

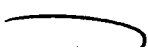
**STRESS AND RELIABILITY ANALYSES OF MULTILAYERED
COMPOSITE CYLINDER UNDER THERMAL AND MECHANICAL LOADS**

by

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(ABSTRACT)

The coupling resulting from the mutual influence of material thermal and mechanical parameters is examined in the thermal stress analysis of a long, hollow, multilayered, isotropic composite cylinder subjected to sudden axisymmetric external and internal temperature. The method of complex frequency response functions together with the Fourier transform technique is utilized.

Because coupling parameters for some composite materials, such as carbon-carbon, are very small, the effect of coupling is neglected in the orthotropic thermal stress analysis. The stress distributions in long, hollow, multilayered orthotropic cylinders subjected to sudden axisymmetric temperature loading combined with dynamic pressure as well as asymmetric temperature loading are also obtained. The method of Fourier series together with Laplace transform is utilized in solving the heat conduction equation and thermal stress analysis. The inertial term is considered and the perturbation technique is applied to cylinders subjected to dynamic pressure loading.

For brittle materials, like carbon-carbon composite, the strength variability is represented by two or three parameter Weibull distributions. The "weakest link" principle which takes into consideration both the applied stresses and the effected volume of

material is used in the reliability analyses for both the isotropic and orthotropic carbon-carbon composite cylinders.

The complex frequency response analysis is performed on a long hollow multilayered orthotropic cylinder under asymmetrical thermal load. Both deterministic and random thermal stress and reliability analyses can be based on the results of this frequency response analysis.

The stress and displacement distributions and reliability of rocket motors under static or dynamic line loads are analyzed by an elasticity approach. Rocket motors are modeled as long hollow multilayered cylinders with an air core, a thick isotropic propellant inner layer and a thin orthotropic kevlar-epoxy case. The case is treated as a single orthotropic layer or a ten layered orthotropic structure. Five material properties and the load are treated as random variables with normal distributions when the reliability of the rocket motor is analyzed by the first-order, second-moment method (FOSM).

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I thank my husband, , for his love and understanding during the past years.

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1.0 INTRODUCTION

1.1 *Literature*

Composite materials with their high strength, light weight and superior mechanical properties at elevated temperatures are utilized in structures subjected to high temperatures and severe thermal gradients such as rocket nozzles and reentry space vehicles. Though the materials retain most of their strength and stiffness when heated, they are brittle and are afflicted by statistically distributed imperfections: matrix poor regions, broken fibers and voids. Such imperfections are more prevalent in large components than in small, laboratory specimens. Hence structural reliability analysis must account for a probability based size effect.

Early works on isotropic cylindrical thin shells under uniform line load along a generator based on shell theory were preformed by several authors (Hoff, Kempner and Pohle 1954; Cooper 1957; Nash and Bridgland 1961; Naghdi 1968). Schwaighofer and Microys (1979) solved the problems of orthotropic cylindrical shells with four fixed

edges subjected to line load based on orthotropic shell theory and experimental work.

Others carried out stress analysis on composite cylinders subjected to mechanical loading. Finite element analysis of general, three-dimensional, axisymmetric laminated anisotropic tubes was done by Rizzo and Vicario (1970). Whitney (1971) applied the Vlasov-Ambartsumyan shell theory to anisotropic laminated cylinders subjected to combined axial load, torsion and internal pressure. A modification made to the shell theory to include the effects of transverse normal strain was shown by numerical results to be necessary for determining stresses induced by free thermal expansion.

Pagano and Whitney (1970) analyzed long anisotropic cylinders under axial traction or internal pressure, redefined the terminology "thin-walled cylinder" for anisotropic materials. Pagano (1972) obtained a general solution for the elastic stress field in a hollow cylinder under two-dimensional surface tractions which do not vary along the axis. The stresses in the cylinder were assumed independent of the axial coordinate x , and the shear stress $\sigma_{r\theta}(r, \theta)$ and $\sigma_{\alpha}(r, \theta)$ are uncoupled with the rest of the four stress components in the above two works. Analysis of thin orthotropic elliptical cylindrical shells under combined bending and pressure loads was done by Spence and Toh (1979).

Sudden changes in the temperature may produce severe thermal stresses in these structures. In such situation, the velocity of the strain induced by the temperature should be considered for some materials with large coupling parameters. Earlier studies on coupled thermoelasticity can be found in references (Biot 1956; Dillon 1965 and 1967). It is observed that when temperature changes at a very high rate, the ef-

fect of volume change cannot be neglected. A coupling term, defined as the contribution of energy resulting from the volume change, is to be added to the classical Fourier heat conduction equation (Boit 1956). As a result, in addition to the temperature function, there is also the volume strain function in the corrected relation. Under this circumstance, the temperature distribution in the structure cannot be independently obtained by considering only the heat conduction problem. The coupled heat conduction equation and the equilibrium equation must be solved simultaneously.

Dillon (1965) worked on analytical solutions of three problems in coupled thermoelasticity for the case when the material coupling parameter equals unity. The problems are: (a) surface temperature step function; (b) surface strain step function; (c) constant velocity impact. Dillon (1967) also introduced a "correction" function to continue the work on several coupled thermoelastic dynamic boundary-value problems for semi-infinite bars. Bahar and Hetnarski (1978) used the method of matrix exponential, which constitutes the basis of the state space approach of a control theory to solve a half-space and a layer coupled dynamic thermoelastic problem in Laplace-transform domain. This method was continued in their work in 1979. They also established a connection between the thermoelastic potential and the state space approach to thermoelasticity in 1979. Analysis of a heated punch moving over the surface of a thermoelastic half-space was carried out by Frydrychowicz and Singh (1982) for temperature and stress distribution in the medium. The authors neglected the coupling term, and considered the inertia term only. Closed-form solutions were obtained for flat, cylindrical, and wedge punches. Sherief and Dhaliwal (1981) also gave the distribution of thermal stresses and temperature in the generalized

thermoelastic semi-space subjected to thermal-shock using a Laplace transform technique.

It is difficult to solve thermoelastic problems by considering coupling and inertia terms together when the structure is more complicated. Takeuti and Furukawa (1981) compared the results of considering the inertia and coupling terms separately. They found that the effect of the coupling term is much more significant than that of the inertia term. Following the above idea, there are some studies on coupled stress analysis where the inertia term is neglected.

A coupled thermal stress problem of an infinite axisymmetric solid cylinder is analyzed by Takeuti and Tanigawa (1981 and 1982). To obtain the solution, a harmonic function is introduced to constitute the relation between the volume strain and temperature functions. The technique of Laplace transforms is used in these papers. Displacement potential and Love's displacement function were introduced to solve a thermal stress problem of a finite circular solid cylinder by Takeuti, Ishida and Tanigawa (1983). Together with perturbation technique, the above method was extended to solve a coupled thermal stress problem of a three-dimensional hollow cylinder.

There are also works reported on thermal stresses in composite material structures. Thermoelastic orthotropic cylindrical shells with fixed ends subjected to an axisymmetric static temperature field were solved by Stavsky and Somolash (1970). Padovan (1976) developed the general solution for cylindrically anisotropic generally laminated cylinders subjected to both static thermal and mechanical loading based on finite and infinite Fourier integral transforms together with complex adjoint differential operators and complex power series expansions.

In 1978, Kalam and Tauchert obtained the solution of stresses in a hollow orthotropic single layer cylinder subjected to an asymmetric steady-state plane temperature distribution by the method of Airy's stress functions in Fourier series form. Thermal stress analysis of layered cylindrical shells was developed based on classical shell theory for shells with all four edges free by Miller, Millavec and Kicher (1981).

In recent years, more work was reported on thermal stresses in this area. Hyer and Cooper (1986) analyzed stresses and deformations in single layer and multiple layer cross-ply composite cylinders due to a circumferential temperature gradient by an elasticity approach. In that paper, the temperature does not vary along the axis and the radius of the cylinders, but is in the form of $\Delta T_0 + \Delta T_1 \cos \theta$. Stress analysis for cross-ply cylindrical panels subjected to mechanical loading and static temperature distribution were done by Huang and Tauchert (1991). Global-Local methods for thermoelastic stress analysis which are efficient for thick laminated cylinders consisting of a repeating sublaminate were introduced by Luo and Sun (1991). In the work, the results for the cylinders subjected to uniform change of temperature and/or constant axial force and torque as well as normal traction on the innermost and outermost surface were obtained.

The temperature analysis is important and sometimes difficult in thermoelastic analysis. Parida and Das (1972) investigated transient plane thermal stresses in a thin circular disc of orthotropic material due to an instantaneous point heat source. The method of separation of variables was applied in solving the heat conduction equation for temperature distribution, and the stress function was applied in the stress and strain analysis.

Tauchert (1989) studied the effects of thermal shock on simply supported, thin orthotropic rectangular plates. The displacements contain the quasistatic part which is an exact Levy-type solution and the dynamic part which was obtained by Galerkin's method and Laplace transformation. Tanigawa, Murakami and Ootao (1989) analyzed transient thermal stresses in a laminated composite beam made of different materials in multilayers. The temperature of the beam was assumed to vary in the direction of the thickness, and the heat conduction equation was solved by Laplace transform. The thermal stresses were obtained by elementary beam theory and Airy's thermal stress functions.

Wang and Chou (1989) worked on transient interlaminar thermal stresses in symmetric angle ply composite laminates with infinite length and finite width. The temperature varies in the width direction. The heat conduction equation was solved by the separation of variables method and the equilibrium equation was analyzed by zeroth-order perturbation technique.

Composite cylinders are of interest to some researchers. Temperature distributions and stresses in isotropic heat radiated tubes induced by an asymmetric heat flux were solved by Fett (1986). In Kardomateas's work (1989), symmetrical thermal stresses in orthotropic composite tubes were analyzed. The heat conduction equation was solved as Bessel functions while the stress and displacement distributions were obtained by an elasticity approach.

Thermal stress analysis of a multilayered isotropic composite hollow cylinder under axisymmetric two-dimensional time dependent thermal loading were done by Ootao, Tanigawa and Fukuda (1991). Fourier cosine transform and Laplace transform were

applied to the temperature and thermoelastic potential functions and Love's displacement function was applied to thermoelastic field.

Stress effects on a multilayered cylinder subjected to a high rate of thermal loading were analyzed by Wang, Thangjitham and Heller (1991). Thermomechanical coupling was considered, but the inertia effect was neglected. The solution procedure used the method of the complex frequency response functions in conjunction with the Fourier transform technique. Different coupling parameters and various loading conditions were considered in the study.

Environmental temperature variations can also produce thermal stresses and strains that causes damage. A large diameter circular cylinder with casing subjected to narrow-band random thermal loading were examined by Heller (1976). The mean and variance of the temperature and thermal stresses were obtained by the method of frequency response functions. This work was extended to reliability analysis by Heller, Kamat and Singh (1979). The complex frequency response functions for temperatures, displacements, and stresses for the general uncoupled thermoelasticity problem in a long multilayered cylinder can be found in Thangjitham, Heller, and Singh (1986). The applications of these functions to stress analysis under steady harmonic thermal loadings are found in references (Heller, Kamat and Singh 1979; Singh, Heller and Thangjitham 1984).

The problem of rocket motor reliability under static and dynamic loads has been treated in USAMICOM Technical Report CR-RD-PR-92-4 by Heller and Janajreh. The report treated the motor case as an isotropic material and utilized a Response Surface Method (SMS) together with a Finite Element Stress Analysis to calculate the reliability of motors.

Experimental results on mechanical properties and on size effect in carbon-carbon have been presented by Heller, Thangjitham, Rantis and Heller (1991) while a structural reliability analysis based on the "weakest link" principle is developed by Heller, Thangjitham and Yeo.

1.2 Objective

In this research, thermal stresses in a long hollow multilayered cylinder subjected to sudden axisymmetric external and internal temperature is analyzed. Coupling resulting from the mutual influence of material thermal and mechanical parameters will be examined while the inertial effect will be neglected. The method of complex frequency response functions together with the Fourier transform technique is utilized here. Due to the linear nature of the problem, the time response functions for temperatures, displacements, and stresses are obtained by applying inverse Fourier transforms to the product of the complex frequency response functions and the Fourier transform of the input temperature. The integrals of the inverse Fourier transform in this problem are found to involve only those of real functions. As a result, standard numerical techniques can be implemented in the inversion procedure. The reliability of the cylinder is analyzed based on the "weakest link" principle and the two parameter Weibull distribution.

The stress distributions in long hollow cylinders with a multilayered orthotropic case and a soft, thick, inner layer subjected to sudden asymmetric temperature and me-

chanical loading are obtained. Due to the assumed linearity, the law of superposition is suitable to the problem. Stresses induced by the thermal and mechanical loading can be solved separately for low frequencies and intermediate temperatures. At high frequencies and temperatures, thermo-mechanical coupling will have to be considered. According to the work of Wang, Thangjitham and Heller (1991), the coupling parameters for some composite materials, like carbon-carbon composite, are very small. For these materials, the effect of coupling can be neglected in the thermal stress analysis. Compared with the coupling effect, the inertial effect is even smaller (Takeuti and Furukawa 1981), and will also be neglected in the thermal stress analysis. The method of Fourier series combined with Laplace transform will be utilized in solving the heat conduction equation and thermal stress analysis, while the inertial term will be considered and the perturbation technique will be applied to stress field subjected to mechanical loading. The "weakest link" principle which takes into consideration both the applied stresses and the effected volume of material is used again in the reliability analysis, but a three parameter Weibull distribution is applied.

During storage and transportation, rocket motors are subjected to line loads and sometimes unexpected patch loads. In this research, the stress and displacement distributions and reliability of rocket motors under such loading situations are analyzed by an elasticity approach. Rocket motors are modeled as long hollow multi-layered cylinders with an air core, a thick isotropic propellant inner layer and a thin orthotropic kevlar-epoxy case. There are two situations considered. In the first one the case is treated as a single orthotropic layer, while in the second the case is treated as a ten layered orthotropic material. The reliability of the rocket motor is analyzed by the first-order, second-moment method (FOSM). Five material properties and the load are treated as random variables with normal distributions.

2.0 COUPLED THERMAL SHOCK PROBLEM

2.1 Governing Equations

The heat conduction equation for a plane axisymmetric coupled thermoelasticity problem expressed in terms of cylindrical coordinates is given as (Boley and Weiner, 1960)

$$\nabla^2 T = \frac{1}{\gamma} \frac{\partial T}{\partial t} + \frac{\alpha E T_R}{\gamma \rho c_v (1 - 2\nu)} \frac{\partial e}{\partial t} \quad (2.1)$$

where $T(r,t)$ and $e(r,t)$ are the temperature and volume strain functions, r and t are the radial coordinate and time, ∇^2 is the Laplacian operator, E and ν are the modulus of elasticity and Poisson's ratio, and α , γ , ρ , and c_v are the coefficients of thermal expansion and of thermal diffusivity, mass density, and specific heat, respectively. T_R is the reference temperature.

The thermo-mechanical coupling parameter, δ , is expressed in terms of the material mechanical and thermal properties as (Boley and Weiner, 1960)

$$\delta = \frac{1 + \nu}{1 - \nu} \frac{\alpha^2 E T_R}{(1 - 2\nu) \rho c_v} \quad (2.2)$$

In terms of the thermo-mechanical coupling parameter, δ , Eq. 2.1 can be rewritten as

$$\nabla^2 T = \frac{1}{\gamma} \frac{\partial T}{\partial t} + \frac{\delta}{\gamma \alpha} \frac{1 - \nu}{1 + \nu} \frac{\partial e}{\partial t} \quad (2.3)$$

where the second term on the right-hand side of the above equation represents the coupling effect of volume strain to the temperature field.

The equation of equilibrium for the case of plane strain under axisymmetric heating is given, in terms of cylindrical coordinates, as (Wang, Thangjitham and Heller, 1991)

$$\frac{\partial e}{\partial r} = \frac{1 + \nu}{1 - \nu} \alpha \frac{\partial T}{\partial r} \quad (2.4)$$

The volume strain function, e , and radial displacement, $u(r,t)$, are connected through the relationship

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} \quad (2.5)$$

In terms of the volume strain, radial displacement, and temperature, the nontrivial stress components are obtained as (Wang, Thangjitham and Heller, 1991)

$$\sigma_r = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} e + \frac{E}{1 + \nu} \frac{\partial u}{\partial r} - \frac{\alpha E}{1 - 2\nu} T \quad (2.6)$$

$$\sigma_{\theta} = \frac{\nu E}{(1+\nu)(1-2\nu)} e + \frac{E}{1+\nu} \frac{u}{r} - \frac{\alpha E}{1-2\nu} T \quad (2.7)$$

where $\sigma_r(r,t)$ and $\sigma_{\theta}(r,t)$ are the radial and tangential stress components, respectively.

The variables in Eqs. 2.3 through 2.7 are written in terms of dimensionless variables with the following transformations:

$$r \rightarrow \frac{r}{r_R}, \quad \nabla^2 \rightarrow r_R^2 \nabla^2, \quad t \rightarrow \frac{\gamma_R t}{r_R^2}$$

$$T \rightarrow \frac{T}{T_R}, \quad e \rightarrow \frac{e}{\alpha_R T_R}, \quad \sigma_r \rightarrow \frac{\sigma_r}{\alpha_R T_R E_R}, \quad \sigma_{\theta} \rightarrow \frac{\sigma_{\theta}}{\alpha_R T_R E_R} \quad (2.8)$$

$$E \rightarrow \frac{E}{E_R}, \quad \gamma \rightarrow \frac{\gamma}{\gamma_R}, \quad \alpha \rightarrow \frac{\alpha}{\alpha_R}$$

where r_R , E_R , α_R , and γ_R are the characteristic radius, reference modulus of elasticity, and reference coefficients of thermal expansion and of thermal diffusivity, respectively.

Integrating Eq. 2.4, the volume strain function is obtained as

$$e = \frac{1+\nu}{1-\nu} \alpha (T + \Phi) \quad (2.9)$$

where $\Phi(t)$ is an unknown function of time.

Substituting Eq. 2.9 into Eq. 2.3, the coupled heat conduction equation becomes

$$\nabla^2 T = \frac{1+\delta}{\gamma} \frac{\partial T}{\partial t} + \frac{\delta}{\gamma} \frac{\partial \Phi}{\partial t} \quad (2.10)$$

where the heat conduction equation is now decoupled. The coupling effect is represented by the coupling parameter, δ , and the unknown function of time, Φ . Because the volume strain function does not appear in the heat conduction equation, the equation involves only the temperature and Φ function.

2.2 Method of Complex Frequency Response Functions

When a sinusoidal temperature input of some amplitude and frequency, ω , is applied to the structure, the responses, such as temperatures, strains, displacements and stresses in the structure, will become sinusoidal with the same frequency as the input but will have different amplitudes and will undergo phase shifts (Heller, Kamat and Singh, 1979; Singh, Heller, and Thangjitham, 1984 and Thangjitham, Heller and Singh, 1986). To analyze the problem under this type of temperature input, the method of complex frequency response functions is expedient. The complex frequency response function is defined as the output response resulting from a sinusoidal temperature input of unit amplitude and frequency, ω . To utilize the method of complex frequency response functions, the following solutions are assumed.

$$e = \tilde{e} \exp(i\omega t), \quad T = \tilde{T} \exp(i\omega t), \quad \Phi = \tilde{\Phi} \exp(i\omega t) \quad (11 \text{ a, b, c})$$

$$u = \tilde{u} \exp(i\omega t), \quad \sigma_r = \tilde{\sigma}_r \exp(i\omega t), \quad \sigma_\theta = \tilde{\sigma}_\theta \exp(i\omega t) \quad (11 \text{ d, e, f})$$

where $\tilde{e}(r, \omega)$, $\tilde{T}(r, \omega)$, $\tilde{\Phi}(\omega)$, $\tilde{u}(r, \omega)$, $\tilde{\sigma}_r(r, \omega)$, and $\tilde{\sigma}_\theta(r, \omega)$ are the corresponding complex frequency response functions for e , T , Φ , u , σ_r , and σ_θ , respectively, and $i = \sqrt{-1}$.

Substituting Eqs. 2.11(b) and 2.11(c) into Eq. 2.10, the coupled heat conduction equation, expressed in terms of the complex frequency response functions, $\tilde{T}$ and $\tilde{\Phi}$ is given as

$$\nabla^2 \tilde{T} = \frac{(1 + \delta)}{\gamma} i\omega \tilde{T} + \frac{\delta}{\gamma} i\omega \tilde{\Phi} \quad (2.12)$$

where the above equation is a complex differential equation with ω as a parameter.

The general solution of Eq. 2.12 is obtained as

$$\tilde{T} = C_1 Br(ar) + C_2 Kr(ar) - \frac{\delta}{1 + \delta} \tilde{\Phi} \quad (2.13)$$

where $C_k(\omega)$, $k = 1, 2$, are the complex constants of integration to be evaluated by applying the appropriate boundary conditions, $a(\omega) = \sqrt{(1 + \delta)\omega/\gamma}$, and $Br(x)$ and $Kr(x)$ are defined as

$$Br(x) = ber(x) + ibei(x), \quad Kr(x) = ker(x) + ikei(x) \quad (2.14)$$

where $ber(x)$, $bei(x)$, $ker(x)$, and $kei(x)$ are the Kelvin functions of order zero (Abramowitz and Stegun, 1972).

Substituting Eq. 2.13 into 2.9 and in conjunction with Eqs. 2.11(a), (b), and (c), the frequency response function for the volume strain, $\tilde{e}$, is given as

$$\tilde{e} = \frac{(1+\nu)}{(1-\nu)} \alpha \left[C_1 Br(ar) + C_2 Kr(ar) + \frac{1}{1+\delta} \tilde{\Phi} \right] \quad (2.15)$$

Upon substituting the above expression into Eq. 2.5, the complex frequency response function for displacement, $\tilde{u}$, is obtained by direct integration as

$$\begin{aligned} \tilde{u} = & \frac{1+\nu}{1-\nu} \alpha \left[C_1 r^{-1} \int^r \xi Br(a\xi) d\xi + C_2 r^{-1} \int^r \xi Kr(a\xi) d\xi \right] + C_3 r^{-1} \\ & + \frac{(1+\nu)\alpha}{2(1-\nu)(1+\delta)} r \tilde{\Phi} \end{aligned} \quad (2.16)$$

Similarly, upon substituting $\tilde{T}$ in Eq. 2.13, $\tilde{e}$ in Eq. 2.15 and $\tilde{u}$ in Eq. 2.16 into Eqs. 2.6 and 2.7, the complex frequency response functions for stress components $\tilde{\sigma}_r$ and $\tilde{\sigma}_\theta$ are obtained as

$$\begin{aligned} \tilde{\sigma}_r = & -\frac{\alpha E}{1-\nu} \left[C_1 r^{-2} \int^r \xi Br(a\xi) d\xi + C_2 r^{-2} \int^r \xi Kr(a\xi) d\xi \right] - \frac{E}{1+\nu} C_3 r^{-2} \\ & + \frac{\alpha E[1+2\delta(1-\nu)]}{2(1-\nu)(1-2\nu)(1+\delta)} \tilde{\Phi} \end{aligned} \quad (2.17)$$

$$\begin{aligned} \tilde{\sigma}_\theta = & \frac{\alpha E}{1-\nu} C_1 \left[r^{-2} \int^r \xi Br(a\xi) d\xi - Br(ar) \right] + \frac{\alpha E}{1-\nu} C_2 \left[r^{-2} \int^r \xi Kr(a\xi) d\xi - Kr(ar) \right] \\ & + \frac{E}{1+\nu} C_3 r^{-2} + \frac{\alpha E[1+2\delta(1-\nu)]}{2(1-\nu)(1-2\nu)(1+\delta)} \tilde{\Phi} \end{aligned} \quad (2.18)$$

where C_k , $k = 1, 2, 3$, are the integration constants.

2.3 Boundary Conditions

For each layer of a multilayered cylinder, there are a total of four unknown constants, namely, C_1 , C_2 , C_3 , and $\tilde{\Phi}$, to be simultaneously evaluated by a set of proper boundary conditions. Consequently, for a J -layered cylinder, there are $4J$ unknown constants to be evaluated. This is accomplished by applying the temperature, heat flux, displacement, and traction boundary conditions at the bounding surfaces and at the interfaces of any two adjacent layers.

In this study, a hollow J -layer cylinder (Figure 1) is considered. Because there exist no induced stresses in the air core (layer 1), only uncoupled temperature analysis is required for this layer. The remaining $(J - 1)$ layers are stressed layers for which the solutions of coupled thermoelasticity are sought. Depending on the location of the applied input temperature, there are two boundary value problems to be considered, i.e., the input temperature is uniformly applied (a) on the outer surface of the outermost (J th) layer and (b) on the inner surface of the innermost stressed (2nd) layer.

Case a: For the case where the input temperature is uniformly applied on the outermost surface of a hollow J -layer cylinder, the following boundary and interface conditions are applied.

At $r = 0$ (center of the bore)

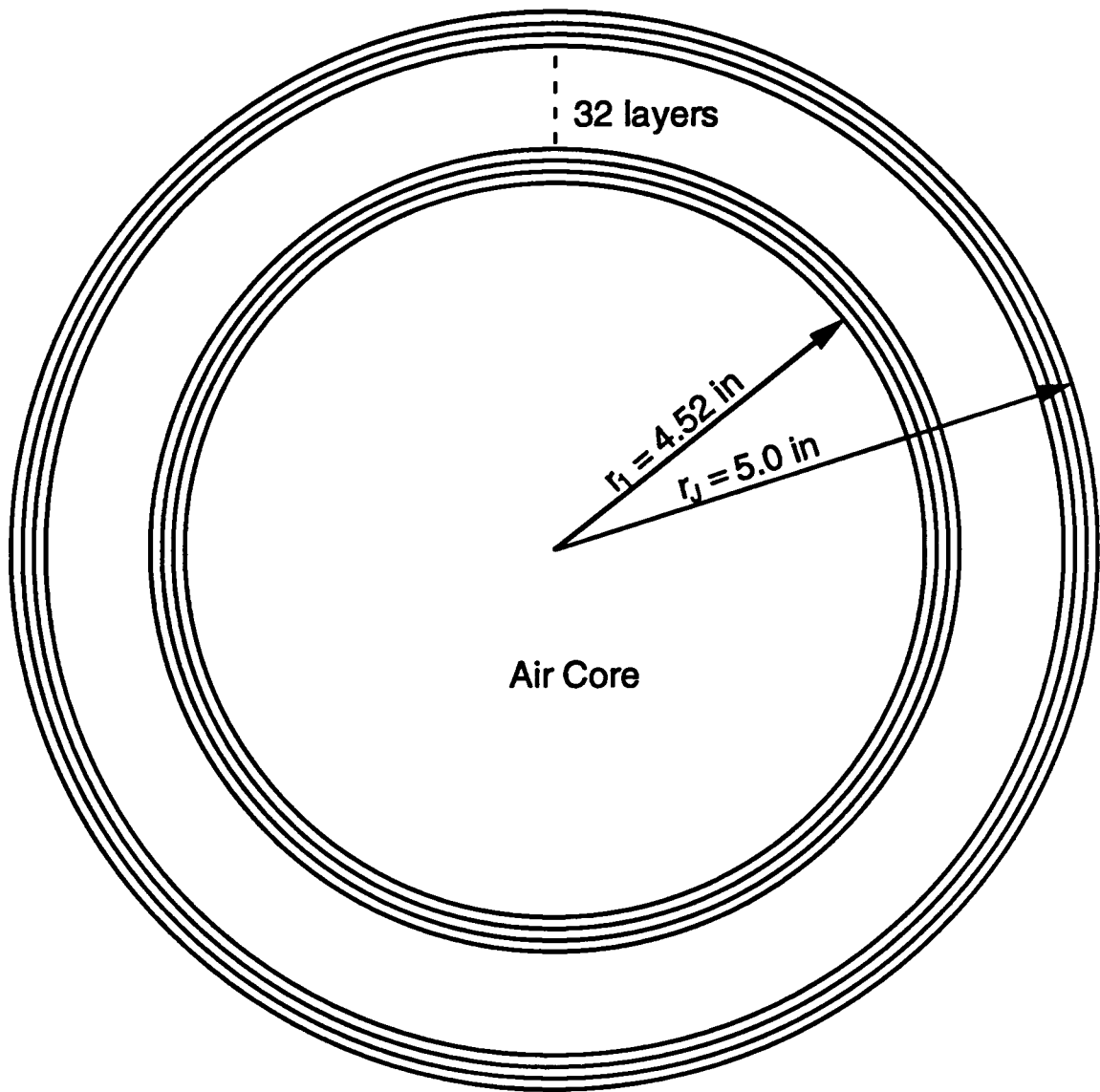


Figure 1. Configuration of the Isotropic Cylinder.

BC.1 $T^1(r,t)$ is finite (superscripts indicate layer numbers)

At $r = r_1$ (on the innermost surface)

$$BC.2 \quad T^1(r,t) = T^2(r,t)$$

$$BC.3 \quad k_1 \frac{\partial T^1(r,t)}{\partial r} = k_2 \frac{\partial T^2(r,t)}{\partial r}$$

$$BC.4 \quad \sigma_r^2(r,t) = 0$$

At $r = r_j$, $j = 2, 3, \dots, J-1$ (on the interfaces between layers)

$$BC.5 \quad T^j(r_j, t) = T^{j+1}(r_j, t)$$

$$BC.6 \quad k_j \frac{\partial T^j(r_j, t)}{\partial r} = k_{j+1} \frac{\partial T^{j+1}(r_j, t)}{\partial r}$$

$$BC.7 \quad u^j(r_j, t) = u^{j+1}(r_j, t)$$

$$BC.8 \quad \sigma_r^j(r_j, t) = \sigma_r^{j+1}(r_j, t)$$

At $r = r_J$ (on the outermost surface)

$$BC.9 \quad T^J(r_J, t) - T_a(t) = 0$$

$$BC.10 \quad \sigma_r^J(r_J, t) = 0$$

where k_j is the thermal conductivity of the j th layer and $T_a(t)$ is the input temperature.

In the above boundary conditions, *BC.2* and *BC.5* and *BC.3* and *BC.6* are the temperature and heat flux continuity conditions across the interfaces of any two adjacent layers, respectively, *BC.4* and *BC.10* imply the traction free conditions at the innermost and outermost surfaces, while *BC.7* and *BC.8* indicate the continuity of displacements and tractions at the layer interfaces, respectively.

Case b: For the case where the input temperature is uniformly applied to the inner surface (bore) of the cylinder, that is at $r = 0$, the boundary condition *BC.1* is not required and at $r = r_1$, *BC.2* is replaced by

$$BC.2a \quad T^1(r_1, t) - T_a(t) = 0$$

while the outermost surface is assumed to be insulated such that *BC. 9* is replaced by

$$BC.9a \quad \frac{\partial T^J(r_J, t)}{\partial r} = 0$$

The other interface boundary conditions are identical to those for the case of external input temperature.

2.4 Reliability Analysis

The structural reliability analysis of the cylinder is based on the “weakest link” principle which takes into consideration both the applied stress and the effected volume

of material. The probability that a reference volume v_r of material survives under the application of a stress, s , is given here in terms of the three parameters Weibull distribution as (Weibull, 1938)

$$L(s) = \exp \left[- \left(\frac{s - R_0}{R_c - R_0} \right)^m \right] \quad (2.19)$$

where $L(s)$ is the probability of survival, R_c , R_0 and m are the characteristic ultimate strength of the reference volume, v_r , the minimum strength established by quality control and the Weibull shape parameter, respectively. The characteristic strength has a probability of survival of $L(R_c) = e^{-1} = 0.3679$. Survival of structural components requires that all volume elements survive. When the elements are independent of each other, the reliability, L , of the component is equal to the product of the individual reliabilities of volume elements (weakest link hypothesis) (Stanley, Sivill and Fessler, 1974).

$$L(s_1, s_2, \dots, s_n) = \exp \left[- \left(\frac{s_1 - R_0}{R_c - R_0} \right)^m \frac{V_1}{v_r} \right] \times \exp \left[- \left(\frac{s_2 - R_0}{R_c - R_0} \right)^m \frac{V_2}{v_r} \right] \\ \times \dots \times \exp \left[- \left(\frac{s_n - R_0}{R_c - R_0} \right)^m \frac{V_n}{v_r} \right] \quad (2.20)$$

or using the common base e

$$L = \exp \left[- \sum_{j=1}^n \left(\frac{s_j - R_0}{R_c - R_0} \right)^m \frac{V_j}{v_r} \right] \quad (2.21)$$

For small volume elements the summation is replaced by integration

$$L = \exp \left[- \int_V \left(\frac{s - R_0}{R_c - R_0} \right)^m \frac{dv}{V_r} \right] \quad (2.22)$$

where the risk of failure, λ , the exponent of e, is called the stress-volume integral (Margetson, 1976). If $S_{\max}$ is the maximum value of the applied stress through the component and V is its total volume, Eq. 2.22 may be written in terms of dimensionless ratios as

$$\ln(1/L) = \lambda = \left(\frac{1}{v_c(1 - 1/v_{\max})} \right)^m \frac{V}{V_r} \int_{s > R_0} \left(\frac{s}{S_{\max}} - \frac{v_c}{v_{\max}} \right)^m \frac{dv}{V} \quad (2.23)$$

The integration is carried out only over the volume where stresses exceed R_0 (Heller, Schmidt and Denninghoff, 1985). Two safety factors are introduced as follow

$$v_c = \frac{R_c}{S_{\max}} \quad \text{and} \quad v_{\max} = \frac{R_c}{R_0}$$

Reliability functions such as Eq. 2.23 can be written for each layer of the carbon-carbon cylinder. After the stress distribution is calculated the reliability of each concentric cylindrical layers is determined.

In cylindrical coordinates, dv is equal to $2\pi r dr$ and hence Eq. 2.23 is transformed to

$$\ln(1/L) = \lambda = 2\pi \left(\frac{1}{v_c(1 - 1/v_{\max})} \right)^m \frac{V}{V_r} \int_{s > R_0} \left(\frac{s}{S_{\max}} - \frac{v_c}{v_{\max}} \right)^m \frac{rdr}{V} \quad (2.24)$$

The probability of failure, P_{fi} , of an individual layer is calculated from Eq. 2.24 as

$$P_{fi} = 1 - L_j \quad (2.25)$$

The reliability of the complete structure is based on the “weakest link” principle and is calculated as the product of layer reliabilities. The probability of failure of the complete structure becomes

$$P_f = 1 - \prod_{j=1}^n L_j \quad (2.26)$$

When risk of failure, λ , in all the layers is very small, then the following equation is quite accurate:

$$P_f = 1 - \prod_{j=1}^n L_j \approx \sum_{j=1}^{n_f} \lambda_j \quad (2.27)$$

where n is the total number of layers.

The three parameters of the Weibull distribution are estimated by the following three equations

$$\bar{R} = (R_c - R_0)\Gamma\left(1 + \frac{1}{m}\right) + R_0 \quad (2.28)$$

$$\sigma_R = (R_c - R_0) \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right) \right]^{1/2} \quad (2.29)$$

$$E[(R - \bar{R})^3] = (R_c - R_0)^3 \left[\Gamma\left(1 + \frac{3}{m}\right) - 3\Gamma\left(1 + \frac{2}{m}\right)\Gamma\left(1 + \frac{1}{m}\right) + 2\Gamma^3\left(1 + \frac{1}{m}\right) \right] \quad (2.30)$$

where $\Gamma(\cdot)$ is the gamma function. $\bar{R}$, σ_R and $E[(R - \bar{R})^3]$ are the average strength, standard deviation and the third central moment, or skewness, respectively and they can be obtained from experimental results.

When the minimum strength, R_0 is zero, the Weibull distribution becomes a two parameter distribution; the shape parameter, m , and the characteristic strength, R_c , of the reference volume are calculated as

$$m \simeq \frac{1.2}{COV} \quad (2.31)$$

$$R_c = \frac{\bar{R}}{\Gamma\left(1 + \frac{1}{m}\right)} \quad (2.32)$$

where COV is the coefficient of variation.

2.5 Illustrative Examples

When the input temperature is a harmonic function with an amplitude A and frequency ω , such that

$$T_a(t) = Ae^{i\omega t} \quad (2.33)$$

the response $X(r,t)$, such as temperature, displacement, and stresses, can be expressed in terms of the complex frequency response function as follows:

$$X = |\tilde{X}| \exp[i(\omega t + \phi)] \quad (2.34)$$

where $|\tilde{X}(r, \omega)|$ and $\phi(r, \omega)$ are the frequency dependent amplitude and phase angle of the response, respectively, and are defined as

$$\tilde{X} = A\tilde{H}, \quad \phi = \tan^{-1}(\tilde{H}_{IM}/\tilde{H}_{RE}) \quad (2.35)$$

with $\tilde{H}(r, \omega) = \tilde{H}_{RE}(r, \omega) + i\tilde{H}_{IM}(r, \omega)$ the corresponding complex frequency response function for the input temperature with a unit amplitude and frequency ω .

If the input temperature is a general function of time, then the responses (temperature, displacement, and stresses) are also some functions of time. In this case, the time response functions for temperature, displacement, and stresses are obtained via the method of Fourier transforms. The Fourier transform pair is defined as (Franklin, 1958 and Crandall and Mark, 1963)

$$\tilde{X}(r, \omega) = \int_{-\infty}^{\infty} X(r,t)e^{-i\omega t} dt, \quad X(r,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{X}(r, \omega)e^{i\omega t} d\omega \quad (2.36)$$

where $\tilde{X}$, in this case, is the complex function representing the product of the frequency response function and the Fourier transform of the input temperature such that

$$\tilde{X} = \tilde{H} \tilde{T}_a \quad (2.37)$$

for which $\tilde{T}_a(\omega)$ is the Fourier transform of the input temperature.

For the current problem of coupled thermoelasticity, the response function $\tilde{H}$ is a complex conjugate function with respect to ω , that is

$$\tilde{H}_{RE}(r, -\omega) = \tilde{H}_{RE}(r, \omega) \quad \text{and} \quad \tilde{H}_{IM}(r, -\omega) = -\tilde{H}_{IM}(r, \omega) \quad (2.38)$$

Furthermore, if the Fourier transform of the input temperature $\tilde{T}_a(\omega)$ is also a conjugate function with respect to the input frequency ω , then $\tilde{X}$, as the product of $\tilde{H}$ and $\tilde{T}_a$, is also a conjugate function with respect to the frequency, ω .

$$\tilde{X}_{RE}(r, -\omega) = \tilde{X}_{RE}(r, \omega) , \quad \tilde{X}_{IM}(r, -\omega) = -\tilde{X}_{IM}(r, \omega) \quad (2.39)$$

In this situation, the inverse Fourier transform of $\tilde{X}$, Eq. 2.36, can be rewritten as

$$X(r, t) = \frac{1}{\pi} \int_0^\infty [\tilde{X}_{RE}(r, \omega) \cos(\omega t) - \tilde{X}_{IM}(r, \omega) \sin(\omega t)] d\omega \quad (2.40)$$

where the above integral involves only real functions and can be integrated using standard numerical methods. After performing the inverse Fourier transformation, the responses, such as temperature, displacement, and stresses at any given point in the cylinder are obtained as functions of time. The advantage of using the Fourier transform technique is that it can directly utilize the existing complex frequency re-

sponse functions. Furthermore, the integral for the inverse Fourier transform is simpler to obtain than that for the Laplace inverse transform.

In this chapter, thermal shocks are applied to a hollow thirty two layer cylinder, with two alternating layers of fiber. The geometric, thermal and mechanical parameters used are listed in Table 1 while the cylinder configuration is shown in Figure 1. The input shock temperature is modeled as

$$T_a(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 2700\beta t e^{1-\beta t} \text{ K} & \text{for } t > 0 \end{cases} \quad (2.41)$$

where β is a constant that regulates the rise time and decay of the shock. The rise time, t_m , is defined as the time required for the input temperature to reach the maximum value. In terms of the constant β , $t_m = 1/\beta$. Thermal shock with rise time, $t_m = 1$ second, is investigated (Figure 2).

The Fourier transform of the input temperature function, Eq. 2.41, is obtained as

$$\tilde{T}_a(\omega) = 2700\beta e \left[\frac{\beta^2 - \omega^2}{(\beta^2 + \omega^2)^2} - i \frac{2\beta\omega}{(\beta^2 + \omega^2)^2} \right] \quad (2.42)$$

where it is obvious that $\tilde{T}_a$ is a complex conjugate function with respect to ω .

The reliability analysis in this chapter is based on the “weakest link” principle and the two parameter Weibull distribution. That means the minimum strength, R_0 , is assumed as zero here. The other two parameters, characteristic ultimate strength, R_c , and Weibull shape parameter, m , are based on the tension tests performed on dog bone shaped specimens in both the warp and fill directions (Heller, Thangjitham, Rantis and Heller, 1991). The specimen size (reference volume) is 2 x 1/4 x 3/8 in,

Table 1
Geometric, Thermal and Mechanical Parameters for Isotropic Materials.

Material, j	Air	Fiber Layers	
		Fill Direction (z)	Warp Direction (θ)
Thickness (in)	$r_1 = 4.52$	0.015	0.015
Conductivity, k_j (Btu/hr ft ^{2} F)	0.0142	12.5	12.5
Diffusivity, γ_j (in ^{2} /hr)	106.28	96.107	96.107
Elastic Modulus, E_j (psi)	---	1.6×10^6	2.55×10^6
Poisson's Ratio, ν_j	---	0.05	0.08
Strength, R_{cj} (psi)	---	1.6×10^3	1.5×10^4
Coefficient of Thermal Expansion, α_j (in./in. $^{\circ}$ F)	---	1.75×10^{-5}	1.75×10^{-5}
Coupling Coefficient, δ_j	---	0.0 0.1 0.3 0.5	0.0 0.1 0.3 0.5

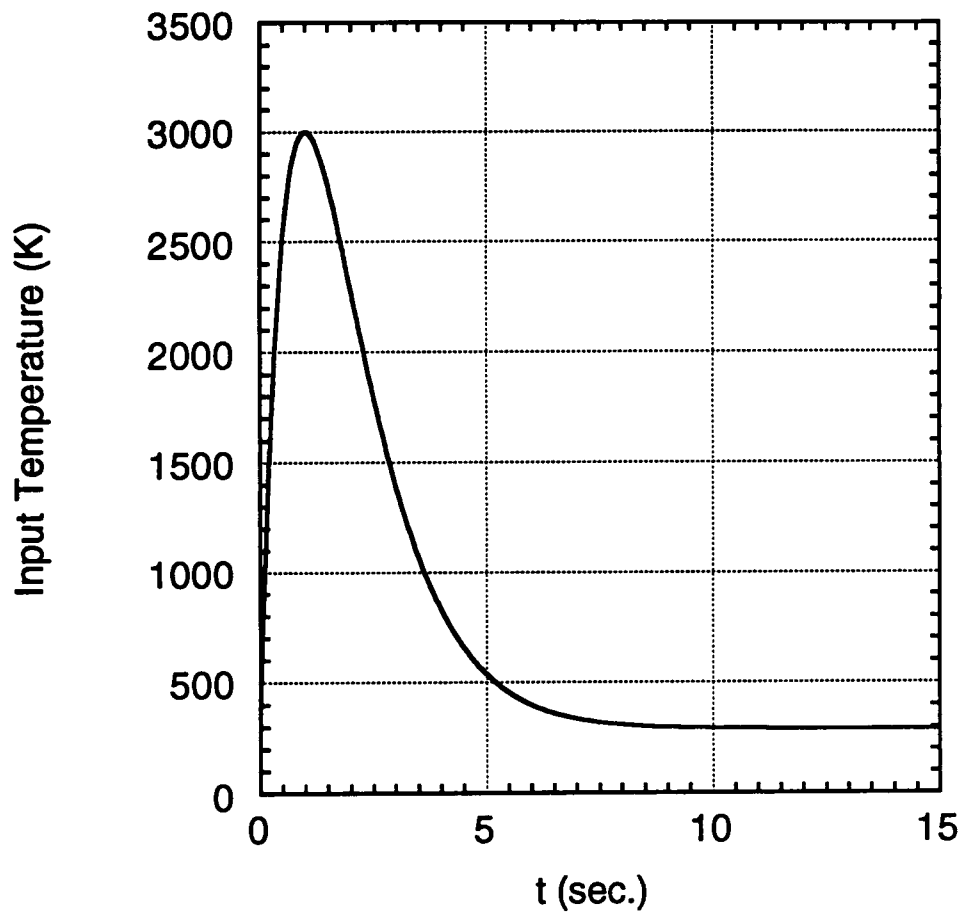


Figure 2. Input Temperature (K).

or 0.1875 in³. It is assumed here that the tensile strength and the compressive strength of the carbon-carbon composite are the same. An average strength of $\bar{R} = 15,000$ psi was measured in the warp direction while the fill strength was 6000 psi for the carbon-carbon composite. The coefficient of variation (COV) in both cases was 6.67%.

From these observations and Eqs. 2.31 and 2.32, the shape parameter, m and the characteristic strength in the warp direction, R_{cw} , and fill direction, R_{cf} , are obtained

$$m \simeq \frac{1.2}{COV} = \frac{1.2}{0.067} = 18$$

and for the characteristic strength in the warp direction, R_{cw} , and fill direction R_{cf}

$$R_{cw} = \frac{15,000}{\Gamma(1 + \frac{1}{m})} = \frac{15,000}{0.9711} = 15,400 \text{ psi}$$

and

$$R_{cf} = \frac{6000}{\Gamma(1 + \frac{1}{m})} = \frac{6000}{0.9711} = 6,180 \text{ psi}$$

Cylinder subjected to an internal shock

For the temperature time history (Figure 2) applied to the innermost surface, the tangential stresses on the innermost and outermost surfaces are shown in Figures 3 and 4 for different coupling coefficients δ . The tangential stresses on the innermost surface reach the highest compressive values first, eventually stress inversion takes place; finally stresses tend to zero while the shock decays. The tangential stresses

on the outermost surface reach the highest tensile values first, then they change sign and reach the highest compressive values; they also tend to zero while the shock decays. The maximum compressive stress in the cylinder occurs at the innermost surface while the maximum tensile stress occurs at the outermost surface.

The probabilities of failure of the structure, the innermost and outermost layers have been calculated as indicated in this chapter with a coupling coefficient, δ , of 0.1 and are presented in Figure 5. These probabilities are influenced most by the stresses on the innermost surface. The effect of the coupling coefficient on the failure probabilities of the cylinder is shown in Figure 6.

Cylinder subjected to an external shock

For the same temperature time history applied to the outermost surface, the tangential stresses are presented in Figures 7 and 8. The tangential stresses on the innermost surface reach the highest tensile values first, then they change sign and subsequently reach the highest compressive values, finally tending to zero while the shock decays. The tangential stresses on the outermost surface reach the highest compressive values first, then they change sign and reach the highest tensile values, again tending to zero while the shock decays. As with the internal shock, the maximum compressive stresses are greater than the tensile ones, but, on the contrary, the maximum compressive stresses occur at the outermost surface while the maximum tensile stresses occur at the innermost surface.

The probabilities of failure of the structure, the innermost and outermost layers have been calculated and plotted in Figure 9 for a coupling coefficient, δ , of 0.1. These probabilities are influenced most by the stresses on the outermost surfaces. It is seen that the two highly stressed layers on the two surfaces of the cylinder contribute most to the probability of failure. The effect of the coupling coefficient on the failure probabilities of the cylinder is shown in Figure 10.

Figures 3, 4, 7 and 8 show the evident differences of tangential stresses for different coupling coefficients. When the coupling coefficients are larger, the maximum tangential stresses are larger. The coupling coefficients have similar effects on the probabilities of failure, hence it is very important to study these effects when the coupling coefficient, δ , is large. Some typical coupling coefficients are presented in Table 2 for various materials.

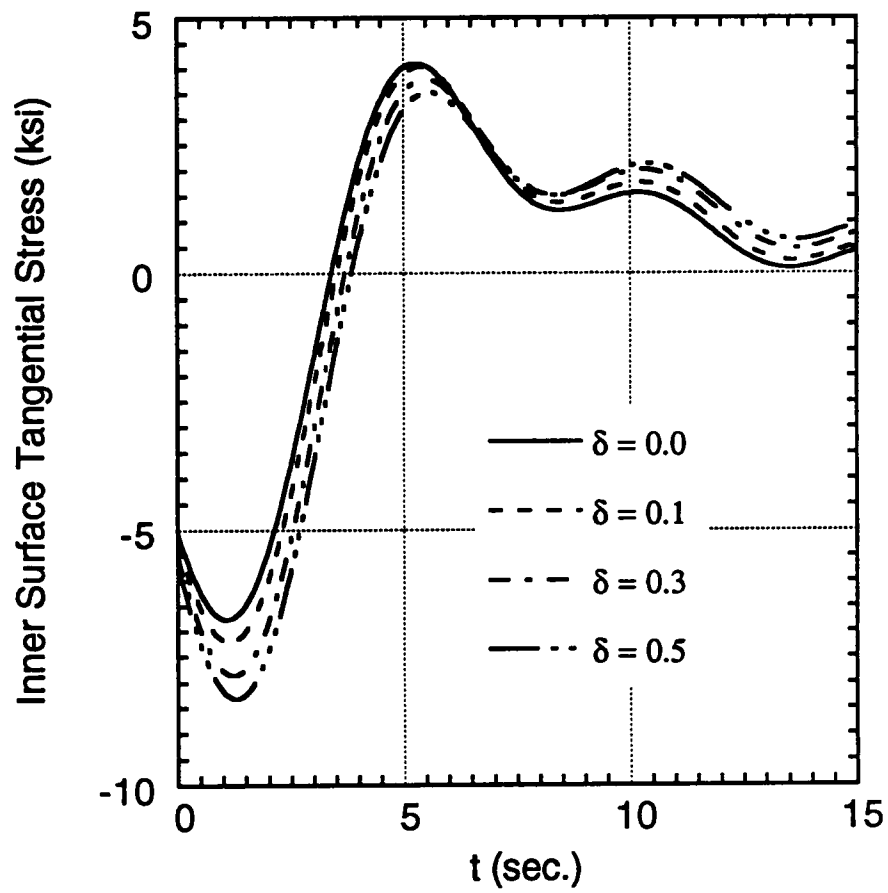


Figure 3. Tangential Stress on the Innermost Surface under Internal Thermal Shock.

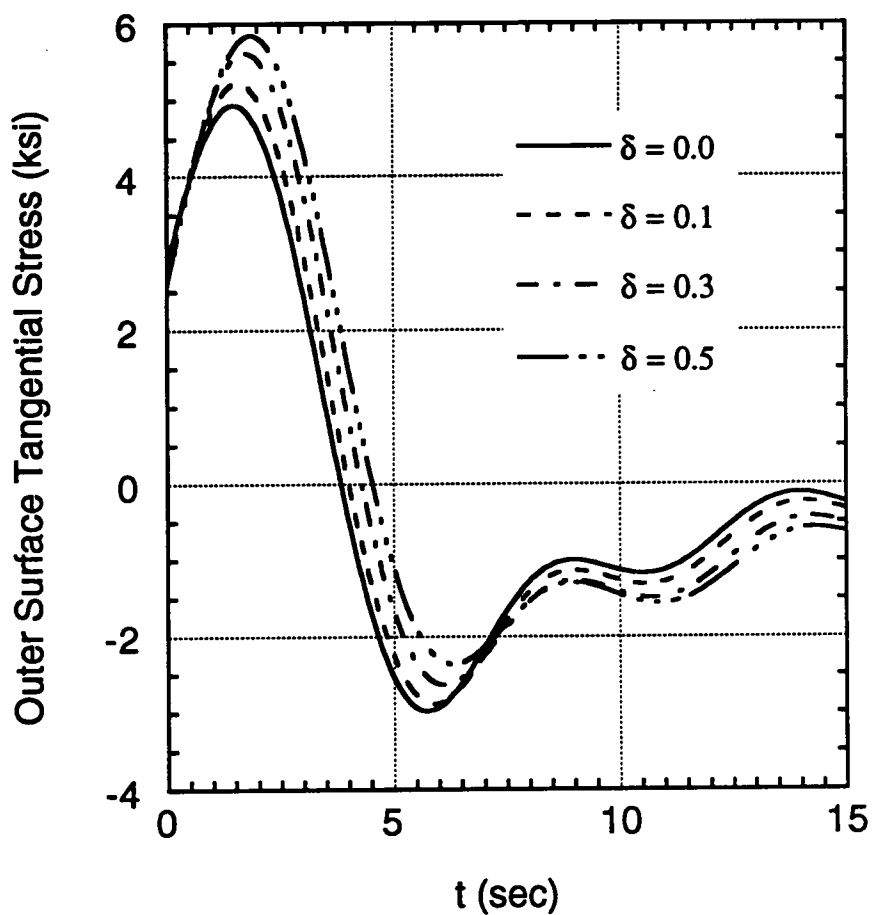


Figure 4. Tangential Stress on the Outermost Surface under Internal Thermal Shock.

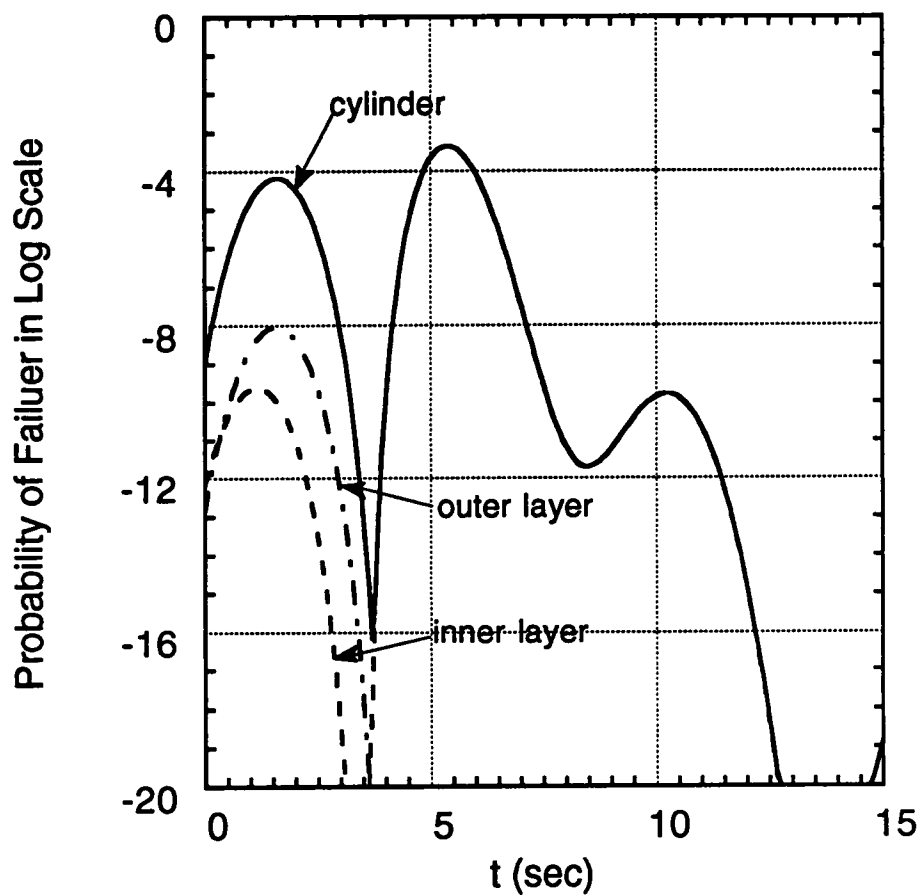


Figure 5. Probability of Failure of the Cylinder, Innermost and Outermost Layer, under Internal Thermal Shock.

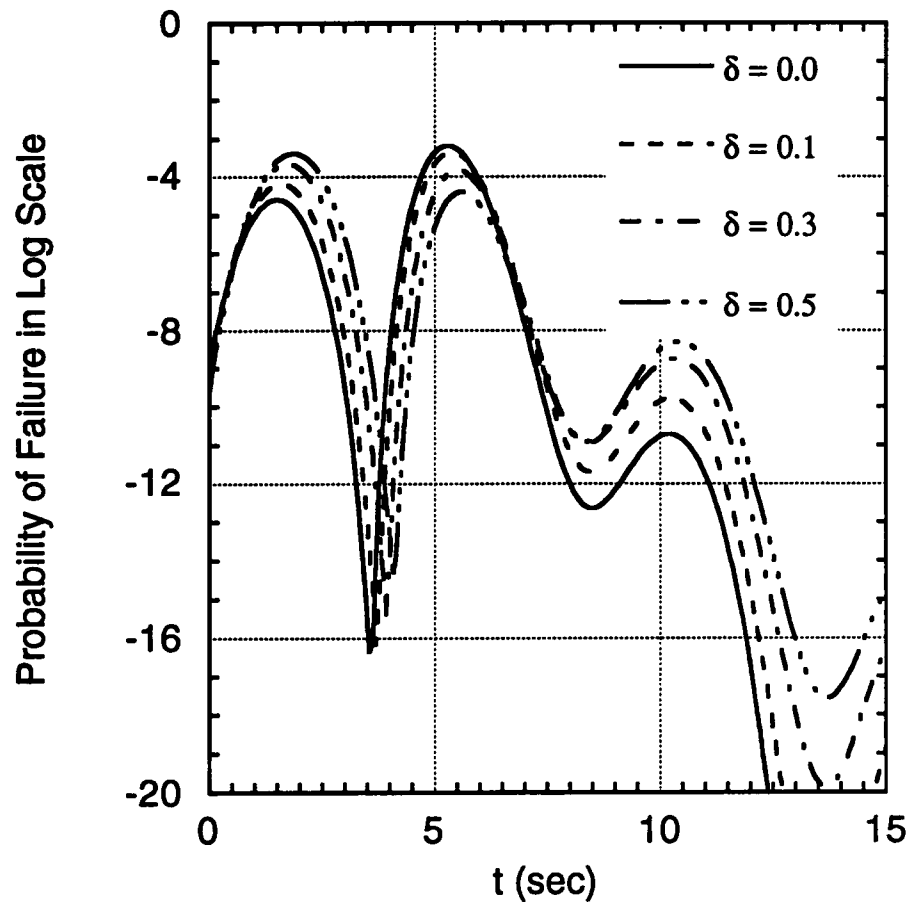


Figure 6. Probability of Failure of the Cylinder, for Different Coupling Coefficients, under Internal Thermal Shock.

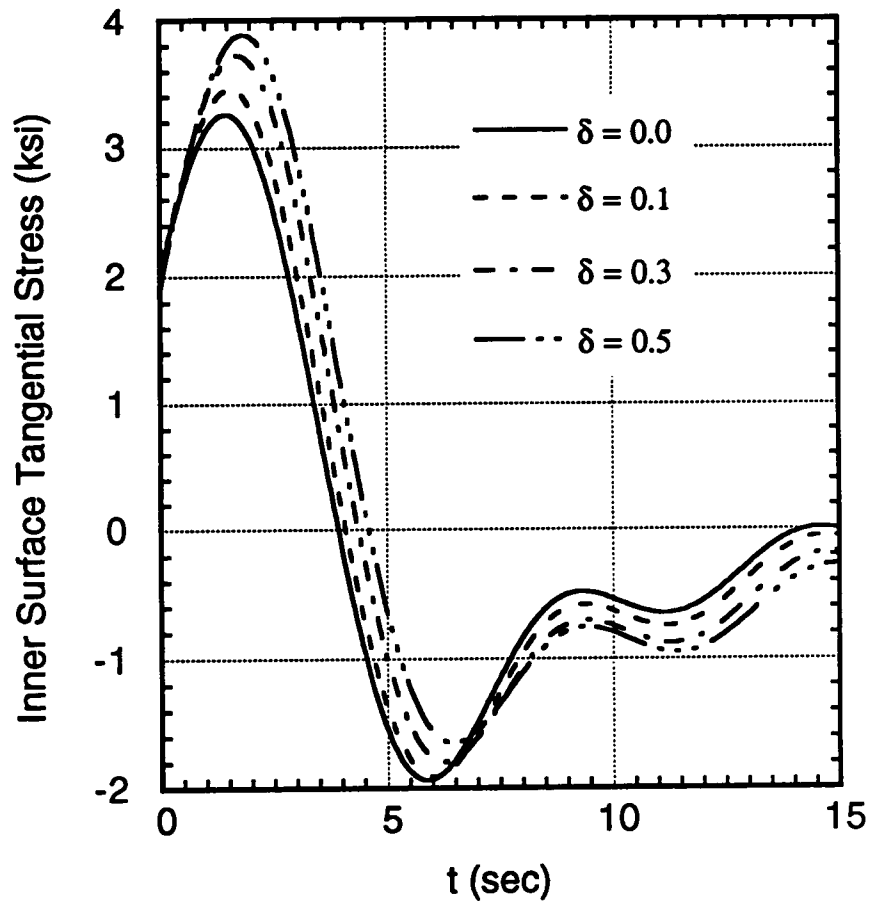


Figure 7. Tangential Stress on the Innermost Surface under External Thermal Shock.

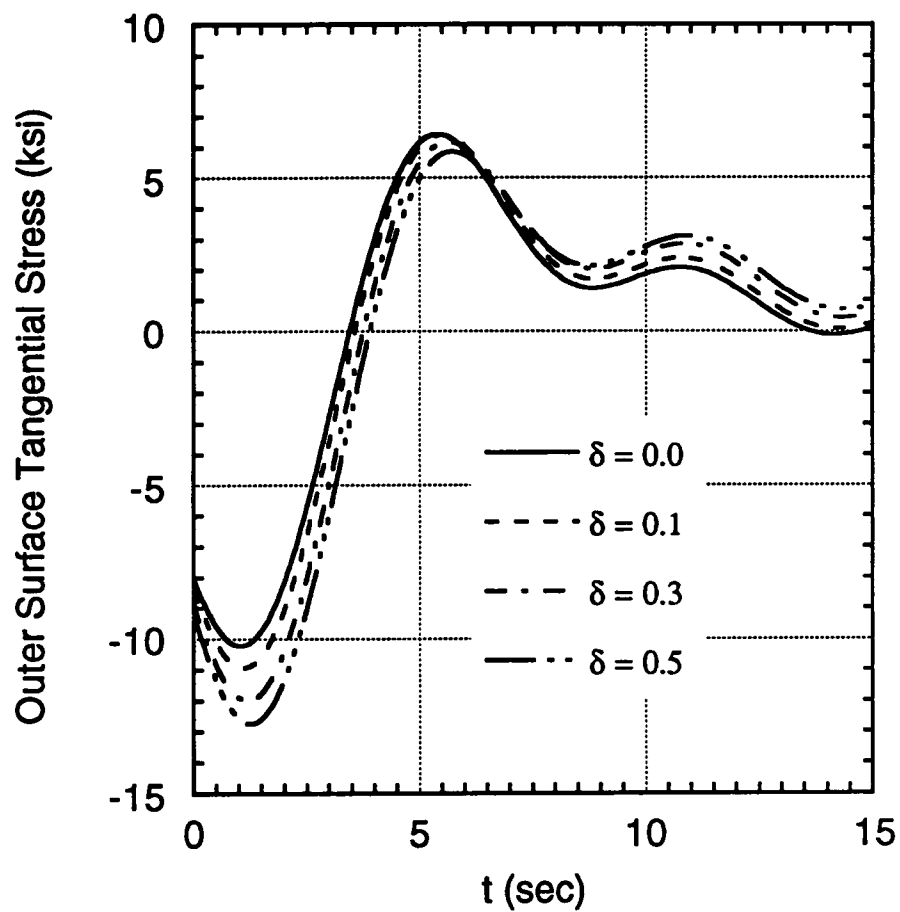


Figure 8. Tangential Stress on the Outermost Surface under External Thermal Shock.

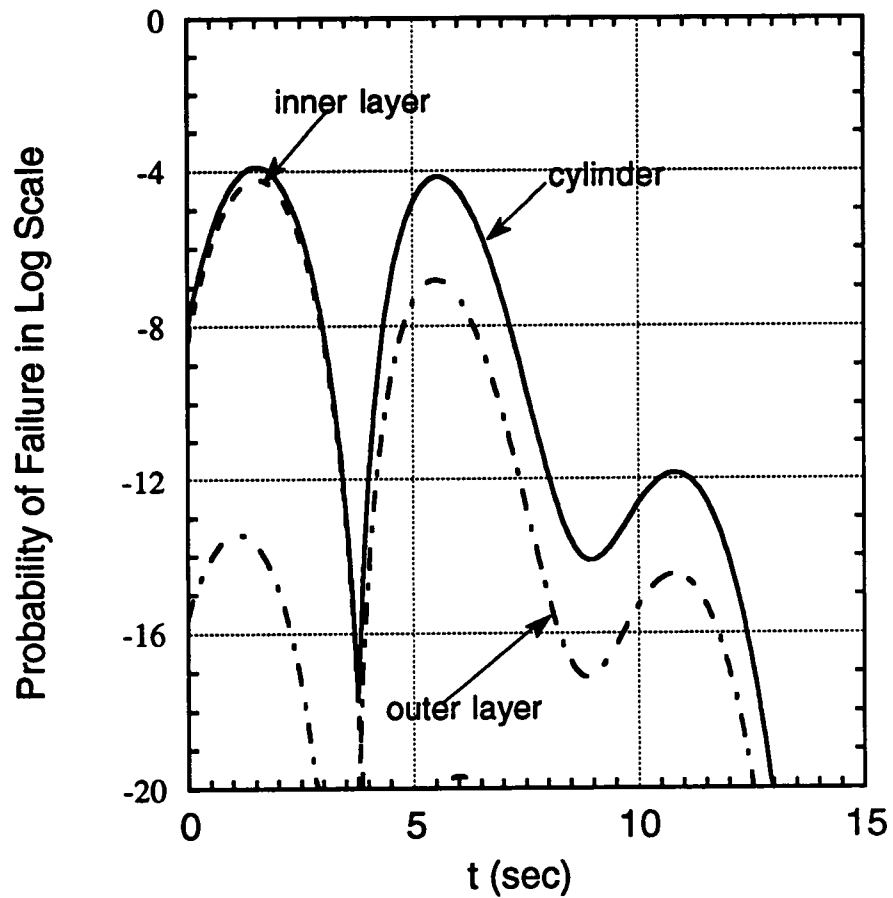


Figure 9. Probability of Failure of the Cylinder, Innermost and Outermost Layer, under External Thermal Shock.

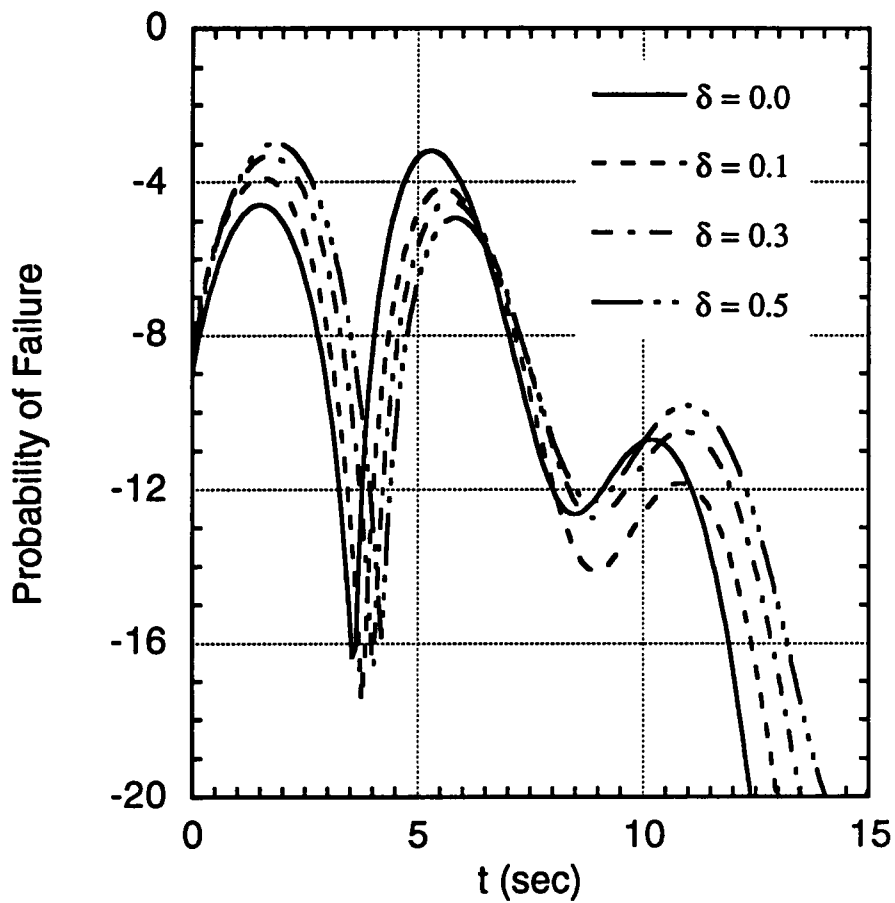


Figure 10. Probability of Failure of the Cylinder, for Different Coupling Coefficients, under External Thermal Shock.

Table 2
Coupling Coefficients of Some Engineering Materials.

Material	Elastic Modulus (GPa)	Poisson's Ratio	Thermal Expansion Coeff. $\times 10^{-6}$: (mm/mm/°K)	Specific Gravity	Specific Heat (cal/kg/°K)	Coupling Coeff.
Aluminum	69.0	0.35	25.2	2.70	215.	0.0374
Copper	124.	0.37	16.6	8.96	92.0	0.0248
Lead	15.9	0.45	28.8	11.4	31.0	0.0706
Steel	207.	0.30	12.1	7.86	108.	0.0118
Polystyrene	4.14	0.45	80.0	1.04	320.	0.15
Polyvinylchloride and Vinylchloride Acetate	4.14	0.40	180.	1.36	200.	0.412
Carbon	17.2	0.08	31.5	1.50	38.0	0.0304
Graphite Fiber	243.	0.32	16.8	0.96	170.	0.162
Kevlar Fiber	152.	0.35	54.0	0.85	250.	0.03
Polyurethane	5.5×10^{-3}	0.49	153.	1.67	368.	0.0022

$$T_R = 300.0^\circ\text{K}$$

$$1 \text{ N}\cdot\text{m} = 1 \text{ J} = 0.239 \text{ cal}$$

$$\text{mass density } \rho \text{ (Kg/m}^3\text{)} = 1000.0 \times \text{Specific Gravity}$$

3.0 ORTHOTROPIC CYLINDERS UNDER THERMAL AND PRESSURE LOADS

3.1 *Temperature Analysis*

It has been illustrated in Chapter 2 that for small coupling coefficients the uncoupled thermal stress analysis is sufficiently accurate. The carbon-carbon composite has a coupling coefficient of $\delta = 0.0304$ (Table 2), hence in the following, uncoupled analysis is used. The separation of the temperature and stress analysis enables the stress and reliability analysis to be extended from the symmetric layered isotropic cylinders under thermal load only in Chapter 2 to both the symmetric and asymmetric layered orthotropic cylinders under thermal and pressure loads in this chapter, even further, the frequency response analysis in Chapter 4.

The heat conduction equation for an orthotropic long cylinder is given as follows:

$$\gamma_{11} \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \gamma_{22} \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = \frac{\partial T}{\partial t} \quad (3.1)$$

where $T(r, \theta, t)$ is the temperature function, r and θ are the radial and tangential coordinates, and t is time, γ_{11} and γ_{22} are the coefficients of radial and tangential thermal diffusivities, respectively.

Expanding the temperature into Fourier series:

$$T(r, \theta, t) = \sum_{n=-\infty}^{\infty} T_n(r, t) e^{in\theta} \quad (3.2)$$

Assuming zero initial condition for the temperature is $T(r, \theta, 0) = 0$, substituting Eq. 3.2 into Eq. 3.1, and applying the Laplace transform to it, the heat conduction equation becomes:

$$\gamma_{11} \left(\frac{\partial^2 \tilde{T}_n}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{T}_n}{\partial r} \right) - \left(\gamma_{22} \frac{n^2}{r^2} + s \right) \tilde{T}_n = 0 \quad (3.3)$$

where $\tilde{T}_n$ is the Laplace transform of T_n in Eq. 3.2. Solving Eq. 3.3, the temperature in Laplace transform domain can be obtained in terms of the modified Bessel functions $I_{\nu_n}(x)$ and $K_{\nu_n}(x)$ as follows:

$$\tilde{T}_n = \tilde{T}_{n1} I_{\nu_n}(ar) + \tilde{T}_{n2} K_{\nu_n}(ar) \quad (3.4)$$

$$a = \sqrt{\frac{s}{\gamma_{11}}} \quad \text{and} \quad \nu_n = n \sqrt{\frac{\gamma_{22}}{\gamma_{11}}}$$

where $\tilde{T}_{n1}$ and $\tilde{T}_{n2}$ are constants to be determined by temperature boundary conditions. Assuming $T_a(\theta, t)$ is input temperature, and it is expanded into Fourier series:

$$T_a = \sum_{n=-\infty}^{\infty} T_{an}(t) e^{in\theta}$$

Then, the temperature boundary conditions can be expressed as following:

Case a: The input temperature is applied on the outermost surface of a hollow J -layered cylinder:

At $r = 0$ (center of the bore)

$$BC.1 \quad T^1(r, \theta, t) \text{ is finite (superscripts indicate layer numbers)}$$

At $r = r_j$, $j = 1, 2, 3, \dots, J - 1$ (on the interfaces between layers)

$$BC.2 \quad T^j(r, \theta, t) = T^{j+1}(r, \theta, t)$$

$$BC.3 \quad k_{rj} \frac{\partial T^j(r, \theta, t)}{\partial r} = k_{r(j+1)} \frac{\partial T^{j+1}(r, \theta, t)}{\partial r}$$

At $r = r_J$ (on the outermost surface)

$$BC.4 \quad T^J(r, \theta, t) - T_a(\theta, t) = 0$$

Case b: The input temperature is applied on the innermost surface of a hollow J -layered cylinder:

At $r = r_1$ (on the innermost surface)

$$BC.1 \quad T^1(r, \theta, t) = T_a(\theta, t)$$

At $r = r_{j+1}$, $j = 1, 2, 3, \dots, J-1$ (on the interfaces between layers)

$$BC.2 \quad T^j(r, \theta, t) = T^{j+1}(r, \theta, t)$$

$$BC.3 \quad k_{rj} \frac{\partial T^j(r, \theta, t)}{\partial r} = k_{r(j+1)} \frac{\partial T^{j+1}(r, \theta, t)}{\partial r}$$

At $r = r_J$ (on the outermost surface)

$$BC.4 \quad \frac{\partial T^J(r, \theta, t)}{\partial r} = 0$$

The temperature distribution can be obtained as the inverse Laplace transform of Eq.

3.4.

3.2 Stress Analysis

3.2.1 Stress Distributions under Thermal Shock

The thermal stress analysis can be carried out by an elasticity analysis for the temperature distribution obtained in the above section. Equilibrium equations for a plane strain problem in cylindrical coordinates are given as:

$$\begin{aligned}\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} (\sigma_r - \sigma_\theta) + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r} \tau_{r\theta} &= 0\end{aligned}\tag{3.5}$$

where $\sigma_r(r, \theta, t)$, $\sigma_\theta(r, \theta, t)$ and $\tau_{r\theta}(r, \theta, t)$ are the radial, tangential and shear stress components, respectively.

The stress-strain relationships for an orthotropic plane strain problem are (Jones, 1975):

$$\begin{aligned}\sigma_r &= C_{11} \varepsilon_r + C_{12} \varepsilon_\theta - \beta_1 T \\ \sigma_\theta &= C_{12} \varepsilon_r + C_{22} \varepsilon_\theta - \beta_2 T \\ \tau_{r\theta} &= C_{66} \gamma_{r\theta}\end{aligned}\tag{3.6}$$

where $\varepsilon_r(r, \theta, t)$, $\varepsilon_\theta(r, \theta, t)$ and $\gamma_{r\theta}(r, \theta, t)$ are radial, circumference, and shear strain, respectively. C_{ij} , $i, j = 1, 2, 3, 6$ are Hook's constants; β_1 , and β_2 are defined in terms of

Hook's constants and coefficients of thermal expansion α_r , α_θ and α_z in the radial, tangential and axial directions respectively, as:

$$\beta_1 = C_{11} \alpha_r + C_{12} \alpha_\theta + C_{13} \alpha_z$$

$$\beta_2 = C_{12} \alpha_r + C_{22} \alpha_\theta + C_{23} \alpha_z \quad (3.7)$$

Hook's constants are defined by the mechanical parameters as follows:

$$C_{11} = \frac{1 - \nu_{\theta z} \nu_{z\theta}}{E_\theta E_z \Delta}$$

$$C_{12} = \frac{\nu_{r\theta} + \nu_{z\theta} \nu_{rz}}{E_r E_z \Delta} \quad (3.8)$$

$$C_{22} = \frac{1 - \nu_{rz} \nu_{zr}}{E_r E_z \Delta}$$

$$C_{66} = G_{r\theta}$$

where

$$\Delta = \frac{1 - \nu_{r\theta} \nu_{\theta r} - \nu_{\theta z} \nu_{z\theta} - \nu_{zr} \nu_{rz} - 2\nu_{\theta r} \nu_{z\theta} \nu_{rz}}{E_r E_\theta E_z}$$

and E_i , ν_{ij} and G_{ij} , $i, j = r, \theta, z$ are elastic moduli, Poisson's ratios and shear moduli, respectively. Elastic moduli, E_{ij} , Poisson's ratios, ν_{ij} , and shear moduli, G_{ij} , satisfy the following equations:

$$\frac{\nu_{r\theta}}{E_r} = \frac{\nu_{\theta r}}{E_\theta} \quad \frac{\nu_{rz}}{E_r} = \frac{\nu_{zr}}{E_z} \quad \frac{\nu_{\theta z}}{E_\theta} = \frac{\nu_{z\theta}}{E_z}$$

$$G_{ij} = G_{ji} \quad (3.9)$$

where i and $j = r, \theta$, and z . As a consequence of Eqs. 3.9, there are only nine independent mechanical parameters.

The strain-displacement relations for a plane strain problem in cylindrical coordinates are:

$$\begin{aligned} \varepsilon_r &= \frac{\partial u}{\partial r}, & \varepsilon_\theta &= \frac{1}{r} \left(\frac{\partial v}{\partial \theta} + u \right) \\ \gamma_{r\theta} &= \frac{1}{r} \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial r} \right) \end{aligned} \quad (3.10)$$

where $u(r, \theta, t)$ is radial displacement and $v(r, \theta, t)$ is circumferential displacement. Substituting Eqs. 3.6 and 3.10 into 3.5, the equilibrium equations for a plane strain problem in cylindrical coordinates can be rewritten in terms of displacements in the following form:

$$\begin{aligned} &C_{11} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + C_{66} \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - C_{22} \frac{u}{r^2} + (C_{12} + C_{66}) \frac{1}{r} \frac{\partial^2 v}{\partial r \partial \theta} \\ &- (C_{22} + C_{66}) \frac{1}{r^2} \frac{\partial v}{\partial \theta} - f_r = 0 \\ &C_{66} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) + C_{22} \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} - C_{66} \frac{v}{r^2} + (C_{12} + C_{66}) \frac{1}{r} \frac{\partial^2 u}{\partial r \partial \theta} \\ &+ (C_{22} + C_{66}) \frac{1}{r^2} \frac{\partial u}{\partial \theta} - f_\theta = 0 \end{aligned} \quad (3.11)$$

where

$$f_r = \beta_1 \frac{\partial T}{\partial r} + (\beta_1 - \beta_2) \frac{T}{r}$$

$$f_\theta = \beta_2 \frac{1}{r} \frac{\partial T}{\partial \theta}$$

Because of f_r and f_θ , radial displacement, u , and tangential displacement, v , contain homogeneous and particular parts of the solutions of Eq. 3.11. Expanding them into Fourier series:

$$u = \sum_{n=-\infty}^{\infty} (u_n + u_{pn}) e^{in\theta}, \quad v = \sum_{n=-\infty}^{\infty} (v_n + v_{pn}) e^{in\theta} \quad (3.12)$$

The homogeneous solutions are as follows

for $n = 0$:

$$u_0 = u_{01} r^{\lambda_{01}} + u_{02} r^{-\lambda_{01}}, \quad v_0 \equiv 0 \quad (3.13)$$

where λ_{01} is defined as

$$\lambda_{01} = \sqrt{\frac{C_{22}}{C_{11}}}$$

for $n = 1$:

$$\begin{aligned} u_1 &= u_{11} + u_{12} \ln r + u_{13} r^{\lambda_{11}} + u_{14} r^{-\lambda_{11}} \\ v_1 &= v_{11} + v_{12} \ln r + v_{13} r^{\lambda_{11}} + v_{14} r^{-\lambda_{11}} \end{aligned} \quad (3.14)$$

where

$$\lambda_{11} = \sqrt{\frac{C_{11}C_{22} - 2C_{12}C_{66} - (C_{12})^2 + (C_{11} + C_{22})C_{66}}{C_{11}C_{66}}}$$

and

$$v_{11} = i \left(u_{11} + \frac{C_{12} + C_{66}}{C_{22} + C_{66}} u_{12} \right)$$

$$v_{12} = i u_{12}$$

$$v_{13} = i \frac{C_{22} + C_{66} - C_{11} (\lambda_{11})^2}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{11}} u_{13}$$

$$v_{14} = i \frac{C_{22} + C_{66} - C_{11} (\lambda_{11})^2}{C_{22} + C_{66} + (C_{12} + C_{66})\lambda_{11}} u_{14}$$

for $n > 1$:

$$u_n = \sum_{m=1}^4 u_{nm} r^{\lambda_{nm}}, \quad v_n = \sum_{m=1}^4 v_{nm} r^{\lambda_{nm}} \quad (3.15)$$

where λ_{nm} are the roots of the following equation

$$C_{11}C_{66}(\lambda_n)^4 - [(C_{11}C_{22} - 2C_{12}C_{66} - (C_{12})^2) n^2 + (C_{11} + C_{22})C_{66}] (\lambda_n)^2 + C_{22}C_{66}(n^2 - 1)^2 = 0$$

and

$$v_n = \frac{i}{n} \sum_{m=1}^4 \frac{C_{22} + C_{66}n^2 - C_{11}(\lambda_{nm})^2}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{nm}} u_{nm} r^{\lambda_{nm}}$$

The particular solutions can be obtained by the method of variation of parameters (Appendix A). Substituting the solutions for displacements back into the strain-displacement relationship Eq. 3.10 and stress-strain relationship Eq. 3.6, the stresses can be obtained as follows:

for $n = 0$:

$$\sigma_{r0} = (C_{12} + C_{11}\lambda_{01})r^{\lambda_{01}-1}u_{01} + (C_{12} - C_{11}\lambda_{01})r^{-\lambda_{01}-1}u_{02}$$

$$+ C_{11} \frac{\partial u_{p0}}{\partial r} + \frac{C_{12}}{r} u_{p0} - \beta_1 T_0$$

$$\sigma_{\theta 0} = (C_{22} + C_{12}\lambda_{01})r^{\lambda_{01}-1}u_{01} + (C_{22} - C_{12}\lambda_{01})r^{-\lambda_{01}-1}u_{02} \quad (3.16)$$

$$+ C_{12} \frac{\partial u_{p0}}{\partial r} + \frac{C_{22}}{r} u_{p0} - \beta_2 T_0$$

$$\tau_{r\theta 0} \equiv 0$$

for $n = 1$:

$$\sigma_{r1} = \left(C_{11} - C_{12} \frac{C_{12} + C_{66}}{C_{22} + C_{66}} \right) \frac{u_{12}}{r}$$

$$\begin{aligned}
& -\lambda_{11} \frac{C_{12}(C_{12} + C_{66}) - C_{11}(C_{22} + C_{66}) + C_{11}C_{66}\lambda_{11}}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{11}} r^{\lambda_{11}-1} u_{13} \\
& + \lambda_{11} \frac{C_{12}(C_{12} + C_{66}) - C_{11}(C_{22} + C_{66}) - C_{11}C_{66}\lambda_{11}}{C_{22} + C_{66} + (C_{12} + C_{66})\lambda_{11}} r^{-\lambda_{11}-1} u_{14} \\
& + C_{11} \frac{du_{p1}}{dr} + inC_{12} \frac{v_{p1}}{r} + C_{12} \frac{u_{p1}}{r} - \beta_1 T_1 \\
\sigma_{\theta 1} = & \left(C_{12} - C_{22} \frac{C_{12} + C_{66}}{C_{22} + C_{66}} \right) \frac{u_{12}}{r} \\
& - \lambda_{11} \frac{C_{66}(C_{22} - C_{12}) + (C_{12}^2 + C_{12}C_{66} - C_{11}C_{22})\lambda_{11}}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{11}} r^{\lambda_{11}-1} u_{13} \\
& + \lambda_{11} \frac{C_{66}(C_{22} - C_{12}) - (C_{12}^2 + C_{12}C_{66} - C_{11}C_{22})\lambda_{11}}{C_{22} + C_{66} + (C_{12} + C_{66})\lambda_{11}} r^{-\lambda_{11}-1} u_{14} \\
& + C_{12} \frac{du_{p1}}{dr} + inC_{22} \frac{v_{p1}}{r} + C_{22} \frac{u_{p1}}{r} - \beta_2 T_1 \\
\tau_{r\theta 1} = & iC_{66} \left(1 - \frac{C_{12} + C_{66}}{C_{22} + C_{66}} \right) \frac{u_{12}}{r} \\
& + i\lambda_{11} \frac{C_{12}(C_{12} + C_{66}) - C_{11}(C_{22} + C_{66}) + C_{11}C_{66}\lambda_{11}}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{11}} r^{\lambda_{11}-1} u_{13} \\
& - i\lambda_{11} \frac{C_{12}(C_{12} + C_{66}) - C_{11}(C_{22} + C_{66}) - C_{11}C_{66}\lambda_{11}}{C_{22} + C_{66} + (C_{12} + C_{66})\lambda_{11}} r^{-\lambda_{11}-1} u_{14}
\end{aligned} \tag{3.17}$$

$$+ C_{66} \left(i \frac{u_{p1}}{r} - \frac{v_{p1}}{r} + \frac{dv_{p1}}{dr} \right)$$

for $n > 1$

$$\sigma_{rn} = \sum_{m=1}^4 \left[C_{11} \lambda_{nm} + C_{12} \left(1 - \frac{C_{22} + C_{66} n^2 - C_{11} (\lambda_{nm})^2}{C_{22} + C_{66} - (C_{12} + C_{66}) \lambda_{nm}} \right) \right] r^{\lambda_{nm}-1} u_{nm}$$

$$+ C_{11} \frac{du_{pn}}{dr} + in C_{12} \frac{v_{pn}}{r} + C_{12} \frac{u_{pn}}{r} - \beta_1 T_n$$

$$\sigma_{\theta n} = \sum_{m=1}^4 \left[C_{12} \lambda_{nm} + C_{22} \left(1 - \frac{C_{22} + C_{66} n^2 - C_{11} (\lambda_{nm})^2}{C_{22} + C_{66} - (C_{12} + C_{66}) \lambda_{nm}} \right) \right] r^{\lambda_{nm}-1} u_{nm} \quad (3.18)$$

$$+ C_{12} \frac{du_{pn}}{dr} + in C_{22} \frac{v_{pn}}{r} + C_{22} \frac{u_{pn}}{r} - \beta_2 T_n$$

$$\tau_{r\theta n} = i C_{66} \sum_{m=1}^4 \left[n + \frac{1 + \lambda_{nm}}{n} \frac{C_{22} + C_{66} n^2 - C_{11} (\lambda_{nm})^2}{C_{22} + C_{66} - (C_{12} + C_{66}) \lambda_{nm}} \right] r^{\lambda_{nm}-1} u_{nm}$$

$$+ C_{66} \left(in \frac{u_{pn}}{r} - \frac{v_{pn}}{r} + \frac{dv_{pn}}{dr} \right)$$

The constants u_{nm} can be defined by the following boundary conditions

At $r = r_1$ (on the innermost surface)

$$BC.1 \quad \sigma_r^2(r_1, \theta, t) = 0 \text{ (superscripts indicate layer numbers)}$$

$$BC.2 \quad \tau_{r\theta}^2(r_1, \theta, t) = 0$$

At $r = r_j$, $j = 2, 3, \dots, J - 1$ (on the interfaces between layers)

$$BC.3 \quad u^j(r_j, \theta, t) = u^{j+1}(r_j, \theta, t)$$

$$BC.4 \quad v^j(r_j, \theta, t) = v^{j+1}(r_j, \theta, t)$$

$$BC.5 \quad \sigma_r^j(r_j, \theta, t) = \sigma_r^{j+1}(r_j, \theta, t)$$

$$BC.6 \quad \tau_{r\theta}^j(r_j, \theta, t) = \tau_{r\theta}^{j+1}(r_j, \theta, t)$$

At $r = r_J$ (on the outermost surface)

$$BC.7 \quad \sigma_r^J(r_J, \theta, t) = 0$$

$$BC.8 \quad \tau_{r\theta}^J(r_J, \theta, t) = 0$$

3.2.2 Stress Distributions under Symmetric Dynamic Pressure Load

When a structure is subjected to a dynamic loading, the inertial term should be considered. The motion equation for an axially symmetric long cylinder in cylindrical coordinates in term of radial displacement, u , is as the follows:

$$C_{11} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) - C_{22} \frac{u}{r^2} = \rho \frac{\partial^2 u}{\partial t^2} \quad (3.19)$$

where $u(r, t)$, t and ρ are radial displacement, time and mass density, respectively. A coefficient of perturbation, ε , is introduced as

$$\varepsilon = \frac{\rho}{C_{11}} \quad (3.20)$$

Normally, mass density, ρ , is very small compared to the Hook' constant, C_{11} , so the ratio of them, the coefficient of perturbation, ε , is very small. In terms of the coefficient of perturbation, Eq. 3.19 can be rewritten as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{C_{22}}{C_{11}} \frac{u}{r^2} = \varepsilon \frac{\partial^2 u}{\partial t^2} \quad (3.21)$$

Expending the displacement, u , into perturbation series:

$$u = \sum_{m=0}^{\infty} u_m(r, t) \varepsilon^m \quad (3.22)$$

Substituting the above equation back into Eq. 3.21, and collecting the like terms, the following set of equations can be obtained

$$\frac{\partial^2 u_m}{\partial r^2} + \frac{1}{r} \frac{\partial u_m}{\partial r} - \frac{C_{22}}{C_{11}} \frac{u_m}{r^2} = \begin{cases} 0 & \text{for } m = 0 \\ \frac{\partial^2 u_{m-1}}{\partial t^2} & \text{for } m \geq 1 \end{cases} \quad (3.23)$$

The solutions of the above equations are as follows

$$u_m = u_{m1}(t) r^{\lambda_{01}} + u_{m2}(t) r^{-\lambda_{01}} + u_{pm}(r, t) \quad (3.24)$$

where λ_{01} is defined the same as in Section 3.2.1

$$\lambda_{01} = \sqrt{\frac{C_{22}}{C_{11}}}$$

$u_{p0} \equiv 0$ and u_{pm} , $m = 1, 2, \dots$ are particular solutions of Eq. 3.23 and are obtained in Appendix B. Substituting the solutions for displacements back into the strain-displacement relationship, Eq. 3.10 and stress-strain relationship, Eq. 3.6, the stresses can be obtained as follows:

$$\begin{aligned} \sigma_{rm} = & (C_{12} + C_{11}\lambda_{01})r^{\lambda_{01}-1}u_{m1} + (C_{12} - C_{11}\lambda_{01})r^{-\lambda_{01}-1}u_{m2} \\ & + C_{11} \frac{\partial u_{pm}}{\partial r} + C_{12} \frac{u_{pm}}{r} \\ \sigma_{\theta m} = & (C_{22} + C_{12}\lambda_{01})r^{\lambda_{01}-1}u_{m1} + (C_{22} - C_{12}\lambda_{01})r^{-\lambda_{01}-1}u_{m2} \\ & + C_{12} \frac{\partial u_{pm}}{\partial r} + C_{22} \frac{u_{pm}}{r} \end{aligned} \quad (3.25)$$

u_{m1} and u_{m2} can be defined by the following boundary conditions

At $r = r_1$ (on the innermost surface)

$$BC.1 \quad \sigma_r^2(r_1, t) = P_a(t) \text{ (superscripts indicate layer numbers)}$$

At $r = r_j$, $j = 2, 3, \dots, J - 1$ (on the interfaces between layers)

$$BC.2 \quad u^j(r_j, t) = u^{j+1}(r_j, t)$$

$$BC.3 \quad \sigma_r^j(r_j, t) = \sigma_r^{j+1}(r_j, t)$$

At $r = r_J$ (on the outermost surface)

$$BC.4 \quad \sigma_r^J(r_J, t) = 0$$

where $P_a(t)$ is the internal pressure.

3.3 *Illustrative Examples*

3.3.1 Symmetric Thermal Loading

For orthotropic materials, there are nine independent mechanical constants, and three coefficients of thermal expansion compared to two mechanical constants and one coefficient of thermal expansion for isotropic materials. The following example shows how the material properties, internal pressure and rise time effect stress and reliability distributions of a hollow cylinder subjected to thermal loading. The cylinder consists of thirty three layers with matrix and fiber layers alternating. The thick-

nesses of layers 1 and 33 are half of the thickness of the rest of the layers. The geometric, mechanical and thermal parameters are listed in Table 3 while the cylinder configuration is shown in Figure 11.

The reliability analysis in this chapter is again based on the "weakest link" principle but with the three parameter Weibull distribution discussed in Section 2.4. The three parameters, characteristic ultimate strength, R_c , minimum strength, R_0 , and Weibull shape parameter, m , are based on the same tests mentioned in Section 2.5. Because there are not enough experimental results to determine the third central moment, the three Weibull parameters cannot be calculated directly from Eqs. 2.28-2.30. The Weibull shape parameter, m , for both fiber layers and matrix layers is estimated as 6.17 (Heller, Thangjitham, Rantis and Heller, 1991). Introducing the normalized strengths r_c and r_0 as follows

$$r_c = \frac{R_c}{\bar{R}} \quad \text{and} \quad r_0 = \frac{R_0}{\bar{R}} \quad (3.26)$$

and the Eqs. 2.28 and 2.29 becomes

$$1 = (r_c - r_0)\Gamma\left(1 + \frac{1}{m}\right) + r_0 \quad (3.27)$$

$$COV = (r_c - r_0)\left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{1/2} \quad (3.28)$$

r_c and r_0 are obtained as 1.03 and 0.61 from the above two equations. The strength of the fiber layers in circumferential direction, $\bar{R}_r = 24,800$ psi, is calculated based on the strengths of the composite (Heller, Thangjitham, Rantis and Heller, 1991), while the strength of the matrix layers, $\bar{R}_m$ is estimated as 2,000 psi. The characteristic strength R_c and the minimum strength R_0 are calculated as follows

$$R_c = r_c \bar{R} \quad \text{and} \quad R_0 = r_0 \bar{R}$$

The characteristic strength for the fiber and the matrix layers are $R_{cf} = 25,544$ psi, and $R_{cm} = 2,060$ psi. The minimum strength for the fiber and the matrix layers are $R_{0f} = 15,128$ psi and $R_{0m} = 1,220$ psi.

The input temperature is applied at the innermost surface of the cylinder and is modeled as

$$T_a(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 3000\beta t e^{1-\beta t} \text{ } ^\circ\text{F} & \text{for } t > 0 \end{cases} \quad (3.29)$$

where the rise time $t_m = 1/\beta = 1$ second in this example.

The effect of the material properties on thermal stress distributions and reliabilities

To save computing time, it is assumed that the temperature is the same as the input temperature everywhere in the cylinder in this parameter study. The radial stresses are very small compared with the tangential stresses in the problem, and will not contribute to the failure of the structure. Figure 12 shows the tangential stresses in layers 1 and 2 at the inner surface and in layers 32 and 33 at the outer surface (Case 1), when both the fiber and matrix materials of the cylinder are isotropic and have the same Possion's ratios and coefficients of thermal expansion ($\nu_{r\theta f} = \nu_{rxf} = \nu_{\theta xf} = \nu_m = 0.33$, $\alpha_{rf} = \alpha_{\theta f} = \alpha_{xf} = \alpha_m = 2.30 \times 10^{-6}$ in/in $^\circ\text{F}$). The only difference between fiber and matrix materials is their elastic moduli ($E_{rf} = E_{\theta f} = E_{xf} = 2.5 \times 10^6$ psi, $E_m = 0.5 \times 10^6$ psi). The tangential stresses in this case are very small and the stresses in the fiber layers are about five times larger than the stresses in the matrix layers, just as the elastic

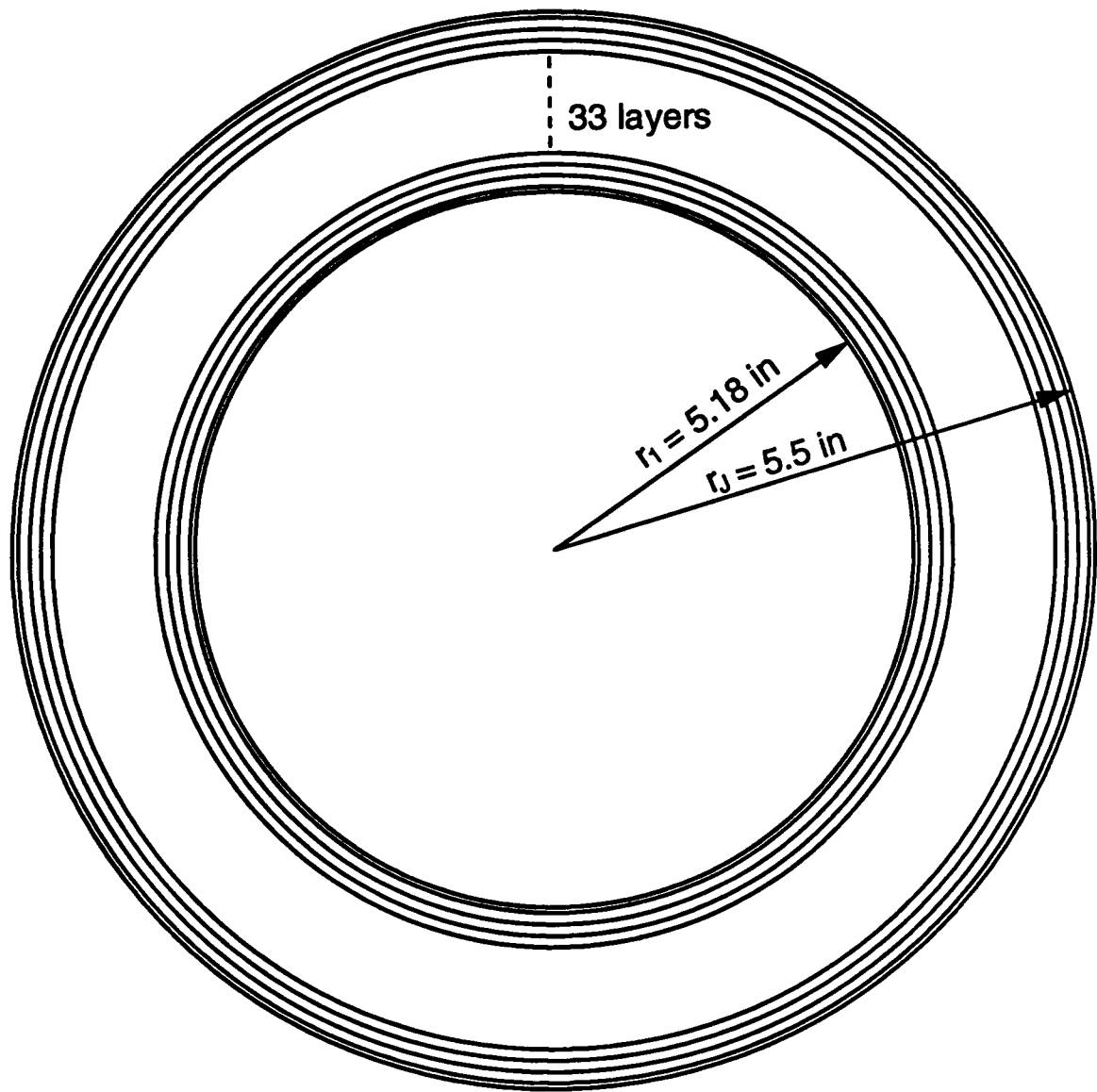


Figure 11. Configuration of the Orthotropic Cylinder.

Table 3
Geometric, Thermal and Mechanical Parameters for Carbon-Carbon.

Material	Air Core	Layer 1 & 33	Odd Layers	Even Layers
Thickness (in)	$r_1 = 5.18$	0.005	0.01	0.01
Conductivity in r direction (Btu/hr ft ² °F)	0.0142	12.5	12.5	12.5
Diffusivity in r direction (in ² /sec)	0.040	0.267	0.267	0.267
Elastic Modulus, (psi)	—	0.5×10^6	0.5×10^6	0.86×10^6
E_r	—	0.5×10^6	0.5×10^6	3.78×10^6
E_θ	—	0.5×10^6	0.5×10^6	2.79×10^6
E_z	—			
Poisson's Ratio	—	0.33	0.33	0.2
$\nu_{r\theta}$	—	0.33	0.33	0.4
ν_{rz}	—	0.33	0.33	0.05
$\nu_{\theta z}$	—			
Coefficient of Thermal Expansion (in./in.°F)	—	2.30×10^{-6}	2.30×10^{-6}	2.22×10^{-6}
$\alpha_{r\theta}$	—	2.30×10^{-6}	2.30×10^{-6}	1.67×10^{-6}
α_{rz}	—	2.30×10^{-6}	2.30×10^{-6}	1.67×10^{-6}
$\alpha_{\theta z}$	—			

modulus in the fiber layers is about five times larger than the modulus in the matrix layers.

As a parametric study, the moduli of the fiber layers are changed to those of carbon-carbon ($E_r = 0.86 \times 10^6$ psi, $E_{\theta r} = 3.78 \times 10^6$ psi, and $E_{zr} = 2.79 \times 10^6$ psi) and the rest of the material properties are kept as in Case 1 above (Case 2). Figure 13 shows that the tangential stresses in the same locations as above are increased a little and the tangential stresses in the fiber layers are still much larger than the stresses in the matrix layers.

Next, the Poisson's ratios of the fiber layers are changed to that of carbon-carbon ($\nu_{\theta r} = 0.2$, $\nu_{rz} = 0.4$, $\nu_{\theta z} = 0.05$), keeping the elastic moduli in the fiber layers as those of carbon-carbon, and the rest of the material properties as in Case 1. In this Case 3, Figure 14 shows that the tangential stresses in the cylinder are much larger than in the above two cases, and the tangential stresses in the matrix layers are almost as large as the tangential stresses in the fiber layers even though the elastic moduli in the fiber layers are still much larger than the modulus in the matrix layers.

Finally, all the material properties including coefficients of thermal expansion in the fiber layers are chosen as those of carbon-carbon (Table 3), and the material properties in the matrix layers are kept as in Case 1. Figure 15 shows that the tangential stress distributions are similar in this Case 4 to Case 3 presented in Figure 14, but the tangential stresses are even larger and the difference between the tangential stresses in the fiber layers and the matrix layers are even smaller.

Assuming that the strengths of the matrix and fiber layers take on those of the carbon-carbon composite (see Table 3) in the parametric study, for the first three

cases, the tangential stresses both in the matrix and the fiber layers are smaller than their minimum strength R_0 , hence the reliability of the cylinder is always one. In the last case, the tangential stresses in the fiber layers are much smaller than the minimum fiber strength, R_{0f} , and do not cause failure, but the tangential stresses in the matrix layers in the time range between 0.4 to 2.0 seconds are larger than the minimum matrix strength, R_{0m} . The reliabilities of those matrix layers are almost 0 and the reliability of the cylinder is also almost 0 (Figure 16).

The effect of internal pressure on reliabilities of the cylinder

To improve the reliability of the structure, internal pressure was applied at the innermost surface of the cylinder while subjected to the thermal loading. The internal pressure, $P_a(t)$, is molded similarly to the temperature loading:

$$P_a(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ -A\beta t e^{1-\beta t} \text{ psi} & \text{for } t > 0 \end{cases} \quad (3.30)$$

where the rise time t_m is chosen as 1 second similarly to that of the thermal loading and A is the maximum value of the internal pressure at $t = t_m = 1/\beta$. When $A = 500$ psi, there is no failure in the matrix layers. The maximum probability of the fiber layers is plotted in Figure 17. In the time range of 0.0 - 0.5 second and for $t > 1.5$ seconds, the failure probability of the cylinder is very small and is neglected. The failure probability of the cylinder as a function of time is plotted in Figure 18.

The above example shows that thermal stress and reliability distributions in a multilayered orthotropic cylinder are sometimes much more complicated than thermal stress and reliability distributions in a multilayered isotropic cylinder. In the above

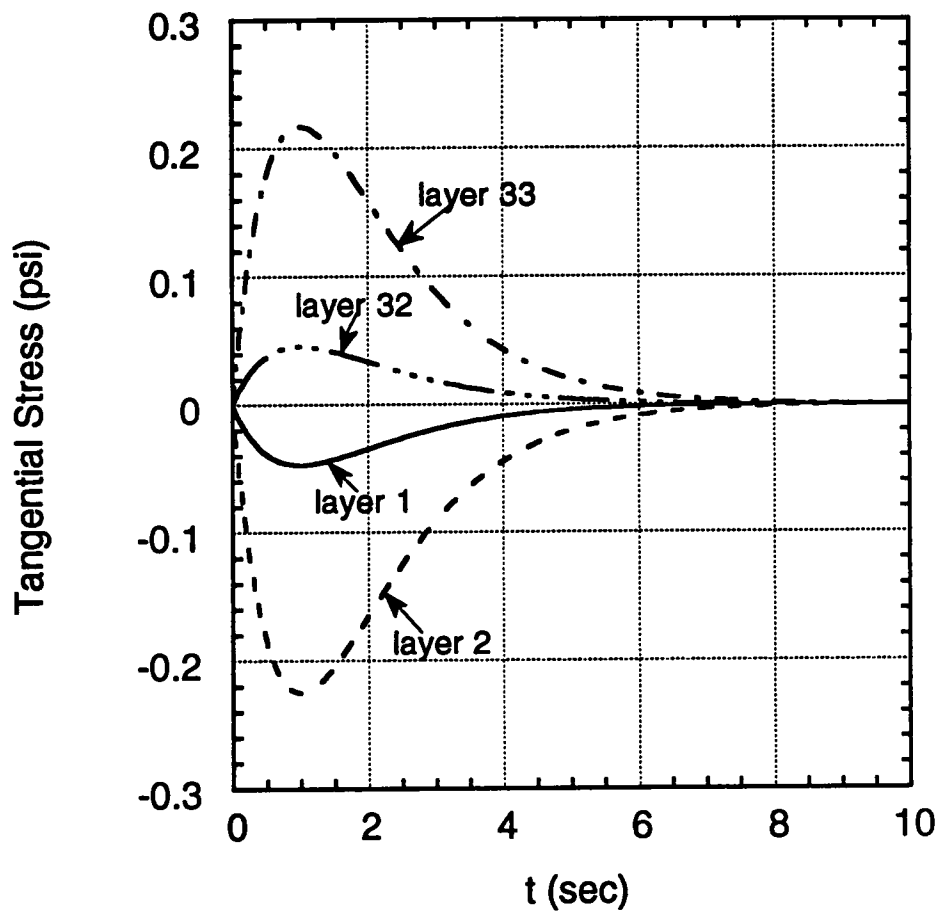


Figure 12. Tangential Stress in Parameter Study (Case 1).

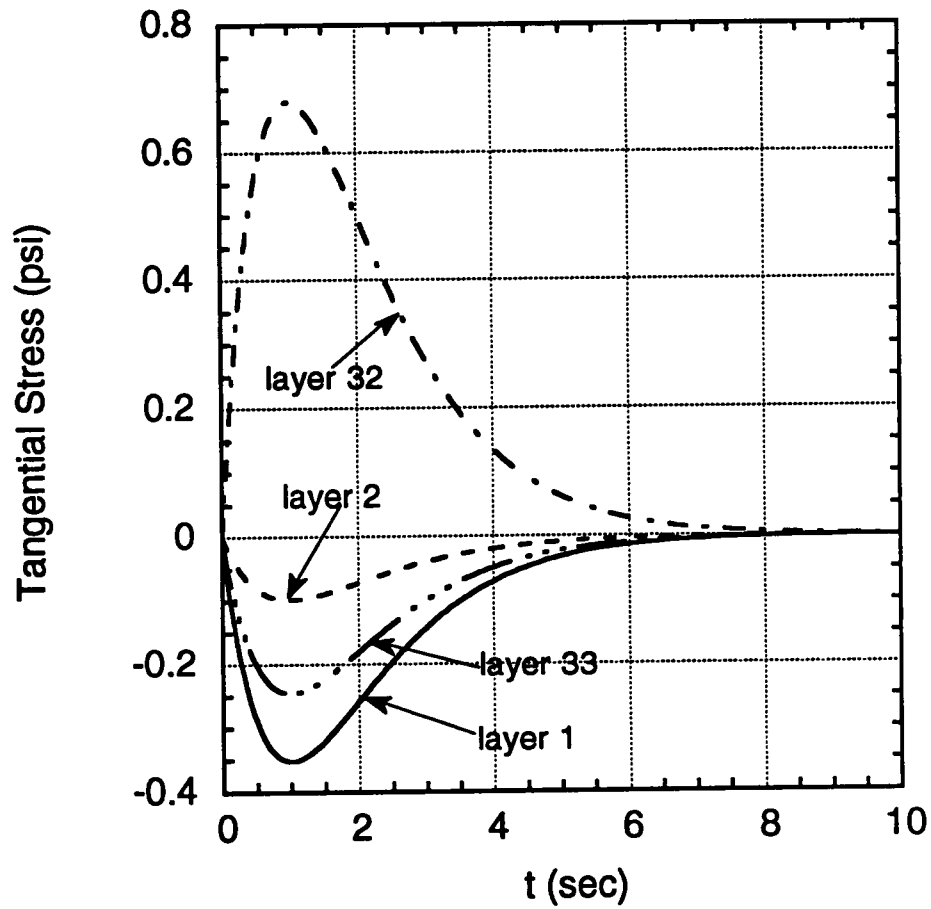


Figure 13. Tangential Stress in Parameter Study (Case 2).

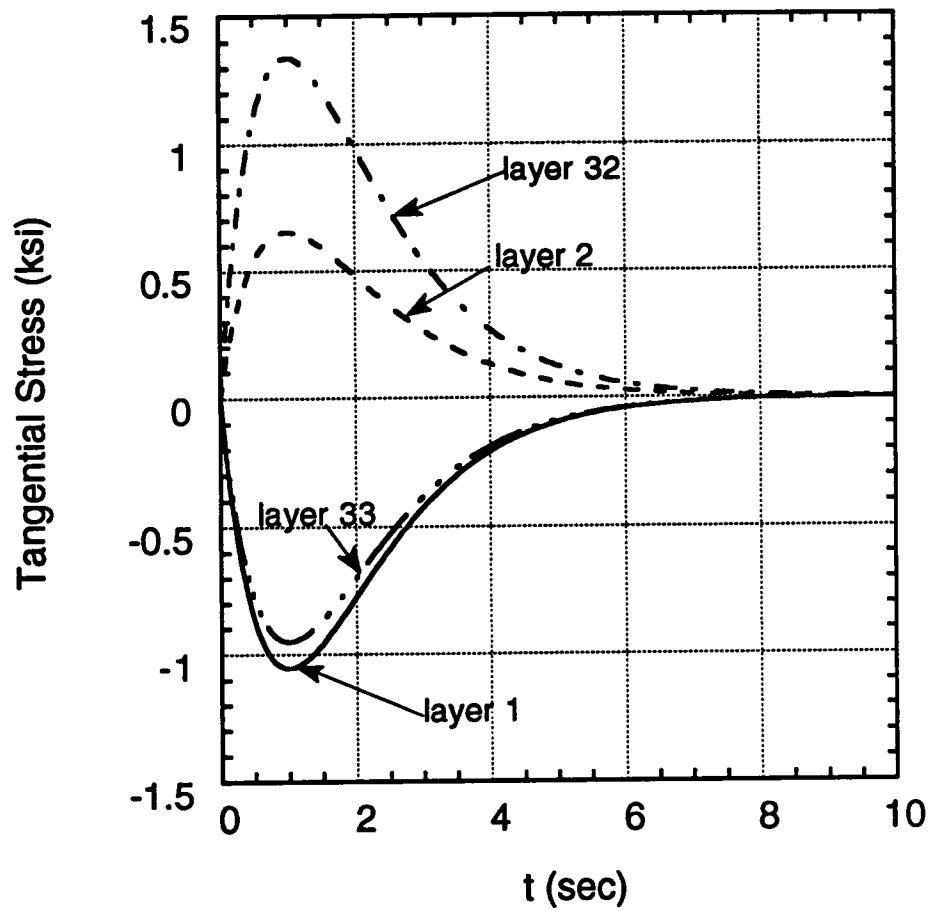


Figure 14. Tangential Stress in Parameter Study (Case 3).

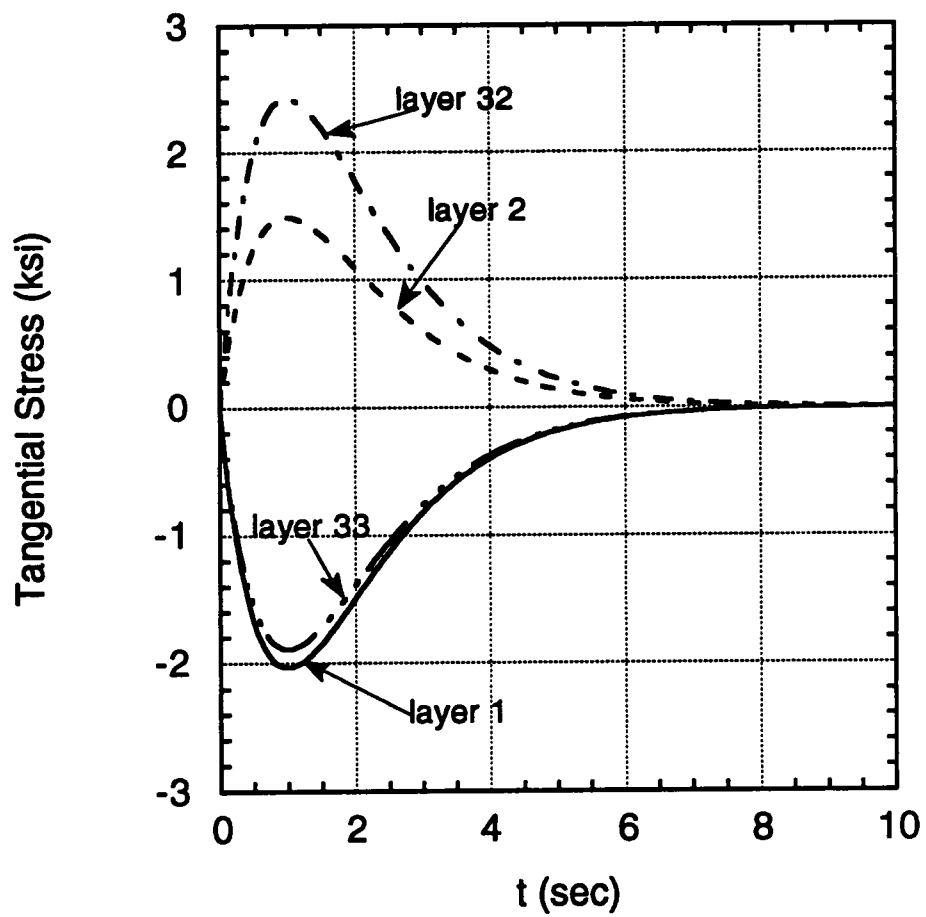


Figure 15. Tangential Stress in Parameter Study (Case 4).

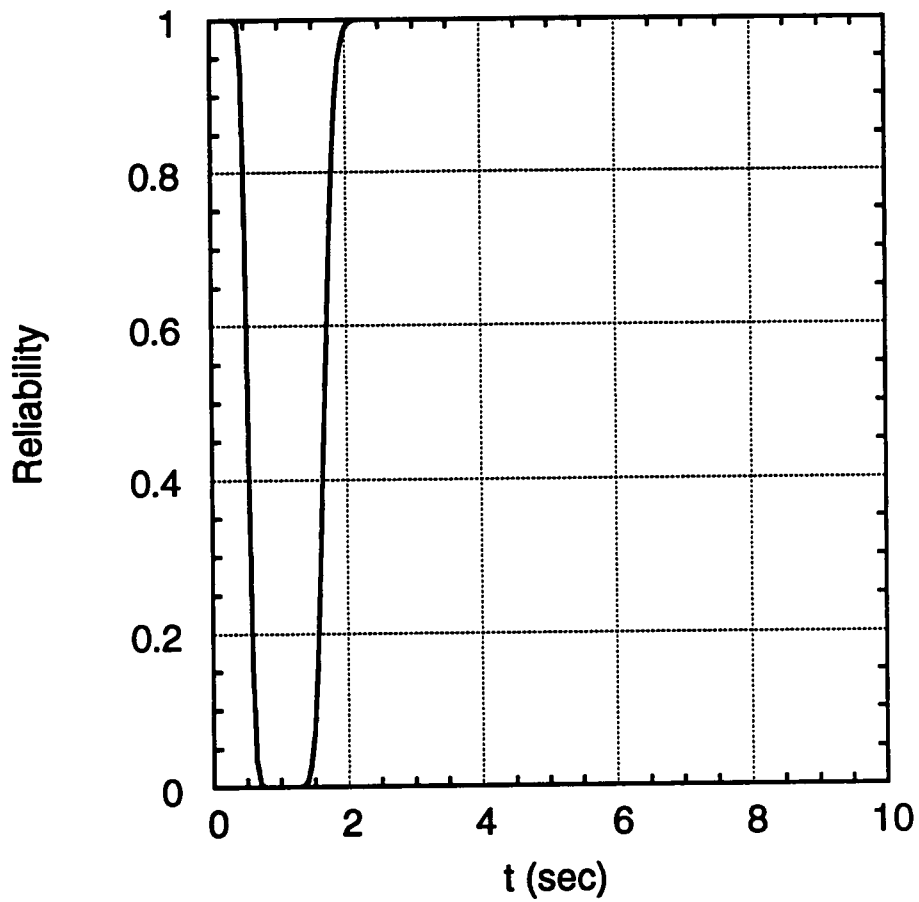


Figure 16. Reliability of the Cylinder in Parameter Study (Case 4).

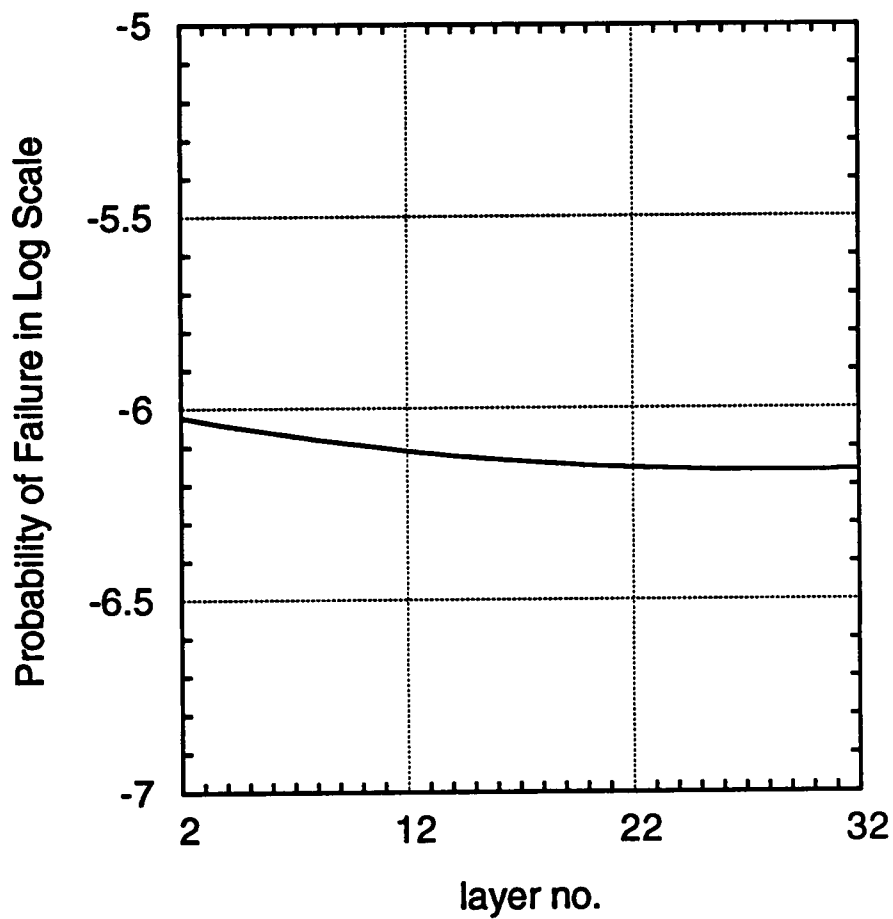


Figure 17. Probability of Failure of Fiber Layers through the Thickness under Combined Thermal and Pressure Loading.

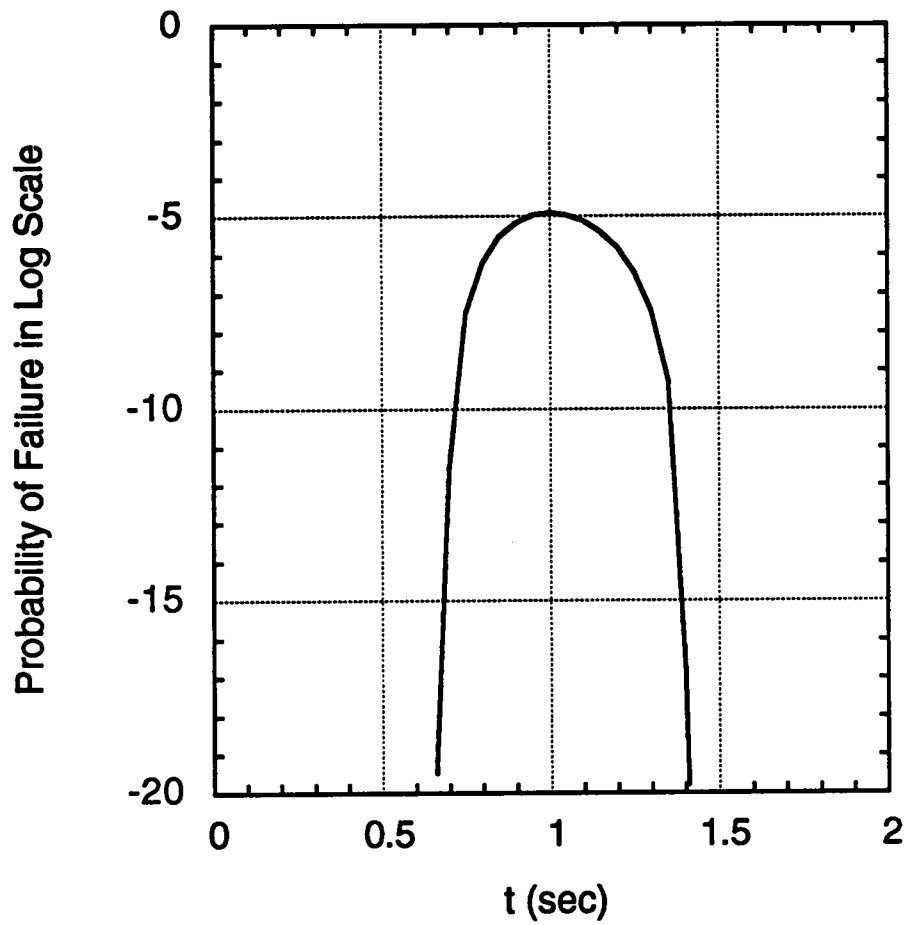


Figure 18. Probability of Failure of the Cylinder under Combined Thermal and Pressure Loading.

four cases, the materials of the matrix are exactly the same, the elastic moduli are almost the same, and the cylinders are subjected to the same thermal loading, but the first three are safe, the last one fails.

The effect of rise time, t_m , on stress distributions

Figures 19-22 show the maximum tangential stresses at the inner surface of layers 1 and 2, and at the outer surface of layers 32 and 33 of the same cylinder as in the last example above, for different rise times, t_m , from 1 second to 4 seconds with internal pressure of 400 psi. The rise time, t_m , has little effect on the maximum tangential stresses at the outer surface of layer 33 when t_m changes from 1 second to 4 seconds. There are obvious differences on maximum tangential stresses at the rest of the locations mentioned above when rise times are between $t_m = 1$ second and 2 seconds. The differences are small when t_m is larger than 2 seconds. This suggests that when rise time, t_m , is smaller than 1 second, the maximum tangential stress in the cylinder could be very large, and should be calculated very carefully. But when t_m is larger than 4 seconds, rise time has little effect on the maximum stress.

The effect of various internal pressure amplitudes and thermal shock

In this example, the cylinder is subjected to internal thermal shock together with internal pressure, $P_s(t)$ both with rise times $t_m = 1$ second. First, $A = 500$ psi is applied. The tangential stresses plotted in solid lines in Figures 23-26 are induced by internal pressure alone, while the tangential stresses plotted in dashed lines are induced by both the combined internal thermal shock and pressure. Figures 27 and 28 show the tangential stresses at different locations in the fiber and matrix layers, while

Figures 29 and 30 show the maximum tangential stress variations through the thickness.

Because the radial stresses in the cylinder are very small, they are neglected in the reliability analysis. It is seen from Figure 29 that all the tangential stresses in the matrix layers are lower than the minimum strength R_{0m} , and have no contribution to the probability of failure. Failures are due to the tangential stresses in the fiber layers. The maximum tangential stresses in the fiber layers are between 15,893 psi and 16,400 psi, this means that the risk of failure, λ , is very small, it is quite accurate to use Eq. 2.27 to calculate the probability of failure.

$$P_f = 1 - \prod_{j=1}^n L_j \approx \sum_{j=1}^{n_f} \lambda_j$$

where n_f is the total number of fiber layers.

The maximum probability failure of the j th fiber layer, P_{fjmax} , occurs when the tangential stress of the layer reaches the maximum value. When P_{fjmax} is small, it is defined as

$$P_{fjmax} = 2\pi \left(\frac{S_{max} - R_0}{R_c - R_0} \right)^m \frac{\bar{r}t}{V_r} \quad (3.31)$$

where $\bar{r}$ is the mean radius of the layer and t is the thickness of the layer while P_{fmax} is defined as

$$P_{fmax} = \sum_{j=1}^{n_f} P_{fjmax} \quad (3.32)$$

These maxima occur at different times in different layers and do not coincide. Hence P_{fmax} represents a worst case and is a conservative estimate. P_{fmax} is larger than the maximum probability of failure of the cylinder. The failure probabilities, P_{fmax} , in fiber layers through the thickness are plotted in Figure 31, P_{fmax} for the cylinder in this case is 2.4136×10^{-5} . The probability of failure for fiber layers across the wall thickness are shown in Figure 32 for various rise times.

Now, the maximum internal pressure A is varied from 0 to 1.0 ksi. The reliabilities of the matrix and fiber layers and of the cylinder under different internal pressures are plotted in Figures 33-35. These Figures show that when A is between 0.3 - 0.55 ksi, the cylinder is safe, when $A < 0.3$ ksi, the cylinder fails because the matrix layers fail, and when $A > 0.55$ ksi, the cylinder fails because the fiber layers fail first.

3.3.2 Asymmetric Thermal Loading

The cylinder investigated here is a model of a rocket motor consisting of an air core, an isotropic propellant layer and thin 5-layer kevlar-epoxy composite case as show in Figure 36. The geometries and material properties are listed in Table 4 (Chamis, 1983). The cylinder is subjected to asymmetric thermal shock loading as follows:

$$T_a(t) = \begin{cases} 250(1 + \cos 2\theta)H(t) & \text{for } -\frac{\pi}{2} \leq \theta < \frac{\pi}{2} \\ 0 & \text{for elsewhere} \end{cases} \quad (3.33)$$

where $H(t)$ is Heaviside's step function. The Input temperature is also shown in Figure 37.

The radial and shear stresses at the interfaces of each layer are shown in Figures 38 and 39, while the tangential stresses in the propellant are shown in Figure 40, in the odd layers of the case in Figure 41 and in the even layers of the case in Figure 42. All the stresses increase very fast right after the asymmetrical thermal loading is applied and tend to constants in 0.03 second. Compared with tangential stresses, the radial and shear stresses are very small.

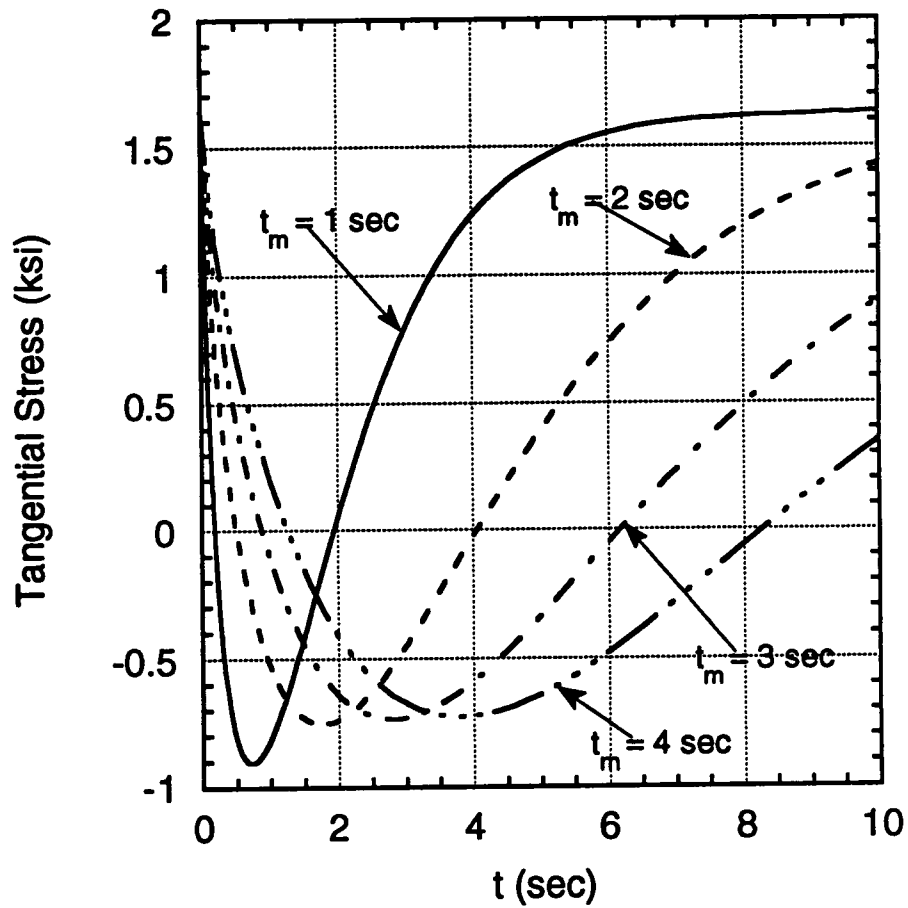


Figure 19. Tangential Stresses On the Innermost Surface (Matrix Layer); Thermal Shock and Pressure.

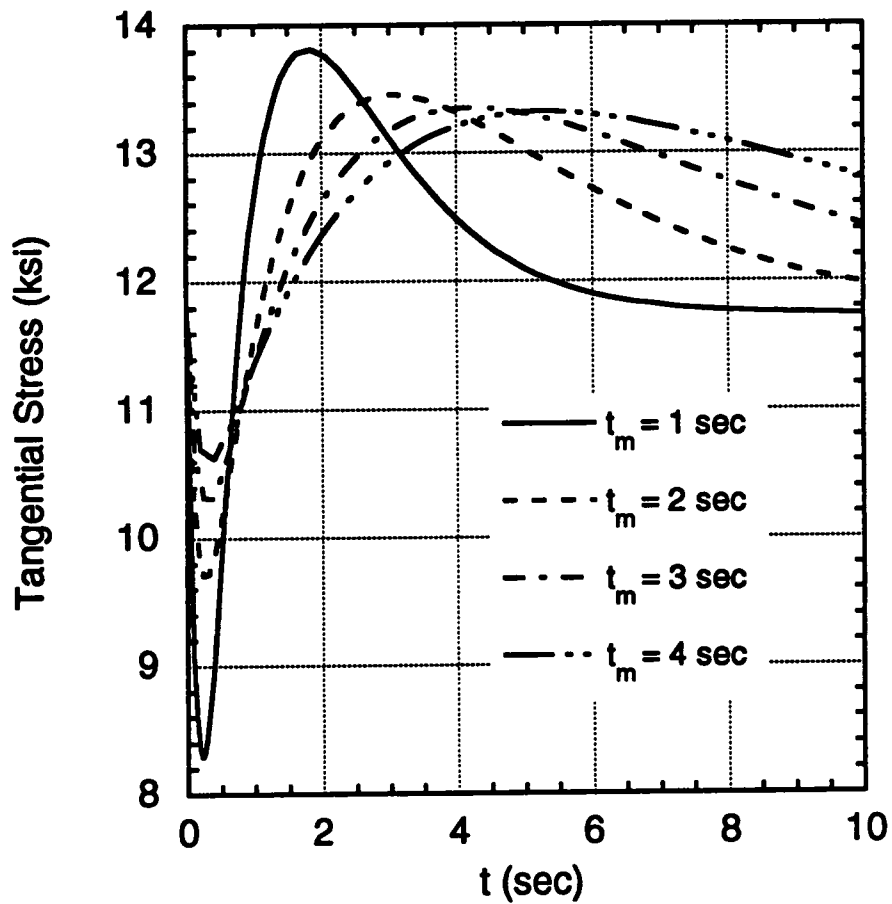


Figure 20. Tangential Stresses on the Inner Surface of Layer 2 (Fiber Layer); Thermal Shock and Pressure.

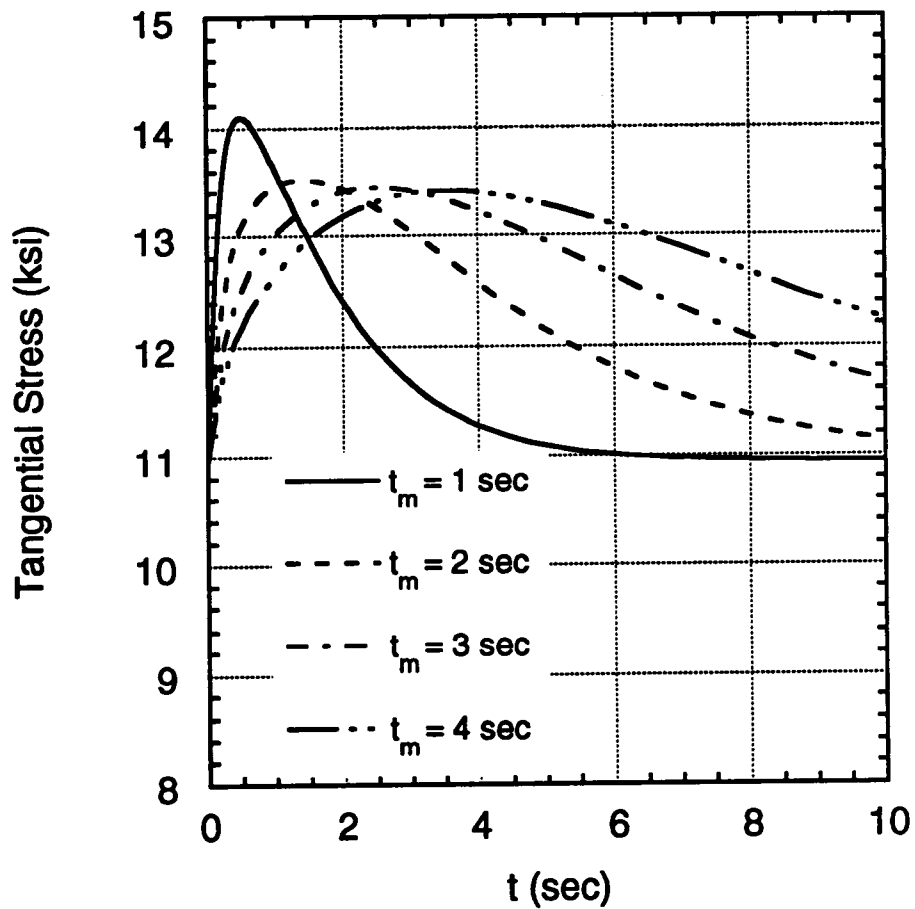


Figure 21. Tangential Stresses on the Outer Surface of Layer 32 (Fiber Layer); Thermal Shock and Pressure.

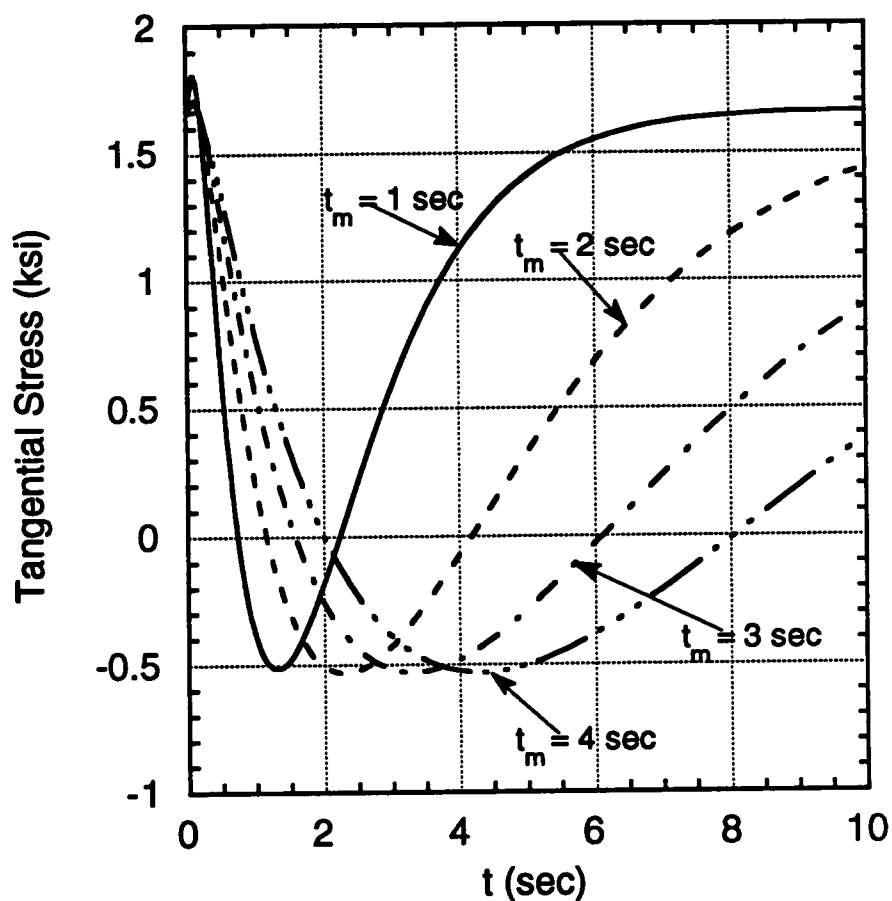


Figure 22. Tangential Stresses on the Outermost Surface (Matrix Layer); Thermal Shock and Pressure.

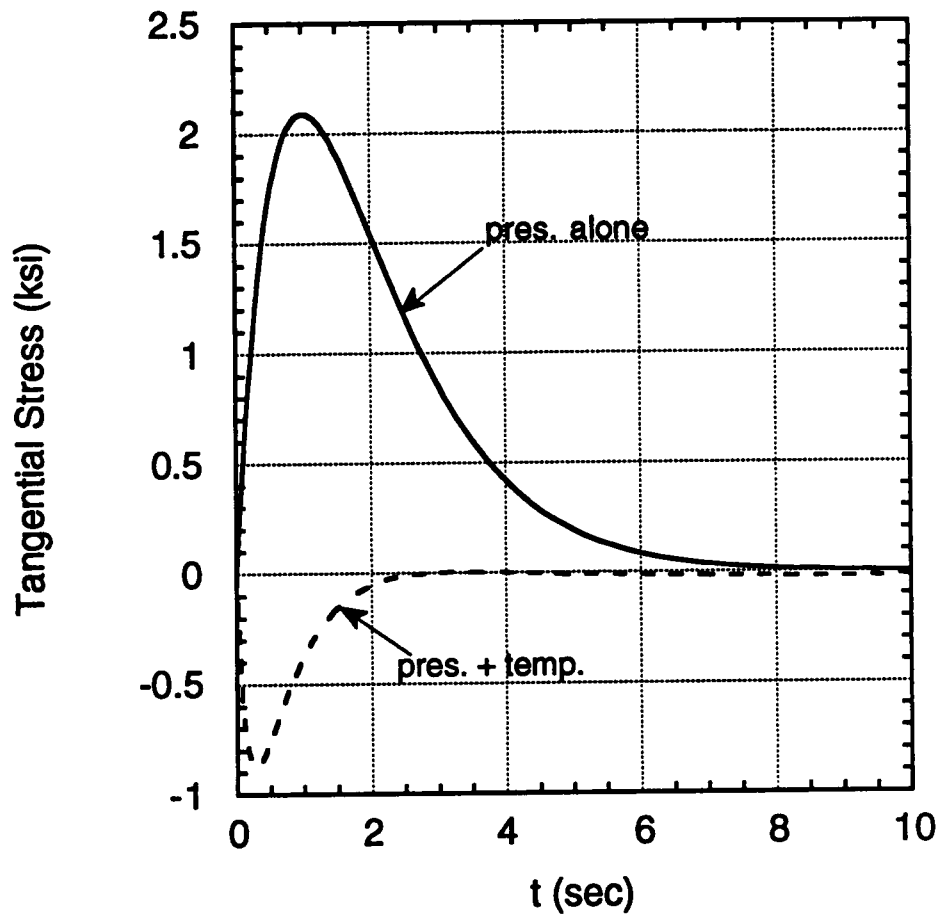


Figure 23. Tangential Stresses on the Innermost Surface (Matrix Layer); Thermal Shock and Pressure.

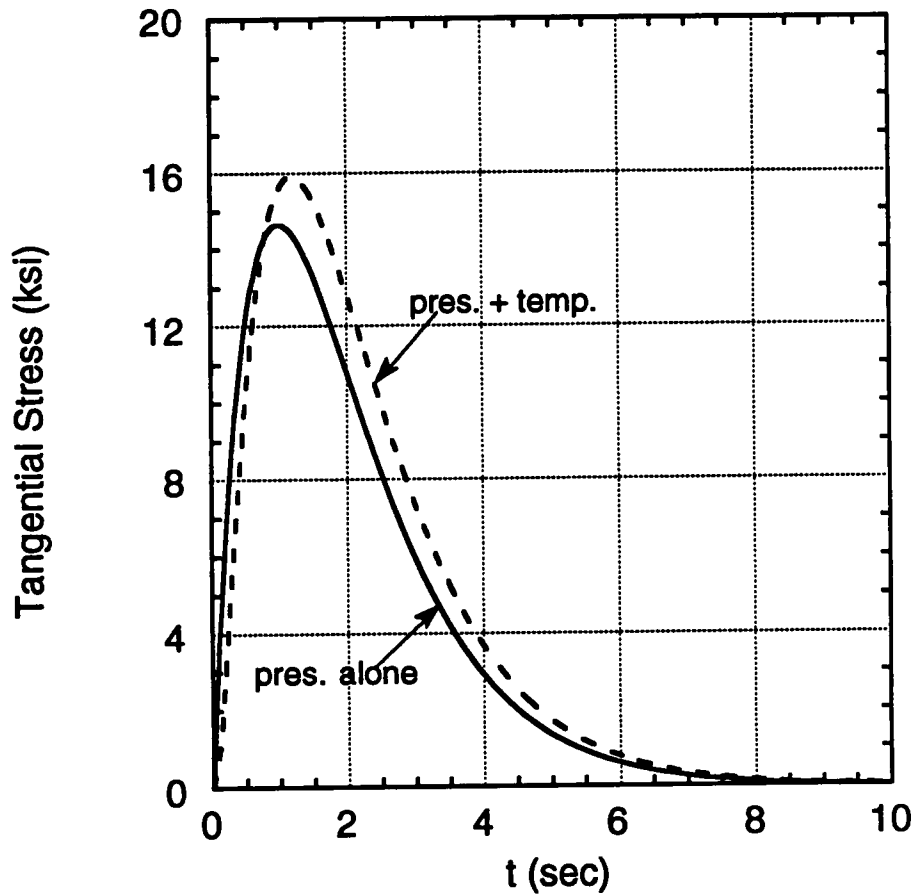


Figure 24. Tangential Stresses on the Inner Surface of Layer 2 (Fiber Layer); Thermal Shock and Pressure.

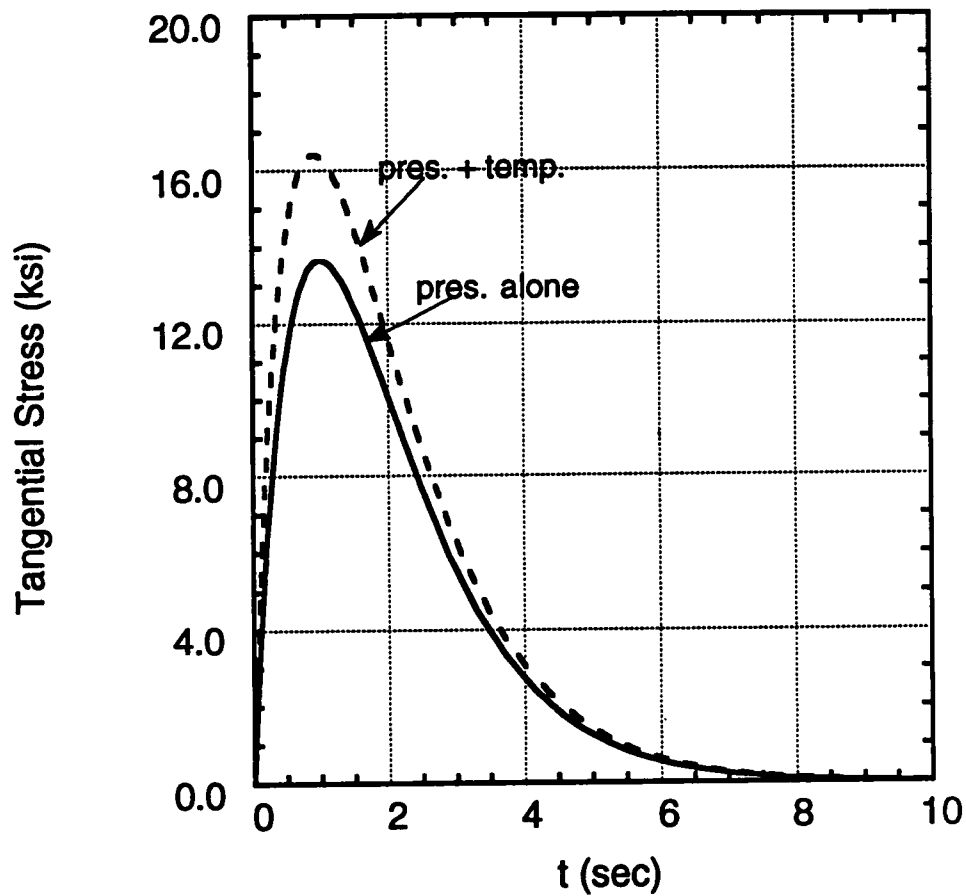


Figure 25. Tangential Stresses on the Outer Surface of Layer 32 (Fiber Layer); Thermal Shock and Pressure.

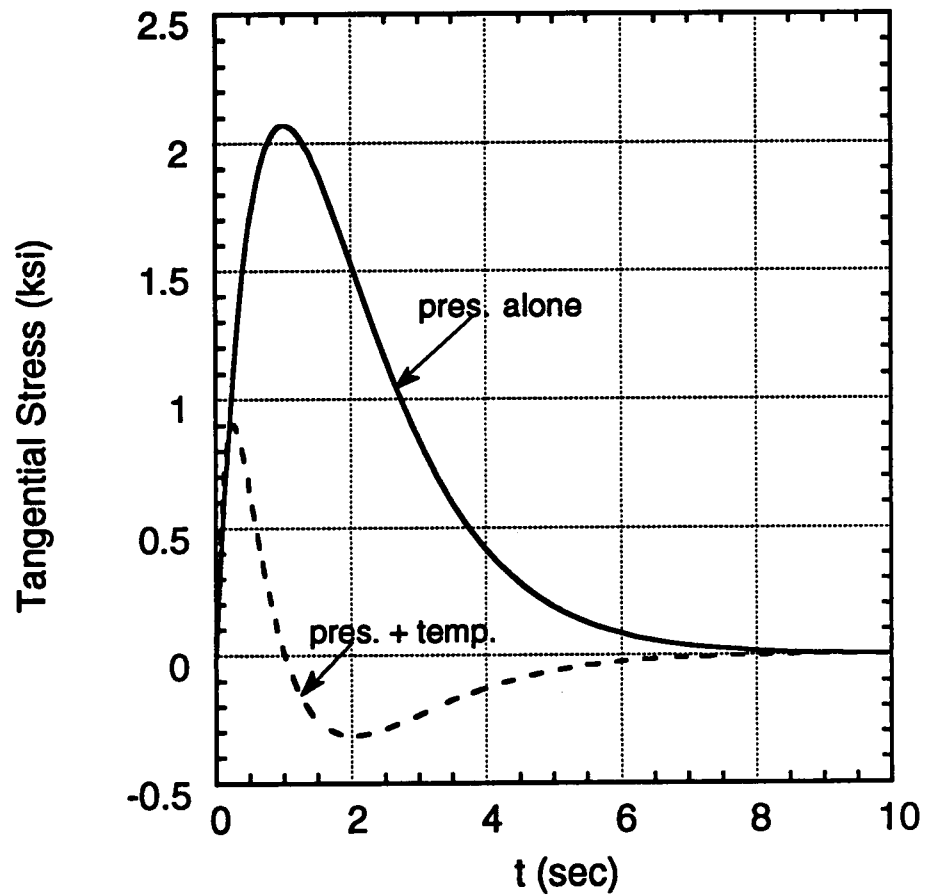


Figure 26. Tangential Stresses on the Outermost Surface (Matrix Layer); Thermal Shock and Pressure.

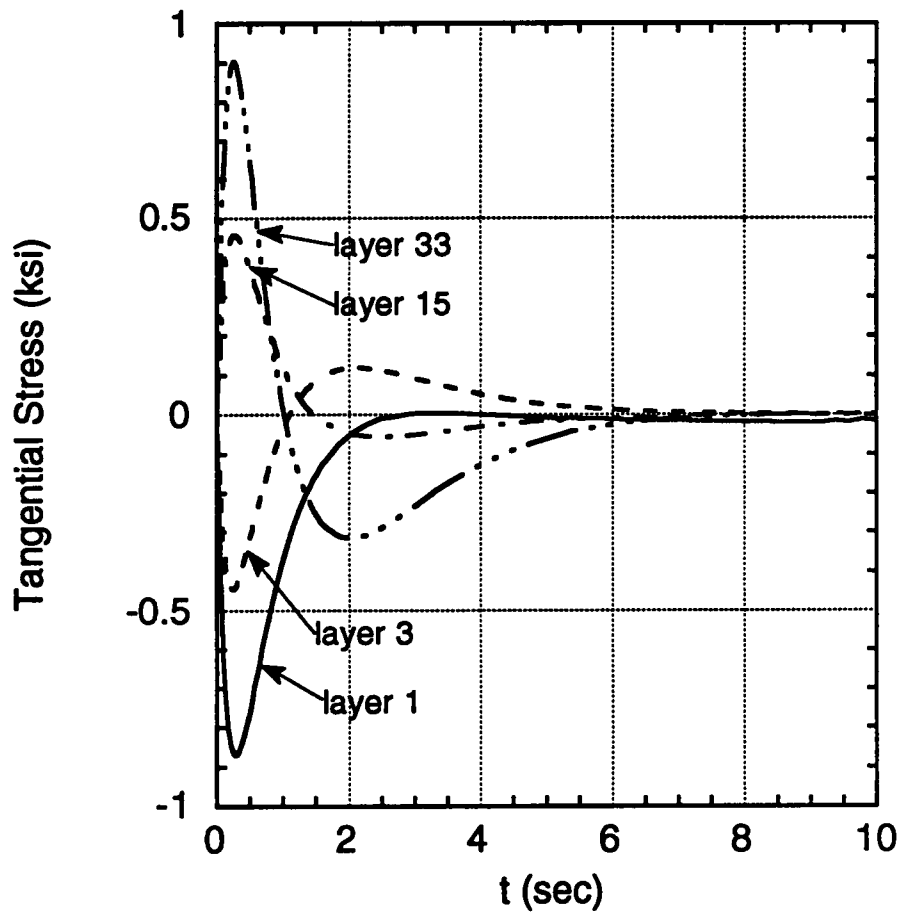


Figure 27. Tangential Stresses in Different Matrix Layers Subjected to Thermal Shock.

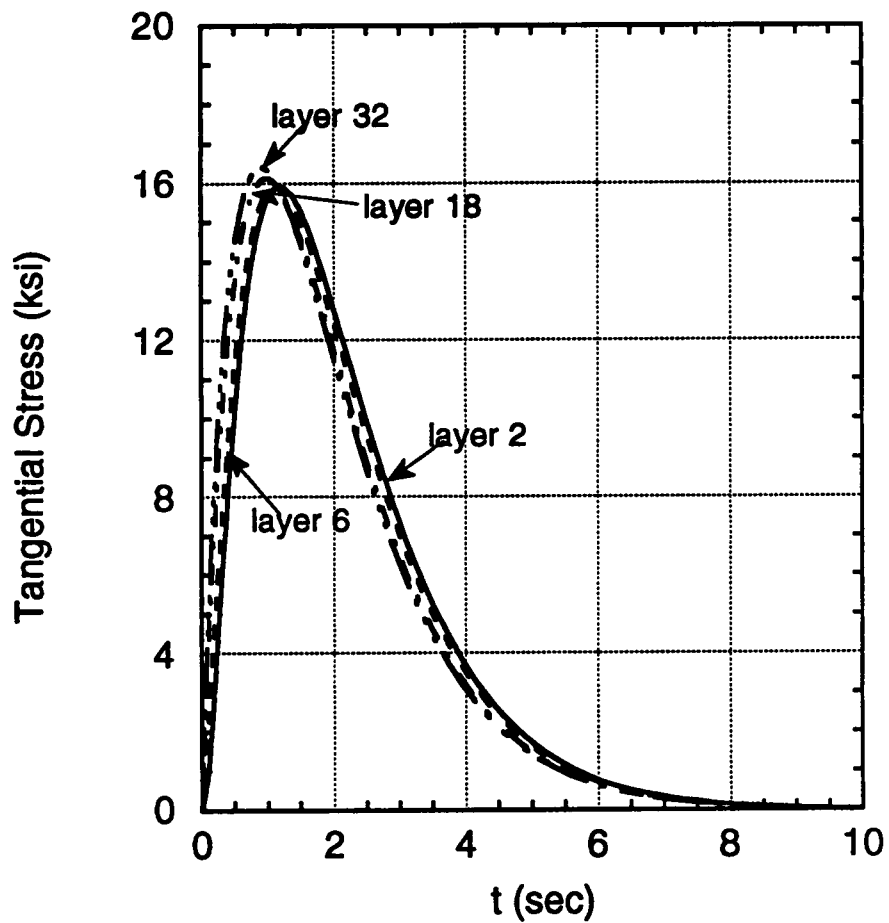


Figure 28. Tangential Stresses in Different Fiber Layers Subjected to Thermal Shock.

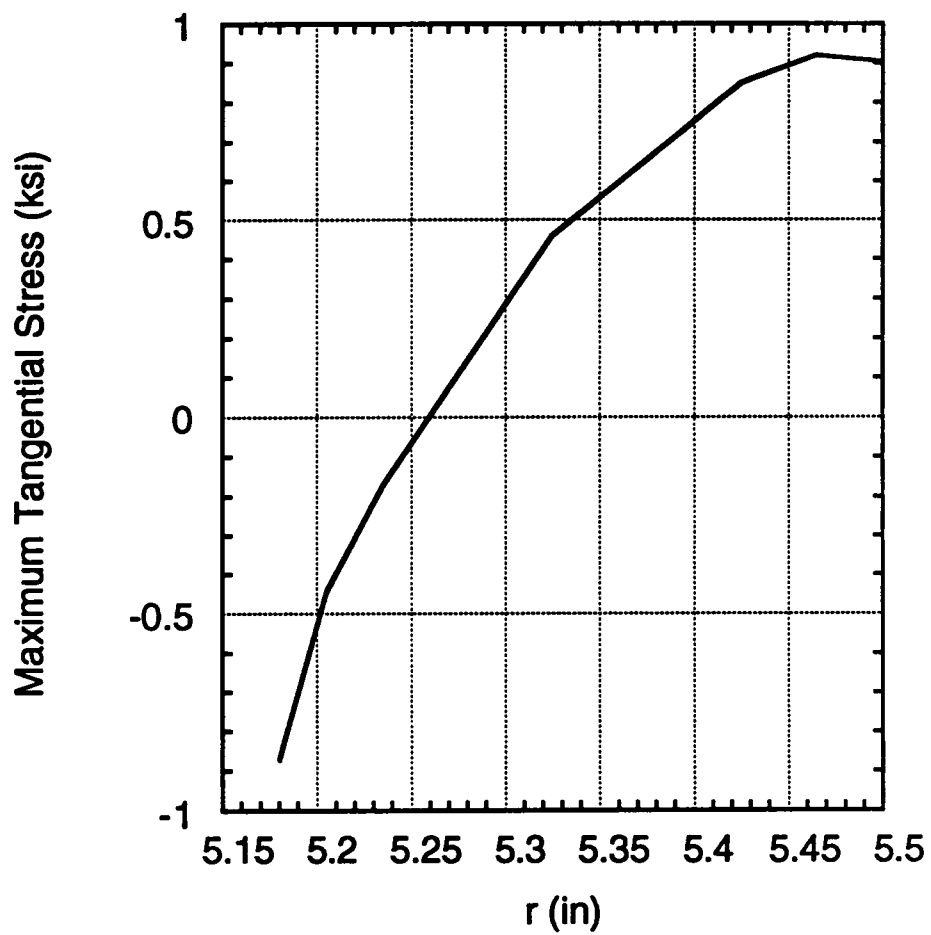


Figure 29. Maximum Tangential Stresses in Matrix Layers through the Thickness of the Cylinder.

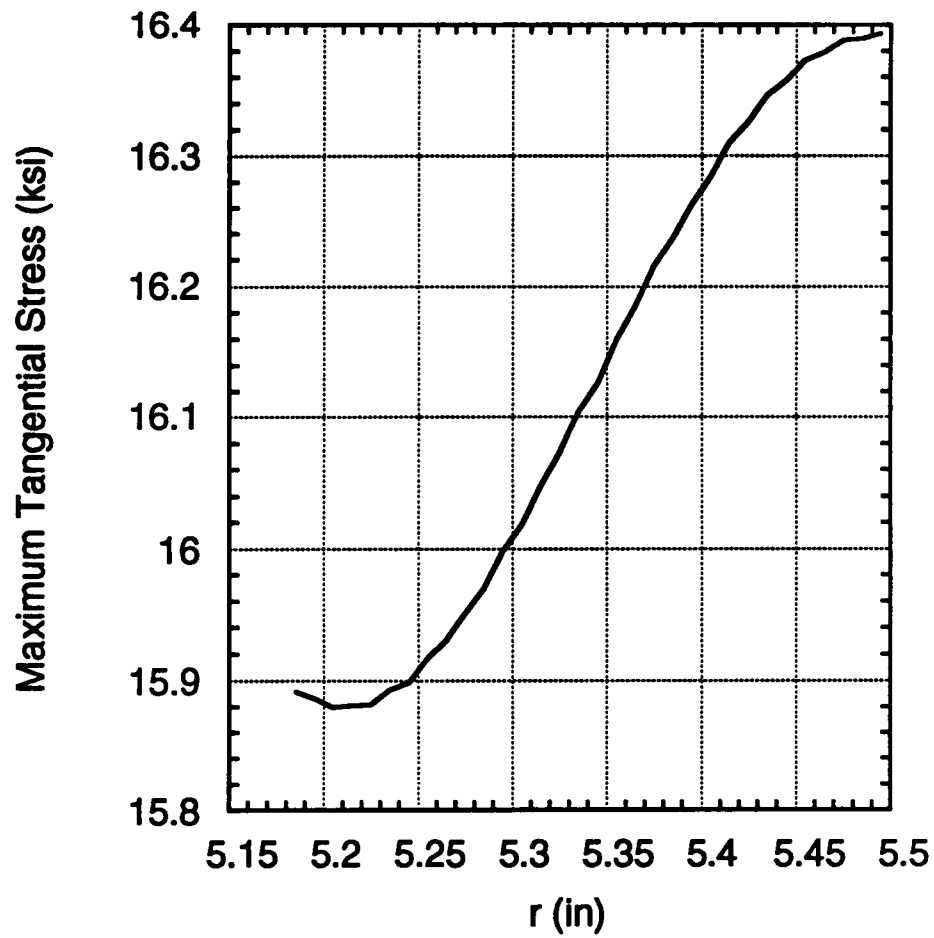


Figure 30. Maximum Tangential Stresses in Fiber Layers through the Thickness of the Cylinder.

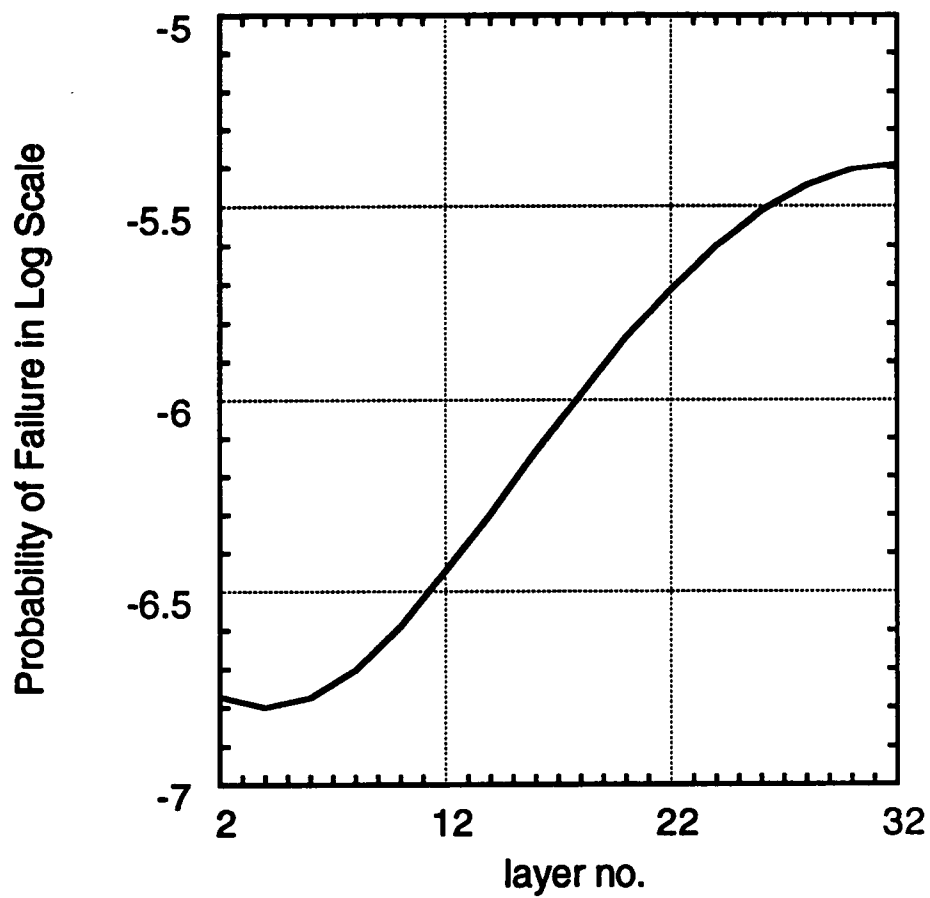


Figure 31. Maximum Probability of Failure of Fiber Layers through the Thickness of the Cylinder.

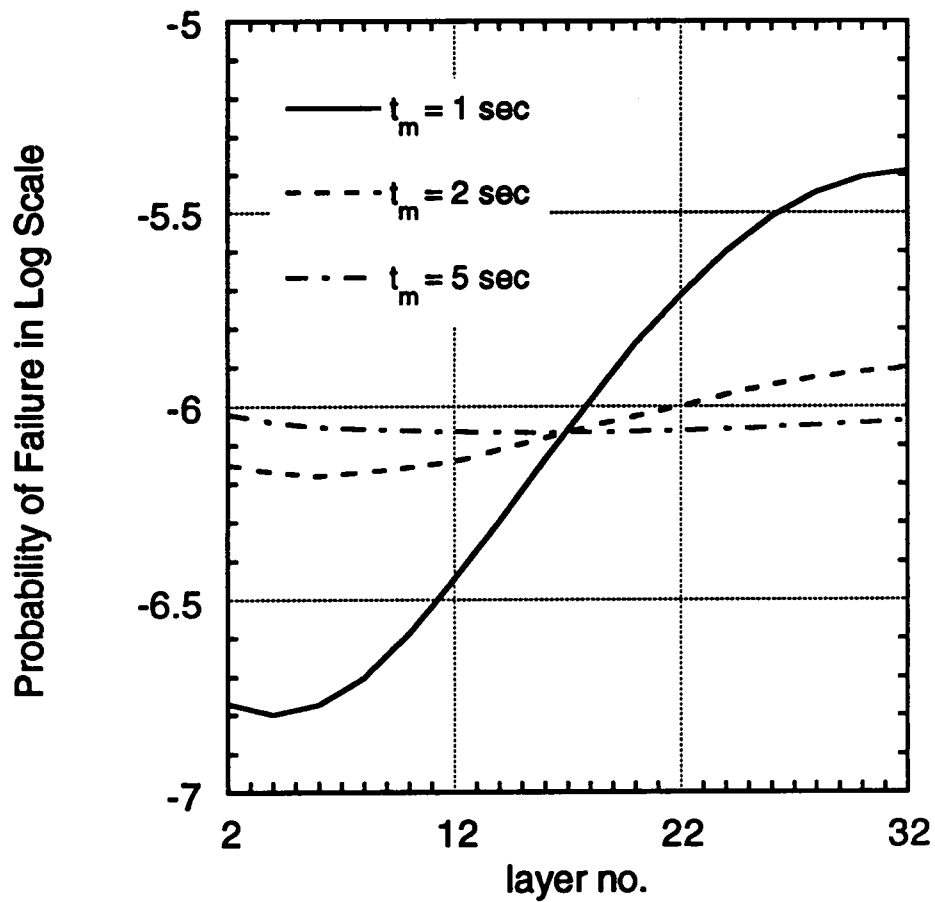


Figure 32. Maximum Probability of Failure of Fiber Layers through the Thickness of the Cylinder for Different Rise Times.

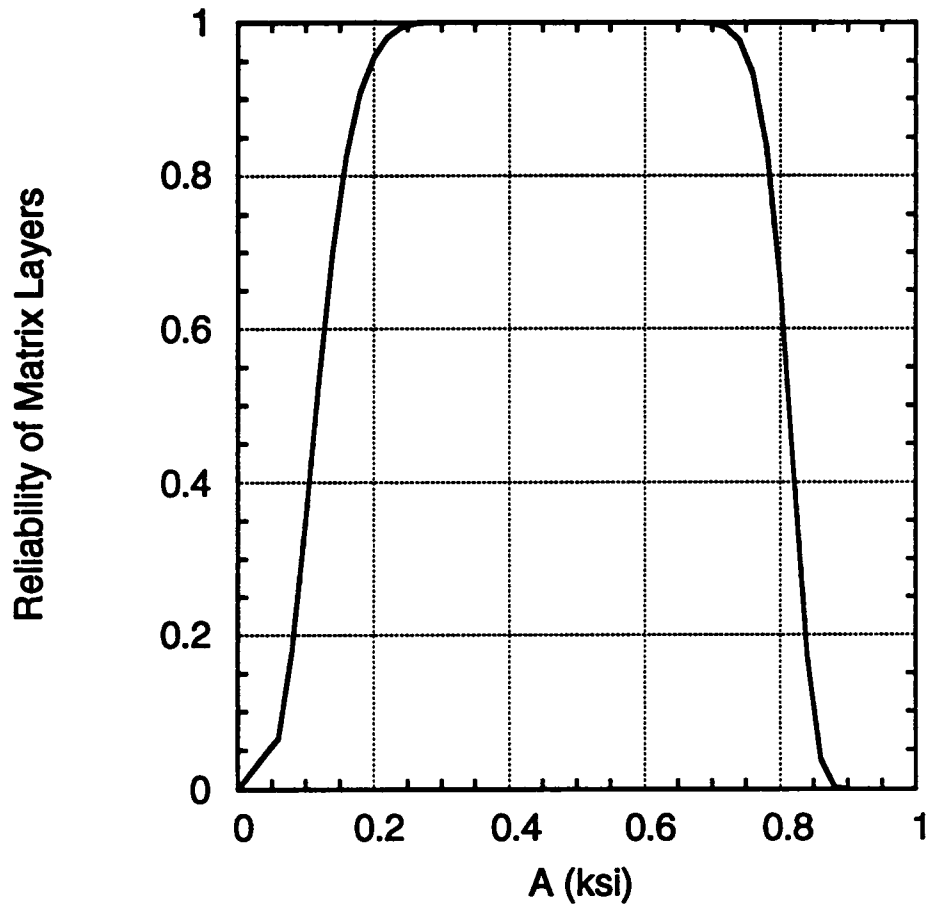


Figure 33. Reliability of the Matrix Layers Subjected to Thermal Shock and Pressure with Different Amplitudes.

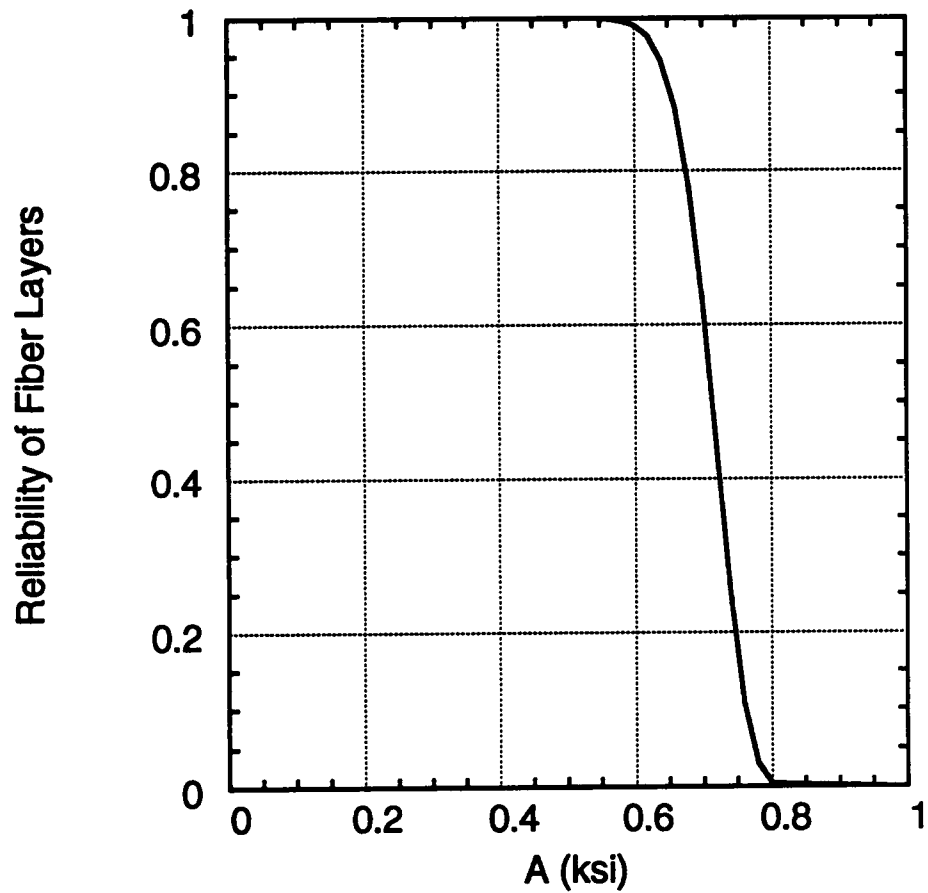


Figure 34. Reliability of the Fiber Layers Subjected to Thermal Shock and Pressure with Different Amplitudes.

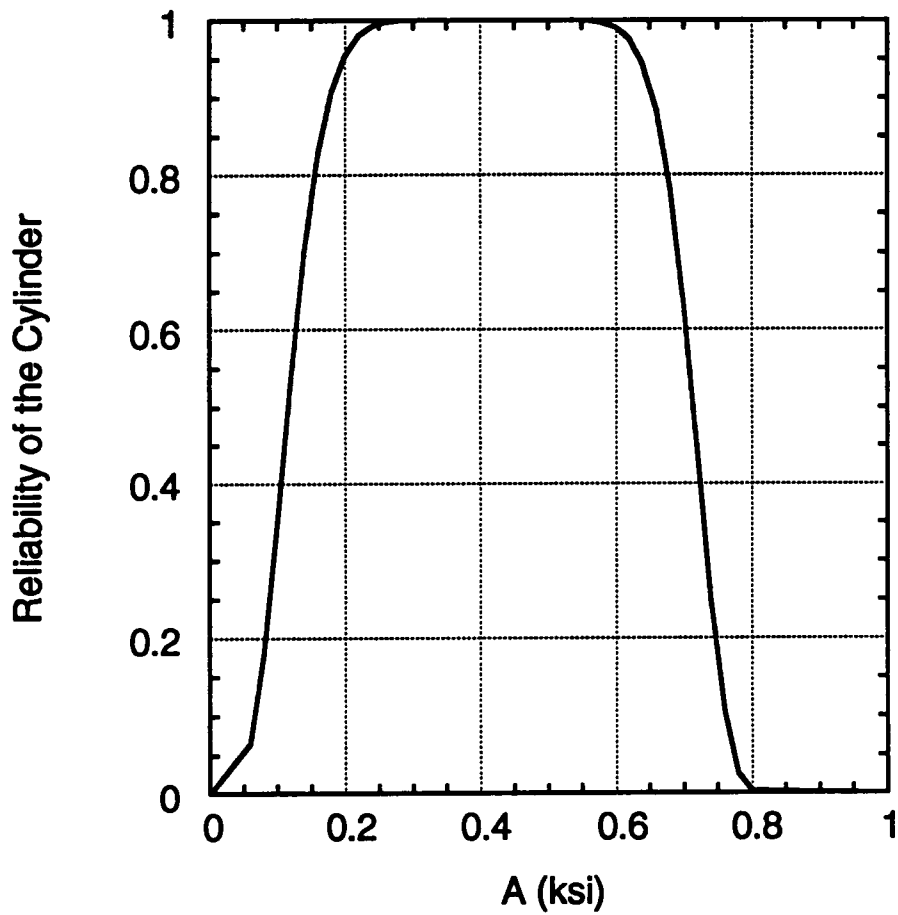


Figure 35. Reliability of the Cylinder Subjected to Thermal Shock and Pressure with Different Amplitudes.

Table 4
Geometric, Thermal and Mechanical Parameters for Rocket Motor with Kevlar Case.

Material	Air Core	Propellant	Odd Layers	Even Layers
Thickness (in)	$r_1 = 0.3$	0.6	0.02	0.02
Conductivity in r direction (Btu/hr ft ² °F)	0.0142	0.271	0.142	0.142
Diffusivity in r direction (in ² /sec)	0.04	0.06	0.267	0.267
Elastic Modulus, (psi)				
E_r	—	300	0.577×10^6	0.577×10^6
E_θ	—	300	0.577×10^6	14.475×10^6
E_z	—	300	14.475×10^6	0.577×10^6
Poisson's Ratio				
$\nu_{r\theta}$	—	0.49	0.212	0.014
ν_{rz}	—	0.49	0.014	0.212
$\nu_{\theta z}$	—	0.49	0.014	0.350
Coefficient of Thermal Expansion (in./in.-°F)				
$\alpha_{r\theta}$	—	6.0×10^{-5}	28.09×10^{-6}	28.09×10^{-6}
α_{rz}	—	6.0×10^{-5}	28.09×10^{-6}	-1.936×10^{-6}
$\alpha_{\theta z}$	—	6.0×10^{-5}	-1.936×10^{-6}	28.09×10^{-6}

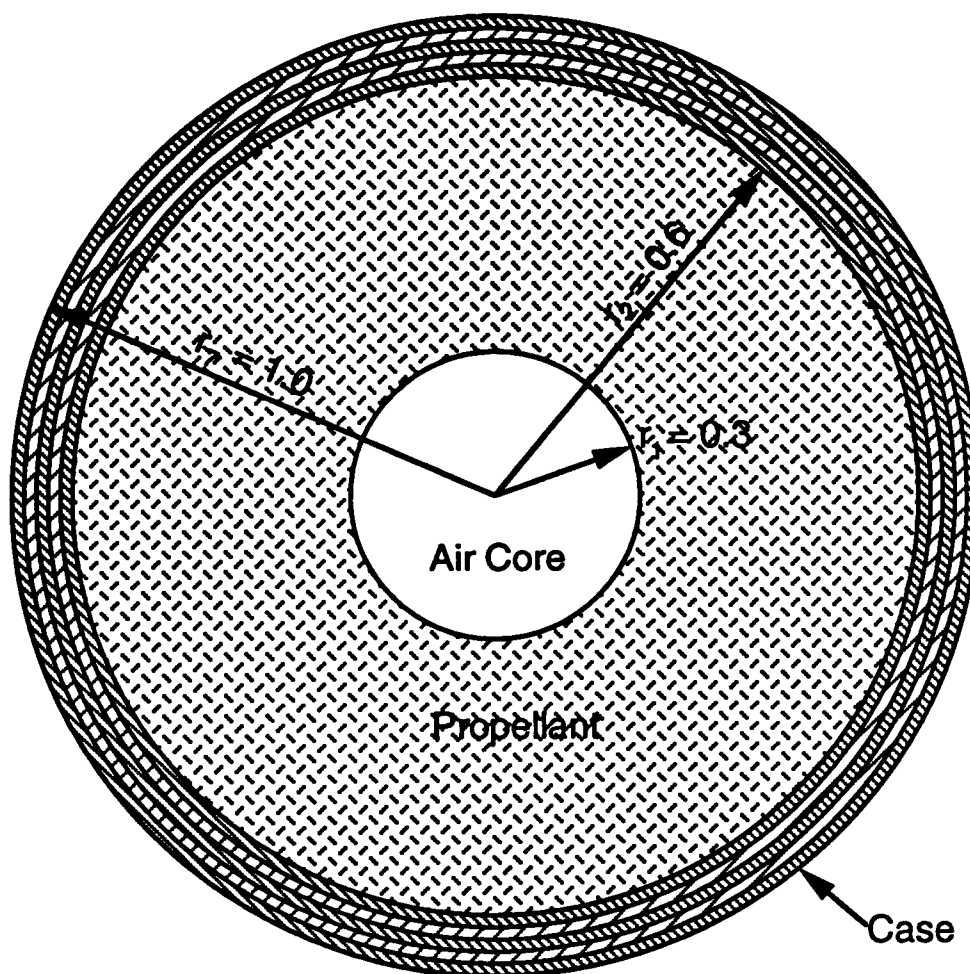


Figure 36. Configuration of the Rocket Motor.

$$T_0 = \begin{cases} 250 (1 + \cos 2\theta) H(t), & -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ 0, & \text{elsewhere} \end{cases}$$

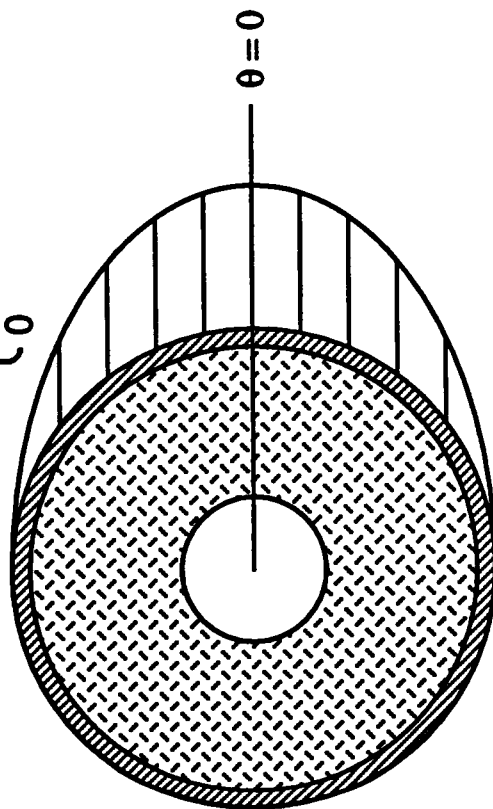


Figure 37. Asymmetric Input Temperature on the Motor.

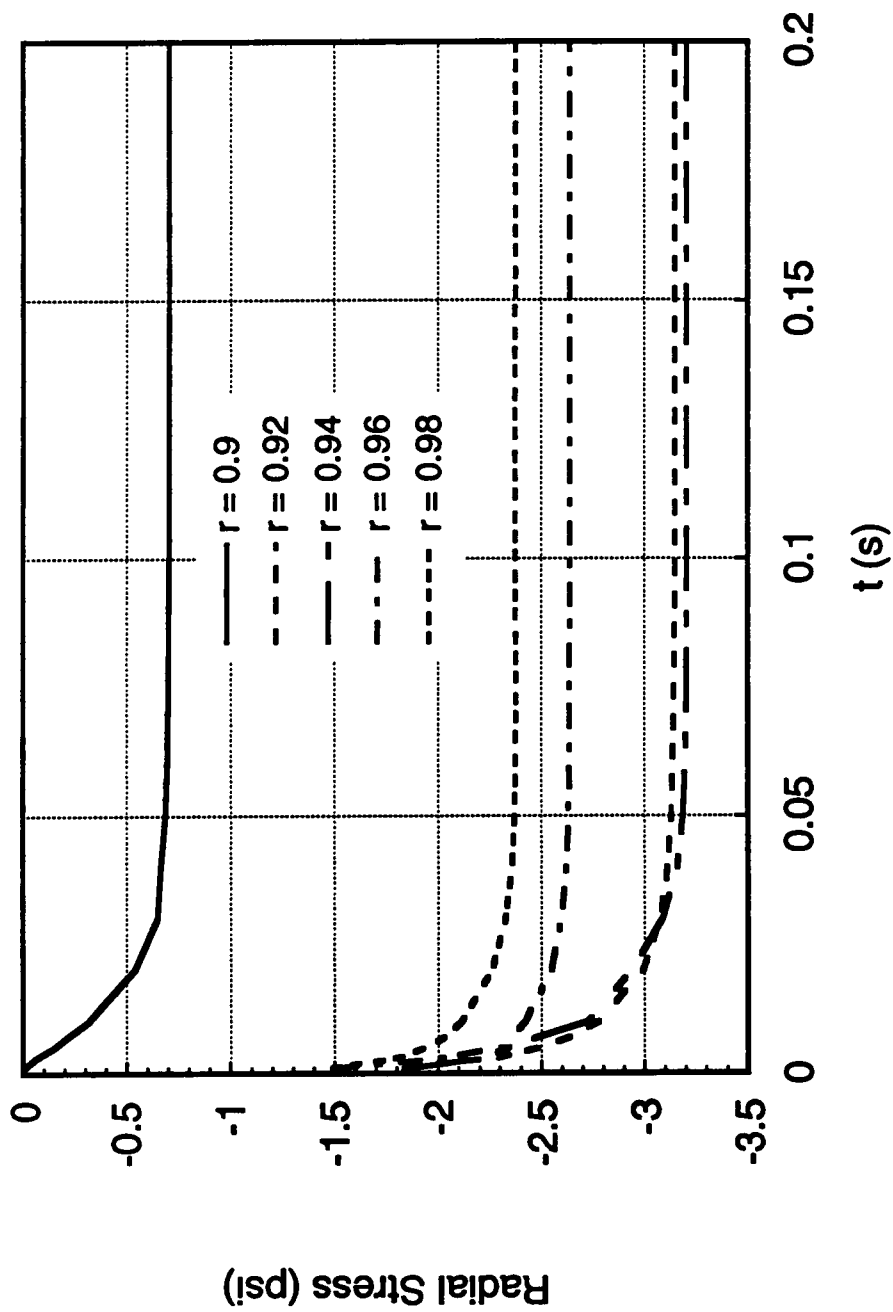


Figure 38. Radial Stresses at the Interfaces.

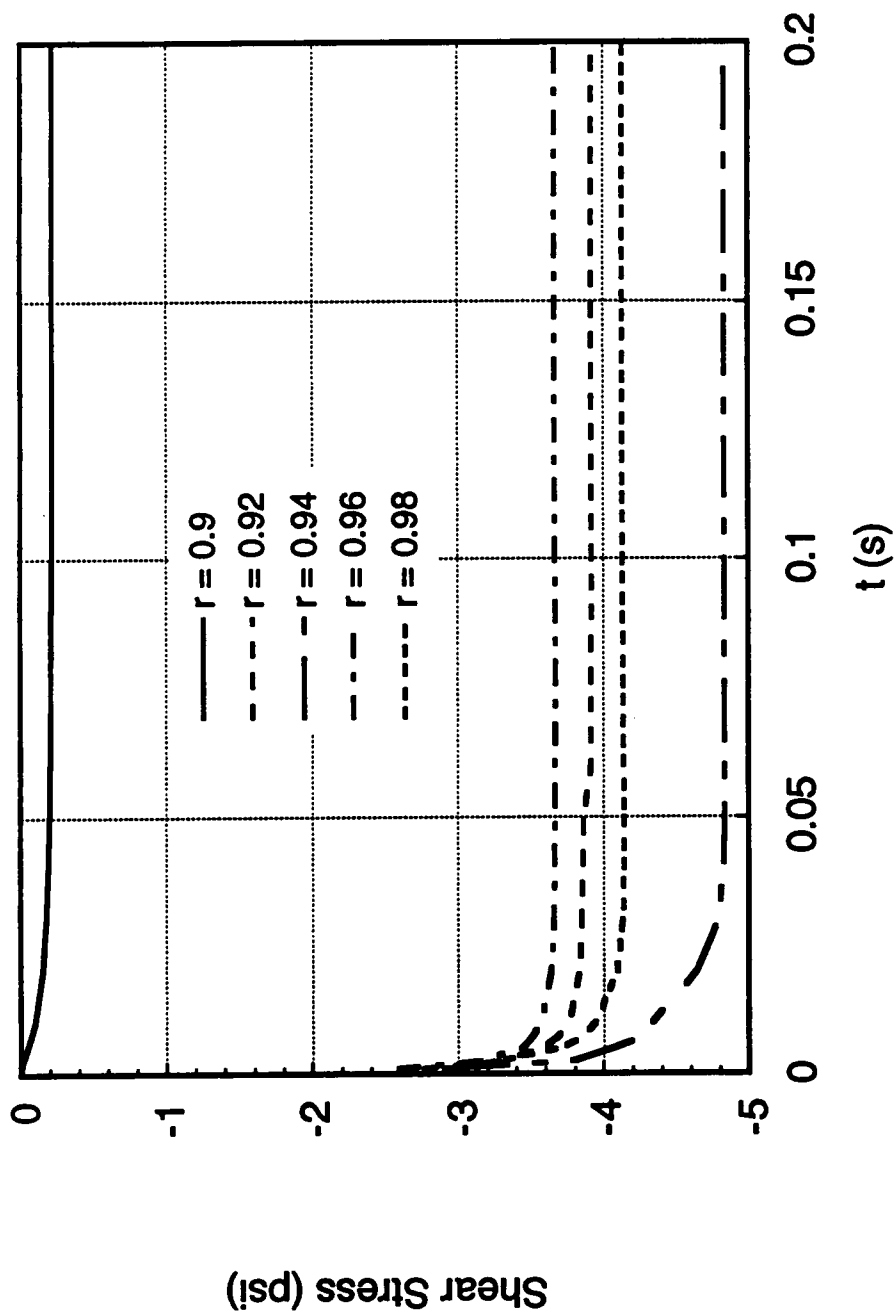


Figure 39. Shear Stresses at the Interfaces.

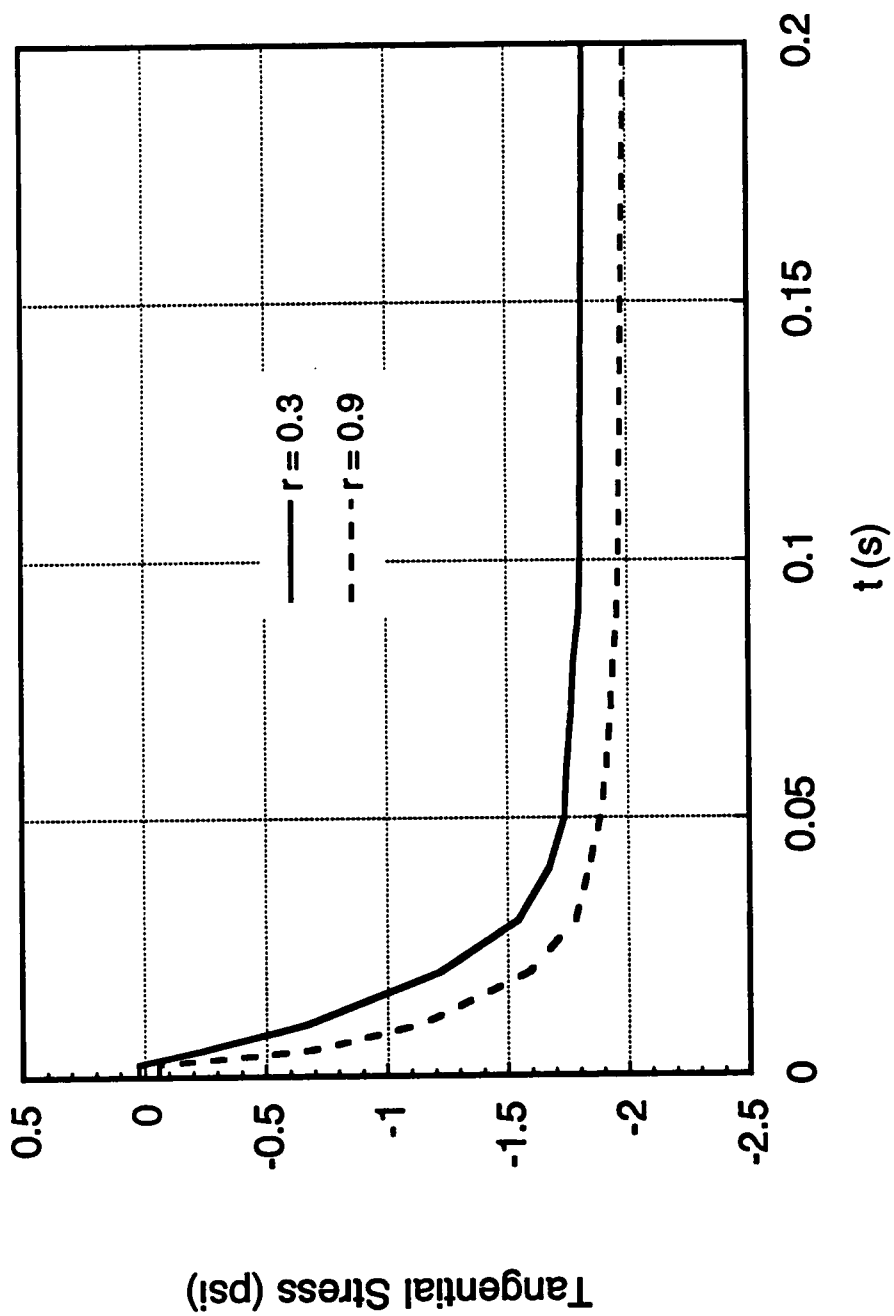


Figure 40. Tangential Stresses in the Propellant.

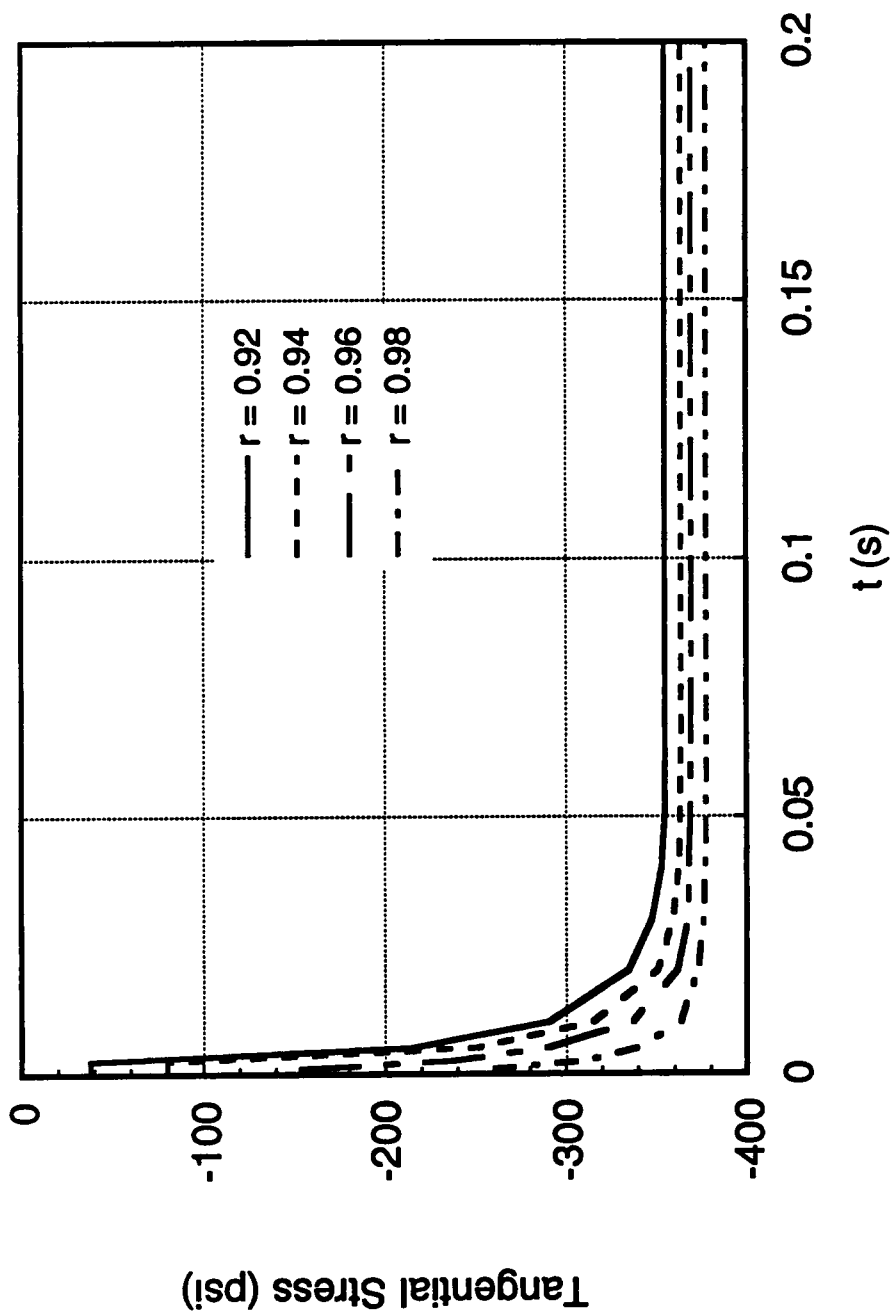


Figure 41. Tangential Stresses in the Case (Odd Layers).

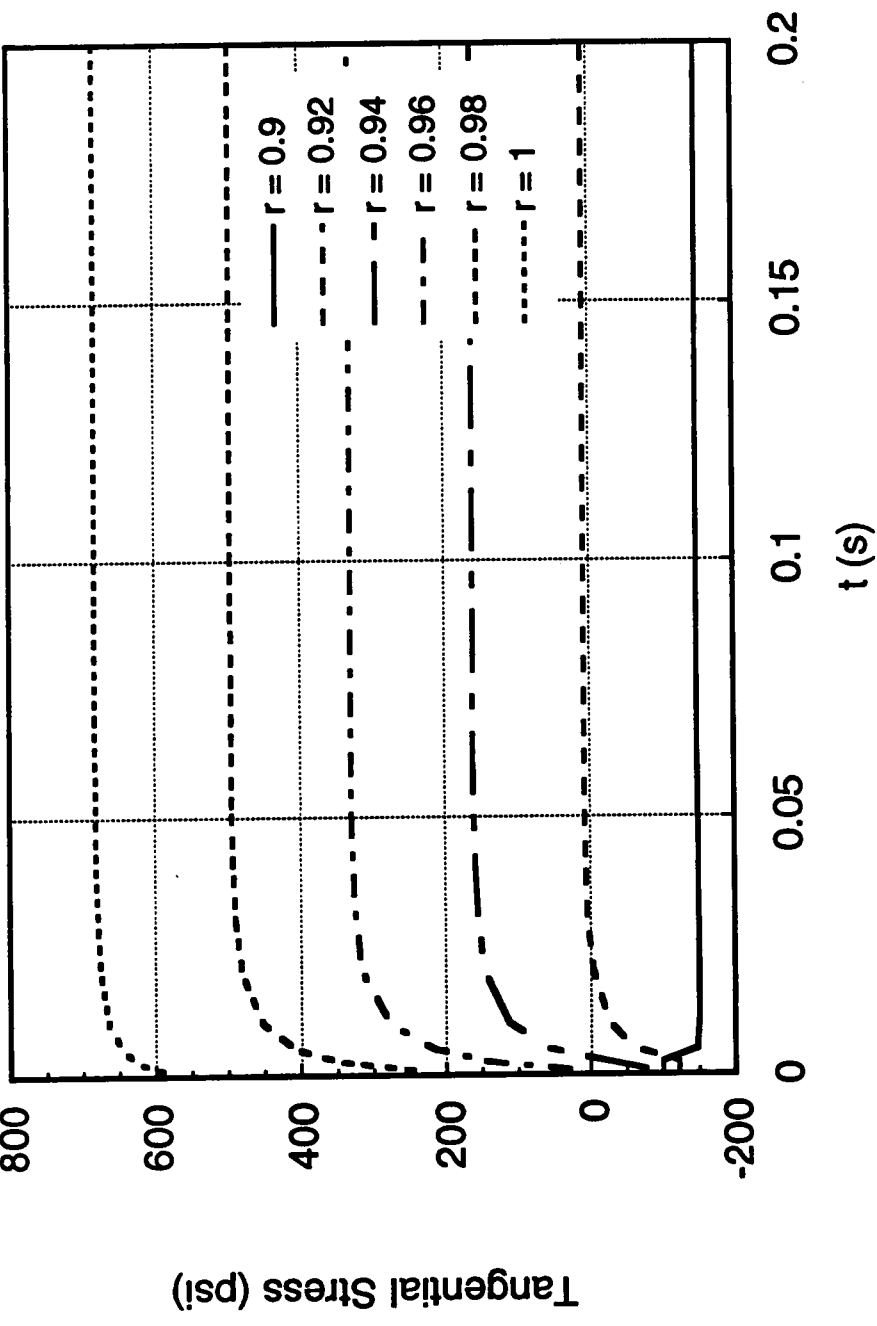


Figure 42. Tangential Stresses in the Case (Even Layers).

4.0 FREQUENCY RESPONSE ANALYSIS OF ORTHOTROPIC CYLINDER

As it was mentioned in Chapter 2, when a sinusoidal temperature input of some amplitude and frequency, ω , is applied to the structure, the responses, such as temperatures, displacements and stresses in the structure, will become sinusoidal with the same frequency as the input but will have different amplitudes and will undergo phase shifts. The frequency response function is defined as the response resulting from a sinusoidal temperature input of unit amplitude and frequency, ω (Heller, 1976; Heller, Kamat and Sing, 1979; Sing, Heller, and Thangjitham, 1984 and Thangjitham, Heller and Sing, 1986). Coupled symmetric thermal shock problems for isotropic materials were analyzed by the aid of complex frequency response functions in Chapter 2. The technique mentioned in Chapter 2 can be extended to orthotropic cylinders. Furthermore, the complex frequency response functions are also useful for evaluating the effect of random stress and reliability (Heller, 1976). A random process can be defined by its mean and variance. When a structure is subjected to a random temperature loading, especially narrow banded temperature loading, both the means

and variances of the thermal stresses in the structure can be obtained by integrating the power spectral density of the displacements and stresses which can be calculated with the aid of these frequency response functions. A reliability analysis can also be performed based on the mean and variance (first-order second-moment) of the stresses obtained above. In this chapter, the complex frequency response functions of temperature stresses for asymmetric orthotropic multilayered cylinder will be developed.

4.1 Frequency Response of Temperature

The heat conduction equation for an orthotropic material is given as Eq. 3.1:

$$\gamma_{11} \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \gamma_{22} \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = \frac{\partial T}{\partial t}$$

where $T(r, \theta, t)$ is the temperature function, r and θ are the radial and tangential coordinates, and t is time, γ_{11} and γ_{22} are the coefficients of radial and tangential thermal diffusivities, respectively. To utilize the method of complex frequency response functions, the following solution is assumed.

$$T = \tilde{T} \exp(i\omega t) \quad (4.1)$$

where $\tilde{T}(r, \theta, \omega)$ is the corresponding complex frequency response function for T , and $i = \sqrt{-1}$.

Substituting Eq. 4.1 into Eq. 3.1, the heat conduction equation, expressed in terms of the complex frequency response function, $\tilde{T}$ is given as

$$\gamma_{11} \left(\frac{\partial^2 \tilde{T}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{T}}{\partial r} \right) + \gamma_{22} \frac{1}{r^2} \frac{\partial^2 \tilde{T}}{\partial \theta^2} - i\omega \tilde{T} = 0 \quad (4.2)$$

where the above equation is a complex partial differential equation with ω as a parameter. Expanding the complex frequency response function of temperature in the j th layer into Fourier series:

$$\tilde{T}(r, \theta, \omega) = \sum_{n=-\infty}^{\infty} \tilde{T}_n(r, \omega) e^{in\theta} \quad (4.3)$$

Substituting Eq. 4.3 into Eq. 4.2, and collecting the coefficients of the like terms, the Eq. 4.2 becomes:

$$\gamma_{11} \left(\frac{\partial^2 \tilde{T}_n}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{T}_n}{\partial r} \right) - \left(\gamma_{22} \frac{n^2}{r^2} + i\omega \right) \tilde{T}_n = 0 \quad (4.4)$$

Solving Eq. 4.4, the complex frequency response function of temperature can be obtained in terms of the Kelvin functions with fractional order $Br_{\nu_n}(x)$ and $Kr_{\nu_n}(x)$ as follows:

$$\tilde{T}_n = \tilde{T}_{n1} Br_{\nu_n}(ar) + \tilde{T}_{n2} Kr_{\nu_n}(ar) \quad (4.5)$$

where

$$a = \sqrt{\frac{\omega}{\gamma_{11}}} \quad \text{and} \quad \nu_n = n \sqrt{\frac{\gamma_{22}}{\gamma_{11}}}$$

$\tilde{T}_{n1}$ and $\tilde{T}_{n2}$ are constants to be determined by the following temperature boundary conditions.

Case a: The input temperature is uniformly applied on the outermost surface of a hollow J -layered cylinder:

At $r = 0$ (center of the bore)

$$BC.1 \quad \tilde{T}_n^1(r, \theta, \omega) \text{ is finite}$$

At $r = r_j$, $j = 1, 2, 3, \dots, J - 1$ (on the interfaces between layers)

$$BC.2 \quad \tilde{T}_n^j(r, \theta, \omega) = \tilde{T}_n^{j+1}(r, \theta, \omega)$$

$$BC.3 \quad k_{rj} \frac{\partial \tilde{T}_n^j(r, \theta, \omega)}{\partial r} = k_{r(j+1)} \frac{\partial \tilde{T}_n^{j+1}(r, \theta, \omega)}{\partial r}$$

At $r = r_J$ (on the outermost surface)

$$BC.4 \quad \tilde{T}_n^J(r, \theta, \omega) - 1 = 0$$

Case b: The input temperature is uniformly applied on the innermost surface of a hollow J -layered cylinder:

At $r = r_1$ (on the innermost surface)

$$BC.1 \quad \tilde{T}_n^1(r, \theta, \omega) - 1 = 0$$

At $r = r_{j+1}$, $j = 1, 2, 3, \dots, J - 1$ (on the interfaces between layers)

$$BC.2 \quad \tilde{T}_n^j(r, \theta, \omega) = \tilde{T}_n^{j+1}(r, \theta, \omega)$$

$$BC.3 \quad k_{rj} \frac{\partial \tilde{T}_n^j(r, \theta, \omega)}{\partial r} = k_{r(j+1)} \frac{\partial \tilde{T}_n^{j+1}(r, \theta, \omega)}{\partial r}$$

At $r = r_j$ (on the outermost surface)

$$BC.4 \quad \frac{\partial \tilde{T}_n^j(r, \theta, \omega)}{\partial r} = 0$$

4.2 Complex Frequency Response of Stresses

To utilize the method of complex frequency response functions, the following solutions are assumed.

$$u = \tilde{u} \exp(i\omega t), \quad v = \tilde{v} \exp(i\omega t) \quad (4.5 \text{ a, b})$$

$$\varepsilon_r = \tilde{\varepsilon}_r \exp(i\omega t), \quad \varepsilon_\theta = \tilde{\varepsilon}_\theta \exp(i\omega t), \quad \gamma_{r\theta} = \tilde{\gamma}_{r\theta} \exp(i\omega t), \quad (4.5 \text{ c, d, e})$$

$$\sigma_r = \tilde{\sigma}_r \exp(i\omega t), \quad \sigma_\theta = \tilde{\sigma}_\theta \exp(i\omega t), \quad \tau_{r\theta} = \tilde{\tau}_{r\theta} \exp(i\omega t), \quad (4.5 \text{ f, g, h})$$

where $\tilde{u}(r, \theta, \omega)$, $\tilde{v}(r, \theta, \omega)$, $\tilde{\varepsilon}_r(r, \theta, \omega)$, $\tilde{\varepsilon}_\theta(r, \theta, \omega)$, $\tilde{\gamma}_{r\theta}(r, \theta, \omega)$, $\tilde{\sigma}_r(r, \theta, \omega)$, $\tilde{\sigma}_\theta(r, \theta, \omega)$, and $\tilde{\tau}_{r\theta}(r, \theta, \omega)$ are the corresponding complex frequency response functions for u , v , ε_r , ε_θ , $\gamma_{r\theta}$, σ_r , σ_θ , and $\tau_{r\theta}$, respectively, and $i = \sqrt{-1}$.

Substituting Eqs. 4.5(a)-(h) into Eqs. 3.6-3.8, the stress-strain, strain-displacement relations and equilibrium equations for a plane strain problem in cylindrical coordinates

in terms of displacements, expressed in terms of the complex frequency response functions, $\tilde{u}$, $\tilde{v}$, $\tilde{\varepsilon}_r$, $\tilde{\varepsilon}_\theta$, $\tilde{\gamma}_{r\theta}$, $\tilde{\sigma}_r$, $\tilde{\sigma}_\theta$, and $\tilde{\tau}_{r\theta}$, are given as

$$\tilde{\sigma}_r = C_{11} \tilde{\varepsilon}_r + C_{12} \tilde{\varepsilon}_\theta - \beta_1 \tilde{T}$$

$$\tilde{\sigma}_\theta = C_{12} \tilde{\varepsilon}_r + C_{22} \tilde{\varepsilon}_\theta - \beta_2 \tilde{T} \quad (4.6)$$

$$\tilde{\tau}_{r\theta} = C_{66} \tilde{\gamma}_{r\theta}$$

$$\tilde{\varepsilon}_r = \frac{\partial \tilde{u}}{\partial r}, \quad \tilde{\varepsilon}_\theta = \frac{1}{r} \left(\frac{\partial \tilde{v}}{\partial \theta} + \tilde{u} \right)$$

$$\tilde{\gamma}_{r\theta} = \frac{1}{r} \left(\frac{\partial \tilde{u}}{\partial \theta} + r \frac{\partial \tilde{v}}{\partial r} \right) \quad (4.7)$$

$$C_{11} \left(\frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{u}}{\partial r} \right) + C_{66} \frac{1}{r^2} \frac{\partial^2 \tilde{u}}{\partial \theta^2} - C_{22} \frac{\tilde{u}}{r^2} + (C_{12} + C_{66}) \frac{1}{r} \frac{\partial^2 \tilde{v}}{\partial r \partial \theta} \\ - (C_{22} + C_{66}) \frac{1}{r^2} \frac{\partial \tilde{v}}{\partial \theta} - \tilde{f}_r = 0$$

$$C_{66} \left(\frac{\partial^2 \tilde{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{v}}{\partial r} \right) + C_{22} \frac{1}{r^2} \frac{\partial^2 \tilde{v}}{\partial \theta^2} - C_{66} \frac{\tilde{v}}{r^2} + (C_{12} + C_{66}) \frac{1}{r} \frac{\partial^2 \tilde{u}}{\partial r \partial \theta} \\ + (C_{22} + C_{66}) \frac{1}{r^2} \frac{\partial \tilde{u}}{\partial \theta} - \tilde{f}_\theta = 0 \quad (4.8)$$

where

$$\tilde{f}_r = \beta_1 \frac{\partial \tilde{T}}{\partial r} + (\beta_1 - \beta_2) \frac{\tilde{T}}{r}$$

$$\tilde{f}_\theta = \beta_2 \frac{1}{r} \frac{\partial \tilde{T}}{\partial \theta}$$

Eqs. 4.6-4.8 are in the same form as Eqs. 3.6-3.8. The solutions can be obtained by the same procedures as in Section 3.2.

4.3 *Illustrative Examples*

The stress frequency response of the cylinder discussed in Section 3.3.2 is analyzed here. The geometries and material properties are listed in Table 4 and the configuration is shown in Figure 36. The amplitudes of the complex frequency response of radial stress for the first four terms, $n = 0, 1, 2, 3$, are shown in Figures 43-47 for different locations. When $n = 0$, the shear stress $\tau_{r\theta} \equiv 0$, the amplitudes of the complex frequency response of shear stress for the other three terms, $n = 1, 2, 3$, are shown in Figures 48-52. While radial and shear stresses are continuous at the interface, there are two different values at the interface of two materials for the tangential stresses. The amplitudes of the complex frequency response of tangential stress for the first four terms, $n = 0, 1, 2, 3$, are shown in Figures 53-64.

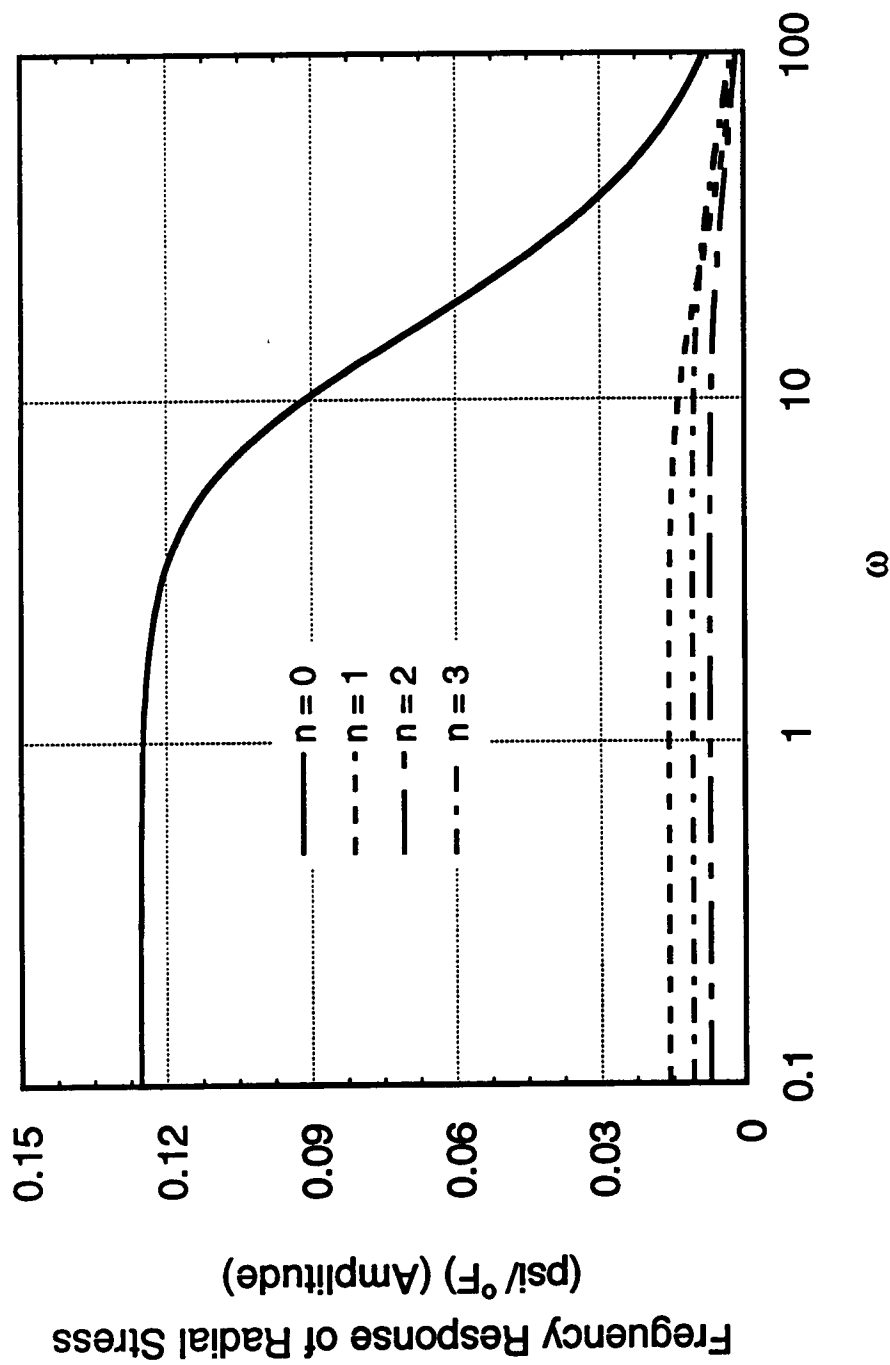


Figure 43. Frequency Responses of Radial Stress at r_2 .

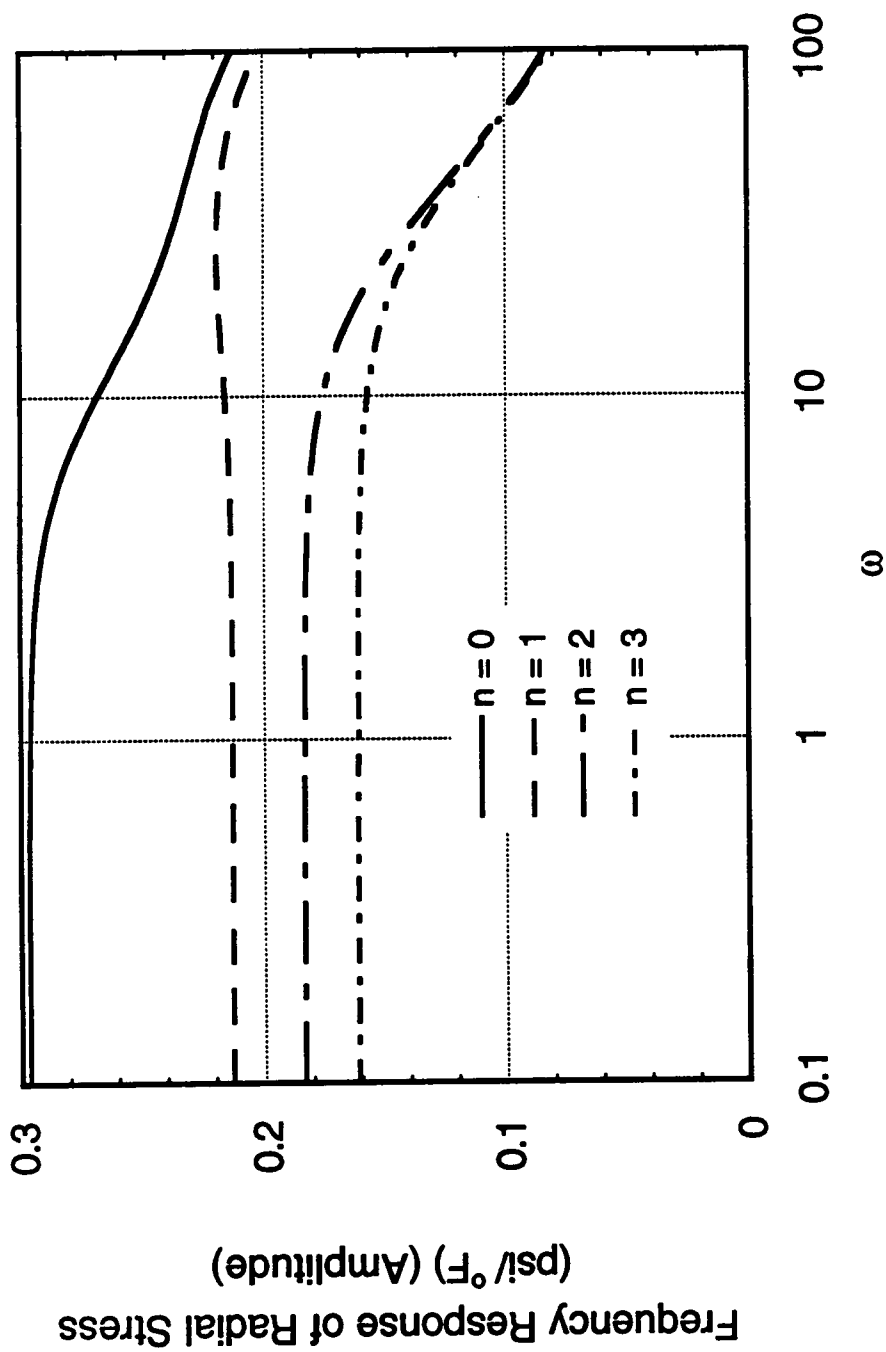


Figure 44. Frequency Responses of Radial Stress at r_3 .

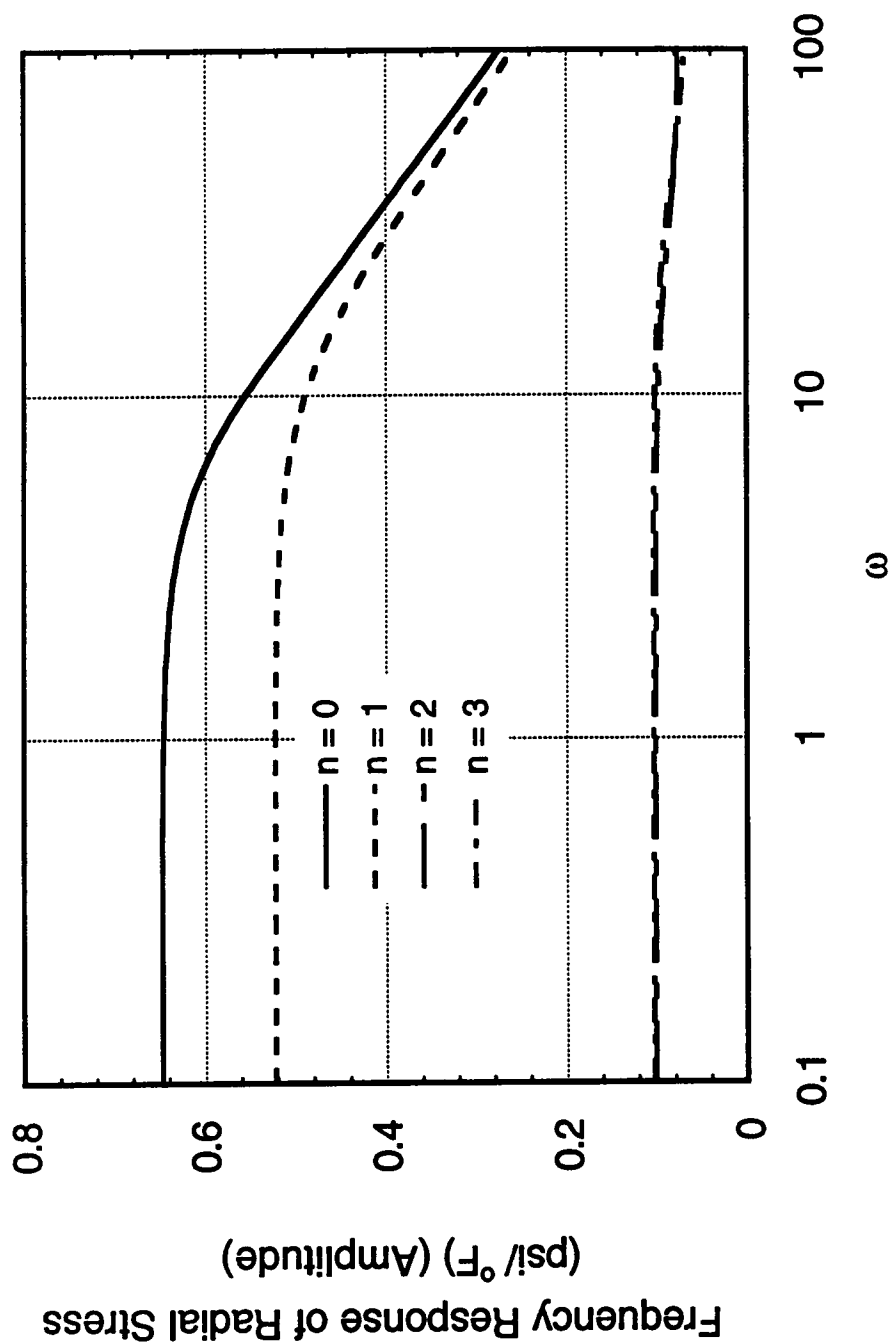


Figure 45. Frequency Responses of Radial Stress at r_4 .

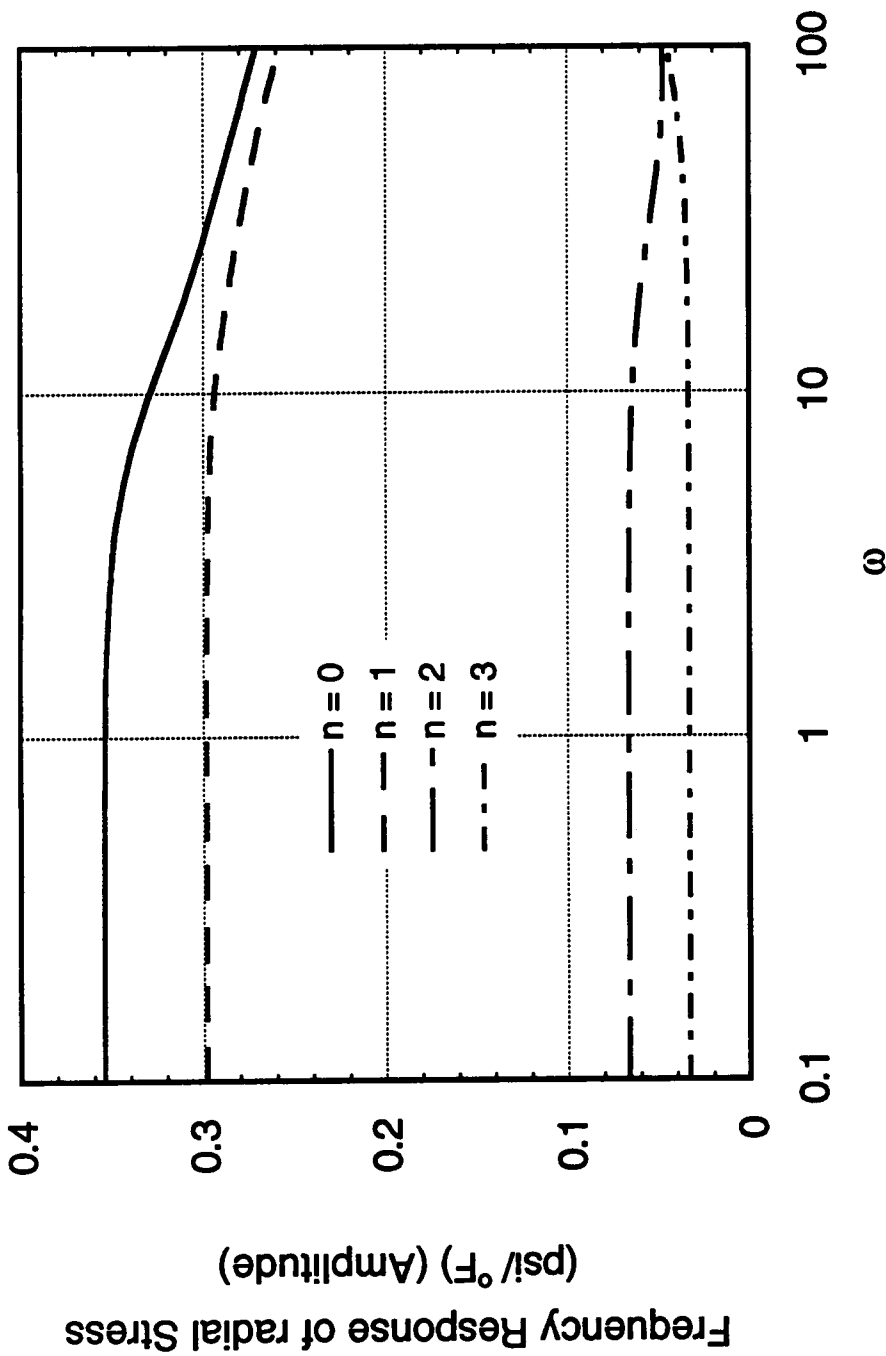


Figure 46. Frequency Responses of Radial Stress at r_5 .

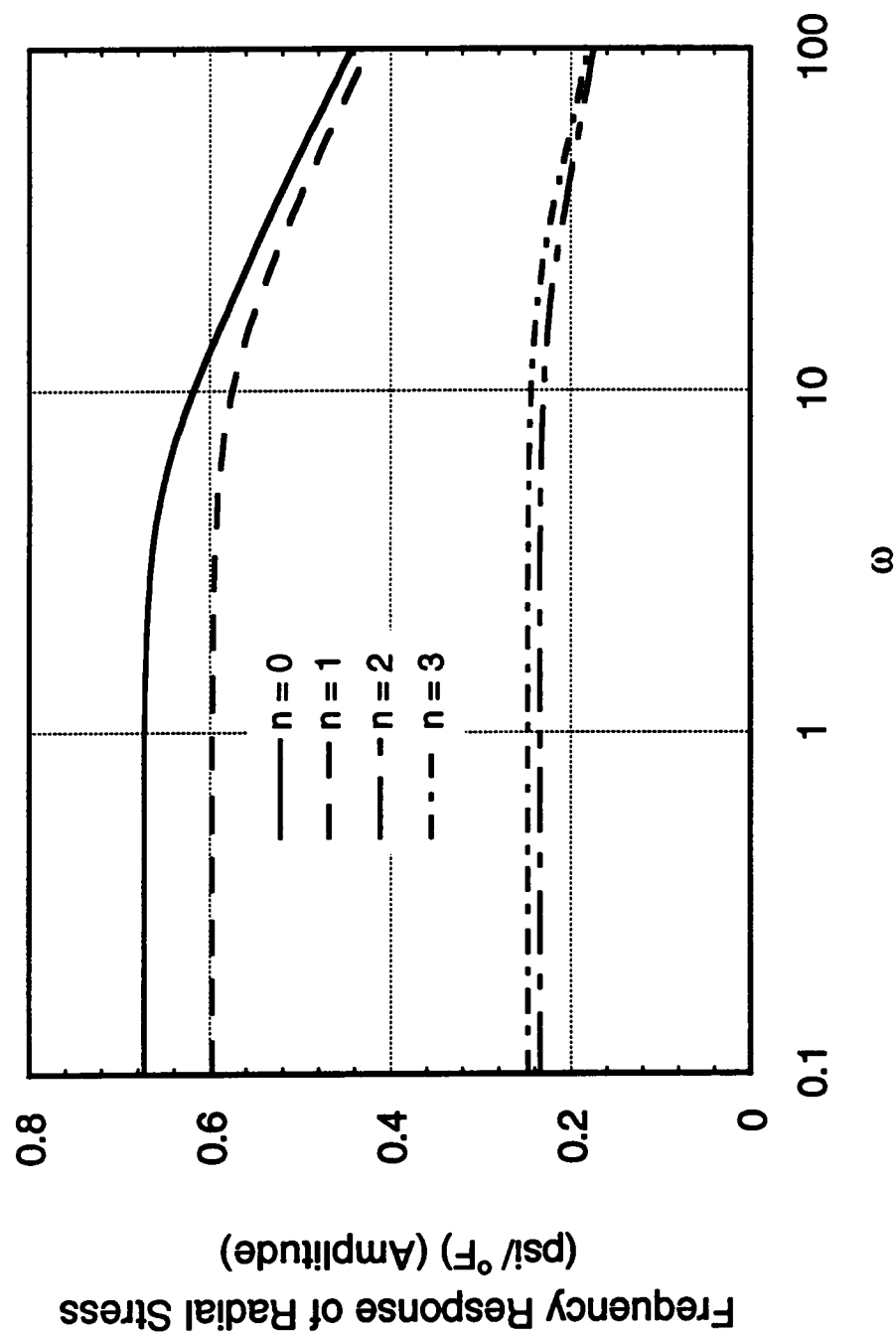


Figure 47. Frequency Responses of Radial Stress at r_6 .

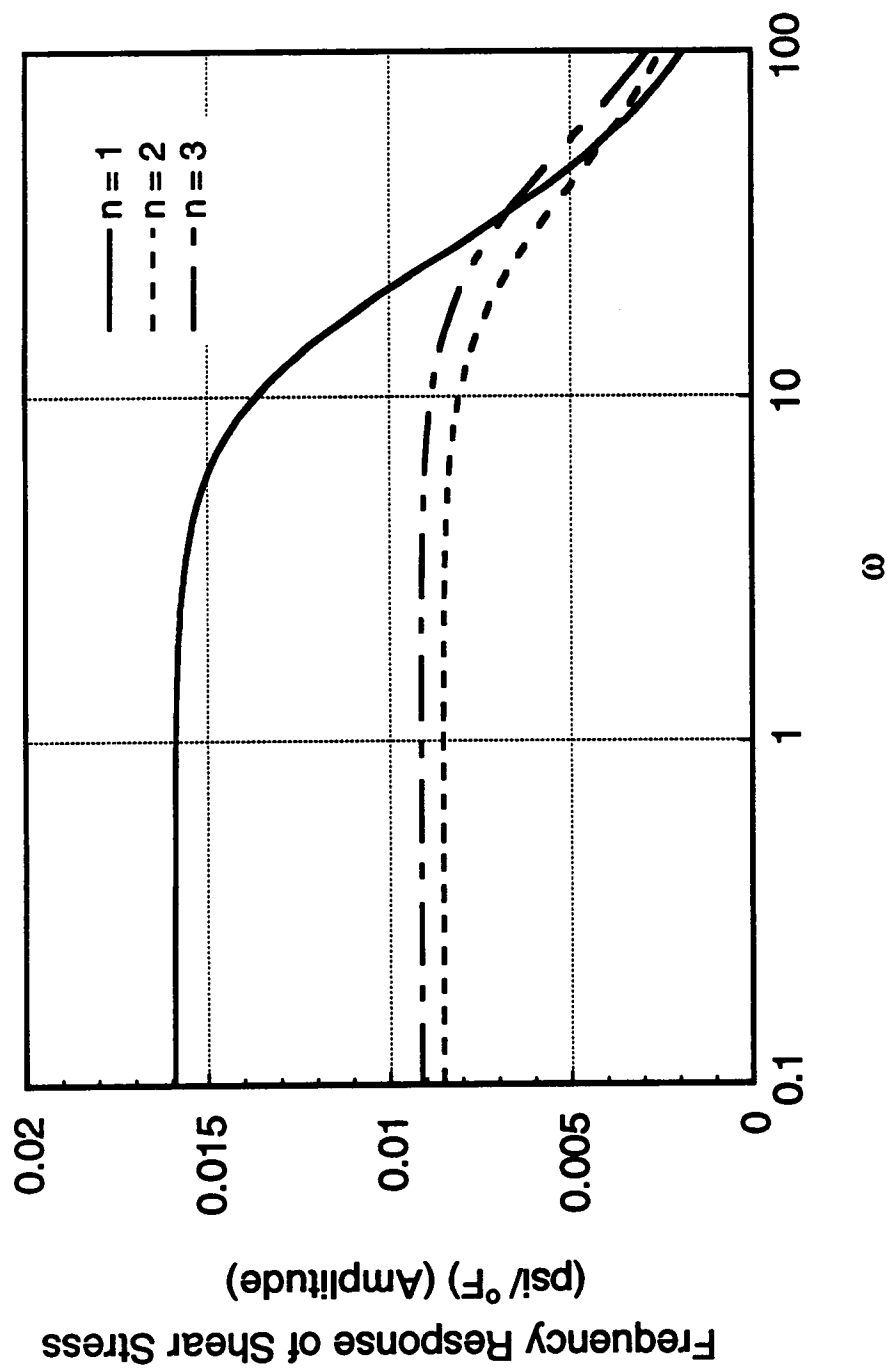


Figure 48. Frequency Responses of Shear Stress at r_2 .

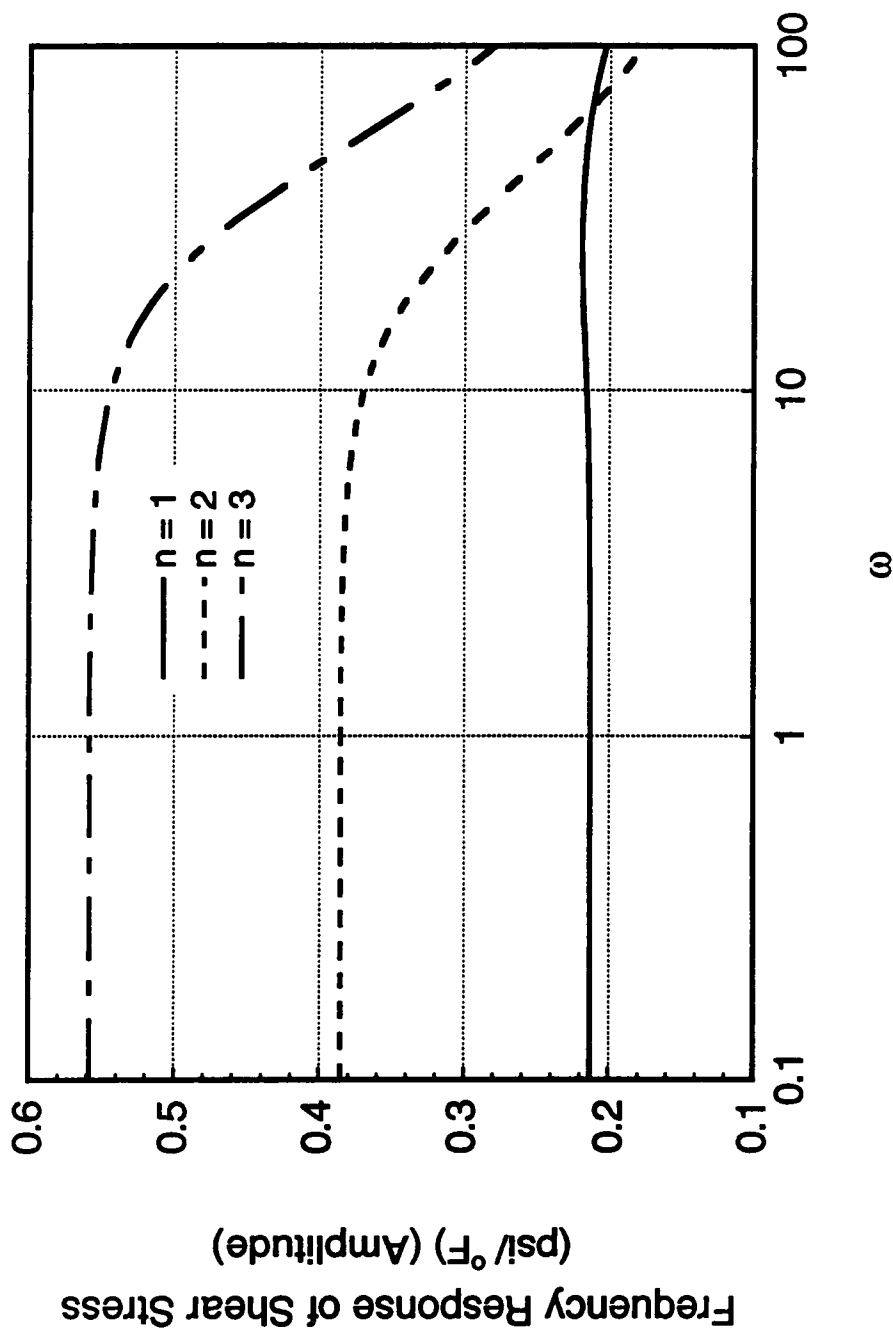


Figure 49. Frequency Responses of Shear Stress at r_3 .

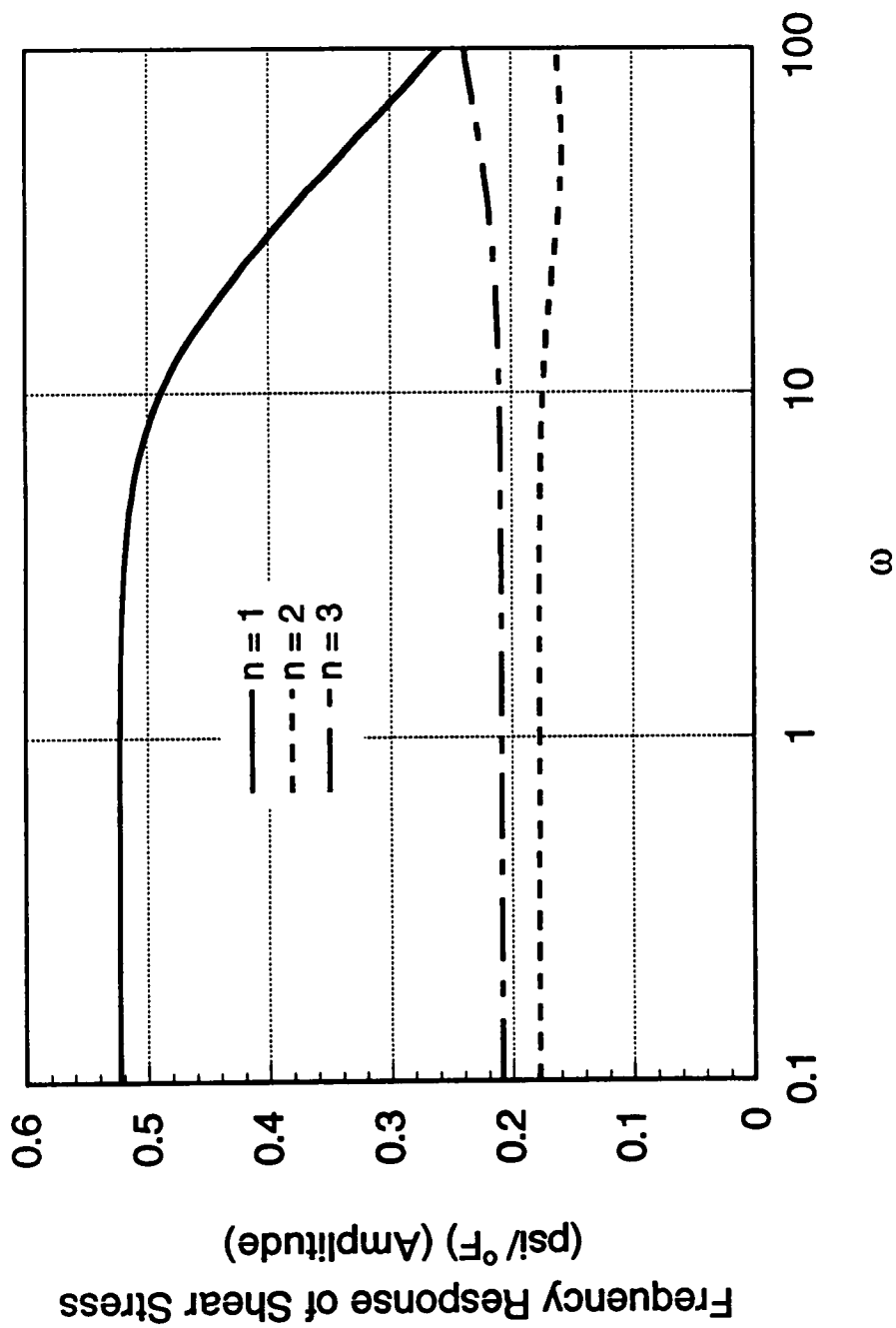


Figure 50. Frequency Responses of Shear Stress at r_4 .

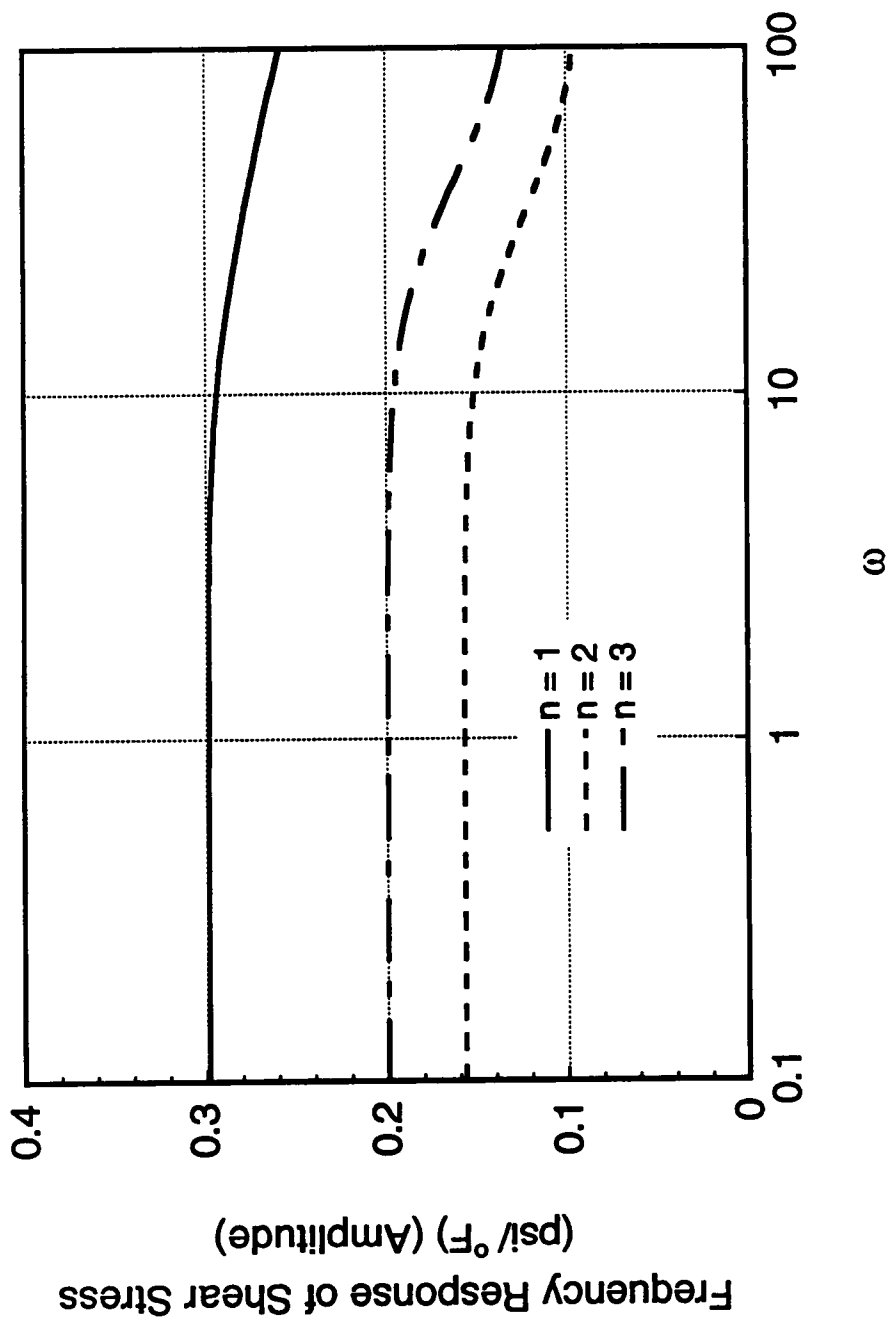


Figure 51. Frequency Responses of Shear Stress at r_5 .

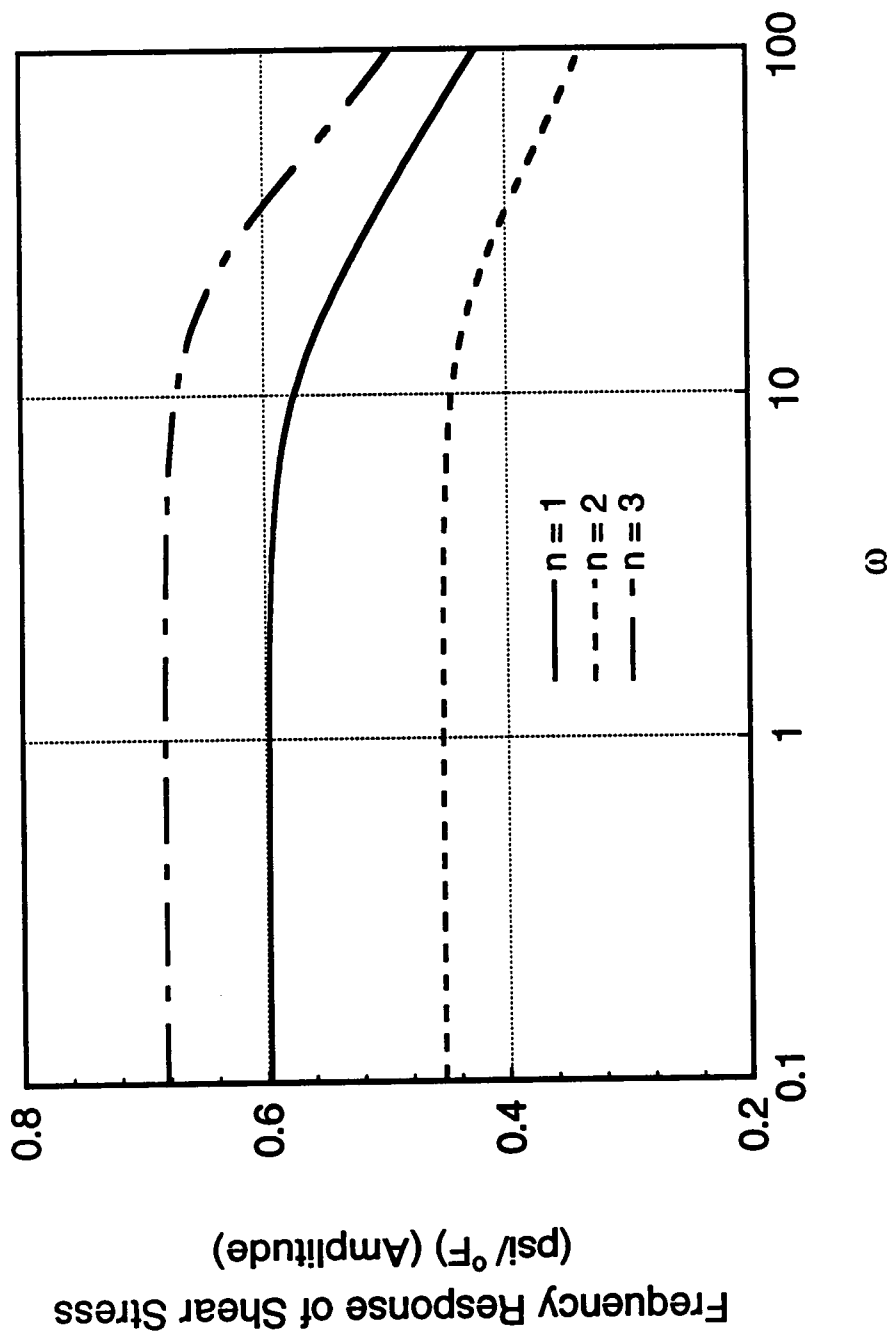


Figure 52. Frequency Responses of Shear Stress at r_6 .

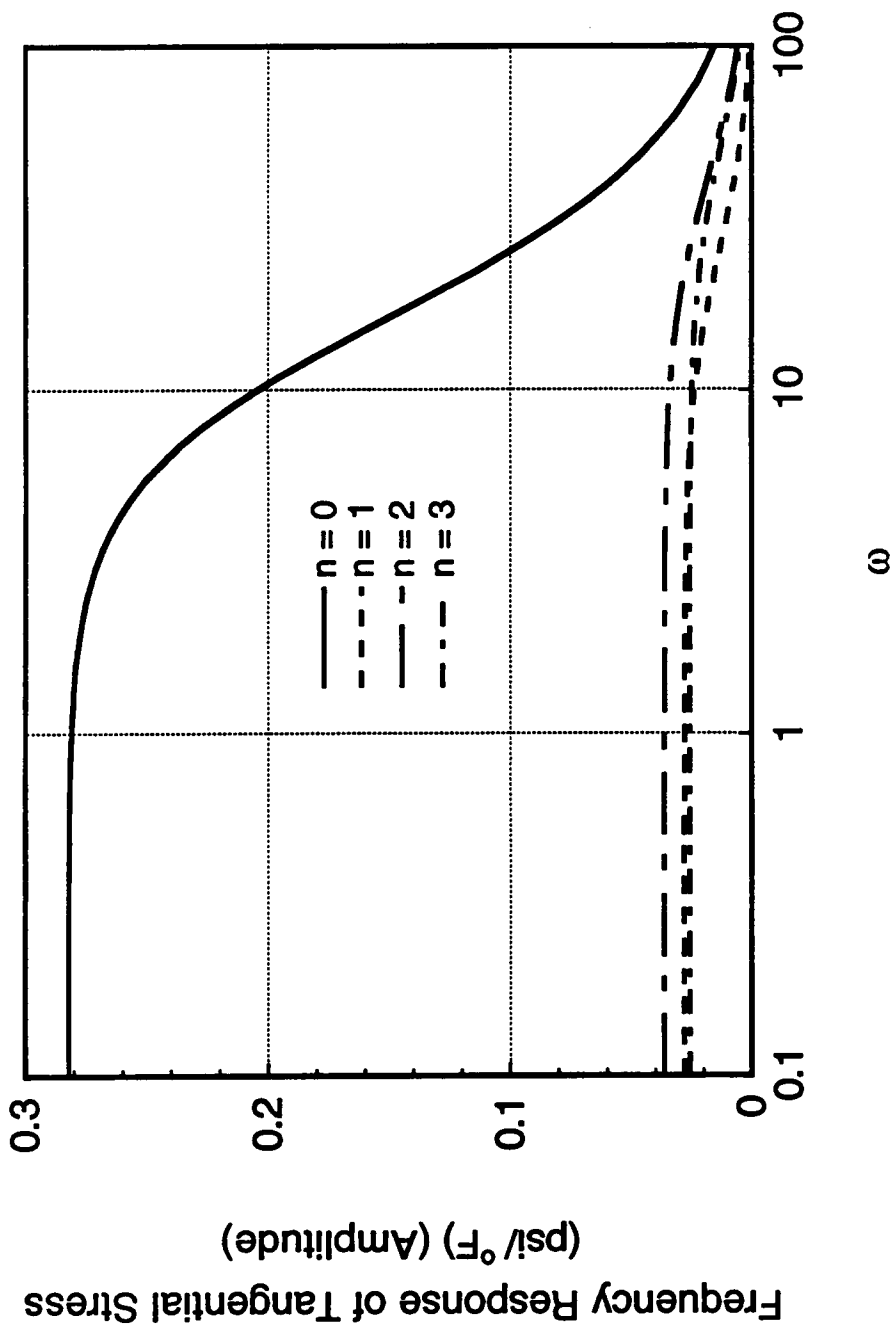


Figure 53. Frequency Responses of Tangential Stress at r_1 in Layer 2.

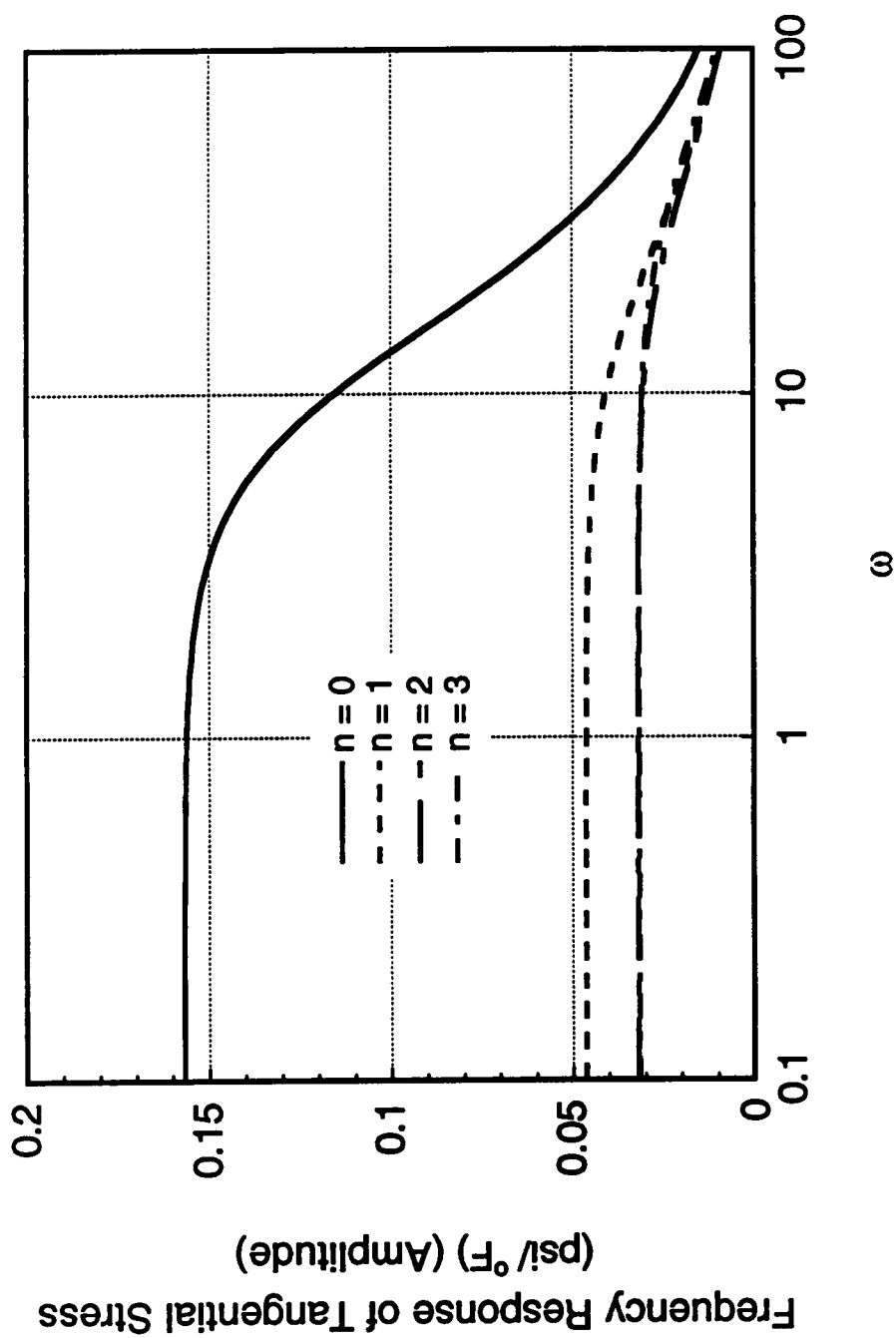


Figure 54. Frequency Responses of Tangential Stress at r_2 in Layer 2.

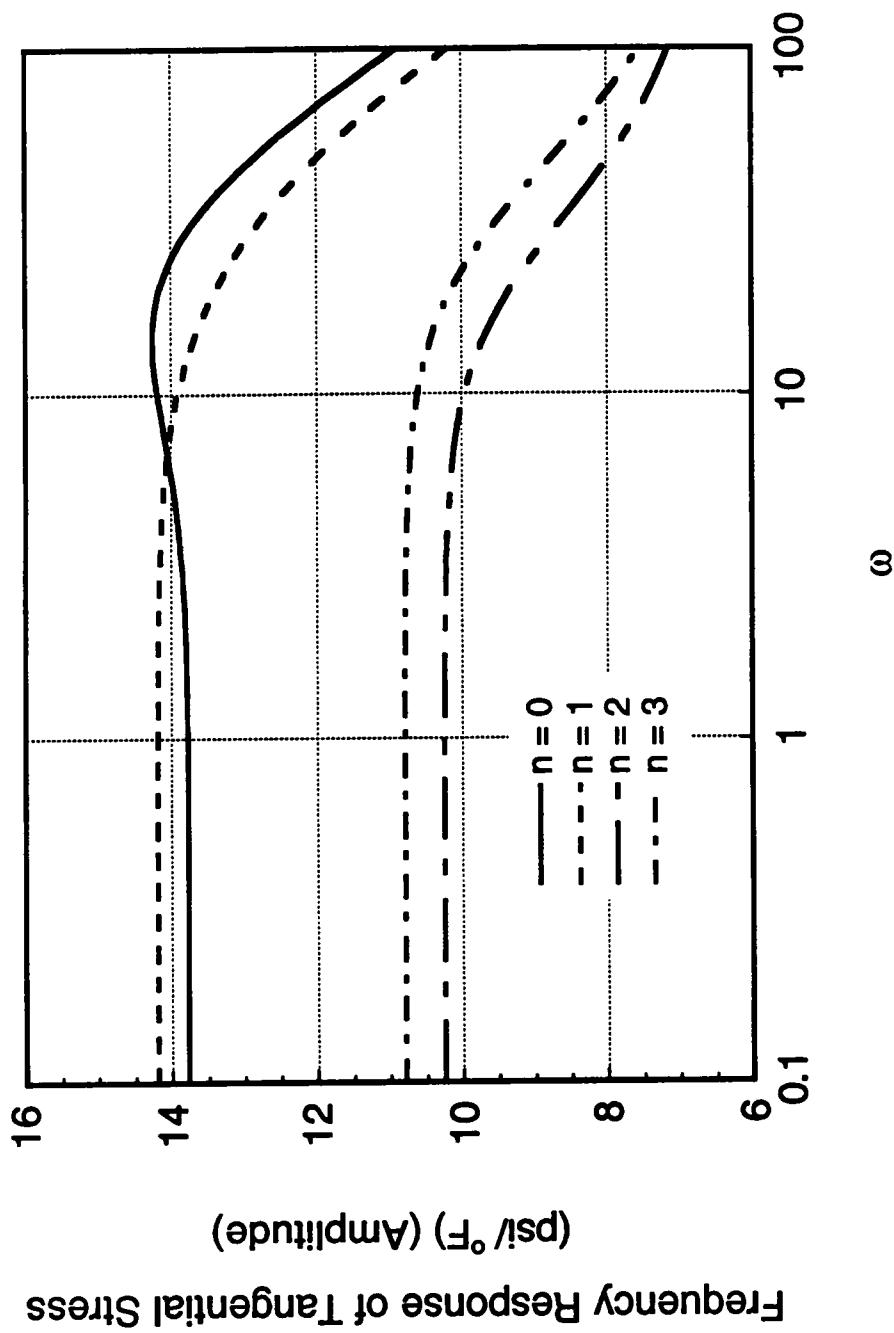


Figure 55. Frequency Responses of Tangential Stress at r_2 in Layer 3.

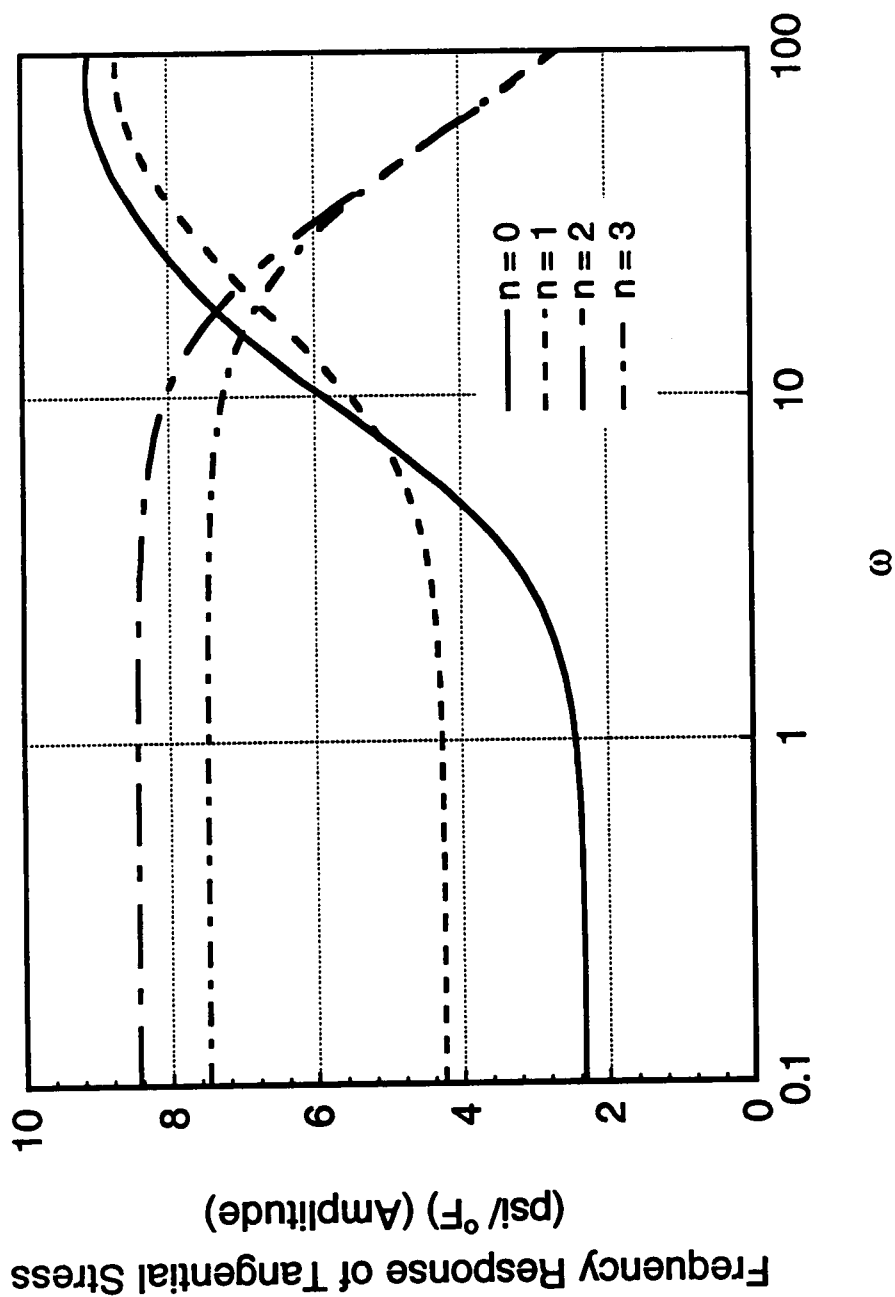


Figure 56. Frequency Responses of Tangential Stress at r_3 in Layer 3.

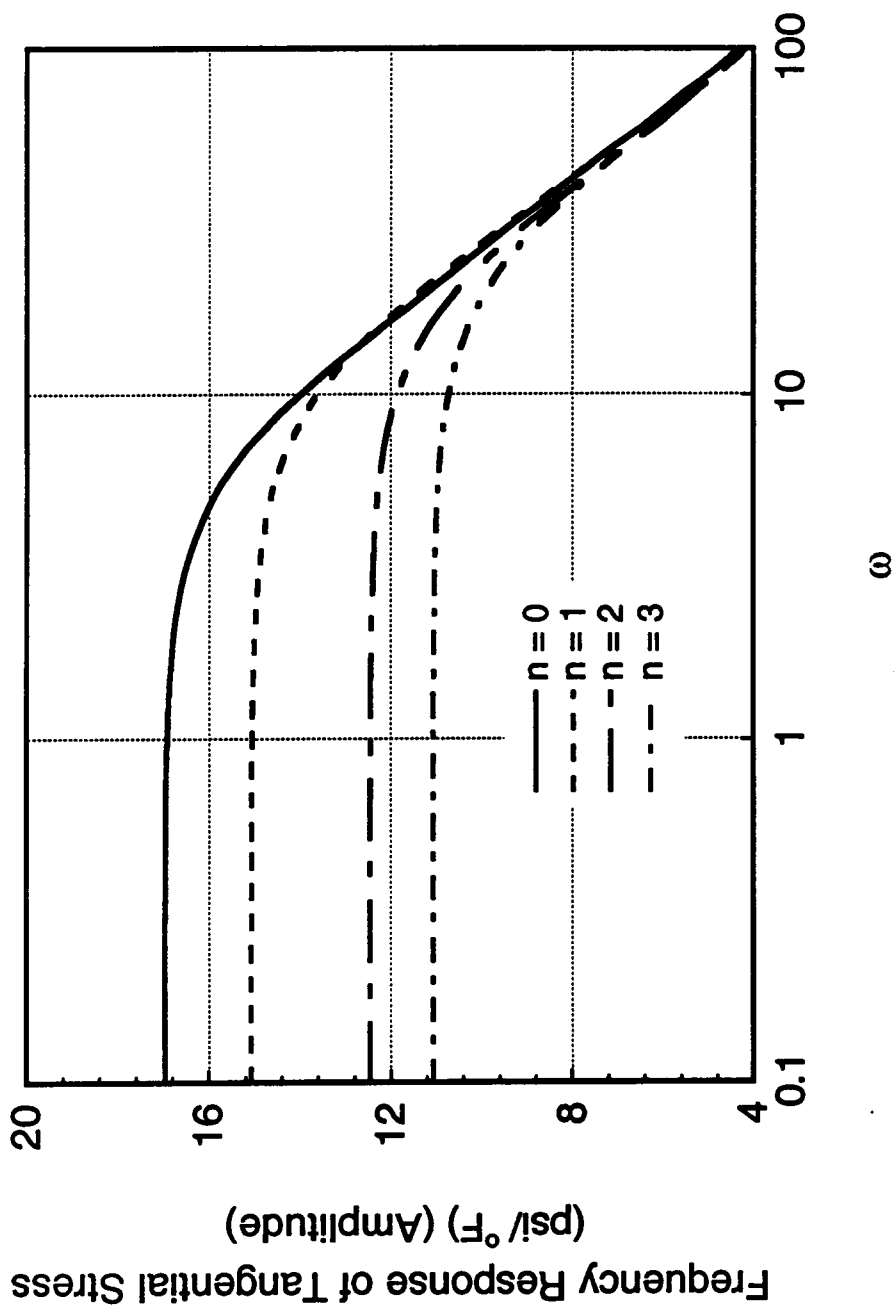


Figure 57. Frequency Responses of Tangential Stress at r_3 in Layer 4.

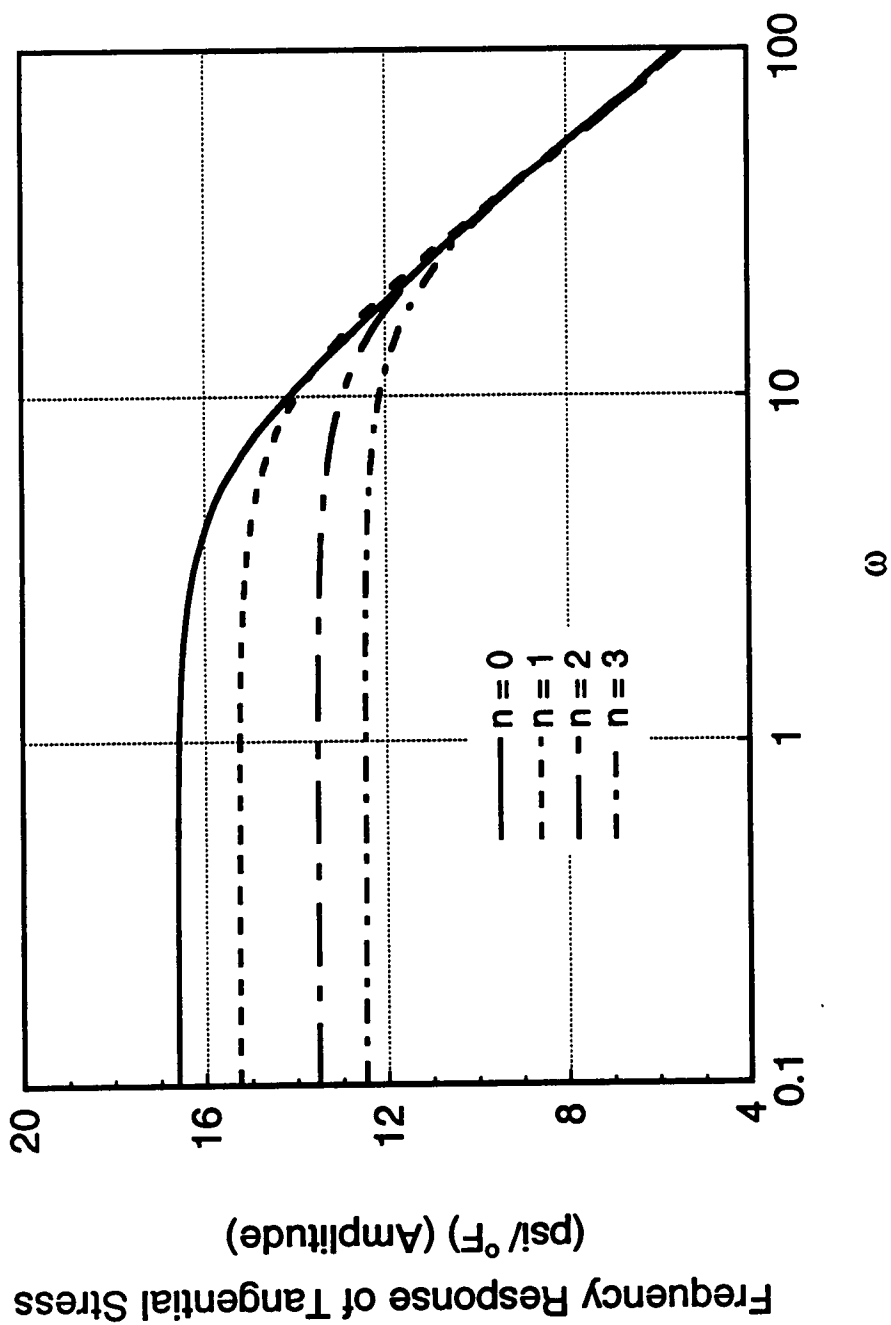


Figure 58. Frequency Responses of Tangential Stress at r_4 in Layer 4.

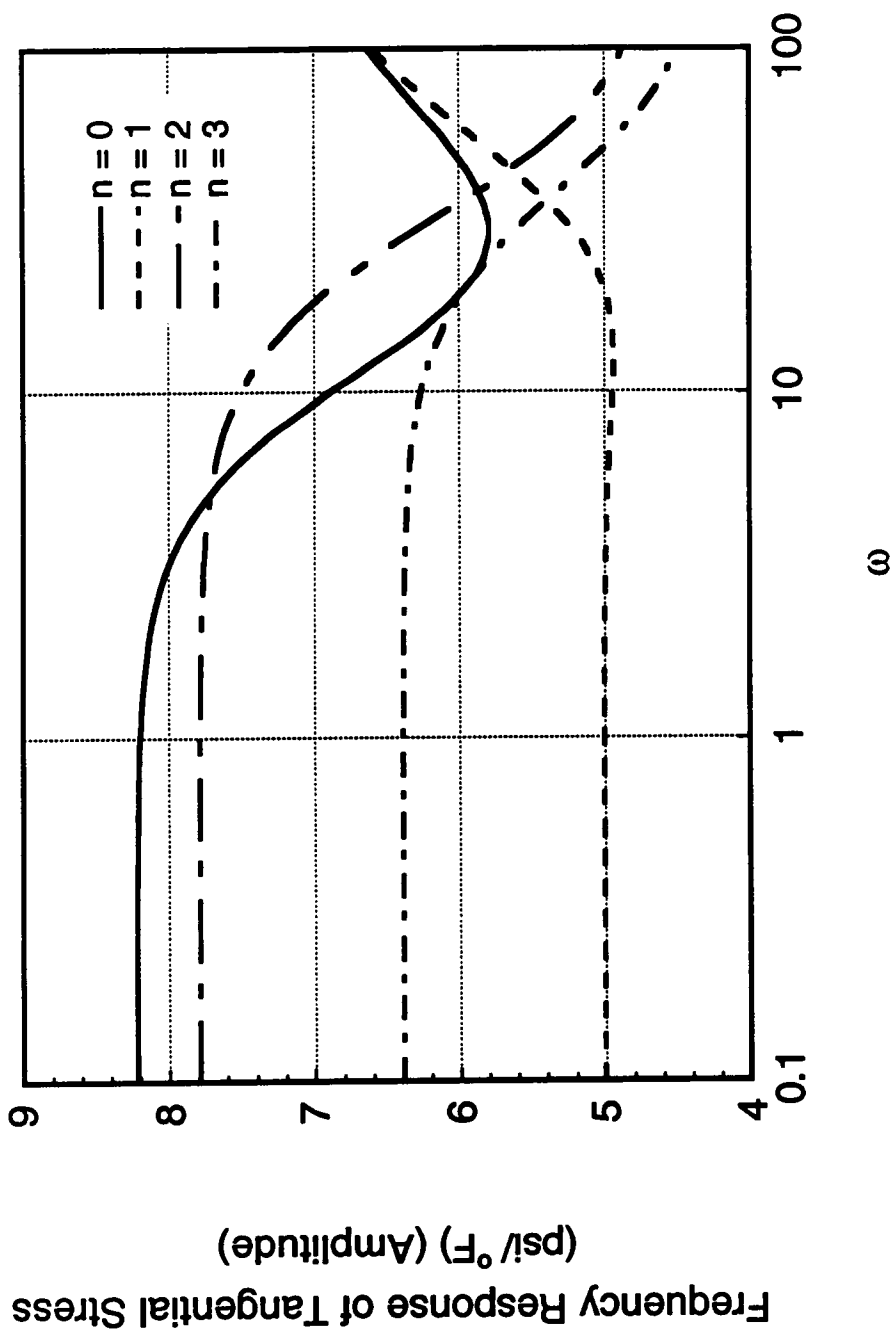


Figure 59. Frequency Responses of Tangential Stress at r_4 in Layer 5.

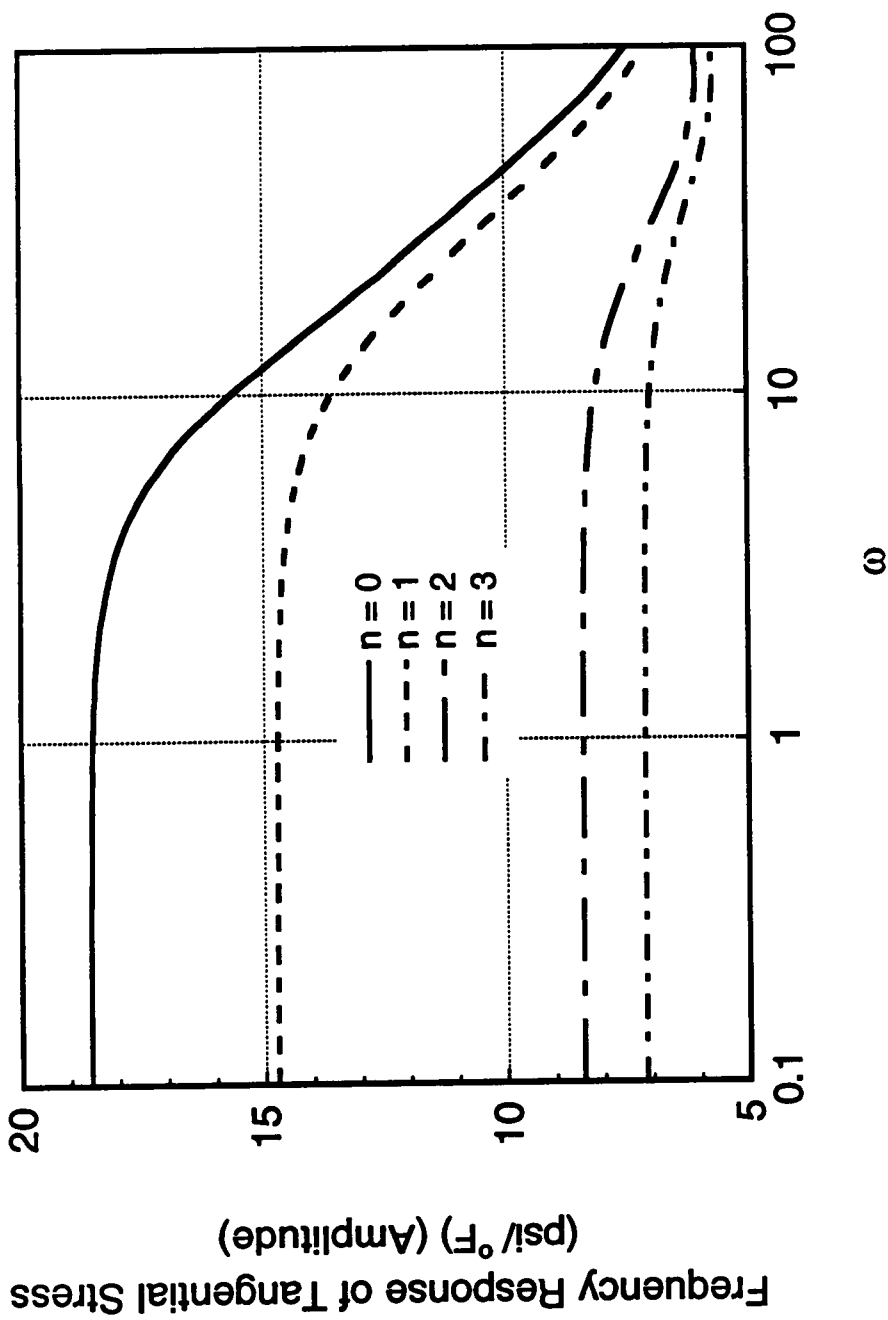


Figure 60. Frequency Responses of Tangential Stress at r_5 in Layer 5.

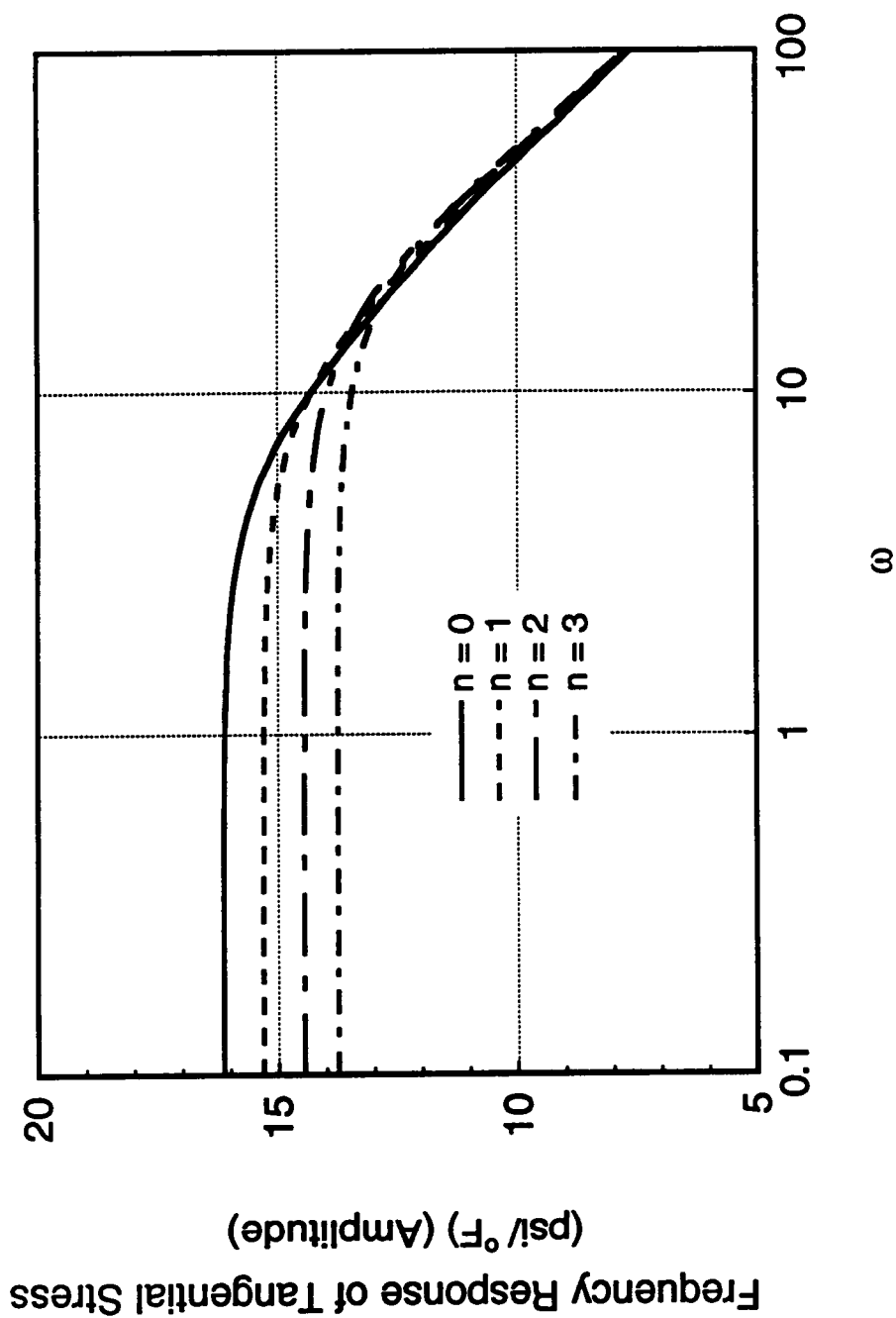


Figure 61. Frequency Responses of Tangential Stress at r_5 in Layer 6.

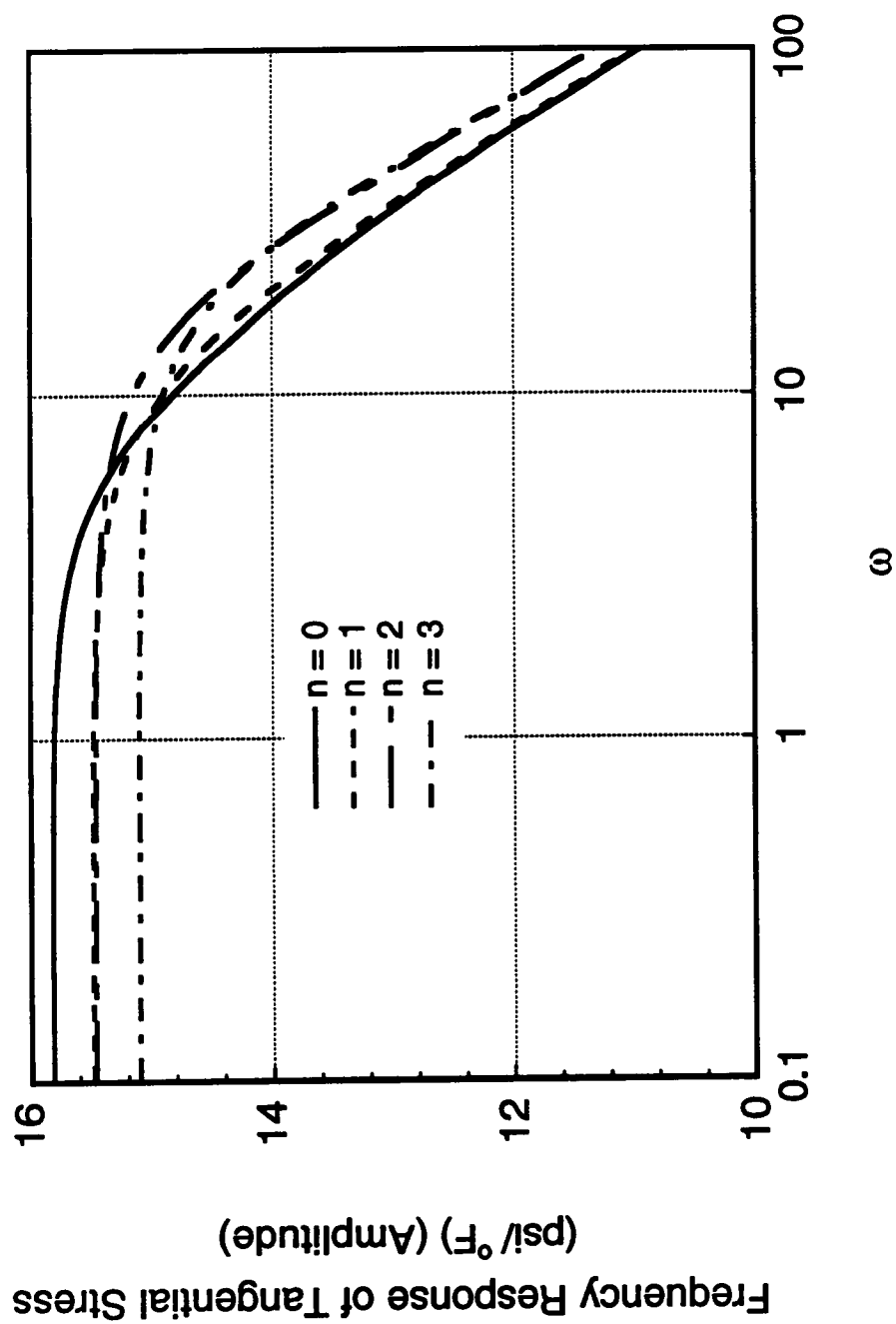


Figure 62. Frequency Responses of Tangential Stress at r_6 in Layer 6.

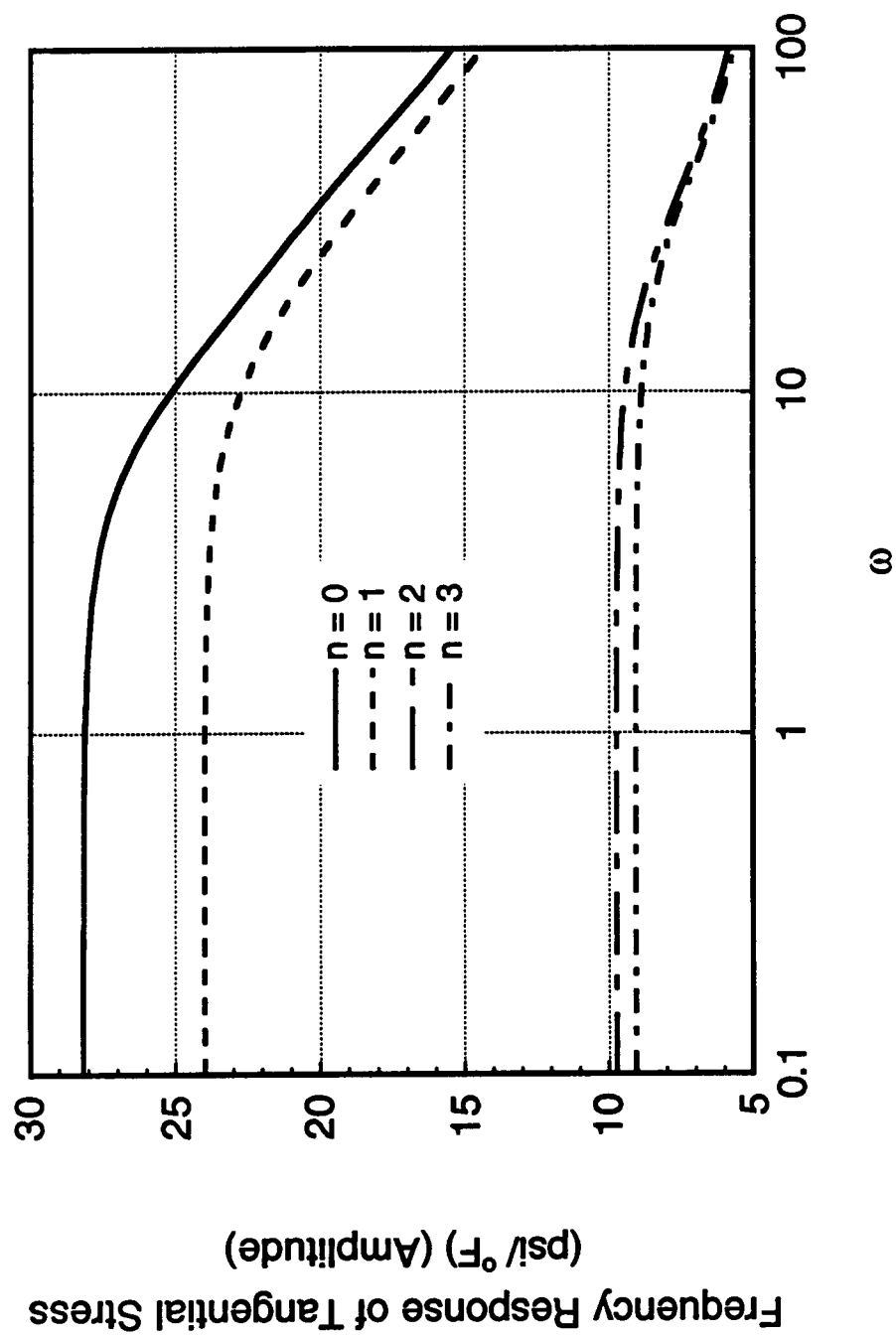


Figure 63. Frequency Responses of Tangential Stress at r_6 in Layer 7.

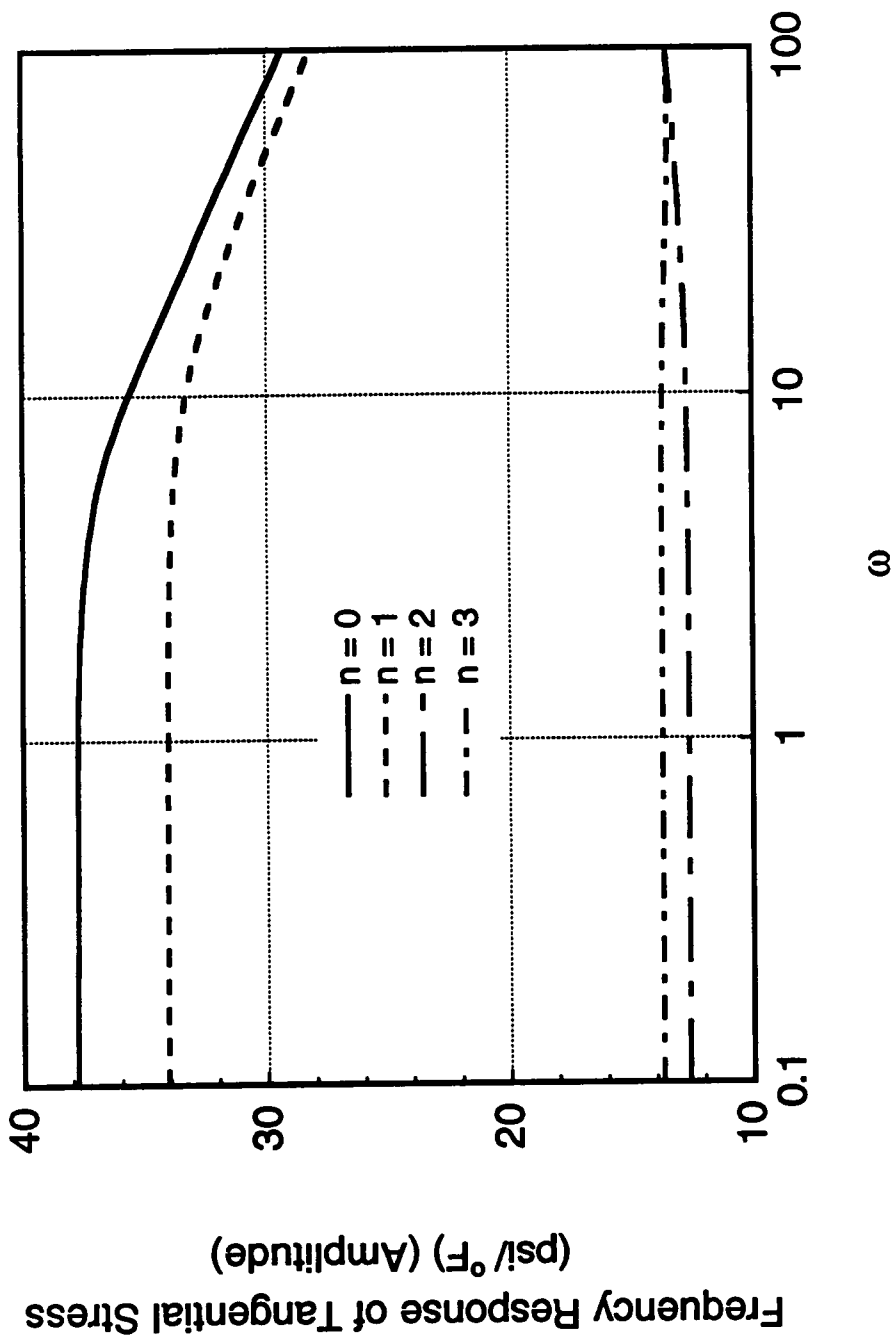


Figure 64. Frequency Responses of Tangential Stress at r_r in Layer 7.

5.0 ROCKET MOTORS UNDER ASYMMETRIC MECHANICAL LOADS

In this chapter, stresses and displacements of rocket motors subjected to line loads are analyzed by an elasticity approach. The motors are modeled as hollow, long, multilayered cylinders with an air core, isotropic propellant inner layer and orthotropic kevlar-epoxy composite case. There are two situations considered. In the first one the case is treated as a single orthotropic layer, while in the second the case is treated as a ten layered orthotropic material. The reliability of the rocket motor is analyzed by the first-order, second-moment method (FOSM). Five material properties and the load are treated as random variables with normal distributions. In order to verify the analytical results, the commercial finite element package ABAQUS is used to reanalyze the displacements and stresses of the cylinder.

5.1 Stress Analysis of Multilayered Orthotropic Cylinder

The rocket motor was modeled here as a long cylinder, that is, a plane strain problem, and the equilibrium equations expressed in terms of stresses are the same as Eq. 3.5

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} (\sigma_r - \sigma_\theta) + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r} \tau_{r\theta} = 0$$

where $\sigma_r(r, \theta, t)$, $\sigma_\theta(r, \theta, t)$ and $\tau_{r\theta}(r, \theta, t)$ are the radial, tangential and shear stress components, respectively.

The cylinders in this chapter are subjected to mechanical loads only, no temperature gradient is involved. By eliminating the temperature terms in Eq. 3.6, the stress-strain relations for an orthotropic plane strain problem in this chapter can be obtained as follows:

$$\sigma_r = C_{11} \varepsilon_r + C_{12} \varepsilon_\theta$$

$$\sigma_\theta = C_{12} \varepsilon_r + C_{22} \varepsilon_\theta \quad (5.1)$$

$$\tau_{r\theta} = C_{66} \gamma_{r\theta}$$

where $\varepsilon_r(r, \theta)$, $\varepsilon_\theta(r, \theta)$ and $\gamma_{r\theta}(r, \theta)$ are the radial, circumferential, and shear strain, respectively. The strain-displacement relations here are exactly the same as Eq. 3.10. The

equilibrium equations for a plane strain problem in cylindrical coordinates can be rewritten in terms of displacements by eliminating the temperature terms in Eq. 3.11

$$\begin{aligned}
 & C_{11} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + C_{66} \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - C_{22} \frac{u}{r^2} + (C_{12} + C_{66}) \frac{1}{r} \frac{\partial^2 v}{\partial r \partial \theta} \\
 & - (C_{22} + C_{66}) \frac{1}{r^2} \frac{\partial v}{\partial \theta} = 0 \\
 & C_{66} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right) + C_{22} \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} - C_{66} \frac{v}{r^2} + (C_{12} + C_{66}) \frac{1}{r} \frac{\partial^2 u}{\partial r \partial \theta} \\
 & + (C_{22} + C_{66}) \frac{1}{r^2} \frac{\partial u}{\partial \theta} = 0
 \end{aligned} \tag{5.2}$$

Eq. 5.2 is a set of homogenous partial differential equations. Displacements u and v are expanded into Fourier series, in a way similar to Section 3.1, with only homegenous parts

$$u = \sum_{n=-\infty}^{\infty} u_n(r) e^{in\theta}, \quad v = \sum_{n=-\infty}^{\infty} v_n(r) e^{in\theta} \tag{5.3}$$

The solutions of displacements are the same as the homogenous solutions obtained in Eqs. 3.13-15. By eliminating the temperature terms in Eqs. 3.16-18, the stresses are obtained as the follows:

for $n = 0$:

$$\sigma_{r0} = (C_{12} + C_{11}\lambda_{01})r^{\lambda_{01}-1}u_{01} + (C_{12} - C_{11}\lambda_{01})r^{-\lambda_{01}-1}u_{02}$$

$$\sigma_{\theta 0} = (C_{22} + C_{12}\lambda_{01})r^{\lambda_{01}-1}u_{01} + (C_{22} - C_{12}\lambda_{01})r^{-\lambda_{01}-1}u_{02} \quad (5.4)$$

$$\tau_{r\theta 0} \equiv 0$$

for $n = 1$:

$$\sigma_{r1} = \left(C_{11} - C_{12} \frac{C_{12} + C_{66}}{C_{22} + C_{66}} \right) \frac{u_{12}}{r}$$

$$- \lambda_{11} \frac{C_{12}(C_{12} + C_{66}) - C_{11}(C_{22} + C_{66}) + C_{11}C_{66}\lambda_{11}}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{11}} r^{\lambda_{11}-1} u_{13}$$

$$+ \lambda_{11} \frac{C_{12}(C_{12} + C_{66}) - C_{11}(C_{22} + C_{66}) - C_{11}C_{66}\lambda_{11}}{C_{22} + C_{66} + (C_{12} + C_{66})\lambda_{11}} r^{-\lambda_{11}-1} u_{14}$$

$$\sigma_{\theta 1} = \left(C_{12} - C_{22} \frac{C_{12} + C_{66}}{C_{22} + C_{66}} \right) \frac{u_{12}}{r}$$

$$- \lambda_{11} \frac{C_{66}(C_{22} - C_{12}) + (C_{12}^2 + C_{12}C_{66} - C_{11}C_{22})\lambda_{11}}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{11}} r^{\lambda_{11}-1} u_{13}$$

(5.5)

$$+ \lambda_{11} \frac{C_{66}(C_{22} - C_{12}) - (C_{12}^2 + C_{12}C_{66} - C_{11}C_{22})\lambda_{11}}{C_{22} + C_{66} + (C_{12} + C_{66})\lambda_{11}} r^{-\lambda_{11}-1} u_{14}$$

$$\begin{aligned}
\tau_{r\theta 1} = & iC_{66} \left(1 - \frac{C_{12} + C_{66}}{C_{22} + C_{66}} \right) \frac{u_{12}}{r} \\
& + i\lambda_{11} \frac{C_{12}(C_{12} + C_{66}) - C_{11}(C_{22} + C_{66}) + C_{11}C_{66}\lambda_{11}}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{11}} r^{\lambda_{11}-1} u_{13} \\
& - i\lambda_{11} \frac{C_{12}(C_{12} + C_{66}) - C_{11}(C_{22} + C_{66}) - C_{11}C_{66}\lambda_{11}}{C_{22} + C_{66} + (C_{12} + C_{66})\lambda_{11}} r^{-\lambda_{11}-1} u_{14}
\end{aligned}$$

for $n > 1$:

$$\sigma_{rn} = \sum_{m=1}^4 \left[C_{11}\lambda_{nm} + C_{12} \left(1 - \frac{C_{22} + C_{66}n^2 - C_{11}(\lambda_{nm})^2}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{nm}} \right) \right] r^{\lambda_{nm}-1} u_{nm}$$

$$\sigma_{\theta n} = \sum_{m=1}^4 \left[C_{12}\lambda_{nm} + C_{22} \left(1 - \frac{C_{22} + C_{66}n^2 - C_{11}(\lambda_{nm})^2}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{nm}} \right) \right] r^{\lambda_{nm}-1} u_{nm} \quad (5.6)$$

$$\tau_{r\theta n} = iC_{66} \sum_{m=1}^4 \left[n + \frac{1 + \lambda_{nm}}{n} \frac{C_{22} + C_{66}n^2 - C_{11}(\lambda_{nm})^2}{C_{22} + C_{66} - (C_{12} + C_{66})\lambda_{nm}} \right] r^{\lambda_{nm}-1} u_{nm}$$

The boundary conditions are the same as in Section 3.2, except that at $r = r_J$, on the outermost surface, BC.7 is replaced by

$$BC.7a \quad \sigma_r^J(r_J, \theta) = P_a(\theta)$$

where P_a is the asymmetric load applied on the outer surface of the cylinder.

5.2 Reliability Analysis

In general, common engineering systems, such as rocket motors, involve multiple random design variables, $X_1, X_2, \dots, X_n$. In the reliability analysis, the joint probability density function, $f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n)$, and the performance function, $g(X_1, X_2, \dots, X_n)$, are needed. When $g > 0$, the system is in a safe state; when $g < 0$, failure occurs. The equation, $g = 0$ is called limit-state equation (Ang and Tang, 1984). The probability of failure of the engineering system, p_f , is equal to the integral of the joint probability density function, $f_{X_1, X_2, \dots, X_n}$, over the region where the failure occurs, $g < 0$

$$p_f = \int \int \dots \int_{g < 0} f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \quad (5.7)$$

However, Eq. 5.7 is seldom used because of the following reasons. First, there are usually insufficient data to determine the joint probability density function and second, even if the joint density function is known, it is usually difficult to evaluate the integration in Eq. 5.7. In many practical situations, only the first two moments, the

means and variances of the random variables are available. The method of first-order second-moment (FOSM) is developed to evaluate the reliability of the engineering system by using these first two moments. The safety index of the system, β , is defined as the ratio of first-order approximations of the mean, μ_g , and the square root of variance (standard deviation), σ_g , of the function g evaluated at the most probable failure point (Ang and Tang, 1984).

$$\beta = \frac{\mu_g}{\sigma_g} \quad (5.8)$$

By introducing a set of reduced variables:

$$Y_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad (5.9)$$

where μ_{X_i} and σ_{X_i} are the mean and standard deviation of X_i , the first order approximations of the mean and variance of the function g can be obtained as

$$\mu_g = - \sum_{i=1}^n \hat{y}_i \partial g / \partial Y_i \quad \text{and} \quad \sigma_g^2 = \sum_{i=1}^n (\partial g / \partial Y_i)^2 \quad (5.10a,b)$$

where the derivatives, $\partial g / \partial Y_i$ are evaluated at $(\hat{y}_1, \hat{y}_2, \dots, \hat{y}_n)$, which is referred to the point $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ on the failure surface $g(X_1, X_2, \dots, X_n) = 0$ (Ang and Tang, 1984).

For a linear performance function, g , when all the random variables, $X_1, X_2, \dots, X_n$, are uncorrelated and normally distributed, the safety index, β , and the probability of failure, p_f , are expressed as standard normal distribution function, $\Phi(\cdot)$ as the follows

$$p_f = 1 - \Phi(\beta) \quad (5.11)$$

5.3 Application to a Particular Motor Geometry

The reliability of a motor consisting of an air core, an isotropic propellant layer, and thin orthotropic kevlar-epoxy composite as the case, subjected to 100 lb/in of line load, will be investigated. The configuration are shown in Figure 65, and the different supports of the Motor are shown in Figure 66.

In the following, two situations will be considered: in the first, the case is assumed to consist of a single, orthotropic, kevlar-epoxy layer, while in the second, the case is a ten layer composite. Geometric and mechanical parameters are shown in Tables 5 and 6. The total thickness of the ten layered case in the second situation is the same as that of the single layer in the first situation.

In contrast to the kevlar-epoxy case, which has essentially the same elastic modulus under both static and dynamic loads, the propellant has different elastic moduli. When the rocket motor is subjected to dynamic load, the elastic modulus of the propellant is larger than when the rocket motor is subjected to a static load. Based on the speed of loading and on the known properties of the propellant (Heller and Janajreh, 1992), the static and dynamic moduli are also listed in Table 5.

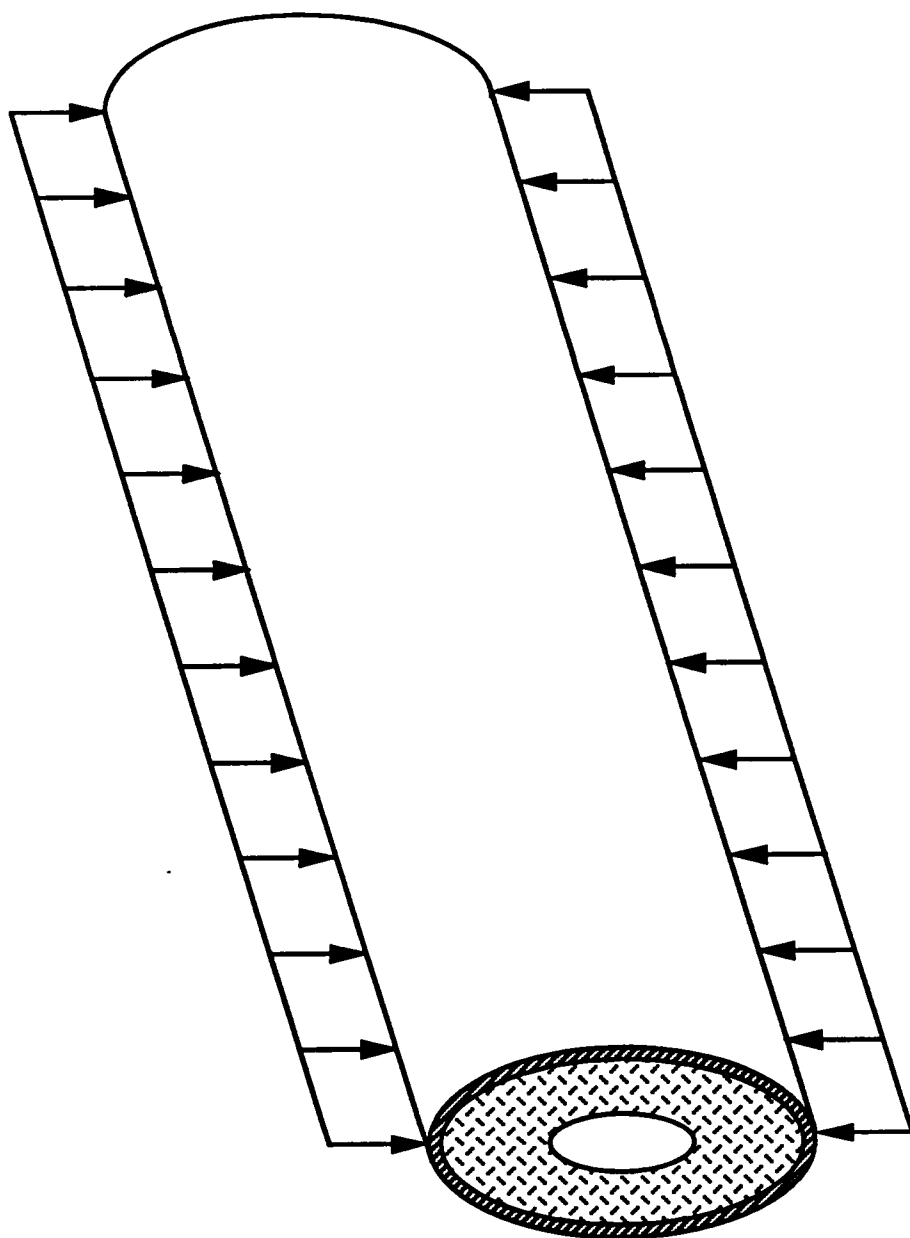


Figure 65. Configuration of Motor under Line Load.

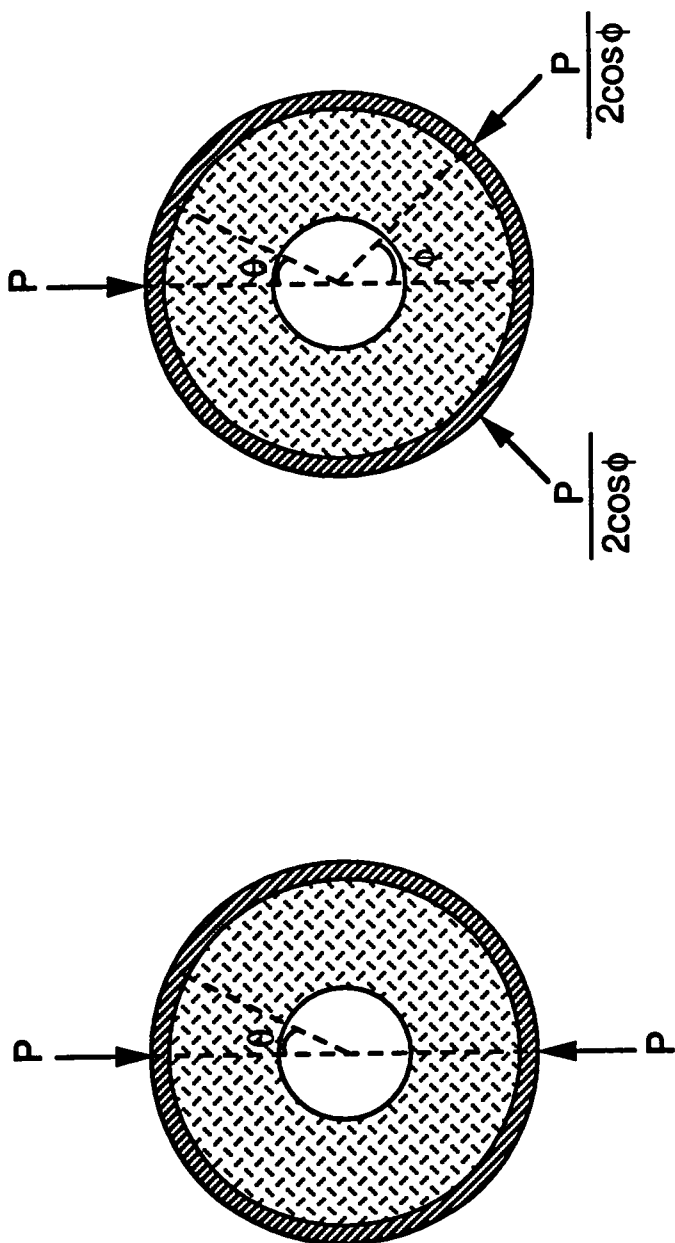


Figure 66. Different Supports for the Motor.

Table 5.
Geometric and Mechanical Parameters for Rocket Motor in the Situation 1.

Material	Air Core	Propellant	Kevlar-Epoxy
Thickness (in)	0.75	0.8775	0.06
Elastic Modulus (Static), (psi)			
E_r	---	300	0.578×10^6
E_θ	---	300	2.452×10^6
E_z	---	300	1.865×10^6
Elastic Modulus (Dynamic), (psi)			
E_r	---	1072	Same as Above
E_θ	---	1072	
E_z	---	1072	
Shear Modulus, (psi)			
$G_{r\theta}$	---	*	0.337×10^6
G_{rz}	---	*	0.211×10^6
$G_{\theta z}$	---	*	0.992×10^6
Poisson's Ratio			
$\nu_{r\theta}$	---	0.49	0.014
ν_{rz}	---	0.49	0.09
$\nu_{\theta z}$	---	0.49	0.18

* For isotropic propellant, the shear modulus, G , can be defined as:

$$G = \frac{E}{2(1 + \nu)}$$

Table 6.
Geometric and Mechanical Parameters for the Case in the Situation 2.

Material	Odd-Layer	Even-Layer
Thickness (in)	0.006	0.006
Elastic Modulus, (psi)		
E_r	0.578×10^6	0.578×10^6
E_θ	1.865×10^6	2.452×10^6
E_z	2.452×10^6	1.865×10^6
Shear Modulus, (psi)		
$G_{r\theta}$	0.211×10^6	0.337×10^6
G_{rz}	0.337×10^6	0.211×10^6
$G_{\theta z}$	0.992×10^6	0.992×10^6
Poisson's Ratio		
$\nu_{r\theta}$	0.090	0.014
ν_{rz}	0.014	0.090
$\nu_{\theta z}$	0.137	0.180

5.3.1 Displacement and Stress Analysis of Motor Subjected to Line Load

The results of the stress and displacement analysis for a two-layered (case and propellant) long cylinder are plotted in Figures 67-74. The displacements, u , in the r direction of the rocket motor at various locations and for different elastic moduli are shown in Figures 67 and 68, where E_s and E_d are the static and dynamic elastic moduli. The maximum displacement, u , occurs at a location right under the load (

$\theta = 0$), and is larger for the static case than for the dynamic case. The radial stress, σ_r , and the shear stress, $\tau_{r\theta}$, shown in Figures 69 and 70 are very small. The shear stress, $\tau_{r\theta}$, is less than 0.21 psi, and can be neglected. Compared with the radial stress, σ_r , and shear stress, $\tau_{r\theta}$, the tangential stress, σ_θ , is much larger. Tangential stresses at various locations are shown in Figures 71-74. The maximum tensile tangential stress, σ_θ , occurs under the load; for the propellant, at the bore, while for the kevlar case, it occurs at the interface. In the propellant, the maximum σ_θ is larger for the dynamic modulus, E_d , than for the static modulus, E_s , but the situation is reversed for the case. The stresses in the case are larger, but of course the case is stronger than the propellant.

Experiments have shown that the motors fail at the bore. Consequently the following reliability analysis can be carried out at the bore. To consider bond line failures, reliability analysis can also be performed at the interface.

The results of the stress and displacement analysis for the eleven-layer cylinder are plotted in Figures 75-82. The displacements at the inner and outer surfaces plotted in Figures 75 and 76 show that, the results are very close to the results of the two-layer cylinder. Similarly to Figures 69 and 70, Figures 77 and 78 show that radial stresses and shear stresses are also very small here. Tangential stresses on the inner and outer surfaces of the propellant are shown in Figures 79 and 80. The maximum positive tangential stresses occur at the inner surface of the third layer. This is the second kevlar-epoxy layer and has a high E_θ . Maximum negative tangential stresses occur at the outermost surface (Figures 81 and 82). All the above plots show that, the results of displacements and stresses for two-layer and eleven-layer cylinders are very close to each other.

To save computing time, the following analysis is, therefore, concentrated on two-layer cylinders.

When the motor support is not opposite to the applied load, but at angles ϕ from the vertical line, as shown in Figure 66, the tangential stresses at the bore and $\theta = 0$ are plotted in Figure 83 for the propellant with static and dynamic elastic moduli. When ϕ is larger than 60° , the tangential stresses at this location change from tension to compression, and the compressive tangential stresses increase very rapidly when ϕ is larger than 70° to infinity when $\phi = 90^\circ$. When motors are stacked, ϕ is 60° . The tangential stresses at the bore are replotted in Figure 84 on an expanded scale for ϕ in the range of 0 to 60° . Figures 85 and 86 show that the maximum tangential stresses for E_s and E_d at the bore still occur at $\theta = 0^\circ$. The graphs indicate that the maximum tangential stress at the bore decreases as ϕ increases when ϕ is in the range from 0 to 60° , which means that the reliability at this location increases with ϕ .

5.3.2 Reliability Analysis of Motor Subjected to Line Load

There are 12 material properties considered in the reliability analysis: elastic modulus and Poisson's ratio for the propellant, nine independent elastic properties for the orthotropic case, and the strengths of the propellant and the case. The variability of some of these material properties does not significantly effect the reliability analysis. If they are treated as random variables in the analysis, a lot of unnecessary computing time will be consumed. In order to decide which material properties play

important roles, and which have minimum effects on reliability and on probability of failure, the following parameter study was performed. It is known, that the elastic modulus and the strength of the propellant play important roles on the reliability of the motor. Assuming the above two material properties as random variables with normal distributions and coefficients of variation (COV) of 0.1, the design points, safety index, reliability and probability of failure are calculated and are listed in Table 7.

Next, assuming the elastic modulus of the case in the r direction also as a random variable with normal distribution, with coefficients of variation (COV) of 0.1, 0.2, 0.3, 0.4, and 0.5, the reliability analysis results in Table 8 show that this modulus won't effect the reliability and, therefore, this material property can be treated as a fixed parameter at its mean value.

By the same procedure, it was found that, only the following five material properties: elastic modulus and strength of the propellant, elastic modulus of the case in the θ and axial directions, shear modulus of case in the $r\theta$ plane, and the load are to be treated as random variables.

All material properties have been obtained from experimental results (Heller and Heller, 1989, Chamis, 1983). The effects of load variations are examined by varying both its range and its statistical properties. Load ranges are best expressed in terms of the Safety Factor, v

$$v = \frac{\bar{R}}{\sigma_{\max}} \quad (5.12)$$

Table 7.
Reliability Analysis by the First-Order Second-Moment Method (Two Random Variables).

Random Variable	Mean (psi)	Variation	Safety Index	Probability of Failure
Strength (Propellant)	319.5	0.1		
E (Propellant)	1072.5	0.1	3.7577	0.8573×10^{-4}

where $\bar{R}$ is the mean of the strength and $\sigma_{\max}$ is the maximum induced stress. According to the above stress analysis, the maximum stress is tangential and occurs in the propellant, at the bore, under the load, and in the case at the interface under the load. Under an applied load of $P_0 = 100$ lb/in, the maximum stress $\sigma_{\max 0}$ for the propellant is 195.96 psi, and 28.749 ksi for the case. For any load, P , and maximum stress, $\sigma_{\max}$, the following relation exists:

$$P_0 : \sigma_{\max 0} = P : \sigma_{\max} \quad (5.13)$$

From Eq. 5.12, the maximum stress can be expressed as

$$\sigma_{\max} = \frac{\bar{R}}{v}$$

Substituting this into Eq. 5.13 the relation between load, P , and safety factor, v , can be obtained as the follows:

Table 8.
Reliability Analysis by the First-Order Second-Moment Method (Three Random Variables).

Random Variable	Mean (psi)	Variation	Safety Index	Probability of Failure
Strength (Propellant)	319.5	0.1		
E (Propellant)	1072.5	0.1		
E_c (Case)	0.5777×10^6	0.1	3.7577	0.8573×10^{-4}
		0.2	3.7577	0.8573×10^{-4}
		0.3	3.7577	0.8573×10^{-4}
		0.4	3.7577	0.8573×10^{-4}
		0.5	3.7577	0.8573×10^{-4}

$$P = P_0 \frac{\sigma_{\max}}{\sigma_{\max 0}} = \frac{P_0 \bar{R}}{\sigma_{\max 0} \nu} \quad (5.14)$$

The mean dynamic strength of the propellant is 319.5 psi. When the safety factor, ν , changes from 1.0 to 4.0, the load, P , changes from 163.04 lb/in to 40.76 lb/in. The mean strength of the case is 67.213 ksi. When the safety factor, ν , varies between 1.0 and 4.0, the load, P , changes from 233.80 lb/in to 58.45 lb/in.

All six random variables are assumed to be normally distributed with the coefficient of variation of the load, P , as 15% and the other five random variables with identical coefficients of variation of 10% or 5%. The mean values of the five material properties are listed in Table 5 and the mean of the load is changed with the Safety Factor.

The safety index, β , and probability of failure, P_f , for propellant and case vary with the safety factor, ν , and the load, P . For different supporting angles, ϕ , these are plotted in Figures 87-94. It can be seen that, the coefficient of variation of the material properties plays an important role in the reliability (probability of failure) of the cylinders. It is shown that when the supporting angle is small, between 0° and 15° , its effect on the reliability is very small, but when this angle is large, between 30° to 45° , the reliability is improved. It is also shown that the safety index and the probability of failure for the propellant is more sensitive to supporting angle, ϕ , than the case.

5.3.3 Displacement and Stress Analysis of Motor Subjected to Line Load by ABAQUS.

In order to check the obtained results, the displacement and stress distributions are reanalyzed by the commercial finite element package ABAQUS. All the geometries, material properties and loads are the same as for the cylinder of situation 1 discussed above, and only the dynamic elastic modulus of the propellant is considered here. Because the structure and the load are symmetrical to the vertical ($\theta = 0$) and horizontal ($\theta = 90^\circ$) axes, a quarter of the cylinder is used in the analysis. The finite element mesh is shown in Figure 95. The displacements and stresses are computed by using four node bilinear elements, which are denoted as CPE4 in ABAQUS. There are a total of 690 node points and 638 elements.

The undeformed and deformed shapes are shown in Figure 96. The ABAQUS results are plotted by its pre-post processing package PATRAN in contour form in Figures 97-104. These figures show that the results of displacements and stresses are in good agreement at most locations with the analytic method.

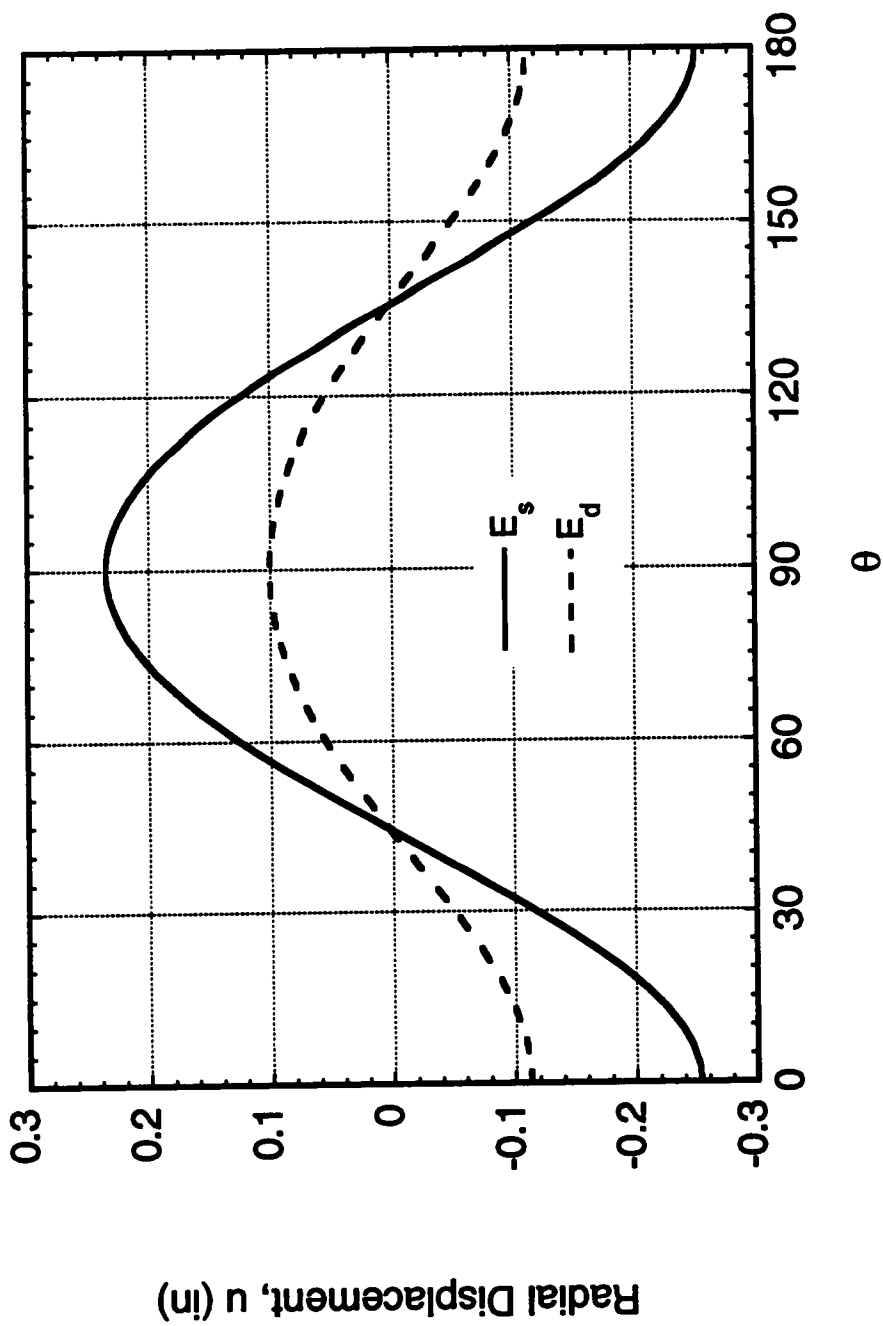


Figure 67. Radial Displacement, u , at the Bore (Situation 1).

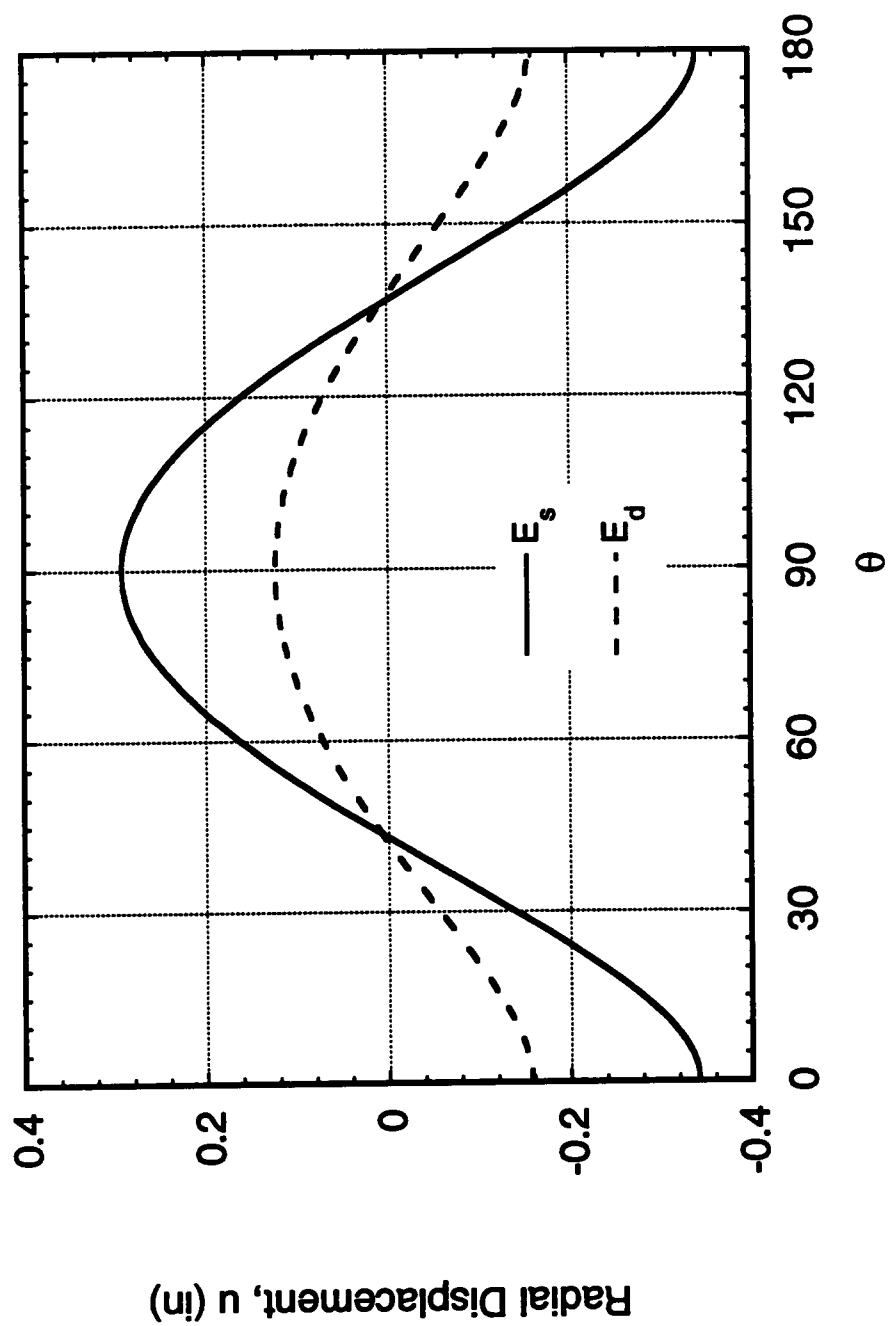


Figure 68. Radial Displacement, u , at the Outer Surface (Situation 1).

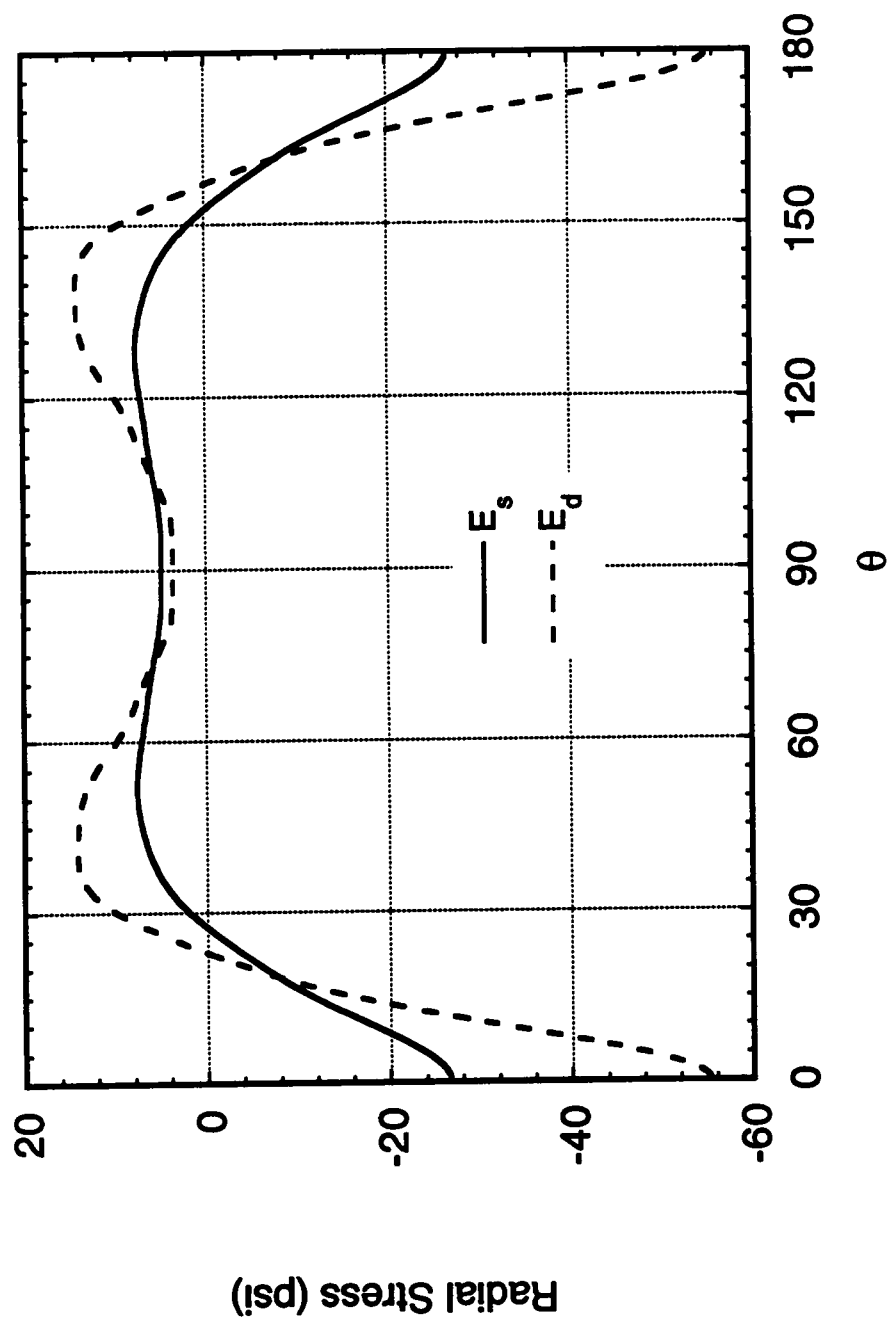


Figure 69. Radial Stresses at the Interface (Situation 1).

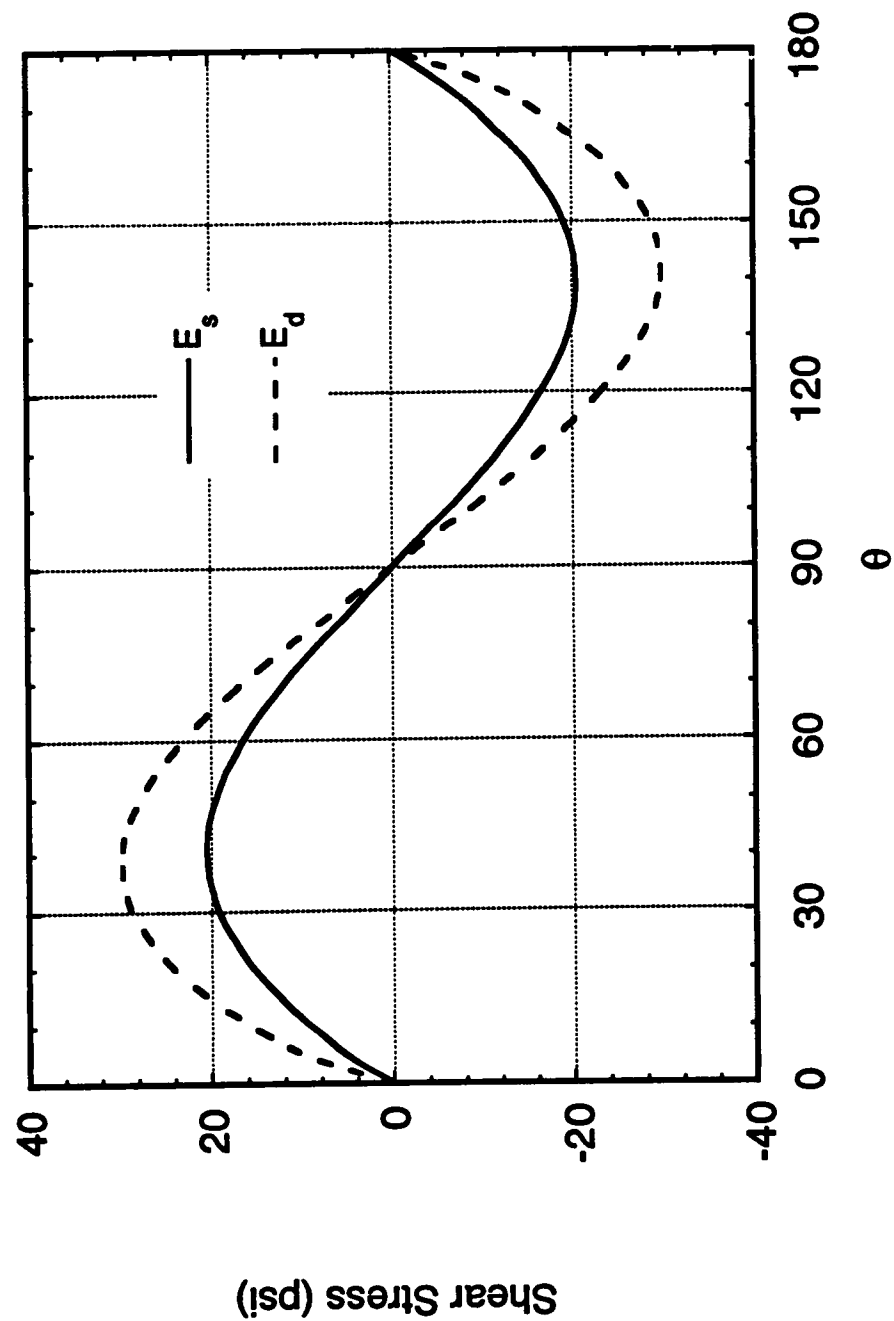


Figure 70. Shear Stresses at the Interface (Situation 1).

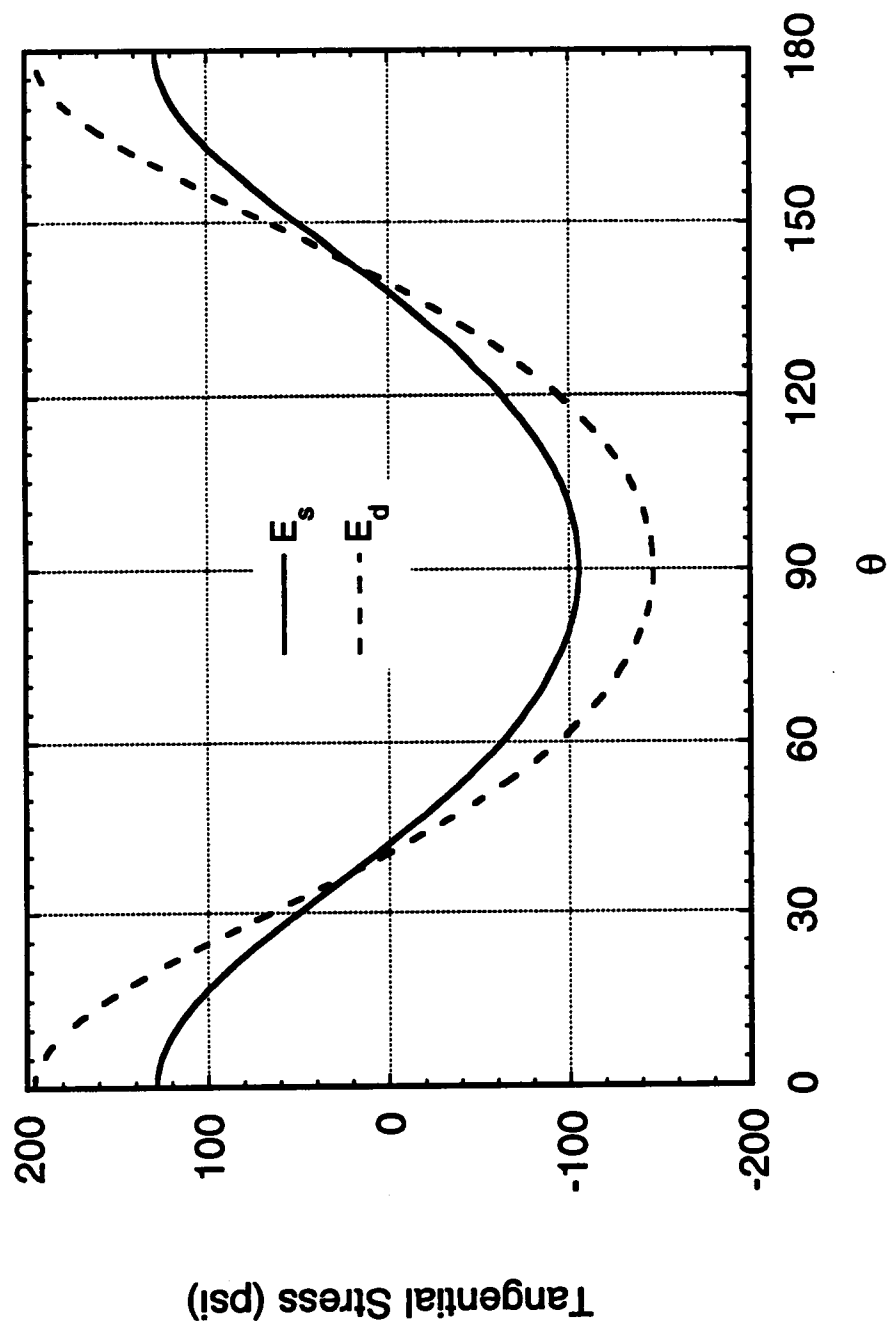


Figure 71. Tangential Stresses at the Bore (Situation 1).

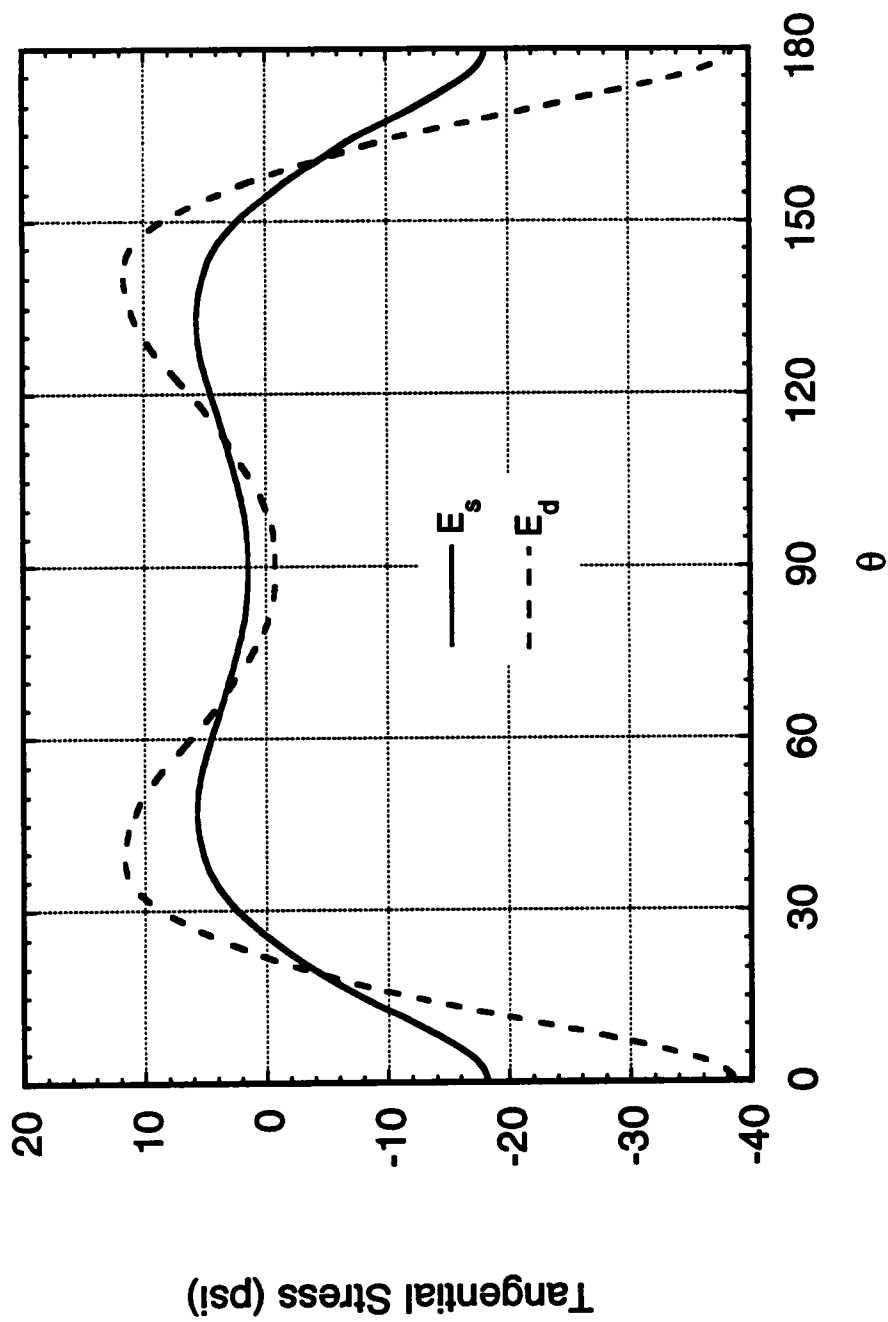


Figure 72. Tangential Stresses at the Interface (in the Propellant, Situation 1).

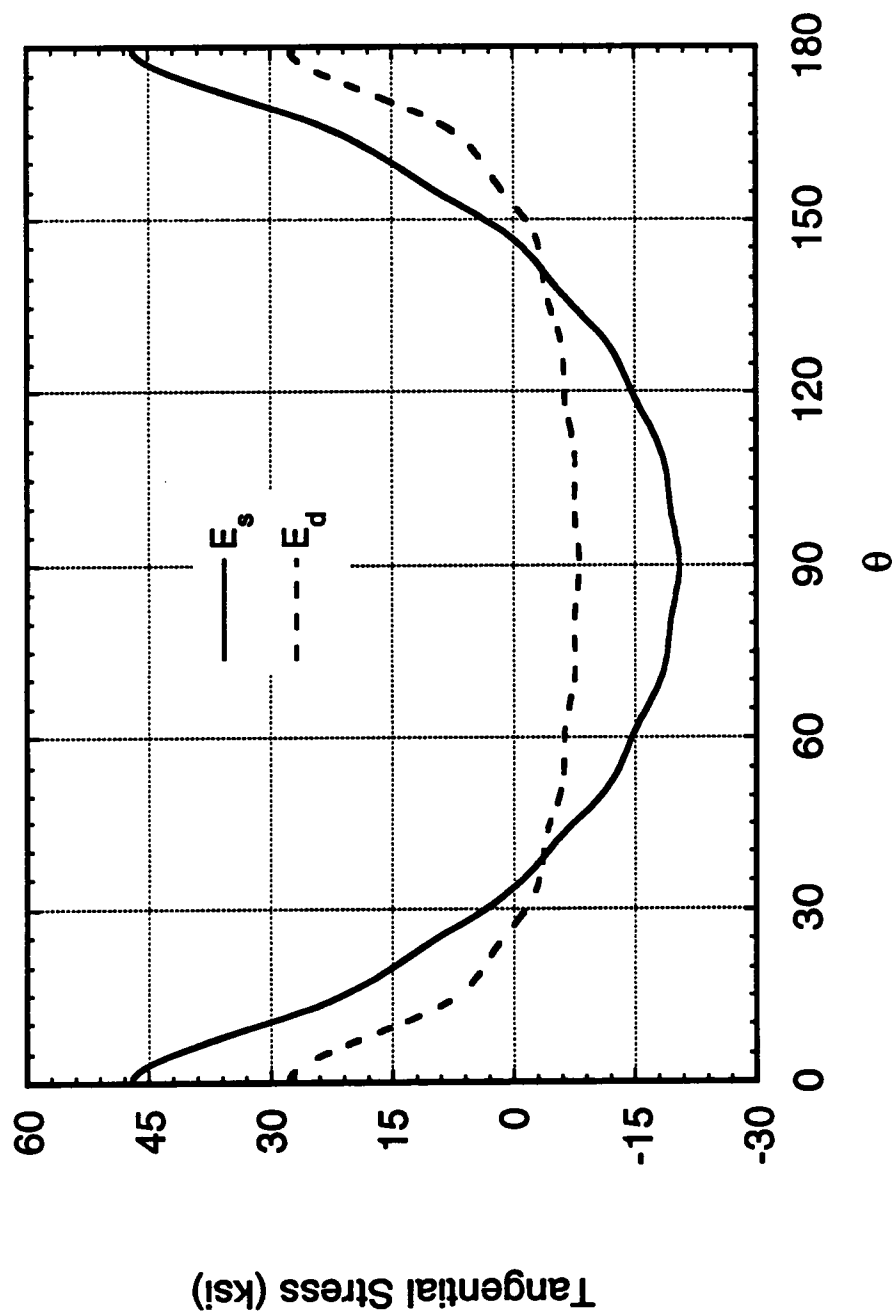


Figure 73. Tangential Stresses at the Interface (in the Case, Situation 1).

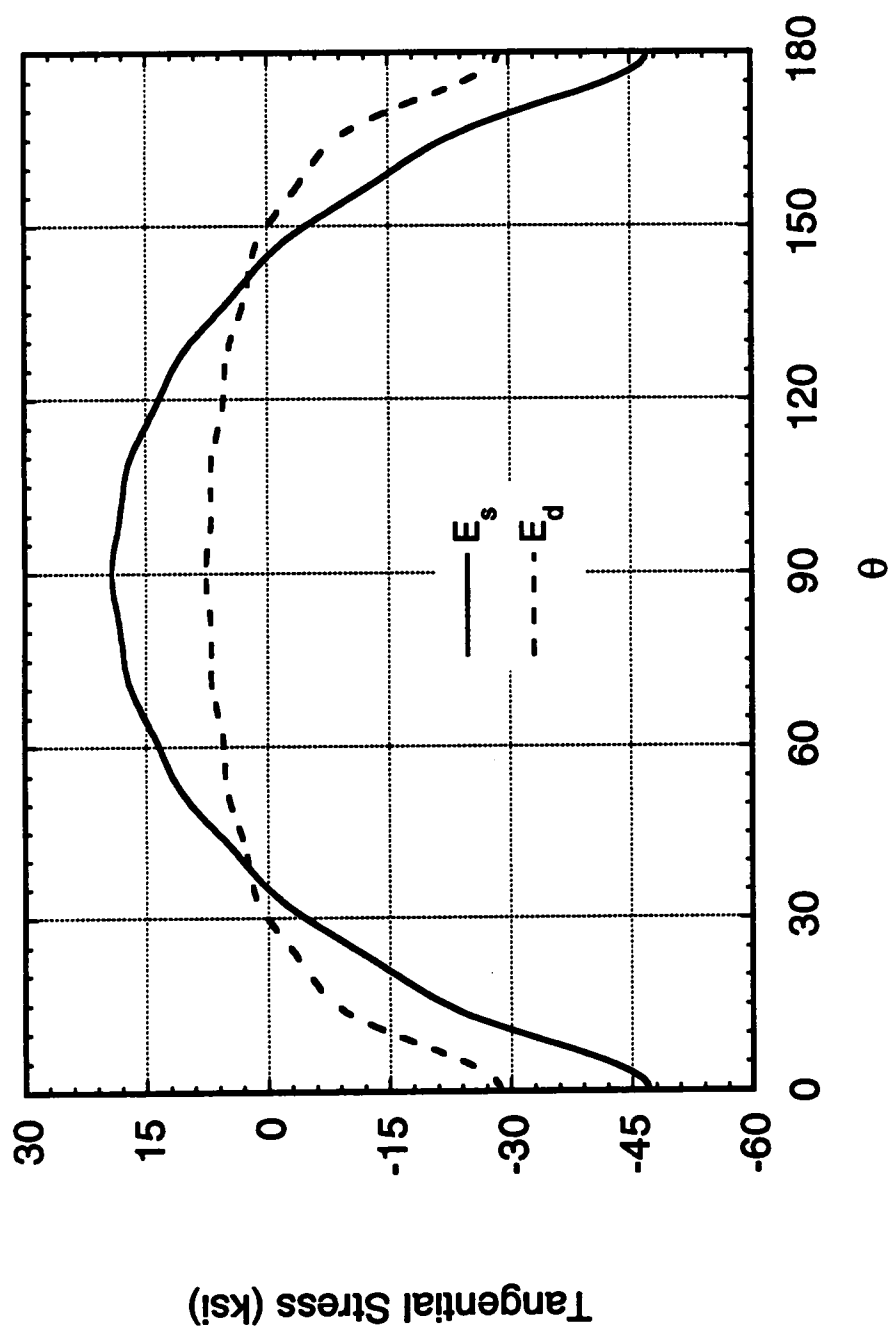


Figure 74. Tangential Stresses at the Outer Surface (Situation 1).

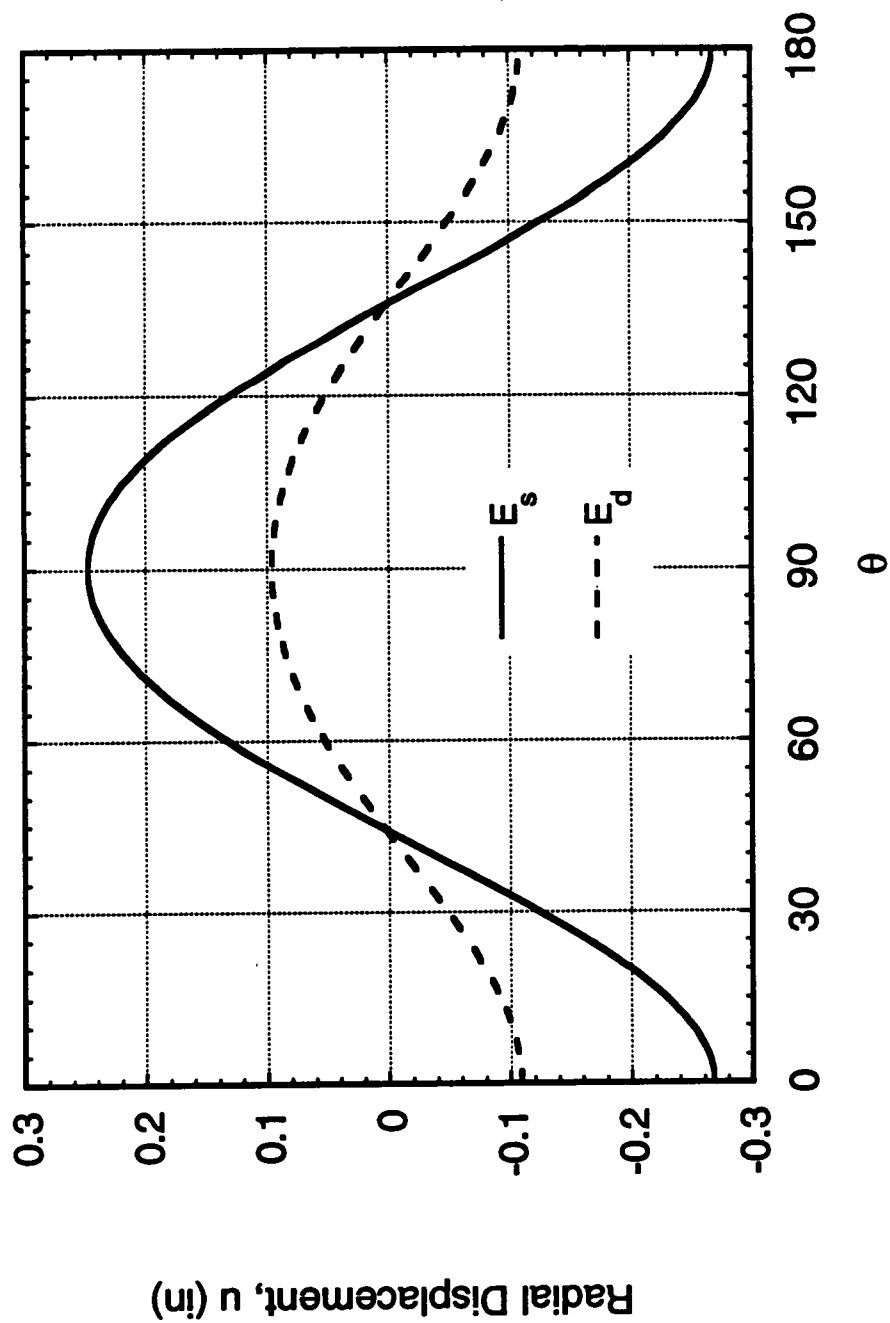


Figure 75. Radial Displacement, u , at the Bore (Situation 2).

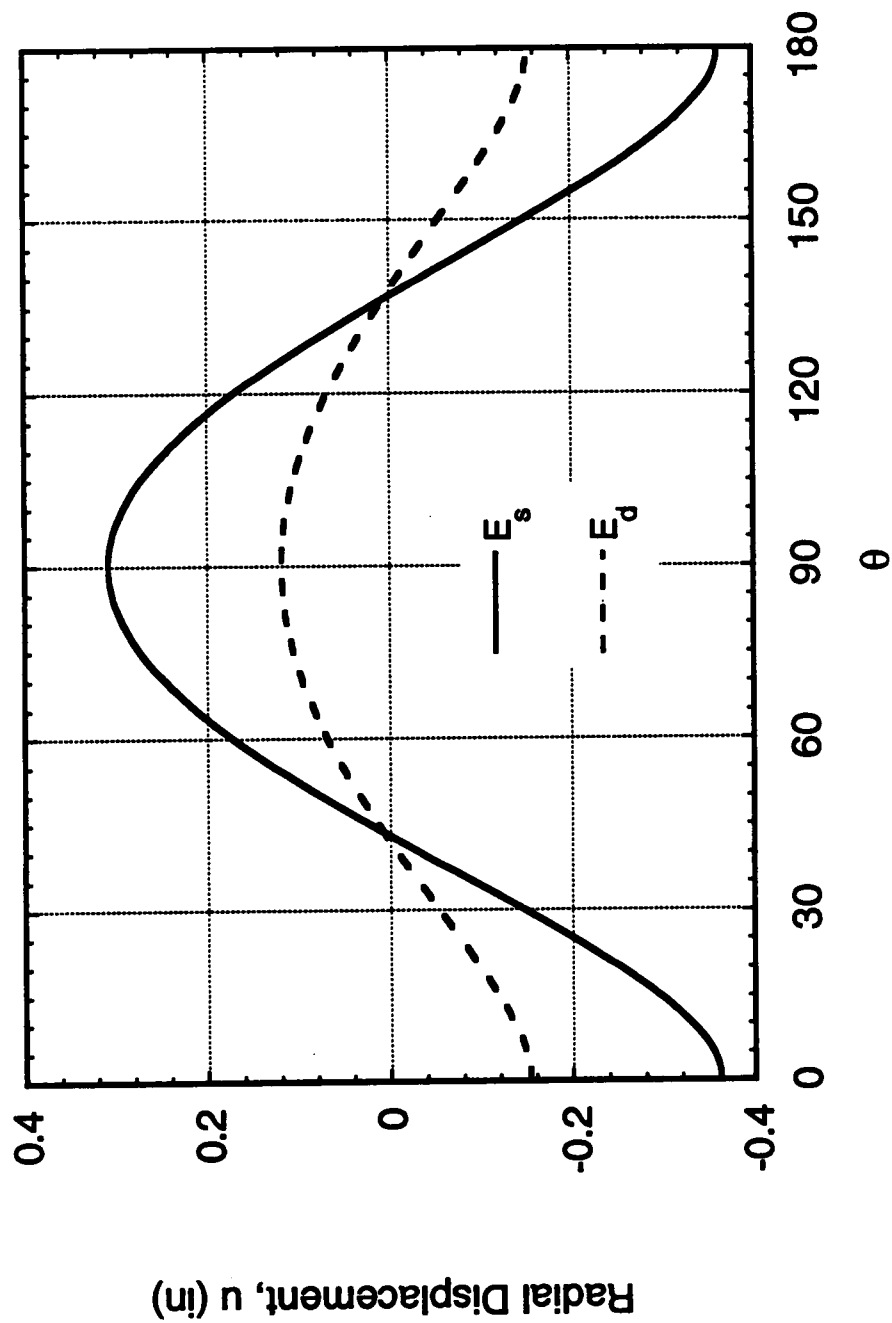


Figure 76. Radial Displacement, u , at the Outer Surface (Situation 2).

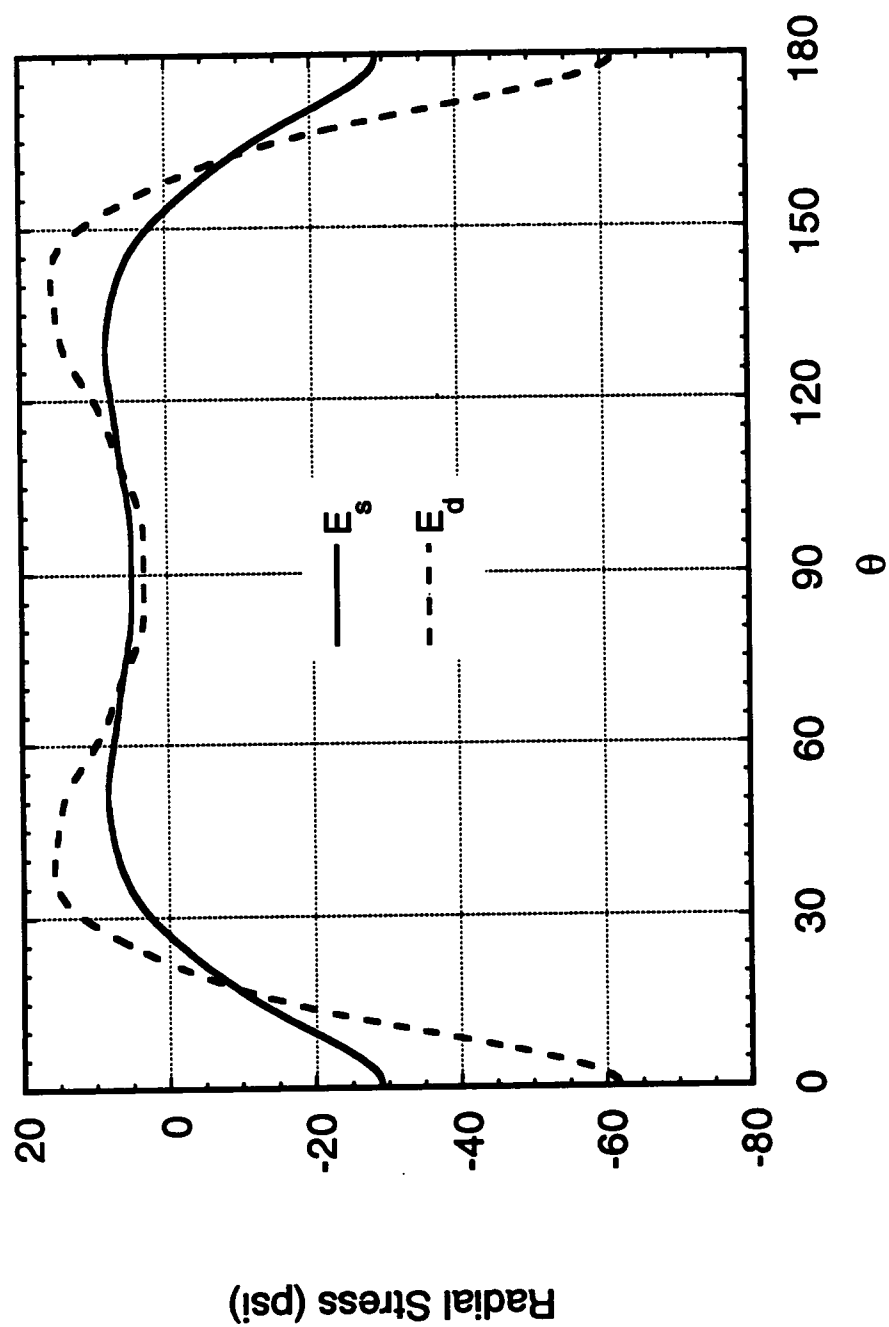


Figure 77. Radial Stresses at the Interface (Situation 2).

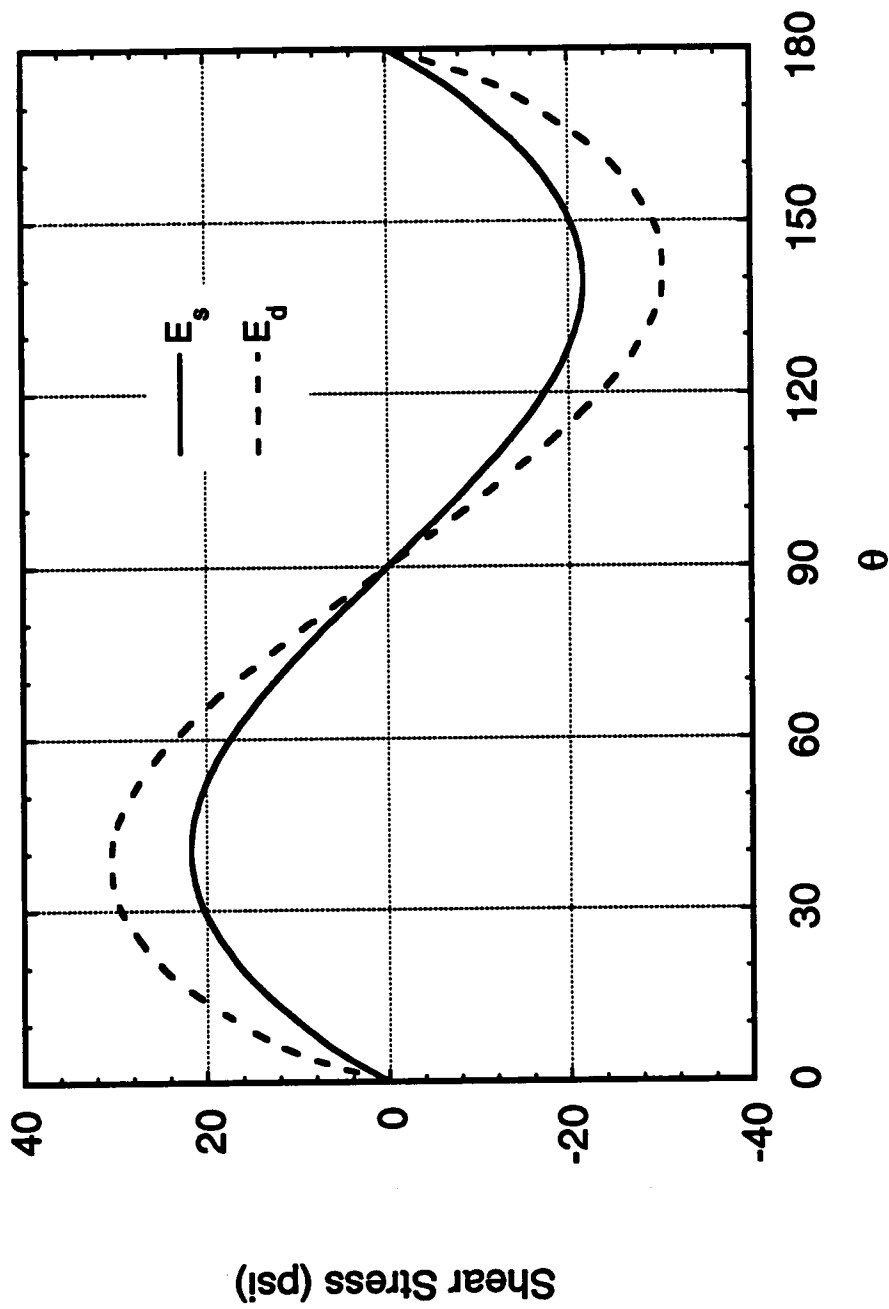


Figure 78. Shear Stresses at the Interface (Situation 2).

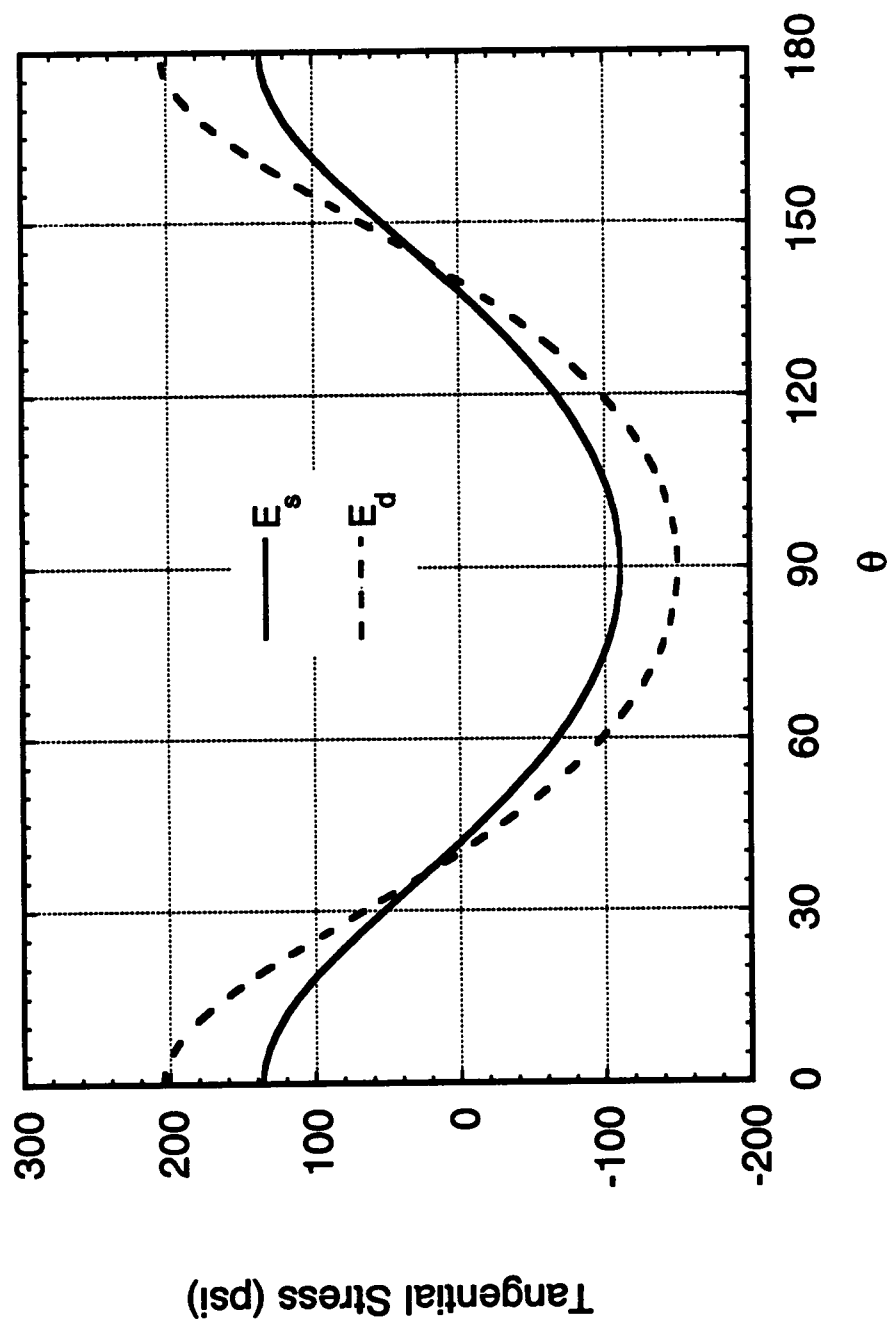


Figure 79. Tangential Stresses at the Bore (Situation 2).

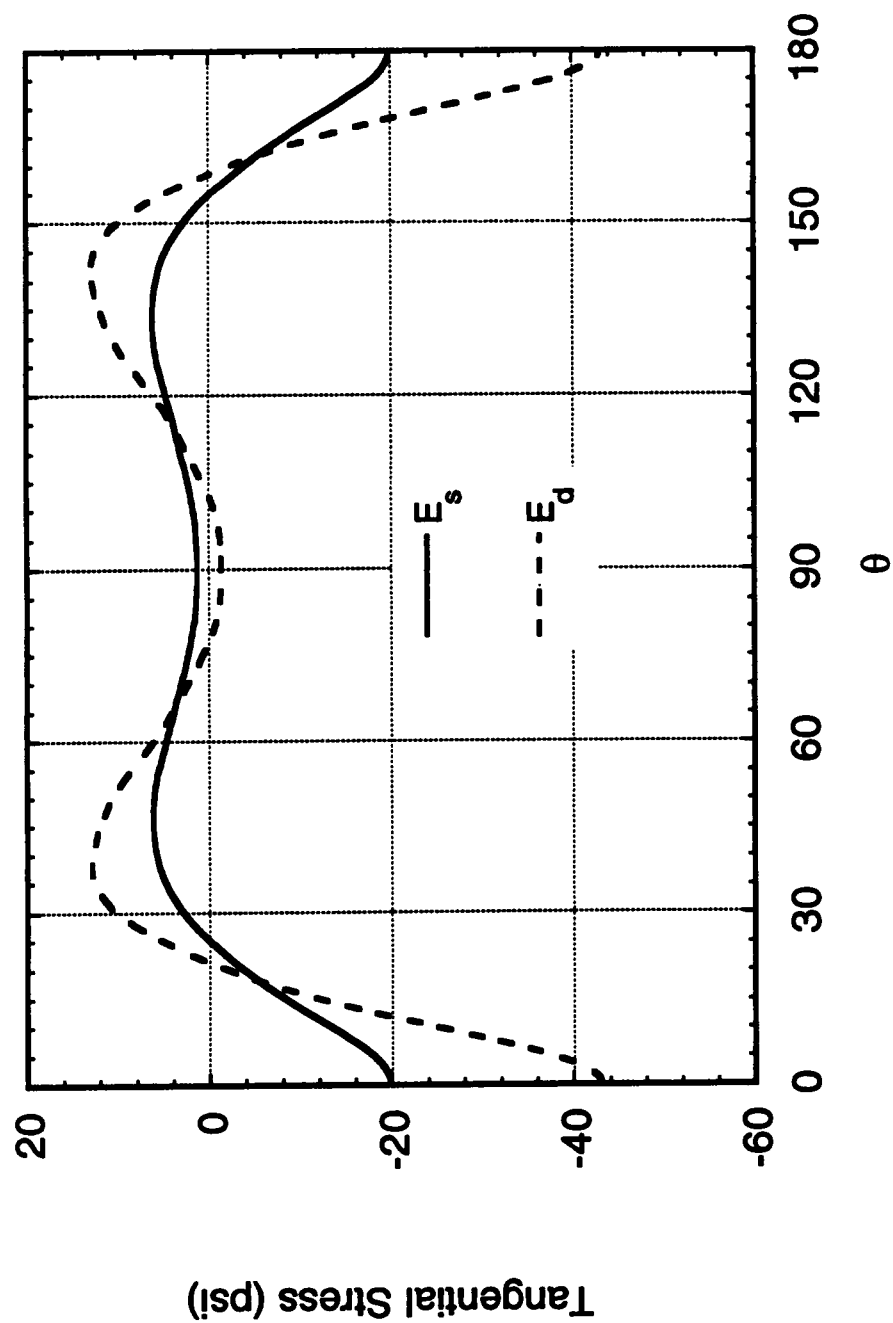


Figure 80. Tangential Stresses in the Propellant at the Interface (Situation 2).

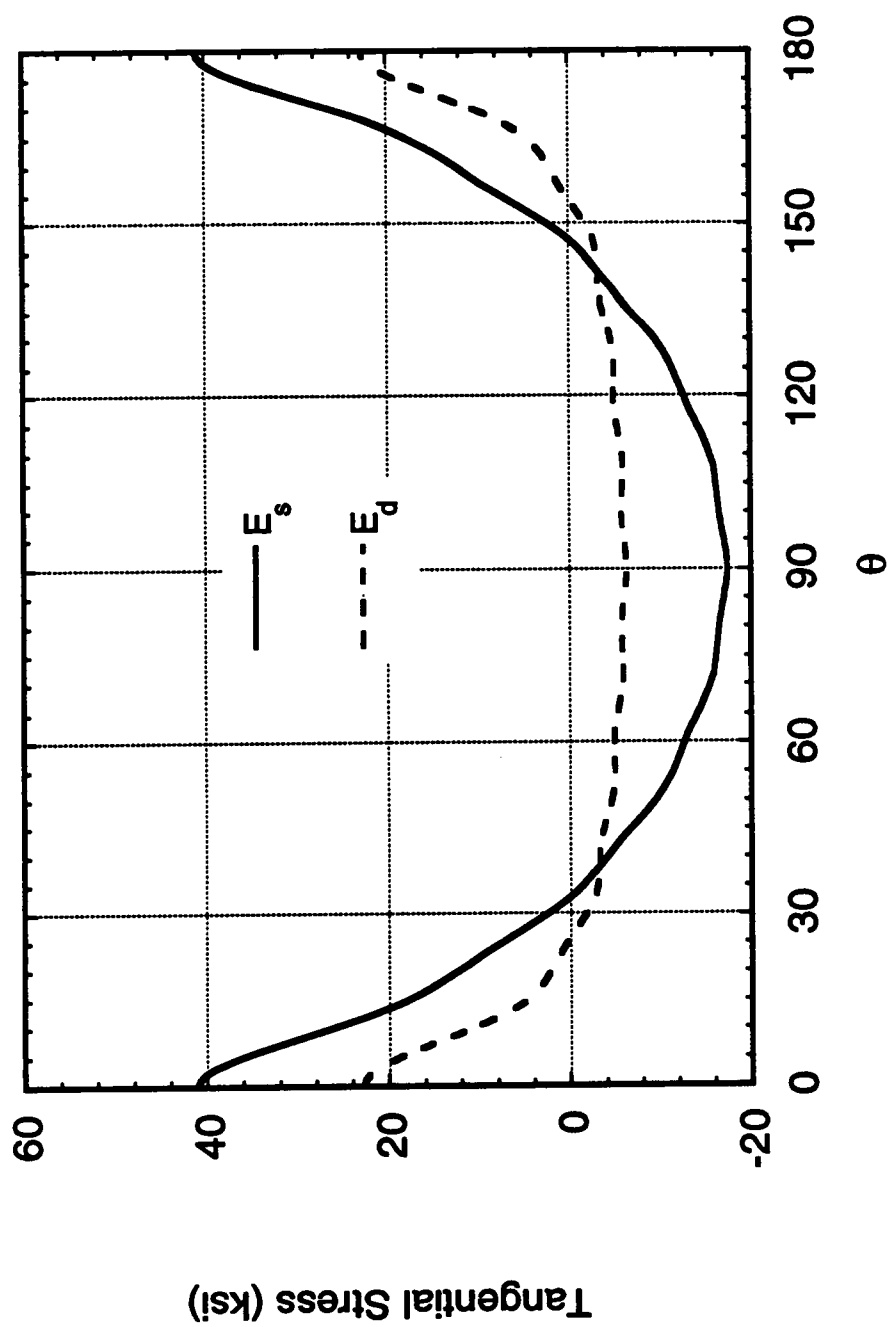


Figure 81. Tangential Stresses in Layer 3 at r_3 (Case, Situation 2).

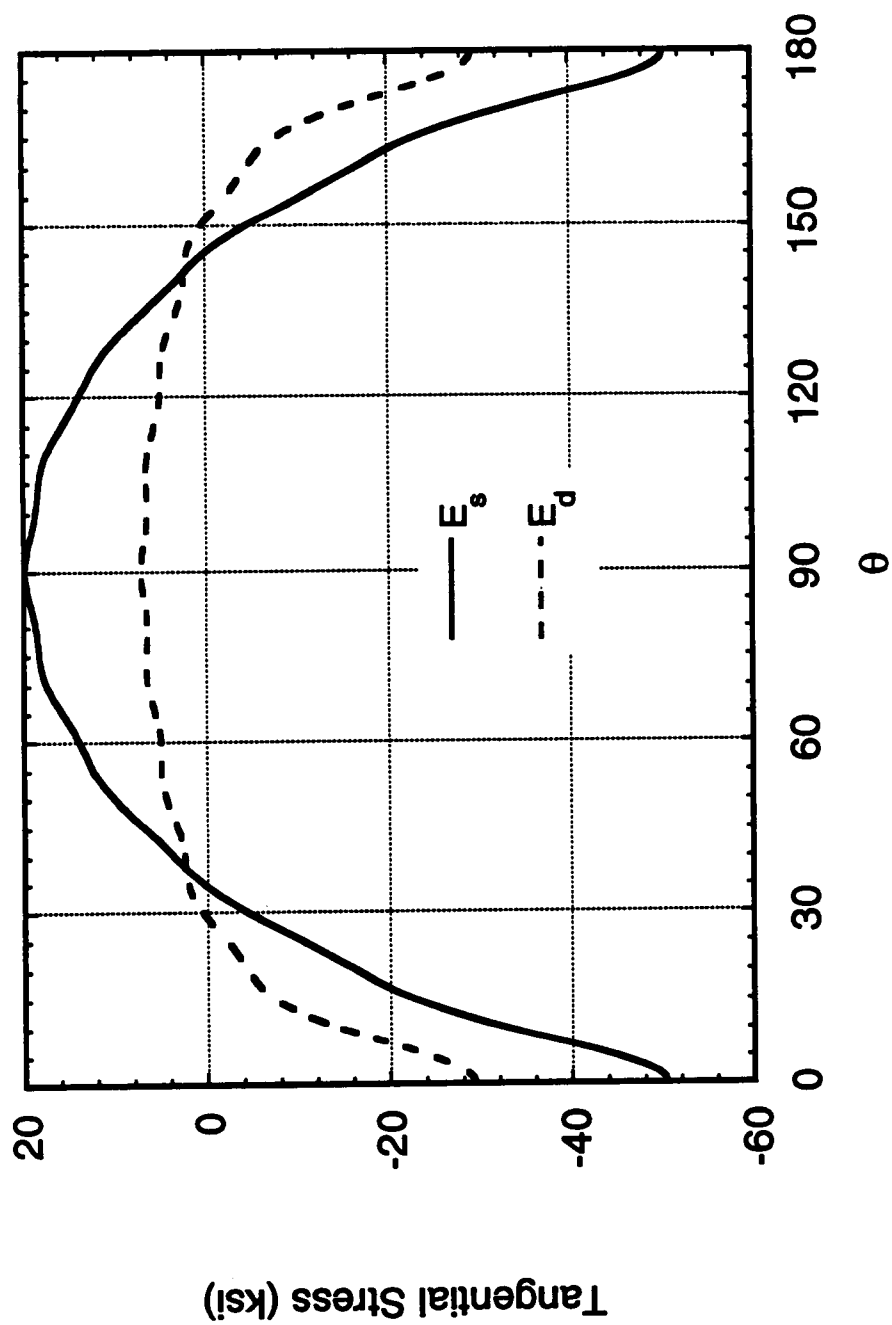


Figure 82. Tangential Stresses at Outer Surface (Situation 2).

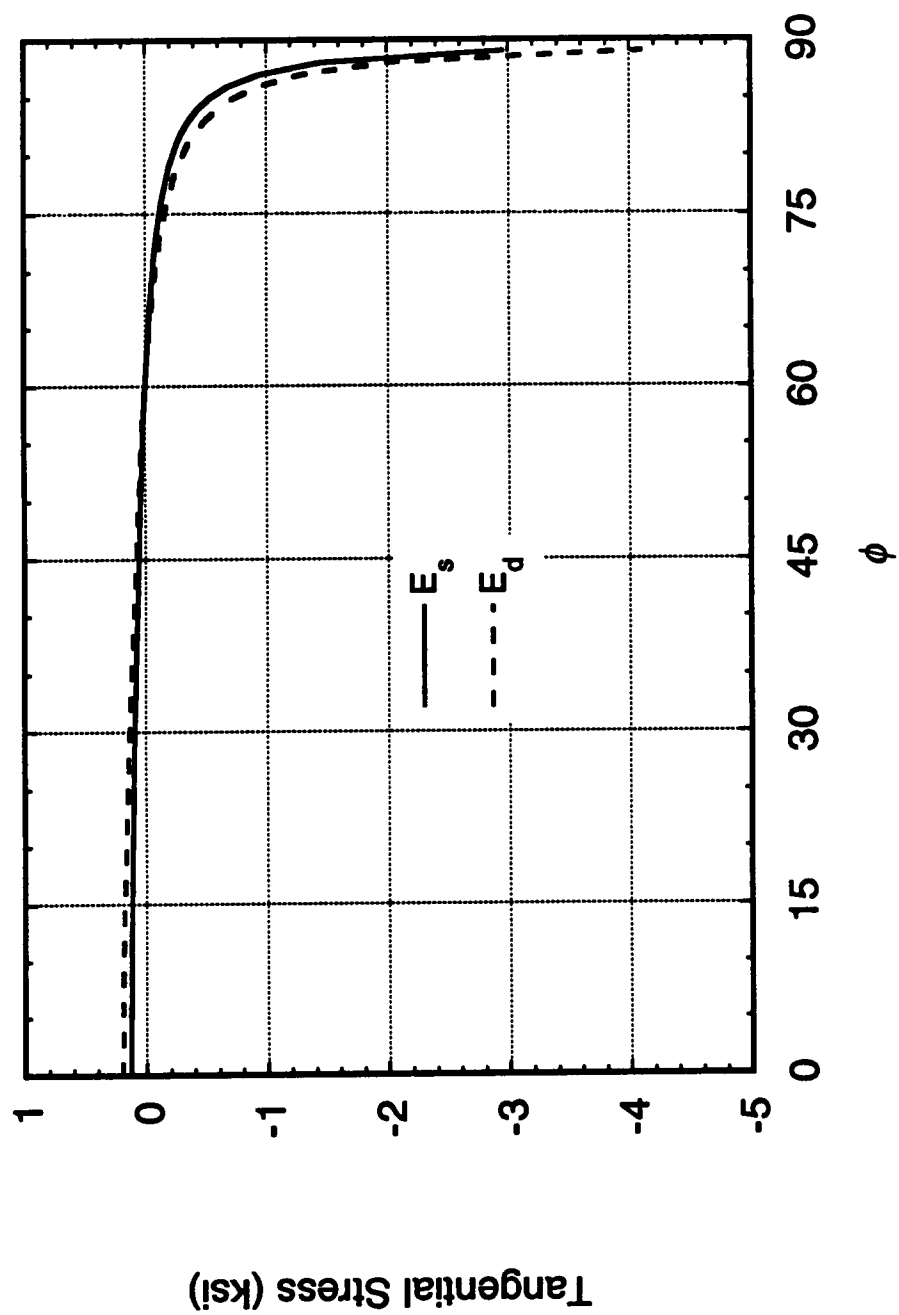


Figure 83. Tangential Stresses at the Bore ($0 - 90^\circ$).

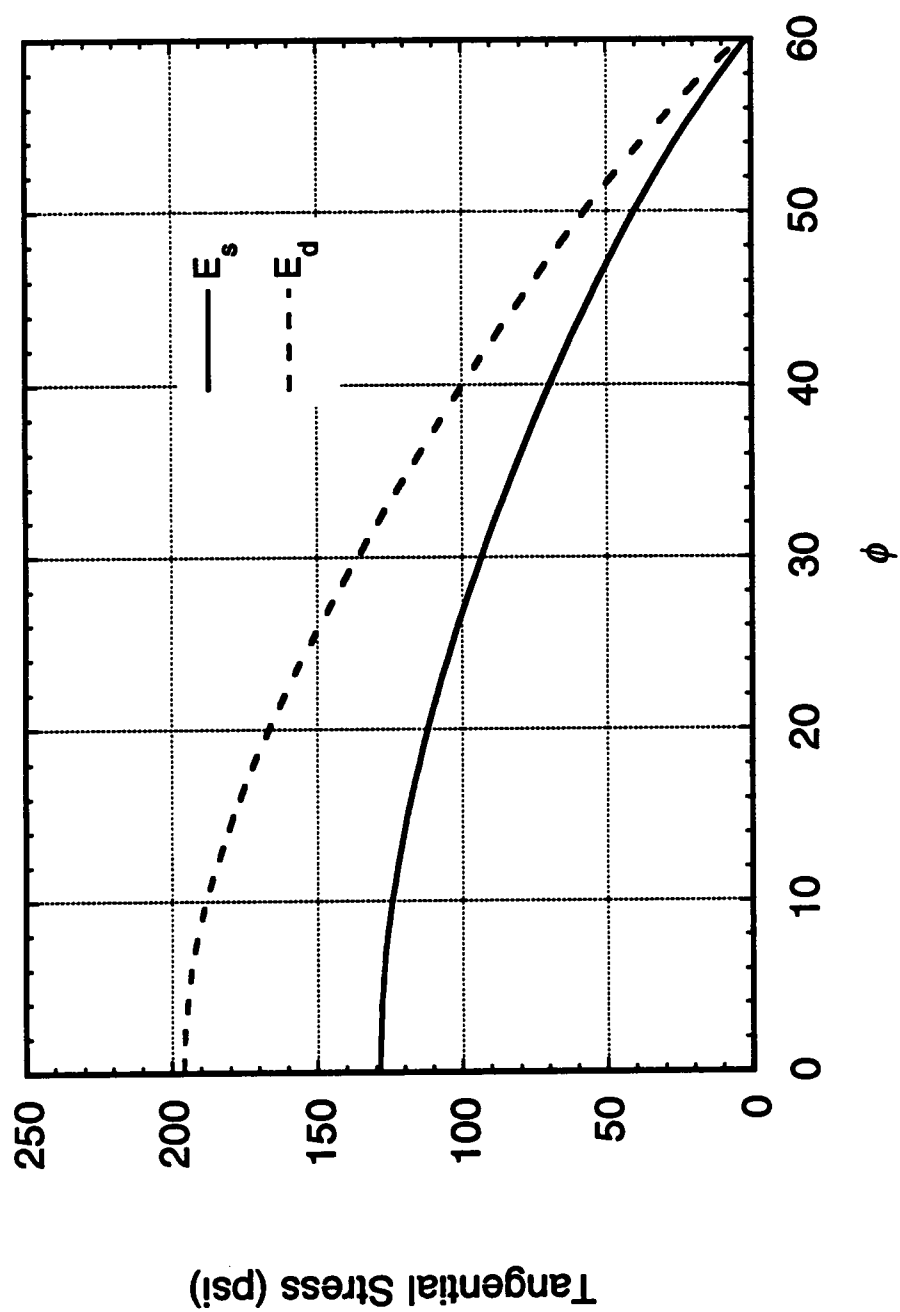


Figure 84. Tangential Stresses at the Bore ($0 - 60^\circ$).

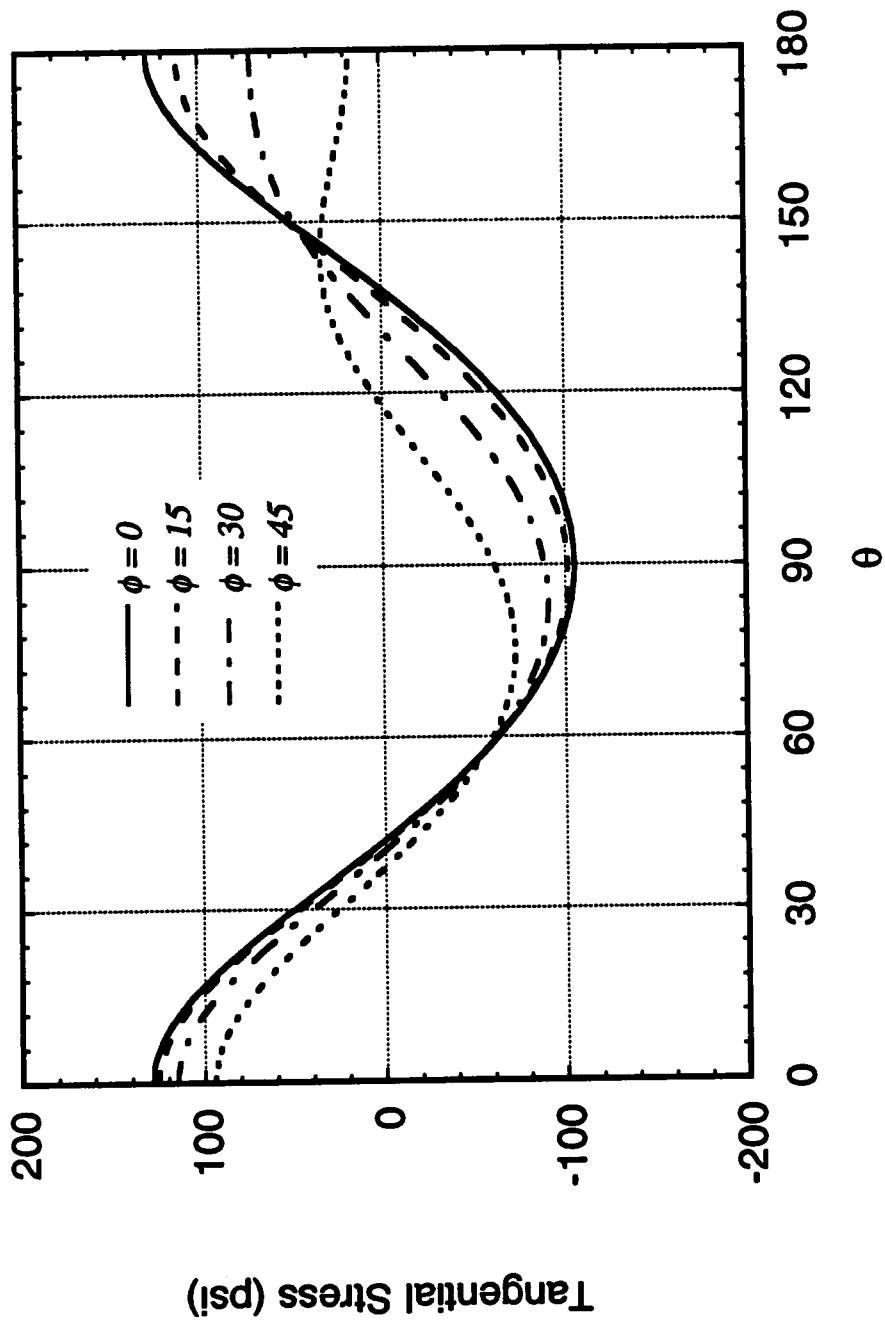


Figure 85. Tangential Stresses at the Bore for E_s .

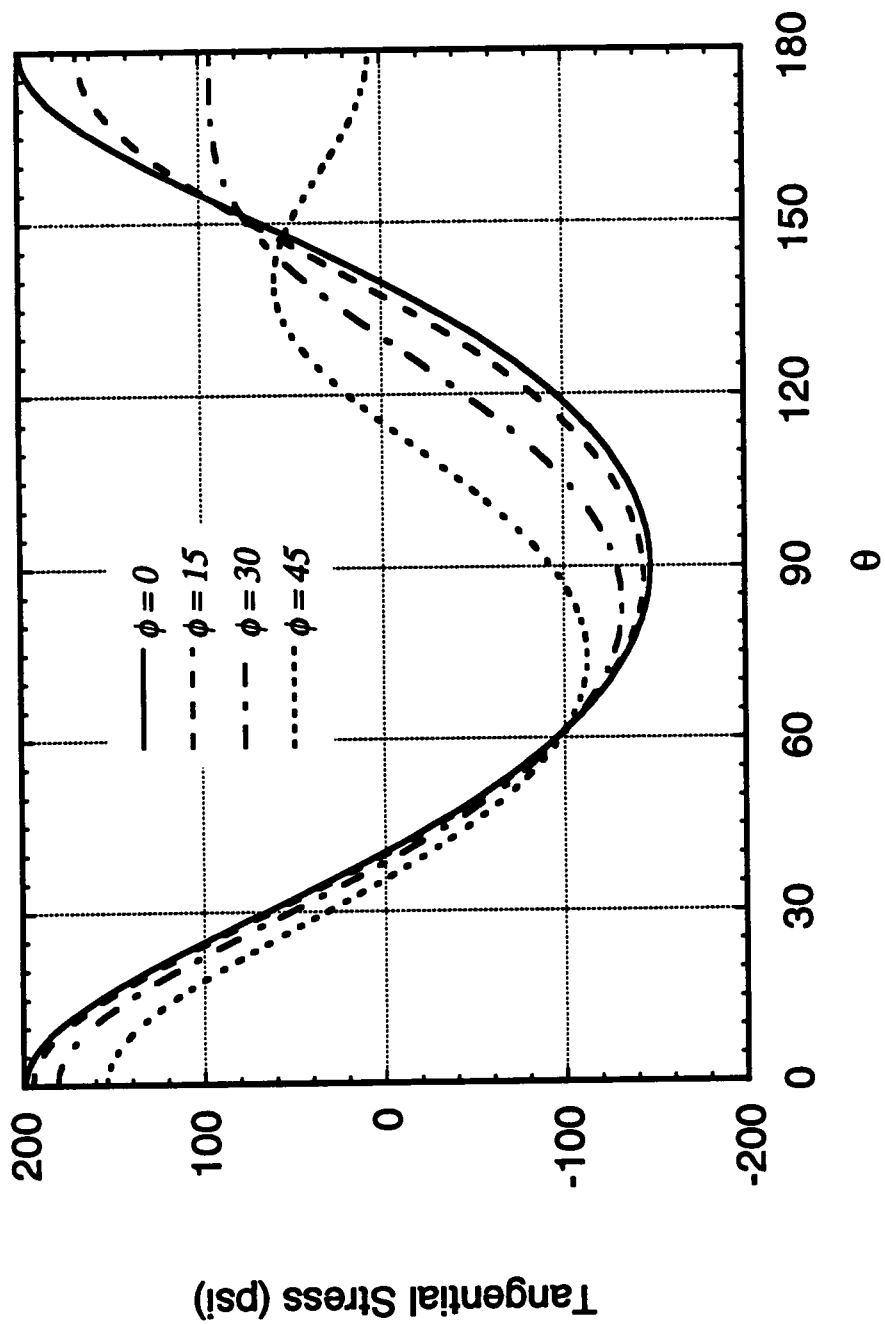


Figure 86. Tangential Stresses at the Bore for E_d .

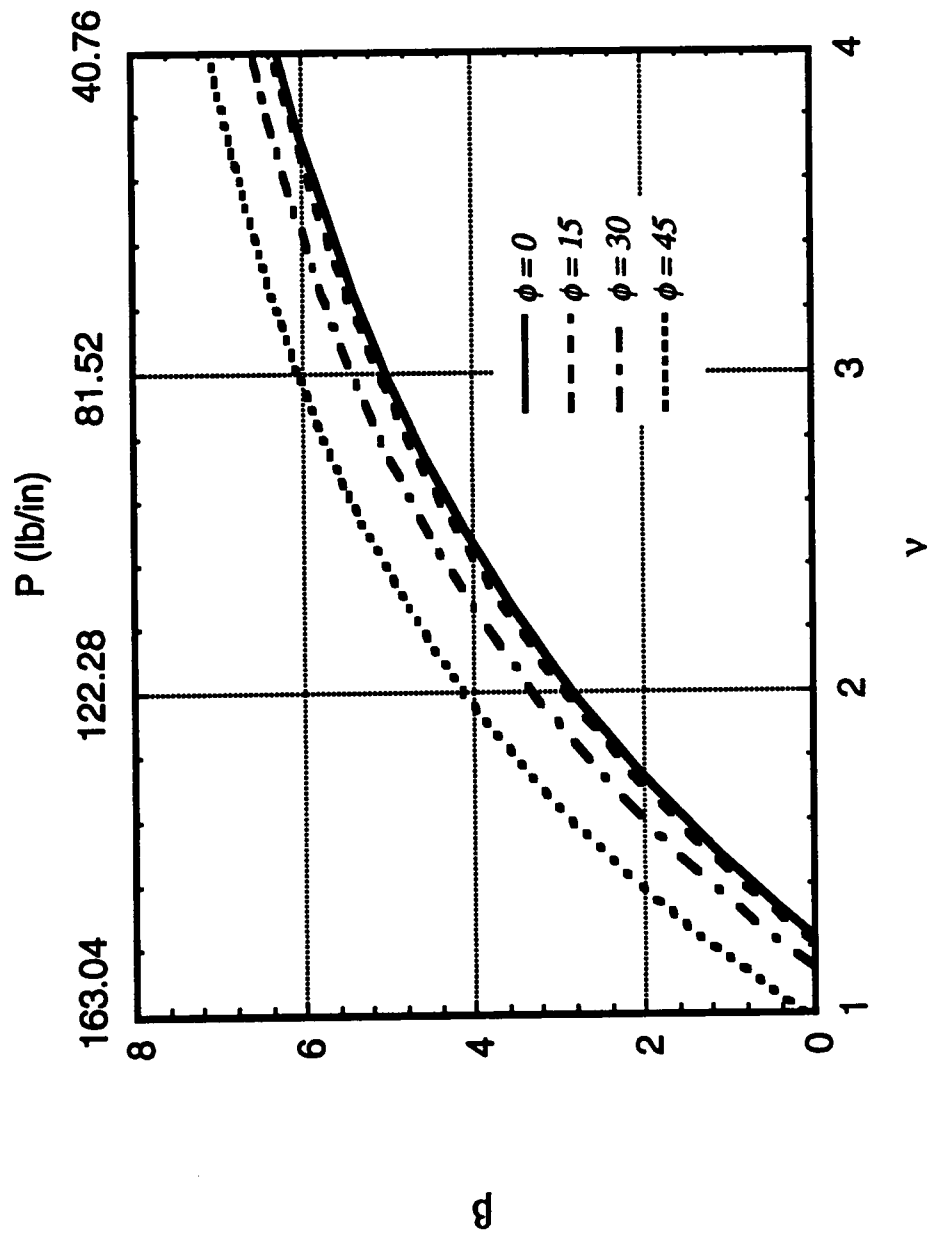


Figure 87. Safety Index of Propellant vs. Safety Factor (COV = 10%, Normal).

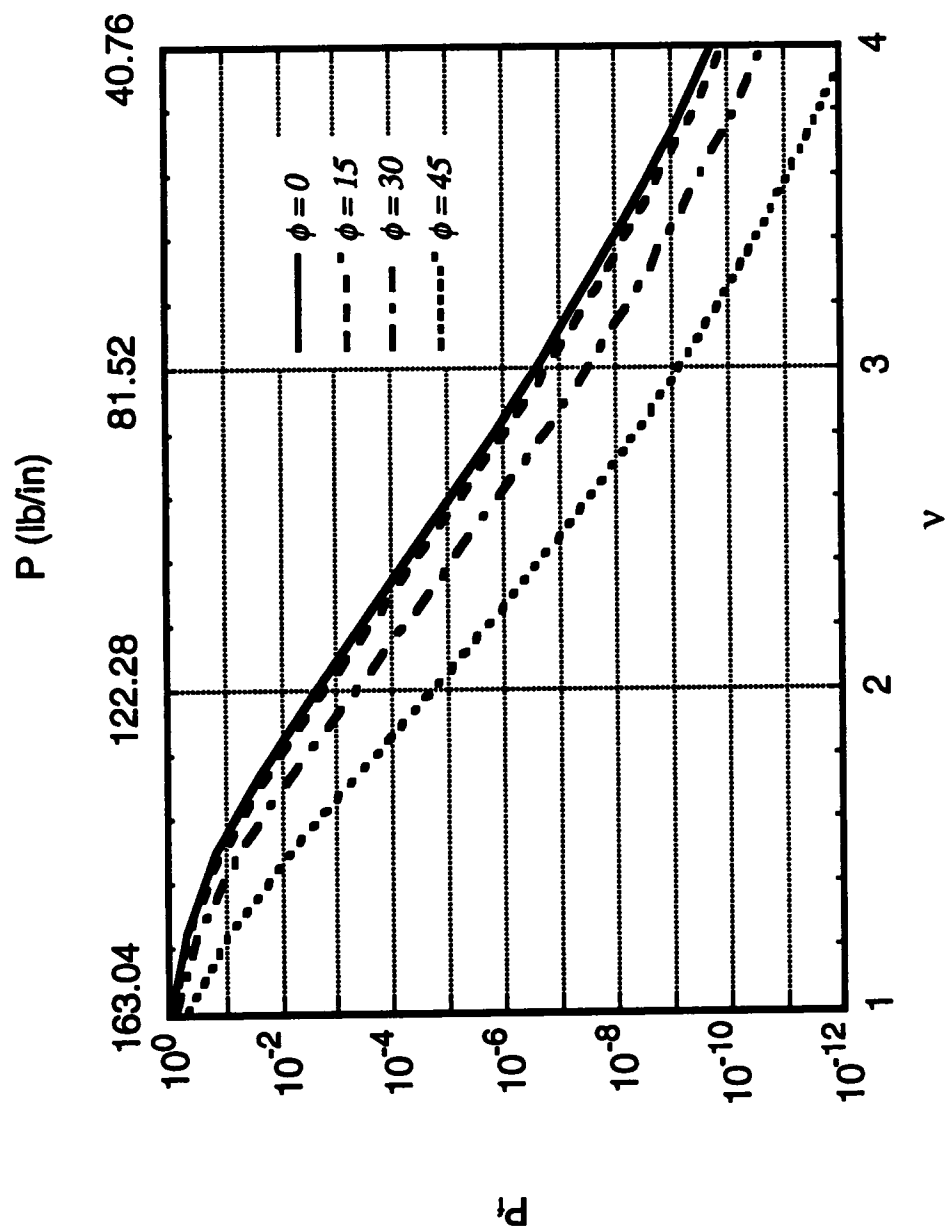


Figure 88. Probability of Failure of Propellant vs. Safety Factor (COV = 10%, Normal).

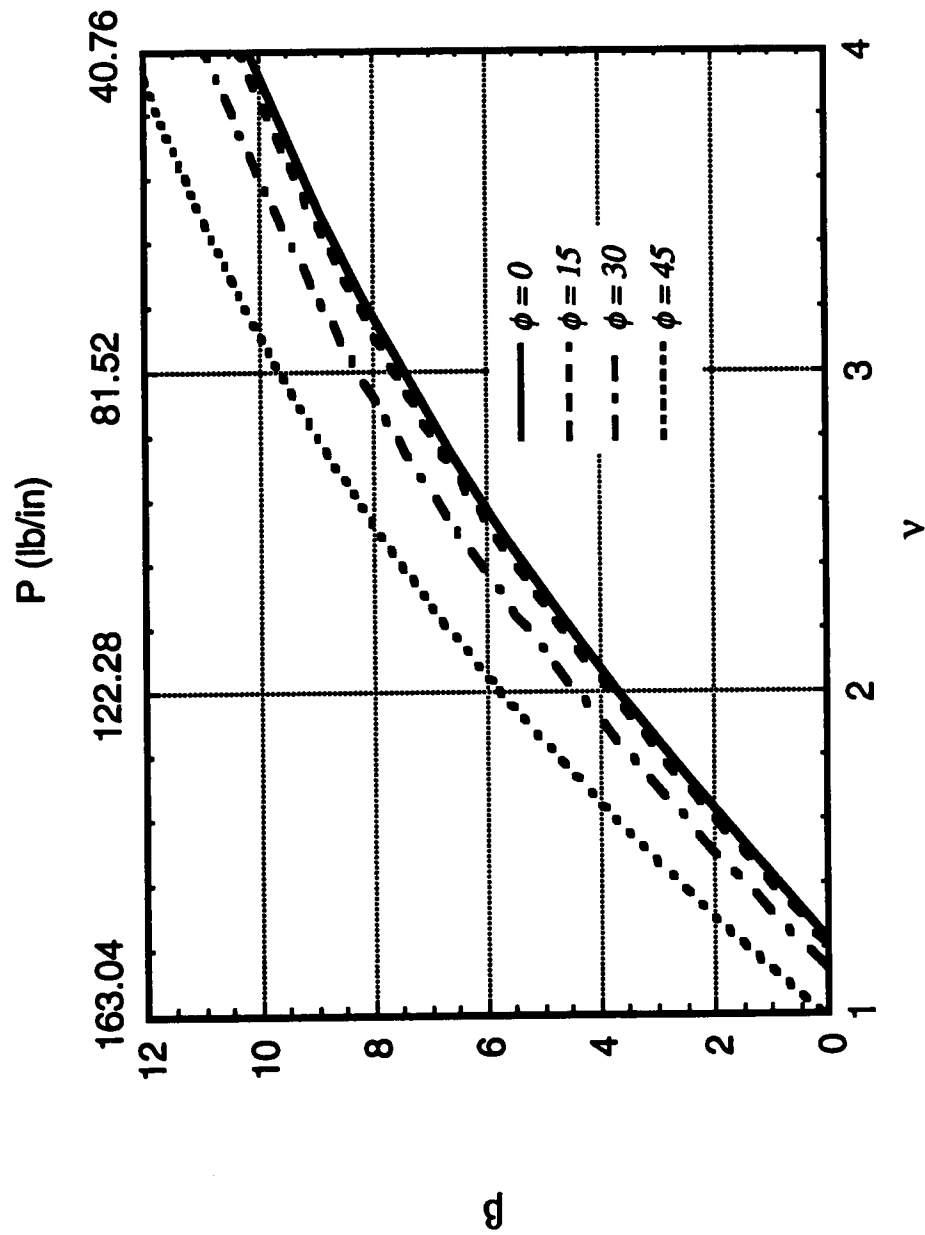


Figure 89. Safety Index of Propellant vs. Safety Factor (COV = 5%, Normal).

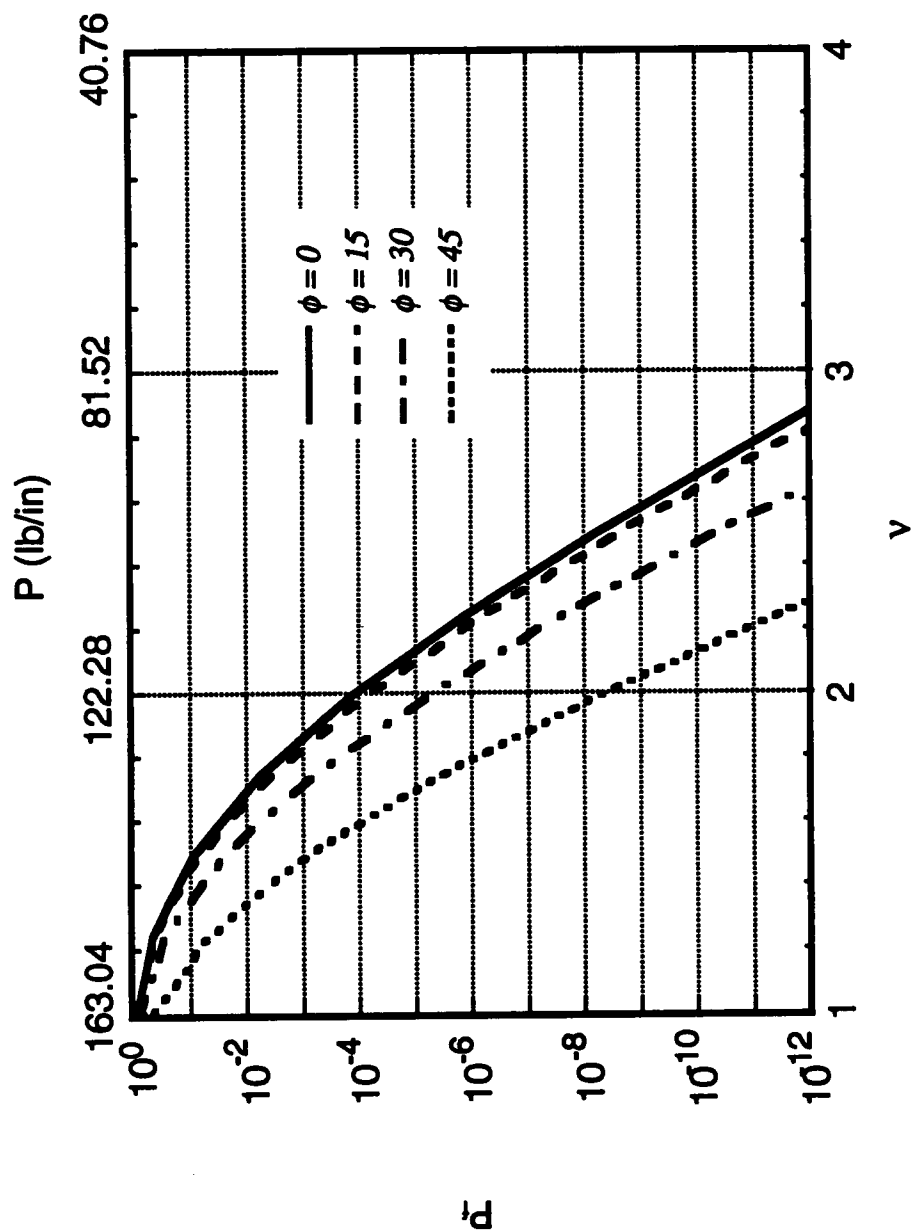


Figure 90. Probability of Failure of Propellant vs. Safety Factor (COV = 5%, Normal).

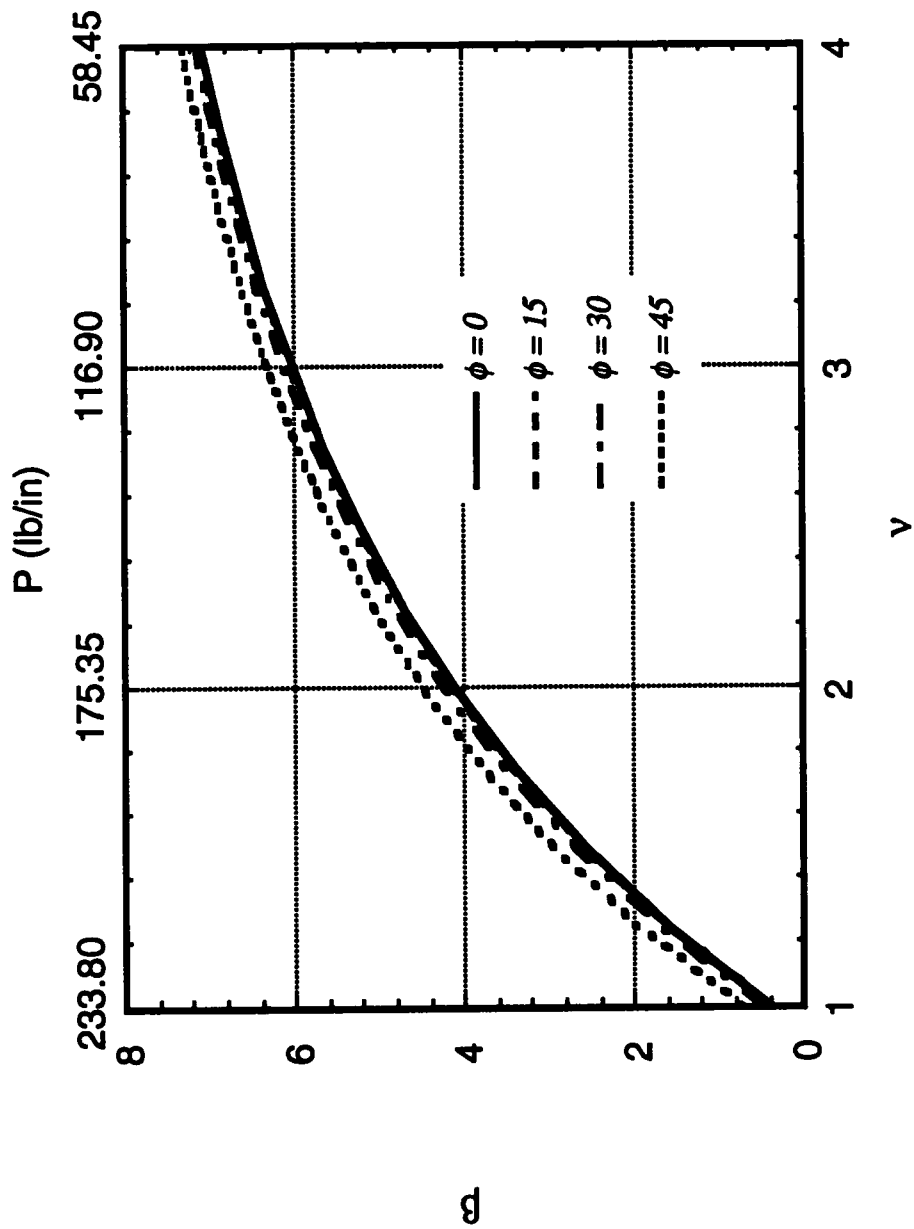


Figure 91. Safety Index of Case vs. Safety Factor (COV = 10%, Normal).

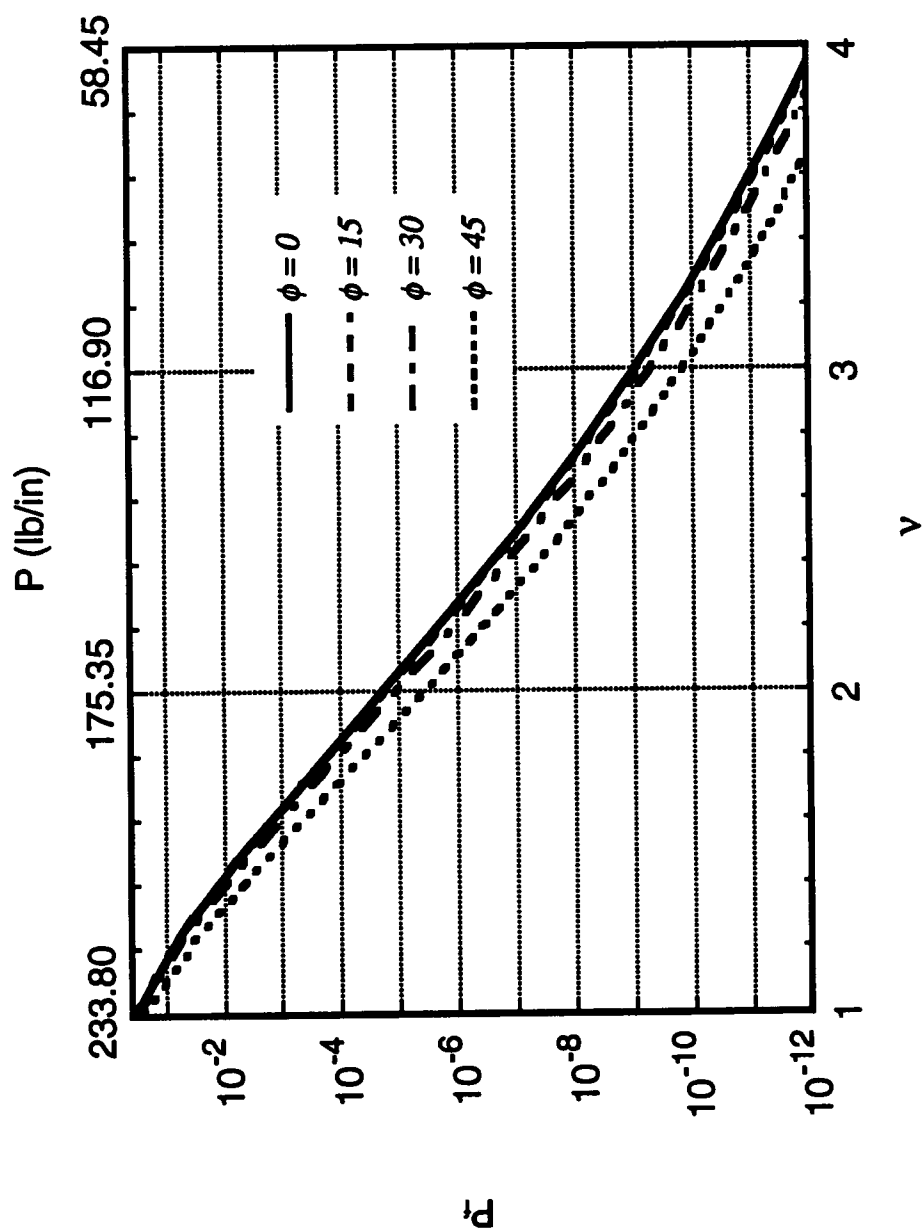


Figure 92. Probability of Failure of Case vs. Safety Index (COV = 10%, Normal).

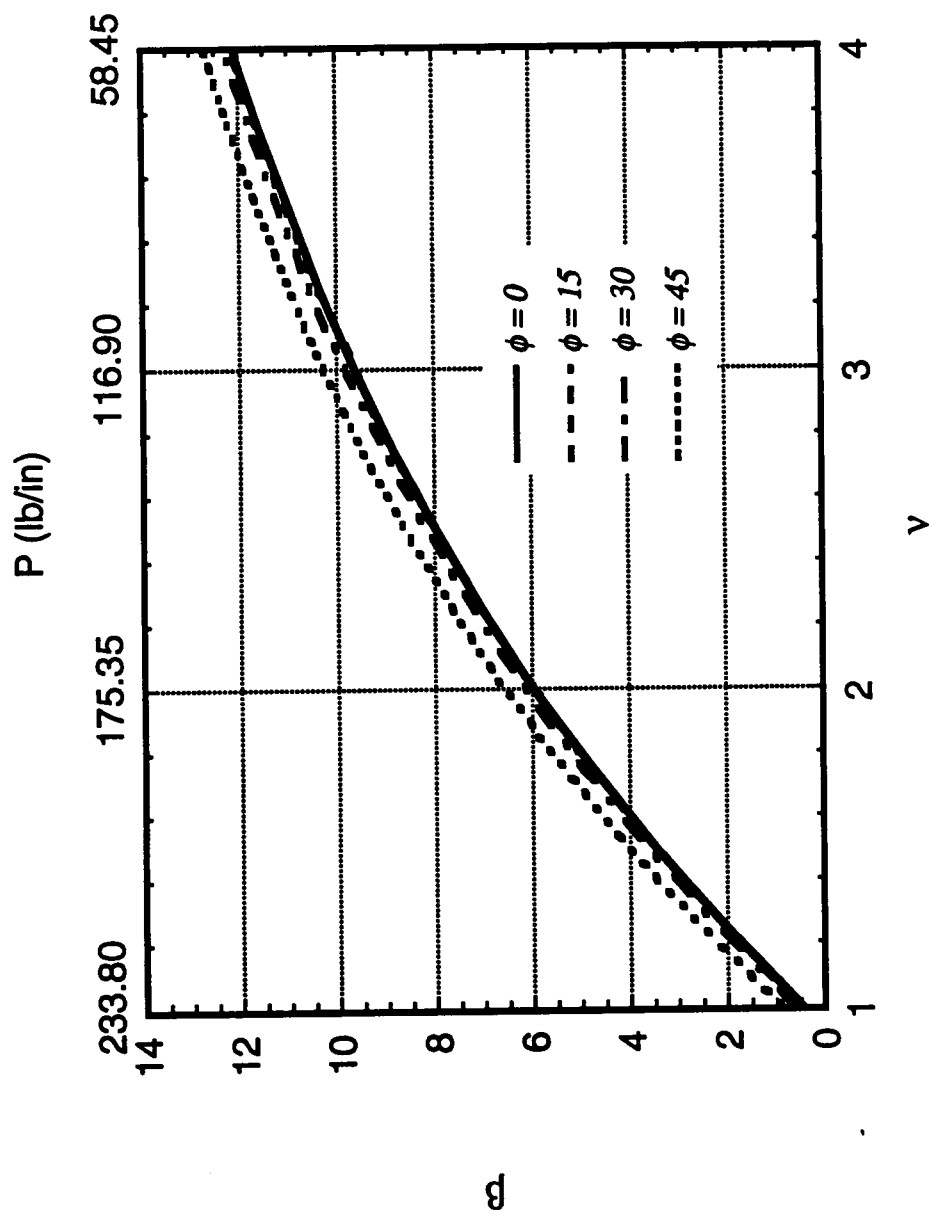


Figure 93. Safety Index of Case vs. Safety Factor (COV = 5%, Normal).

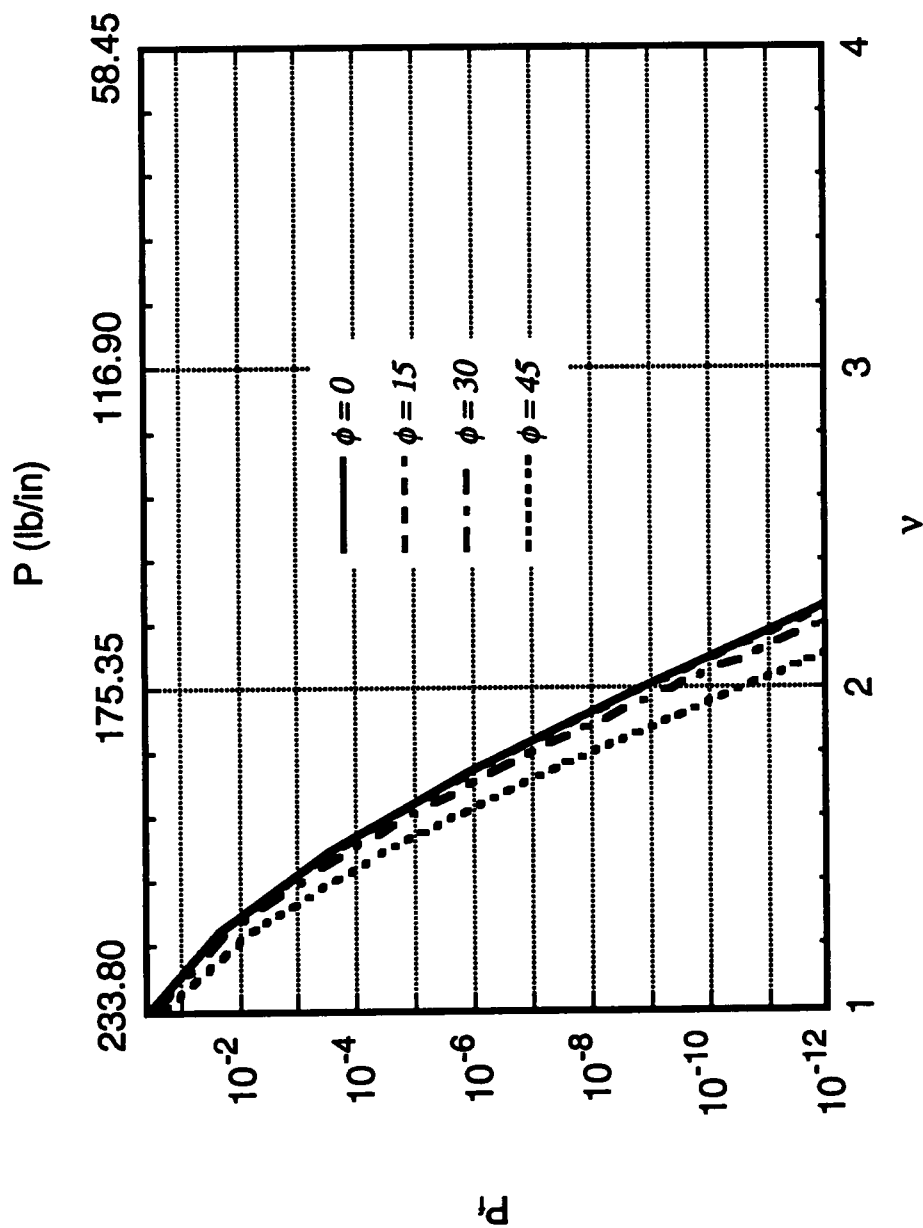


Figure 94. Probability of Failure of Case vs. Safety Index (COV = 5%, Normal).

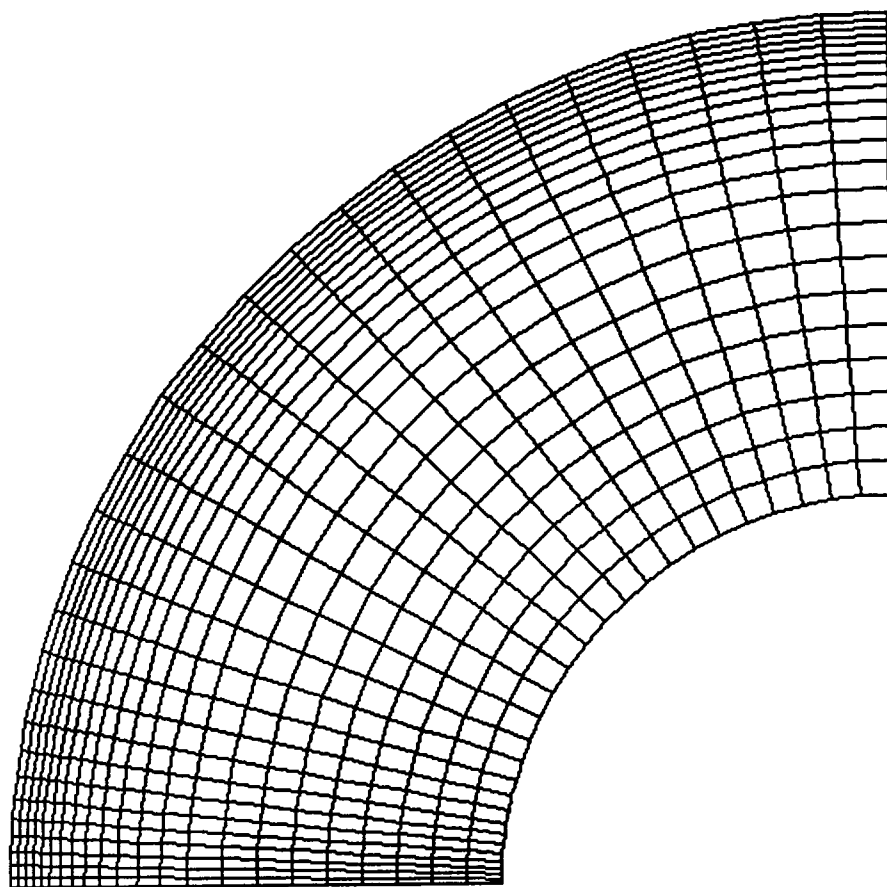


Figure 95. The Finite Element Mesh.

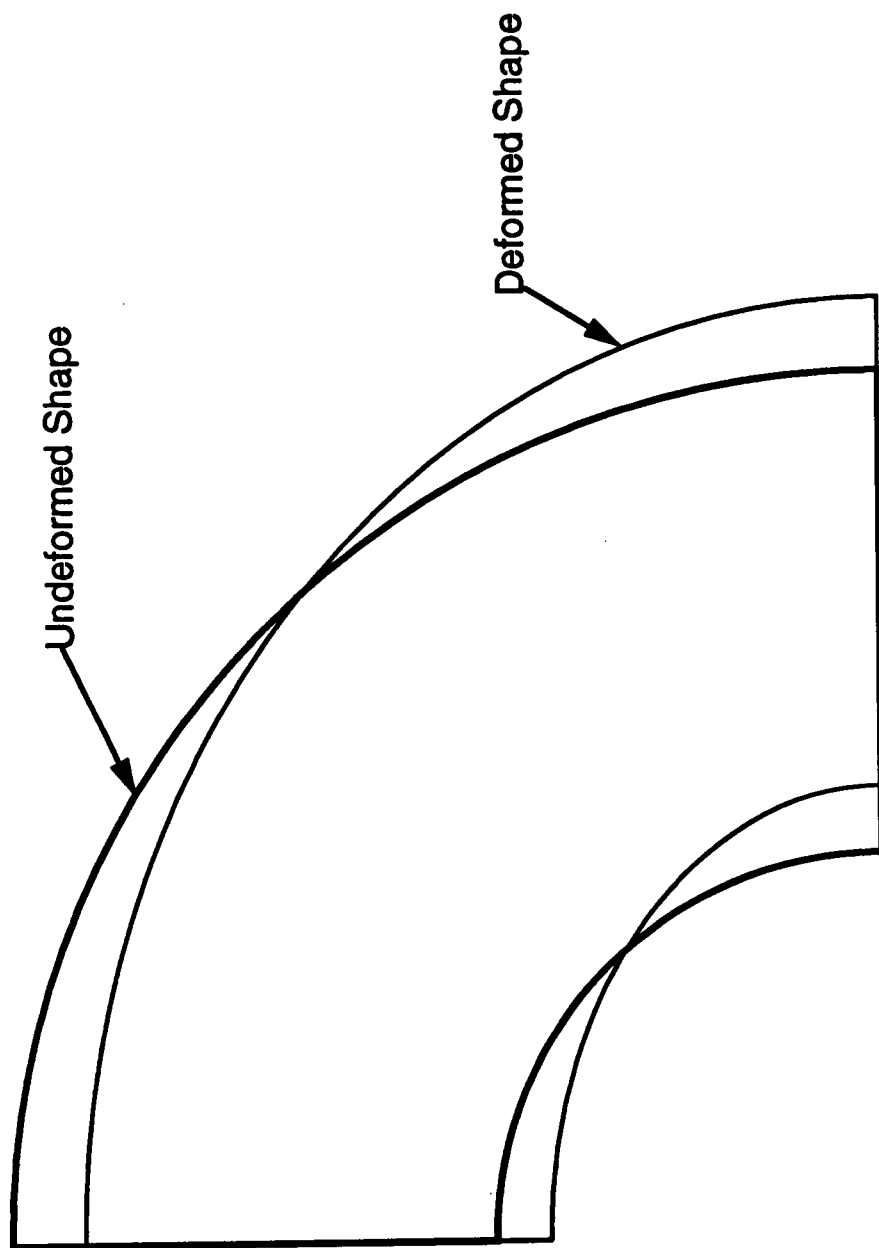


Figure 96. The Undeformed and Deformed Shapes of the Cylinder.

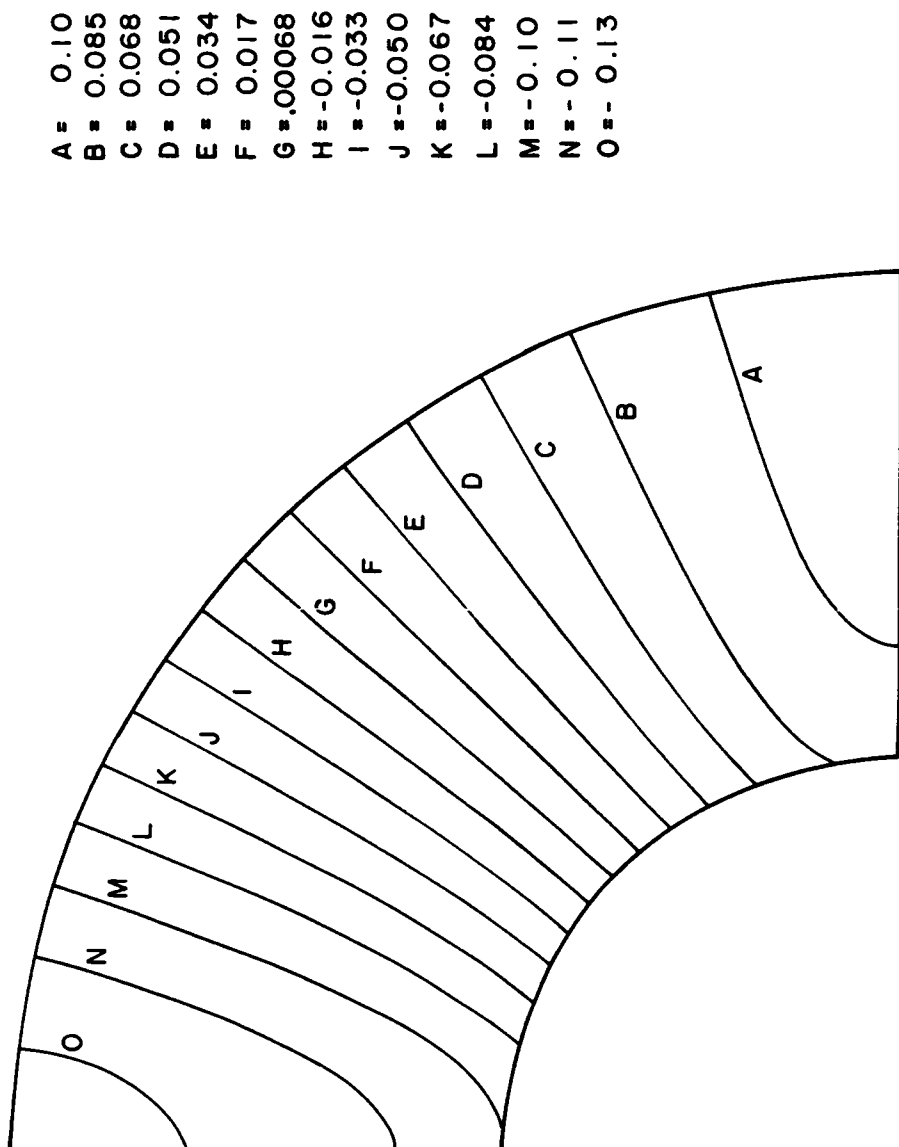


Figure 97. Contours of Radial Displacements (in) (Long Cylinder, by ABAQUS).

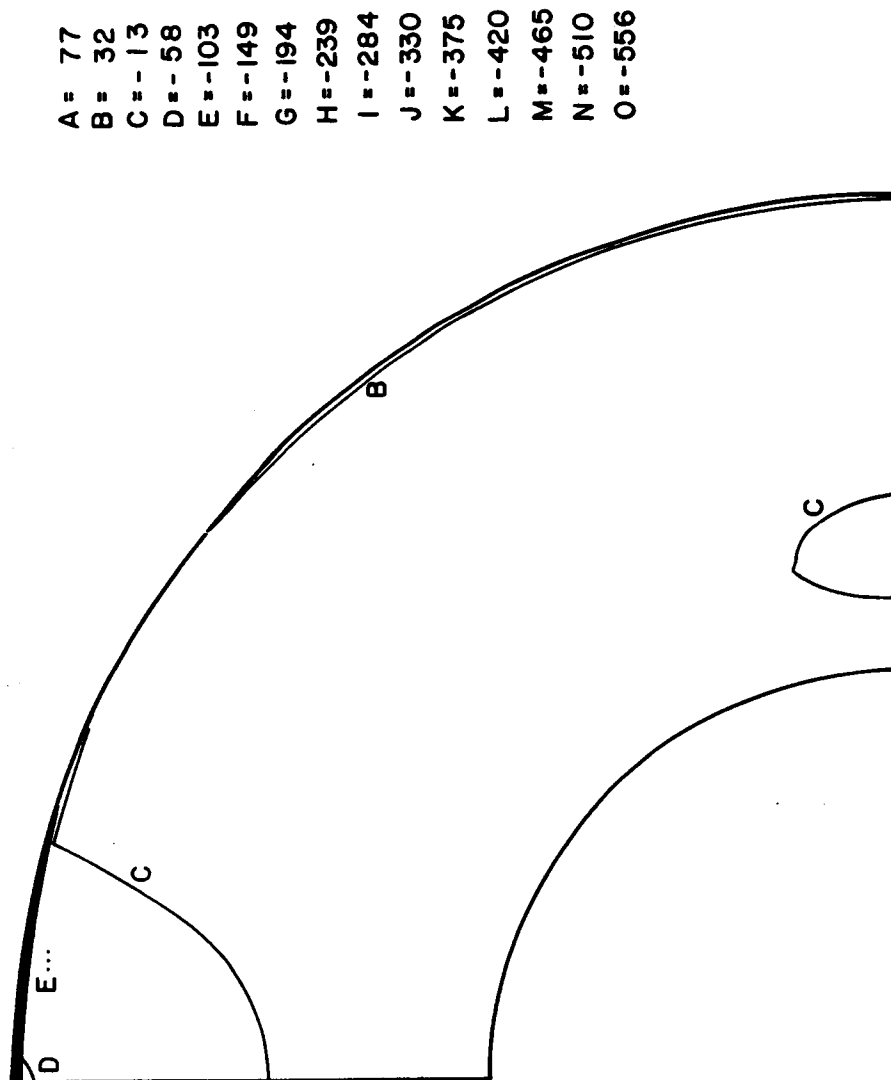


Figure 98. Contours of Radial Stresses (psi) in the Propellant (Long Cylinder, by ABAQUS).

A = 77
 B = 32
 C = -13
 D = -58
 E = -103
 F = -149
 G = -194
 H = -239
 I = -284
 J = -330
 K = -375
 L = -420
 M = -465
 N = -510
 O = -556

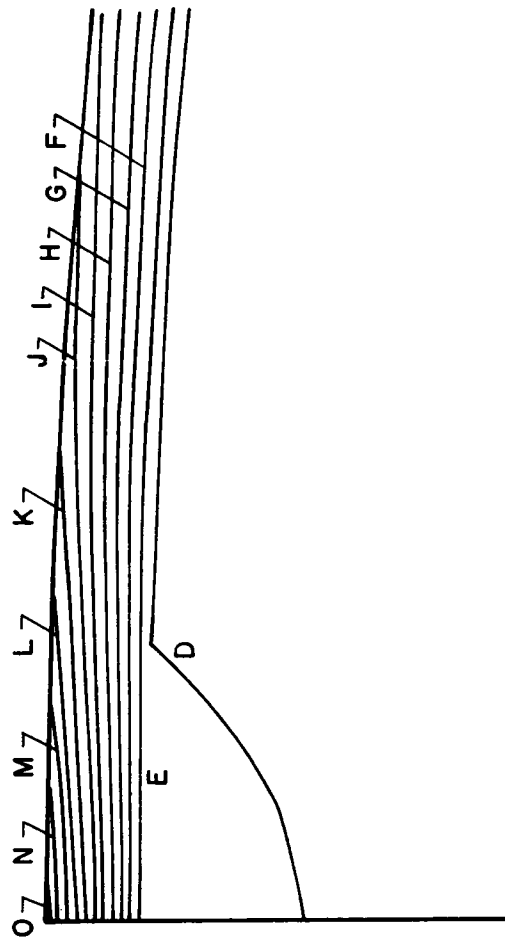


Figure 99. Enlargement of the Left-upper Corner of Figure 98.

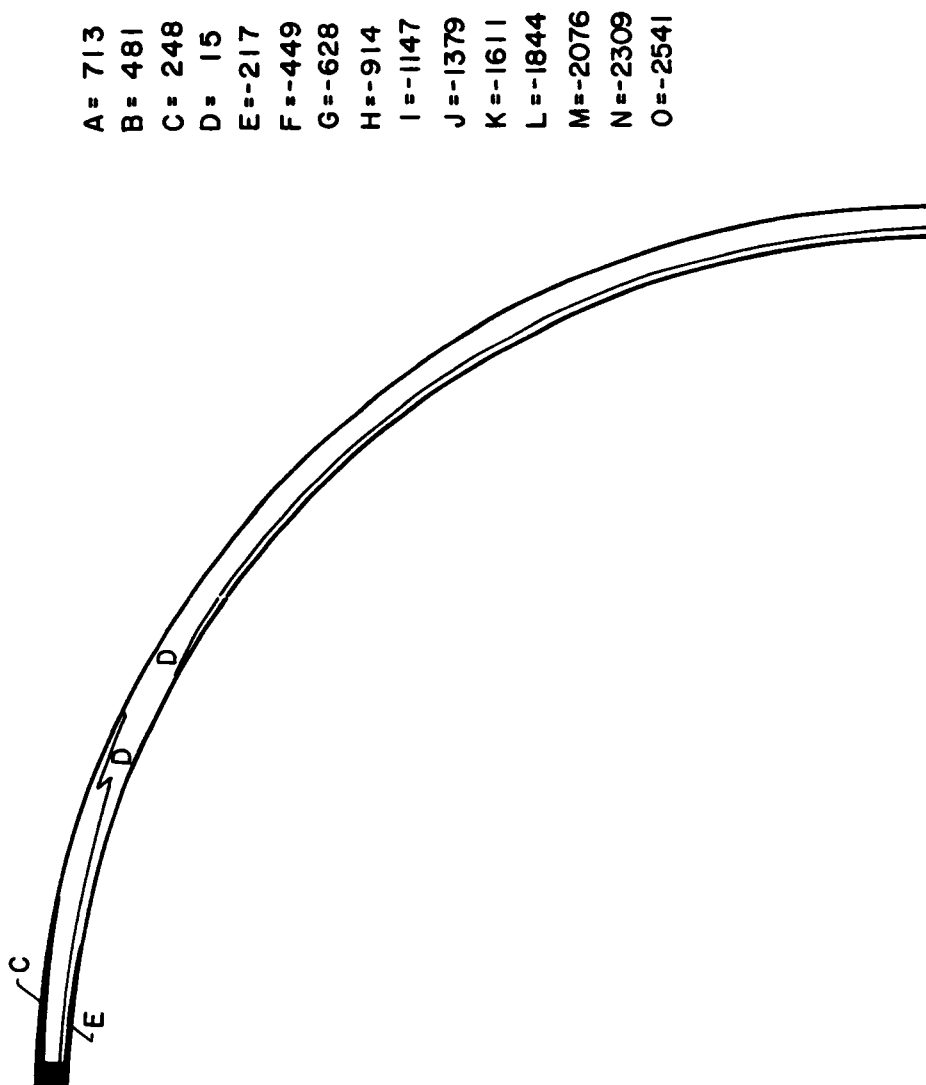


Figure 100. Contours of Radial Stresses (psi) in the Case (Long Cylinder, by ABAQUS).

A = 713
 B = 481
 C = 248
 D = 15
 E = -217
 F = -449
 G = -628
 H = -914
 I = -1147
 J = -1379
 K = -1611
 L = -1844
 M = -2076
 N = -2309
 O = -2541

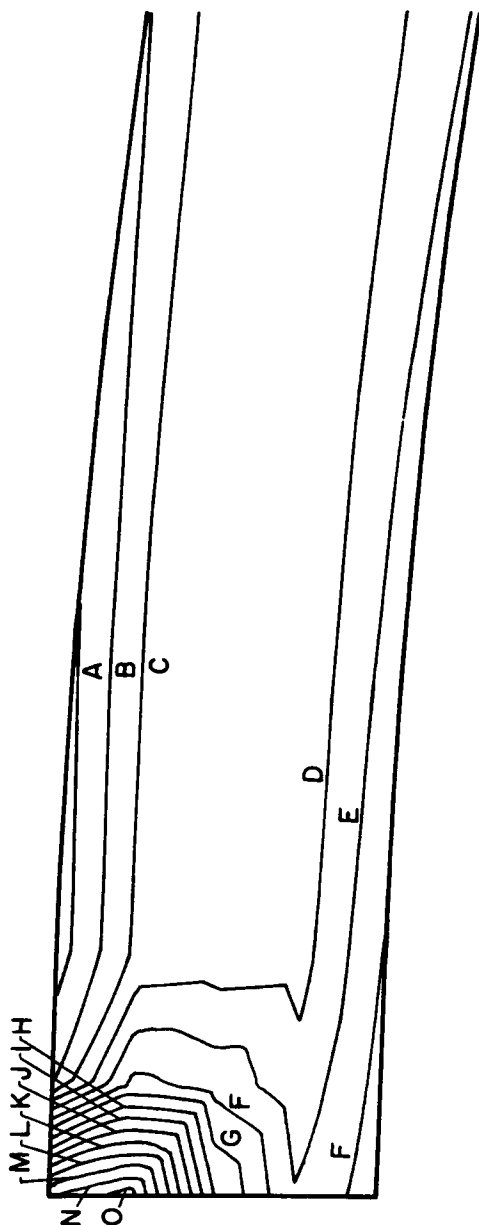


Figure 101. Enlargement of the Left-upper Corner of Figure 100.

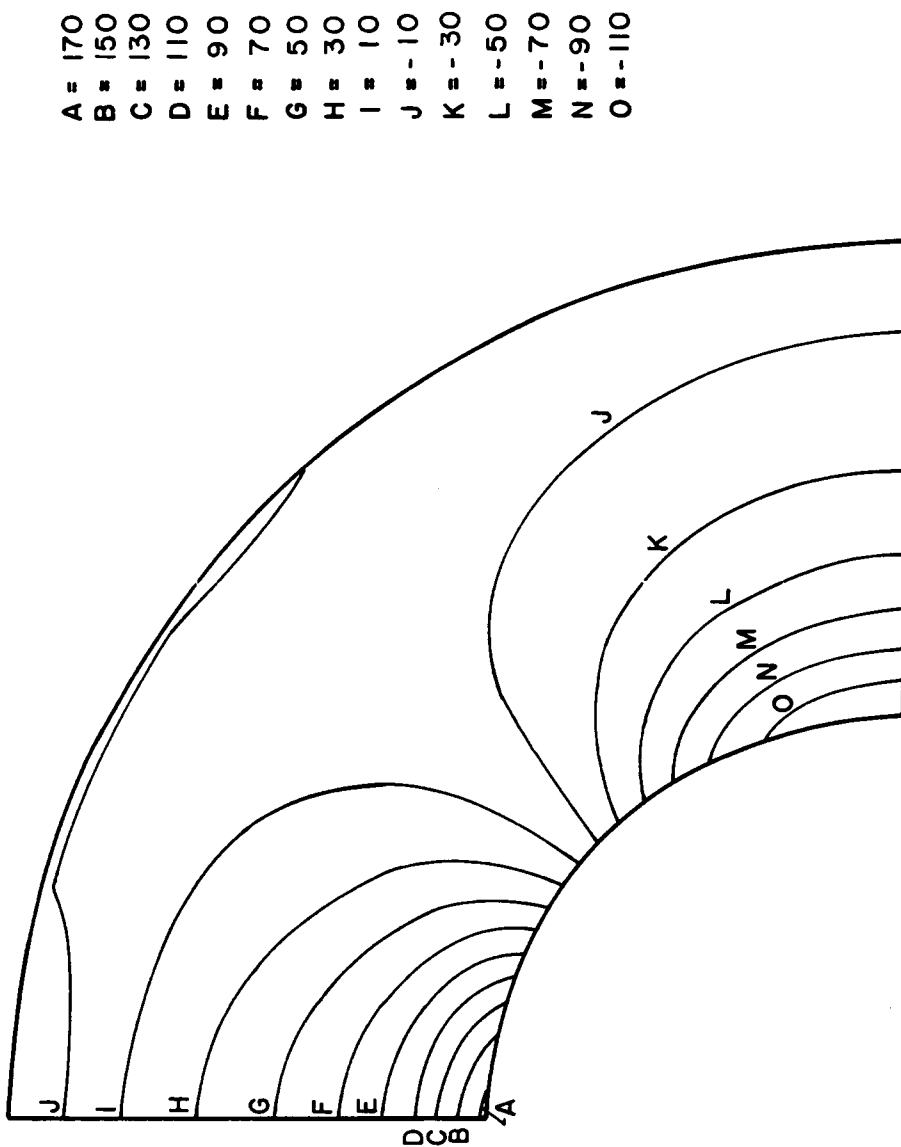


Figure 102. Contours of Tangential Stresses (psi) in the Propellant (Long Cylinder, by ABAQUS).

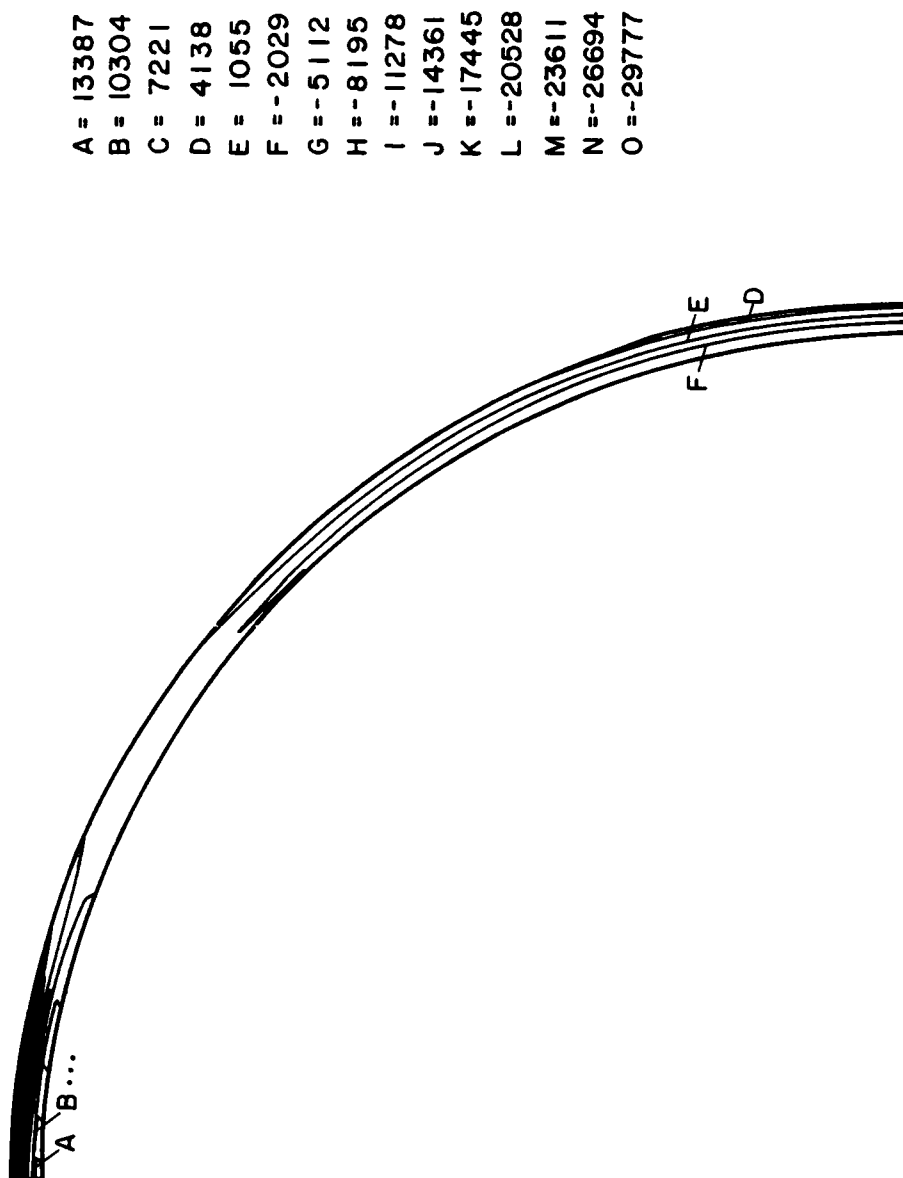


Figure 103. Contours of Tangential Stresses (psi) in the Case (Long Cylinder, by ABAQUS).

A = 13387
 B = 10304
 C = 7221
 D = 4138
 E = 1055
 F = -2029
 G = -5112
 H = -8195
 I = -11278
 J = -14361
 K = -17445
 L = -20528
 M = -23611
 N = -26694
 O = -29777

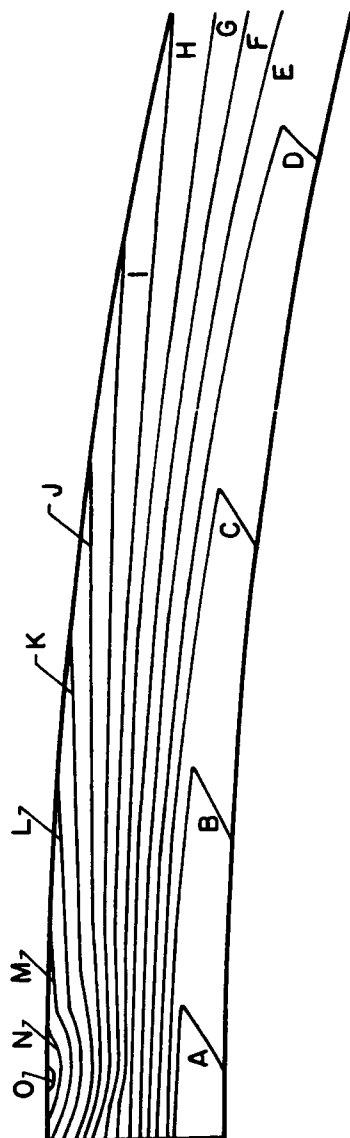


Figure 104. Enlargement of the Left-upper Corner of Figure 103.

6.0 CONCLUSIONS

The analysis of coupled thermal stresses resulting from thermal shocks applied to the internal and external surfaces of an isotropic multilayered hollow cylinder by the method of complex frequency response functions together with the Fourier transform technique have been developed. It has been seen that a large coupling parameter has a significant influence on structural response. Values of the parameter have considerable variations for engineering materials as indicated in Table 2. Because mechanical and thermal properties vary with temperature, the coupling parameter also undergoes thermal changes. Unfortunately information on this temperature dependence is largely unavailable, hence a range of values between zero and 0.5 have been arbitrarily chosen. Coupling affects the results to a lesser extent when the rise time of the input function is longer, that is, when the input temperature varies slowly. In this study, temperature changes rapidly, (rises from room temperature to 3000 °K within only several seconds). For high rates of temperature rise, the coupling effects of thermal and mechanical parameters are important but inertial effects can be neglected as long as rates are significantly lower than the speed of dilatational waves in the material.

The "weakest link" principle together with the Weibull distribution has been used to calculate the probability of failure for the structure. This method is applicable to brittle materials and is useful for structural components in the present problem. Once the stresses in each element were evaluated, the reliability calculations were carried out on an element by element basis. The reliability of the structure was then obtained from the product rule.

An orthotropic carbon-carbon multilayered hollow cylinder subjected to thermal shock and combined with mechanical loads has been investigated by an elasticity approach together with Laplace transform technique. Because for this material the coupling parameter is small, coupling effects were neglected. The illustrations presented indicate that thermal stress and reliability distributions in a multilayered orthotropic cylinder are sometimes much more complicated than thermal stress and reliability distributions in a multilayered isotropic cylinder. For orthotropic materials with same moduli, but different Poisson's ratios, the response stresses and reliability to the same load may be largely different. The illustrations presented also indicate that for the particular geometry and thermal shock applied, the structure would fail but the application of an appropriate internal pressure would reduce the induced stresses and the probability of failure as well.

The complex frequency response functions of stresses in a multilayered orthotropic cylinder subjected to asymmetric thermal loading have also been developed. The results can be used in random stress and reliability analyses later.

The elasticity approach has also been utilized to determine the states of stress and deformation in solid propellant motors contained in a single or multilayered, orthotropic, Kevlar-epoxy composite case and subjected to static and dynamic line

loads. The reliability of the motors is calculated based on the first-order, second-moment (FOSM) method. It has been found that neither the state of stress nor the probability failure are significantly affected by lumping multiple orthotropic layers into a single, thick, orthotropic layer. The most critical failure region is at the bore of the cylinder though calculations can be carried out for bond-line failures if the properties of the materials at this region are available. The results of the stress analysis were compared to those obtained with the aid of the commercial finite element code ABAQUS. Good agreement has been found.

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Appendix A. Particular Solutions for Displacements under Thermal Loading

for $n = 0$

$$u_{p0} = \frac{1}{2C_{11} \lambda_{01}} \left(r^{\lambda_{01}} \int \rho^{-\lambda_{01}+1} f_r d\rho - r^{-\lambda_{01}} \int \rho^{\lambda_{01}+1} f_r d\rho \right)$$

$$v_{p0} \equiv 0$$

$$\frac{du_{p0}}{dr} = \frac{1}{2C_{11}} \left(r^{\lambda_{01}-1} \int \rho^{-\lambda_{01}+1} f_r d\rho + r^{-\lambda_{01}-1} \int \rho^{\lambda_{01}+1} f_r d\rho \right)$$

$$\frac{dv_{p0}}{dr} \equiv 0$$

for $n = 1$

$$\begin{aligned}
u_{p1} = & \frac{1}{C_{11}C_{66}(\lambda_{11})^2} \left\{ - (C_{22} + C_{66}) \left(\int \rho \ln \rho f_r d\rho - \ln r \int \rho f_r d\rho \right) \right. \\
& - (C_{12} + C_{66}) \int \rho f_\theta d\rho \\
& \left. - (C_{22} + C_{66}) \left(\int \rho \ln \rho f_\theta d\rho - \ln r \int \rho f_\theta d\rho \right) \right\} \\
& + \frac{1}{2C_{11}C_{66}(\lambda_{11})^3} \left\{ (C_{66}\lambda_{11}^2 - C_{22} - C_{66}) r^{\lambda_{11}} \int \rho^{1-\lambda_{11}} f_r d\rho \right. \\
& - (C_{66}\lambda_{11}^2 - C_{22} - C_{66}) r^{-\lambda_{11}} \int \rho^{1+\lambda_{11}} f_r d\rho \\
& - [C_{22} + C_{66} - \lambda_{11}(C_{12} + C_{66})] r^{\lambda_{11}} \int \rho^{1-\lambda_{11}} f_\theta d\rho \\
& \left. + [C_{22} + C_{66} + \lambda_{11}(C_{12} + C_{66})] r^{-\lambda_{11}} \int \rho^{1+\lambda_{11}} f_\theta d\rho \right\}
\end{aligned}$$

$$\begin{aligned}
v_{p1} = & \frac{in}{C_{11}C_{66}(\lambda_{11})^2} \left\{ - (C_{22} + C_{66}) \left(\int \rho \ln \rho f_\theta d\rho - \ln r \int \rho f_\theta d\rho \right) \right. \\
& + (C_{12} + C_{66}) \int \rho f_r d\rho \\
& \left. + (C_{22} + C_{66}) \left(\ln r \int \rho f_r d\rho - \int \rho \ln \rho f_r d\rho \right) \right\} \\
& + \frac{in}{2C_{11}C_{66}(\lambda_{11})^3} \left\{ (C_{11}\lambda_{11}^2 - C_{22} - C_{66}) r^{\lambda_{11}} \int \rho^{1-\lambda_{11}} f_\theta d\rho \right. \\
& - (C_{11}\lambda_{11}^2 - C_{22} - C_{66}) r^{-\lambda_{11}} \int \rho^{1+\lambda_{11}} f_\theta d\rho \\
& - [(C_{12} + C_{66})\lambda_{11} + C_{22} + C_{66}] r^{\lambda_{11}} \int \rho^{1-\lambda_{11}} f_r d\rho \\
& \left. - [(C_{12} + C_{66})\lambda_{11} - C_{22} - C_{66}] r^{-\lambda_{11}} \int \rho^{1+\lambda_{11}} f_r d\rho \right\}
\end{aligned}$$

$$\begin{aligned}
\frac{du_{p1}}{dr} = & \frac{1}{C_{11}C_{66}(\lambda_{11})^2} \frac{(C_{22} + C_{66})}{r} \left(\int \rho f_r d\rho + \int \rho f_\theta d\rho \right) \\
& + \frac{1}{2C_{11}C_{66}(\lambda_{11})^2} \left\{ (C_{66}\lambda_{11}^2 - C_{22} - C_{66}) r^{\lambda_{11}-1} \int \rho^{1-\lambda_{11}} f_r d\rho \right. \\
& \quad + (C_{66}\lambda_{11}^2 - C_{22} - C_{66}) r^{-\lambda_{11}-1} \int \rho^{1+\lambda_{11}} f_r d\rho \\
& \quad - [C_{22} + C_{66} - \lambda_{11}(C_{12} + C_{66})] r^{\lambda_{11}-1} \int \rho^{1-\lambda_{11}} f_\theta d\rho \\
& \quad \left. - [C_{22} + C_{66} + \lambda_{11}(C_{12} + C_{66})] r^{-\lambda_{11}-1} \int \rho^{1+\lambda_{11}} f_\theta d\rho \right\}
\end{aligned}$$

$$\begin{aligned}
\frac{dv_{p1}}{dr} = & \frac{in}{C_{11}C_{66}(\lambda_{11})^2} \frac{(C_{22} + C_{66})}{r} \left(\int \rho f_\theta d\rho + \int \rho f_r d\rho \right) \\
& + \frac{in}{2C_{11}C_{66}(\lambda_{11})^2} \left\{ (C_{11}\lambda_{11}^2 - C_{22} - C_{66}) r^{\lambda_{11}-1} \int \rho^{1-\lambda_{11}} f_\theta d\rho \right. \\
& \quad + (C_{11}\lambda_{11}^2 - C_{22} - C_{66}) r^{-\lambda_{11}-1} \int \rho^{1+\lambda_{11}} f_\theta d\rho \\
& \quad - [(C_{12} + C_{66})\lambda_{11} + C_{22} + C_{66}] r^{\lambda_{11}-1} \int \rho^{1-\lambda_{11}} f_r d\rho \\
& \quad \left. + [(C_{12} + C_{66})\lambda_{11} - C_{22} - C_{66}] r^{-\lambda_{11}-1} \int \rho^{1+\lambda_{11}} f_r d\rho \right\}
\end{aligned}$$

for $n > 1$

$$\begin{aligned}
u_{pn} = & \frac{1}{2C_{11}C_{66}\lambda_1(\lambda_2^2 - \lambda_1^2)} \left\{ -[(\lambda_1^2 - 1)C_{66} - C_{22}n^2] r^{\lambda_1} \int \rho^{1-\lambda_1} f_r d\rho \right. \\
& + [(\lambda_1^2 - 1)C_{66} - C_{22}n^2] r^{-\lambda_1} \int \rho^{1+\lambda_1} f_r d\rho \\
& + n^2 [C_{22} + C_{66} - \lambda_1(C_{12} + C_{66})] r^{\lambda_1} \int \rho^{1-\lambda_1} f_\theta d\rho \\
& \left. - n^2 [C_{22} + C_{66} + \lambda_1(C_{12} + C_{66})] r^{-\lambda_1} \int \rho^{1+\lambda_1} f_\theta d\rho \right\} \\
& + \frac{1}{2C_{11}C_{66}\lambda_2(\lambda_2^2 - \lambda_1^2)} \left\{ [(\lambda_2^2 - 1)C_{66} - C_{22}n^2] r^{\lambda_2} \int \rho^{1-\lambda_2} f_r d\rho \right. \\
& - [(\lambda_2^2 - 1)C_{66} - C_{22}n^2] r^{-\lambda_2} \int \rho^{1+\lambda_2} f_r d\rho \\
& - n^2 [C_{22} + C_{66} - \lambda_2(C_{12} + C_{66})] r^{\lambda_2} \int \rho^{1-\lambda_2} f_\theta d\rho \\
& \left. + n^2 [C_{22} + C_{66} + \lambda_2(C_{12} + C_{66})] r^{-\lambda_2} \int \rho^{1+\lambda_2} f_\theta d\rho \right\} \\
v_{pn} = & \frac{in}{2C_{11}C_{66}\lambda_1(\lambda_2^2 - \lambda_1^2)} \left\{ -(\lambda_1^2 C_{11} - C_{22} - C_{66}n^2) r^{\lambda_1} \int \rho^{1-\lambda_1} f_\theta d\rho \right. \\
& + (\lambda_1^2 C_{11} - C_{22} - C_{66}n^2) r^{-\lambda_1} \int \rho^{1+\lambda_1} f_\theta d\rho \\
& + [C_{22} + C_{66} + \lambda_1(C_{12} + C_{66})] r^{\lambda_1} \int \rho^{1-\lambda_1} f_r d\rho \\
& \left. - [C_{22} + C_{66} - \lambda_1(C_{12} + C_{66})] r^{-\lambda_1} \int \rho^{1+\lambda_1} f_r d\rho \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{in}{2C_{11}C_{66}\lambda_2(\lambda_2^2 - \lambda_1^2)} \left\{ (\lambda_2^2 C_{11} - C_{22} - C_{66}n^2)r^{\lambda_2} \int \rho^{1-\lambda_2} f_\theta d\rho \right. \\
& \quad - (\lambda_2^2 C_{11} - C_{22} - C_{66}n^2)r^{-\lambda_2} \int \rho^{1+\lambda_2} f_\theta d\rho \\
& \quad - [C_{22} + C_{66} + \lambda_2(C_{12} + C_{66})]r^{\lambda_2} \int \rho^{1-\lambda_2} f_r d\rho \\
& \quad \left. + [C_{22} + C_{66} - \lambda_2(C_{12} + C_{66})]r^{-\lambda_2} \int \rho^{1+\lambda_2} f_r d\rho \right\}
\end{aligned}$$

$$\begin{aligned}
\frac{du_{pn}}{dr} = \frac{1}{2C_{11}C_{66}(\lambda_2^2 - \lambda_1^2)} \left\{ -[(\lambda_1^2 - 1)C_{66} - C_{22}n^2]r^{\lambda_1-1} \int \rho^{1-\lambda_1} f_r d\rho \right. \\
& \quad - [(\lambda_1^2 - 1)C_{66} - C_{22}n^2]r^{-\lambda_1-1} \int \rho^{1+\lambda_1} f_r d\rho \\
& \quad + [(\lambda_2^2 - 1)C_{66} - C_{22}n^2]r^{\lambda_2-1} \int \rho^{1-\lambda_2} f_r d\rho \\
& \quad + [(\lambda_2^2 - 1)C_{66} - C_{22}n^2]r^{-\lambda_2-1} \int \rho^{1+\lambda_2} f_r d\rho \\
& \quad + n^2[C_{22} + C_{66} - \lambda_1(C_{12} + C_{66})]r^{\lambda_1-1} \int \rho^{1-\lambda_1} f_\theta d\rho \\
& \quad - n^2[C_{22} + C_{66} - \lambda_2(C_{12} + C_{66})]r^{\lambda_2-1} \int \rho^{1-\lambda_2} f_\theta d\rho \\
& \quad + n^2[C_{22} + C_{66} + \lambda_1(C_{12} + C_{66})]r^{-\lambda_1-1} \int \rho^{1+\lambda_1} f_\theta d\rho \\
& \quad \left. - n^2[C_{22} + C_{66} + \lambda_2(C_{12} + C_{66})]r^{-\lambda_2-1} \int \rho^{1+\lambda_2} f_\theta d\rho \right\}
\end{aligned}$$

$$\begin{aligned}
\frac{dv_{pn}}{dr} = \frac{in}{2C_{11}C_{66}(\lambda_2^2 - \lambda_1^2)} \left\{ & -(\lambda_1^2 C_{11} - C_{22} - C_{66}n^2) r^{\lambda_1 - 1} \int \rho^{1 - \lambda_1} f_\theta d\rho \right. \\
& -(\lambda_1^2 C_{11} - C_{22} - C_{66}n^2) r^{-\lambda_1 - 1} \int \rho^{1 + \lambda_1} f_\theta d\rho \\
& +(\lambda_2^2 C_{11} - C_{22} - C_{66}n^2) r^{\lambda_2 - 1} \int \rho^{1 - \lambda_2} f_\theta d\rho \\
& +(\lambda_2^2 C_{11} - C_{22} - C_{66}n^2) r^{-\lambda_2 - 1} \int \rho^{1 + \lambda_2} f_\theta d\rho \\
& + [C_{22} + C_{66} + \lambda_1 (C_{12} + C_{66})] r^{\lambda_1 - 1} \int \rho^{1 - \lambda_1} f_r d\rho \\
& - [C_{22} + C_{66} + \lambda_2 (C_{12} + C_{66})] r^{\lambda_2 - 1} \int \rho^{1 - \lambda_2} f_r d\rho \\
& + [C_{22} + C_{66} - \lambda_1 (C_{12} + C_{66})] r^{-\lambda_1 - 1} \int \rho^{1 + \lambda_1} f_r d\rho \\
& \left. - [C_{22} + C_{66} - \lambda_2 (C_{12} + C_{66})] r^{-\lambda_2 - 1} \int \rho^{1 + \lambda_2} f_r d\rho \right\}
\end{aligned}$$

$$r^{\lambda_{11}} \int \rho^{1 - \lambda_{11}} f_r d\rho = \beta_1 r T_n + (\beta_1 \lambda_{11} - \beta_2) r^{\lambda_{11}} \int \rho^{-\lambda_{11}} T_n d\rho$$

$$r^{-\lambda_{11}} \int \rho^{1 + \lambda_{11}} f_r d\rho = \beta_1 r T_n - (\beta_1 \lambda_{11} + \beta_2) r^{-\lambda_{11}} \int \rho^{\lambda_{11}} T_n d\rho$$

$$r^{\lambda_{11}} \int \rho^{1 - \lambda_{11}} f_\theta d\rho = r^{\lambda_{11}} \beta_2 \int \rho^{-\lambda_{11}} T_n d\rho$$

$$r^{-\lambda_{11}} \int \rho^{1 + \lambda_{11}} f_\theta d\rho = r^{-\lambda_{11}} \beta_2 \int \rho^{\lambda_{11}} T_n d\rho$$

$$\int \rho f_r d\rho = \beta_1 r T_n - \beta_2 \int T_n d\rho$$

$$\int \rho \ln \rho f_r d\rho = \beta_1 r \ln r T_n - \beta_2 \int \ln \rho T_n d\rho - \beta_1 \int T_n d\rho$$

$$\int \rho f_\theta d\rho = \beta_2 \int T_n d\rho$$

$$\int \rho \ln \rho f_\theta d\rho = \beta_2 \int \ln \rho T_n d\rho$$

Appendix B. Particular Solutions for Displacements under Dynamic Loading

When $m = 1$, Eq. 3.23 becomes

$$\begin{aligned}\frac{\partial^2 u_{p1}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{p1}}{\partial r} - \frac{C_{22}}{C_{11}} \frac{u_{p1}}{r^2} &= \frac{\partial^2 u_0}{\partial t^2} \\ &= \frac{d^2 u_{01}}{dt^2} r^{\lambda_{01}} + \frac{d^2 u_{02}}{dt^2} r^{-\lambda_{01}}\end{aligned}$$

The solution is as the follows:

$$u_{p1} = \frac{1}{4(\lambda_{01} + 1)} \frac{d^2 u_{01}}{dt^2} r^{\lambda_{01} + 2} - \frac{1}{4(\lambda_{01} - 1)} \frac{d^2 u_{02}}{dt^2} r^{-\lambda_{01} + 2}$$

When $m = 2$, Eq. 3.23 becomes

$$\begin{aligned}\frac{\partial^2 u_{p2}}{\partial r^2} + \frac{1}{r} \frac{\partial u_{p2}}{\partial r} - \frac{C_{22}}{C_{11}} \frac{u_{p2}}{r^2} &= \frac{\partial^2 u_1}{\partial t^2} \\ &= \frac{d^2 u_{11}}{dt^2} r^{\lambda_{01}} + \frac{d^2 u_{12}}{dt^2} r^{-\lambda_{01}} + \frac{\partial^2 u_{p1}}{\partial t^2} \\ &= \frac{d^2 u_{11}}{dt^2} r^{\lambda_{01}} + \frac{d^2 u_{12}}{dt^2} r^{-\lambda_{01}} + \frac{1}{4(\lambda_{01} + 1)} \frac{d^4 u_{01}}{dt^4} r^{\lambda_{01} + 2} - \frac{1}{4(\lambda_{01} - 1)} \frac{d^4 u_{02}}{dt^4} r^{-\lambda_{01} + 2}\end{aligned}$$

The solution is as the follows:

$$\begin{aligned}
 u_{p2} = & \frac{1}{4(\lambda_{01} + 1)} \frac{d^2 u_{11}}{dt^2} r^{\lambda_{01} + 2} - \frac{1}{4(\lambda_{01} - 1)} \frac{d^2 u_{12}}{dt^2} r^{-\lambda_{01} + 2} \\
 & + \frac{1}{32(\lambda_{01} + 1)(\lambda_{01} + 2)} \frac{d^4 u_{01}}{dt^4} r^{\lambda_{01} + 4} + \frac{1}{32(\lambda_{01} - 1)(\lambda_{01} - 2)} \frac{d^4 u_{02}}{dt^4} r^{-\lambda_{01} + 4}
 \end{aligned}$$

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