APPENDIX C   AREA AND MEMBRANE CORRECTIONS

APPENDIX C.1   Parabolic Area Correction

Derivation of Parabolic Radius and Area Correction for Drained Loading

Equation for a parabola with vertex = V(a,b) $(z - b)^2 = 4 \cdot p \cdot (r - a)$

For specimen: $b = \frac{h}{2}$

Solve for r: $r = \frac{1}{16} \cdot \frac{\left(4 \cdot z^2 - 4 \cdot z \cdot h + h^2 + 16 \cdot p \cdot a\right)}{p}$

BC's: 1)at z=0,h $r=r_0$

2)at z=h/2 r=a

1.) $r_0 = \frac{1}{16} \cdot \frac{\left(h^2 + 16 \cdot p \cdot a\right)}{p}$

Solve for p: $p = \frac{h^2}{\left(16 \cdot r_0 - 16 \cdot a\right)}$

Substitute p into expression for r: $r = \frac{1}{16} \cdot \frac{\left(4 \cdot z^2 - 4 \cdot z \cdot h + h^2 + 16 \cdot p \cdot a\right)}{p}$

$$r = r_0 + 4 \cdot \left(\frac{z^2}{h^2} - \frac{z}{h}\right) \cdot \left(r_0 - a\right)$$

Define the volume in terms of initial volume and volumetric strain:

initial volume: $Vol_0 = \pi \cdot r_0^2 \cdot h_0$

$$Volume = Vol_0 \cdot (1 - \varepsilon_v)$$

$$Volume = \pi \cdot r_0^2 \cdot h_0 \cdot (1 - \varepsilon_v)$$

$$Volume = \int_0^h \pi \cdot r^2 \, dz$$

$$Volume = \frac{1}{15} \cdot \pi \cdot h \cdot \left(8 \cdot a^2 + 3 \cdot r_0^2 + 4 \cdot r_0 \cdot a\right)$$

$$\pi \cdot r_0^2 \cdot h_0 \cdot (1 - \varepsilon_v) = \frac{1}{15} \cdot \pi \cdot h \cdot \left(8 \cdot a^2 + 3 \cdot r_0^2 + 4 \cdot r_0 \cdot a\right)$$

Solve for a (two roots):

$$\begin{bmatrix} \dfrac{r_0}{4 \cdot \sqrt{h}} \cdot \left[-\sqrt{h} + \sqrt{30 \cdot h_0 \cdot (1 - \varepsilon_v) - 5 \cdot h}\right] \\ \dfrac{r_0}{4 \cdot \sqrt{h}} \cdot \left[-\sqrt{h} - \sqrt{30 \cdot h_0 \cdot (1 - \varepsilon_v) - 5 \cdot h}\right] \end{bmatrix}$$

Root 1 is the correct root.

The complete equation for the radius is:

$$r = r_0 + 4\cdot\left(\frac{z^2}{h^2} - \frac{z}{h}\right)\cdot\left(r_0 - a\right)$$

$$r = r_0 + \left(4\cdot\frac{z^2}{h^2} - 4\cdot\frac{z}{h}\right)\cdot\left[r_0 - \frac{1}{4}\cdot\frac{r_0}{\sqrt{h}}\cdot\left[-\sqrt{h} + \sqrt{30\cdot h_0\cdot\left(1 - \varepsilon_v\right) - 5\cdot h}\right]\right]$$

$$r = r_0\cdot\left[1 + \left(\frac{z^2}{h^2} - \frac{z}{h}\right)\cdot\left[5 - \sqrt{30\cdot\frac{h_0}{h}\cdot\left(1 - \varepsilon_v\right) - 5}\right]\right]$$

Define axial strain: $\varepsilon_a = \dfrac{h_0 - h}{h_0}$ or $\varepsilon_a = 1 - \dfrac{h}{h_0}$

$$h_0 = \frac{h}{\left(1 - \varepsilon_a\right)}$$

Substitute into expression for r:

$$r = r_0 \cdot \left[1 + \frac{\left(z^2 - z \cdot h\right)}{h^2} \cdot \left[5 - \sqrt{30 \cdot \frac{h_0}{h} \cdot \left(1 - \varepsilon_v\right) - 5} \right] \right]$$

$$r = r_0 \cdot \left[1 + \frac{\left(z^2 - z \cdot h\right)}{h^2} \cdot \left[5 - \sqrt{\frac{30 \cdot \left(1 - \varepsilon_v\right)}{\left(1 - \varepsilon_a\right)} - 5} \right] \right]$$

$$\frac{r}{r_0} = \left[1 + \frac{\left(z^2 - z \cdot h\right)}{h^2} \cdot \left[5 - \sqrt{\frac{30 \cdot \left(1 - \varepsilon_v\right)}{\left(1 - \varepsilon_a\right)} - 5} \right] \right]$$

For corrected area: $A_c = A_0 \cdot k$

$$k = \frac{A_c}{A_0} = \frac{r^2}{{r_0}^2}$$

$$k = \left[1 + \frac{\left(z^2 - z \cdot h\right)}{h^2} \cdot \left[5 - \sqrt{\frac{30 \cdot \left(1 - \varepsilon_v\right)}{\left(1 - \varepsilon_a\right)} - 5} \right] \right]^2$$

At the mid-height of the specimen $z = h/2$ so :

$$\frac{r}{r_0} = \frac{-1}{4} + \frac{1}{4} \cdot \sqrt{\frac{30 \cdot \left(1 - \varepsilon_v\right)}{\left(1 - \varepsilon_a\right)} - 5}$$

$$k = \frac{A_c}{A_0} = \left[\frac{-1}{4} + \frac{1}{4} \cdot \sqrt{\frac{30 \cdot \left(1 - \varepsilon_v\right)}{\left(1 - \varepsilon_a\right)} - 5} \right]^2$$

$$k = \frac{15 \cdot \left(1 - \varepsilon_v\right)}{8 \cdot \left(1 - \varepsilon_a\right)} - \frac{1}{4} - \frac{1}{8} \cdot \sqrt{\frac{30 \cdot \left(1 - \varepsilon_v\right)}{\left(1 - \varepsilon_a\right)} - 5}$$

APPENDIX C.2   Variation of Strain for Parabolic Area Correction

Variation of Strain for Parabolic Area Correction

BC's: 1)at z=0,h $r=r_0$
2)at z=h/2 $r=a$

$r_0 := 1.4$

$h := 5.1$

$h_0 := 6$

$\varepsilon_{vAVG} := 0.0$

$zz := 20$

$i := 0 .. zz$

$z_i := \frac{i}{zz} \cdot h$

The complete equation for the radius is: $r_i := r_0 \cdot \left[1 + \left[\frac{(z_i)^2}{h^2} - \frac{z_i}{h} \right] \cdot \left[5 - \sqrt{30 \cdot \frac{h_0}{h} \cdot \left(1 - \varepsilon_{vAVG}\right) - 5} \right] \right]$

The radial strain at height h is: $\varepsilon_r = \frac{r_0 - r}{r_0}$ $\qquad \varepsilon_{r_i} := 1 - \frac{r_i}{r_0}$

$$\varepsilon_r = \left(\frac{z^2}{h^2} - \frac{z}{h} \right) \cdot \left[5 - \sqrt{30 \cdot \frac{h_0}{h} \cdot \left(1 - \varepsilon_{vAVG}\right) - 5} \right]$$

Define average axial strain: $\varepsilon_{aAVG} = \frac{h_0 - h}{h_0}$ or $\qquad \varepsilon_{aAVG} := 1 - \frac{h}{h_0}$

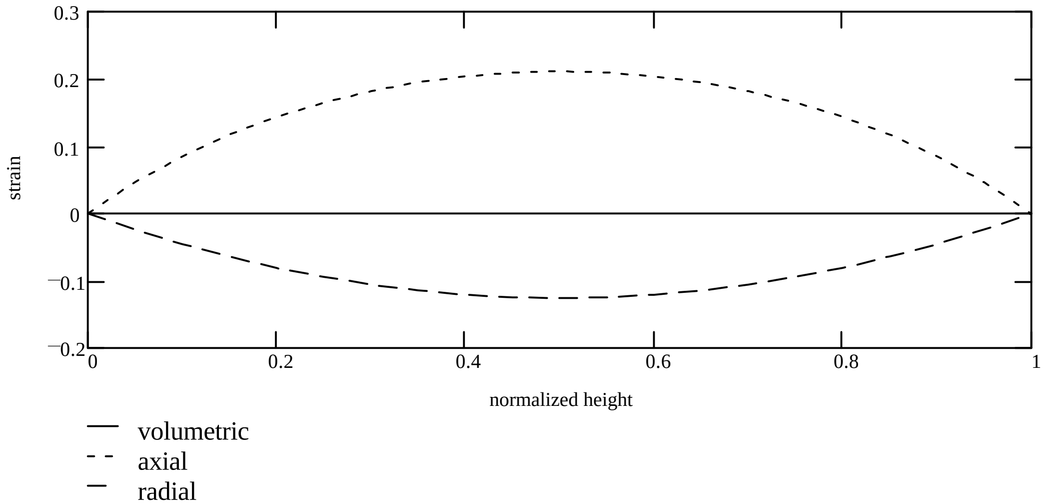
Radial strain: $\varepsilon_r = \left(\frac{z^2}{h^2} - \frac{z}{h} \right) \cdot \left[5 - \sqrt{30 \cdot \frac{\left(1 - \varepsilon_{vAVG}\right)}{\left(1 - \varepsilon_{aAVG}\right)} - 5} \right]$

All strain are constant in horizontal plane, so strains can only vary with z. Radial strain varies with z, and is determined from equation above.

$$1 - \varepsilon_v = \left(1 - \varepsilon_r\right)^2 \cdot \left(1 - \varepsilon_a\right)$$

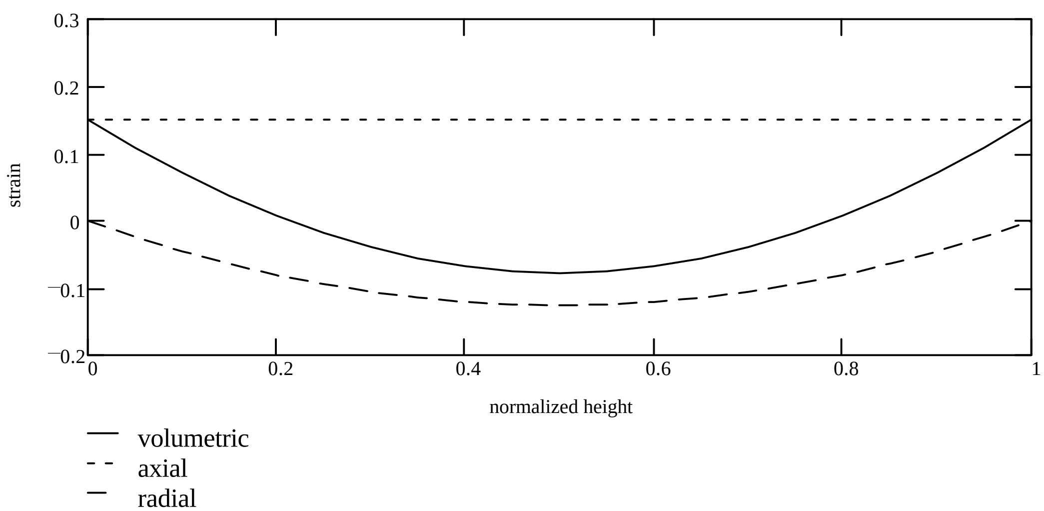
1.) Assume that volumetric strain is distributed uniformly with height, so that ε_v is equal to ε_{vAVG}

$$1-\varepsilon_v = \left(1-\varepsilon_r\right)^2 \cdot \left(1-\varepsilon_a\right) \qquad \varepsilon_{v_i} := \varepsilon_{vAVG} \qquad \varepsilon_{a_i} := 1 - \frac{1-\varepsilon_{vAVG}}{\left(1-\varepsilon_{r_i}\right)^2}$$



2.) Assume that axial strain is distributed uniformly with height, so that ε_a is equal to ε_{aAVG}

$$1-\varepsilon_v = \left(1-\varepsilon_r\right)^2 \cdot \left(1-\varepsilon_a\right) \qquad \varepsilon_{a_i} := \varepsilon_{aAVG} \qquad \varepsilon_{v_i} := 1 - \left(1-\varepsilon_{r_i}\right)^2 \cdot \left(1-\varepsilon_{aAVG}\right)$$



3.) Assume that variation of axial strain and volumetric strain is consistent with the variation of radial strain with height.

$$\left(1-\varepsilon_{vAVG}\right)=\left(1-\varepsilon_{rAVG}\right)^{2}\cdot\left(1-\varepsilon_{aAVG}\right)$$

$$\left(1-\varepsilon_{rAVG}\right)=\sqrt{\frac{\left(1-\varepsilon_{vAVG}\right)}{\left(1-\varepsilon_{aAVG}\right)}}$$

Apply variation in radial strain equally to axial and volumetric strain.

$$\left[\frac{\left(1-\varepsilon_{rAVG}\right)}{\left(1-\varepsilon_{r}\right)}\right]^{2}\cdot\left(1-\varepsilon_{r}\right)^{2}=\frac{1-\varepsilon_{vAVG}}{1-\varepsilon_{aAVG}}$$

$$\left(1-\varepsilon_{r}\right)^{2}=\left[\frac{1-\varepsilon_{vAVG}}{\frac{\left(1-\varepsilon_{rAVG}\right)}{\left(1-\varepsilon_{r}\right)}}\right]\cdot\left[\frac{1}{\left(1-\varepsilon_{aAVG}\right)\cdot\frac{\left(1-\varepsilon_{rAVG}\right)}{\left(1-\varepsilon_{r}\right)}}\right]$$

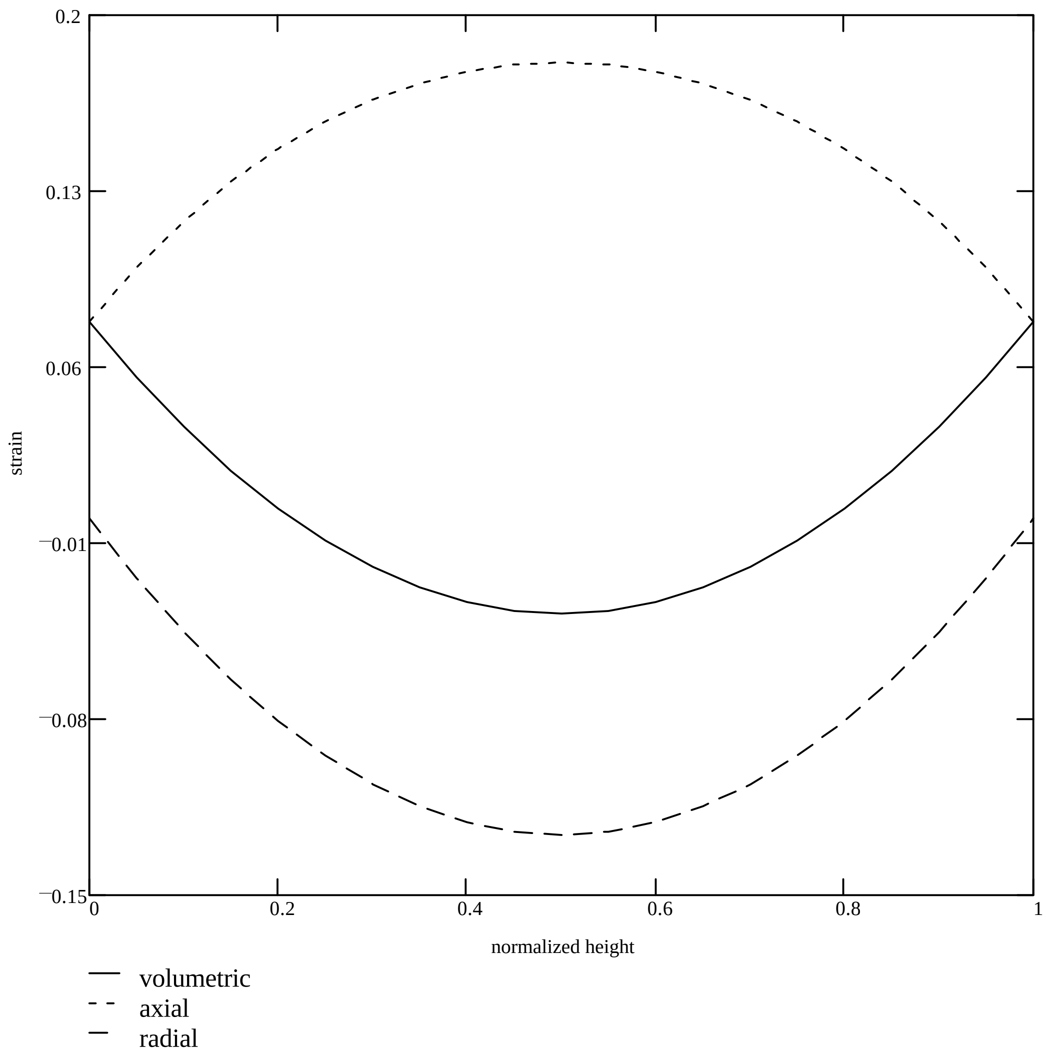
Since: $\left(1-\varepsilon_{r}\right)^{2}=\frac{\left(1-\varepsilon_{v}\right)}{\left(1-\varepsilon_{a}\right)}$ then the axial and volumetric strain can be defined.

$$\left(1-\varepsilon_{v}\right)=\frac{1-\varepsilon_{vAVG}}{\frac{\left(1-\varepsilon_{rAVG}\right)}{\left(1-\varepsilon_{r}\right)}}\qquad\left(1-\varepsilon_{a}\right)=\left(1-\varepsilon_{aAVG}\right)\cdot\frac{\left(1-\varepsilon_{rAVG}\right)}{\left(1-\varepsilon_{r}\right)}$$

$$\left(1-\varepsilon_{v}\right)=\frac{1-\varepsilon_{vAVG}}{\left[\frac{\sqrt{\frac{\left(1-\varepsilon_{vAVG}\right)}{\left(1-\varepsilon_{aAVG}\right)}}}{\left(1-\varepsilon_{r}\right)}\right]}\qquad\left(1-\varepsilon_{a}\right)=\left(1-\varepsilon_{aAVG}\right)\cdot\frac{\sqrt{\frac{\left(1-\varepsilon_{vAVG}\right)}{\left(1-\varepsilon_{aAVG}\right)}}}{\left(1-\varepsilon_{r}\right)}$$

$$\left(1-\varepsilon_{v}\right)=\sqrt{\left(1-\varepsilon_{vAVG}\right)\cdot\left(1-\varepsilon_{aAVG}\right)}\cdot\left(1-\varepsilon_{r}\right)\qquad\left(1-\varepsilon_{a}\right)=\frac{\sqrt{\left(1-\varepsilon_{vAVG}\right)\cdot\left(1-\varepsilon_{aAVG}\right)}}{\left(1-\varepsilon_{r}\right)}$$

$$\varepsilon_{v_i} := 1 - \sqrt{\left(1-\varepsilon_{vAVG}\right)\cdot\left(1-\varepsilon_{aAVG}\right)\cdot\left(1-\varepsilon_{r_i}\right)} \qquad \varepsilon_{a_i} := 1 - \frac{\sqrt{\left(1-\varepsilon_{vAVG}\right)\cdot\left(1-\varepsilon_{aAVG}\right)}}{\left(1-\varepsilon_{r_i}\right)}$$



APPENDIX C.3   Sinusoidal Area Correction

Derivation for Sinusoidal Radius and Area Correction

Drained Loading

Equation for a sinusoid with:

1.) half period = h
2.) amplitude = b

$$(r - a) = b \cdot \sin\left(\frac{\pi \cdot z}{h}\right)$$

For specimen:

$$a = r_0$$

Solving for r:

$$r = r_0 + b \cdot \sin\left(\frac{\pi \cdot z}{h}\right)$$

BC's: 1)at z=0,h $r=r_0$
2)at z=h/2 $r=r_0+b$

1.)

$$r_0 = r_0 + b \cdot \sin(0) \qquad r_0 = r_0 + b \cdot \sin(\pi)$$

$$r_0 = r_0 \qquad r_0 = r_0$$

2.)

$$r_0 + b = r_0 + b \cdot \sin\left(\frac{\pi}{2}\right)$$

$$r_0 + b = r_0 + b$$

To find b, the volume relationships are examined:

$$\text{Volume} = \int_0^h \pi \cdot r^2 \, dz$$

$$\text{Volume} = \int_0^h \pi \cdot \left(r_0 + b \cdot \sin\left(\frac{\pi \cdot z}{h}\right)\right)^2 dz$$

$$\text{Volume} = r_0^{\,2} \cdot \pi \cdot h + 4 \cdot b \cdot r_0 \cdot h + \frac{1}{2} \cdot b^2 \cdot \pi \cdot h$$

$$\text{Volume} = \left(r_0^{\,2} + \frac{4}{\pi} \cdot b \cdot r_0 + \frac{1}{2} \cdot b^2\right) \cdot \pi \cdot h$$

Define average axial strain: $\varepsilon_a = \dfrac{h_0 - h}{h_0}$ or $\varepsilon_a = 1 - \dfrac{h}{h_0}$

$$h = h_0 \cdot (1 - \varepsilon_a)$$

Substiute for h based on average axial strain:

$$\text{Volume} = \left(r_0^{\,2} + \frac{4}{\pi}\cdot b\cdot r_0 + \frac{1}{2}\cdot b^2\right)\cdot\pi\cdot h_0\cdot(1-\varepsilon_a)$$

Substiute for Volume based on average volumetric strain:

$$\text{Vol} = \text{Vol}_0\cdot(1-\varepsilon_v)$$

$$\left(r_0^{\,2} + \frac{4}{\pi}\cdot b\cdot r_0 + \frac{1}{2}\cdot b^2\right)\cdot\pi\cdot h_0\cdot(1-\varepsilon_a) = \text{Vol}_0\cdot(1-\varepsilon_v)$$

Initial volume: $\text{Vol}_0 = \pi\cdot r_0^{\,2}\cdot h_0$

$$\left(r_0^{\,2} + \frac{4}{\pi}\cdot b\cdot r_0 + \frac{1}{2}\cdot b^2\right)\cdot\pi\cdot h_0\cdot(1-\varepsilon_a) = \pi\cdot r_0^{\,2}\cdot h_0\cdot(1-\varepsilon_v)$$

$$\left[1 + \frac{4}{\pi}\cdot\frac{b}{r_0} + \frac{1}{2}\cdot\left(\frac{b}{r_0}\right)^2\right] - \frac{(1-\varepsilon_v)}{(1-\varepsilon_a)} = 0$$

$$\frac{1}{2}\cdot\left(\frac{b}{r_0}\right)^2 + \frac{4}{\pi}\cdot\frac{b}{r_0} + 1 - \frac{(1-\varepsilon_v)}{(1-\varepsilon_a)} = 0$$

Solve for b:

$$\begin{bmatrix} r_0\cdot\left[\dfrac{-4}{\pi} + \sqrt{2}\cdot\sqrt{\dfrac{8}{\pi^2} - 1 + \dfrac{(1-\varepsilon_v)}{(1-\varepsilon_a)}}\right] \\ r_0\cdot\left[\dfrac{-4}{\pi} - \sqrt{2}\cdot\sqrt{\dfrac{8}{\pi^2} - 1 + \dfrac{(1-\varepsilon_v)}{(1-\varepsilon_a)}}\right] \end{bmatrix}$$

Check to see which is the appropriate root:

$$\varepsilon_a := .1 \qquad \varepsilon_v := 0 \qquad r_0 := 1.4 \qquad h := 5.4 \qquad z := \frac{h}{2}$$

$$b_1 := r_0 \cdot \left[\frac{-4}{\pi} + \sqrt{2} \cdot \sqrt{\frac{8}{\pi^2} - 1 + \frac{(1 - \varepsilon_v)}{(1 - \varepsilon_a)}} \right] \qquad b_1 = 0.118$$

$$b_2 := r_0 \cdot \left[\frac{-4}{\pi} - \sqrt{2} \cdot \sqrt{\frac{8}{\pi^2} - 1 + \frac{(1 - \varepsilon_v)}{(1 - \varepsilon_a)}} \right] \qquad b_2 = -3.683$$

$$r_1 := r_0 + b_1 \cdot \sin\left(\frac{\pi \cdot z}{h}\right) \qquad r_2 := r_0 + b_2 \cdot \sin\left(\frac{\pi \cdot z}{h}\right)$$

$$r_1 = 1.518 \qquad b_2 = -3.683$$

Root 1 is the correct root.

Now we have a complete equation for the radius:

$$r = r_0 + r_0 \cdot \left[\frac{-4}{\pi} + \sqrt{2} \cdot \sqrt{\frac{8}{\pi^2} - 1 + \frac{(1 - \varepsilon_v)}{(1 - \varepsilon_a)}} \right] \cdot \sin\left(\pi \cdot \frac{z}{h}\right)$$

Rewriting:

$$r = r_0 \cdot \left[1 + \left[\sqrt{\frac{16}{\pi^2} - 2 + 2 \cdot \frac{(1 - \varepsilon_v)}{(1 - \varepsilon_a)}} - \frac{4}{\pi} \right] \cdot \sin\left(\pi \cdot \frac{z}{h}\right) \right]$$

For corrected area:

$$\frac{A_c}{A_0} = \frac{r^2}{r_0^{\,2}}$$

$$\frac{A_c}{A_0} = \left[1 + \left[\sqrt{\frac{16}{\pi^2} - 2 + 2 \cdot \frac{(1-\varepsilon_v)}{(1-\varepsilon_a)}} - \frac{4}{\pi}\right] \cdot \sin\left(\pi \cdot \frac{z}{h}\right)\right]^2$$

At the mid-height of the specimen z = h/2 so :

$$\frac{r}{r_0} = 1 + \sqrt{\frac{16}{\pi^2} - 2 + 2 \cdot \frac{(1-\varepsilon_v)}{(1-\varepsilon_a)}} - \frac{4}{\pi}$$

$$\frac{A_c}{A_0} = \left[1 + \sqrt{\frac{16}{\pi^2} - 2 + 2 \cdot \frac{(1-\varepsilon_v)}{(1-\varepsilon_a)}} - \frac{4}{\pi}\right]^2$$

$$\frac{A_c}{A_0} = \frac{32}{\pi^2} - \frac{8}{\pi} - 1 + 2 \cdot \frac{ev}{ea} + \left(2 - \frac{8}{\pi}\right) \cdot \sqrt{2} \cdot \sqrt{\frac{8}{\pi^2} - 1 + \frac{(1-\varepsilon_v)}{(1-\varepsilon_a)}}$$

APPENDIX C.4 Superposed Membrane-Specimen Stress-Displacement Solution

Superposed Solution for Stresses Under Combined Loading of Membrane and Specimen

Superposed Solution for Thick-Walled Cylinder:

$$D = \frac{P_o \cdot b^2 - P_i \cdot a^2}{b^2 - a^2} \qquad C = \frac{a^2 \cdot b^2}{b^2 - a^2} \cdot \left(P_o - P_i\right) \qquad A = P_a$$

Stress Vector:

$$\sigma = \begin{bmatrix} D - \frac{C}{r^2} \\ A \\ D + \frac{C}{r^2} \end{bmatrix} \qquad \sigma = \begin{bmatrix} \frac{P_o \cdot b^2 - P_i \cdot a^2}{b^2 - a^2} - \frac{a^2 \cdot b^2}{r^2 \cdot \left(b^2 - a^2\right)} \cdot \left(P_o - P_i\right) \\ A \\ \frac{P_o \cdot b^2 - P_i \cdot a^2}{b^2 - a^2} + \frac{a^2 \cdot b^2}{r^2 \cdot \left(b^2 - a^2\right)} \cdot \left(P_o - P_i\right) \end{bmatrix}$$

Strain Vector:

$$\varepsilon = \frac{1}{E} \cdot \begin{bmatrix} D \cdot (1 - \nu) - A \cdot \nu - \frac{C}{r^2} \cdot (1 + \nu) \\ A - D \cdot (2 \cdot \nu) \\ D \cdot (1 - \nu) - A \cdot \nu + \frac{C}{r^2} \cdot (1 + \nu) \end{bmatrix}$$

Displacement Vector:

$$u = \frac{-1}{E}\cdot\begin{bmatrix} r\cdot\left[D\cdot(1-\nu) - A\cdot\nu + \frac{C}{r^2}\cdot(1+\nu)\right] \\ z\cdot(A - D\cdot(2\cdot\nu)) \\ 0 \end{bmatrix}$$

For Specimen, a = 0

$$D = P_{os} \qquad A = P_{as} \qquad C = 0$$

$$\sigma = \begin{bmatrix} P_{os} \\ P_{as} \\ P_{os} \end{bmatrix} \qquad \varepsilon = \frac{1}{E_s}\cdot\begin{bmatrix} P_{os}\cdot\left(1-\nu_s\right) - P_{as}\cdot\nu_s \\ P_{as} - P_{os}\cdot\left(2\cdot\nu_s\right) \\ P_{os}\cdot\left(1-\nu_s\right) - P_{as}\cdot\nu_s \end{bmatrix} \qquad u = \frac{-1}{E_s}\cdot\begin{bmatrix} r\cdot\left[P_{os}\cdot\left(1-\nu_s\right) - P_{as}\cdot\nu_s\right] \\ z\cdot\left[P_{as} - P_{os}\cdot\left(2\cdot\nu_s\right)\right] \\ 0 \end{bmatrix}$$

For Membrane

$$D = \frac{P_o\cdot b^2 - P_i\cdot a^2}{b^2 - a^2} \qquad C = \frac{a^2\cdot b^2}{b^2 - a^2}\cdot\left(P_o - P_i\right) \qquad A = P_{am}$$

$$\sigma = \begin{bmatrix} \dfrac{P_o \cdot b^2 - P_i \cdot a^2}{b^2 - a^2} - \dfrac{a^2 \cdot b^2}{r^2 \cdot \left(b^2 - a^2\right)} \cdot \left(P_o - P_i\right) \\ P_{am} \\ \dfrac{P_o \cdot b^2 - P_i \cdot a^2}{b^2 - a^2} + \dfrac{a^2 \cdot b^2}{r^2 \cdot \left(b^2 - a^2\right)} \cdot \left(P_o - P_i\right) \end{bmatrix}$$

$$\varepsilon = \frac{1}{E_m} \cdot \begin{bmatrix} \dfrac{P_o \cdot b^2 - P_i \cdot a^2}{b^2 - a^2} \cdot \left(1 - \nu_m\right) - P_{am} \cdot \nu_m - \dfrac{\dfrac{a^2 \cdot b^2}{b^2 - a^2} \cdot \left(P_o - P_i\right)}{r^2} \cdot \left(1 + \nu_m\right) \\ P_{am} - \dfrac{P_o \cdot b^2 - P_i \cdot a^2}{b^2 - a^2} \cdot \left(2 \cdot \nu_m\right) \\ \dfrac{P_o \cdot b^2 - P_i \cdot a^2}{b^2 - a^2} \cdot \left(1 - \nu_m\right) - P_{am} \cdot \nu_m + \dfrac{\dfrac{a^2 \cdot b^2}{b^2 - a^2} \cdot \left(P_o - P_i\right)}{r^2} \cdot \left(1 + \nu_m\right) \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \left(1 - \dfrac{a^2}{r^2}\right) \cdot \dfrac{b^2}{\left(b^2 - a^2\right)} \cdot P_o - \left(1 - \dfrac{b^2}{r^2}\right) \cdot \dfrac{a^2}{\left(b^2 - a^2\right)} \cdot P_i \\ P_{am} \\ \left(1 + \dfrac{a^2}{r^2}\right) \cdot \dfrac{b^2}{\left(b^2 - a^2\right)} \cdot P_o - \left(1 + \dfrac{b^2}{r^2}\right) \cdot \dfrac{a^2}{\left(b^2 - a^2\right)} \cdot P_i \end{bmatrix}$$

$$\varepsilon = \frac{1}{E_m}\cdot\left[\frac{\left[\left(1-\nu_m\right)-\frac{a^2}{r^2}\cdot\left(1+\nu_m\right)\right]\cdot\frac{b^2}{\left(b^2-a^2\right)}\cdot P_o-\left[\left(1-\nu_m\right)-\frac{b^2}{r^2}\cdot\left(1+\nu_m\right)\right]\cdot\frac{a^2}{\left(b^2-a^2\right)}\cdot P_i-P_{am}\cdot\nu_m}{\dfrac{P_{am}-\dfrac{P_o\cdot b^2-P_i\cdot a^2}{b^2-a^2}\cdot\left(2\cdot\nu_m\right)}{\left[\left(1-\nu_m\right)+\frac{a^2}{r^2}\cdot\left(1+\nu_m\right)\right]\cdot\frac{b^2}{\left(b^2-a^2\right)}\cdot P_o-\left[\left(1-\nu_m\right)+\frac{b^2}{r^2}\cdot\left(1+\nu_m\right)\right]\cdot\frac{a^2}{\left(b^2-a^2\right)}\cdot P_i-P_{am}\cdot\nu_m}}\right]$$

P_{as} P_{am} P_{os} and P_i are unknown, so boundary conditions are applied.

1.) At r = a $\quad \sigma_{rm} = \sigma_{rs}$

$$\left(1-\frac{a^2}{a^2}\right)\cdot\frac{b^2}{\left(b^2-a^2\right)}\cdot P_o-\left(1-\frac{b^2}{a^2}\right)\cdot\frac{a^2}{\left(b^2-a^2\right)}\cdot P_i = P_{os}$$

$$P_{os} = P_i$$

2.) At r = a and $u_{rm} = u_{rs}$ or $\varepsilon_{\theta m} = \varepsilon_{rs}$

$$\frac{1}{E_m}\cdot\left[\frac{P_o\cdot b^2 - P_i\cdot a^2}{b^2 - a^2}\cdot\left(1 - \nu_m\right) - P_{am}\cdot\nu_m + \frac{\frac{a^2\cdot b^2}{b^2 - a^2}\cdot\left(P_o - P_i\right)}{a^2}\cdot\left(1 + \nu_m\right)\right] = \frac{1}{E_s}\cdot\left[P_{os}\cdot\left(1 - \nu_s\right) - P_{as}\cdot\nu_s\right]$$

$$\frac{1}{E_m}\cdot\left[\left(\frac{2\cdot b^2}{b^2 - a^2}\right)\cdot P_o - \left[\frac{a^2\cdot\left(1 - \nu_m\right) + b^2\cdot\left(1 + \nu_m\right)}{b^2 - a^2}\right]\cdot P_i - P_{am}\cdot\nu_m\right] = \frac{1}{E_s}\cdot\left[P_i\cdot\left(1 - \nu_s\right) - P_{as}\cdot\nu_s\right]$$

$$P_i\cdot\left[\frac{1}{E_s}\cdot\left(1 - \nu_s\right) + \frac{1}{E_m}\cdot\frac{a^2\cdot\left(1 - \nu_m\right) + b^2\cdot\left(1 + \nu_m\right)}{b^2 - a^2}\right] = \frac{1}{E_s}\cdot P_{as}\cdot\nu_s - \frac{1}{E_m}\cdot P_{am}\cdot\nu_m + \frac{1}{E_m}\cdot\left(\frac{2\cdot b^2}{b^2 - a^2}\cdot P_o\right)$$

$$P_i\cdot\left[\frac{E_m\cdot\left(b^2 - a^2\right)\cdot\left(1 - \nu_s\right) + E_s\cdot\left[a^2\cdot\left(1 - \nu_m\right) + b^2\cdot\left(1 + \nu_m\right)\right]}{E_s\cdot E_m\cdot\left(b^2 - a^2\right)}\right] = \frac{1}{E_s}\cdot P_{as}\cdot\nu_s + \frac{1}{E_m}\cdot\left(\frac{2\cdot b^2}{b^2 - a^2}\cdot P_o - P_{am}\cdot\nu_m\right)$$

$$P_i = \frac{E_m\cdot\left(b^2 - a^2\right)\cdot P_{as}\cdot\nu_s - E_s\cdot\left(b^2 - a^2\right)\cdot P_{am}\cdot\nu_m + E_s\cdot\left(2\cdot b^2\cdot P_o\right)}{\left[E_m\cdot\left(b^2 - a^2\right)\cdot\left(1 - \nu_s\right) + E_s\cdot\left[a^2\cdot\left(1 - \nu_m\right) + b^2\cdot\left(1 + \nu_m\right)\right]\right]}$$

3.) $\varepsilon_{zm} = \varepsilon_a$

$$\varepsilon_a = \frac{1}{E_m}\cdot\left[P_{am} - \frac{\left(P_o\cdot b^2 - P_i\cdot a^2\right)}{b^2 - a^2}\cdot\left(2\cdot\nu_m\right)\right]$$

$$P_{am} = \varepsilon_a\cdot E_m + \frac{P_o\cdot b^2 - P_i\cdot a^2}{b^2 - a^2}\cdot\left(2\cdot\nu_m\right)$$

4.) $\varepsilon_{zs} = \varepsilon_a$

$$\varepsilon_a = \frac{1}{E_s}\cdot\left[P_{as} - P_{os}\cdot\left(2\cdot\nu_s\right)\right]$$

$$P_{as} = \varepsilon_a\cdot E_s + P_i\cdot\left(2\cdot\nu_s\right)$$

Substitute 3.) and 4.) into 2.)

$$P_i = \frac{E_m\cdot\left(b^2 - a^2\right)\cdot P_{as}\cdot\nu_s - E_s\cdot\left(b^2 - a^2\right)\cdot P_{am}\cdot\nu_m + E_s\cdot\left(2\cdot b^2\cdot P_o\right)}{\left[E_m\cdot\left(b^2 - a^2\right)\cdot\left(1 - \nu_s\right) + E_s\cdot\left[a^2\cdot\left(1 - \nu_m\right) + b^2\cdot\left(1 + \nu_m\right)\right]\right]}$$

$$P_i = \frac{\left[E_m\cdot\left(b^2 - a^2\right)\cdot\left(\varepsilon_a\cdot E_s + 2\cdot P_i\cdot\nu_s\right)\cdot\nu_s - E_s\cdot\left(b^2 - a^2\right)\cdot\left[\varepsilon_a\cdot E_m + 2\cdot\frac{\left(P_o\cdot b^2 - P_i\cdot a^2\right)}{\left(b^2 - a^2\right)}\cdot\nu_m\right]\cdot\nu_m + 2\cdot E_s\cdot P_o\cdot b^2\right]}{\left[E_m\cdot\left(b^2 - a^2\right)\cdot\left(1 - \nu_s\right) + E_s\cdot\left[a^2\cdot\left(1 - \nu_m\right) + b^2\cdot\left(1 + \nu_m\right)\right]\right]}$$

$$P_i\cdot\left[1-\frac{E_s\cdot a^2\cdot\left(2\cdot\nu_m{}^2\right)+E_m\cdot\left(b^2-a^2\right)\cdot\left(2\cdot\nu_s{}^2\right)}{\left[E_m\cdot\left(b^2-a^2\right)\cdot\left(1-\nu_s\right)+E_s\cdot\left[a^2\cdot\left(1-\nu_m\right)+b^2\cdot\left(1+\nu_m\right)\right]\right]}\right]=\frac{\left[\varepsilon_a\cdot E_s\cdot E_m\cdot\left(b^2-a^2\right)\cdot\left(\nu_s-\nu_m\right)+2\cdot E_s\cdot P_o\cdot b^2\cdot\left(1-\nu_m{}^2\right)\right]}{\left[E_m\cdot\left(b^2-a^2\right)\cdot\left(1-\nu_s\right)+E_s\cdot\left[a^2\cdot\left(1-\nu_m\right)+b^2\cdot\left(1+\nu_m\right)\right]\right]}$$

$$P_i\cdot\left[E_m\cdot\left(b^2-a^2\right)\cdot\left(1-\nu_s-2\cdot\nu_s{}^2\right)+E_s\cdot\left[a^2\cdot\left(1-\nu_m-2\cdot\nu_m{}^2\right)+b^2\cdot\left(1+\nu_m\right)\right]\right]=\varepsilon_a\cdot E_s\cdot E_m\cdot\left(b^2-a^2\right)\cdot\left(\nu_s-\nu_m\right)+2\cdot E_s\cdot P_o\cdot b^2\cdot\left(1-\nu_m{}^2\right)$$

$$P_i=\frac{\varepsilon_a\cdot E_m\cdot\left(b^2-a^2\right)\cdot\left(\nu_s-\nu_m\right)+2\cdot P_o\cdot b^2\cdot\left(1+\nu_m\right)\cdot\left(1-\nu_m\right)}{\left[\frac{E_m}{E_s}\cdot\left(b^2-a^2\right)\cdot\left(1+\nu_s\right)\cdot\left(1-2\cdot\nu_s\right)+\left[a^2\cdot\left(1+\nu_m\right)\cdot\left(1-2\cdot\nu_m\right)+b^2\cdot\left(1+\nu_m\right)\right]\right]}$$

If $\nu_s=\nu_m$ and $E_s=E_m$

$$P_i=P_o$$

If $\nu_s=\nu_m=\frac{1}{2}$ and $E_s\neq E_m$

$$P_i=P_o$$

If $\nu_s=\nu_m$ and $E_s\neq E_m$

$$P_i=\frac{\left(1-\nu_m\right)\cdot\left(2\cdot b^2\right)}{\left[\frac{E_m}{E_s}\cdot\left(b^2-a^2\right)\cdot\left(1-2\cdot\nu_m\right)+a^2\cdot\left(1-2\cdot\nu_m\right)+b^2\right]}\cdot P_o$$