CHAPTER II

Background and Literature Review

In this chapter we present a brief review of the extensive literature on location theory, and of the contributions to the Reformulation-Linearization Technique (RLT) method. Our literature review will be grouped into three main topics. First, we discuss the pure location problem since this is a principal subproblem even for solving the more general location-allocation problems (LAP). Second, we discuss the rectilinear distance location-allocation problem, the Euclidean distance and the squared Euclidean distance location-allocation problems, and fixed-charge discrete location-allocation problems. Following this, we give a brief account of location-allocation problems defined on networks along with other variants that consider stochastic elements, area destinations, and sequential location-allocation decisions. Finally, as a third main topic , we discuss the RLT method since we adopt it as an approach for developing linear or convex relaxations in our proposed algorithm for solving the Euclidean distance location-allocation problem.

2.1 Pure Location Problems

The (discrete demand) multifacility location problem (MFLOC) is a problem of optimally locating *n* new service facilities to serve the demand at *m* existing customer locations or facilities. The general l_p distance model for this problem can be stated as follows:

$$\text{Minimize} \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}\ l_p(X_i, P_j) + \sum_{i=1}^{n-1}\sum_{k=i+1}^{n} v_{ik}\ l_p(X_i, X_k)$$

where,

w_{ij} is a non-negative parameter that represents the annual cost of separating new facility *i* and existing facility *j* by a unit distance,

v_{ik} is a non-negative parameter that represents the annual cost of separating new facilities *i* and *k* by a unit distance by a unit distance,

$X_i = (x_i, y_i)$ are the location coordinates of new facility *i*, and

$P_j = (a_j, b_j)$ are the location coordinates of existing facility *j*.

Here, for $1 \leq p \leq 2$, the l_p distance measure is given by $l_p(X_i,P_j)=[|x_i-a_j|^p + |y_i- b_j|^p]^{1/p}$. When $n = 1$, the MFLOC becomes a single facility location problem (SFLOC) (Love and Dowling, 1989). Selecting different distance measures, by substituting different values of *p* in MFLOC and SFLOC, we obtain different variants of SFLOC and MFLOC. Furthermore, we can introduce additional variants by restricting new facility locations to specific sites, and by letting demands be either stochastic or distributed over areas. These types of problems are discussed below.

2.1.1 Rectilinear Distance Location Problems

The rectilinear distance location problem is a variant of MFLOC in which p=1. Among the reasons for considering the rectilinear distance measure is that for a grid of city streets or in a network of aisles in a factory or a warehouse, it is the best applicable distance measure (Francis, 1963). When the l_1 norm is used in Model (2.1), the objective function separates into *x* and *y* coordinate subproblems, and these can be posed and solved as specially structured linear programming problems.

The single facility rectilinear distance location Problem SFLOC was first solved by Francis (1963), who characterized a simple median location optimal solution. Later, Francis (1964) solved a special case of the multifacility rectilinear distance location problem, where he dealt with equal weights. In 1970, Cabot *et al*. proposed a network flow solution procedure, while Pritsker and Ghare (1970) suggested a gradient technique. However, Rao (1973) demonstrated that this latter approach was basically a primal simplex-based linear programming approach, and in the presence of degeneracy, the optimality conditions were not sufficient. Also, Wesolowsky and Love (1971 b) and Morris (1975) showed that the problem with linear locational constraints could be solved by linear programming. A thorough set of necessary and sufficient optimality conditions were finally developed by Juel and Love (1976).

Among other approaches for solving the rectilinear distance Problem MFLOC are the nonlinear approximation method developed by Wesolowsky and Love (1972), where any number of linear and (or) nonlinear constraints defining a convex feasible region can be included, and the hyperboloid approximation procedure for solving the perturbed rectilinear distance MFLOC that was proposed by Eyster *et al*. (1973). Another specialized simplex based-algorithm is derived by Sherali and Shetty (1977 a). Picard and Ratliff (1978) solved the problem via at most (m-1) minimum cut problems on derived networks containing at most (n-2) vertices. Subsequently, Kolen (1981) exhibited the equivalence of the method of Sherali and Shetty and Picard and Ratliff and that the main difference between these two procedures was principally in the computational implementation. Moreover, this type of approach is known to be the most effective way of solving the rectilinear distance Problem MFLOC. A modified version of the method of Picard and Ratliff (1978) was proposed by Cheung (1980). Dax (1986 a) gave a new method that, as he stated, handles efficiently the rectilinear distance Problem MFLOC having large clusters, i.e. where several new facilities are located together at one point. Guccione and Gillen (1991) provided an economic interpretation of a dual problem concerned with maximizing the revenue when using rectilinear distances.

2.1.2 Euclidean Distance Problems

When $p = 2$ in Model (2.1), the resulting problem is called the Euclidean distance multifacility location problem (EMFL). This problem becomes the Euclidean single facility location problem (ESFL) if $n = 1$. According to Rosen and Xue (1993), there is always an optimal solution for Problem ESFL when the existing facilities are collinear in which the new facility location coincides with one of the existing facilities. Therefore, the literature typically assumes that the existing facilities are non-collinear. For this kind of problem, Weiszfeld (1937) was the first to propose a fixed-point iterative scheme, that has come to be known as the Weiszfeld procedure. Also, Cooper (1963) and Kuhn and

Kuenne (1962) solved the same problem using this concept. Later, Cooper and Katz (1981) proposed an optimal gradient method with inexact quadratic fit based line-search for Problem ESFL, and showed its superiority to the Weiszfeld algorithm in most cases. However, Rado (1988) pointed out that when the Weiszfeld procedure is used instead of a gradient-based method, global convergence is achieved and the computation of step-sizes is circumvented.

The objective function of the Problem ESFL is convex and nondifferentiable. The nondifferentiability occurs only at a finite number of points where the location of the new facility matches one of the existing facility locations (Rosen and Xue, 1993). Therefore, the convergence of the Weiszfeld procedure is expected to be very slow when the minimizer coincides with one of the existing facilities (Wang *et al*. 1975). To avoid this behavior, Ostresh (1978) and Balas and Yu (1982) developed a version of the Weiszfeld procedure that contains a step to perturb the solution if it coincides with one of the existing facilities in order to ensure the convergence to an optimum. With this step, the Problem ESFL can be viewed as a smooth problem (Rosen and Xue, 1993). Eckardt (1980) studied the problem in general spaces. In 1987 and 1989, Xue developed a second-order method to solve the constrained single facility location problem and proved that for any initial point, the algorithm either stops at an optimal solution or generates a sequence that quadratically converges to the optimal solution. Drezner (1985) conducted some sensitivity analyses for the Problem ESFL and studied the cases where the existing facilities are restricted to small areas and the weights are restricted to given ranges. There is a large amount of literature on the ESFL problem. For further details on this class of problems, see the books of Francis *et al*.(1991) and of Love *et al*.(1988).

For the multifacility Problem EMFLP, Francis and Cabot (1972) have proven that the objective function is convex, and it is strictly convex if for $i = 1,..., n$, the set $S_i = \{ j: w_{ij} >0 \}$ is non-empty and the points of S_i are non-collinear. Furthermore, they have shown that the optimal solution of problem EMFLP exists and lies in the convex hull of the existing facilities. Several researchers such as Hansen *et al*. (1980), and Juel and Love (1983) have also discussed the existence of the optimal solution in the convex hull of the existing facilities. Miehle (1958) was the first to propose an extension of the Weiszfeld algorithm. Ostresh (1977) proved that Miehle's algorithm is a descent scheme. Rado (1988) slightly changed Miehle's algorithm and proved that this algorithm always converges to the minimizer of the Problem EMFL for certain well-structured cases in which the objective function is strictly convex. However, Rosen and Xue (1992 a) constructed a counter example showing that Miehle's algorithm may converge to a non-optimal point even for such well-structured problems.

The multifacility problem basically suffers from the nondifferentiability of the objective function. These points of nondifferentiability of the objective function occur not only when new and existing facility locations coincide, but also occur on linear subspaces where the new facilities themselves coincide. In addition, since the objective function is not strictly convex, multiple minimizers of the problem are resulted to which iterative schemes might tend to converge.

To overcome the difficulty of having nondifferentiable objective function in Problem EMFL, Eyster *et al*. (1973) used an extension of the Weiszfeld algorithm. In this procedure, they approximated the objective function by a hyperboloid, which is a smooth function, and derived the associated extremal equations. This procedure is labeled the hyperboloid approximation procedure (HAP) and is probably the most common procedure for solving the multifacility location problem, using Euclidean distances or even rectilinear distances. In 1977, Ostresh proved that HAP is a descent algorithm under certain conditions. In 1985, Charalambous developed a method to accelerate the rate of convergence of HAP. However, it was only recently that Xue (1991) and Rosen and Xue (1993) proved the global convergence of HAP when applied to the Problems EMFL. Unfortunately, it is well known that the HAP approach suffers from ill-conditioning effects if the point of convergence is nondifferentiable (Charalambous, 1985). As a result, other methods that directly focus on this issue of nondifferentiability have been developed. For example, Calamai and Charalambous (1980) have proposed a pseudo-gradient technique that classifies the new facilities into distinct categories based on their coincidence with other facilities in order to derive a descent method for solving EMFLP. However, Juel (1982) showed that this algorithm could terminate at a suboptimal solution. Chatelon *et al*. (1978) have also approached EMFLP by using a general ε-subgradient method in which search directions are generated based on the subdifferential of the objective function over a neighborhood of the current iterate. Sequential unconstrained minimization techniques used by Love (1969) and the Weiszfeld fixed-point iterative method as utilized by Rado (1988), are also among other efforts to solve EMFLP.

Several second-order methods have also been designed to solve the Problem EMFLP. Calamai and Conn (1980) were the first to propose a projected gradient-based algorithm. Various quadratic convergence approaches have also been developed by Calamai and Conn (1982, 1987) and Overton (1983), in which specialized line-searches are used in conjunction with projected second-order techniques. Rosen and Xue (1992 b) developed an algorithm which, from any initial point, generates a sequence of points that converges to the closed convex set of optimal solutions to the Problem EMFLP.

Since in some multifacility location problems, the optimal solution coincides with one of the existing facilities, researchers have derived necessary and sufficient conditions for optimality to avoid the nondifferentiability difficulty of the objective function associated with such a coincidence. For a single facility problem with three existing facilities, Juel and Love (1986) proved that it is possible to determine which existing facility is the optimal location by means of a simple geometrical construction. For the multifacility location problem with no constraints on the location of the new facilities, Juel and Love (1980) derived some sufficient conditions for the coincidence of facilities that are valid in a general symmetric metric. These results were later extended by Lefebvre *et al*. (1991) to be applicable to some location problems having certain locational constraints. Examples of other works on this subject are Francis and Cabot (1972), Calamai and Charalambous (1980), Calamai and Conn (1980) and (1982), Dax (1986 b), Overton (1983), Lefebvre *et al*. (1990)**,** and Plastria (1992). Recently, Mazzerella and Pesamosco (1996) have used the optimality conditions of EMFLP as a tool for obtaining both stopping rules for some computational algorithms such as the projected Newton procedure of Calamai and Conn (1987), and the analytical solution of many simple problems.

Many other contributions for solving this problem have appeared in the literature. Love (1969) applied convex programming to a problem in three dimensions. Recently, Carrizosa *et al*. (1993) derived the geometrical characterizations for the set of efficient, weakly efficient, and properly efficient solutions of the Problem EMFLP when it includes certain convex locational constraints. In addition, Love and Yoeng (1981) explored the bounding method that continuously updates a lower bound on the optimal objective function value during each iteration. This method is based on the idea that the convex hull and the current value of the gradient determine an upper bound on the objective function's improvement. Among other works that have been proposed in deriving such bounds are those due to Elzinga and Hearn (1983), Juel (1984), and Love and Dowling (1989). Also, Wendell and Peterson (1984) have derived a lower bound from the dual to EMFLP.

Many papers have used the dual as an approach for solving this problem, starting with Witzgall (1964) and Kuhn (1967) who independently addressed the dual problem. Love (1974) developed the dual problem corresponding to a hyperbolic approximation of the constrained multifacility location problem with l_p distances. White (1976) gave a Varignon frame interpretation of the dual problem. Using Sinha's (1966) duality results involving general quadratic forms, Francis and Cabot (1972) derived a differentiable, convex quadratically constrained dual optimization problem, and achieved several useful relationships between the dual and EMFLP. However, they considered the actual use of this dual problem to solve EMFLP as an open problem. More recently, Xue *et al*. (1996) have suggested the use of polynomial-time interior point algorithms to solve this dual problem. Based on this idea, they presented a procedure in which an approximate optimum to EMFLP can be recovered by solving a sequence of linear equations, each associated with an iterate of the interior point algorithm used to solve the dual problem. Also, other papers dealing with duality of various constrained versions of this problem have appeared, such as the paper of Idrissi *et al*. (1989) where a primal-dual method to solve the constrained multifacility location problem with mixed norms was presented, and the paper of Love and Kraemer (1973) where a dual decomposition method for solving the constrained EMFLP was given.

Extensions of results and algorithms for the Euclidean distance problem to the general l_p distance problem have also been studied. In 1993, Brimberg and Love utilized the Weiszfeld algorithm to solve the single facility problem with l_p distance measures. Other examples of such extensions are Drezner and Wesolowsky (1978), Morris and Verdini

(1979), and Juel and Love (1981). For the constrained multifacility location problem with l_p distance, Love (1974) developed the dual problem corresponding to a hyperbolic approximation of the objective function.

2.1.3 Squared-Euclidean Distance Problems

If we define the facility-customer separation penalty to be proportional to the square of the Euclidean distance, then the resulting problem is called a squared-Euclidean distance problem. This problem is separable in the x and y variables. Eyster and White (1973) cite some special applications for this class of problems. In this problem, it is obvious that the function that is to be minimized is strictly convex, and unlike the Euclidean distance case, it has continuous first partial derivatives with respect to x and y. Consequently, the optimal solution of both the single and the multifacility problems is unique and can be obtained by simple calculus techniques. However, for the multifacility case, one needs to solve two systems of n linear equations in n variables (Francis *et al.*, 1992). Consequently, the solution of the squared-Euclidean distance problem has been used to obtain a good starting solution for the corresponding EMFLP (Francis *et al.*, 1992).

2.1.4 Variants and Extensions

The literature on the pure location problem is quite extensive and several variants and extensions have been proposed. One type of variant enforces a separation of facilities by adding a set of metric side-constraints to the problem, as in Schaefer and Hurter (1974). In another variant, the components of the location problem are considered to be stochastic, as in the papers by Cooper (1974) and Katz and Cooper (1974, 1976), where the coordinates of the destination locations are specified by probability distributions. A version of this problem in which the transportation costs are random variables was considered by Seppala (1975) and Drezner and Wesolowsky (1981). For the multifacility case, Aly and White (1978) solved the Euclidean distance problem where the existing facilities and the interactions between the facilities are random variables. Rao and Varma

(1985) also studied this same problem, while Wesolowsky (1971 b) considered randomness in a single facility location problem using rectilinear distances instead of Euclidean distances.

Models where the destinations are permitted to be uniformly distributed over regions have also been considered in the literature. Wesolowsky and Love (1971a) solved such a rectilinear distance multifacility location problem by developing a gradient search algorithm. For the single facility case of the former problem, Love (1972) used a nonlinear optimization technique. Bennet and Mirakhor (1974) solved an approximation of the same problem by replacing the areas with their center of gravity. The extension of this problem to a similar problem, but with general l_p distances and general demand area shapes, was considered by Drenzner and Wesolowsky (1980). In their paper, they explored a two-stage iterative procedure, based on the Weiszfeld procedure, to solve the single facility problem. Drezner (1986) considers the problem EMFLP and the squared-Euclidean distance location problem when both new and existing facilities have circular shapes and demand and service has a uniform probability density inside each shapes. Cavalier and Sherali (1986a) also consider Euclidean problems having uniform demand distributions over convex polygons.

Another type of variant includes models that require the relocation of the sources over a multiperiod horizon due to the changes in demand weights or locations of the destinations. Among the contributions to this variant is the dynamic approach for solving the rectilinear distance single facility location problem by Wesolowsky (1973), and for solving the multifacility case by Wesolowsky and Truscott (1976), Scott (1971), and Cavalier and Sherali (1985). Several papers such as Katz and Cooper (1981), Aneja and Parlar (1991) and Butt and Cavalier (1996) have considered a nonconvex variant of the problem ESFL when there are forbidden regions present in the plane.

A new variant of the location problem that removes the restriction of having non-negative demand weights has also been treated more recently by Tellier and Polanski (1989) and Drezner and Wesolowsky (1990). In addition, Tuy *et al*.(1995) developed an algorithm to solve the single facility problem that is based on a representation of the objective function as a difference of two convex functions.

In another variant, different spaces where the location can take place, other than Euclidean plane, have also been considered by some researchers. Love (1969) has extended the location problem to three dimensions. Other researchers have considered the spherical distance in treating large region location problems because the surface of the earth is geodesic, rather than Euclidean. Aly *et al*. (1979), Dhar and Rao (1982), Plastria (1987), and Aykin and Babu (1987) are among those who have studied this problem.

There are still further variants of the location problem, such as those which involve different types of distance norms, as in Ward and Wendell (1980) and Wu (1994), others that require the transportation costs to be increasing and continuous functions of distance as in Hansen *et al*. (1985). Klincewicz *et al*.(1986) proposed optimal and heuristic algorithms for a variant of location problems in which customers require several different products. Abdelmalek (1985) developed a method to optimally position a single facility that provides a service to a set of moving facilities over a time horizon. Sherali and Kim (1992) considered a more general problem of determining the optimal paths of a service facility that moves through a region containing some existing facilities, while Kim *et al*. (1992) studied the same problem but where the existing facilities are distributed over a network. For a review of the formulation and solutions of several classes of location problems see Francis *et al.* (1983).

2.2.1 Location and Location-Allocation Problems on Networks

Consider a network $G(N, A)$ having n nodes v_i, i=1,..., n, each with a demand of h_i $i \in N$, and having links a_j, $j \in A$, each containing a uniform spread of demand of total weight w_j, j = 1,..., n. Let the locations, which are to be determined on G, of some p facilities be represented by $X = \{x_1, x_2,..., x_p\}$, and let $d(v, X) = \min \{d(v, x_1), d(v, x_2),..., d(v, x_p)\}$ represent the shortest path distance on G from any point v in G to a facility location.

2.2.1.a *P*-Median Problems

The p-median Problem (p-M) is to locate p new facilities, called medians, on the network G in order to minimize the sum of the weighted distances from each node to its nearest new facility (Francis *et al*., 1992). This problem can be mathematically stated as follows:

$$\underset{X \text{ on } G}{\text{Minimize}} \; f(X) = \sum_{i=1}^{n} h_i \, d(v_i, X) \,.$$

If $p \geq 2$, then this problem can be viewed as a location-allocation problem (LAP). This is because the location of the new facilities will determine the allocation of their service in order to best satisfy the nodal demands. Hakimi (1964) proved that in networks, a set of optimal locations will always coincide with the vertices. He also proposed an enumerative-graph theoretic approach for the problem. Revelle and Swain (1970) proposed other procedures to solve this problem after reformulating it as an integer programming (IP) problem. Jarvinen *et al*. (1972) also used this IP formulation and proposed a branch-and-bound algorithm for this problem. Due to the NP-hardness of the problem, several heuristic procedures have been developed, such as those of Maranzana (1964) and Teitz and Bart (1968). Beasly (1993) has also developed Lagrangian heuristics for this p-median location problem, based on Lagrangian relaxation and subgradient optimization concepts.

Several variants and extensions of the p-median problem have been addressed in the literature. One type of variant, studied by Pesamosca (1991), considers the interaction weights between the new facilities as well as the connection scheme as a tree. This case was treated as a problem EMFLP on a tree and its optimality conditions were then obtained using the optimality conditions of p problems of the type ESFL. Accordingly, for solving the problem EMFLP, a fixed point algorithm was developed to iteratively solve Problem ESFL using the Weiszfeld algorithm if differentiability is met, and otherwise, the algorithm switches over to Miehle’s algorithm. Another type of variant involves placing the capacity restrictions on the facilities to be located. When the capacity is finite, the resulting problem is called a capacitated problem; otherwise the problem is uncapacitated. Cavalier and Sherali (1986b) presented exact algorithms to solve the p-median problem on a chain graph and the 2-median problem on a tree graph, where the demand density functions are assumed to be piecewise uniform. For the uncapacitated *p*-median problem, Chiu (1987) addressed the 1- median problem on a general network as well as on a tree network. Dynamic location considerations on networks are addressed by Sherali (1991). Recently, Francis *et al*. (1993) developed a median-row-column aggregation algorithm to solve large-scale rectilinear distance *p*-median problems. On the other hand, Sherali and Nordai (1988) gave certain localization results and algorithms for solving the capacitated *p*-median problem on a chain graph and the 2- median problem on a tree graph. Another variant involves the treatment of a continuous demand over the network, which arises in some situations such as the location of public service facilities or in probabilistic distributions of demand. Among the contributions on this variant are Minieka (1978), Handler and Mirchandani (1979), Chiu (1987) and Derardo *et al*. (1982). Combining the two last variants, Sherali and Rizzo (1991) solved an unbalanced, capacitated *p-median* problem on a chain graph with a continuum of link demands. For solving this problem, they considered two unbalanced cases, the deficit and over-capacitated cases, provided a first-order characterization of optimality for these two problems and developed an enumerative algorithm based on a partitioning of the dual space. There are still further variants that include capacity restrictions on links, probabilistic travel times on links, and maximum distance constraints. For surveys of research done on location problems on networks, see Handler and Mirchandani (1979), Kariv and Hakimi (1979 b), and Tansel *et al*. (1983).

2.2.1.b *P*-Center Problems

The objective of the *p*-center problem is to locate *p* new facilities, called centers, on G in order to minimize the maximum weighted distance between a node and its nearest facility. Mathematically, this can be stated as follows:

$$\underset{X \text{ on } G}{\text{Minimize}} \; f(X) = \max_{1 \le i \le n} \; h_i \; d(v_i, X).$$

We briefly review only the absolute *p*-center problem, and for more details, and extensions of this problem, the reader is referred to Minieka (1978), Halpern and Maimon (1982), Kariv and Hakimi (1979 a), and Handler and Mirchandani (1979). The methods developed for solving this problem are quite different from those for the *p-median* problem, even though the two problems are related. The first approach developed was by Hakimi (1964) who proposed an enumerative approach for *p*=1 to specifically locate a local center on each link, and thereby to determine the overall optimal location. A more effective method was suggested by Christofides (1975), who showed that one needs to consider only a subset of the links for an optimal location. However, this approach is unable to solve general *p*-center problems. Erkut *et al*. (1992) presented a polynomial-time, binary search algorithm to solve the distance-constrained *p*-center problem. More methods have been proposed and tested to solve the *p*-center problem, such as the exact algorithms due to Christofides and Viola (1971), Granfinkel *et al*. (1977), and various heuristics proposed by Singer (1968).

2.2.2 Continuous Location-Allocation Problems with Discrete Demand Points

Continuous location-allocation problems are concerned with determining the location of *n* supply centers in a plane to serve *m* customers having fixed locations, and simultaneously, determining the flow allocations of these supply centers, so as to minimize the total distribution costs. This problem can be formulated as follows.

$$\text{(LAP)} \qquad \text{Minimize} \quad \sum_{j=1}^{m} \sum_{i=1}^{n} c_{ij} \, w_{ij} \, d(X_i, P_j)$$

subject to $\sum_{j=1}^{m} w_{ij} = s_i \qquad i = 1,\ldots,n$

$$\sum_{i=1}^{n} w_{ij} = d_j \qquad j = 1,\ldots,m$$

$$w_{ij} \geq 0 \qquad i = 1,\ldots,n; \quad j = 1,\ldots,m$$

where m = the number of customers

n = the number of supply centers

w_{ij}= flow from supply center i to customer j

$X_i = (x_i,y_i) \equiv$ location of supply center i

$P_j = (a_j,b_j)$ location of customer j

$d(X_i,P_j)$ = distance between the supply center i and customer j

s_i = (annual) capacity of supply center i

d_j = (annual) demand of customer j

c_{ij} = cost of unit flow per unit distance from supply center i to customer j.

The decision variables in the problem are $X_i \ \forall\ i=1,\ldots,n,$ and $w_{ij} \ \forall\ i=1,\ldots,n,\ \forall j =1,\ldots,m$. Depending on which type of decision variable is fixed, special cases of LAP result. For a fixed set of allocations $w \equiv (w_{ij},\ i =1,\ldots,\ n\ ; j = 1,\ \ldots,\ m)$, LAP simplifies to a pure location problem, whereas for a fixed set of locations $X_i = \{(x_i,y_i),\ i =1,\ldots,n\}$ the problem becomes the ordinary transportation problem.

As mentioned earlier, for the location problem that results when the allocation variable w is fixed, the distance measures or penalties most frequently used are the Euclidean, rectilinear, and squared-Euclidean distances.

The above formulation of LAP represents the capacitated version, wherein the capacity restriction s_i of each supply center is finite, while if $s_i = \infty\ \forall i =1,\ 2,\ldots,n$, the problem reduces to an uncapacitated LAP in which service is provided simply by the closest supply center. Shetty and Sherali (1977) have also formulated and analyzed a more general version of this problem, treating multiple commodities along with interactions between new facilities, as well. The work of Aykin and Brown (1992) is among other works that considers the interaction between the new facilities in the location-allocation problem.

2.2.2.a Rectilinear Distance Location-Allocation Problems

If the distance measure in LAP is represented by the rectilinear norm, the resulting problem is a nonconvex bilinear programming problem (Vaish, 1974). Love and Morris (1975) solved the uncapacitated case of this problem by proposing a two-stage procedure. In the first stage of their procedure, they used a set reduction algorithm to reduce the set of all possible optimal locations of the new facilities. The resulting problem was equivalent to a *p*-median problem on a weighted connected graph. Consequently, in the second stage, they used a technique for solving the *p*-median problem to obtain an optimal location and allocation solution. Following this, Sherali and Shetty (1977b) solved the capacitated version of this problem by developing a convergent cutting plane algorithm. This algorithm was then extended to a more efficient algorithm by Shetty and Sherali (1977b), who proposed deeper cutting planes. Moreover, they extended this model to include more general problems which address multiple commodities along with interactions between the new facilities. More recently, Sherali *et al*. (1994) have proposed a more effective algorithm for this class of problems, based on a new technique applied to a mixed integer-bilinear programming formulation of this problem. This technique applies the Reformulation-Linearization Technique (RLT), proposed by Sherali and Tuncbilek (1992), to provide tight lower bounds on the problem. A cutting plane procedure is employed to strengthen the developed linear programming relaxation, and a Lagrangian dual based lower bound is then computed to obtain a quick lower bound on the linear programming problem. This lower bounding technique is embedded within a branch-and-bound algorithm that (partially) enumerates over the locational decision space.

2.2.2 b Euclidean Distance Location-Allocation Problems

Cooper (1972) was the first who addressed the capacitated version of Problem LAP using Euclidean distances. For solving this problem, he proposed an exact, total enumeration algorithm and provided two heuristic methods. In the exact solution, all basic feasible transportation flows are enumerated, knowing that an optimum solution lies at an extreme point of the feasible region, and for each of them, the optimum location is determined. The first heuristic procedure, has the basic idea of alternatively solving the location and the allocation problems. The procedure takes some of the destination locations as initial locations for the sources. The resulting transportation problem is then solved in order to find the optimum allocations. With this known allocation, the new optimal locations are determined. This process continues until no further improvement can be obtained. The second approach uses a heuristic method that ignores the capacity constraints and solves the uncapacitated problem. When the capacity constraints are violated, the sources having surplus capacity are selected to serve the destinations that have uncovered demand. This allocation process is achieved in an appropriate way so that the resulting change in the objective function is relatively small. With this new set of allocations, the algorithm employs the aforementioned alternating technique. Murtagh and Niwattisyawong (1982) treat both the capacitated and the uncapacitated LAP problems as a nonlinear programming instance, and suggest a heuristic procedure using the commercial software package MINOS along with a good starting solution. They use an initial estimate of the source locations and obtain the corresponding optimal allocations in order to derive a starting solution. No bound on the quality of solutions produced by this method is available. Avriel (1980) proposes a geometric programming approach by exploiting the special structure of the location-allocation problem.

The literature lacks any exact algorithm to solve the capacitated version of the Euclidean distance LAP, other than the total enumeration approach of Cooper (1972) and an unpublished work of Selim (1979). Selim (1979) proposed an exact biconvex programming cutting plane procedure which is more tractable, even though not very efficient.

The uncapacitated version of LAP was first considered by Cooper (1963) who suggested a total enumeration algorithm to obtain an optimal solution. Since there is no restriction on the capacity of the sources, each customer is served by a single source. Given the assignment of destinations to each customer, the algorithm finds optimum locations using the first order optimality conditions. Practically, this algorithm turns out to be prohibitive when applied to other than small problems. Cooper (1964, 1967) also proposed some heuristic solution procedures that are based on alternately solving the location and the allocation problems. The first heuristic procedure restricts the source location to a subset of the destination sites, while, the second procedure assigns a subset of the destinations to a single source, and then for each new facility, a single source location problem is solved using an exact location method. Eilon *et al*. (1971) improved one of Cooper's heuristic method to make it computationally more effective. Following this, Kuenne and Soland (1972) proposed another heuristic procedure along with a branch-and-bound type of exact solution procedure to optimally solve the problem. At some iteration of the branch-and-bound algorithm, the algorithm obtains a partial solution by assigning a subset of destinations to the sources. At the branching step, a free destination is then included in the partial solution. Chen (1983) provided a differentiable approximation to the objective function of the problem and solved it using a quasi-Newton based method. However, the resulting solution is not necessarily a global minimum.

For $n = 2$, Ostresh (1975) utilized the property of nonoverlapping convex hulls to find a global minimum. Drezner (1984) has developed similar technique to solve the problem. Rosing (1990) generalized the procedure of Ostresh (1975) to solve the multifacility version of problem LAP. The extension of the problem LAP to the general l_p distance metric has also received attention in the literature such as in the work of Love and Juel (1982) in which the uncapacitated LAP problem is solved. Love and Juel show that this problem is equivalent to a concave minimization problem, and accordingly, develop five heuristic strategies that differ from each other in the manner in which they perturb a given local optimum solution. They define a local optimum solution as one for which the current locations are optimal when the allocations are fixed, and vice versa. In the perturbation step, the current allocations are changed to obtain a better local optimum. Chen (1984) ties Cooper's exact solution to a steepest descent approach, and accordingly proposes a modified step-length to improve the iterative process. Recently, Bongartz *et* *al*. (1994) have developed a methodology to solve the general l_p distance LAP. This method relaxes the {0,1} constraints on the allocations, and simultaneously, solves for both the locations and the allocations. The algorithm involves both a step to compute a good starting solution and a specialized linesearch procedure that retains the feasibility of the allocation and recognizes the discontinuity of the first derivative along the search direction. A set of necessary and sufficient conditions for the local minima of the relaxed problem are then given, which in turns leads to an efficient algorithm involving the use of an active set strategy and orthogonal projections.

2.2.2.c Squared- Euclidean Distance Location-Allocation Problems

Sherali and Tuncbilek (1992) have solved the capacitated version of a problem when the separation penalty is proportional to the square of the Euclidean distance. Using calculus, they showed that this problem can be transformed to a quadratic convex maximization problem by projecting it onto the space of allocation variables. An efficient exact algorithm is developed based on applying the RLT method to obtain an Upper Bounding Linear Program (UBLP). The UBLP is embedded in a finitely convergent branch-and-bound framework that partially enumerates the extreme points of W. Their algorithm handles up to 20 facilities and 120 customers.

2.2.3 Fixed-Charge Location/Discrete Location-Allocation Problems

The fixed-charge or discrete location-allocation problem is a variant of LAP that restricts the location of the n supply centers to be determined to certain n preselected sites (Davis and Ray, 1969). The "fixed-charge" is the cost of establishing a supply center at a given location (Ellwein and Gray, 1971).

The uncapacitated version of this problem has received a significant amount of attention. Kuehn and Hamburger (1963), Manne (1964), and Feldman *et al*. (1966) have proposed heuristic methods to obtain "good" solutions (Effroymson and Ray, 1966). A branch-and-bound algorithm was developed by Effroymson and Ray (1966) to solve the capacitated fixed-charge problem by formulating it as an integer problem having an associated continuous subproblem that can be optimized to within certain bounds. Curry and Skeith (1969) proposed a dynamic programming algorithm for this problem. A dual ascent enumerative algorithm was also developed by Erlenkotter (1978) to solve this problem. Guignard and Spielberg (1979), and Sherali and Adams (1982) have demonstrated that procedures utilizing dual ascent and employing cutting planes in an enumerative framework are most effective for this class of problems.

The capacitated version of this problem has received attention by researchers working on exact and heuristic methods since 1960. We briefly address some selected works in this variant. Sa (1969) developed the first exact algorithm based on a weak linear programming formulation, and this was later strengthened by Davis and Ray (1969). Among other contributors are Geoffrion and McBride (1977) who applied Lagrangian relaxation on the improved disaggregated formulation to derive lower bounds on the problem, and showed that this procedure attacks the capacitated location problems quite successfully. Also for this problem, Beasley (1993) developed a heuristic that is based upon Lagrangean relaxation and subgradient optimization. Other heuristic techniques, such as those by Khumawala (1974), were also developed to solve this problem, based on various approximations to the optimal transportation problem. Several variants of this fixed-charge problem have also been considered. A variant for the capacitated fixed-charge location problem is addressed by El-Shaieb (1973), in which he considered the customer locations alone for potential new facility sites. Feldman *et al*. (1966) developed a heuristic for a situation in which the establishment and operation costs of the new facilities are continuous concave functions. One noteworthy variant, which is addressed by Sherali and Adams (1984), seeks the location of n facilities on a set of n potential sites in a one-to-one fashion and the allocation of the products to customers. They transformed the problem to a fixed-charge location problem with assignment side constraints and developed an efficient Lagrangian relaxation, enumeration and decomposition composite algorithm. Furthermore, a multistage variant of the capacitated fixed-charge problem has been considered by Nagelhout and Thompson (1981). Erlenkotter (1981) gave a comparative study for the behavior of seven dynamic approaches for solving the former problem and suggested strategies that combine different methods into possibly more effective techniques.

2.2.4 Stochastic Location-Allocation Problems

Several versions of the stochastic location-allocation problems have appeared in the literature. One version, due to Marucheck and Aly (1981), uses the rectilinear distance metric in an uncapacitated formulation having demands that are characterized by bivariate uniform and normal probability density functions over rectangular regions. They developed a branch-and-bound algorithm in order to minimize an expected weighted distance objective function. Among other versions, is the stochastic Euclidean distance LAP that appeared in the dissertation of Selim (1979), where the customer demands are considered to be independent random variables having known probability distributions. A deterministic equivalent is derived for this problem via chance-constraints, and a two-stage with recourse formulation is developed. Based on this, a cutting plane algorithm is proposed to solve the problem.

2.2.5 Location-Allocation Problems with Area Destinations

There are some realistic cases where the demand is spread over an area instead of being concentrated at discrete points. Therefore, some researchers were motivated to focus on mathematically formulating such scenarios. Leamer (1968) was the first to explicitly consider an area destination multifacility uncapacitated LAP. In this problem, he used the Euclidean distance metric and assumed the demand to be uniformly distributed over an area, such as that confined by a square, a circle, or an equilateral triangle. He then utilized a heuristic perturbational algorithm to obtain a solution that could not be improved by perturbing any single facility location. Additionally, he made an interesting observation, namely, that as the number of facilities increases, the areas served by each facility tends to be hexagonal, except for distorting effects on the boundaries of the region. Cavalier and Sherali (1986a) improved on Leamer's procedure and provided exact solutions for the various test problems considered. Wesolowskey and Love (1984) and many others have also considered this problem. For more detail on this issue, see Bennett and Mirakhor (1974), Maruchek and Aly (1981), and Cavalier and Sherali (1986a).

2.2.6 Sequential and Dynamic Location-Allocation Problems

In practice, the time horizon plays an important role in describing some location-allocation Problems. When analyzing such problems, the time horizon may be partitioned into a finite number of periods, or in some problems, it is found that the variations with respect to time is continuous. Also, due to financial considerations, the construction of new proposed facilities may not take place simultaneously, and this might also entail the relocation of previously constructed facilities. We briefly present some contributions in this area. Scott (1971) presented a good example of sequential types of decisions. He solved an uncapacitated Euclidean distance LAP, in which one new facility must be located in every period over a given time horizon. He approached the problem by studying two strategies. In one strategy, a myopic strategy, a new facility is located in every period to minimize the costs of the present period. In another strategy, which is a long-range strategy, the new facility locations are determined via an optimal multifacility solution, but then, one must decide on the order in which these new facilities are sited in a period by period basis.

Dynamic programming has been widely used to solve many variants of LAP, such as the 1-center problem studied by Megiddo (1986) in which a global optimum is achieved, and capacitated or uncapacitated discrete or fixed-charge versions in which the demand is distributed over a finite, discrete horizon. For such problems, Khachaturov and Astukhov (1976) have proposed a combinatorial heuristic, and Truscott (1980) has developed an implicit enumeration algorithm for a zero-one linear integer programming formulation of this problem.

In one version of the above problem, the relocation of facilities is needed as in the cases where constructing a new facility on a site is linked with concurrently closing previously constructed facilities at other sites. Examples of research conducted in this area are the works of Sweeny and Tatham (1976), Erlenkotter (1981), Van Roy and Erlenkotter (1982), Cavalier and Sherali (1985) and Hormozi and Khumawala (1996). Sherali (1991) considered this problem with both deterministic and probabilistic components of the demand, requiring the location of one additional capacitated facility in each of the *p* specified periods and determining the service allocation variables so as to optimally satisfy the demand on chain graphs and tree networks. He addressed two location strategies. The first is a myopic strategy, as described above, and the second is a discounted present worth strategy. Love (1976) used a dynamic programming approach procedure to solve a one-dimensional uncapacitated rectilinear location-allocation problem. Suzuki *et al.* (1991) examined the sequential location-allocation of public facilities in one-and-two-dimensional spaces, by developing several solution strategies.

A dynamic location-allocation problem with continuous time variations, as in Tapiero (1971), and a discrete site LAP, as in Rao and Rutenberg (1977), are among variants of dynamic LAP that have received attention in the literature. For more references in this area, the reader is referred to Erlenkotter (1981).

2.3 Reformulation-Linearization Technique

The Reformulation-Linearization Technique (RLT) developed by Sherali and Adams (1990, 1994), and by Sherali and Tuncbilek (1992) is a new method that can be utilized to generate tight linear or convex programming representations which can be used not only for constructing exact solution algorithms, but also for designing powerful heuristics for various classes of nonconvex problems (Sherali and Adams, 1996). Problems of this type arise in production planning, location-allocation, economics, and game theory contexts.

Recently, the development of branch-and cut algorithms for discrete optimization problems, and polyhedral outer-approximation methods for continuous programming problems, have received considerable attention by many researchers such as Nemhauser and Wolsey (1988), Hoffman and Padberg (1991), Sherali (1995), and Horst and Tuy (1993). At the heart of their works is a sequence of linear programming problems that drive the solution process. The success of such approaches is strongly tied to the strength or tightness of the linear programming representations employed.

The RLT essentially generates a hierarchy of progressively tighter, higher dimensional, linear programming or convex relaxations for the underlying mixed-integer, polynomial 0-1 and continuous problems that can assist in solving the given problem within the context of a partitioning scheme. This is achieved within two phases. In the first phase, which is the Reformulation Phase, additional nonlinear implied inequalities are generated and added to the problem. In the second phase, which is the Linearization Phase, the problem resulting from the first phase is transformed into a linear program by substituting a single variable for each distinct variable-product term. (Sometimes, additional convex constraints are added to the relaxation in the case of nonlinear polynomial programming problems.)

The RLT has been successfully employed for solving discrete as well as continuous nonconvex problems. Sherali and Adams (1990, 1994) addressed mixed-integer 0-1 linear programming problems. The RLT method first converts this problem into a nonlinear, polynomial mixed-integer 0-1 problem by multiplying the constraints with some suitable d-degree ($d = 0,..., n$) polynomials involving the n binary variables, and subsequently linearizes the resulting problem using variable transformations. Relaxing integrality, the problem is recast into a higher dimensional space. As d varies from 0 to n, a hierarchy of relaxations ranging from the ordinary linear programming relaxation to the convex hull of feasible solutions is realized. Even the first order relaxation has been shown to provide algorithmic computational advantages beyond the continuous relaxation (see the discussion in Sherali and Adams, 1996).

The RLT method has also been applied using variations of the first level implementation to the quadratic assignment problem (Adams and Johnson, 1994), continuous and discrete bilinear programming problems (Adams and Sherali, 1993; Sherali and Alameddine, 1992), rectilinear distance location-allocation problems (Sherali *et al*. 1994), nonconvex quadratic programming (Sherali and Tunbilek 1995), airline gate assignment problems (Sherali and Brown, 1994), and to many other applications arising in network design telecommunications (Sherali *et al*., 1996).

Specialized RLT algorithms have also been derived by Sherali and Tuncbilek (1992) to globally optimize polynomial programming problems. They developed an RLT method for directly generating linear programming relaxations for the polynomial program, and have designed globally convergent branch-and-bound algorithms using these relaxations. The resulting relaxations have been shown to theoretically dominate the relaxations that would be obtained by applying RLT to an equivalent quadratic polynomial problem (see Sherali and Tunbilek, 1995). More, recently, Sherali (1996) has extended the former work of RLT to handle polynomial programming problems that have rational exponents on variables. For more detailed discussion on the applications and results regarding the RLT technique, see Sherali and Adams (1996).