} = \\ = \int_s^t d\tau \sum_{i=E}^{4} C_i^{+}(\tau, f)^{\#}(\vec{r}, \vec{v}, \tau) \exp\left\{ -\sum_{i=E}^{4} (F_i^{\#}(t) - F_i^{\#}(\tau)) \right\} \end{aligned} \tag{4.5}$$

for any $0 < s < t \leq T$ and a.e. in $(\vec{r}, \vec{v}) \in R^3 \times R^3$.

In order to get familiar with some equivalent relationships among the solutions defined above, in the following, we give some lemmas and explain them in detail.

Theorem 4.1. *(see Theorem A in ref.[21]) Let* $f, h \in L^1_{loc}(R^3 \times R^3 \times R)$ *and assume that*

$$Tf \equiv \partial f + \vec{v} \cdot \nabla_{\vec{r}} f = h \qquad in \qquad D'(R^3 \times R^3 \times R) \tag{4.6}$$

Then, for almost all $\vec{r}, \vec{v} \in R^3, f^{\#}(\vec{r}, \vec{v}, t)$ *is absolutely continuous with respect to* t, $h^{\#}(\vec{r}, \vec{v}, t) \in L^1_{loc}(R)$ *and*

$$f^{\#}(t_2) - f^{\#}(t_1) = \int_{t_1}^{t_2} h^{\#}(s) ds, \qquad for\, all\, t_1,\, t_2 \in R. \tag{4.7}$$

Conversely, if $f, h \in L^1_{loc}(R^3 \times R^3 \times R)$ *are such that, for almost all* $\vec{r}, \vec{v} \in R^3, f^{\#}(\vec{r}, \vec{v}, t)$ *is absolutely continuous with respect to* t, $h^{\#}(\vec{r}, \vec{v}, t) \in L^1_{loc}(R)$ *and (4.7) holds then (4.6) holds.*

Proof: Multiplying (4.6) by $\psi(\vec{r} - t\vec{v}, \vec{v})\zeta(t)$ where $\psi \in D(R^3 \times R^3), \zeta \in D(R)$ we find immediately

$$- \int_R dt \iint_{R^3 \times R^3} d\vec{r} d\vec{v} \psi(\vec{r} - t\vec{v}, \vec{v}) \zeta'(t) f = \int_R dt \int_{R^3 \times R^3} d\vec{r} d\vec{v} \psi(\vec{r} - t\vec{v}, \vec{v}) \zeta(t) h, \tag{4.8}$$

and changing variables $(\vec{r}, \vec{v}, t) \rightarrow (\vec{r} + t\vec{v}, \vec{v}, t)$, we deduce

$$\iint_{R^3 \times R^3} d\vec{r} d\vec{v} \psi(\vec{r}, \vec{v}) \{ \int_R dt (\zeta'(t) f^{\#} + \zeta(t) h^{\#}) \} = 0. \tag{4.9}$$

Therefore, using the fact that C^1 functions with compact support span a separable dense subspace, we see that for almost all $\vec{r}, \vec{v} \in R^3$

$$\int_R dt (\zeta'(t) f^{\#} + \zeta(t) h^{\#}) = 0 \tag{4.10}$$

for all $\zeta \in C^1(R)$ with compact support. Now, in the distribution sense, we have

$$(f^{\#})' - h^{\#} = 0.$$

Conversely, one deduced from (4.7) taking $t_1 = t, t_2 = t + h$, dividing by h and passing to the limit as $h \rightarrow 0_+$ in the sense of distributions,

$$\frac{\partial f^{\#}}{\partial t} = h^{\#} \qquad in \qquad D'(R^3 \times R^3 \times R) \tag{4.11}$$

One recovers easily (4.6) by multiplying smooth test function at both sides and changing variables.

Lemma 4.2. *Assuming* $\beta \in C^1(0,\infty), |\beta'(t)| < \frac{c}{1+t}$. *If*

$$\frac{\partial}{\partial t}\beta + \vec{v}\cdot\nabla_{\vec{r}}\beta = \beta'(f)E(f,f) \tag{4.12}$$

holds in the sense of distributions and $C_i^{\pm}(f,f) \in L^1_{loc}$ *then* f *solves (1.2) in the sense of distributions.*

Proof: It suffices to take $\beta_\delta(f) = \frac{1}{\delta}\log(1+\delta f)$ and observe that by (4.12) we have

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}}\right)\beta_\delta(f) = \frac{1}{1+\delta f}E(f,f). \tag{4.13}$$

We note, for any $\psi \in D$,

$$\int \psi\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}}\right)\beta_\delta(f) = \int \frac{1}{1+\delta f}E(f,f)\psi, \tag{4.14}$$

$$\int \beta_\delta(f)\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}}\right)^*\psi = \int \frac{1}{1+\delta f}E(f,f)\psi, \tag{4.15}$$

hence, letting $\delta \to 0^+$ and passing to the limit in the sense of distributions, we get

$$\int f\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}}\right)^*\psi = \int E(f,f)\psi, \tag{4.16}$$

i.e.,

$$\int \psi\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}}\right)f = \int E(f,f)\psi. \tag{4.17}$$

Lemma 4.3. *If* $f \in L^1_{loc}$ *solves (1.2) in the sense of distributions with* $C_i^{\pm}(f,f) \in L^1_{loc}$ *then* f *is a solution of (1.2) in the renormalized sense. If* f *is a solution of (1.2) in renormalized sense then (4.12) holds in the sense of distributions.*

Proof: Assume $\varphi \in L^1_{loc}(R^3\times R^3\times(0,\infty))$ and $T\varphi \in L^1_{loc}(R^3\times R^3\times(0,\infty)$, using Theorem 4.1, for almost all $\vec{r},\vec{v} \in R^3$, we see $\varphi^{\#}(\vec{r},\vec{v},t)$ is absolutely continuous with respect to t. We observe that for all Lipschitz maps β

$$\beta(\varphi)^{\#}(t) - \beta(\varphi)^{\#}(s)$$

$$
\begin{aligned}
&= \int_s^t (\beta(\varphi)^{\#})'\, d\sigma \\
&= \int_s^t (\beta'(\varphi)^{\#})\, (\varphi^{\#})' d\sigma \\
&= \int_s^t (\beta'(\varphi)\, T\varphi)^{\#}\, d\sigma. \qquad (4.18)
\end{aligned}
$$

Hence

$$T\,\beta(\varphi) \in L^1_{loc}; \qquad T\,\beta(\varphi) = \beta'(\varphi)\, T\varphi \qquad a.e. \qquad (4.19)$$

If $f \in L^1_{loc}$ solves (1.2) with $C_i^{\pm}(f,f) \in L^1_{log}$ in the sense of distributions, then letting $g(f) = \log(1+f)$, we have

$$T\,g = g'(f)\,Tf = \frac{1}{1+f}\, E(f,f) \qquad \text{a.e. in the sense of distributions,} \qquad (4.20)$$

This implies that f is a solution of (1.2) in the renormalized sense.

If f is a solution of (1.2) in the renormalized sense, one has, noting that $\beta'\,E = \{\beta'(f)(1+f)\}\,\{(1+f)^{-1}\,E\}$ and $\beta'(f)(1+f) \in L^{\infty}$,

$$T\,\beta(f) = \beta'(f)\,Tf = \beta'(f)\,\frac{1}{1+f} E(f,f) = \beta'(f)E(f,f) \qquad (4.21)$$

in the sense of distributions.

Lemma 4.4. *Let* $F_i^{\#}(\vec{r},\vec{v},t) = \int_0^t L_i(f)^{\#}(\vec{r},\vec{v},\sigma)d\sigma$, $i = E, 2, 3, 4$, *and assume that for any* $R, T < \infty$, $L_i(f) \in L^1(B_R \times B_R \times (0,T))$.*Then* f *is a mild solution of (1.2) if and only if it is a solution of the exponential multiplier form of (1.2).*

$$f^{\#}(\vec{r},\vec{v},t) - f^{\#}(\vec{r},\vec{v},s)\exp[-\sum_{i=E}^{4}(F_i^{\#}(t) - F_i^{\#}(s))] =$$

$$= \int_s^t \sum_{i=E}^{4} C_i^{+}(f,f)^{\#}(\vec{r},\vec{v},t)\cdot \exp[-\sum_{i=E}^{4}(F_i^{\#}(t) - F_i^{\#}(s))]d\sigma, \qquad (4.22)$$

for any $0 < s < t < \infty$, a.e. $\vec{r}, \vec{v}$.

Proof: We observe that if $L_i(f) \in L^1_{loc}(R^3 \times R^3 \times (0,T))$, then $F_i \in L^1_{loc}(R^3 \times R^3 \times (0,T))$, and for any $\psi \in D$,

$$\iiint (\lim_{t_2 \to t_1} \frac{F_i^\#(t_2) - F_i^\#(t_1)}{t_2 - t_1}) \psi(\vec{r}, \vec{v}, t_1) d\vec{r} d\vec{v} dt_1 \tag{4.23}$$

$$= \int \lim_{t_2 \to t_1} \frac{\int_{t_1}^{t_2} (\iint L_i(\vec{r} + \sigma \vec{v}, \vec{v}, \sigma) \psi(\vec{r}, \vec{v}, t_1) d\vec{r} d\vec{v}) d\sigma}{t_2 - t_1} dt_1 \tag{4.24}$$

$$= \iiint L_i(\vec{r} + t_1 \vec{v}, \vec{v}, t_1) \psi(\vec{r}, \vec{v}, t_1) d\vec{r} d\vec{v} dt_1. \tag{4.25}$$

Therefore,

$$TF_i = L_i \qquad D'(R^3 \times R^3 \times (0,T)). \tag{4.26}$$

Due to the Theorem 4.1, $F_i^\#$ is absolutely continuous w.r.t. t for almost all $\vec{r}, \vec{v} \in R^3$. So, the remaining part of the proof is easy. Indeed, we see

$$\frac{\partial}{\partial t}(f^\#(\vec{r}, \vec{v}, t) \exp(\sum_{i=E}^{4} F_i^\#(\vec{r}, \vec{v}, t))) =$$

$$= \frac{\partial f^\#}{\partial t} \cdot \exp(\sum_{i=E}^{4} F_i^\#(\vec{r}, \vec{v}, t)) + f^\#(\vec{r}, \vec{v}, t) \exp(\sum_{i=E}^{4} F_i^\#(\vec{r}, \vec{v}, t)) \cdot (\sum_{i=E}^{4} L_i^\#(\vec{r}, \vec{v}, t))$$

$$= \sum_{i=E}^{4} (C_i^+(f)^\# - C_i^-(f)^\#) \exp(\sum_{i=E}^{4} F_i^\#(\vec{r}, \vec{v}, t)) + \sum_{i=E}^{4} C_i^-(f)^\# \exp(\sum_{i=E}^{4} F_i^\#(\vec{r}, \vec{v}, t))$$

$$= \sum_{i=E}^{4} C_i^+(f)^\#(\vec{r}, \vec{v}, t)) \exp(\sum_{i=E}^{4} F_i^\#(\vec{r}, \vec{v}, t))$$

for almost all $\vec{r}, \vec{v}$.

Conversely, if the exponential multiplier form (4.22) holds for f, then

$$f^\# \exp(\sum_{i=E}^{4} F_i^\#(\vec{r}, \vec{v}, t))$$

is absolutely continuous w.r.t. t, so the absolutely continuity of $F_i^{\#}(\vec{r},\vec{v},t))$ implies the absolutely continuity of $f^{\#}$, hence, from (4.22), we get

$$T f = E(f,f) \qquad in \qquad D'(R^3 \times R^3 \times (0,T)).$$

Lemma 4.5. *f is a renormalized solution of (1.2) if and only if it is a mild solution of (1.2) and $\frac{C_i^{\pm}}{1+f} \in L^1_{loc}$, $\quad i = E, 2, 3, 4$.*

Proof: If f is a solution of (1.2) in the renormalized sense, then, by Theorem 4.1 and Lemma 4.3, for almost $\vec{r},\vec{v} \in R^3$, and for all $\delta > 0$, $g_\delta^{\#} = \frac{1}{\delta}\log(1+\delta f^{\#})$ is absolutely continuous w.r.t. t, and $\frac{C_i^{\pm}(f,f)^{\#}}{1+f^{\#}} \in L^1_{loc}(0,\infty)$. Since $f^{\#} = \exp(g_1^{\#}) - 1$, $f^{\#}$ is also absolutely continuous w.r.t. t and thus $C_i^{\pm}(f,f)^{\#} \in L^1_{loc}(0,\infty)$ for almost all $\vec{r},\vec{v}$. In addition, for all $t > s > 0$,

$$g_\delta^{\#}(t) - g_\delta^{\#}(s) = \int_s^t \frac{1}{1+\delta f^{\#}} E(f,f)^{\#} d\sigma, \qquad a.e.\ \vec{r},\vec{v} \tag{4.27}$$

and we conclude by letting $\delta \to 0$ and using lebesque's lemma.

Conversely, if f is a mild solution then $f^{\#}$ is absolutely continuous w.r.t. t for almost all $\vec{r},\vec{v} \in R^3$, and so is $g^{\#} = \log(1+f^{\#})$. Furthermore, we have, for all $t > s > 0$,

$$g^{\#}(t) - g^{\#}(s) = \int_s^t \frac{1}{1+f^{\#}} E(f,f)^{\#} d\sigma, \qquad a.e.\ \vec{r},\vec{v}. \tag{4.28}$$

Therefore, if $\frac{C_i^{\pm}(f,f)}{1+f} \in L^1_{loc}$, we conclude that f is a solution of (1.2) in the renormalized sense using Theorem 4.1.

§V. Approximate solutions

(5.1). Construction of approximate equations

let X_n, W_n be symmetric about x and y. and continuous in $(x, y) \in R^3 \times R^3$ such that

$$X_n(x.y) = W_n(x,y) = \begin{cases} 1 & x^2+y^2 \leq n^2 \\ 0 & x^2+y^2 \geq (n+1)^2 \\ \delta(x^2+y^2) & \text{otherwise} \end{cases} \tag{5.1}$$

for some $0 < \delta(x) < 1$.

Lemma 5.1. *For the correlation function $g(\mu.\nu)$, there exists a sequence $\{g_k\}$ such that*

(i) For each $k \geq 1$, $g_k(\mu,\nu) \geq 0$ for $0 \leq f \in L^1(R^3 \times R^3)$, $g_k(\mu,\nu)$ is symmetric under the interchange of $\mu, \nu \in R$, and jointly continuous;

(ii) For any compact set $K \subset R^2$, as $k \to \infty$, $g_k(\mu,\nu) \to g(\mu,\nu)$, uniformly;

(iii) There is a constant C_k such that

$$|g_k(\mu_1,\nu_1) - g_k(\mu_2,\nu_2)| \leq C_k(|\mu_1-\mu_2| + |\nu_1-\nu_2|) \tag{5.2}$$

Proof: Let $g'_k : R \times R \to R$ be continuous and symmetric under interchange of x_1 and x_2, with

$$g'_k = \begin{cases} g(x_1,x_2), & x_1^2+x_2^2 \leq (k+1)^2, \\ 0, & x_1^2+x_2^2 \geq (k+2)^2, \\ \textit{continuous}, & \textit{otherwise}. \end{cases} \tag{5.3}$$

then, for integer $k \geq 1$, $x \in R^2$ define

$$\begin{aligned} g_k(x) &= \int_{R^2} g'_k(x-y)\alpha_k(y)\,dy \\ &= \int_{R^2} g'_k(y)\alpha_k(x-y)\,dy \end{aligned}$$

where $\alpha_k(x) = k^2\alpha(kx)$ and

$$\alpha(x) = \begin{cases} C\exp(|x|^2-1)^{-1}, & |x|<1 \\ 0, & |x| \geq 1. \end{cases} \tag{5.4}$$

with $\int_{R^2} \alpha(x)\,dx = 1$. Check

$$g_k(x) - g(x) = \int_{R^2} (g'_k(x-y) - g(x))\alpha_k(y)dy,$$

when $x \in$ some compact set $K \subset B_{k_0} = \{(x_1, x_2) : x_1^2 + x_2^2 \leq k_0^2\}$, due to *supp* $\alpha_k(y) \subset B_{\frac{1}{k}}(y)$, $x - y, x \in B_{k_0+1} = \{(x_1, x_2) : x_1^2 + x_2^2 \leq (k_0+1)^2\}$. So, when $k > k_0$,

$$\begin{aligned}
|g_k(x) - g(x)| &\leq \int_{R^2} |g'_k(x-y) - g(x)|\alpha_k(y)dy \\
&= \int_{R^2} |g(x-y) - g(x)|\alpha_k(y)dy \\
&\leq \max_{x, x-y \in B_{k_0+1}, \|y\| \leq \frac{1}{k}} |g(x-y) - g(x)| \int_{R^2} \alpha_k(y)dy \\
&\leq \max_{x, x-y \in B_{k_0+1}, \|y\| \leq \frac{1}{k}} |g(x-y) - g(x)|
\end{aligned}$$

We see that, as $k \to \infty$, $g_k(x) \to g(x)$, uniformly in K, hence, $g_k(x) \to g(x)$ for any $x \in R^2$.

Because $g_k(x) = \int_{R^2} g'_k(x-y)\alpha_k(y)\,dy$ and *supp*(g_k) is compact, hence, there exists a $C_k > 0$ such that $\max|\partial g_k(\mu, \nu)| \leq C_k < \infty$. Therefore, we get (5.2).

Next, let us define $C^{\pm}_{in}(t, \varphi)(i = E, 2, 3, 4)$, for φ a nonnegative measurable function on $R^3 \times R^3 \times R^1_+$, as

$$\begin{aligned}
C^+_{En}(t, \varphi) = a^2 \iint\limits_{R^3 \times S^2} & d\vec{v}_2 d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p})W_n(\vec{v}_1', \vec{v}_2')g_n(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}|n(\varphi)) \\
& \times X_n(\vec{r}_1, \vec{r}_1 + a\vec{\sigma})\varphi(\vec{r}_1, \vec{v}_1', t)\varphi(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2', t) \\
& \times [1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1, \vec{v}_1, t)|d\vec{v}_1]^{-1}[1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1 + a\vec{\sigma}, \vec{v}_1, t)|d\vec{v}_1]^{-1} \\
C^-_{En}(t, \varphi) = a^2 \iint\limits_{R^3 \times S^2} & d\vec{v}_2 d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p})W_n(\vec{v}_1, \vec{v}_2)g_n(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n(\varphi)) \\
& \times X_n(\vec{r}_1, \vec{r}_1 - a\vec{\sigma})\varphi(\vec{r}_1, \vec{v}_1, t)\varphi(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t) \\
& \times [1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1, \vec{v}_1, t)|d\vec{v}_1]^{-1}[1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1 - a\vec{\sigma}, \vec{v}_1, t)|d\vec{v}_1]^{-1} \\
C^+_{2n}(t, \varphi) = R^2 a^2 \iint\limits_{R^3 \times S^2} & d\vec{v}_2 d\vec{\sigma}(\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p})W_n(\vec{v}_1'', \vec{v}_2'')g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(\varphi))
\end{aligned}$$

$$
\begin{aligned}
&\times X_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})\varphi(\vec{r}_1, \vec{v}_1'')\varphi(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2'', t) \\
&\times [1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1, \vec{v}_1, t)|d\vec{v}_1]^{-1}[1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_1, t)|d\vec{v}_1]^{-1} \\
C_{2n}^{-}(t, \varphi) = {}& R^2a^2 \iint\limits_{R^3\times S^2} d\vec{v}_2 d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})W_n(\vec{v}_1, \vec{v}_2)g_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(\varphi)) \\
&\times X_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma})\varphi(\vec{r}_1, \vec{v}_1, t)\varphi(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) \\
&\times [1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1, \vec{v}_1, t)|d\vec{v}_1]^{-1}[1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_1, t)|d\vec{v}_1]^{-1} \\
C_{3n}^{+}(t, \varphi) = {}& R^2a^2 \iint\limits_{R^3\times S^2} d\vec{v}_2 d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta((\vec{\sigma}\cdot\vec{p}) - \sqrt{4\epsilon})W_n(\vec{v}_1''', \vec{v}_2''')g_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(\varphi)) \\
&\times X_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma})\varphi(\vec{r}_1, \vec{v}_1''', t)\varphi(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2''', t) \\
&\times [1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1, \vec{v}_1, t)|d\vec{v}_1]^{-1}[1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1 - Ra\vec{\sigma}, v_1, t)|d\vec{v}_1]^{-1} \\
C_{3n}^{-}(t, \varphi) = {}& R^2a^2 \iint\limits_{R^3\times S^2} d\vec{v}_2 d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta((\vec{\sigma}\cdot\vec{p}) - \sqrt{4\epsilon})W_n(\vec{v}_1, \vec{v}_2)g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(\varphi)) \\
&\times X_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})\varphi(\vec{r}_1, \vec{v}_1, t)\varphi(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t) \\
&\times [1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1, \vec{v}_1, t)|d\vec{v}_1]^{-1}[1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1 + Ra\vec{\sigma}, v_1, t)|d\vec{v}_1]^{-1} \\
C_{4n}^{+}(t, \varphi) = {}& R^2a^2 \iint\limits_{R^3\times S^2} d\vec{v}_2 d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon} - (\vec{\sigma}\cdot\vec{p}))W_n(\vec{v}_1', \vec{v}_2')g_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n(\varphi)) \\
&\times X_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma})\varphi(\vec{r}_1, \vec{v}_1', t)\varphi(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2', t) \\
&\times [1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1, \vec{v}_1, t)|d\vec{v}_1]^{-1}[1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1 - Ra\vec{\sigma}, v_1, t)|d\vec{v}_1]^{-1} \\
C_{4n}^{-}(t, \varphi) = {}& R^2a^2 \iint\limits_{R^3\times S^2} d\vec{v}_2 d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon} - (\vec{\sigma}\cdot\vec{p}))W_n(\vec{v}_1, \vec{v}_2)g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(\varphi)) \\
&\times X_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})\varphi(\vec{r}_1, \vec{v}_1, t)\varphi(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t) \\
&\times [1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1, \vec{v}_1, t)|d\vec{v}_1]^{-1}[1 + \frac{1}{n}\int_{R^3} |\varphi(\vec{r}_1 + Ra\vec{\sigma}, v_1, t)|d\vec{v}_1]^{-1}
\end{aligned} \tag{5.5}
$$

Let us consider the truncated kinetic equation

$$
\frac{\partial f_n}{\partial t} + \vec{v}\cdot\frac{\partial f_n}{\partial x} = \epsilon_n(t, f_n) = \sum_{i=E}^{4} \left[C_{in}^{+}(t, f_n) - C_{in}^{-}(t, f_n)\right]
$$

$$f_n(\vec{r}_1, \vec{v}_1, t=0) = f_0^n(\vec{r}_1, \vec{v}_1) \tag{5.6}$$

where

$$f_0^n = \tilde{f}_0^n + \frac{1}{n}\exp\left(-\frac{1}{2}|\vec{r}|^2 - \frac{1}{2}|\vec{v}|^2\right). \tag{5.7}$$

Here, $\tilde{f}_0^n = f_{cut} * \chi$ for $\chi \in C^\infty$ with compact support, and

$$f_{cut}(r, v, 0) = \begin{cases} \min\{f_0(r,v), n\}, & \text{for } |r|^2 \le n^2, |v|^2 \le n^2 \\ 0, & \text{otherwise} \end{cases} . \tag{5.8}$$

We note

$$\iint_{R^3 \times S^2} d\vec{r} d\vec{v} |f_0 - \tilde{f}_0^n| (1 + |\vec{r}|^2 + |\vec{v}|^2) \xrightarrow[n \to \infty]{} 0, \tag{5.9}$$

and

$$\iint_{R^3 \times S^2} d\vec{r} d\vec{v} \tilde{f}_0^n |\log \tilde{f}_0^n| \le C, \tag{5.10}$$

where C is a constant independent of n.

By directly computing, we obtain,

Lemma 5.2. *For $i = E, 2, 3, 4$, any $T \ge 0$, and $\varphi \in L^1(R^3 \times R^3)$ for almost all $t \in (0, T)$,*

$$\int |C_{in}^{\pm}(t, \varphi)| d\vec{v}_1 \le C_n \int |\varphi(\vec{r}_1, \vec{v}_1, t)| d\vec{v}_1. \tag{5.11}$$

These C_n are independent of φ.

Lemma 5.3. *for $i = E, 2, 3, 4$, and $T \ge 0$, and $\varphi \in L^1(R^3 \times R^3)$ for almost all $t \in (0, T)$,*

$$\|C_{in}^-(t, \varphi)\|_{L^\infty(R^3 \times R^3)} \le C_n \|\varphi\|_{L^\infty(R^3 \times R^3)}, \tag{5.12a}$$

and for $i = 2, 3$,

$$\|C_{in}^+(t, \varphi)\|_{L^\infty(R^3 \times R^3)} \le C_n \|\varphi\|_{L^\infty(R^3 \times R^3)}. \tag{5.12b}$$

These C_n are independent of φ .

Proof: For $C_{in}^-(t, \varphi)$, the proof is immediate by (3.45). By the method first introduced by Carlemann (p.32 of ref.[26]), for $C_{2n}^+(\varphi, \varphi)$, we have

$$\begin{aligned}
C_{2n}^{+}(\varphi,\varphi) = R^2a^2 &\iint\limits_{R^3\times S^2} d\vec{v}_2(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})W_n(\vec{v}_1'',\vec{v}_2'')g_n(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n(\varphi)) \\
&\times X_n(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma})\varphi(\vec{r}_1,\vec{v}_1'')\varphi(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2'',t) \\
&\times [1+\frac{1}{n}\int_{R^3}|\varphi(\vec{r}_1,\vec{v}_1,t)|d\vec{v}_1]^{-1}\cdot[1+\frac{1}{n}\int_{R^3}|\varphi(\vec{r}_1+Ra\vec{\sigma},\vec{v}_1,t)|d\vec{v}_1]^{-1} \\
\le R^2a^2 &\iint\limits_{R^3\times S^2} d\vec{v}_2(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})W_n(\vec{v}_1'',\vec{v}_2'')g_n(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n(\varphi)) \\
&\times X_n(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma})\varphi(\vec{r}_1,\vec{v}_1'')\varphi(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2'',t).
\end{aligned}$$

Let $\vec{\sigma}=(l,m,n)$ where

$$\begin{aligned}
l &= \sin\theta\cos\varphi \\
m &= \sin\theta\sin\varphi \qquad (5.13) \\
n &= \cos\theta
\end{aligned}$$

Let $d\sigma_q = \sin\theta d\theta d\varphi$ be the surface element. Hence

$$\begin{aligned}
C_{2n}^{T}(\varphi,\varphi) \le R^2a^2 &\iint\limits_{R^3\times S^2} (\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})W_n(\vec{v}_1'',\vec{v}_2'')g_n(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma}|n(\varphi)) \\
\times X_n(\vec{r}_1,\vec{r}_1+Ra\vec{\sigma})&\varphi(\vec{r}_1,\vec{v}_1'')\varphi(\vec{r}_1+Ra\vec{\sigma},\vec{v}_2'',t)\ \sin\theta d\theta d\varphi dv_{2x}dv_{2y}dv_{2z}
\end{aligned}$$

We define

$$\begin{aligned}
B &= [(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}} \\
A &= 1-(\vec{\sigma}\cdot\vec{p})\,B^{-1} \\
\vec{\sigma} &= (l,\,m,\,n\,), \\
M_1 &= \frac{1}{2}\cos^2\theta A \\
M_2 &= \frac{1}{2}[\cos\theta\frac{\partial W}{\partial\theta} + \sin\theta B]\,A \\
M_3 &= \frac{1}{2}\cos\theta\frac{\partial W}{\partial\varphi}\,A \\
M_4 &= -\frac{1}{2}\sin\theta\cos\varphi\cos\theta\,A
\end{aligned}$$

$$\begin{aligned}
M_5 &= \frac{1}{2}[\cos\theta\cos\varphi B - \sin\theta\cos\varphi\frac{\partial W}{\partial\theta}]\,A \\
M_6 &= \frac{1}{2}[-\sin\theta\sin\varphi B - \sin\theta\cos\varphi\frac{\partial W}{\partial\theta}]\,A \\
M_7 &= -\frac{1}{2}\sin\theta\sin\varphi\cos\theta\,A \\
M_8 &= \frac{1}{2}[\cos\theta\sin\varphi B - \sin\theta\sin\varphi\frac{\partial W}{\partial\theta}]\,A \\
M_9 &= \frac{1}{2}[\sin\theta\cos\varphi\, B - \sin\theta\sin\varphi\frac{\partial W}{\partial\theta}]\,A
\end{aligned}$$

where $W = (\vec{\sigma}\cdot\vec{p}) = l(v_{2x} - v_{1x}) + m(v_{2y} - v_{1y}) + n(v_{2z} - v_{1z})$ and $l = \sin\theta\cos\varphi$, $m = \sin\theta\sin\varphi$, $n = \cos\theta$. We note

$$\begin{aligned}
v''_{1z} &= v_{1z} + \frac{1}{2}\cos\theta(\vec{\sigma}\cdot\vec{p}) - \frac{1}{2}\cos\theta B \\
v''_{2x} &= v_{2x} - \frac{1}{2}\sin\theta\cos\varphi(\vec{\sigma}\cdot\vec{p}) + \frac{1}{2}\sin\theta\cos\varphi B \\
v''_{2y} &= v_{2y} - \frac{1}{2}\sin\theta\sin\varphi(\vec{\sigma}\cdot\vec{p}) + \frac{1}{2}\sin\theta\sin\varphi B,
\end{aligned} \tag{5.14}$$

and

$$(\vec{v}_1'' - \vec{v}_1)\cdot(\vec{v}_2'' - \vec{v}_1) = -\epsilon,$$

so,

$$\begin{aligned}
&\frac{\partial(v''_{1x}, v''_{1y}, v''_{1z}, v''_{2x}, v''_{2y})}{\partial(v_{2x}, v_{2y}, v_{2z}, \theta, \varphi)} \\
&= \frac{\partial(v''_{1x} + v''_{2x}, v''_{1y} + v''_{2y}, v''_{1z}, v''_{2x}, v''_{2y})}{\partial(v_{2x}, v_{2y}, v_{2z}, \theta, \varphi)} \\
&= \frac{\partial(v''_{1z}, v''_{2x}, v''_{2y})}{\partial(v_{2z}, \theta, \varphi)} \\
&= \begin{vmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{vmatrix} \\
&= \frac{1}{8}A^3\cos\theta \begin{vmatrix} \cos\theta & \sin\theta B & 0 \\ -\sin\theta\cos\varphi & \cos\theta\cos\varphi B & -\sin\theta\sin\varphi B \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi B & \sin\theta\cos\varphi B \end{vmatrix} \\
&= \frac{1}{8}A^3B^2\sin\theta\cos\theta
\end{aligned}$$

$$= \frac{1}{8}\left(1 - \frac{(\vec{\sigma}\cdot\vec{p})}{[(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}}}\right)^3[(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]\sin\theta\cos\theta. \tag{5.15}$$

Similarly, we can get

$$\frac{\partial(v'''_{1x}, v'''_{1y}, v'''_{1z}, v'''_{2x}, v'''_{2y})}{\partial(v_{2x}, v_{2y}, v_{2z}, \theta, \varphi)} = \frac{1}{8}\left(1 - \frac{(\vec{\sigma}\cdot\vec{p})}{[(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{\frac{1}{2}}}\right)^3[(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]\sin\theta\cos\theta. \tag{5.16}$$

So,

$$\frac{\partial(v_{2x}, v_{2y}, v_{2z}, \theta, \varphi)}{\partial(v'''_{1x}, v'''_{1y}, v'''_{1z}, v'''_{2x}, v'''_{2y})} = \frac{8\,[(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{\frac{1}{2}}}{\sin\theta\cos\theta(([(\vec{\sigma}\cdot\vec{p})^2 - 4\epsilon]^{\frac{1}{2}} - (\vec{\sigma}\cdot\vec{p}))^3} \tag{5.17}$$

$$\frac{\partial(v_{2x}, v_{2y}, v_{2z}, \theta, \varphi)}{\partial(v''_{1x}, v''_{1y}, v''_{1z}, v''_{2x}, v''_{2y})} = \frac{8\,[(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}}}{\sin\theta\cos\theta(([(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}} - (\vec{\sigma}\cdot\vec{p}))^3}. \tag{5.18}$$

The integral area for $dv''_{2x}dv''_{2y}$ is on the plane $E_{\vec{v}_1\vec{v}''_1}$ through which the vector $\overrightarrow{v_1v''_1}$ perpendicularly passes at the point:

$$(v_{1x} - \frac{v''_{1x} - v_{1x}}{||\vec{v}''_1 - \vec{v}_1||}\epsilon,\ v_{1y} - \frac{v''_{1y} - v_{1y}}{||\vec{v}''_1 - \vec{v}_1||}\epsilon,\ v_{1z} - \frac{v''_{1z} - v_{1z}}{||\vec{v}''_1 - \vec{v}_1||}\epsilon).$$

Let q be any point of the plane $E_{\vec{v}_1\vec{v}''_1}$, one has (see Figure 5.1)

$$dv''_{2x}dv''_{2y} = |\cos\theta|\, d\sigma_q.$$

Having these preparations, we may obtain

$$\begin{aligned}
|C^{+}_{2n}(\varphi,\varphi)| &\le C_n \int_{R^3} g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(\varphi))\,\varphi(\vec{r}_1, \vec{v}''_1, t)\, d\vec{v}''_1 \\
&\quad \times \int_{E_{\vec{v}_1\vec{v}''_1}} \varphi(\vec{r}_1 + Ra\vec{\sigma}, q, t)\, W_n\, \frac{8\,|(\vec{\sigma}\cdot\vec{p})|[(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}}}{([(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}} - (\vec{\sigma}\cdot\vec{p}))^3} d\sigma_q \\
&\le C_n\, ||\varphi||_{L^\infty} \int_{R^3} g_n(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}|n(\varphi))\varphi(\vec{r}_1, \vec{v}''_1, t)\, d\vec{v}''_1 \\
&\quad \times \int_{E_{\vec{v}_1\vec{v}''_1}} W_n \frac{8\,|(\vec{\sigma}\cdot\vec{p})|[(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}}}{([(\vec{\sigma}\cdot\vec{p})^2 + 4\epsilon]^{\frac{1}{2}} - (\vec{\sigma}\cdot\vec{p}))^3}\, d\sigma_q \\
&= C_n\, ||\varphi||_{L^\infty} \int_{R^3} g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n(\varphi))\varphi(\vec{r}_1, \vec{v}''_1, t)\, d\vec{v}''_1
\end{aligned}$$

$$\leq C_n \, ||\varphi||_{L^\infty}. \tag{5.19}$$

Hence,

$$||C^+_{2n}(t,\varphi)||_{L^\infty(R^3 \times R^3)} \leq C_n ||\varphi||_{L^\infty(R^3 \times R^3)}. \tag{5.20}$$

By the same method, we can get for $i = 3$,

$$||C^+_{3n}(t,\varphi)||_{L^\infty(R^3 \times R^3)} \leq C_n ||\varphi||_{L^\infty(R^3 \times R^3)}. \tag{5.21}$$

(5.2). Lipschitz condition of approximate equations

Lemma 5.4. *For each $T \geq 0$, there is a constant C_n such that*

$$\iint\limits_{R^3 \times R^3} |\epsilon_n(t,\varphi) - \epsilon_n(t,\psi)| d\vec{r}_1 d\vec{v}_1 \leq C_n \iint\limits_{R^3 \times R^3} |\varphi - \psi| d\vec{r}_1 d\vec{v}_1$$

$$\iint\limits_{R^3 \times R^3} |C^\pm_{in}(t,\varphi) - C^\pm_{in}(t,\psi)| d\vec{r}_1 d\vec{v}_1 \leq C_n \iint\limits_{R^3 \times R^3} |\varphi - \psi| d\vec{r}_1 d\vec{v}_1 \tag{5.22}$$

hold for all nonnegative $\varphi, \psi \in L^1(R^3 \times R^3)$, where C_n are independent of φ, ψ. (5.22) is just a Lipschitz condition.

Proof: We only check C^+_{En}, since for the other terms, our argument is the same. Set

$$\begin{aligned} x_1 &= g_n(\vec{r}_1, \vec{r}_1 + a\vec{\sigma} | n(\varphi)) \\ x_2 &= \varphi(\vec{r}_1, \vec{v}_1', t) \\ x_3 &= \varphi(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2', t) \\ x_4 &= 1 + \frac{1}{n} \int_{R^3} |\varphi(\vec{r}_1, \vec{v}_1, t)| d\vec{v}_1 \\ x_5 &= 1 + \frac{1}{n} \int_{R^3} |\varphi(\vec{r}_1 + a\vec{\sigma}, \vec{v}_1, t)| d\vec{v}_1 \end{aligned} \tag{5.23}$$

and, by the change of function, $\varphi \to \psi$, define $y_i, \quad i = 1,2,3,4,5$. We compute

$$\frac{x_1 x_2 x_3}{x_4 x_5} - \frac{y_1 y_2 y_3}{y_4 y_5} = (x_4 x_5 y_4 y_5)^{-1} [x_1 x_2 x_3 y_4 y_5 - y_1 y_2 y_3 x_4 x_5]$$

$$= (x_4x_5y_4y_5)^{-1}[(x_1-y_1)x_2x_3y_4y_5 + y_1(x_2-y_2)x_3y_4y_5 + y_1y_2(x_3-y_3)y_4y_5$$
$$+y_1y_2y_3(y_4-x_4)y_5 + y_1y_2y_3x_4(y_5-x_5)] \tag{5.24}$$

Now, it is easy to get the inequality (5.22) by using (3.45) and the expressions (5.1)-(5.5). For example, the term corresponding to the first term above is

$$\iiiint_{R^3\times R^3\times R^3\times S^2} X_nW_n|W|d\vec{r}_1d\vec{v}_1d\vec{v}_2d\vec{\sigma}$$
$$\times\frac{[g_n(n(\vec{r}_1,\varphi),n(\vec{r}_1+a\vec{\sigma},\varphi))-g_n(n(\vec{r}_1,\psi),n(\vec{r}_1+a\vec{\sigma},\psi))]\varphi(\vec{r}_1,\vec{v}_1',t)\varphi(\vec{r}_1+a\vec{\sigma},\vec{v}_2',t)}{[1+\frac{1}{n}n(\vec{r}_1,\varphi)][1+\frac{1}{n}n(\vec{r}_1+a\vec{\sigma},\varphi)]}$$
$$\leq C_n\iint_{R^3\times R^3}|\varphi(\vec{r}_1,\vec{v}_1,t)-\psi(\vec{r}_1,\vec{v}_1,t)|d\vec{r}_1d\vec{v}_1.$$

(5.3). Existence and nonnegativity of approximate solutions

We give a standard result from semigroup theory[27]. Here X is a Banach space.

Theorem 5.5. *For any $T \geq 0$, let $f : [0,T]\times X \longrightarrow X$ be continuous in $t \in [0,T]$ and uniformly Lipschitz continuous (with constant L) on X. If $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$ on X, then for every $u_0 \in X$, the initial value problem*

$$\frac{du(t)}{dt} + Au(t) = f(t,u(t)). \qquad t > 0$$
$$u(0) = u_0 \tag{5.25}$$

has a unique mild solution $u \in C([0,T] : X)$.

Note that $A = -\vec{v}_1 \cdot \nabla_{\vec{r}}$ is a generator of a C_0 semigroup $T(t)$ in $L^1(R^3\times R^3)$, where $T(t)f(\vec{r},\vec{v},t) = f(\vec{r}-\vec{v}t,\vec{v},t)$. Introducing $f^{\#}(\vec{r},\vec{v},t) = f(\vec{r}+\vec{v}t,\vec{v},t)$, equation (5.6) has the following mild form:

$$f_n^{\#} = f_0^n + \int_0^t \epsilon_n(s,f_n)^{\#}ds \tag{5.26}$$

Theorem 5.6. *For each natural number n and for any $T \geq 0$, the approximation equation (5.6) has a unique mild solution in $C([0,T], L^1(R^3 \times R^3))$.*

Proof: Based on (5.22) and the above theorem, the proof is easy.

Lemma 5.7. *The approximation solution is nonnegative.*

Proof: For $0 \leq t \leq T$, define

$$\begin{aligned}
L^+_{En}(f_n) &= a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2 d\vec{\sigma} (\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) W_n(\vec{v}_1,\vec{v}_2) g_n(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n) X_n(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}) \\
&\quad \times f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)[1 + \frac{1}{n}N(\vec{r}_1, t)]^{-1}[1 + \frac{1}{n}N(\vec{r}_1 - a\vec{\sigma})]^{-1} \\
L^+_{2n}(f_n) &= R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2 d\vec{\sigma} (\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) W_n(\vec{v}_1,\vec{v}_2) g_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n) X_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}) \\
&\quad \times f_n(\vec{r}_1 - Ra\vec{\sigma}, v_2, t)[1 + \frac{1}{n}N(\vec{r}_1, t)]^{-1}[1 + \frac{1}{n}N(\vec{r}_1 - Ra\vec{\sigma}, t)]^{-1} \\
L^+_{3n}(f_n) &= R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2 d\vec{\sigma} (\vec{\sigma}\cdot\vec{p})\theta((\vec{\sigma}\cdot\vec{p}) - \sqrt{4\epsilon}) W_n(\vec{v}_1,\vec{v}_2) g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n) \\
&\quad \times X_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}) f_n(\vec{r}_1 + Ra\vec{\sigma}, v_2, t)[1 + \frac{1}{n}N(\vec{r}_1, t)]^{-1}[1 + \frac{1}{n}N(\vec{r}_1 + Ra\vec{\sigma}, t)]^{-1} \\
L^+_{4n}(f_n) &= R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2 d\vec{\sigma} (\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon} - (\vec{\sigma}\cdot\vec{p})) W_n(\vec{v}_1,\vec{v}_2) g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n) \\
&\quad \times X_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}) f_n(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)[1 + \frac{1}{n}N(\vec{r}_1, t)]^{-1}[1 + \frac{1}{n}N(\vec{r}_1 + Ra\vec{\sigma}, t)]^{-1}
\end{aligned} \tag{5.27}$$

Let $F^{\#}_{in}(\vec{r}, \vec{v}, t) = \int_0^t L^+_{in}(f_n)(\vec{r}, \vec{v}, \tau)d\tau$, then the functions f_n satisfy, for $0 \leq s \leq t \leq T$,

$$f^{\#}_n(\vec{r}, \vec{v}, t) - f^{\#}_n(\vec{r}, \vec{v}, s) \exp\left\{ -\sum_{i=E}^{4} (F^{\#}_{in}(t) - F^{\#}_{in}(s)) \right\} =$$

$$= \int_s^t \sum_{i=E}^{4} C^+_{in}(\tau, f_n)^{\#}(r, v, \tau) \exp\left\{ -\sum_{i=E}^{4} (F^{\#}_{in}(t) - F^{\#}_{in}(\tau)) \right\} d\tau. \tag{5.28}$$

Consider the equation

$$h(\vec{r} + t\vec{v}, \vec{v}, t) = \exp[-\sum_{i=E}^{4} F^{\#}_{in}(t)] \cdot f^n_0(\vec{r}, \vec{v})$$

$$+\int_0^t \exp[-\sum_{i=E}^{4} F^{\#}_{in}(t) + \sum_{i=E}^{4} F^{\#}_{in}(\tau)] \cdot \sum_{i=E}^{4} C^{+}_{in}(\tau, h)^{\#}(\vec{r}, \vec{v}, \tau) d\tau \tag{5.29}$$

We see that $C^{+}_{in}(\tau, h) \geq 0$, if $h \geq 0$ and that f_n is the solution of (5.29), for $t \in [0, T]$. Because $L^{+}_{in}(f_n)$ is bounded due to condition (3.45), we have, for $0 \leq t \leq T$, there is a $C_n(T)$, such that

$$0 < \exp[-\sum_{i=1}^{4} F^{\#}_{in}(t) + \sum_{i=1}^{4} F^{\#}_{in}(\tau)] < C_n(T). \tag{5.30}$$

So, by (5.22), f_n is the unique solution of (5.29). But for $0 \leq t \leq T$, we can construct the following convergent iteration scheme in L^1:

$$\begin{gathered} h_1 = 0 \\ h_j(\vec{r} + t\vec{v}, \vec{v}, t) = \exp[-\sum_{i=1}^{4} F^{\#}_{in}(t)] \cdot f^{n}_{0}(\vec{r}, \vec{v}) \\ +\int_0^t \exp[-\sum_{i=1}^{4} F^{\#}_{in}(t) + \sum_{i=1}^{4} F^{\#}_{in}(\tau)] \cdot \sum_{i=1}^{4} C^{+}_{in}(\tau, h_{j-1})^{\#} d\tau \end{gathered} \tag{5.31}$$

for $j = 1, 2, \ldots$. It follows that $h_2 \geq h_1 \geq 0$, and by induction, $\{h_j\}$ is a non-negative increasing sequence, hence $h = \lim_{j\to\infty} h_j \geq 0$. As f_n is the unique solution, it follows that $h = f_n \geq 0$ for $0 \leq t \leq T$.

Following DiPerna and Lions' work [21], we can show that

Lemma 5.8. *For $t \in [0, T]$,*

$$f_n(\vec{r}, \vec{v}, t) \geq \frac{\rho}{n} \exp(-C_n t - \frac{1}{2}|\vec{r} - t\vec{v}|^2 - \frac{1}{2}|\vec{v}|^2) \equiv g_n \qquad \text{a.e. in } (\vec{r}, \vec{v}) \in R^3 \times R^3. \tag{5.32}$$

Proof: We note, $C^{-}_{in}(t, f_n) \leq \frac{1}{4} C_n f_n$ on $R^3 \times R^3 \times R^1_+$, for some constant C_n [the same constant appearing in (5.32)]. We can see that

$$\frac{\partial f_n}{\partial t} + \vec{v} \cdot \frac{\partial f_n}{\partial x} + C_n f_n \geq 0 \qquad \text{in } R^3 \times R^3 \times [0, T]. \tag{5.33}$$

One may see directly for g_n,

$$\frac{\partial g_n}{\partial t} + \vec{v} \cdot \frac{\partial g_n}{\partial x} + C_n g_n = 0 \qquad \text{in} R^3 \times R^3 \times [0, T]. \tag{5.34}$$

Let $y = f_n - g_n$. Since $y(0) = f_n(\vec{r}, \vec{v}, 0) - g_n(\vec{r}, \vec{v}, 0) > 0$ for t in some neighborhood of $t = 0$, we have

$$\frac{dy}{dt} \geq -C_n y \implies \log\ y \geq -C_n t + C_0 \tag{5.35}$$

or

$$y \geq C_n e^{-c_n t}. \tag{5.36}$$

Hence, the inequality (5.32) is proved. This fact is used to show that $f_n \log f_n \in L^1$.

As for the Enskog equation, symmetries of the collision integrals lead to identities for

$$\iint\limits_{R^3 \times R^3} \varphi(\vec{r}_1, \vec{v}_1, t) C_{in}(f, f) d\vec{r}_1, d\vec{v}_1,$$

for $i = E, 2, 3, 4$ and φ a measurable function on $R^3 \times R^3 \times R^1_+$. Indeed, construction of the approximate equations (note the various arguments of W_n) has been arranged so that these symmetries can be exploited even though energy conservation fails, e.g., $\vec{v}_1^2 + \vec{v}_2^2 \neq (\vec{v}_1'''')^2 + (\vec{v}_2'''')^2$. One result is the estimate

$$\begin{aligned}
&\iint\limits_{R^3 \times R^3} (C_{En} + C_{2n} + C_{3n} + C_{4n}) \log f_n \leq \\
&\leq \frac{1}{2} a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) g_n(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}) W_n(\vec{v}_1, \vec{v}_2) X_n(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}) \\
&\qquad \times [f_n(\vec{r}_1, v_1', t)) f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2', t) - f_n(\vec{r}_1, \vec{v}_1, t) f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_1, t)] \\
&- \frac{1}{2} R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) \theta(\sqrt{4\epsilon} - (\vec{\sigma} \cdot \vec{p})) \\
&\qquad \times g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}) W_n(\vec{v}_1, \vec{v}_2) X_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}) \\
&\qquad \times [f_n(\vec{r}_1, \vec{v}_1, t) f_n(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t) - f_n(\vec{r}_1, \vec{v}_1', t) f_n(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2', t)] \\
&- \frac{1}{2} R^2 a^2 \int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) g_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}) W_n(\vec{v}_1, \vec{v}_2) X_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma})
\end{aligned}$$

$$
\begin{aligned}
&\times [f_n(\vec{r}_1,\vec{v}_1,t)f_n(\vec{r}_1 - Ra\vec{\sigma},\vec{v}_2,t) - f_n(\vec{r}_1, v_1'',t)f_n(\vec{r}_1 - Ra\vec{\sigma},\vec{v}_2'',t)] \\
-\frac{1}{2}R^2a^2 &\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta((\vec{\sigma}\cdot\vec{p}) - \sqrt{4\epsilon}) \\
&\times g_n(\vec{r}_1,\vec{r}_1 + Ra\vec{\sigma})W_n(\vec{v}_1,\vec{v}_2)X_n(\vec{r}_1,\vec{r}_1 + Ra\vec{\sigma}) \\
&\times [f_n(\vec{r}_1,\vec{v}_1,t)f_n(\vec{r}_1 + Ra\vec{\sigma},\vec{v}_2,t) - f_n(\vec{r}_1,\vec{v}_1''',t)f_n(\vec{r}_1 + Ra\vec{\sigma},\vec{v}_2''',t)].
\end{aligned}
$$

Therefore, defining

$$
\begin{aligned}
&2I_n(t) = \\
&a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g_n(\vec{r}_1,\vec{r}_1 + a\vec{\sigma})f_n(\vec{r}_1,\vec{v}_1,t)f_n(\vec{r}_1 + a\vec{\sigma},\vec{v}_2,t) \\
&+ R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon} - (\vec{\sigma}\cdot\vec{p})) \\
&\qquad \times g_n(\vec{r}_1,\vec{r}_1 - Ra\vec{\sigma})f_n(\vec{r}_1,\vec{v}_1,t)f_n(\vec{r}_1 - Ra\vec{\sigma},\vec{v}_2,t) \\
&+ R^2a^2\int d\vec{r}_1 d\vec{v}_1 d\vec{v}_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta((\vec{\sigma}\cdot\vec{p}) - \sqrt{4\epsilon}) \\
&\qquad \times g_n(\vec{r}_1,\vec{r}_1 - Ra\vec{\sigma})f_n(\vec{r}_1,\vec{v}_1,t)f_n(\vec{r}_1 - Ra\vec{\sigma},\vec{v}_2,t) \\
&+ R^2a^2\int d\vec{r}_1 d\vec{v}_1 dv_2 \int d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p}) \\
&\qquad \times g_n(\vec{r}_1,\vec{r}_1 + Ra\vec{\sigma})f_n(\vec{r}_1,\vec{v}_1,t)f_n(\vec{r}_1 + Ra\vec{\sigma},\vec{v}_2,t),
\end{aligned}
$$

we can define the Liapunov functional (H-function)

$$
H_n(t) = \iint\limits_{R^3\times S^2} f_n(\vec{r}_1,\vec{v}_1,t)\log f_n(\vec{r}_1,\vec{v}_1,t)d\vec{r}_1 d\vec{v}_1 - \int_0^t I_n(s)ds
$$

which satisfies

$$
\frac{dH_n}{dt} \le 0.
$$

(5.4). Compactness of approximate solutions

As we note that the estimates (3.58)-(3.67) carry over in an obvious fashion to the approximate equations, we have

$$
\sup_{t\in[0,T]} \iint\limits_{R^3\times R^3} f_n(\vec{r},\vec{v},t)(1 + |\vec{r}|^2 + |\vec{v}|^2 + |\log f_n(\vec{r},\vec{v},t)|)d\vec{r}d\vec{v} \le C(T, f_0, M_g) < \infty. \quad (5.39)
$$

In the following, we show that sequence $\{f_n\}$ is weakly compact in $L^1(R^3 \times R^3 \times (0.T))$ for all $T > 0$. We state

Lemma. *(Dunford-Pettis Theorem 4.21.2, p. 274 of [28]) If* $X = L^1(\Omega, d, \mu)$ *for* μ *a positive Radon measure on a locally compact space* Ω*, then* $\{f_n\}_1^\infty \subset X$ *is weakly compact if and only if*

(i) $\|f_n\| \leq M < \infty$, $\forall n$;

(ii) $\forall \epsilon > 0$, $\exists \delta > 0$ *such that* $\int_B |f_n| d\mu < \epsilon$ *for all* $B \subset \Omega$ *with* $\mu(B) < \delta$.

(iii) $\forall \epsilon > 0$, $\exists$ *compact set* K *such that*

$$\int_{\Omega \setminus K} |f_n| d\mu < \epsilon, \ \forall n.$$

Lemma 5.9. *.The sequence* $\{f_n\}$ *is weakly compact in the case of (5.39).*

Proof: (i) is obvious

For (ii) we note that for any measurable set $B \subset R^3 \times R^3$ and any real $K > 1$, if A is defined by $A = \{(x,y) \in R^3 \times R^3 | f_n \leq K\}$, then

$$\iint_B f_n d\vec{r} d\vec{v} = \iint_{B \cap A} f_n d\vec{r} d\vec{v} + \iint_{B \cap A^c} f_n d\vec{r} d\vec{v} \leq \iint_{B \cap A} f_n d\vec{r} d\vec{v} + \iint_{B \cap A^c} f_n \frac{\log f_n}{\log K} d\vec{r} d\vec{v}$$

By taking K large enough, we observe that for all $\epsilon > 0$ there exists $\delta > 0$ such that $\int_B |f_n| d\vec{r} d\vec{v} < \epsilon$ for all B with $\mu(B) < \delta$.

Finally, to see (iii), let $B_R = \{(x,y) \in R^3 \times R^3 | x^2 + y^2 \leq R^2\}$ and estimate

$$\iint_{B_R^c} f_n d\vec{r} d\vec{v} \leq \iint_{B_R^c} f_n \frac{|\vec{r}|^2 + |\vec{v}|^2}{R^2} d\vec{r} d\vec{v} \leq \frac{1}{R^2} \iint_{R^3 \times R^3} f_n (|\vec{r}|^2 + |\vec{v}|^2) d\vec{r} d\vec{v}.$$

Thus, (iii) is proved.

§VI. Some Results Due to Weak Limit

In the following, we first give a compactness lemma due to Golse et al.[22]. When we study velocity average convergence of approximate solution and collision kernels which plays an important role in our proof.

(6.1). Compactness lemma

Golse Lemma. *(see Theorem 54 in ref. [22]) Suppose* $f_n, g_n \in L^1_{loc}(R^3 \times R^3 \times (0,T))$ *satisfy*

$$Tf_n = \frac{\partial f_n}{\partial t} + \vec{v} \cdot \frac{\partial f_n}{\partial t} = g_n$$

in $\mathcal{D}'(R^3 \times R^3 \times (0,T))$, *and for each compact set* $K \subset R^3 \times R^3 \times (0,T)$, *the sequences* $\{f_n\}$ *and* $\{g_n\}$ *are weakly compact in* $L^1(R^3 \times R^3 \times (0,T))$ *and* $L^1(K)$, *respectively. Then for all* $\varphi \in L^\infty(R^3 \times R^3 \times (0,T))$. *the set* $\{\int_{R^3} \varphi f_n \, d\vec{v}\} = \{\int_{R^3} \varphi T^{-1} g_n d\vec{v}\}$ *is compact in* $L^1(R^3 \times (0,T))$.

Lemma 6.1. *(see Theorem 4.21.10, p. 279 of ref.[28]) Assuming that* g_n *and* h_n *are measurable on* $\Sigma = R^3 \times R^3 \times (0,T)$, *one has*

(i) $\int_\Sigma h_n g_n \to \int_\Sigma hg$ *if* $g_n \to g$ *weakly in* L^1, $\underset{n\geq 1}{sup} \|h_n\|_{L^1} < \infty$, *and* $h_n \to h$ *a.e.*

(ii) $\int_\Sigma h_n g_n \to \int_\Sigma hg$ *if* $h_n \to h$ *weakly in* L^1, $g_n \to g$ *strongly in* L^1, *and* $\sup_{n\geq 1} \|h_n\|_{L^\infty} < \infty$.

(6.2). Weak convergence of approximate solutions

In the sections above we have already proved that the approximation solution sequence $\{f_n\}$ is weakly compact in $L^1(R^3 \times R^3 \times (0,T))$ for all $T > 0$. We may assume, without lost of generality, $f_n \to f$ weakly in $L^1(R^3 \times R^3 \times (0,T))$, for some function $0 \leq f \in L^1(R^3 \times R^3 \times (0,T))$.

Proposition 6.2. *For any* $T > 0$,

$$\underset{t\in[0,T]}{sup} \int\int_{R^3 \times R^3} [1 + |\vec{r}|^2 + |\vec{v}|^2 + |\log f|] f(\vec{r}, \vec{v}, t) \, d\vec{r} \, d\vec{v} \leq C(T) < \infty \tag{6.1}$$

Theorem 6.3. *[28-29] Let X be a Banach space, $g : M \subseteq X \to R_1 \cup \{+\infty\}$, be a convex function on the weakly closed convex set M. Then, on the set M, lower weak semicontinuity of g is equivalent to lower semicontinuity.*

Lemma 6.4. *Let $M = \{0 \le f \in L^1(R^3 \times R^3) : \iint\limits_{R^3 \times R^3} f(\vec{r}, \vec{v})(1 + |\vec{r}| + |\vec{v}|) d\vec{r} d\vec{v} \le C\}$. Then, M is a weak closed convex subset in $L^1(R^3 \times R^3)$.*

Proof: Suppose $f_n \to f$ weakly, and let $\psi = \chi_S$ such that S is compact. It is easy to see

$$\iint\limits_{R^3 \times R^3} f_n(1 + |\vec{r}| + |\vec{v}|)\psi d\vec{r} d\vec{v}$$

$$\to \iint\limits_{R^3 \times R^3} f(1 + |\vec{r}| + |\vec{v}|)\psi d\vec{r} d\vec{v} \le C$$

Because S is an arbitrary compact set, we can get

$$\iint\limits_{R^3 \times R^3} f(1 + |\vec{r}| + |\vec{v}|) d\vec{r} d\vec{v} \le C$$

So, $f \in M$, and the lemma is proved.

Proof of Proposition 6.2: By Fatou's lemma, for any bounded measurable set $E \subset R^3 \times R^3$,

$$\int\!\!\int\limits_{E} (1 + |\vec{r}|^2 + |\vec{v}|^2) f(\vec{r}, \vec{v}, t)\, d\vec{r}\, d\vec{v} \le C(T) \qquad a.e.\ in\ t \in [0, T] \tag{6.2}$$

Since the E is arbitrary, we have

$$\int\!\!\int\limits_{R^3 \times R^3} (1 + |\vec{r}|^2 + |\vec{v}|^2) f(\vec{r}, \vec{v}, t)\, d\vec{r}\, d\vec{v} \le C(T) \qquad a.e.\ in\ t \in [0, T] \tag{6.3}$$

Next, we want to prove

$$\int\!\!\int\limits_{R^3 \times R^3} f |\log f|\, d\vec{r}\, d\vec{v} \le C(T) \qquad a.e.\ in\ t \in [0, T] \tag{6.4}$$

Since $z \log z$ is convex, lower weak semicontinuity of $f_n \log f_n$ is equivalent to lower semicontinuity on weakly closed convex sets. Hence, we assume f_n converges strongly to f in $L^1(R^3 \times R^3)$. Because of (5.39) and (6.3), we can also assume that

$$(1 + |\vec{r}| + |\vec{v}|) f_n \rightarrow (1 + |\vec{r}| + |\vec{v}|) f \qquad in\ L^1(R^3 \times R^3). \tag{6.5}$$

Since $f_n \log^- f_n \leq (1 + |\vec{r}| + |\vec{v}|) f_n + \exp(-1 - |\vec{v}| - |\vec{r}|)$,

$$f_n \log^- f_n \rightarrow f \log^- f \tag{6.6}$$

in $L^1(R^3 \times R^3)$. Hence

$$\liminf_{n \rightarrow \infty} \int\limits_{R^3 \times R^3}\!\!\int f_n \log^+ f_n \, d\vec{r}\, d\vec{v} \leq \sup_{n \geq 1} \int\limits_{R^3 \times R^3}\!\!\int f_n \log f_n \, d\vec{r}\, d\vec{v} + \int\limits_{R^3 \times R^3}\!\!\int f \log^- f \, d\vec{r}\, d\vec{v} \tag{6.7}$$

By Fatou's lemma, $\int\limits_{R^3 \times R^3}\!\!\int f \log^+ f \, d\vec{r}\, d\vec{v} \leq \liminf_{n \rightarrow \infty} \int\limits_{R^3 \times R^3}\!\!\int f_n \log^+ f_n \, d\vec{r}\, d\vec{v}$, and one has

$$\int\limits_{R^3 \times R^3}\!\!\int f |\log f| \, d\vec{r}\, d\vec{v} \leq C(T) \qquad a.e.\ in\ t \in [0, T] \tag{6.8}$$

So, combining (6.2) and (6.8), it follows that, for any $T > 0$,

$$\sup_{t \in [0,T]} \int\limits_{R^3 \times R^3}\!\!\int [1 + |\vec{r}|^2 + |\vec{v}|^2 + |\log f|] f(\vec{r}, \vec{v}, t) \, d\vec{r}\, d\vec{v} \leq C(T) < \infty \tag{6.1}$$

In order to apply the compactness lemma to the square-well kinetic equation, we need to prove

Proposition 6.5. *For each* $\delta > 0$ *and* $R > 0$, *the sequence* $\{C_i^{\pm}(t, f_n)/(1 + \delta f_n)\}$, $(i = E, 2, 3, 4)$ *is weakly compact in* $L^1(R^3 \times B_R \times (0, T))$, *where* $B_R = \{\vec{v} \in R^3 : |\vec{v}| \leq R\}$.

Proof: For any $\delta > 0$,

$$\begin{aligned} &\frac{1}{1 + \delta f_n} a^2 (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) g_n(\vec{r}_1, \vec{r}_1 - a\vec{\sigma} | n(f_n)) f_n(\vec{r}_1, \vec{v}_1, t) f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t) \\ &\leq C(\delta) M_g' f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)(|\vec{v}_1| + |\vec{v}_2|) \end{aligned} \tag{6.9}$$

where $M'_g = \sup_{(x,y)\in R^3\times R^3} g(x,y)$. Hence, if the right side is weakly compact, the left side is weakly compact in $L^1(R^3\times B_R\times(0,T)\times R^3\times S^2)$, where we consider $(\vec{r}_1, \vec{v}_1, t, \vec{v}_2, \vec{\sigma}) \in (R^3 \times B_R \times (0,T) \times R^3 \times S^2)$.

Let us check the conditions of the Dunford–Pettis Theorem.

(i) is obvious.

(ii) Let $A_1 = \{(\vec{r}_1, \vec{v}_1, t, \vec{v}_2, \vec{\sigma}) : \vec{v}_2 \le K\}$. One has, for any measurable set $B_1 \subset R^3 \times B_R \times (0,T) \times R^3 \times S^2$,

$$
\begin{aligned}
&\int\!\!\int\!\!\underset{B_1}{\int}\!\!\int\!\!\int f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)(|\vec{v}_1| + |\vec{v}_2|)\, d\vec{r}_1\, d\vec{v}_1\, dt\, d\vec{v}_2\, d\vec{\sigma} \\
\le& R\int\!\!\int\!\!\underset{B_1}{\int}\!\!\int\!\!\int f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)\, d\vec{r}_1\, d\vec{v}_1\, dt\, d\vec{v}_2\, d\vec{\sigma} \\
&+ \int\!\!\int\!\!\underset{B_1\cap A_1}{\int}\!\!\int\!\!\int f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)|\vec{v}_2|\, d\vec{r}_1\, d\vec{v}_1\, dt\, d\vec{v}_2\, d\vec{\sigma} \\
&+ \int\!\!\int\!\!\underset{B_1\cap A_1^C}{\int}\!\!\int\!\!\int f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)|\vec{v}_2|\frac{|\vec{v}_2|}{K}\, d\vec{r}_1\, d\vec{v}_1\, dt\, d\vec{v}_2\, d\vec{\sigma}
\end{aligned}
\tag{6.10}
$$

Choosing K large enough, letting the measure of B_1 be small enough and using the inequality (5.39) ,we get (ii).

(iii) Let $B_R^1 = \{(\vec{r}_1, \vec{v}_2, t, \vec{v}_2, \vec{\sigma}) : |\vec{r}_1| + |\vec{v}_1| + |\vec{v}_2| \le R\}$ and note

$$
\begin{aligned}
&\underset{(R^3\times R^3_B\times(0,T)\times R^3\times S^2)\cap B_R^{1C}}{\int\!\!\int\!\!\int\!\!\int\!\!\int} f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)(|\vec{v}_1| + |\vec{v}_2|)\, d\vec{r}_1\, d\vec{v}_1\, dt\, d\vec{v}_2\, d\vec{\sigma} \le \\
\le& \underset{R^3\times R^3_B\times(0,T)\times R^3\times S^2}{\int\!\!\int\!\!\int\!\!\int\!\!\int} f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)(|\vec{v}_1| + |\vec{v}_2|)\frac{|\vec{r}_1| + |\vec{v}_1 + |\vec{v}_2|}{R}\, d\vec{r}_1\, d\vec{v}_1\, dt\, d\vec{v}_2\, d\vec{\sigma}
\end{aligned}
\tag{6.11}
$$

Thus (iii) is satisfied. Therefore, $f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)(|\vec{v}_1| + |\vec{v}_2|)$ is weakly compact in $L^1(R^3 \times B_R \times (0,T) \times R^3 \times S^2)$. Moreover, because of (6.9), one concludes that the sequence composed of the left side of (6.9) is weakly compact. Since the operator

$\int\int_{R^3\times S^2} \theta(\vec{\sigma}\cdot\vec{p})\cdot d\vec{v}_2\, d\vec{\sigma}$ is a bounded operator from $L^1(R^3\times B_R\times(0,T)\times R^3\times S^2)$ to $L^1(R^3\times B_R\times(0,T))$, we obtain, for any $\delta>0$ and $R>0$, $C^-_{En}(t,f_n)/(1+\delta f_n)$ is weakly compact in $L^1(R^3\times B_R\times(0,T))$.

Similarly, we can obtain $C^-_{in}(t,f_n)/(1+\delta f_n)$ $(i=2,3,4)$ is weakly compact in $L^1(R^3\times B_R\times(0,T))$. Also, we see that the gain–loss estimation (3.58)–(3.61), together with bound (3.67) implies that for any $\delta>0$ and $R>0$ the sequence $\{C^+_{in}(t,f_n)/(1+\delta f_n)\}$, $(i=E,2,3,4)$, is weakly compact in $L^1(R^3\times B_R\times(0,T))$. Indeed, by (3.58)-(3.61), $\{C^+_{in}(t,f_n)/(1+\delta f_n)\}$ $(i=E,2,3,4)$, is bounded in $L^1(R^3\times B_R\times(0,T))$, and for any measurable sets $S\subset R^3\times R^3\times(0,T)$ and for all $R>0$,

$$\iiint_{R^3\times R^3\times(0,T)} d\vec{r}\, d\vec{v}\, dt \frac{1}{1+f_n} C^+_{in}(f_n,f_n)(\chi_S+\chi_{|\vec{r}|^2+|\vec{v}|^2\geq R})$$

$$\leq M \iiint_{R^3\times R^3\times(0,T)} d\vec{r}\, d\vec{v}\, dt \frac{1}{1+f_n} C^-_{in}(f_n,f_n)(\chi_S+\chi_{|\vec{r}|^2+|\vec{v}|^2\geq R}) + \frac{C(T,f_0,M_g)}{\log M}$$

Hence, as $R\to\infty$ and the measure of S goes to 0,

$$\sup_{n\geq 1} \iiint_{R^3\times R^3\times(0,T)} d\vec{r}\, d\vec{v}\, dt \frac{1}{1+f_n} C^+_{in}(f_n,f_n)(\chi_S+\chi_{|\vec{r}|^2+|\vec{v}|^2\geq R}) \to 0.$$

(6.3). Velocity average convergence of approximate solutions

In the remainer of this thesis, we shall exploit Golse lemma to obtain velocity average convergence of approximate solutions and collision kernels, then extend DiPerna and Lions's arguments[21] to prove existence of solution of the square-well kinetic equation.

Proposition 6.6. *For each φ with $(1+|\vec{r}_1|^k+|\vec{v}_1|^k)^{-1}\varphi\in L^\infty(R^3\times R^3\times(0,T))$ and $0\leq k<2$, one has*

$$\int_{R^3} f_n\varphi\, d\vec{v} \to \int_{R^3} f\varphi\, d\vec{v} \tag{6.12}$$

as $n \to \infty$ *in* $L^1(R^3 \times (0,T))$. *and for any* $R > 0$,

$$L^+_{in}(f_n) \to L^+_i(f) \tag{6.13}$$

in $L^1(R^3 \times B_R \times (0,T))$.

Proof: we define, for $\delta > 0$,

$$f_n^\delta = \frac{1}{\delta}\log(1 + \delta f_n). \tag{6.14}$$

Since $0 \le f_n^\delta \le f_n$, $\{f_n^\delta\}_{n=1}^\infty$ is weakly compact in $L^1(R^3 \times R^3 \times (0,T))$. We observe that f_n^δ satisfies

$$\frac{\partial}{\partial t} f_n^\delta + \vec{v} \cdot \frac{\partial}{\partial x} f_n^\delta = \frac{1}{1 + \delta f_n} \epsilon_n(t, f_n) \qquad in\ \mathcal{D}'(R^3 \times R^3 \times (0,\infty)). \tag{6.15}$$

Golse lemma implies that for all $\varphi \in L^\infty(R^3 \times R^3 \times (0,T))$, after passing to a subsequence if necessary,

$$\int_{R^3} f_n^\delta \varphi \, dv \underset{n\to\infty}{\longrightarrow} \int_{R^3} f^\delta \varphi \, d\vec{v} \qquad in\ L^1(R^3 \times (0,T)) \tag{6.16}$$

where f^δ is the weak limit of $\{f_n^\delta\}$ in $L^1(R^3 \times R^3 \times (0,T))$.

Let us write, for any $\varphi \in L^\infty(R^3 \times R^3 \times (0,T))$

$$\int_{R^3} f_n \varphi \, dv = \int_{R^3} (f_n - f_n^\delta)\varphi \, d\vec{v} + \int_{R^3} (f_n^\delta - f^\delta)\varphi \, d\vec{v} + \int_{R^3} f^\delta \varphi \, d\vec{v}. \tag{6.17}$$

Then we will obtain

$$\int_{R^3} f_n \varphi \, d\vec{v} \underset{n\to\infty}{\longrightarrow} \int_{R^3} f \varphi \, d\vec{v} \qquad in\ L^1(R^3 \times (0,T)) \tag{6.18}$$

if we can show that

$$\sup_{n \ge 1} \sup_{t \in [0,T]} \|f_n - f_n^\delta\|_{L^1(R^3 \times R^3)} \underset{\delta\to 0^+}{\longrightarrow} 0. \tag{6.19}$$

It is obvious from the lower weak semicontinuity of the norm and (6.19).

$$\|f - f^{\delta}\|_{L^1(R^3 \times R^3 \times (0,T))} \leq T \sup_{t \in [0,T]} \liminf \|f_n - f_n^{\delta}\|_{L^1(R^3 \times R^3)} \longrightarrow 0 \tag{6.20}$$

Now, we prove (6.19), obviously,

$$0 \leq f_n - \frac{1}{\delta}\log(1 + \delta f_n) \leq f_n \left[\left(1 - \frac{\log(1 + \delta f_n)}{\delta f_n} \right) \chi_{\{f_n \leq R\}} \right] + f_n \chi_{\{f_n \geq R\}}. \tag{6.21}$$

Since $[(1 - \frac{\log(1+\delta f_n)}{\delta f_n})\chi_{\{f_n \leq R\}}] \overset{\delta \to 0^+}{\longrightarrow} 0$ uniformly, and

$$\sup_{n \geq 1} \sup_{t \in [0,T]} \iint_{R^3 \times R^3} f_n \chi_{\{f_n \geq R\}}\, d\vec{r}\, d\vec{v} \underset{R \to \infty}{\longrightarrow} 0, \tag{6.22}$$

we can get (6.19) easily. From the proof of (6.8), φ in (6.18) can be replaced by any φ such that

$$(1 + |\vec{r}|^k + |\vec{v}|^k)^{-1}\varphi \in L^{\infty}(R^3 \times R^3 \times (0.T)), \tag{6.23}$$

where $0 \leq k < 2$.

Now, let us prove that, for $i = E, 2, 3, 4$,

$$L_{in}^{+}(f_n) \longrightarrow L_i^{+}(f) \qquad in\ L^1(R^3 \times B_R \times (0,T)), \tag{6.24}$$

where

$$L_E^{+}(h) = a^2 \iint_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p}) g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n) h(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t) \tag{6.25}$$

$$L_2^{+}(h) = R^2 a^2 \iint_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p}) g(\vec{r}_1 . \vec{r}_1 - Ra\vec{\sigma}|n) h(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t) \tag{6.26}$$

$$L_3^{+}(h) = R^2 a^2 \iint_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon}) g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n) h(\vec{r}_1 + Ra\vec{\sigma}. \vec{v}_2, t) \tag{6.27}$$

$$L_4^{+}(h) = R^2 a^2 \iint_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p})\theta(\vec{\sigma} \cdot \vec{p})\theta(\sqrt{4\epsilon} - \vec{\sigma} \cdot \vec{p}) g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n) h(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t) \tag{6.28}$$

First, we have, by (6.18), (6.23) and the boundedness of the operator $\int\limits_{S^2} \theta(\vec\sigma\cdot\vec p)\cdot d\vec\sigma$, which is a bounded operator from $L^1(R^3\times B_R\times(0,T)\times S^2)$ to $L^1(R^3\times B_R\times(0.T))$,

$$a^2\iint\limits_{R^3\times S^2} d\vec v_2\, d\vec\sigma(\vec\sigma\cdot\vec p)\theta(\vec\sigma\cdot\vec p)g(\vec r_1,\vec r_1-a\vec\sigma|n)f_n(\vec r_1-a\vec\sigma,\vec v_2,t)$$

is compact in $L^1(R^3\times B_R\times(0.T))$. Without loss of generality, we assume

$$a^2\iint\limits_{R^3\times S^2} d\vec v_2\, d\vec\sigma(\vec\sigma\cdot\vec g)\theta(\vec\sigma\cdot\vec p)g(\vec r_1,\vec r_1-a\vec\sigma|n)f_n(\vec r_1-a\vec\sigma,\vec v_2 . t)\longrightarrow g_1^+ \tag{6.29}$$

in $L^1(R^3\times B_R\times(0,T))$ for some measurable function g_1^+ in $L^1(R^3\times B_R\times(0,T))$.

On the other hand, using the same argument as for the left side sequence in (6.9), one obtains

$$a^2\iint\limits_{R^3\times S^2} d\vec v_2\, d\vec\sigma(\vec\sigma\cdot\vec g)\theta(\vec\sigma\cdot\vec p)g(\vec r_1,\vec r_1-a\vec\sigma|n)f_n(\vec r_1-a\vec\sigma,\vec v_2,t)\underset{n\to\infty}{\longrightarrow} L_E^+(f) \tag{6.30}$$

weakly in $L^1(R^3\times B_R\times(0,T))$. Now, (6.29) and (6.30) implies

$$a^2\iint\limits_{R^3\times S^2} d\vec v_2\, d\vec\sigma(\vec\sigma\cdot\vec g)\theta(\vec\sigma\cdot\vec p)g(\vec r_1,\vec r_1-a\vec\sigma|n)f_n(\vec r_1-a\vec\sigma,\vec v_2,t)\underset{n\to\infty}{\longrightarrow} L_E^+(f) \tag{6.31}$$

in $L^1(R^3\times B_R\times(0,T))$. Therefore, changing $g(\vec r_1,\vec r_1-a\vec\sigma$ to $g_nW_nX_n$ with no complication, we claim that

$$L_{E_n}^+(f_n)\underset{n\to\infty}{\longrightarrow} L_E^+(f) \qquad \textit{in } L^1(R^3\times B_R\times(0,T)). \tag{6.32}$$

Similarly, we have

$$L_{in}^+(f_n)\underset{n\to\infty}{\longrightarrow} L_i^+(f) \qquad \textit{in } L^1(R^3\times B_R\times(0.T)) \tag{6.33}$$

for $i=2,3,4$.

(6.4). Velocity average convergence of collision kernels

One may now show

Proposition 6.7. *For all $R > 0$ and all $0 \leq \varphi \in L^\infty(R^3 \times R^3 \times (0,T))$.*

$$\int_{R^3} C^\pm_{in}(f_n)\varphi \, d\vec{v} \underset{n\to\infty}{\longrightarrow} \int_{R^3} C^\pm_i(f)\varphi \, d\vec{v} \tag{6.34}$$

in measure on $B_R \times (0,T)$. In addition, $C^\pm_i(f)(\vec{r},\cdot,t) \in L^1(R^3_v)$ a.e. in $(\vec{r},t) \in R^3 \times (0,T)$ and

$$\frac{1}{1+f} C^-_i(f) \in L^\infty((0,T); L^1(R^3 \times B_R))$$

Proof: We first check (6.34) for $C^-_{En}(f_n)$. Notice that (6.18) and (6.23) also imply that

$$\int_{R^3} f_n \psi_n \, d\vec{v} \underset{n\to\infty}{\longrightarrow} \int_{R^3} f\psi \, d\vec{v} \qquad \text{in } L^1(R^3 \times (0,T)) \tag{6.35}$$

for $\psi_n \underset{n\to\infty}{\longrightarrow} \psi$ a.e. in $(\vec{r},\vec{v},t) \in R^3 \times R^3 \times (0,T)$, and

$$\sup_{n\geq 1} \left\| \frac{\psi_n}{1+|v|} \right\|_{L^\infty(R^3 \times R^3 \times (0,T))} < \infty. \tag{6.36}$$

Because

$$\int_{R^3} f_n \psi_n \, d\vec{v} = \int_{R^3} f_n (1+|\vec{v}|) \frac{\psi_n - \psi}{1+|\vec{v}|} \, d\vec{v} + \int_{R^3} f_n \psi \, d\vec{v}, \tag{6.37}$$

by using (6.36), (6.18), (6.23) and the weak compactness of $f_n(1+|\vec{v}_1|)$, we can complete the proof of (6.35).

Next, because of (6.18) and (6.24), the sequence

$$\left\{ \frac{1}{1+L_E(f_n)} L^+_{En}(f_n) \right\}$$

with $L_E(f_n) = \underset{R^3 \times S^2}{\int\int} (1+|\vec{v}_2|) f_n(\vec{r} - a\vec{\sigma}, \vec{v}_2, t) \, d\vec{\sigma} \, d\vec{v}_2$ converges to $\frac{1}{1+L_E(f)} L^+_E(f)$ pointwise a.e. in $(\vec{r},\vec{v},t) \in R^3 \times R^3 \times (0,T)$. Thus, for any $\varphi \in L^\infty(R^3 \times R^3 \times (0,T))$, the convergence (6.35) with

$$\psi_n = \frac{1}{1+L_E(f_n)} L^+_{En}(f_n)\varphi$$

implies that

$$\frac{1}{1+L_E(f_n)}\int_{R^3} C^-_{En}(f_n)\varphi\, d\vec{v} \xrightarrow[n\to\infty]{} \frac{1}{1+L_E(f)}\int_{R^3} C^-_{E}(f)\varphi\, d\vec{v} \qquad (6.38)$$

in $L^1(R^3 \times (0,T))$. Because

$$\begin{aligned}
&||L_E(f_n) - L_E(f)||_{L(R^3\times(0,T)}\\
&= \iint d\vec{r}_1 dt| \iint\limits_{R^3\times S^2} (1+|\vec{v}_2|)f_n(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2, t)d\vec{v}_2 d\vec{\sigma} - \iint\limits_{R^3\times S^2} (1+|\vec{v}_2|)f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2, t)d\vec{v}_2 d\vec{\sigma}|\\
&= \iint d\vec{r}_1 dt| \int\limits_{S^2} d\vec{\sigma}(\int_{R^3} (1+|\vec{v}_2|)f_n(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2, t)d\vec{v}_2 - \int\limits_{R^3} (1+|\vec{v}_2|)f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2, t)d\vec{v}_2)\\
&\le \int d\vec{\sigma} \iint\limits_{R^3\times(0,T)} | \int (1+|\vec{v}_2|)f_n(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2, t)d\vec{v}_2 - \int\limits_{R^3} (1+|\vec{v}_2|)f(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2, t)d\vec{v}_2|d\vec{r}_1 dt\\
&= \int d\vec{\sigma} \iint\limits_{R^3\times(0,T)} | \int (1+|\vec{v}_2|)f_n(\vec{r}_1, \vec{v}_2, t)d\vec{v}_2 - \int\limits_{R^3} (1+|\vec{v}_2|)f(\vec{r}_1, \vec{v}_2, t)d\vec{v}_2|d\vec{r}_1 dt\\
&\to 0 \qquad (6.39)
\end{aligned}$$

by (6.35), therefore,

$$L_E(f_n) \to L_E(f), \qquad a.e. \qquad \vec{r}_1, \vec{v}_1, t. \qquad (6.40)$$

Because

$$\frac{1}{1+L_E(f_n)}\int_{R^3} C^-_{E}(f_n)\varphi\, d\vec{v} \xrightarrow[n\to\infty]{} \frac{1}{1+L_E(f)}\int_{R^3} C^-_{E}(f)\varphi\, d\vec{v} \qquad (6.38)$$

in $L^1(R^3 \times (0,T))$, we have, for any $R > 0$

$$\frac{1}{1+L_E(f_n)}\int_{R^3} C^-_{E}(f_n)\varphi\, d\vec{v} \xrightarrow[n\to\infty]{} \frac{1}{1+L_E(f)}\int_{R^3} C^-_{E}(f)\varphi\, d\vec{v} \qquad (6.41)$$

a.e in $B^R \times (0,T)$, which implies

$$(1+L_E(f_n))\frac{1}{1+L_E(f_n)}\int_{R^3} C^-_{E}(f_n)\varphi\, d\vec{v} \xrightarrow[n\to\infty]{} \int_{R^3} C^-_{E}(f)\varphi\, d\vec{v} \qquad (6.42)$$

a.e. in $B^R \times (0,T)$. So, after passing to a subsequence of $\{f_n\}$, one gets

$$\int_{R^3} C_E^-(f_n)\varphi\, d\vec{v} \underset{n\to\infty}{\longrightarrow} \int_{R^3} C_E^-(f)\varphi\, d\vec{v} \tag{6.43}$$

in measure on $B_R \times (0,T)$. This completes the proof of (6.34) for $C_E^-(f_n)$.

By the change of the variables $(\vec{v}_1, \vec{v}_2) \leftrightarrow (\vec{v}_1{}', \vec{v}_2{}')$ and $\vec{\sigma} \to -\vec{\sigma}$, one has

$$\begin{aligned}\int_{R^3} \varphi C_E^+(f_n)\, d\vec{v}_1 = \int_{R^3} f_n\Big(\iint\limits_{R^3\times S^2} & a^2\vec{\sigma}\cdot\vec{p}\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1, r - a\vec{\sigma}|n)f_n(r - a\vec{\sigma}, \vec{v}_2, t) \\ & \times \varphi(r, v_1', t)\, d\vec{v}_2\, d\vec{\sigma}\Big)\, d\vec{v}_1.\end{aligned} \tag{6.44}$$

Thus, the proof of (6.34) for $C_E^+(f_n)$ will be reduced to the case of (6.34) for $C_E^-(f_n)$ if

$$\begin{aligned}&\iint\limits_{R^3\times S^2} \varphi(\vec{r}_1, \vec{v}_1{}', t)\vec{\sigma}\cdot\vec{p}\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n)f_n(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)\, d\vec{\sigma}\, d\vec{v}_2 \\ &\longrightarrow \iint\limits_{R^3\times S^2} \vec{\sigma}\cdot\vec{p}\varphi(\vec{r}_1, \vec{v}_1{}', t)g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n)f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)\theta(\vec{\sigma}\cdot\vec{p})\, d\vec{\sigma}\, d\vec{v}_2\end{aligned} \tag{6.45}$$

in $L^1(R^3 \times B^R \times (0,T))$.

But the convergence in (6.45) is obtained easily by using arguments very similar to those in the proof of (6.24).

Finally, the convergence in $L^1(R^3 \times (0,T))$ of (6.38) and the finiteness of $L_E(f)$ a.e. in $(\vec{r}_1, t)$ imply that $C_E^{\pm}(f)(\vec{r}, \cdot, t) \in L^1(R^3_v)$ a.e. in $(r,t) \in R^3 \times (0,T)$. Since $f \geq 0$, we have

$$\frac{1}{1+f} C_E^-(f) \in L^\infty((0,T); L^1(R^3 \times B_R)). \tag{6.46}$$

Note

$$\frac{1}{1+f} C_E^-(f) = \frac{f}{1+f} a^2 \iint\limits_{R^3\times S^2} d\vec{v}_2 d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n)f(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t).$$

For $C_i^{\pm}$, $i = 2, 3, 4$, by using the changes of $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1'', \vec{v}_2'')$ and $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1''', \vec{v}_2''')$, similarly, we can show the analogous results. In those cases,

$$L_2(f_n) = \iint\limits_{R^3 \times S^2} (1 + |\vec{v}_2|) f_n(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)\, d\vec{\sigma}\, d\vec{v}_2 \tag{6.47}$$

$$L_3(f_n) = \iint\limits_{R^3 \times S^2} (1 + |\vec{v}_2|) f_n(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)\, d\vec{\sigma}\, d\vec{v}_2 \tag{6.48}$$

$$L_4(f_n) = \iint\limits_{R^3 \times S^2} (1 + |\vec{v}_2|) f_n(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)\, d\vec{\sigma}\, d\vec{v}_2 \tag{6.49}$$

§VII. Existence of Mild Solutions

In this section, we first prove two inequalities, and then we will demonstrate the existence of mild solutions of (1.2).

Lemma 7.1. *For all* $0 \le s \le t \le T$,

$$f^{\#}(\vec{r},\vec{v},t) - f^{\#}(\vec{r},\vec{v},s)\exp\left\{-\sum_{i=E}^{4}(F_i^{\#}(t) - F_i^{\#}(s))\right\} \ge$$

$$\ge \int_s^t d\tau \sum_{i=E}^{4} C_i^{+}(\tau,f)^{\#}(\vec{r},\vec{v},\tau)\exp\left\{-\sum_{i=E}^{4}(F_i^{\#}(t) - F_i^{\#}(\tau))\right\} \tag{7.1}$$

a.e. in $(\vec{r},\vec{v}) \in R^3 \times R^3$, *where* $F_i^{\#}(\vec{r},\vec{v},t) = \int_0^t L_i^{+}(f)(\vec{r},\vec{v},\tau)\,d\tau$.

Proof: We already have, for any $0 \le s \le t \le T$ and a.e. in $(\vec{r},\vec{v}) \in R^3 \times R^3$,

$$f_n^{\#}(\vec{r},\vec{v},t) - f_n^{\#}(\vec{r},\vec{v},s)\exp\left\{-\sum_{i=E}^{4}(F_{in}^{\#}(t) - F_{in}^{\#}(s))\right\} =$$

$$= \int_s^t d\tau \sum_{i=E}^{4} C_{in}^{+}(\tau,f_n)^{\#}(\vec{r},\vec{v},\tau)\exp\left\{-\sum_{i=E}^{4}(F_{in}^{\#}(t) - F_{in}^{\#}(\tau))\right\}. \tag{7.2}$$

Since

$$\iint \left| \int_0^t L_{in}^{+}(f_n)^{\#}(\vec{r}_1,\vec{v}_1,t)d\tau - \int_0^t L_i^{+}(f)^{\#}(\vec{r}_1,\vec{v}_1,t)d\tau \right| d\vec{r}_1\vec{v}_1$$

$$\le \int_0^t \iint |L_{in}^{+}(f_n)^{\#} - L_i^{+}(f)^{\#}|\,d\vec{r}_1 d\vec{v}_1, \tag{7.3}$$

we see that the sequence $F_{in}^{\#}(\vec{r},\vec{v},t) = \int_0^t L_i^{+}(f_n)^{\#}(\vec{r},\vec{v},\tau)\,d\tau$ converges to

$$F_i^{\#}(r,v,t) = \int_0^t L_i^{+}(f)^{\#}(\vec{r},\vec{v},\tau)\,d\tau$$

in $C([0,T], L^1_{loc}(R^3 \times R^3))$. Because

$$\exp\left\{-\sum_{i=E}^{4}(F_{in}^{\#}(t) - F_{in}^{\#}(s))\right\} \le 1 \tag{7.4}$$

and

$$\exp\left\{-\sum_{i=E}^{4}(F_{in}^{\#}(t) - F_{in}^{\#}(s))\right\} \xrightarrow[n\to\infty]{} \exp\left\{-\sum_{i=E}^{4}(F_i^{\#}(t) - F_i^{\#}(s))\right\} \tag{7.5}$$

in $L^1_{loc}(R^3 \times R^3)$, uniformly in $0 \le s \le t \le T$, in order to prove (7.1), it is enough to show that for all $0 \le \varphi \in L^\infty(R^3 \times R^3 \times [0,T] \times [0,T])$,

$$\int_0^T \int_0^T \iint_{R^3 \times R^3} \varphi \left(\int_s^t \sum_{i=E}^4 C_i^+(f)^\#(\vec{r},\vec{v},\tau) \exp\left\{ -\sum_{i=E}^4 (F_i^\#(t) - F_i^\#(\tau)) \right\} d\tau \right) d\vec{r}\, d\vec{v}\, ds\, dt$$

$$\le \liminf_{n\to\infty} \int_0^T \int_0^T \iint_{R^3 \times R^3} \varphi Z(\vec{r},\vec{v},s,t) d\vec{r}\, d\vec{v}\, ds\, dt, \tag{7.6}$$

where $Z(\vec{r},\vec{v},s,t) = \int_s^t \sum_{i=E}^4 C_{in}^+(f_n)^\#(\vec{r},\vec{v},\tau) \exp\left\{ -\sum_{i=E}^4 (F_{in}^\#(t) - F_{in}^\#(\tau)) \right\} d\tau$

By a change of integration variables, we have

$$\iint_{R^3 \times R^3} \varphi \left(\int_s^t \sum_{i=E}^4 C_{in}^+(f_n)^\#(\vec{r},\vec{v},\tau) \exp\left\{ -\sum_{i=E}^4 (F_{in}^\#(t) - F_{in}^\#(\tau)) \right\} d\tau \right) dr\, dv$$

$$= \int_s^t \iint_{R^3 \times R^3} \varphi(\vec{r}_1 - \vec{v}_1 \tau, \vec{v}_1, s, t) \sum_{i=E}^4 C_{in}^+(f_n)(\vec{r}_1 + \vec{v}_1(t-\tau), \vec{v}_1, t)$$

$$\times \exp\left\{ -\sum_{i=E}^4 (F_{in}(\vec{r}_1 + \vec{v}_1(t-\tau), \vec{v}_1, t) - F_{in}(\vec{r}_1, \vec{v}_1, \tau)) \right\} d\vec{r}_1\, d\vec{v}_1\, d\tau$$

$$= \int_s^t \iint_{R^3 \times R^3} \varphi(\vec{r} - \vec{v}\tau, \vec{v}, s, t) \sum_{i=E}^4 C_i^+(f_n) G_n(\vec{r},\vec{v},\tau,t)\, d\vec{r}\, d\vec{v}\, d\tau \tag{7.7}$$

where

$$G_n(\vec{r},\vec{v},\tau,t) = \exp\left\{ -\sum_{i=E}^4 (F_{in}(\vec{r} + \vec{v}(t-\tau), \vec{v}, t) - F_{in}(\vec{r},\vec{v},\tau)) \right\}, \tag{7.8}$$

$0 \le G_n \le 1$, and $G_n \to G$ pointwise a.e. in $\vec{r},\vec{v},\tau,t$ with

$$G(\vec{r},\vec{v},\tau,t) = \exp\left\{ -\sum_{i=E}^4 (F_i(\vec{r} + \vec{v}(t-\tau), \vec{v}, t) - F_i(\vec{r},\vec{v},\tau)) \right\}. \tag{7.9}$$

We consider the sequence $f_n^R = \min\{f_n, R\}$ for $0 < R < \infty$. Since $f_n^R \le f_n$, $\{f_n^R\}$ is weakly compact in $L^1(R^3 \times R^3 \times (0,T))$. Then, since $0 \le f_n - f_n^R \le f_n \chi_{\{f_n \ge R\}}$, $\mu(f_n \ge R) = \iint_{f_n \ge R} d\vec{r} d\vec{v} \le \iint \frac{f_n}{R} d\vec{r} d\vec{v} \le \frac{1}{R} \iint f_n d\vec{r} d\vec{v}$ and $\{f_n\}$ is weakly compact, one has

$$\sup_{n \ge 1} \sup_{t \in [0,T]} \left\| f_n - f_n^R \right\|_{L^1(R^3 \times R^3)} \xrightarrow[R \to \infty]{} 0. \tag{7.10}$$

Let f^R be the weak limit of the sequence $\{f_n^R\}$. From the lower weak semicontinuity of the norm and (7.10), with the same argument as in (6.20) and (6.23), we have that

$$||f - f^R||_{L^1(R^3 \times R^3 \times (0,T))} \le T \sup_{t \in (0,T)} \liminf_{n \to \infty} ||f_n - f_n^R||_{L^1(R^3 \times R^3)} \to 0 \tag{7.11}$$

i.e. $f^R \xrightarrow[R \to \infty]{} f$ in $L^\infty((0,T); L^1(R^3 \times R^3))$, where f^R is the weak limit of the sequence $\{f_n^R\}$.

Define

$$C_{En}^R(f_n) = a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) w_R(\vec{v}_1, \vec{v}_2) g_n(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}|n)$$
$$\times X_R(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}) f_n^R(\vec{r}_1, \vec{v}_1', t) f_n(\vec{r}_1 + a\vec{\sigma}, v_2', t) \tag{7.12}$$

$$C_{2n}^R(f_n) = R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) w_R(\vec{v}_1'', \vec{v}_2'') g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n) X_R(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})$$
$$\times f_n^R(\vec{r}_1, \vec{v}_1'', t) f_n(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2'', t) \tag{7.13}$$

$$C_{3n}^R(f_n) = R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p} - \sqrt{4\epsilon}) w_R(\vec{v}_1''', \vec{v}_2''') g_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n)$$
$$\times X_R(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}) f_n^R(\vec{r}_1, \vec{v}_1''', t) f_n(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2''', t) \tag{7.14}$$

$$C_{4n}^R(f_n) = R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma} (\vec{\sigma} \cdot \vec{p}) \theta(\vec{\sigma} \cdot \vec{p}) \theta(\sqrt{4\epsilon} - \vec{\sigma} \cdot \vec{p}) w_R(\vec{v}_1, \vec{v}_2) g_n(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n)$$
$$\times X_R(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}) f_n^R(\vec{r}_1, \vec{v}_1', t) f_n(\vec{r}_1 - a\vec{\sigma}, v', t) \tag{7.15}$$

Similarly, define $\tilde{C}_i^R(f)$ with f_n^R in $C_{in}^R(f_n)$ replaced by f^R.

We claim that for a fixed $R < \infty$, $0 \le s \le t \le T$, and $0 \le \varphi \in L^\infty(R^3 \times R^3 \times (0,T) \times (0,T))$ with $\tilde{\varphi} = \varphi(\vec{r}_1 - \vec{v}_1 \tau, \vec{v}_1, s, t)$, one has for $i = E, 2, 3, 4$,

$$\lim_{n \to \infty} \int_s^t \iint\limits_{R^3 \times R^3} \tilde{\varphi} C_{in}^R(f_n) G_n \, d\vec{r}\, d\vec{v}_1\, d\tau = \int_s^t \iint\limits_{R^3 \times R^3} \tilde{\varphi} \tilde{C}_i^R(f) G \, d\vec{r}\, d\vec{v}_1\, d\tau \tag{7.16}$$

Indeed, the change of variables $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1', \vec{v}_2')$ and $\vec{\sigma} \to -\vec{\sigma}$, $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1'', \vec{v}_2'')$ and $(\vec{v}_1, \vec{v}_2) \to (\vec{v}_1''', \vec{v}_2''')$, respectively, combined with similar arguments as in the proof of (6.24), leads to the following convergence for $0 \leq s \leq t \leq T$:

$$a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})w_R(\vec{v}_1, \vec{v}_2)g_n(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n)X_R(\vec{r}_1, \vec{r}_1 - a\vec{\sigma})$$

$$\times f_n(\vec{r} - a\vec{\sigma}, \vec{v}_2, t)G_n(\vec{r}, \vec{v}_1', s, t)\varphi(\vec{r} - \vec{v}_1'\tau, \vec{v}_1', s, t)$$

$$\xrightarrow[n\to\infty]{} a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})w_R(\vec{v}_1, \vec{v}_2)g(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}|n)X_R(\vec{r}_1, \vec{r}_1 - a\vec{\sigma})$$

$$\times f(\vec{r}_1 - a\vec{\sigma}, v_2, t)G(\vec{r}_1, v_1', s, t)\varphi(\vec{r}_1 - \vec{v}_1'\tau, \vec{v}_1', s, t) \tag{7.17}$$

$$R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})w_R(\vec{v}_1, \vec{v}_2)g_n(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n)X_R(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma})$$

$$\times f_n(\vec{r} - Ra\vec{\sigma}, \vec{v}_2, t)G_n(\vec{r}, \vec{v}_1'', s, t)\varphi(\vec{r} - \vec{v}_1''\tau, \vec{v}_1'', s, t)$$

$$\xrightarrow[n\to\infty]{} R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})w_R(\vec{v}_1, \vec{v}_2)g(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma}|n)X_R(\vec{r}_1, \vec{r}_1 - Ra\vec{\sigma})$$

$$\times f(\vec{r} - Ra\vec{\sigma}, \vec{v}_2, t)G(\vec{r}, \vec{v}_1''', s, t)\varphi(\vec{r} - \vec{v}_1'''\tau, \vec{v}_1''', s, t) \tag{7.18}$$

$$R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p} - \sqrt{4\epsilon})w_R(\vec{v}_1, \vec{v}_2)g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma} : n)X_R(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})$$

$$\times f_n(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)G_n(\vec{r}_1, \vec{v}_1''', s, t)\varphi(\vec{r}_1 - \vec{v}_1'''\tau, \vec{v}_1''', s, t)$$

$$\xrightarrow[n\to\infty]{} R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p} - \sqrt{4\epsilon})w_R(\vec{v}_1, \vec{v}_2)g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma} : n)X_R(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})$$

$$\times f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)G(\vec{r}_1, v_1''', s, t)\varphi(r - v_1'''\tau, v_1''', s, t) \tag{7.19}$$

$$R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon} - \vec{\sigma}\cdot\vec{p})w_R(\vec{v}_1, \vec{v}_2)g_n(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n)X_R(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})$$

$$\times f_n(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)G_n(\vec{r}_1, \vec{v}_1', s, t)\varphi(\vec{r}_1 - \vec{v}_1'\tau, \vec{v}_1', s, t)$$

$$\xrightarrow[n\to\infty]{} R^2 a^2 \iint\limits_{R^3 \times S^2} d\vec{v}_2\, d\vec{\sigma}(\vec{\sigma}\cdot\vec{p})\theta(\vec{\sigma}\cdot\vec{p})\theta(\sqrt{4\epsilon} - \vec{\sigma}\cdot\vec{p})w_R(\vec{v}_1, \vec{v}_2)g(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma}|n)X_R(\vec{r}_1, \vec{r}_1 + Ra\vec{\sigma})$$

$$\times f(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)G(\vec{r}_1, \vec{v}_1', s, t)\varphi(\vec{r}_1 - \vec{v}_1'\tau, v_1', s, t) \tag{7.20}$$

in $L^1(R^3 \times R^3 \times (s,t))$. Since $f_n^R \underset{n\to\infty}{\longrightarrow} f^R$ weakly in $L^1(R^3 \times R^3 \times (0,T))$, we obtain the limit in (7.16) from (ii) of the Lemma 6.1.

For a fixed $R < \infty$ and $0 \leq s \leq t \leq T$, $i = E, 2, 3, 4$

$$\begin{aligned} &\liminf_{n\to\infty} \int_s^t \iint\limits_{R^3 \times R^3} \tilde{\varphi} C_{in}^+(f_n) G_n \, d\vec{r}_1 \, d\vec{v}_1 \, d\tau \\ &\geq \lim_{n\to\infty} \int_s^t \iint\limits_{R^3 \times R^3} \tilde{\varphi} C_{in}^R G_n \, d\vec{r}_1 \, d\vec{v}_1 \, d\tau \\ &= \int_s^t \iint\limits_{R^3 \times R^3} \tilde{\varphi} \tilde{C}_i^R(f) G \, d\vec{r}_1 \, d\vec{v}_1 \, d\tau \qquad (7.21) \\ &\to \int_s^t \iint\limits_{R^3 \times R^3} \tilde{\varphi} C_i^+(f) G \, d\vec{r}_1 \, d\vec{v}_1 \, d\tau, \qquad (7.22) \end{aligned}$$

since $f^R \uparrow f$ and $\tilde{C}_i^R(f) \uparrow C_i^+(f)$ pointwise a.e. as $R \uparrow \infty$, the monotone convergence theorem, integration with respect to t and s, and Fatou's lemma complete the proof of (7.6) by combining the four terms related to $i = E, 2, 3, 4$.

Lemma 7.2. *For any $T > 0$ and a.e. in $(\vec{r}, \vec{v}) \in R^3 \times R^3$, $C_i^+(f) \in L^1(0,T)$.*

Proof: Obviously,

$$\int_0^T C_i^+(f)^\# \, ds \leq \exp\left[\sum_{i=E}^{4} F_i^\#(T)\right] \int_0^T C_i^+(f)^\# \exp\left\{-\sum_{i=E}^{4}(F_i^\#(T) - F_i^\#(s))\right\} \qquad (7.23)$$

But $F_i^\#(s) \leq F_i^\#(t)$ a.e. in $(\vec{r}, \vec{v}) \in R^3 \times R^3$ and for $0 \leq s \leq t \leq T$. Since $F_i^\#(T) \in L^1(R^3 \times B_R \times (0,T))$ for any $R > 0$, inequality (7.1) completes the proof.

lemma 7.3. *For all $0 \leq s \leq t \leq T$ and a.e. in $(\vec{r}, \vec{v}) \in R^3 \times R^3$, f satisfies the inequality*

$$f^\#(r,v,t) - f^\#(r,v,s) \leq \int_s^t \sum_{i=E}^{4} C_i(f)^\#(\vec{r}, \vec{v}, \tau) \, d\tau. \qquad (7.24)$$

Proof: We have already seen that for all $0 \leq s \leq t \leq T$ and a.e. in $(\vec{r}, \vec{v}) \in R^3 \times R^3$,

f_n^δ satisfies

$$f_n^\delta(\vec{r},\vec{v},t)^\# - f_n^\delta(\vec{r},\vec{v},s)^\# = \int_s^t \sum_{i=E}^{4} \left(\left[\frac{C_{in}^+(f_n)}{1+\delta f_n}\right]^\# - \left[\frac{f_n}{1+\delta f_n}\right]^\# L_{in}^+(f_n)^\# \right) d\tau \quad (7.25)$$

Using (6.24) and (ii) of lemma (7.2), the weak convergence of f_n^δ to f^δ, and the weak convergence of $C_{in}^+(f_n)/(1+\delta f_n)$ and $h_n^\delta = f_n/(1+\delta f_n)$ to some $C_{i\delta}^+$ and h^δ, respectively, we obtain for $0 \le s \le t \le T$ and a.e. in $(\vec{r},\vec{v}) \in R^3 \times R^3$,

$$f^{\delta\#}(\vec{r},\vec{v},t) - f^{\delta\#}(\vec{r},\vec{v},s) = \int_s^t \sum_{i=E}^{4} \left[C_{i\delta}^{+\#} - h^{\delta\#} L_i^+(f)^\#\right] d\tau, \qquad (7.26)$$

By (6.19),(6.20) and (6.23), $f^{\delta\#}$ converges to $f^\#$ in $L^1(R^3 \times R^3)$ uniformly in $t \in [0,T]$. Now, since

$$0 \le z - \frac{z}{1+\delta z} \le \delta z R + z\chi_{\{z \ge R\}}$$

and $\{f_n\}$ is weakly compact, one also has, due to the lower weak semicontinuity of the norm,

$$\sup_{t\in[0,T]} \|f - h^\delta\|_{L^1(R^3\times R^3)} \le \sup_{t\in[0,T]} \liminf_{n\to\infty} \|f_n - h_n^\delta\|_{L^1(R^3\times R^3)} \xrightarrow{\delta\to 0^+} 0 \qquad (7.27)$$

Furthermore, $h^\delta \uparrow f$ as $\delta \downarrow 0^+$, and (7.24) follows from (7.26) by letting $\delta \to 0^+$ and using the monotone convergence theorem, if it can be shown that $C_{i\delta}^+ \le C_i^+(f)$ a.e. in $(\vec{r},\vec{v},t) \in R^3 \times R^3 \times (0,T)$. The proof follows from (6.34). Indeed, after passing to a subsequence if necessary, one has, for $0 \le \psi \in L^\infty(R^3 \times R^3 \times (0,T))$, in the case that C_i^- were replaced by C_i^+,

$$\frac{\int_{R^3} C_{in}^+(f_n)\varphi \, d\vec{v}_1}{1+L_i(f_n)} \xrightarrow[n\to\infty]{} \frac{\int_{R^3} C_i^+(f)\varphi \, d\vec{v}_1}{1+L_i(f)}$$

a.e. in $(r,v,t) \in R^3 \times R^3 \times (0,T)$. Since

$$\frac{\int_{R^3} C_{in}^+(f_n)\varphi \, dv}{1+L_i(f_n)} \le \|\varphi\|_{L^\infty(R^3\times R^3\times(0,T))} M_g' \int_{R^3} (1+|v|) f_n \, dv \qquad (7.28)$$

the sequence $\left\{\frac{\int_{R^3} C_{in}^+(f_n)\varphi\, dv}{1+L_i(f_n)}\right\}$ is weakly compact in $L^1(R^3 \times (0,T))$. Therefore,

$$\begin{aligned}
&\iiint\limits_{R^3\times R^3\times(0,T)} \frac{C_{i\delta}^+\varphi}{1+L_i(f)}\, d\vec{r}\, d\vec{v}_1\, dt \\
&\leq \liminf_{n\to\infty} \iiint\limits_{R^3\times R^3\times(0,T)} \frac{C_{in}^+(f_n)\varphi}{1+L_i(f_n)}\, d\vec{r}\, d\vec{v}\, dt \\
&= \iiint\limits_{R^3\times R^3\times(0,T)} \frac{C_i^+(f)\varphi}{1+L_i(f)}\, dr\, dv\, dt. \qquad (7.29)
\end{aligned}$$

Since $0 \leq \varphi \in L^\infty(R^3 \times R^3 \times (0,T))$ is arbitrary,

$$\frac{C_{i\delta}^+}{1+L_i(f)} \leq \frac{C_i^+(f)}{1+L_i(f)} \qquad (7.30)$$

a.e. in $(r,v,t) \in R^3 \times R^3 \times (0,T)$. Hence, $C_{i\delta}^+ \leq C_i^+(f)$. a.e. in $(r,v,t) \in R^3 \times R^3 \times (0,T)$. This completes the proof of (7.24).

We note that by (7.23), $C_i^+(f)^\# \in L^1(0,T)$, a.e. in $(r,v) \in R^3 \times R^3$. Thus (7.24) implies that $C_i^-(f)^\# \in L^1(0,T)$, a.e. in $(\vec{r},\vec{v}) \in R^3 \times R^3$.

We now come to the main result of this paper. The square-well kinetic equation (1.2) with initial condition f_0 satisfying (1.5) has a mild solution in $L^1(R^3 \times R^3 \times [0,T])$ for all $T > 0$.

Theorem 7.4. *The weak limit f of the sequence $\{f_n\}$ is a mild solution of (1.2) satisfying*

$$\iint\limits_{R^3\times S^2} (1+|\vec{r}_1|^2+|\vec{v}_1|^2+|\log f|)f(\vec{r}_1,\vec{v}_1,t)d\vec{r}_1 d\vec{v}_1 \leq C < \infty, \qquad 0 < t < T. \qquad (7.35)$$

Proof: We shall show that the weak limit f of the sequence $\{f_n\}$ is a mild solution of (1.2). From the equality $F_i^\#(r,v,t) = \int_0^t L_i^+(f)^\#(\vec{r},\vec{v},\tau)\, d\tau$, one has $F_i^\#(t)$ is absolutely continuous for almost all $\vec{r}$, $\vec{v}$, and

$$\frac{dF_i^\#}{dt} = L_i^+(f)^\# \qquad a.e.\ in\ t. \qquad (7.31)$$

By (7.1) and (7.24), $f^{\#}$ is absolutely continuous in t for almost all $\vec{r}$, $\vec{v}$. Therefore, $f^{\#}\exp\{\sum_{i=E}^{4} F_i^{\#}\}$ is absolutely continuous in t for almost all $\vec{r}$, $\vec{v}$. (7.1) implies that

$$\frac{d}{dt}\{f^{\#}\exp\{\sum_{i=E}^{4} F_i^{\#}\}\} \geq \sum_{i=E}^{4} C_i^{+}(f)^{\#}\exp\{\sum_{i=E}^{4} F_i^{\#}\} \qquad a.e.\ in\ t, \tag{7.32}$$

for almost all $\vec{r}$, $\vec{v}$. Thus,

$$\frac{d}{dt}f^{\#} \geq \sum_{i=E}^{4} C_i(f)^{\#} \qquad a.e.\ in\ t \tag{7.33}$$

and

$$f^{\#}(t) - f^{\#}(s) \geq \int_s^t \sum_{i=E}^{4} C_i(f)^{\#}\, d\tau \qquad for\ 0 \leq s \leq t \tag{7.34}$$

for almost all $\vec{r}, \vec{v}$.

Now (7.24) and (7.34) complete the proof of the theorem.

We conclude with some observations on extensions of these results. First, let us note that the arguments presented herein apply as well to the Cauchy problem for the square shoulder kinetic equation. defined by the intermolecular potential (2.16) with $\epsilon > 0$, and given explicitly by (2.17). Indeed, some of the complications related to non-conservation of energy are simpler in this case.

It is also clear that for a step function potential (piecewise constant), the arguments and the results herein can also be extended. In this manner, one might consider a piecewise constant approximation to a realistic (finite range) van der Waals force. For the case of a piecewise step function potential, the kinetic equation becomes

$$\frac{\partial}{\partial t}f + \vec{v}_1 \cdot \nabla_{\vec{r}_1} f = E(f,f) + \sum_{i=1}^{n} Q_i(f,f) \tag{7.35}$$

where E is the Enskog term, and Q_i, $i = 1, 2, \ldots, n$, is either a square-well term or a square-shoulder term. Our previous arguments show that, under the same assumptions as for the square-well or square-shoulder kinetic equation, there exists a mild solution for the piecewise step potential equation (7.35).

It would be interesting to know if the existence of solutions to the Cauchy problem can be proved for the square well kinetic equation under the assumption $g \equiv 1$. Although this is the dilute gas limit, it is also the first term in the Mayer series expansion for the correlation function g. Unfortunately, we do not know at this time how to treat the case $g \equiv 1$.

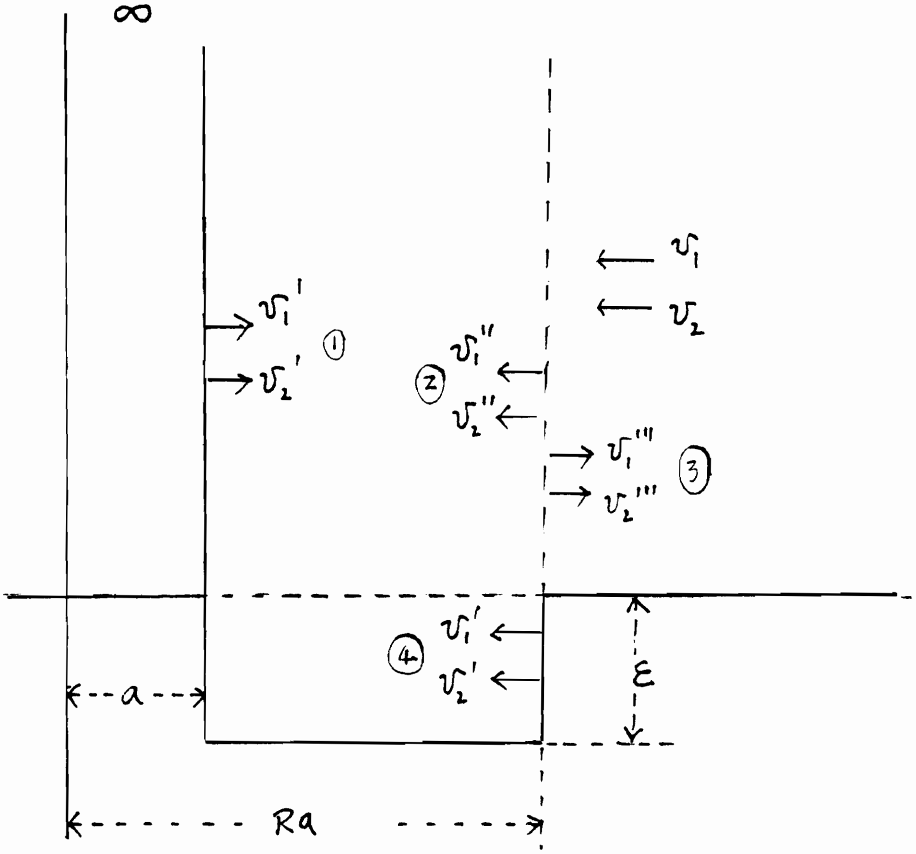

Fig. 1.1 Collision Configuration for the Square Well Potential

1 Hard core collision
2 Entering collision
3 Escaping collision
4 Bound state collision

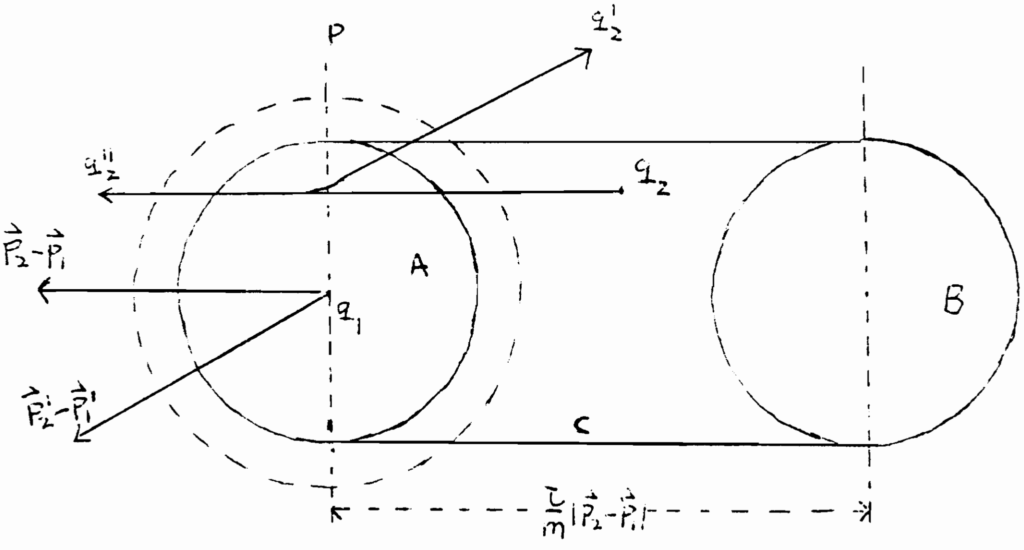

Fig. 2.1 Configuration Space for a Binary Collision

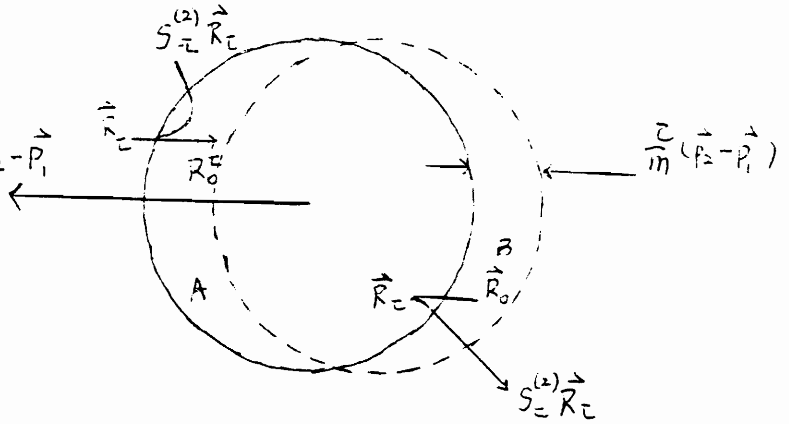

Fig. 2.2 Precollision Configuration for the Repulsive Power Law Potential

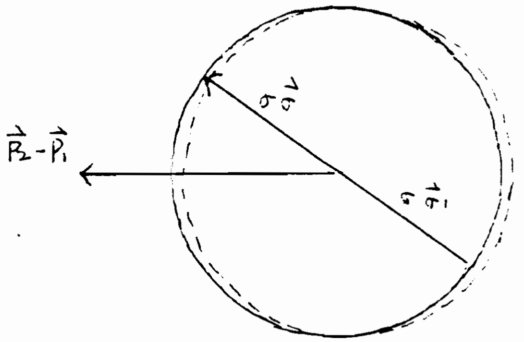

Fig. 2.3 Geometry of Precollision Configuration in Hard-Sphere Limit

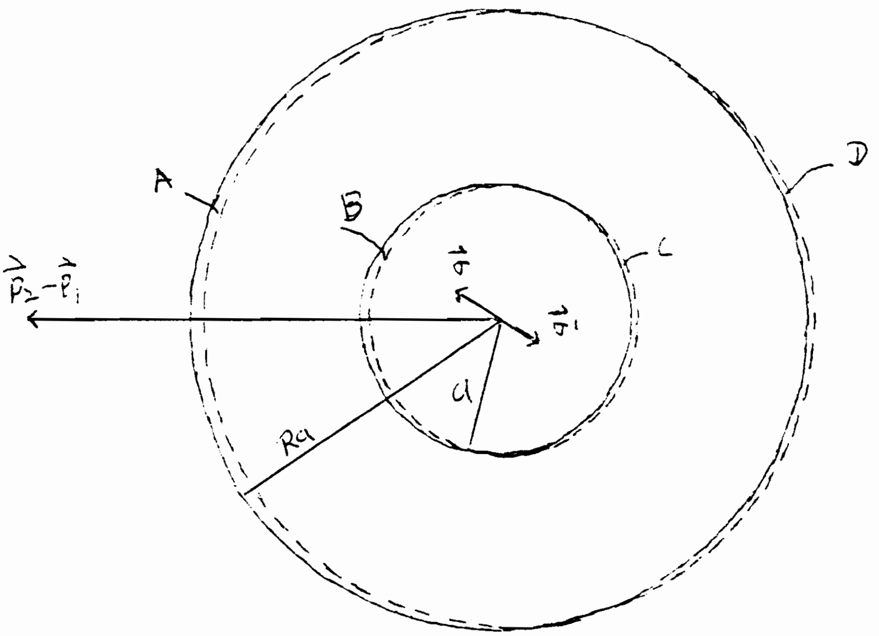

Fig. 2.4 Geometry of Precollision Configuration for the Square Well Potential

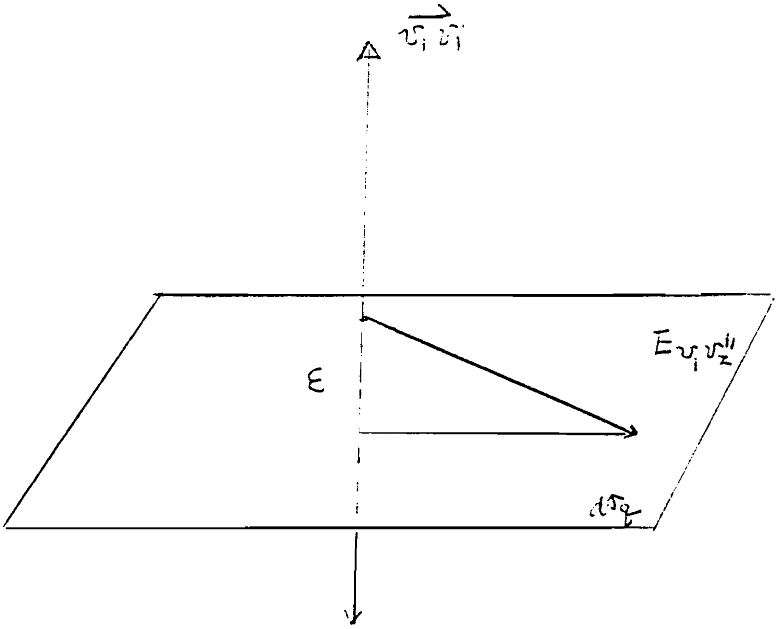

Fig. 5.1 Configuration for integral area of $d\vec{v}''_{2x} d\vec{v}''_{2y}$

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PUBLICATIONS

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