Methods for Computing Functional Gains for LQR Control of Partial Differential Equations

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(ABSTRACT)

This work focuses on a comparison of numerical methods for linear quadratic regulator (LQR) problems defined by parabolic partial differential equations. In particular, we study various methods for computing functional gains to boundary control problems for the heat equation. These methods require us to solve various equations including the algebraic Riccati equation, the Riccati partial differential equation and the Chandrasekhar partial differential equations. Numerical results are presented for control of a one-dimensional and a two-dimensional heat equation with Dirichlet or Robin's boundary controls.

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Chapter 1

Introduction

During the past twenty-five years considerable effort has been devoted to distributed parameter control, i.e., control of systems governed by partial differential equations. Feedback design via linear quadratic control is an established methodology with numerous applications. Linear quadratic design is understood for finite-dimensional systems and computational tools are now available for reasonably large systems. Although theoretical results for infinite-dimensional systems are well documented, the development of computational tools that can handle problems with two and three-dimensional spatial domains has lagged behind. In this work, we focus on the development and evaluation of algorithms that help bridge the gap between these theories and practical computational tools.

We concentrate on parabolic systems with boundary control. Lions provides a classical reference [30] for the control of parabolic partial differential equations. More recently, one can find theoretical treatments regarding boundary control of parabolic systems in Balakrishnan [3], Lasiecka and Triggiani [28], Pritchard and Salamon [38], and Sorine [42].

The two-volume collection by Bensoussan, Da Prato, Delfour and Mitter [7] has an extensive list of references for general linear quadratic regulator problems and unbounded controls.

We restrict our focus to problems where the feedback law has an integral representation of the form

$$u(t,\xi) = -\int_{\Omega} h(t,\xi,\mathbf{x})w(t,\mathbf{x})d\mathbf{x}$$

where $w(t,\mathbf{x})$ is the state of the system. The function $h(t,\xi,\mathbf{x})$ (if it exists) is called the functional gain. Functional gains offer insight into issues such as sensor/actuator placement [12] and controller reduction [13]. Applications involving sensor placement and design include, but are not limited to, piezoceramic/piezoelectric sensors or magnetostrictive sensors in the field of smart materials (see [5], [29] and [34]). To be practical, one must be able to compute these functions for a wide variety of partial differential equations in two and three spatial dimensions. Standard "approximate-then-design" approaches have been very useful for small one-dimensional problems. However, in two and three-dimensional problems, the size of the approximating systems limits this method as a practical computational tool. Therefore, alternative methods are needed.

This thesis deals with the computation of functional gains for an α-shifted linear quadratic regulator (α-LQR) problem (see Burns and Kang [11]) applied to the heat equation with boundary control. In particular, we compute the feedback operators defined by the optimal control and its kernel (the functional gain) corresponding to the heat equation with Dirichlet boundary control, or Robin boundary control. We concentrate on "direct" approaches for computing functional gains and consider certain theoretical issues concerning the existence and regularity properties of these functional gains. The approaches we consider include solving (i) an operator algebraic Riccati equation, (ii) a Riccati partial differential equation and (iii) Chandrasekhar partial differential equations.

The next three chapters cover the theoretical aspect of these equations and the functional gains. In Chapter 2, we review the derivation of the three-dimensional heat equation and describe several physical systems and their associated boundary conditions. In addition, we formulate the mathematical framework for evaluating the system of equations for the boundary control problems. The α-LQR problem is introduced in Chapter 3. We discuss the operator algebraic Riccati equation and derive a Riccati partial differential equation, which characterizes the functional gains. In Chapter 4, we consider similar topics for Chandrasekhar equations. Existence and regularity of the functional gains are established and Chandrasekhar partial differential equations are derived.

In Chapters 5 and 6, we discuss and evaluate the numerical methods for approximating the functional gains via the approaches discussed in Chapters 3 and 4. In Chapter 5, we discuss finite element methods used to approximate the appropriate equations from which we compute the functional gains. We also briefly cover some important convergence issues and, in addition, we discuss the use of Nitsche's approximations for approximating functional gains for Dirichlet boundary control problems. Chapter 6 presents numerical results for one-dimensional and two-dimensional problems. We compare the accuracy and efficiency of the various methods using these numerical results and discuss the benefits and drawbacks of each method. Finally, we conclude with an overview of the new results obtained in this thesis and indicate future research directions.

Chapter 2

The Heat Equation

The diffusion equation is often used to model thermal processes, describing the distribution of temperature in a body. Many applications involve the need to accurately measure and/or control the temperature of a specific spatial domain. The control of the temperature distribution is a problem involving a cost function as well as certain constraints.

We study an α-shifted linear quadratic regulator (α-LQR) problem for the heat equation. Before discussing the α-LQR problem, we begin with an introduction to the heat equation. We review the derivation so that it is clear how and when a particular input control arises in the model. In the early 1800s, Fourier used experimental observations in his work [20] to derive the basic law of heat conduction for his analytic heat theory. In the following sections, we review the derivation of the heat equation based on this law and formulate the appropriate boundary value problem. Although this material is well known, it is worthwhile to summarize the results so that the correct "physics" matches the boundary control problem formulated below.

2.1 Derivation of the Heat Equation

In this section, we review the derivation of the heat equation in three space dimensions, which can then be reduced to one or two dimensions. We refer the reader to the standard references [8, 19, 22, 36].

Consider an infinitesimal block as illustrated in Figure 2.1 with lengths Δx, Δy, Δz and where F_x is the heat flow into the volume in the x-direction. The rate of heat flow into the volume at the surface at x_0 is

$$F_x \equiv f_x(t, x_0, y, z)\Delta y\Delta z,$$

where $f_x(t, x_0, y, z)$ is the heat flux in the x-direction. The heat flux is defined to be the rate of transfer of energy across a given surface. The equation for the heat flux is dependent on the physical phenomenon being studied. The rate of heat flow out of the surface at $x_0+\Delta x$ is then

$$F_x + \frac{\partial F_x}{\partial x}\Delta x.$$

Similarly, the rates of flow into and out of the volume at the surfaces y_0 and z_0 are given by

$$F_y \equiv f_y(t, x, y_0, z)\Delta x\Delta z, \quad F_y + \frac{\partial F_y}{\partial y}\Delta y$$

and

$$F_z \equiv f_z(t, x, y, z_0)\Delta x\Delta y, \quad F_z + \frac{\partial F_z}{\partial z}\Delta z,$$

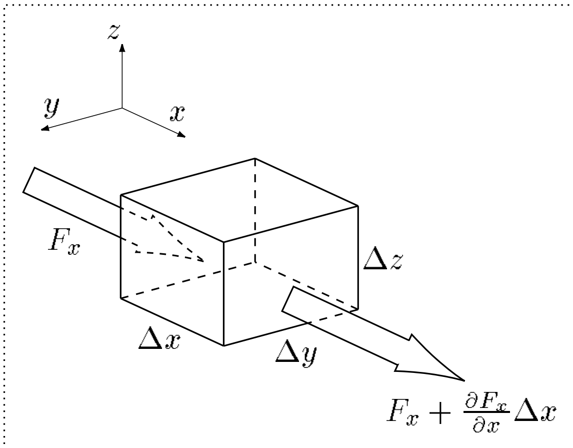

Figure 2.1: Rate of Heat Flow in x-Direction

respectively.

By conservation of energy, the amount of heat that enters the volume and is generated inside the volume must equal the amount of increase in energy in the element. The equation for energy balance in the infinitesimal volume element, $\Delta x \Delta y \Delta z$, is written as

$$I_1 + G = I_2,$$

where I_1 is the net rate of heat entering into the element, G is the rate of energy generated inside the element, and I_2 is the rate of increase of internal energy of the element.

To obtain I_1, we study the heat flows into and out of the block. The net rate of heat entering the volume in the x-direction is the difference between the entering and leaving rates of heat flow. Therefore, it follows that the net rate of heat entering in the x-direction is given by

$$-\frac{\partial F_x}{\partial x}\Delta x = -\frac{\partial}{\partial x} f_x(t, x_0, y, z)\Delta x \Delta y \Delta z.$$

Similarly, the net rates of heat entering the volume in the y-direction and the z-direction are given by

$$-\frac{\partial}{\partial y} f_y(t, x, y_0, z)\Delta x \Delta y \Delta z$$

and

$$-\frac{\partial}{\partial z} f_z(t, x, y, z_0)\Delta x \Delta y \Delta z.$$

Thus, the total rate of energy entering the element is

$$I_1 = -\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right)\Delta x \Delta y \Delta z.$$

The term G depends on sources of heat generation inside the block. If there are distributed energy sources generating heat in the medium, then the rate of energy generation in the element is given by

$$G \equiv g(t, x, y, z)\Delta x \Delta y \Delta z,$$

where $g(t, x, y, z)$ is a function representing the rate of internal heat generation per unit time per unit volume. If there are no internal heat sources, then $G = 0$.

To obtain the rate of increase of heat energy in the volume, we first define $T(t, x, y, z)$ to be the temperature function. The rate of increase in temperature at each point is given by

$$\frac{\partial}{\partial t} T(t, x, y, z). \tag{2.1}$$

Hence, the rate of change of internal energy in the infinitesimal volume is given by

$$I_2 \equiv \rho s \frac{\partial T}{\partial t}\Delta x \Delta y \Delta z$$

where ρ is the density of the medium and s is the ratio of specific heat to constant pressure in the material.

Making the requisite substitutions for I_1, G and I_2, we obtain the energy balance equation:

$$-\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) + g \;=\; \rho s \frac{\partial T}{\partial t}.$$

To complete the heat equation, we need to obtain an accurate representation for the heat flux. The mathematics that describe the flux of the domain depends on the process we are studying.

In classic physics, thermal processes in certain materials can be described by Newton's Law of Cooling. Newton's Law of Cooling states that the average amount of heat that passes through a small distance is proportional to the difference of the temperatures at the two ends. The relation is given as

$$\text{amount of heat per unit time} \;=\; \kappa \frac{|T_1 - T_0|}{\delta},$$

where δ is the distance between the two ends with temperatures, T_0 and T_1, and κ is the thermal conductivity function. The thermal conductivity κ is usually a function of the spatial variables x, y and z. Newton's Law of Cooling also states that the heat flow will pass from warm areas to cool areas.

Using Newton's Law of Cooling, the heat flux $f_x(t, x_0, y, z)$ is the instantaneous rate of energy transfer from left to right across x_0 and hence

$$\begin{aligned} f_x(t, x_0, y, z) &= -\lim_{\delta \to 0} \frac{\kappa(x_0,y,z)\Big(T(t,x_0+\delta,y,z)-T(t,x_0-\delta,y,z)\Big)}{\delta} \\ &= -\kappa(x_0, y, z)\frac{\partial}{\partial x}T(t, x_0, y, z). \end{aligned}$$

The negative sign indicates that a positive flow of heat occurs from left to right only if the temperature is greater to the left of x_0 than to the right. That is, $\partial T/\partial x$ would be negative and hence $f_x(t, x_0, y, z)$ is positive. Similarly, we obtain heat fluxes in the other directions across y_0 and z_0. These fluxes are given by

$$f_y(t, x, y_0, z) = -\kappa(x, y_0, z)\frac{\partial}{\partial y}T(t, x, y_0, z)$$

and

$$f_z(t, x, y, z_0) = -\kappa(x, y, z_0)\frac{\partial}{\partial z}T(t, x, y, z_0),$$

respectively.

Substituting the flux terms into the energy balance equation, we obtain the equation

$$\frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\kappa\frac{\partial T}{\partial z}\right) + g = \rho s\frac{\partial T}{\partial t}, \tag{2.2}$$

where $T \equiv T(t, x, y, z)$, $\kappa \equiv \kappa(x, y, z)$ and $g \equiv g(t, x, y, z)$. Equation (2.2) is called the heat equation. This equation can be reduced to one or two dimensions by the elimination of the appropriate spatial terms.

We assume there are no internal heat sources in the material and the thermal conductivity is uniform throughout the substance, i.e. κ is a constant. Thus, the heat equation reduces to

$$\frac{\partial}{\partial t}T(t,x,y,z) = \epsilon\nabla^2 T(t,x,y,z) = \epsilon\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right), \tag{2.3}$$

where $\epsilon = \kappa/(\rho s)$ is the thermal diffusivity constant and depends only on the material. The dimensions of ϵ are $(\text{length})^2/\text{time}$. Typical values for ϵ taken from [8] are given in Table 2.1.

Table 2.1: Thermal Diffusivity Constants for Various Materials

Material	cm^2/sec
Silver	1.71
Copper	1.14
Aluminum	0.86
Cast Iron	0.12
Granite	0.011
Brick	0.0038
Water	0.00144

If the geometry of the domain is spherical in shape, one may wish to derive the heat equation in terms of polar coordinates. Rubio uses polar coordinates for the equations governing fluid flow and thermal convection in her dissertation [41]. The derivations for the heat equation in polar coordinates can also be found in many textbooks (see, e.g., [22]).

2.2 Boundary Conditions and Controls

In addition to the basic equation, physics requires that specific conditions be imposed on the boundaries. In particular, the distribution of temperature throughout the domain depends not only on sources inside the domain but also on how the temperature is affected on the boundaries. This section will review the boundary conditions that occur in various physical systems.

In order to formulate the boundary conditions, we must understand how heat flows through the boundaries. The conditions on the boundary must take into account the material on either side of the boundary. That is, the temperature along the boundary of a domain depends on the source with which the domain is in contact. Since we are interested in controlling the temperature at the boundaries, we control the temperature of the heat source in contact with the domain. For example, we could control heat lamps placed along the boundaries of a metal plate. We briefly discuss how the Robin boundary condition can be described mathematically.

Consider a cube in $\mathbb{R}^3$ given by $[0,1] \times [0,1] \times [0,1]$. Suppose an infinitesimal part of a face of the cube, Γ_0 where $x = 0$, is in contact with a second body of small thickness L in the x-direction and with thermal conductivity $\kappa_0(x, y, z)$, as shown in Figure 2.2. Let $T(t, x, y, z)$ be the function representing the temperature distribution of the cube at time t. Similarly, let $T_0(t, x, y, z)$ be the function representing the temperature distribution of the contacting material. With perfect contact, as in Figure 2.2, the conditions at the boundary must agree and hence the temperature of the two materials are the same:

$$T(t, x, y, z) \;=\; T_0(t, x, y, z),$$

and the flux are the same:

$$f(t,x,y,z) \;=\; f_0(t,x,y,z)$$

for all $t > 0$ and $(x,y,z) \in \Gamma_0$. At the other end of the contacting material, we have

$$T_0(t,-L,y,z) \;=\; u(t,y,z), \tag{2.4}$$

where $u(t,y,z)$ is the temperature function which we control.

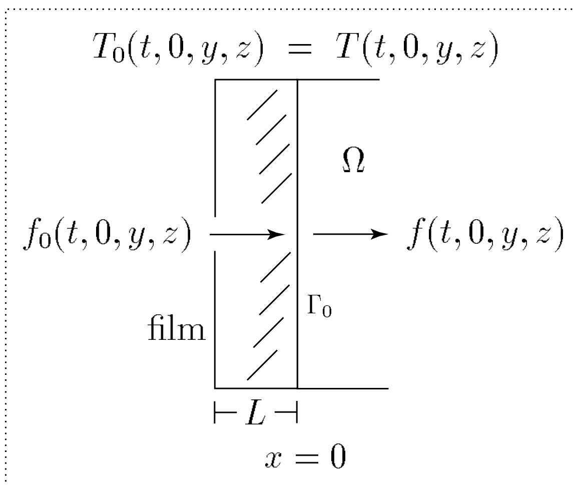

Figure 2.2: Conditions for Perfect Contact at $x = 0$

We derive the fluxes at Γ_0 using Newton's Law of Cooling. The flux into Ω from the boundary is given by

$$\begin{aligned} f(t,0,y,z) &= -\lim_{\delta \to 0^+} \frac{\kappa(0,y,z)(T(t,\delta,y,z) - T(t,0,y,z))}{\delta} \\ &= -\kappa(0,y,z)\frac{\partial}{\partial x}T(t,0,y,z), \end{aligned}$$

where $\kappa(x,y,z)$ is the thermal conductivity of the domain. The flux into the boundary is given by

$$\begin{aligned} f_0(t,0,y,z) &= \lim_{\delta\to 0^-} \frac{\kappa_0(0,y,z)(T_0(t,0,y,z)-T_0(t,\delta,y,z))}{\delta} \\ &= -\kappa_0(0,y,z)\frac{\partial}{\partial x}T_0(t,0,y,z), \end{aligned}$$

where $\kappa_0(x,y,z)$ is the thermal conductivity of the contacting material. Assuming small L, we approximate the derivative $\partial T_0/\partial x$ by the difference quotient

$$\begin{aligned} \kappa_0\frac{\partial}{\partial x}T_0(t,0,y,z) &= \kappa_0(x,y,z)\frac{T_0(t,-L,y,z)-T_0(t,0,y,z)}{-L} \\ &= -\frac{1}{L}\kappa_0(x,y,z)\big(u(t,y,z)-T(t,0,y,z)\big). \end{aligned}$$

Thus, the boundary condition at $x=0$ is

$$f(t,x,y,z) = \kappa(x,y,z)\frac{\partial}{\partial x}T(t,x,y,z) = -\frac{1}{L}\kappa_0(x,y,z)\big(u(t,y,z)-T(t,x,y,z)\big) \quad (2.5)$$

for all $t>0$ and $(x,y,z)\in\Gamma_0$.

This kind of boundary condition, where the rate of heat transfer through the boundary is proportional to the temperature along that portion of the boundary, is known as a Robin boundary condition. In general, a Robin boundary condition with control on the boundary Γ of a domain Ω is given by

$$\kappa(x,y,z)\frac{\partial}{\partial \eta}T(t,x,y,z) = -\frac{\kappa_0(x,y,z)}{L}\big(u(t,x,y,z)-T(t,x,y,z)\big) \quad (2.6)$$

for all $(x,y,z)\in\Gamma$, where η is the outward normal to Γ and $u(t,x,y,z)$ is a control function defined on the boundary.

If the thermal conductivity of the main body at the boundary is zero (i.e., $\kappa(0,y,z)=0$) or if the length of the film is zero (i.e., $L\to 0$), then the temperature at the boundary Γ

is maintained by the control function $u(t,x,y,z)$. This special case of a Robin boundary condition is given by:

$$T(t,x,y,z)\big|_{\Gamma} = u(t,x,y,z)\big|_{\Gamma}$$

for all $t > 0$. This is the Dirichlet boundary control.

Suppose the boundary is insulated so that no heat passes through it. Then the rate of heat transfer (the flux) at the boundary is 0, and this is reflected (for this special case where the flux is given as above) by the equation

$$\frac{\partial}{\partial\eta}T(t,x,y,z)\big|_{\Gamma} = 0,$$

for all $t > 0$, where η is the outward normal to the boundary Γ. This is the homogeneous Neumann boundary condition. This condition is also a special case of a Robin boundary condition, namely, when the thermal conductivity κ_0 at the boundary is zero. To obtain a Neumann boundary control, we would need to control the flux, i.e.,

$$\frac{\partial}{\partial\eta}T(t,x,y,z)\big|_{\Gamma} = u(t,x,y,z)\big|_{\Gamma},$$

for all $t > 0$.

Boundary control problems have been studied extensively for Dirichlet boundary controls and Neumann boundary controls (e.g., in [28, 42, 7]). However, the Robin boundary condition reflect the physics of problems and boundary control problems that are a mixture of Dirichlet and Neumann boundary conditions may be viewed as limiting cases of the Robin model. We concentrate on Robin boundary control problems. However, due to the complexity of the Dirichlet boundary control, we will also touch upon the Dirichlet problem as well.

2.3 Boundary Value Problem

With an initial condition to determine the flow of heat in a spatial domain Ω completely, we obtain a boundary value problem for temperature distribution with no internal heat source and constant thermal diffusivity ϵ:

$$\left.\begin{aligned} \frac{\partial}{\partial t}T(t,\mathbf{x}) &= \epsilon\nabla^2 T(t,\mathbf{x}) && \forall\ t>0,\ \ \mathbf{x}\in\Omega \\ T(t,\mathbf{x}) &= 0 && \forall\ t>0,\ \ \mathbf{x}\in\Gamma_0 \\ T(t,\mathbf{x}) + \frac{L\kappa}{\kappa_0}\frac{\partial}{\partial\eta}T(t,\mathbf{x}) &= u(t,\mathbf{x}) && \forall\ t>0,\ \ \mathbf{x}\in\Gamma_c \\ T(0,\mathbf{x}) &= T_0(\mathbf{x}) && \forall\ \mathbf{x}\in\Omega, \end{aligned}\right\} \tag{2.7}$$

where Γ is the boundary of the domain Ω, homogeneous Dirichlet boundary conditions are specified on Γ_0, and Robin boundary control is placed on Γ_c. From here on, we study the control problem involving the system of equations (2.7) such as in the following examples.

EXAMPLE 1. Suppose we look at a thin rod that is insulated on the sides and is made of a homogeneous material. One end is positioned at $x = 0$ and the rod is of length l. We place the end $x = 0$ in contact with a thin film with which we control the temperature. The other end of the rod is set at constant temperature of 0. Then the one-dimensional equation for this situation is given by

$$\left.\begin{aligned} \frac{\partial}{\partial t}T(t,x) &= \epsilon\frac{\partial^2}{\partial x^2}T(t,x) && \forall\ t>0,\quad 0<x<l \\ T(t,l) &= 0 && \forall\ t>0 \\ T(t,0)-\frac{L\kappa}{\kappa_0}\frac{\partial}{\partial x}T(t,0) &= u(t) && \forall\ t>0 \\ T(0,x) &= T_0(x) && \forall\ 0<x<l, \end{aligned}\right\} \tag{2.8}$$

where ϵ is the thermal diffusivity of the rod, L is the length of the contact film, and κ and κ_0 are the thermal conductivities of the rod and contact film, respectively.

EXAMPLE 2. Consider a square metal plate defined by the dimensions $[0,1]\times[0,1]$ in $\mathbb{R}^2$. Suppose the boundary, $\Gamma_c = \{(x,y)|x=0, 0<y<1\}$, is in contact with another body of film, or suppose it is being heated by a series of lamps. Suppose the remaining boundary of the plate has zero temperature. The two-dimensional initial boundary value problem is given by

$$\left.\begin{aligned} \frac{\partial}{\partial t}T(t,x,y) &= \epsilon\nabla^2 T(t,x,y) && \forall\ t>0,\quad (x,y)\in\Omega \\ T(t,x,y) &= 0 && \forall\ t>0,\quad (x,y)\in\Gamma_0 \\ T(t,0,y)-\frac{L\kappa}{\kappa_0}\frac{\partial}{\partial x}T(t,0,y) &= u(t,y) && \forall\ t>0,\quad 0<y<1 \\ T(0,x,y) &= T_0(x,y) && \forall\ (x,y)\in\Omega, \end{aligned}\right\} \tag{2.9}$$

where $\Omega \equiv [0,1]\times[0,1]$ and Γ_0 is the part of the boundary where $x\neq 0$. In general, $u(t,y)$ is considered to be an continuous control function of time and space. For this example, we consider the face of the plate at $x=0$ being heated or cooled by a series of lamps. Then we can write

$$u(t,y)=\sum_{i=1}^{N}u_i(t)g_i(y)$$

where $g_i(y)$ is a given distribution, $u_i(t)$ is the control for lamp i in time, and N is the number of lamps.

In the next section, we formulate the system of equations (2.7) in an abstract form. The abstract form of the heat equation will enable us to use results obtained for LQR problems.

2.4 Abstract Form of B.V.P.

Let Ω be a bounded open set in $\mathbb{R}^n$ $(n = 1, 2, 3)$ with boundary Γ. We consider the system of equations (2.7), where $T_0 \in L^2(\Omega)$ and $u(t) \in L^2(\Gamma_c)$. We formulate (2.7) as an abstract system.

Define $H = L_2(\Omega)$,

$$V = H^1_{\Gamma_0}(\Omega) \equiv \{\phi \in L_2(\Omega) | \phi' \in L_2(\Omega), \phi|_{\Gamma_0} = 0\}$$

and let V' be the dual space of V. Denote $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ the inner product and the norm in H, respectively. $\langle \cdot, \cdot \rangle_X$ and $|| \cdot ||_X$ denotes the same operations on a Hilbert space X. The inner product and norm of the space $L_2(\Omega)$ of square-integrable functions on a domain Ω are given by

$$\begin{aligned} \langle \phi, \psi \rangle &= \int_\Omega \phi(\mathbf{x}) \psi(\mathbf{x}) d\Omega, \\ ||\phi|| &= \left(\int_\Omega |\phi(\mathbf{x})|^2 d\Omega \right)^{1/2}, \end{aligned}$$

respectively, where ϕ and ψ are functions in $L_2(\Omega)$.

The space $\mathcal{L}(X, Y)$ is the space of all bounded linear operators from one Hilbert space X to another Hilbert space Y and $\mathcal{L}(X, X)$ is denoted by $\mathcal{L}(X)$.

Let $A_0 : \mathcal{D}(A_0) \to L^2(\Omega)$ be the elliptic, linear operator defined by

$$[A_0\varphi](\mathbf{x}) \;=\; \epsilon\nabla^2\varphi(\mathbf{x}) \tag{2.10}$$

on the domain

$$\mathcal{D}(A_0) \;\equiv\; \left\{\varphi \in H^2(\Omega) \;\middle|\; \beta\varphi + \frac{\partial\varphi}{\partial\eta} = 0 \text{ on } \Gamma_c,\; \varphi = 0 \text{ on } \Gamma_0\right\}, \tag{2.11}$$

where $\beta = \kappa_0/(\kappa L)$. We establish that A_0 is self-adjoint and that $-A_0$ is a strongly elliptic operator of order 2. These are necessary for some assumptions to be satisfied in later chapters.

Theorem 2.1. *The operator A_0 defined by* (2.10)-(2.11) *is self-adjoint and strongly elliptic.*

Proof. From ([44], page 242), $\psi \in \mathcal{D}(A_0^*)$ if and only if there exists some $\omega \in [\mathcal{D}(A_0)]^*$ such that

$$\langle A_0\varphi, \psi\rangle \;=\; \langle\varphi, \omega\rangle$$

for all $\varphi \in \mathcal{D}(A_0)$. Such an ω is unique, if it exists, and $\omega = A_0^*\psi$ (see [44]). Using this and the definition of generalized derivatives, the proof is standard. Since A_0 is elliptic and self-adjoint, then A_0 is strongly elliptic (Theorem 10.7, [46]). □

We define $A : V \to V'$ by

$$[A\varphi](\psi) \;= \langle A\varphi, \psi\rangle_V \;=\; a(\varphi, \psi) + c(\varphi, \psi) \tag{2.12}$$

for all $\psi \in V$, where $a(\varphi, \psi)$ is the bilinear form defined by

$$a(\varphi, \psi) = \epsilon \int_{\Omega} \nabla\varphi \cdot \nabla\psi d\mathbf{x}$$

and $c(\varphi, \psi)$ is the boundary form defined by

$$c(\varphi, \psi) = -\frac{\epsilon\beta_1}{\beta_0} \int_{\Gamma_c} \varphi\psi d\Gamma_c.$$

Lions defined the Dirichlet map and the Neumann map in his work [30] to formulate the boundary control problems as an abstract system. For the Robin problem, we define the control operator $B : L^2(\Gamma_c) \to V'$ by

$$[Bu(t)](\psi) = -c(u(t), \psi) = \frac{\epsilon\kappa_0}{\kappa L} \int_{\Gamma_c} u(t)\psi d\Gamma_c, \tag{2.13}$$

for all $\psi \in V$. From the definition of the boundary form above, one can show

$$\langle u(t), B^*\psi\rangle_{L^2(\Gamma_c)} = \langle Bu(t), \psi\rangle_{L^2(\Gamma_c)} = \int_{\Gamma_c} u(t) \left[\frac{\epsilon\kappa_0}{\kappa L}\psi\right] d\Gamma_c. \tag{2.14}$$

Thus, B^* is defined by

$$[B^*\varphi](\mathbf{x}) = \frac{\epsilon\kappa_0}{\kappa L}\varphi(\mathbf{x})\big|_{\Gamma_c}. \tag{2.15}$$

Defining A and B as shown above, the initial boundary value problem is written in an abstract form as follows:

$$\left.\begin{aligned} \frac{d}{dt}\omega(t) &= A\omega(t) + Bu(t), \\ \omega(0) &= \omega_0, \end{aligned}\right\} \tag{2.16}$$

where $\omega(t) \in V$, $u(t) \in L^2(\Gamma_c)$ and d/dt is defined in the distributional sense as defined by Lions [30]. It can be shown that solutions of (2.16) are equivalent to weak solutions of the partial differential equation given by (2.7) (see [7], [18], [46] and the references therein).

2.5 Note on Dirichlet Boundary Control

Notice that in special instances (i.e., when $L = 0$ in Section 2.3), the Robin boundary condition becomes the Dirichlet boundary condition:

$$T(t, \mathbf{x}) \;=\; u(t, \mathbf{x}) \;\text{ on }\; \Gamma_c.$$

For the problem involving Dirichlet boundary controls, the formulation of the abstract system becomes a bit more complex. The variational form as shown in the preceding section will not work for this situation.

For this section, we define the space $W = \mathcal{D}(A_0^*)$ and thus, the dual space is given by $W' = [\mathcal{D}(A_0^*)]'$. We extend the operator A defined by (2.10) to the space $H = L_2(\Omega)$. Define the operator $A_1 : H \to W'$ by

$$[A_1 f](\psi) \;= \langle f, A_0^* \psi \rangle \;=\; \epsilon \int_\Omega f \nabla^2 \psi d\mathbf{x} \tag{2.17}$$

for all $\psi \in \mathcal{D}(A_0)$. Using the Dirichlet map D as defined by Lasiecka [27], we define the B operator by $B = -A_1 D$ and we have $B : L^2(\Gamma_c) \to W'$. The boundary value problem can be formulated as the system in W'

$$\left.\begin{aligned} \frac{d}{dt}\omega(t) \;&=\; A_1 T(t) + Bu(t), \\ T(0) \;&=\; T_0, \end{aligned}\right\} \tag{2.18}$$

where $T(t) \in H$ and $u(t) \in L^2(\Gamma_c)$. The variational equation, which is the basis for finite element methods, is

$$\left.\begin{aligned} \langle \dot{T}(t), \varphi \rangle \;&=\; \langle T(t), A_0^* \varphi \rangle + \langle u(t), B^* \varphi \rangle, \\ T(0) \;&=\; T_0. \end{aligned}\right\} \tag{2.19}$$

The main goal of this problem is not only to find the solution $T(t)$ that will satisfy (2.16) or (2.18), but to find a control that gives the optimum performance subject to the linear quadratic criteria. Thus, we discuss the α-LQR problem and the algebraic operator Riccati equation (ARE) in the next section. We will then use the ARE to derive a Riccati partial differential equation.

Chapter 3

The Riccati Equation

This chapter contains a discussion on the α-LQR problem and the corresponding Riccati equation. The standard Neumann and Dirichlet problems may be found in Banks and Kunisch [5], Lasiecka and Triggiani [28], and Lions [30]. Bensoussan, et. al., [7], provide an extensive overview of these problems. However, the Robin problem is not as well documented.

3.1 Notation and Definitions

Let H, U and Y be separable Hilbert spaces. We assume that V and W are Hilbert spaces such that

$$W \subseteq V \subseteq H = H' \subseteq V' \subseteq W'$$

where the embeddings are continuous and dense. The case $W = V$ is not ruled out and, as noted in the previous chapter, corresponds to the case with Robin boundary control.

Dirichlet boundary control will correspond to the case with $W = \mathcal{D}(A_0^*)$ and $W' = [\mathcal{D}(A_0^*)]'$.

Consider the dynamical system

$$\left.\begin{aligned} \frac{d}{dt}\omega(t) &= A\omega(t) + Bu(t), \\ \omega(0) &= \omega_0, \end{aligned}\right\} \tag{3.1}$$

where A is an operator that maps V to V' (or H to W', e.g., see Chapter 2) and $\omega_0 \in H$. We consider operators B that are unbounded. In particular, B is a control operator that arises from boundary control problems and maps U into the dual space W' of W, where A_0^* is the adjoint of A_0 with respect to the topology of H.

The state equation (3.1) can be represented by an integral equation

$$\omega(t) = e^{At}\omega_0 + \int_0^t e^{(t-\tau)A} Bu(\tau)d\tau, \tag{3.2}$$

where $e^{(t-\tau)A}$ is defined in Bensoussan, et. al., [7]. The formula (3.2) is a meaningful expression and it can be shown that the solution ω is in $L^2(0,T;H)$ (see [45], page 653). We present some general theory for the α-LQR problem.

3.2 The α-LQR Problem: General Theory

The following assumptions (see [28]) are made in order to use the results discussed in this section:

(A1) $A : \mathcal{D}(A) \subset H \to H$ is the generator of a strongly continuous, analytic semigroup e^{At} on H for all $t > 0$;

(A2) $B \in \mathcal{L}(U, \mathcal{D}(A^*)')$ and moreover,

$$A^{-\gamma}B \;\in\; \mathcal{L}(U, H) \tag{3.3}$$

for some constant $0 \leq \gamma < 1$;

(A3) $Q \equiv C^*C$ is bounded, where C is the observation operator that acts on $\omega(t)$ and maps from H to Y;

(A4) $R \in \mathcal{L}(U)$ such that $R = R^*$ and for all $u \in U$

$$\langle Ru, u \rangle_U \;\geq\; \nu ||u||_U^2 \tag{3.4}$$

for some constant $\nu > 0$, where R is some weighted operator that acts on the control $u(t)$.

Although the general theory is valid for unbounded Q, we focus on the case with Q bounded and assume Q is Hilbert-Schmidt (see [25]) so that we can derive Chandrasekhar partial differential equations.

The linear quadratic regulator problem is to minimize the quadratic functional

$$J(u) \;=\; \int_0^T \{\langle y(t), y(t)\rangle + \langle Ru(t), u(t)\rangle_U\}\, dt, \tag{3.5}$$

over all $u(t) \in U$ subject to the system (3.1), where $y(t) \in Y$ is the controlled output function defined by

$$y(t) \;=\; C\omega(t), \tag{3.6}$$

for some $C : H \rightarrow Y$, and $R > 0$ in $\mathcal{L}(U)$ (see [28]). Lasiecka and Triggiani [28] consider this problem for parabolic and hyperbolic systems, and for cases where $T < \infty$ and where $T = \infty$.

For the case $T < \infty$, we first introduce the operator G and its adjoint G^* as defined by

$$[Gu]\,(t) = \int_0^t e^{(t-\tau)A} B u(\tau) d\tau, \tag{3.7}$$

$$[G^*\omega]\,(t) = \int_t^T B^* e^{(t-\tau)A^*} \omega(\tau) d\tau, \tag{3.8}$$

where $\langle [Gu](t), \omega(t)\rangle_H = \langle u(t), [G^*\omega](t)\rangle_U$. This operator is defined by the term found in the "variation of constants" formula to derive (3.2). With this definition, we then have the following theorem [28].

Theorem 3.1. *There exists a unique optimal solution $(u^o(t), \omega^o(t))$ to the LQR problem explicitly given by*

$$u^o(t) = -[\Lambda^{-1} G^* C^* C(e^{At}\omega_0)](t), \tag{3.9}$$

$$\omega^o(t) = e^{tA}\omega_0 + [Gu^o](t), \tag{3.10}$$

$$\Lambda = I + G^* C^* C G. \tag{3.11}$$

Moreover, there exists a unique non-negative, self-adjoint operator $P(t) \in \mathcal{L}(L_2(\Omega))$, which is a solution of the differential Riccati equation (DRE):

$$\left.\begin{aligned} \tfrac{d}{dt}\langle P(t)\varphi, \psi\rangle &= -\langle C^* C\varphi, \psi\rangle - \langle P(t)A\varphi, \psi\rangle \\ &\qquad -\langle P(t)\varphi, A\psi\rangle + \langle B^* P(t)\varphi, R^{-1} B^* P(t)\psi\rangle_U, \\ P(T) &= 0 \end{aligned}\right\} \tag{3.12}$$

for all $\varphi, \psi \in \mathcal{D}(A)$. In fact, $P(t)$ is given constructively by

$$P(t)\varphi \;=\; \int_t^T e^{A^*(\tau-t)} C^* C \omega^o(\tau, t) d\tau \tag{3.13}$$

for $\varphi \in L^2(\Omega)$. Furthermore, $P(t)$ has the property that $B^* P(t)$ is a bounded linear operator from $L^2(\Omega)$ to U. For feedback synthesis, the optimal pair is related by

$$u^o(t) \;=\; -R^{-1} B^* P(t) \omega^o(t) \tag{3.14}$$

for $0 \leq t \leq T$, and

$$\left.\begin{aligned} \langle P(t)\varphi, \psi \rangle \;&=\; \textstyle\int_t^\tau \langle C^* C \omega^o(\tau, t), \omega^o(\tau) \rangle d\tau \\ &\quad + \textstyle\int_t^\tau \langle B^* P(\tau)\omega^o(\tau), R^{-1} B^* P(\tau) \omega^o(\tau) \rangle_U d\tau \end{aligned}\right\} \tag{3.15}$$

from which the optimal cost is

$$J(u^o(t), \omega^o(t)) \;=\; \langle P(t)\omega_0, \omega_0 \rangle, \tag{3.16}$$

where $\omega_0 \in L^2(\Omega)$.

Although we focus on the infinite-time horizon where $T = \infty$, the result for differential Riccati equations (DRE) is important for deriving the results of the infinite-time problem. Moreover, it is necessary to understand the DRE when we discuss the Chandrasekhar equations which are covered in the next chapter.

We wish to control the rate of decay of the cost function as discussed in the classic work by Anderson and Moore [1] (see also [11]). We consider the α-shifted linear quadratic control problem with infinite-time ($T = \infty$) horizon:

Find $u^o(\cdot) \in L^2(0,\infty;U)$ *minimizing the cost function*

$$J(u) = \int_0^\infty \{\langle y(t), y(t)\rangle + \langle Ru(t), u(t)\rangle_U\} e^{2\alpha t} dt \tag{3.17}$$

subject to the constraint

$$\left.\begin{aligned} \frac{d}{dt}\omega(t) &= A\omega(t) + Bu(t), \\ \omega(0) &= \omega_0, \end{aligned}\right\} \tag{3.18}$$

where $y(t)$ *is a given output function defined in* (3.6).

If we introduce the "α-shifted" system considered by Burns' and Kang in [12], i.e., $\hat{\omega}(t) = \omega(t)e^{\alpha t}$, $\hat{u}(t) = u(t)e^{\alpha t}$ and $\hat{y}(t) = y(t)e^{\alpha t}$, the corresponding shifted regulator problem becomes:

For any given $\alpha > 0$, *find* $\hat{u}^o(t)$ *that minimizes the cost functional*

$$\hat{J}(\hat{u}(\cdot)) = \int_0^\infty \{\langle \hat{y}(t), \hat{y}(t)\rangle + \langle Ru(t), u(t)\rangle_U\} dt \tag{3.19}$$

subject to the shifted (weak) system

$$\left.\begin{aligned} \frac{d}{dt}\hat{\omega}(t) &= (A + \alpha I)\hat{\omega}(t) + B\hat{u}(t), \\ \hat{\omega}(0) &= \omega_0, \end{aligned}\right\} \tag{3.20}$$

where the output function is given by

$$\hat{y}(t) = C\hat{\omega}(t). \tag{3.21}$$

Burns and Kang proved the existence of an unique, optimal control to this α-LQR problem for the one-dimensional Dirichlet boundary control problem. We apply the same results to be sure that the infinite-time problem for the Robin boundary control has an unique optimal control.

The linear quadratic problem given by the cost function (3.19) subject to the system (3.20) will have a unique control if the conditions in [28] are satisfied for $(A + \alpha I)$ and B. Since we are studying the infinite-time quadratic cost function, we need two other hypotheses to be satisfied (see [28]):

Finite Cost Condition (F.C.C.): For every $\omega(t) \in L^2(\Omega)$, there exists $u \in L^2(0, \infty; U)$ such that the corresponding functional (3.19) satisfies $J(u, \omega) < \infty$.

Detectability Condition (D.C.): There exists $F \in \mathcal{L}(Y, H)$ such that the strongly-continuous (s.c.) semigroup $e^{t(A+\alpha I+FC)}$ generated by $A + \alpha I + FC$ is exponentially and uniformly stable on H.

Once the above conditions and assumptions (A1)-(A4) are satisfied, then the following result from [28] can be applied.

Theorem 3.2. *If the assumptions (A1)-(A4), F.C.C. and D.C. are satisfied, then there exists an unique optimal control*

$$u^o(t) \;=\; -R^{-1}B^*\Pi\omega(t) \tag{3.22}$$

for all $t > 0$, where $\Pi = \Pi^* \in \mathcal{L}(L^2(\Omega))$ is the unique, nonnegative-definite solution to the algebraic Riccati equation (α-ARE):

$$\langle (A+\alpha)\varphi, \Pi\psi\rangle + \langle \Pi\varphi, (A+\alpha)\psi\rangle - \langle B^*\Pi\varphi, R^{-1}B^*\Pi\psi\rangle_U + \langle C^*C\varphi, \psi\rangle \;=\; 0 \tag{3.23}$$

for all $\varphi, \psi \in H$. Furthermore, the following are true:

(i) $A^*\Pi \in \mathcal{L}(H)$;

(ii) the optimal cost is $J(u^o) = \langle \Pi\omega_0, \omega_0 \rangle_H$; *and*

(iii) the s.c. semigroup $e^{A_\Pi t}$ *generated by* $A_\Pi = \tilde{A} - BR^{-1}B^*\Pi$ *is uniformly stable on* H:

$$||e^{A_\Pi t}||_{\mathcal{L}(H)} \leq Me^{-\nu t} \tag{3.24}$$

for all $t > 0$ and for some constants $M, \nu > 0$.

We apply the general theorems to the specific α-LQR problem (3.17) subject to the heat equation (2.16) or (2.18), where the output function $y(t)$ is given by

$$y(t) = CT(t) = \int_\Omega c(x)\omega(t,x)dx. \tag{3.25}$$

The operator $C : H \to \mathbb{R}$ acts on $T(t)$ for a specific reason, i.e., more weight on the observation of the temperature of Ω at specific areas or on locations we wish to control. The function $c(x)$ is called the weight or observation function. For this problem, our spaces are $H = L_2(\Omega)$, $U = L_2(\Gamma)$ (or $\mathbb{R}^n$) and $Y = \mathbb{R}$. It can be shown that the abstract system (2.16) or (2.18) from Chapter 2 satisfy the assumptions so that the above theorems apply (see [28]). Using these spaces and the theorem for the α-ARE (3.23), we derive a partial differential equation in the next section which could be used to compute the functional gain in a more direct way.

3.3 The Riccati PDE

Conceptually, there are several ways to compute the functional gain. One basic approach is to approximate the α-ARE (3.23) and obtain an approximate Riccati operator Π^N. The approximate feedback operator is given by

$$K^N = [R^{-1}]^N [B^*]^N \Pi^N, \tag{3.26}$$

and this feedback law yields an approximation $h^N(\cdot)$ to the functional gain $h(\cdot)$.

A more direct approach is to first obtain a representation for the Riccati operator of the form

$$[\Pi\varphi](x) = \int_0^l p(x,\xi)\varphi(\xi)d\xi, \tag{3.27}$$

and then to solve for $p(x,\xi)$. For certain systems (distributed control and Neumann boundary control), Lions [30] uses the Schwartz Kernel Theorem to obtain a representation of the form (3.27). Lions' result does not apply directly to the Dirichlet or Robin boundary control problem (see pages 157-167 in [30]). However, we can apply King's results [26] to the case considered here. The following theorem follows directly from a proposition on embeddings of fractional Sobolev spaces (see [26]).

Theorem 3.3. *If $\beta > 0$ and $T : L_2(\Omega) \to H^{\beta+N/2}(\Omega)$ is a bounded linear operator, then there exists a kernel $p(\cdot,\cdot)$ defined on Ω such that*

(i) $p(\cdot,\cdot) \in L_2(\Omega \times \Omega)$;

(ii) $[T\phi](\xi) = \int_\Omega p(\xi, y)\phi(y)dy$;

(iii) $||T||_2^2 = \int_{\Omega\times\Omega} |p(\xi, y)|^2 d\xi dy$.

Although King's results are for parabolic systems with Dirichlet boundary controls, one can extend this to the Neumann and Robin boundary control. Let Π_R denote the Riccati operator for the Robin boundary control problem (2.16) with cost function (3.19):

Theorem 3.4. *If* $\Pi_R = \Pi_R^*$ *is the unique non-negative solution of the ARE for the parabolic control problem* (2.16), (3.19), *then* Π_R *is Hilbert-Schmidt. In particular, there exists a kernel function* $p_R(\cdot, \cdot)$ *such that*

(a) $p_R(\cdot, \cdot) \in L_2(\Omega \times \Omega)$;

(b) $p_R(x, \xi) = p_R(\xi, x)$ *for* $x, \xi \in \Omega$;

(c) $[\Pi_R \varphi](x) = \int_\Omega p_R(x, \xi)\varphi(\xi)d\xi$ *for all* $\phi \in L_2(\Omega)$.

In order to apply this result, we assume that $C : L_2(\Omega) \to \mathbb{R}$ has the form

$$C\varphi = \int_\Omega c(x)\varphi(x)dx, \tag{3.28}$$

where $c(x) \in L_2(\Omega)$. We define the operator C to map to the real space $\mathbb{R}$ so that we can derive our Chandrasekhar partial differential equations in the next chapter. Note that the operator $Q = C^*C$ has the form

$$\begin{aligned}[Q\varphi](x) &= c(x)\int_\Omega c(\xi)\varphi(\xi)d\xi \\ &= \int_\Omega q(x, \xi)\varphi(\xi)d\xi,\end{aligned}$$

where $q(x,\xi)$ is the kernel for Q. In addition, for the rest of this chapter and the next, we assume $R = I$ since the problem can be normalized for R.

The following theorem extends the result in [30] to the Robin boundary control problem considered here.

Theorem 3.5. *The kernel $p_R(x,\xi)$ of the operator Π_R for the Robin problem is a weak solution to the Riccati partial differential equation (R-PDE)*

$$(A_x + A_\xi + 2\alpha I)p_R(x,\xi) - [B_{nl}p_R](x,\xi) + q(x,\xi) \;=\; 0. \tag{3.29}$$

In particular, $p_R(\cdot,\cdot) \in L_2(\Omega \times \Omega)$ satisfies the Dirichlet boundary conditions

$$\left.\begin{aligned} p_R(x,\xi) &= 0 \quad a.e.\ for\ \ x \in \Gamma_0,\ \ \xi \in \Omega, \\ p_R(x,\xi) &= 0 \quad a.e.\ for\ \ x \in \Omega,\ \ \xi \in \Gamma_0, \end{aligned}\right\} \tag{3.30}$$

the Robin's boundary conditions (distributional sense),

$$\left.\begin{aligned} \tfrac{\kappa_0}{L}p_R(x,\xi) + \kappa\frac{\partial}{\partial\eta_x}p_R(x,\xi) &= 0 \quad for\ \ x \in \Gamma_c,\ \ \xi \in \Omega \\ \tfrac{\kappa_0}{L}p_R(x,\xi) + \kappa\frac{\partial}{\partial\eta_\xi}p_R(x,\xi) &= 0 \quad for\ \ x \in \Omega,\ \ \xi \in \Gamma_c \end{aligned}\right\} \tag{3.31}$$

and the symmetry condition

$$p_R(x,\xi) \;=\; p_R(\xi,x) \;\;\forall\;\; x,\xi \in \Omega. \tag{3.32}$$

Proof. Property (3.32) follows from $\Pi_R^* = \Pi_R$; that is,

$$[\Pi_R\varphi](x) \;=\; \int_\Omega p_R(x,\xi)\varphi(\xi)d\xi \;=\; \int_\Omega p_R(\xi,x)\varphi(\xi)d\xi \;=\; [\Pi_R^*\varphi](x).$$

To obtain (3.29), we substitute the representation of the Riccati operator into the weak form of the α-ARE (3.23) defined by

$$\begin{aligned}\langle \Pi_R\varphi, (A^*+\alpha)\psi\rangle + &\langle (A^*+\alpha)\varphi, \Pi_R\psi\rangle \\ &- \langle B^*\Pi_R\varphi, B^*\Pi_R\psi\rangle + \langle C\varphi, C\psi\rangle \;=\; 0,\end{aligned} \tag{3.33}$$

for each $\varphi, \psi \in \mathcal{D}(A)$. In particular, the first term in (3.33) is given by (using Fubini's Theorem [40])

$$\begin{aligned}\langle \Pi_R\varphi, (A^*+\alpha)\psi\rangle &= \int_\Omega \left[\int_\Omega p_R(x,\xi)\varphi(\xi)d\xi\right](\psi''(x)+\alpha\psi)dx \\ &= \int_{\Omega\times\Omega}\big(p_R(x,\xi)\psi''(x)\varphi(\xi) + \alpha p_R(x,\xi)\psi\varphi\big)dxd\xi.\end{aligned}$$

Let

$$V \;\equiv\; \{v \in H^2(\Omega\times\Omega) \mid v \text{ satisfies (3.30) and (3.31)}\}.$$

Define $A_x : L_2(\Omega\times\Omega) \to V^*$ by

$$[A_x p_R](v(\cdot,\cdot)) \;=\; \int_{\Omega\times\Omega} p_R(x,\xi)\frac{\partial^2}{\partial x^2}v(x,\xi)dxd\xi. \tag{3.34}$$

The operator A_ξ is defined similarly. The term $\langle C\varphi, C\psi\rangle$ takes the form

$$\begin{aligned}\langle C^*C\varphi, \psi\rangle &= \langle Q\varphi, \psi\rangle \\ &= \int_\Omega\left[\int_\Omega q(x,\xi)\varphi(\xi)d\xi\right]\psi(x)dx \\ &= \int_{\Omega\times\Omega} q(x,\xi)\psi(x)\varphi(\xi)dxd\xi.\end{aligned}$$

Finally, it follows that

$$\begin{aligned}\langle B^*\Pi_R\varphi, B^*\Pi_R\psi\rangle &= \left(B^*\int_\Omega p_R(x,\xi)\varphi(\xi)d\xi\right)\left(B^*\int_\Omega p_R(x,\xi)\psi(x)dx\right) \\ &= \left(\beta\int_\Omega p_R(x,\xi)\varphi(\xi)d\xi\right)\Big|_{\Gamma_c}\left(\beta\int_\Omega p_R(x,\xi)\psi(x)dx\right)\Big|_{\Gamma_c} \\ &= \beta^2\left(\int_{\Omega\times\Omega} p_R(\zeta,\xi)p_R(x,\zeta)\psi(x)\varphi(\xi)dxd\xi\right)\Big|_{\Gamma_c}\end{aligned}$$

where $\beta = (\epsilon\kappa_0)/(\kappa L)$. We define $B_{nl} : L_2(\Omega \times \Omega) \to L_2(\Omega \times \Omega)'$ by

$$[B_{nl}p_R](v(\cdot,\cdot)) \;=\; \beta^2 \left(\int_{\Omega\times\Omega} p_R(x,\xi)p_R(\zeta,x)v(\xi,\zeta)d\xi d\zeta \right) \Big|_{\Gamma_c}. \tag{3.35}$$

Therefore, $p_R(\cdot,\cdot) \in L_2(\Omega \times \Omega)$ satisfies the variational problem: Find $p_R(\cdot,\cdot)$ such that for all $v(\cdot,\cdot) \in V$

$$\begin{aligned}\int_{\Omega\times\Omega} p_R(x,\xi)\frac{\partial^2}{\partial x^2}v(x,\xi)dxd\xi + \int_{\Omega\times\Omega} p_R(x,\xi)\frac{\partial^2}{\partial \xi^2}v(x,\xi)dxd\xi \\ + 2\alpha \int_{\Omega\times\Omega} p_R(x,\xi)v(x,\xi)dxd\xi - \beta^2 \left(\int_{\Omega\times\Omega} p_R(x,\xi)p_R(\zeta,x)v(\xi,\zeta)d\xi d\zeta \right) \Big|_{\Gamma_c} \\ + \int_{\Omega\times\Omega} q(x,\xi)v(x,\xi)dxd\xi \;=\; 0.\end{aligned} \tag{3.36}$$

Thus the kernel $p_R(x,\xi)$ is a weak solution to (3.29).

To derive the boundary conditions, we must take care since the regularity property of $p_R(\cdot,\cdot)$ is not strong. We only have $p_R(\cdot,\cdot) \in L_2(\Omega \times \Omega)$. Following the method used in Lions' [30], we utilize Green's formula on the first term of (3.36):

$$\begin{aligned}&\textstyle\int_{\Omega\times\Omega} p_R(x,\xi)\frac{\partial^2}{\partial x^2}v(x,\xi)dxd\xi \\ &= \textstyle\int_{\Gamma\times\Omega} p_R(x,\xi)\frac{\partial}{\partial \eta_x}v(x,\xi)d\Gamma d\xi - \int_{\Omega\times\Omega} \nabla_x p_R(x,\xi)\cdot\nabla_x v(x,\xi)dxd\xi \\ &= \textstyle -\int_{\Gamma_c\times\Omega} \beta p_R(x,\xi)v(x,\xi)d\Gamma_c d\xi + \int_{\Gamma_0\times\Omega} p_R(x,\xi)\frac{\partial}{\partial \eta_x}v(x,\xi)d\Gamma_0 d\xi \\ &\quad \textstyle - \int_{\Gamma\times\Omega} \frac{\partial}{\partial \eta_x}p_R(x,\xi)v(x,\xi)d\Gamma d\xi + \int_{\Omega\times\Omega} \frac{\partial^2}{\partial x^2}p_R(x,\xi)v(x,\xi)dxd\xi \\ &= \textstyle -\int_{\Gamma_c\times\Omega} \left(\beta p_R(x,\xi) + \frac{\partial}{\partial \eta_x}p_R(x,\xi)\right) v(x,\xi)d\Gamma_c d\xi \\ &\quad \textstyle + \int_{\Gamma_0\times\Omega} p_R(x,\xi)\frac{\partial}{\partial \eta_x}v(x,\xi)d\Gamma_0 d\xi + \int_{\Omega\times\Omega} \frac{\partial^2}{\partial x^2}p_R(x,\xi)v(x,\xi)dxd\xi.\end{aligned}$$

Similarly, for the second term, we obtain

$$
\begin{aligned}
&\int_{\Omega\times\Omega} p_R(x,\xi)\frac{\partial^2}{\partial\xi^2}v(x,\xi)dxd\xi \\
&\quad = -\int_{\Omega\times\Gamma_c}\left(\beta p_R(x,\xi)+\frac{\partial}{\partial\eta_\xi}p_R(x,\xi)\right)v(x,\xi)dxd\Gamma_c \\
&\qquad + \int_{\Omega\times\Gamma_0} p_R(x,\xi)\frac{\partial}{\partial\eta_\xi}v(x,\xi)dxd\Gamma_0 + \int_{\Omega\times\Omega}\frac{\partial^2}{\partial\xi^2}p_R(x,\xi)v(x,\xi)dxd\xi.
\end{aligned}
$$

Substituting these back into (3.36) and since the final equation must be satisfied for all $v(x,\xi)\in V$, we have that

$$
\begin{aligned}
\int_{\Gamma_0\times\Omega} p_R(x,\xi)\frac{\partial}{\partial\eta_x}v(x,\xi)d\Gamma_0 d\xi &= 0, \\
\int_{\Omega\times\Gamma_0} p_R(x,\xi)\frac{\partial}{\partial\eta_\xi}v(x,\xi)dxd\Gamma_0 &= 0, \\
\int_{\Gamma_c\times\Omega}\left(\beta p_R(x,\xi)+\frac{\partial}{\partial\eta_x}p_R(x,\xi)\right)v(x,\xi)d\Gamma_c d\xi &= 0, \\
\int_{\Omega\times\Gamma_c}\left(\beta p_R(x,\xi)+\frac{\partial}{\partial\eta_\xi}p_R(x,\xi)\right)v(x,\xi)dxd\Gamma_c &= 0.
\end{aligned}
$$

Hence, the first two equations imply (3.30) which is satisfied almost everywhere in the space $L_2(\Omega\times\Omega)$. The latter two equations imply (3.31) which is only satisfied in the distributional sense. □

EXAMPLE 1. In the one-dimensional case, we can get stronger regularity results for the Riccati kernel $p_R(\cdot,\cdot)$. King obtained a result in [25] that can be extended to parabolic systems with Robin boundary control.

Theorem 3.6. *Assume $Q: L_2(0,l)\to L_2(0,l)$ is bounded. If $\Pi_R=\Pi_R^*$ is the unique, nonnegative-definite solution to the α-ARE* (3.23)*, then Π_R is Hilbert-Schmidt. Moreover,* *there exists a function* $p_R(x,\xi)$ *such that* Π_R *has the representation* (3.27) *where the kernel* $p_R(x,\xi)$ *satisfy the following conditions:*

(1) $p_R(x,\xi) \in C^1([0,l]\times[0,l])$.

(2) For each $\varphi \in L_2(0,l)$,

$$\frac{d}{dx}[\Pi_R\varphi](x) \;=\; \frac{\partial}{\partial x}\int_0^l p_R(x,\xi)\varphi(\xi)d\xi \;=\; \int_0^l \frac{\partial}{\partial x}p_R(x,\xi)\varphi(\xi)d\xi. \tag{3.37}$$

We look at the one-dimensional problem as described in Chapter 2. Using the above theorem and obtaining the kernel $q(x,\xi)$ for Q as shown above, we obtain the following theorem.

Theorem 3.7. *The kernel* $p_R(x,\xi)$ *of the operator* Π_R *is a weak solution to the Riccati partial differential equation (R-PDE)*$_R$

$$(A_x + A_\xi + 2\alpha I)p_R(x,\xi) - \beta^2 p_R(0,\xi)p_R(x,0) + q(x,\xi) \;=\; 0. \tag{3.38}$$

In particular, $p_R(\cdot,\cdot) \in C^1([0,l]\times[0,l])$ *satisfies the Dirichlet boundary conditions*

$$p_R(l,\xi) \;=\; p_R(x,l) \;=\; 0, \tag{3.39}$$

the Robin boundary conditions,

$$\left.\begin{aligned} \tfrac{\kappa_0}{L}p_R(0,\xi) - \kappa\tfrac{\partial}{\partial x}p_R(0,\xi) \;&=\; 0, \\ \tfrac{\kappa_0}{L}p_R(x,0) - \kappa\tfrac{\partial}{\partial \xi}p_R(x,0) \;&=\; 0, \end{aligned}\right\} \tag{3.40}$$

and the symmetry condition

$$p_R(x,\xi) \;=\; p_R(\xi,x), \quad \forall\, x,\xi \in (0,l). \tag{3.41}$$

Proof. To obtain (3.38) and (3.41), we follow the same steps as shown in the proof of Theorem 3.5. Since we have stronger regularity results for the Riccati kernel, we can prove that the boundary conditions are satisfied in a strong sense. From Theorem 3.6, $p_R(\cdot,\cdot)$ is in $C^1([0,l]\times[0,l])$. Suppose that, without loss of generality, $p_R(l,\xi_l) > 0$ for some fixed value of $\xi_l \in (0,l)$. Since $p_R(l,\cdot) \in C^1(0,l)$, there exists a neighborhood $N_l \subset (0,l)$ containing ξ_l such that $p_R(l,\xi) > 0$ for all $\xi \in N_l$. We know that $\Pi_R\varphi \in \mathcal{D}(A_0)$ and hence $[\Pi_R\varphi](l) = 0$ for all $\varphi \in L_2(0,l)$. If $\varphi(x) = 1$ on N_l and $\varphi(x) = 0$ elsewhere, then

$$[\Pi_R\varphi](l) \;=\; \int_0^l p_R(l,\xi)\varphi(\xi)d\xi \;=\; \int_{N_l} p_R(l,\xi)d\xi \;>\; 0, \qquad (3.42)$$

which contradicts the equality $[\Pi_R\varphi](l) = 0$. Therefore, $p_R(l,\xi) = 0$ for all $\xi \in (0,l)$. The other boundary condition in (3.39) follows from the symmetry condition.

Since $p_R(\cdot,\cdot) \in C^1([0,l]\times[0,l])$,

$$f(0,\cdot) \;\equiv\; \frac{\kappa_0}{L}p_R(0,\cdot) - \kappa\frac{\partial}{\partial x}p_R(0,\cdot) \;\in\; C(0,l).$$

Suppose that, without loss of generality, $f(0,\xi_0) > 0$ for some fixed $\xi_0 \in (0,l)$. There there exists a neighborhood $N_0 \subset (0,l)$ containing ξ_0 such that $f(0,\xi) > 0$ for all $\xi \in N_0$. Since $\Pi_R\varphi \in \mathcal{D}(A_0)$,

$$\frac{\kappa_0}{L}[\Pi_R\varphi](0) - \kappa\frac{d}{dx}[\Pi_R\varphi](0) \;=\; 0 \qquad (3.43)$$

for all $\varphi \in L_2(0,l)$. Using Theorem 3.6, we know that

$$\frac{d}{dx}[\Pi_R\varphi](0) \;=\; \int_0^l \frac{\partial}{\partial x}p_R(0,\xi)\varphi(\xi)d\xi.$$

If $\varphi(x) = 1$ on N_0 and $\varphi(x) = 0$ elsewhere, then

$$\begin{aligned}
\frac{\kappa_0}{L}[\Pi_R\varphi](0) - \kappa\frac{d}{dx}[\Pi_R\varphi](0) &= \frac{\kappa_0}{L}\int_0^l p_R(0,\xi)\varphi(\xi)d\xi - \kappa\int_0^l \frac{\partial}{\partial x}p_R(0,\xi)\varphi(\xi)d\xi \\
&= \int_0^l \left(\frac{\kappa_0}{L}p_R(0,\xi) - \kappa\frac{\partial}{\partial x}p_R(0,\xi)\right)\varphi(\xi)d\xi \\
&= \int_{N_0} \left(\frac{\kappa_0}{L}p_R(0,\xi) - \kappa\frac{\partial}{\partial x}p_R(0,\xi)\right)d\xi \\
&= \int_{N_0} f(0,\xi)d\xi \;>\; 0,
\end{aligned}$$

which contradicts (3.43). The other boundary condition of (3.40) follows from the symmetry condition. □

The feedback gain operator has the form

$$\begin{aligned}
K\varphi &= B^*\Pi_R\varphi \\
&= B^*\int_0^1 p_R(x,\xi)\varphi(\xi)d\xi \\
&= \frac{\epsilon\kappa_0}{\kappa L}\int_0^1 p_R(0,\xi)\varphi(\xi)d\xi.
\end{aligned}$$

Thus, the functional gain $h(\xi)$ is given by

$$h(\xi) \;=\; \frac{\epsilon\kappa_0}{\kappa L}p_R(0,\xi). \tag{3.44}$$

In Chapter 6, we use the finite element method to compute the Riccati kernel using (3.38) in the one-dimensional case. The computed approximate Riccati kernel then yields the approximate functional gain. However, note that, as the dimension of the physical domain increases, so does the number of spatial terms involved in the Riccati kernel. In particular, for $\Omega = \mathbb{R}^n$, the number of spatial terms defining the Riccati kernel is $2n$. Thus solving the Riccati PDE is time-consuming for higher dimensional spaces.

However, as noted in [12], [25], [26], the functional gains are often local in space and this observation opens the possibility of employing adaptive finite elements to reduce the computational complexity. Another possibility is described in the next chapter where we consider Chandrasekhar partial differential equations.

Chapter 4

Chandrasekhar Equations

The Chandrasekhar equations provide an alternate approach to the problem of computing optimal feedback gains. In certain cases it can dramatically reduce the number of equations to be solved. This is especially true for distributed parameter systems with a finite number of inputs and outputs. The origin of these equations date back to the astrophysicist S. Chandrasekhar and his work on radiative tranfser in the atmosphere [17].

The Chandrasekhar equations (also known as generalized X and Y functions, see [15]) can easily be derived from the matrix Riccati equation by differentiating these equations with respect to time (see Kailath [24]). Similarly, the Chandrasekhar equations were derived for a finite-dimensional system of equations involved in a control or filtering problem (see [14], [24], [32]). For infinite-dimensional systems, the generalized X and Y functions have been discussed in [16] using Lions' framework [30], but in a distributional sense. Sorine [42] derived a set of Chandrasekhar equations satisfied in a strong sense for parabolic systems relying on the analyticity of the semigroup generated by the operator A. Ito and

Powers [23] derived the same set of equations for a more general case using a sequence of approximating optimal control problems and the methods of [15] and [24]. Their discussion has been restricted to the LQR problem and the operator B being bounded. We review the method used by Sorine [42] to obtain the Chandrasekhar equations for the parabolic problem with Robin boundary control.

4.1 The General Theory

We consider two problems: the regular α-shifted LQR problem as described by the cost (3.19) and the abstract system (2.16), and the dual problem which is given by:

Find the optimal control $q^o \in L^2(0,\infty;Y)$ that minimizes the infinite-horizon cost function

$$\tilde{J}(q) = \int_0^\infty \left\{ \langle B^* z(t), B^* z(t) \rangle_U + |q|_Y^2 \right\} e^{\alpha t} dt \tag{4.1}$$

subject to the dual system of (2.16)

$$\left.\begin{aligned} \frac{d}{dt} z(t) &= (A^* + \alpha) z(t) + C^* q \\ z(0) &= z_0. \end{aligned}\right\} \tag{4.2}$$

We state the following definition from Wloka [46]:

Definition. *The bilinear form $a(\phi,\psi)$ is* **V-coercive** *if*

1. $|a(\phi,\psi)| \leq c||\phi||_V ||\psi||_V$ *for* $\phi, \psi \in V$, *and*

2. *there exist constants* λ *and* $\alpha > 0$ *with*

$$|a(\phi, \phi) + \lambda ||\phi||_H^2| \ \geq \ \alpha ||\phi||_V^2 \tag{4.3}$$

for all $\phi \in V$ *(Gårding's Inequality).*

We suppose that $A \in \mathcal{L}(V, V')$ such that $\langle A\phi, \psi \rangle$ satisfies Gårding's inequality, $B \in \mathcal{L}(E, V')$ and $C \in \mathcal{L}(V', F)$, where E and F are real, separable Hilbert spaces. Furthermore, we assume that the pair (A, B) is C-stabilizable and the pair (A, C) is B-detectable. That is,

Definition. *A pair* (A, B) *is* C-**stabilizable** *if for all* $\phi(t) \in H$, $C\phi(\cdot) \in L^2(0, \infty; F)$, *where* ϕ *is the corresponding solution to* (2.16).

Definition. *The pair* (A, C) *is* B-**detectable** *if the pair* (A^*, C^*) *is* B^**-stabilizable with respect to* (4.2).

From Sorine [42], we know that the problems (3.19) and (4.1) have unique solutions and these solutions can be found via solving the α-ARE (3.23) for Π and the dual algebraic Riccati equation (DARE)

$$\Lambda(A^* + \alpha) + (A + \alpha)\Lambda - \Lambda CC^*\Lambda + BB^* \ = \ 0 \tag{4.4}$$

for the dual Riccati operator Λ (see [28]) in $\mathcal{L}(V, V')$. Furthermore, we assume that the following regularity hypothesis is satisfied on the triple $\{V \times V, H \times H, (\mathcal{A}\phi, \psi)\}$ where

$$\mathcal{A} \ = \ \begin{bmatrix} A & BB^* \\ CC^* & -A^* \end{bmatrix}.$$

(R) There exists a Hilbert space D, $D \subset \mathcal{H}$ such that

(a) $\mathcal{V}$ is a closed subspace of $[D, \mathcal{H}]_{1/2}$, a fractional space as defined precisely in [31], and

(b) $\mathcal{D}(\mathcal{A}) \subset D \times D$, $\mathcal{D}(\mathcal{A}^*) \subset D \times D$.

With these conditions and definitions, we can extend the propositions and theorems of Sorine's paper to the α-LQR problem.

Sorine [42] was able to obtain the following result using semigroup methods.

Theorem 4.1. *Under the hypothesis* **(H)**, *the Riccati operator $P(t)$ satisfies the differential Riccati equation (DRE):*

$$\frac{d}{dt}P(t) = P(t)(A+\alpha) + (A^*+\alpha)P(t) - P(t)BB^*P(t) + Q, \tag{4.5}$$

$$P(T) = 0, \tag{4.6}$$

in the space $\mathcal{L}(V, V')$. Moreover, $t \to P(t)$ is the only solution of the DRE in the class

$$P(\cdot) \in C([0,\infty); \mathcal{L}_S(H,H)) \cap C^1((0,\infty); \mathcal{L}_S(H,H)) \tag{4.7}$$

($\mathcal{L}_S$ is the space $\mathcal{L}$ with the topology of strong convergence) and in fact, for all $t > 0$,

$$P(t) \in \mathcal{C}^+ \cap \mathcal{L}(V, V) \tag{4.8}$$

where $\mathcal{C}^+$ is the cone of $\mathcal{L}(H,H)$ consisting of the self-adjoint, positive semidefinite operators.

This theorem led to the following proposition, which is used to obtain the Chandrasekhar equations. Note that to derive this proposition and the following results, Sorine used the semigroup $e^{-t(A^*-CC^*\Lambda)}$ generated by the operator $A^* - CC^*\Lambda$ where Λ is the solution of the DARE (4.4) (see [42]). Again, we extend Sorine's work to account for the α-shifting of the LQR problem by replace A with $A+\alpha$.

Proporsition 4.1. *Under the hypotheses* **(H)** *and by the definitions*

$$\left.\begin{aligned} P(t) &= \mathcal{P}(t)(0), \\ \phi(t) &= e^{-t(A^*+\alpha-CC^*\Lambda)}(I+\Lambda P(t)), \end{aligned}\right\} \tag{4.9}$$

then for all $t>0$, $P(t)$ and $\phi(t)$ satisfy

$$\left.\begin{aligned} \tfrac{d}{dt}P(t) &= \phi^*(t)C^*C\phi(t), \\ \tfrac{d}{dt}\phi(t) &= -\phi(t)(A+\alpha-BB^*P(t)), \\ P(T) &= 0, \qquad \phi(T) = I, \end{aligned}\right\} \tag{4.10}$$

in the space $\mathcal{L}(V,V')$.

In order to obtain the spaces that contain $K(\cdot)$ and $L(\cdot)$, we need another proposition that Sorine put forth in his paper. The proposition is as follows:

Proporsition 4.2. *Under the hypothesis* **(H)***,*

$$[\mathcal{P}(\cdot)](0) \in C^\infty((0,\infty);\mathcal{L}_S(V,V)) \tag{4.11}$$

and

$$(I+\Lambda[\mathcal{P}(\cdot)](0))^{-1} \in C^\infty((0,\infty);\mathcal{L}_U(V,V)). \tag{4.12}$$

The optimal gain for the control problem associated with (2.16) is given by

$$K(t) \;=\; B^*P(t). \tag{4.13}$$

If one also defines an operator $L(\cdot)$ by

$$L(t) \;=\; Ce^{-t(A^*+\alpha-CC^*\Lambda)}(I+\Lambda P(t)), \tag{4.14}$$

one may obtain a differential system in $P(\cdot)$ and $L(\cdot)$, or for this thesis, a system in $K(\cdot)$ and $L(\cdot)$ which is given in the following result.

Theorem 4.2. *Under the hypothesis* **(H)**, $P(\cdot)$, $K(\cdot)$ *and* $L(\cdot)$ *defined by* (4.9), (4.13) *and* (4.14), *respectively, satisfy:*

$$K \;\in\; C^\infty((0,\infty);\mathcal{L}(V,E)), \quad L \;\in\; C^\infty((0,\infty);\mathcal{L}(V',F)), \tag{4.15}$$

$$\left.\begin{aligned} \frac{d}{dt}P(t) &= L^*(t)L(t), \\ \frac{d}{dt}L(t) &= -L(t)(A+\alpha-BB^*P(t)), \\ P(T) &= 0, \quad L(T) \;=\; C, \end{aligned}\right\} \tag{4.16}$$

in $\mathcal{L}(V,V')$ *for all* $t>0$, *and*

$$\left.\begin{aligned} \frac{d}{dt}K(t) &= B^*L^*(t)L(t), \\ \frac{d}{dt}L(t) &= -L(t)(A+\alpha-BK(t)), \\ K(T) &= 0, \quad L(T) \;=\; C, \end{aligned}\right\} \tag{4.17}$$

in $\mathcal{L}(V,V')$ *for all* $t>0$.

Proof. (4.15) is a direct consequence of Proposition 4.2. Since $B^* \in \mathcal{L}(V,E)$, then by the definition of $K(\cdot)$ and Proposition 4.2, we obtain

$$K \;\in\; C^\infty((0,\infty);\mathcal{L}(V,E)).$$

By the definition of $L(\cdot)$ and Propostion 3.2, we then obtain

$$L \in C^{\infty}((0,\infty);\mathcal{L}(V',F)).$$

The system of equations (4.16) follows from the Proposition 4.1 and the definitions of $K(\cdot)$ and $L(\cdot)$. □

This theorem provides a system of equations with which one could compute the feedback gain directly, that is, without the need to compute the Riccati operator. In finite dimensional cases, the number of computations needed to derive the solution to (4.17) could be much less than those needed to solve (3.23). The next chapter will touch on this issue. The system of equations (4.17) is known as the Chandrasekhar equations. However, solving the equations (4.17) does not neccesarily produce a unique solution. With the satisfaction of the regularity hypothesis **(R)**, uniqueness can be established as shown by Sorine [42].

Theorem 4.3. *Under the hypothesis* **(H)** *and* **(R)***, the Chandrasekhar equations defined by* (4.17) *have a unique solution in the class of* $K(\cdot)$ *and* $L(\cdot)$ *satisfying*

$$K(\cdot) \in C^0([0,\infty);\mathcal{L}_S(V,E)) \cap C^1((0,\infty);\mathcal{L}_S(V,E)), \tag{4.18}$$

$$L(\cdot) \in C^0([0,\infty);\mathcal{L}_S(V,F)) \cap C^1((0,\infty);\mathcal{L}_S(V',F)) \tag{4.19}$$

and

$$P(t) = \int_0^t L^*(\tau)L(\tau)d\tau \in C^1([0,\infty);\mathcal{L}(V,V')) \cap C^1((0,\infty);\mathcal{L}(H,H)). \tag{4.20}$$

Since we are considering the infinite-time control problem, we wish to compute the time-invariant feedback gain K. Under suitable conditions, one can show that $K(t) \to K$ as $t \to -\infty$. This is important to know so that we can intuit that the same idea would apply to the functional gains. In the following section, we present a new method for computing the functional gain of the α-LQR problem directly using the Chandrasekhar partial differential equations.

4.2 Chandrasekhar PDE's

We can apply Sorine's result to the heat problem (2.16) and (3.19). For the problem considered here with the assumption that $C \in \mathcal{L}(H, \mathbb{R})$, the following proposition can be shown.

Proporsition 4.3. *Under the hypotheses and assumption that $C \in \mathcal{L}(H, \mathbb{R})$, there exist unique functions $h(t, \cdot)$, $l(t, \cdot)$ such that the operators $K(t)$ and $L(t)$ have unique integral representations:*

$$[K(t)\varphi](x) = \int_\Omega h(t, x, \xi)\big|_{\Gamma_c \times \Omega} \varphi(\xi) d\xi, \qquad (4.21)$$

$$L(t)\varphi = \int_\Omega l(t, \xi)\varphi(\xi) d\xi. \qquad (4.22)$$

Proof. By the previous section which shows the results of Sorine [42], we have that $K(t) \in \mathcal{L}(H, L_2(\Gamma))$ and $L(t) \in \mathcal{L}(H, \mathbb{R})$. By the Riesz Representation Theorem, there exists a unique kernel $l(t,\cdot) \in L_2(\Omega)$ such that

$$L(t)\varphi = \int_\Omega l(t,\xi)\varphi(\xi)d\xi$$

and $||L(t)||_{\mathcal{L}(H,\mathbb{R})} = ||l(t,\cdot)||$. Then, using the fact that

$$K(t)\varphi = \int_0^t B^* L^*(\tau)L(\tau)\varphi d\tau$$

from (4.17), we substitute the integral representation (4.22) into the above equation. Thus, we get

$$\begin{aligned} K(t)\varphi &= \int_0^t [B^* l](\tau,\cdot)\left(\int_\Omega l(\tau,\xi)\varphi(\xi)d\xi\right) d\tau \\ &= \int_0^t \frac{\epsilon\kappa_0}{\kappa L} l(\tau,x)\big|_{\Gamma_c}\left(\int_\Omega l(\tau,\xi)\varphi(\xi)d\xi\right) d\tau \\ &= \int_\Omega \int_0^t \frac{\epsilon\kappa_0}{\kappa L} l(\tau,\xi)l(\tau,x)\big|_{\Gamma_c}\varphi(\xi)d\tau d\xi. \end{aligned}$$

Thus, the feedback operator has an integral representation with the kernel

$$h(t,x,\xi)\big|_{\Gamma_c\times\Omega} = \frac{\epsilon\kappa_0}{\kappa L}\int_0^t l(\tau,\xi)l(\tau,x)\big|_{\Gamma_c}.$$

Note that the kernel of $K(t)$ is unique due to the uniqueness of $l(t,\cdot)$ and is differentiable with respect to time. Hence, the proposition is proved. □

With the integral representations (4.21) and (4.22) of the operators of (4.17), we can now prove the following theorem.

Theorem 4.4. *The functional gain $h(t,\cdot,\cdot)$ of $K(t)$ and the kernel $l(t,\cdot)$ of $L(t)$ are weak solutions to the Chandrasekhar partial differential equation (C-PDE)*

$$\frac{\partial}{\partial t}h(t,x,\xi) = [B^*l(t)]l(t,\xi), \tag{4.23}$$

$$\frac{d}{dt}l(t,\xi) = (A+\alpha I)l(t,\xi) + \int_{\Gamma_c}[B^*l(t)]h(t,x,\xi)dx. \tag{4.24}$$

Moreover, the kernels satisfy the following conditions:

the Dirichlet boundary conditions

$$h(t,x,\xi)\big|_{\Gamma_c\times\Gamma_0} = 0, \quad l(t,\xi)\big|_{\Gamma_0} = 0, \tag{4.25}$$

Robin boundary conditions

$$\beta_1 h(t,x,\xi)\big|_{\Gamma_c\times\Gamma_c} + \beta_0\frac{\partial}{\partial\eta_\xi}h(t,x,\xi)\big|_{\Gamma_c\times\Gamma_c} = 0, \tag{4.26}$$

$$\beta_1 l(t,\xi)\big|_{\Gamma_c} + \beta_0\frac{\partial}{\partial\eta_\xi}l(t,\xi)\big|_{\Gamma_c} = 0, \tag{4.27}$$

and the initial conditions

$$h(t_f,x,\xi) = 0, \quad l(t_f,\xi) = c(\xi). \tag{4.28}$$

Proof. The equation (4.23) follows immediately from the definition of the functional gain given in the proof of Proposition 4.3. That is, B^* is defined by

$$B^*l(t,\mathbf{x}) = \frac{\epsilon\kappa_0}{\kappa L}l(t,\mathbf{x})\big|_{\Gamma_c}.$$

To obtain (4.24), we use the same method as shown in the proof of Theorem 3.5. The weak form of the second equation of (4.17) is given as

$$\langle\dot{L}(t)\varphi,u\rangle = -\langle A\varphi,L^*(t)u\rangle - \alpha\langle L(t)\varphi,u\rangle + \langle K(t)\varphi,B^*L^*(t)u\rangle. \tag{4.29}$$

Substituting the representations of $K(t)$ and $L(t)$ (and dividing out u), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega l(t,\xi)\varphi(\xi)d\xi &= -\epsilon\int_\Omega \frac{\partial}{\partial\xi}l(t,\xi)\varphi'(\xi)d\xi - \alpha\int_\Omega l(t,\xi)\varphi(\xi)d\xi \\ &\quad -\frac{\epsilon\kappa_0}{\kappa L}\int_{\Gamma_c} l(t,\xi)\big|_{\Gamma_c}\varphi(\xi)\big|_{\Gamma_c}d\Gamma_c \\ &\quad +\frac{\epsilon\kappa_0}{\kappa L}\int_{\Gamma_c} l(t,x)\big|_{\Gamma_c}\left(\int_\Omega h(t,x,\xi)\varphi(\xi)d\xi\right)d\Gamma_c,\end{aligned}$$

for $\varphi \in H^1(\Omega)$. The conditions (4.28), follow from the fact that $K(t_f) = 0$ and $L(t_f) = C$. In particular, for all $\phi \in H$,

$$K(t_f)\phi = \int_\Omega h(t_f,x,\xi)\phi(\xi)d\xi = 0$$

which implies that $h(t_f,x,\xi) = 0$. Similarly, we have for all $\phi \in H$,

$$L(t_f)\phi = \int_\Omega l(t_f,\xi)\phi(\xi)d\xi = C\phi = \int_\Omega c(\xi)\phi(\xi)d\xi,$$

and thus, it follows that $l(t_f,\xi) = c(\xi)$. The boundary conditions follow from the boundary conditions on $P(t)\phi$. □

For the one-dimensional case discussed in Chapter 2, the Riesz Representation Theorem yields a functional gain $h(\cdot)$ so that

$$K\varphi = \int_0^l h(\xi)\varphi(\xi)d\xi. \tag{4.30}$$

From Sorine's paper [42], we have that $K(t) \in \mathcal{L}(L_2(0,l),\mathbb{R})$ and $L(t) \in \mathcal{L}(L_2(0,l),\mathbb{R})$. Therefore, by Riesz Representation Theorem, the feedback operator K and the operator L have the forms

$$K(t)\varphi = \int_0^l h(t,\xi)\varphi(\xi)d\xi, \tag{4.31}$$

and

$$L(t)\varphi \;=\; \int_0^l l(t,\xi)\varphi(\xi)d\xi, \tag{4.32}$$

respectively, where $h(t,\cdot),\ l(t,\cdot) \in L_2(0,l)$.

Substituting the representations of K and L into (4.17), the first equation is

$$\begin{aligned}\dot{K}(t)\varphi \;=\; -B^*L^*(t)L(t)\varphi \quad \Rightarrow \quad \frac{d}{dt}\int_0^l h(t,\xi)\varphi(\xi)d\xi \;&=\; -B^*L^*(t)\int_0^l l(t,\xi)\varphi(\xi)d\xi \\ &=\; -B^*l(t,x)\int_0^l l(t,\xi)\varphi(\xi)d\xi \\ &=\; -\big(\beta l(t,0)\big)\int_0^l l(t,\xi)\varphi(\xi)d\xi.\end{aligned}$$

The variational problem becomes: Find $h(t,\cdot),\ l(t,\cdot) \in L_2(0,l)$ such that

$$\tfrac{d}{dt}\textstyle\int_0^l h(t,\xi)\varphi(\xi)d\xi \;=\; -\beta l(t,0)\int_0^l l(t,\xi)\varphi(\xi)d\xi,$$

$$\tfrac{d}{dt}\textstyle\int_0^l l(t,\xi)\varphi(\xi)d\xi \;=\; -\int_0^l l(t,\xi)\varphi''(\xi)d\xi - \alpha\int_0^l l(t,\xi)\varphi(\xi)d\xi + \beta l(t,0)\int_0^l h(t,\xi)\varphi(\xi)d\xi,$$

for all $\varphi \in \mathcal{D}(A_0)$.

From the previous section, we can use the same idea to obtain a representation of the Riccati operator $P(t)$, which is the (weak) solution of the differential Riccati equation (DRE):

$$\dot{P}(t) \;=\; -A^*P(t) - P(t)A - Q + P(t)BB^*P(t). \tag{4.33}$$

The integral representation of $P(t)$ has the form

$$[P(t)\varphi](x) \;=\; \int_0^l p(t,x,\xi)\varphi(\xi)d\xi, \tag{4.34}$$

where $\varphi \in L_2(0,l)$. $p(t,\cdot,\cdot) \in C^1([0,l]\times[0,l])$ will satisfy the same boundary conditions as the kernel for Π.

The feedback operator $K(t)$ is related to $P(t)$ by

$$K(t) \;=\; B^*P(t).$$

Hence, we have the functional gain $h(t,\cdot)$ is defined by

$$h(t,\xi) \;=\; p(t,0,\xi).$$

Since $p(t,\cdot,\cdot) \in C^1([0,l]\times[0,l])$ and $p(t,l,\xi) = 0$ for all $\xi \in (0,l)$, then $h(t,l) = 0$. It is known that

$$[P(t)\varphi](x) \;=\; \int_0^t [L^*(s)L(s)\varphi](x)ds.$$

Thus, we have the following by substitution of representations:

$$\begin{aligned}\int_0^l p(t,x,\xi)\varphi(\xi)d\xi \;&=\; \int_0^t l(s,x)\left(\int_0^l l(s,\xi)\varphi(\xi)d\xi\right)ds \\ &=\; \int_0^l \left(\int_0^t l(s,x)l(s,\xi)ds\right)\varphi(\xi)d\xi.\end{aligned}$$

Hence,

$$p(t,x,\xi) \;=\; \int_0^t l(s,x)l(s,\xi)ds. \tag{4.35}$$

Using the boundary conditions for $p(t,x,\xi)$ we have

$$p(t,l,\xi) \;=\; \int_0^t l(s,l)l(s,\xi)ds \;=\; 0.$$

Since $l(t,\xi)$ must be nontrivial over time (from nontriviality of $p(t,x,\xi)$ over time), $l(t,l) = 0$. Similarly, we can show that $h(t,\xi)$ and $l(t,\xi)$ must satisfy the Robin boundary condition at $\xi = 0$.

Chapter 5

Approximations

In this chapter, we discuss the finite element method and some convergence results for the numerical algorithm to compute the functional gains. Four different methods were presented in the last two chapters for computing the functional gains. Before we discuss the difference in these methods, we present the scheme used to approximate the equations and hence compute the functional gains.

5.1 Finite Element Method

We approximate the space V by an appropriate subspace, V_n, where n, $0 < n < \infty$, is a discretizing parameter which tends to infinity. For the finite element scheme, these spaces are normally constructed as a space of linear combinations of splines.

EXAMPLE 1. For the one-dimensional problem, the approximate subspace of $H^1(0, l)$ would be a space of linear combinations of a basis of polynomials, possibly b-splines. Our approximating subspace is given by

$$V_n = \{\sum_{i=0}^{n+1} a_i h_i(x) \mid a_i \in \mathbb{R}\}.$$

The functions, $h_i(x)$, are usually "hat" functions. Other splines that are commonly used are parabolic splines or cubic splines, but for these cases we would have to reorder the summation in V_n. For the numerical results in the next chapter, we use "hat" functions:

$$h_i(x) = \begin{cases} (x - x_{i-1})/h, & x_{i-1} \leq x < x_i \\ (x_{i+1} - x)/h, & x_i \leq x \leq x_{i+1} \\ 0, & otherwise, \end{cases} \tag{5.1}$$

where $h = (il)/(n+1)$ for $i = 1, \ldots, n$, and n is the number of subintervals of $(0, l)$. At $x = 0$ and $x = l$, $h_0(x)$ and $h_{n+1}(x)$ are given by

$$h_0(x) = \begin{cases} (h - x)/h, & 0 \leq x < h \\ 0, & otherwise, \end{cases}$$

and

$$h_{n+1}(x) = \begin{cases} (x - x_n)/h, & x_n \leq x \leq l \\ 0, & otherwise, \end{cases}$$

respectively.

EXAMPLE 2. For the two- and higher dimensional problems, the exercise of discretizing V becomes more complex. First, we must be concerned with how the domain will be discretized. Common practices include triangularization or separating the domain into rectangular parts. Rubio [41] discretizes her space using the rectangles and bilinear functions. However, in this thesis, we use triangularization to discretize the two-dimensional square (see Figure 5.1) since it is the common method of discretization.

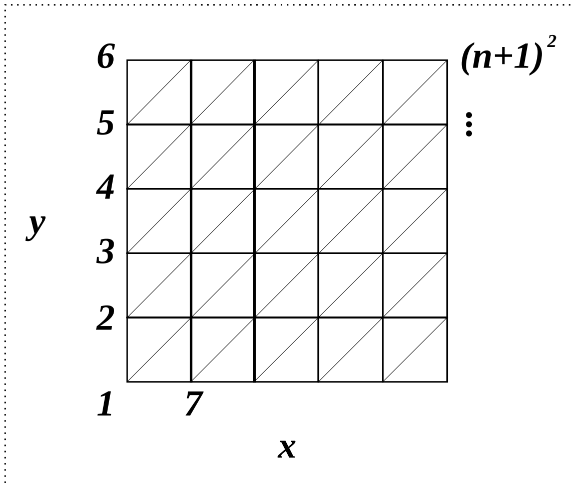

Figure 5.1: Discretization of Square

Note that ordering of the nodes is sometimes important in finite elements. To keep things simple, we number the nodes from the origin and then go up the y-axis, as can be seen in Figure 5.1. Using the elements, we construct basis functions like those in the first example. The basis functions can be linear, quadratic or have an even higher degree of freedom. We use functions of the following form:

$$\phi_i(x, y) \;=\; a + bx + cy + dxy,$$

where the constants a, b, c and d are determined by the elements and their nodes. (See, e.g., [10, 21, 39] and the references therein for details on constructing finite element spaces in two dimenions.) Our approximating space is given by

$$V_n \;=\; \{ \sum_{i=0}^{(n+1)^2} a_i\phi_i(x, y) \mid a_i \in \mathbb{R}\},$$

where the unit square is divided into $n \times n$ triangular elements.

Once a finite element subspace is selected, the next approach is to build matrix representations of the operators of the system (2.16) or (2.18) with cost function (3.19). In particular, we obtain $A^n \approx A,\ B^n \approx B,\ C^n \approx C$ and $R^n \approx R$. (For all of the examples, we let $R = cI$ where c is some constant.) These matrix representations yield approximate Riccati equations, Riccati PDE and Chandrasekhar PDEs, each of which can be solved using the appropriate numerical algorithm.

It is clear that one can generate finite element approximations to the various equations by projecting these systems onto the various finite element spaces. In particular, the equation (2.16) can be put in variational form as follows:

$$\langle \dot{z(t)}, \varphi \rangle \;=\; \langle Az(t), \varphi \rangle + \langle Bu(t), \varphi \rangle, \tag{5.2}$$

for all $\varphi \in H^1(\Omega)$ and where A and B are operators defined at the end of Chapter 2. Further, we then obtain the following form for (5.2):

$$\begin{aligned} \int_\Omega \dot{z}(t,\mathbf{x})\phi(\mathbf{x})d\mathbf{x} \;&=\; -\beta \int_{\Gamma_c} z(\mathbf{x})\phi(\mathbf{x})d\Gamma_c - \epsilon \int_\Omega \nabla z(t,\mathbf{x})\nabla\phi(\mathbf{x})d\mathbf{x} \\ &\qquad -\alpha \int_\Omega z(t,\mathbf{x})\phi(\mathbf{x})d\mathbf{x} + \beta \int_{\Gamma_c} u(t)\phi(\mathbf{x})d\Gamma_c \end{aligned} \tag{5.3}$$

for all $\phi \in H^1(\Omega)$ and where $\beta = (\epsilon\kappa_0)/(\kappa L)$. We will apply the standard finite element method to solve the control problem.

We approximate the solution $z(t,\cdot) \in H^1(\Omega)$ with the above splines that make up the basis for the finite element spaces. The approximation $z^N(t,\cdot)$ to $z(t,x)$ is given by

$$z^n(t,\mathbf{x}) \;=\; \sum_{i=0}^{n+1} z_i^n(t)\phi_i^n(\mathbf{x}), \tag{5.4}$$

where ϕ_i^n are "hat" functions in the one-dimensional problem or bilinear functions in the two-dimensional problem. When we substitute this into the variational form (5.3) we obtain the following:

$$\begin{aligned}
\sum_{i=0}^{N} z_i^n(t) \int_\Omega \phi_i^n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} &= -\beta \sum_{i=0}^{N_l} z_i^n(t) \int_{\Gamma_c} \phi_i^n(\mathbf{x})\phi(\mathbf{x})d\Gamma_c \\
&\quad -\epsilon \sum_{i=0}^{N} z_i^n(t) \int_\Omega \nabla\phi_i^n(\mathbf{x})\nabla\phi(\mathbf{x})d\mathbf{x} \\
&\quad -\alpha \sum_{i=0}^{N} z_i^n(t) \int_\Omega \phi_i^n(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} \\
&\quad +\beta \sum_{i=0}^{N_l} u_i(t) \int_{\Gamma_c} \phi_i^n(\mathbf{x})\phi(\mathbf{x})d\Gamma_c,
\end{aligned} \tag{5.5}$$

where N is the number of vertices in the grid of the domain and N_l is the number of vertices describing the control along the face of the domain. The test function $\phi(\cdot)$ ranges over the basis functions. We then compute the integrals and we obtain the following ordinary differential equation

$$\dot{\mathbf{z}}(t) = A^N\mathbf{z}(t) + B^N\mathbf{u}(t) \tag{5.6}$$

where our initial condition is given by

$$\mathbf{z}(0) = \mathbf{z}_0 \tag{5.7}$$

where $\mathbf{z}_0$ is the vector of coefficients of the approximate initial condition $z_0^n(x)$, the projection of $z_0(x)$ onto the subspace V_n.

EXAMPLE 1. We find the solution to the problem in the subspace $V_n \subseteq H^1(0,l)$, where V_n is the space spanned by $h_i(x)$, $i = 0, \ldots, n+1$, as shown above.

In V_n, the matrix representation for A is $A^n = (M^n)^{-1}S^n$ where M^n and S^n are the mass matrix and stiffness matrix, respectively. These matrices are given by

$$M^n = \frac{h}{6}\begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ 0 & 1 & 4 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4 \end{bmatrix}, \qquad S^n = \frac{\epsilon}{h}\begin{bmatrix} 1+\beta h & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix},$$

The topmost row and leftmost column are truncated for the Dirichlet problem. The approximation B^n has the representation $(M^n)^{-1}\tilde{B}^n$ where

$$\tilde{B}^n = \frac{\epsilon}{h}\begin{bmatrix} \beta \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{5.8}$$

EXAMPLE 2. In 2-dimensional situations, the construction of finite element approximations to the system of equations becomes more complex. For two-dimensional examples, we do not look at the computation of the functional gain via the Riccati PDE's because (3.29) is an equation of four spatial variables, which can be time consuming and complex in approximating and solving. We focus on the ARE method and the direct method by using the Chandrasekhar PDE's. Gaussian quadrature is used to evaluate the integrals of (5.5). For more details about constructing these matrices and using Gaussian quadrature, see [43].

We compute the functional gains by projecting= the α-ARE, R-PDE and C-PDE onto the appropriate spaces. Once we obtain $A^n \approx A$, $B^n \approx B$, $Q^n \approx Q$ and $R^n = R = cI$, we then have an approximate Riccati equation:

$$\Pi^n(A^n + \alpha I^n) + (A^n + \alpha I^n)\Pi^n - \Pi^n B^n (B^n)^T \Pi^n + Q^n = 0. \tag{5.9}$$

Using *MatLab* and the `lqr` command, we can solve for Π^n. The approximate solution Π^N yields an approximation $h^N(\cdot)$ of $h(\cdot)$ as follows:

$$h^n = (M^n)^{-1}K^n$$

where K^n is the approximate solution to the feedback operator K and

$$K^n = R^n(B^n)^T\Pi^n.$$

The number of unknowns to solve for the approximate α-ARE (5.9) is $n(n+1)/2$ due to the symmetry of the Riccati operator. However, this is an indirect way of obtaining the approximate functional gain and the number of unknowns to be solved can be reduced.

We can obtain the approximation of K directly by using the approximations of A, B and C. We solve the approximate Chandrasekhar equations

$$\left.\begin{aligned} \dot{\mathbf{K}}^N(t) &= [\mathbf{B}^N]^*[\mathbf{L}^N]^*(t)\mathbf{L}^N(t), \\ \dot{\mathbf{L}}^N(t) &= \mathbf{L}^N(t)\big(\mathbf{A}^N - \mathbf{B}^N\mathbf{K}^N(t)\big). \end{aligned}\right\} \tag{5.10}$$

We can use a numerical algorithm and asymptotically achieve a steady-state solution to K. This will yield the approximation to the functional gain. An algorithm that proves to be very efficient and fast is the one derived by Banks and Ito in [4], known as the hybrid method. However, we are most interested in using the finite element method to approximate the Riccati and Chandrasekhar PDE's.

It is clear that one can generate finite element approximations to the Riccati and Chandrasekhar partial differential equations by projecting these systems onto the various finite element spaces. For the approximate Riccati PDE, we project the various functions onto the subspace, $V_n \times V_n$, in $H^1((0,l)\times(0,l))$ spanned by the bilinear basis functions $h_i(x)h_i(y)$ for $i = 0, 1, \ldots, n$ where h_i's are defined above. Notice that we only do this for the one-dimensional case, because the Riccati PDE gets too complicated in higher dimensions. For the one-dimensional problem, we are solving for a function of two dimensions. Thus, the Riccati kernel $p(\cdot,\cdot)$ is approximated as

$$p^n(x,y) \;=\; \sum_{i,j=0}^{n} p_{ij} h_i(x) h_j(y).$$

Our test functions will be the basis functions of V_n, and hence we derive the approximate Riccati PDE:

$$\mathbf{A}^N \mathbf{p}^N + [\mathbf{A}^*]^N \mathbf{p}^N - (\mathbf{B}_1^N \mathbf{p}^N) \cdot (\mathbf{B}_2^N \mathbf{p}^N) \;=\; \mathbf{q}^N, \tag{5.11}$$

where $\mathbf{p}^N = [p_{ij}]$ is an $n(n-1)$ vector and $\mathbf{q}^N$ is the approximation to the kernel of the Q operator. Using Newton's method, we solve for $\mathbf{p}^N$. This will yield the approximate functional gain $\mathbf{h}^N(\cdot)$. We know that the functional gain can be represented by the Riccati kernel as follows

$$h(\cdot) \;=\; [B^* p](\cdot,\cdot) \;=\; \beta p(0,\cdot). \tag{5.12}$$

Hence, the matrix representation $\mathbf{h}^N$ of the functional gain $h(\cdot)$ is given by a multiple of the coefficients p_{ij} where $h_i(0) = 1$.

Using the Chandrasekhar PDE's, we can compute the approximate functional gain $\mathbf{h}^N(\cdot)$ directly without solving for the Riccati kernel. Here we project the system of equations onto the space spanned by linear functions as shown at the beginning of this section. Then we obtain the set of ODE's as follows:

$$\left.\begin{aligned}\dot{\mathbf{k}}^N(t) &= \big([\tilde{\mathbf{B}}^*]^N \mathbf{l}^N(t)\big)\mathbf{l}^N(t),\\ \dot{\mathbf{l}}^N(t) &= \mathbf{A}^N\mathbf{l}^N(t) + \big([\tilde{\mathbf{B}}^*]^N \mathbf{l}^N(t)\big)\mathbf{k}^N(t).\end{aligned}\right\} \tag{5.13}$$

This system of equations can be solved using a Runge-Kutta method, or perhaps the method suggested by Banks and Ito (see [4]). One advantage of using this is that we retain sparsity in the matrices used. For example, the mass matrix in the first equation is eliminated all together, while in the second equation, we use the mass matrix for the computation of $\mathbf{A}^N$. We solve this set of equations in forward time until a steady-state solution is reached or until $\mathbf{l}^N(\cdot)$ is the zero function.

We need to show that the solutions to these approximate equations converge to the actual solution of the original equations. However, showing convergence is a complex issue especially for the Chandrasekhar PDEs. We will save some of the details for future research. For now, we review some general results from [28].

5.2 General Approximation Theory

Given a family of approximating subspaces $V_n \subset V$ where n, $0 < n < \infty$, is a discretizing parameter which tends to infinity, we let P_n be the orthogonal projection operator with the property

$$||P_n\varphi - \varphi||_H \;\to\; 0, \quad \varphi \in H. \tag{5.14}$$

Let $A_n : V_n \to V_n$ be an approximation of A which satisfies the following requirements:

$$\left|A_n e^{A_n t}\right|_{\mathcal{L}(H)} \;\leq\; \frac{c}{t} e^{(\omega+\varepsilon)t}, \quad t > 0, \tag{5.15}$$

and

$$\left|P_n A^{-1} - A_n^{-1} P_n\right|_{\mathcal{L}(H)} \;\leq\; ch^s \;\text{ for some }\; s > 0. \tag{5.16}$$

Assume $B : U \to V'$ and $B_n : U \to V_n$ satisfy the following properties:

$$||B^* \varphi_n||_U + ||B_n^* \varphi_n||_U \;\leq\; ch^{-\gamma s} ||\varphi_n||_H, \quad \forall\, \varphi_n \in V_n, \tag{5.17}$$

$$||B^*(P_n - I)\varphi||_U \;\leq\; ch^{s(1-\gamma)} ||\varphi||_V, \quad \varphi \in V, \tag{5.18}$$

$$||B^*\varphi - B_n^* P_n \varphi||_U \;\leq\; ch^{s(1-\gamma)} ||\varphi||_V, \quad \varphi \in V, \tag{5.19}$$

and

$$||B^* P_n \varphi||_U \;\leq\; c||(A^*)^\gamma \varphi||_H, \quad \varphi \in \mathcal{D}((A^*)^\gamma). \tag{5.20}$$

Lasiecka and Triggiani [28] note that assumptions (5.16)-(5.18) are satisfied by typical approximation schemes, in particular, by finite elements.

Given the approximate α-ARE,

$$\langle (A_n + \alpha)\varphi, \Pi_n \psi\rangle + \langle \Pi_n \varphi, (A_n + \alpha)\psi\rangle - \langle B_n^* \Pi_n \varphi, B_n^* \Pi_n \psi\rangle_U + \langle C^* C \varphi, \psi\rangle \;=\; 0, \tag{5.21}$$

one obtains the following theorem from Lasiecka and Triggiani [28].

Theorem 5.1. *Given that the above assumptions are satisfied, there exists an $n_0 > 0$ such that for all $n < n_0$, the solution Π_n to the equation (α-ARE)$_n$* (5.21) *exists, is unique and the following holds:*

$$|\Pi - \Pi_n|_{\mathcal{L}(H)} \;\rightarrow\; 0, \tag{5.22}$$

and

$$|B_n^*\Pi_n - B^*\Pi|_{\mathcal{L}(H)} \;\rightarrow\; 0. \tag{5.23}$$

Using this theorem, we can obtain an approximate solution to the Riccati operator via (3.23). This computation would then yield the approximate functional gain $h_n(\cdot)$ to $h(\cdot)$ via the approximate feedback operator. It follows from the above theorem that $h_n \rightarrow h$. The application of the approximating theory is applied to the Dirichlet boundary control problem in [28]. We shall now apply it to the Robin problem.

5.3 Heat Equation with Robin Boundary Control

We select as the approximating space $V_n \subset V$ to be a space of linear splines, i.e., we are using the finite element method of approximation. This complies with the following properties [28]:

$$||P_n\varphi - \varphi||_{H^l(\Omega)} \;\leq ch^{s-l}||\varphi||_{H^s(\Omega)}, \quad s \leq 2, \quad s-l \geq 0, \quad 0 \leq l \leq 1, \tag{5.24}$$

and

$$||\varphi_n||_{L^2(\Gamma)} + h||\frac{\partial \varphi_n}{\partial \nu}||_{L_2(\Gamma)} \;\leq\; ch^{-1/2}||\varphi_n||_{L^2(\Omega)}, \quad \varphi_n \in V_n. \tag{5.25}$$

We define $A_n : V_n \to V_n$ by

$$\langle A_n\varphi_n, \psi_n\rangle \;=\; \langle A\varphi_n, \psi_n\rangle, \quad \varphi_n, \psi_n \in V_n. \tag{5.26}$$

We define $B_n : U \to V_n$ by

$$\langle B_n u, \varphi_n\rangle \;=\; \int_{\Gamma_c} u\varphi_n\big|_{\Gamma_c} d\Gamma_c. \tag{5.27}$$

Hence, we have

$$B_n^*\varphi_n \;=\; \varphi_n\big|_{\Gamma_c}.$$

The optimal feedback control for the approximating problem is

$$u_n(t) \;=\; -B_n^*\Pi_n T_n(t)$$

where Π_n satisfies (α-ARE)$_n$.

Assumption (5.15) is satisfied because of the results on Galerkin approximations of elliptic operators from [9] for self-adjoint operators. The standard elliptic approximation is

$$||\Pi_n A^{-1} - A_n^{-1}\Pi_n||_{\mathcal{L}(L_2(\Omega))} \;\leq\; ch^2$$

[2, 28], so that assumption (5.16) holds with $s = 2$.

By the definition of the trace operator B^* and (5.25), we obtain with $U = L^2(\Gamma_c)$ and $H = L^2(\Omega)$,

$$||B^*\varphi_n||_U \;=\; ||B_n^*\varphi_n||_U \;=\; ||\varphi_n||_{L^2(\Gamma_c)} \;\leq\; ch^{-1/2}||\varphi_n||_{L^2(\Omega)}. \tag{5.28}$$

Therefore, assumption (5.17) is satisfied with $s = 2$ and $\gamma = 1/4 + \varepsilon$.

By the definition of the trace operator and (5.24)-(5.25), we have

$$||B^*(P_n\varphi - \varphi)||_{L^2(\Gamma_c)} \;=\; ||P_n\varphi - \varphi||_{L^2(\Gamma_c)} \;\le\; ch^{-1/2}||P_n\varphi - \varphi||_{L^2(\Omega)} \;\le\; ch^{3/2}||\varphi||_{H^2(\Omega)}. \tag{5.29}$$

Therefore, assumption (5.18) is satisfied with $s = 2$. Assumption (5.19) follows from (5.18) since $B^*\varphi_n \;=\; B_n^*\varphi_n$.

Finally, from trace theory [46] and the paragraph above,

$$\begin{aligned} ||B^*P_n\varphi||_{L^2(\Gamma_c)} &= ||P_n\varphi||_{L^2(\Gamma_c)}\ le\ ||(P_n - I)\varphi||_{L^2(\Gamma)} + ||\varphi||_{L^2(\Gamma)} \\ &\le ch^{3/2}||\varphi||_{H^2(\Omega)} + c||\varphi||_{H^{1/2}(\Omega)}. \end{aligned}$$

Assumption (5.20) is satisfied for $\gamma = 1/2$.

Thus, with all of the assumptions verified, Theorem 5.1 gives us convergence of the approximate Riccati operator Π_n to Π. Hence, we have convergence of the functional gains.

5.4 Note on Nitsche's Approximation

Lasiecka and Triggiani [28] suggested that Nitsche's approximation would be useful in the LQR problem because of improved accuracy of the Dirichlet boundary conditions. This scheme produces optimal convergence rates in the solution of the open-loop system (see [9] and [35]). As we shall see in the next chapter, the Nitsche scheme produces excellent approximations to the functional gains. Nitsche's approximation A_β^N of the A operator is defined by

$$\langle A_\beta^N z^N, \phi^N\rangle \;=\; a(z^N, \phi^N) - \langle \frac{d}{dx} z^N, \phi^N\rangle_\Gamma - \langle z^N, \frac{d}{dx}\phi^N\rangle_\Gamma - \beta h^{-1}\langle z^N, \phi^N\rangle_\Gamma, \tag{5.30}$$

where $\beta > 0$ is sufficiently large, Γ is the boundary of the domain and $a(\cdot,\cdot)$ is the bilinear form shown in the Chapter 2.

EXAMPLE 1. For the one-dimensional problem with Dirichlet boundary control on the left end of the rod, only one term in a couple of matrices is affected. Rearranging the terms of Nitsche's approximation A^N_β and using the fact that $z^N(0) = u_0$, the only matrices that are changed are the stiffness and the control matrices. The resulting stiffness matrix is the $n \times n$ matrix

$$S^N_\beta = \frac{\epsilon}{h}\begin{bmatrix} -1-\beta & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{bmatrix}, \tag{5.31}$$

and the control matrix is given by

$$\tilde{B}^N_\beta = \frac{\epsilon}{h}\begin{bmatrix} 1+\beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.32}$$

A similar addition gets added to the boundary nodes for the two-dimensional problem. However, constructing such matrices is more complex. The study of Nitsche's approximation for higher-dimensional problems is necessary in future research. In the next chapter, we study the results of using the various schemes for computing the functional gain.

Chapter 6

Numerical Results

In this chapter, we present some numerical results obtained using the results of previous chapters. The first section contains results of various one-dimensional problems. The second section presents numerical computations of a few two-dimensional problems with a domain given by the unit square.

6.1 One-Dimensional Examples

The one-dimensional examples shown in this section use a rod of length l with control placed on the left end of the rod. The heat equation for these examples is given by

$$\frac{\partial}{\partial t}T(t,x) \;=\; \varepsilon\frac{\partial^2}{\partial x^2}T(t,x), \tag{6.1}$$

with initial condition

$$T(0,x) \;=\; T_0(x), \tag{6.2}$$

boundary condition

$$T(t,l) \;=\; 0, \tag{6.3}$$

on the right end of the rod and, if we have Robin boundary control,

$$\frac{\partial}{\partial x}T(t,0) \;=\; -\frac{l\kappa_0}{\kappa}\big(u(t)-T(t,0)\big). \tag{6.4}$$

If we study Dirichlet boundary control, then the condition on the left end of the rod is given by

$$T(t,0) \;=\; u(t). \tag{6.5}$$

The constant ε is the thermal conductivity of the rod. The constants κ and κ_0 are the thermal diffusivity constants of the rod and contact material, respectively. We now look at the numerical results of our first example:

EXAMPLE 1. In this example, we consider a thin rod made of copper ($\varepsilon = 1.14\ cm^2/sec$) and 10 cm long. The left end is in contact with an aluminum film that is 1 cm long. The thermal diffusivity constants κ and κ_0 are 0.93 and 0.55, respectively. The weight function $c(x)$ is defined by

$$c(x) \;=\; \begin{cases} 1, & 0 \le x \le 5 \\ 0, & 5 < x \le 10. \end{cases} \tag{6.6}$$

Using the finite element method as discussed in Chapter 5, we compute the approximate functional gain using the ARE, R-PDE and C-PDE methods. *MatLab* was used on a Sun machine. The first figure 6.1 shows that the solution to the problem (6.1)-(6.4) is asymptotically stable - the temperature approaches zero as time reaches about 20 seconds. Figure 6.2 contains the approximate Riccati kernels from the R-PDE method. One can see how the kernels converge to a "true" solution of the R-PDE. The first plot is for $N = 4$, the next for $N = 8$ and the final two at the bottom are for $N = 16$ and $N = 32$ respectively.

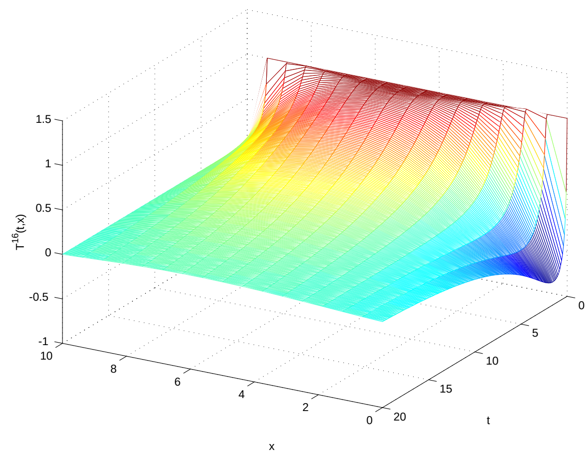

Figure 6.1: Approximate Solution to Heat Equation

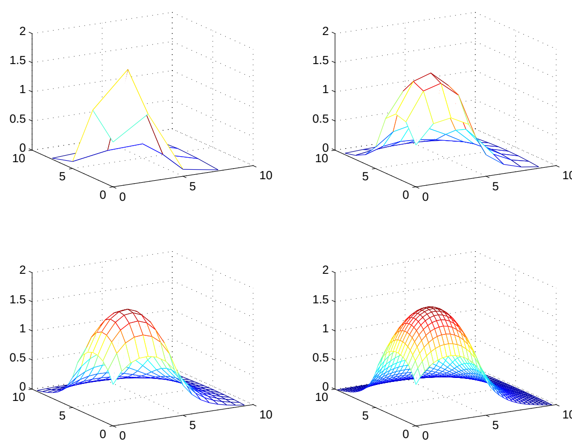

Figure 6.2: Convergence of Riccati Kernels

Figure 6.3 shows a comparison of the three methods of finding the functional gain at $N = 4$ to the "exact" solution of the functional gain which was found using the ARE method with $N = 256$. One will find virtually no difference between the three approximations.

This shows that the C-PDE and R-PDE methods produce good approximations for the functional gain. Moreover, they stay together as N approaches infinity.

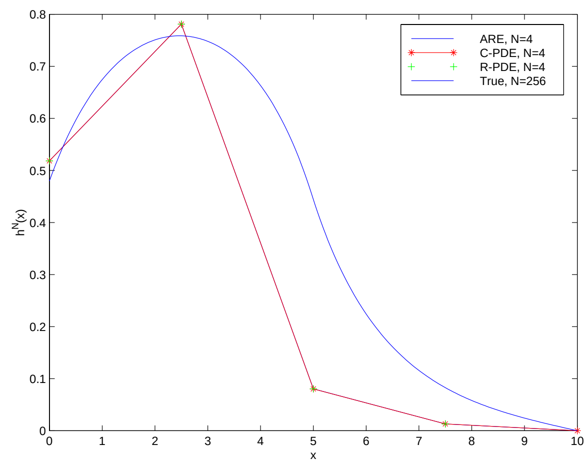

Figure 6.3: Comparison of Three Methods

Table 6.1 shows the CPU times to run the algorithms and compute the functional gains. One will notice that as N tends toward infinity, the speed of approximating the functional gain via the Riccati PDE's gets exponentially slower. The main reason for this is that we are solving for a function in two spatial terms. There appears to be no advantage to using the C-PDE timewise, either. We move on to the next example.

EXAMPLE 2. In this example, we use the same problem as in the previous example. This time, we incorporate the shift constant in the weighted cost function. The constant α is set to 0.25. Thus the amount of time it takes for the solution to reach zero is now close to 50 seconds, as shown by Figure 6.4. The time it takes to compute the functional gains are similar to those shown in the last example and any difference in using the ARE method and the C-PDE method is indistinguishable. However, Figure 6.5 shows that functional

Table 6.1: CPU Times

N	ARE	C-PDE	R-PDE
4	0.19	0.51	0.93
8	0.21	0.84	1.81
16	0.30	2.23	19.59
32	0.80	8.05	7,118.1
64	2.97	35.31	
128	13.90	193.93	
256	80.39		

gain found via the R-PDE is different from the others. It appears to be sensitive to the use of α in its equation.

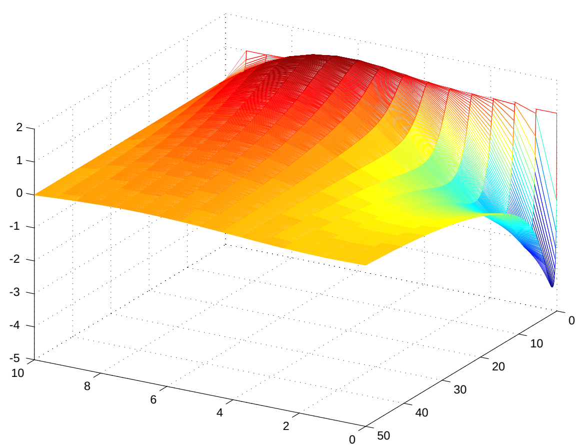

Figure 6.4: Approximate Solution to Heat Equation

If one were to look at the graphs of the Riccati kernels computed for $N = 4$, $N = 8$ and $N = 16$, one would notice that the approximations do not appear to converge. In fact, it appears to become unstable as N goes to infinity. The use of the R-PDE does not appear to be robust and is sensitive to changes in α.

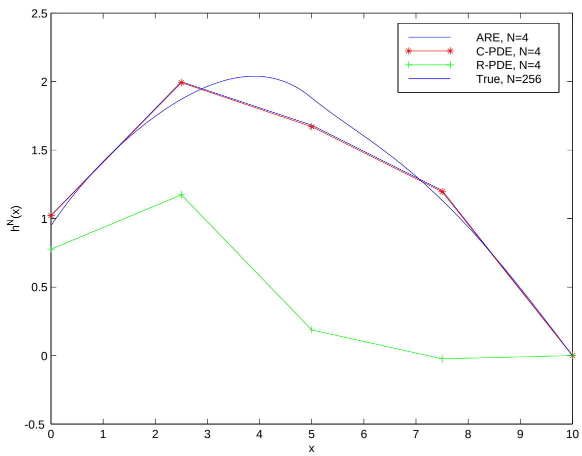

Figure 6.5: Comparison of Three Methods

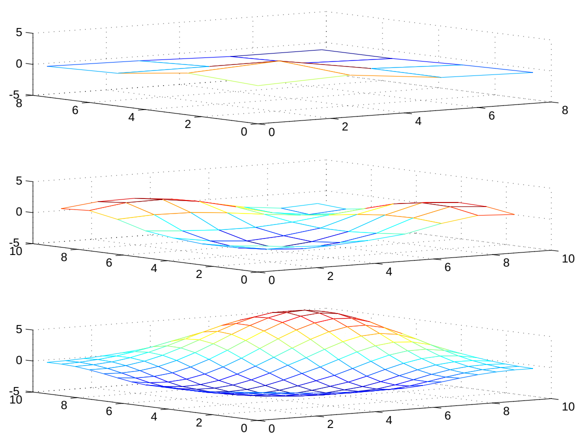

Figure 6.6: Convergence of Riccati Kernels

The rest of the examples will no longer compare methods to that of the R-PDE because of the time factor and the number of spatial terms needed to solve the equation. In particular, we would need to work with a function of 4 space dimensions in order to solve a two-dimensional problem. In the next example, we look at a problem and compare the functional gains found via the ARE, C-PDE and Nitsche's approximation.

EXAMPLE 3. In this example, we consider a thin rod made of aluminum ($\varepsilon = 0.86$

cm^2/sec) and 1 cm long. The weight function $c(x)$ is defined by

$$c(x) = \begin{cases} 10, & 0 \leq x \leq 0.25 \\ -1, & 0.25 < x \leq 0.5 \\ 1, & 0.5 < x \leq 0.75 \\ 0, & 0.75 < x \leq 1. \end{cases} \tag{6.7}$$

We introduce drastic changes in $c(x)$ so that we may hopefully see differences between using Nitsche's approximation and standard finite elements. A comparison of the functional gains to the problem (6.1)-(6.5) is shown in Figure 6.7. One notices that the difference between the functional gain found by the ARE is still no different from that found via the C-PDE. In Table 6.2, we see that the CPU times to compute the functional gains is similar to those in the first example.

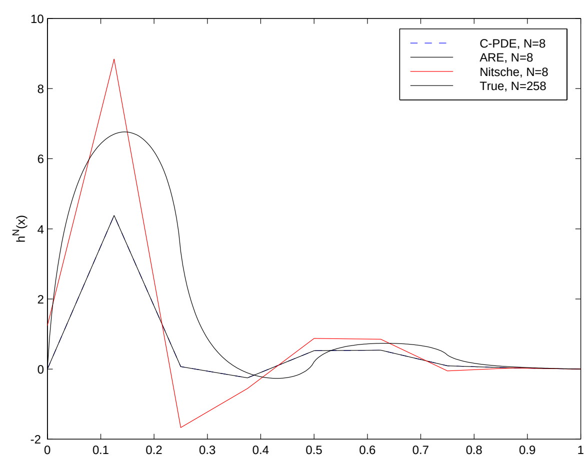

Figure 6.7: Comparison of Three Methods

For Nitsche's approximation, we set $\beta = 100$. We notice in Figure 6.7 that the approximation of the functional gain using Nitsche's approximation is better on the boundary. Table 6.2 shows that the CPU times using Nitsche's approximation is comparable to the

Table 6.2: CPU Times

N	ARE	C-PDE	Nitsche
4	0.18	0.43	0.19
8	0.19	0.82	0.21
16	0.29	3.45	0.35
32	0.74	13.85	0.83
64	2.80	59.61	3.11
128	13.25	272.78	14.19
256	74.66		80.79

ARE method. A good approximation of a function on the boundary can be essential to the problem, and so, the use of Nitsche's appromximation can be important, especially if there is a neccesity for knowing the solution at the boundary of the domain. We now move on to some examples of two-dimensional problems.

6.2 Two-Dimensional Examples

The two-dimensional examples shown in this section consist of a metal square plate ($[0,1]\times[0,1]$) with control placed on the left face ($x=0$) of the plate. The heat equation for these examples is given by

$$\frac{\partial}{\partial t}T(t,x,y) \;=\; \varepsilon\left(\frac{\partial^2}{\partial x^2}T(t,x,y)+\frac{\partial^2}{\partial y^2}T(t,x,y)\right), \tag{6.8}$$

with initial condition

$$T(0,x,y) \;=\; T_0(x,y), \tag{6.9}$$

boundary conditions

$$T(t, x, 0) \;=\; T(t, x, 1) \;=\; T(t, 1, y) \;=\; 0 \tag{6.10}$$

and, if we have Robin's boundary control,

$$\frac{\partial}{\partial x}T(t, 0, y) \;=\; -\Big(\sum_{i=1}^{l} u_i(t)g_i(y) - T(t, 0, y)\Big), \tag{6.11}$$

where l is the number of "lamps" placed along the boundary of the plate and $g_i(y)$ are the "lamp" functions defined as semi-ellipses. That is, $g_i(y)$ are defined by

$$g_i(y) \;=\; l^2\left(y - \frac{i-1}{l}\right)\left(\frac{i}{l} - y\right). \tag{6.12}$$

If we study Dirichlet boundary control, then the condition on the left end of the rod is given by

$$T(t, 0, y) \;=\; \sum_{i=1}^{l} u_i(t)g_i(y). \tag{6.13}$$

The constant ε is the thermal conductivity of the plate. We now look at the numerical results of our first example:

EXAMPLE 1. In this example, we consider a unit square plate made of silver ($\varepsilon = 0.00173$ ft^2/sec). The face of the square where $x = 0$ is heated by a series of 4 lamps as described above. The weight function $c(x, y)$ is defined by

$$c(x) \;=\; \begin{cases} 75, & \frac{1}{3} \le x, y \le \frac{2}{3} \\ 10, & otherwise. \end{cases} \tag{6.14}$$

In addition, we weigh the controls in the cost function by setting $r = 1/1000$.

Using the finite element method as discussed in Chapter 5, we compute the approximate functional gain using the ARE and C-PDE methods. The first figure 6.8 shows the functional gains for the controls $u_1(t)$, $u_2(t)$, $u_3(t)$ and $u_4(t)$, respectively, all of which were computed using the C-PDE method. Figure 6.9 shows the functional gain $h_2(x, y)$ for $N = 12$, 16, 20 and 24, all of which were computed using the C-PDE method. The ARE method is not shown because its computed functional gains look exactly the same as those in Figures 6.8 and 6.9. In fact, comparing the two methods by a specific y-coordinate, we would see no difference just as encountered in the previous section. Figure 6.10 shows a comparison of the two methods for the functional gain $h_2(x, y)$, where y is fixed at 3/8.

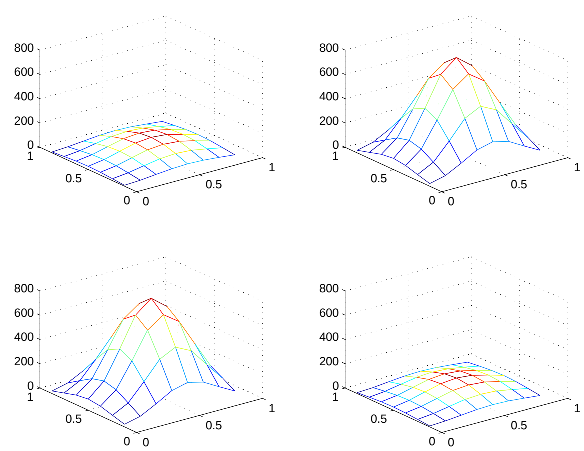

Figure 6.8: Functional Gains $h_1(x, y)$, $h_2(x, y)$, $h_3(x, y)$ and $h_4(x, y)$

An advantage appears in the time Table 6.3. One will notice that as N tends toward infinity, the speed of approximating the functional gain via the Chandrasekhar PDE's is significantly smaller than that for the ARE method. This appears to be a great advantage in that larger systems can be solved more quickly. This remains true as long as the number of controls is fixed and is smaller than the number of states. This feature is passed on to

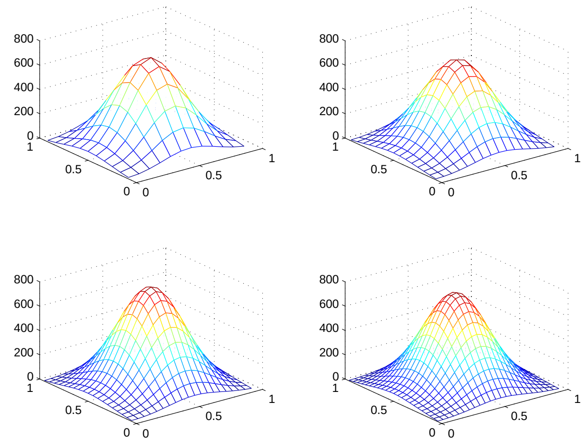

Figure 6.9: Convergence of $h_2^N(x, y)$

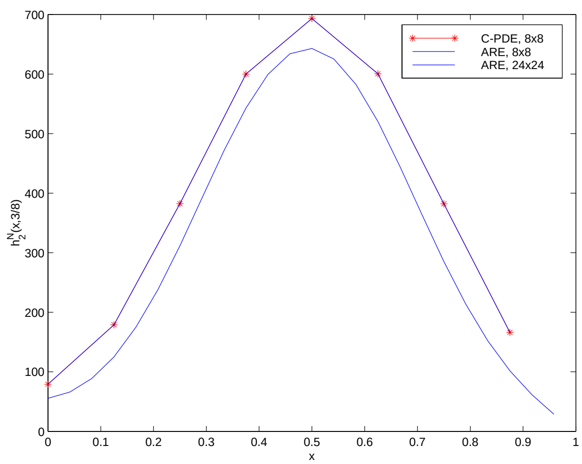

Figure 6.10: Comparison of Methods for $h_2^N(x, 3/8)$

the C-PDE from the original Chandrasekhar equations.

EXAMPLE 2. We consider the same example as shown above, but this time we set $\alpha = 0.40$. To compensate for this, we had to change the length of time to solve the C-PDE in the *MATLAB* program.

The first figure 6.11 shows the functional gains for the controls $u_1(t)$, $u_2(t)$, $u_3(t)$ and $u_4(t)$,

Table 6.3: CPU Times

N	ARE	C-PDE
8	3.12	6.2
12	38.24	16.61
16	419.07	43.46
20	4,443.5	115.42
24	21,434	435.23

respectively, all of which were computed using the C-PDE method. Figure 6.12 shows the functional gain $h_2(x, y)$ for $N = 8$, 12, 16 and 20, all of which were computed using the C-PDE method. The ARE method is not shown because its computed functional gains look exactly the same as those in Figures 6.11 and 6.12. In fact, comparing the two methods by a specific y-coordinate, we would see no difference just as encountered in the previous section. Figure 6.13 shows a comparison of the two methods for the functional gain $h_2(x, y)$, where y is fixed at 3/8.

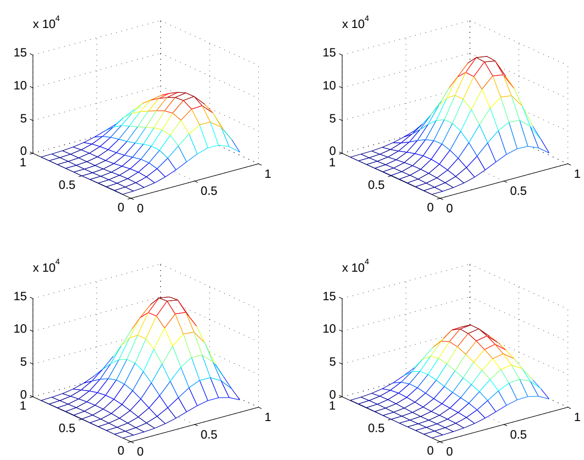

Figure 6.11: Functional Gains $h_1(x, y)$, $h_2(x, y)$, $h_3(x, y)$ and $h_4(x, y)$

The time Table 6.4 shows that the C-PDE method is significantly faster than that of the

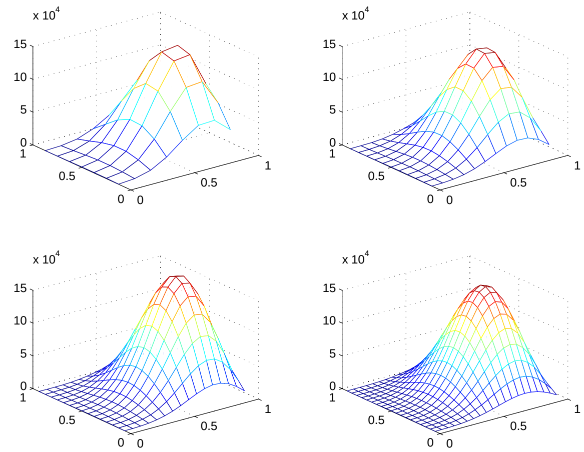

Figure 6.12: Convergence of $h_2^N(x, y)$

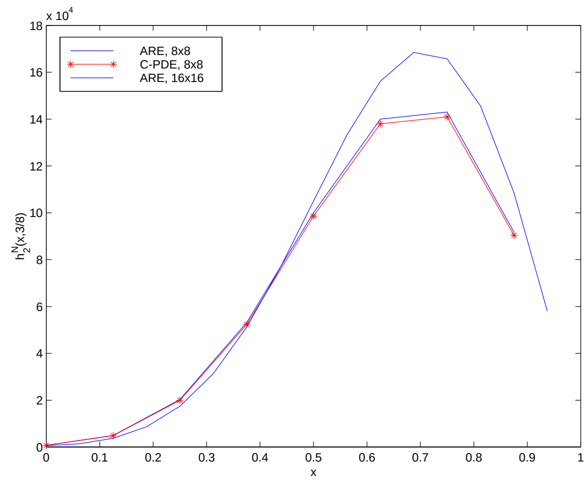

Figure 6.13: Comparison of Methods for $h_2^N(x, 3/8)$

ARE method as N gets larger. The next example will involve the Dirichlet boundary control and will have more "lamps" along the side of the plate.

EXAMPLE 3. In this example, we consider a unit square plate made of silver ($\varepsilon = 0.00173$ ft^2/sec). The face of the square where $x = 0$ is heated by a series of 9 lamps as described

Table 6.4: CPU Times

N	ARE	C-PDE
8	3.93	14.5
12	37.47	47.17
16	408.98	130.36
20	4,001	342.45

above. The weight function $c(x, y)$ is defined by

$$c(x) = \begin{cases} 75, & \frac{1}{3} \leq x \leq \frac{2}{3},\ \frac{5}{8} \leq y \leq \frac{7}{8} \\ 10, & otherwise. \end{cases} \tag{6.15}$$

In addition, we weigh the controls in the cost function by setting $r = 1/10$.

Figure 6.14 shows the functional gains of the nine lamps on a 20×20 grid. If we compare the ARE method to the C-PDE method, then we would see no difference in the approximation as shown in Figure 6.15.

The time Table 6.5 shows that the C-PDE method is significantly faster than the ARE method as N gets larger.

These examples allow us to make some comparisons of the schemes presented. We end the thesis with our conclusions and discuss future research for the Chandrasekhar PDE's.

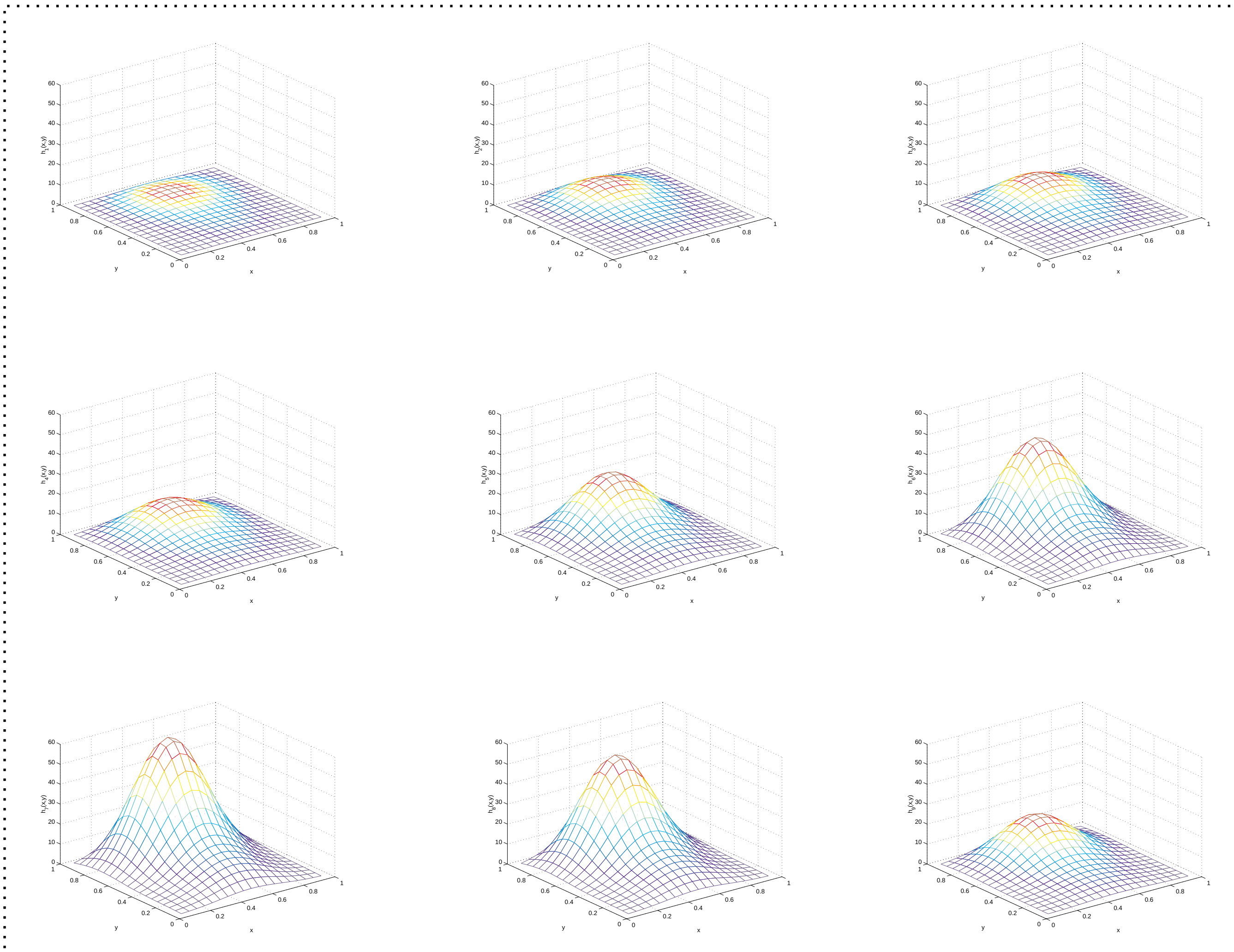

Figure 6.14: Functional Gains $h_1(x, y)$ to $h_9(x, y)$ and $h_4(x, y)$

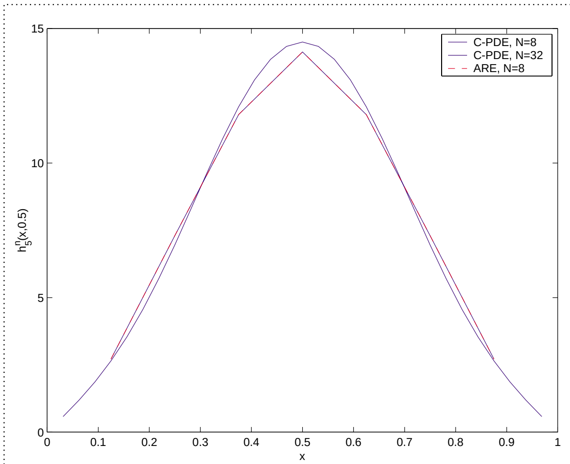

Figure 6.15: Comparison of Methods for $h_5^N(x, 0.5)$

Table 6.5: CPU Times for 2D Dirichlet Problem

N	ARE	C-PDE
8	2.02	8.34
12	20.42	32.24
16	86.41	60.37
20	313.24	147.48
24	1,882.1	994.84
28		1,107.6
32		2,208.3

Chapter 7

Conclusions and Future Research

The main goal of this dissertation was to find fast and more efficient methods of computing functional gains. The derivation of Chandrasekhar partial differential equations was an essential development toward this goal. We have seen that this method accurately computes the functional gain. The major improvement came in time for two-dimensional problems. For a fixed number of controls along a face of an object, the C-PDE could compute an approximate functional gain on a much finer mesh in less time than any other methods. Using the R-PDE method did not appear to be as beneficial. The time it takes to compute an approximate functional gain using the R-PDE method is slow compared to the other methods. It is not feasible to use it for high-dimensional problems, including 2D problems, because of the amount of spatial terms for the Riccati kernel. We also breifly discuss Nitsche's approximation (used in a one-dimensional Dirichlet problem) which can give better approximations for functional gains on the boundary. Both the C-PDE and Nitsche's approximations need more exploration and hence we have our future research.

There are many issues that will hopefully be resolved about the C-PDE in future research. We have seen how well the approximate functional gains compare to those obtained using the ARE method. A major reason we studied C-PDE is the possible use of adaptive finite element method. With the implementation of adaptive finite elements, we might see better approximation of the functional gains using the C-PDE. In addition, for Dirichlet problems, I hope to see the use of Nitsche's approximation with Chandrasekhar partial differential equations. This could lead to even better approximations of the functional gains. Thus, we have a lot of issues to explore. Other issues that need to be resolved include getting a bound on convergence of the solutions to the C-PDE, studying numerical results for the C-PDE and ARE in three-dimensional problems and exploring Nitsche's approximation in two- and three-dimensional problems. The possibilities are exciting and just waiting to be explored.

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Vita

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