

AN INCREMENTAL TOTAL LAGRANGIAN FORMULATION FOR
GENERAL ANISOTROPIC SHELL-TYPE STRUCTURES

by

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ABSTRACT

Based on the principle of virtual displacements, the incremental equations of motion of a continuous medium are formulated by using the total Lagrangian description. After linearization of the incremental equations of motion, the displacement finite element model is obtained, which is solved iteratively. From this displacement finite element model, four different elements, i.e. degenerated shell element, degenerated curved beam element, 3-D continuum element and solid-shell transition element, are developed for the geometric nonlinear analysis of general shell-type structures, anisotropic as well as isotropic. Compatibility and completeness requirements are stressed in modelling the general shell-type structures in order to assure the convergence of the finite-element analysis. For the transient analysis Newmark scheme is adopted for time discretization. An iterative solution procedure, either Newton-Raphson method or modified Riks/Wempner method, is employed to trace the nonlinear equilibrium path. The latter is also used to perform post-buckling analysis. A variety of numerical examples are presented to demonstrate the validity and efficiency of various elements separately and in combination. The effects of boundary conditions, lamination scheme, transverse shear deformations and geometric nonlinearity on static and transient responses are also investigated. Many of the numerical results of general shell-type structures presented here could serve as references for future investigations.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

In the practical analysis of shell-type structures, such as stiffened plates and shells, folded plates, arch dams and foundations, turbine blades mounted on a shaft etc., by finite element method we need to employ shell elements in conjunction with three-dimensional beam elements and/or 3-D continuum elements to model the structures effectively and accurately. Also transition elements are preferred to model shell intersections and solid-to-shell transition regions without invoking the constraint equations. These elements must satisfy compatibility and completeness requirements in order to assure the convergence of the finite-element analysis.

Traditionally the formulations of shell and beam elements have been based on various shell and beam theories. The detailed governing equations may vary considerably depending upon the approximation used. This accounts for the existence of several different beam and shell finite elements with varying degree of accuracy and applicability. These elements are not useful for nonlinear, large displacement, analysis of structures since the elements become distorted and the changes in the structural geometries cannot be accounted for accurately. Remedial formulations of shell and beam elements [11,16] have been proposed in which no specific shell and beam theories are employed; instead, the geometry and the displacement fields of the structure are directly discretized and interpolated as in the analysis of continuum problems. These general shell elements and three-dimensional beam elements are degenerated from a three-dimensional isoparametric element by imposing some geometric and static constraints to satisfy the

assumptions of a shell or beam theory. Since the same interpolation functions are employed for the beam element as in the formulation of the shell element, the beam and shell elements are compatible and can be used together effectively to model stiffened plates and shells. Both shell and beam elements can have a variable number of nodes, and the shell element can be modified as transition elements to model shell intersections or solid-to-shell transition regions. Such formulations would appear to be especially applicable to material and geometrical nonlinear analysis of shell-type structures in which large displacements and rotations are experienced.

A brief review of the literature is presented below to indicate the directions of recent research efforts in the analysis of general shell-type structures. The review is intended to provide a background for the present study. Due to the increasing applications of layered composite shell-type structures in engineering the present study will be oriented particularly toward the analysis of the laminated composite structures.

1.2 Review of the Literature

A large number of different finite elements have been formulated for the static and dynamic analyses of plate and shell problems, anisotropic as well as isotropic. In these developments, basically three approaches have been followed. In the first approach a plate or shell theory is used as the starting point of the finite element formulation. This plate or shell theory is derived from the three-dimensional continuum mechanics equations by various static and kinematic assumptions. Using variational formulations based on these theories various finite element models have been developed; namely, displacement, hybrid and mixed models [1-5]. In the second approach 3-D elements based on three dimensional elasticity are used [6-10]. In the third approach, isoparametric elements with independent rotational and translational displacements degrees of freedom

are employed and the geometry and displacement fields are directly discretized and interpolated as in the analysis of continuum problems without applying any specific shell or plate theory. The third approach was originally introduced by Ahmad et al. [11] for the linear analysis of moderately thick and thin shells and has been applied to the non-linear analysis of plates and shells by Ramm [12], Krakeland [13], Chang [14], Bathe [15], Bolourchi [16], Chao and Reddy [17], among others. The advantage of elements based on the third approach is their inherent generality compared to 2-D shell or plate elements in the first approach and simplicity compared to fully 3-D elements. They can account for full geometric non-linearities in contrast to 2-D elements based on shell or plate theory in the first approach, while possessing computational simplicity over a fully 3-D element, where the use of several nodes across the shell thickness ignores the well-known fact that even for thick shells the 'normals' to the middle surface remain almost straight after deformation.

Curved beams are used extensively nowadays as stand alone or as reinforcing members of thin shells. When solving the problems of shells with stiffeners by the finite element method, a beam element whose displacement pattern is compatible with that of the shell is required. In analyzing eccentrically stiffened cylindrical shell, Kohnke et al. [18] proposed a 16 d.o.f. isotropic beam finite element which has displacements compatible with the cylindrical shell element from which the beam element is reduced. Rao and Venkatesh have presented a laminated anisotropic curved stiffener element with 16 d.o.f. [19] which is degenerated from a laminated anisotropic rectangular shallow thin shell element with 48 d.o.f. [20] in order to achieve compatibility all along the shell-stiffener junction line. This compatible curved stiffener element [19] and the rectangular shell element [20] have been used in Ref.[21] to solve problems of laminated anisotropic shells stiffened by laminated anisotropic stiffeners. Rao and Venkatesh [22] later presented the

analysis of laminated anisotropic shells of revolution reinforced by laminated anisotropic stiffener using 48 d.o.f. doubly curved quadrilateral shell of revolution element [23], where the stiffener elements again are degenerated from the shell element in [23]. The element matrices of the beam element and the element matrices of the shell are superposed after both have been suitably transformed. Thus no any additional degrees of freedom are introduced when compared with non-stiffened shell structures.

One alternative method of taking into account the presence of stiffening beams and ribs attached to shell structures is to approximate these members by the same element types as used for the shell [24-25]. This procedure has the disadvantage of introducing a substantial number of additional nodes and nodal displacement unknowns.

A variable thickness curved beam and shell stiffening element with transverse shear deformation was developed by Ferguson and Clark [26]. In this work, a family of 2-D and 3-D beam elements are presented which are double degeneration of a fully 3-D isoparametric continuum element. They exhibit such characteristics as displacement compatibility with Ahmad thick shell elements [11], transverse shear and variable thickness properties. Also, with the introduction of offsets in the basic element formulation, 2-D and 3-D stiffeners and curved beams of general cross-section can be modelled.

In References [18], [21] and [22] for the analysis of stiffened shell structures, the shell and beam elements are all based on the classical thin shell and beam theories. Therefore, the transverse shear deformation is neglected. Bathe and Bolourchi [16] employed Ahmad thick shell elements in conjunction with degenerated 3-D beam elements to model an isotropic stiffened plate in which the transverse shear deformations are included. But in all of these works only linear static analysis is considered.

The development of the governing finite element equations for the nonlinear response of a solid body under external loads has been given by numerous investigators [14-17,27-33]. Finite element analyses of the large displacement theories are based on the principle of virtual work or the associated principle of stationary potential energy. Horrigmoe and Bergan [27] presented classical variational principles for non-linear problems by considering incremental deformations of a continuum. A survey of various principles in incremental form in different reference configurations, such as the total Lagrangian and the updated Lagrangian formulations, is presented by Wunderlich [28]. In the total Lagrangian description, all static and kinematic variables are referred to the initial configuration. In the updated Lagrangian description all variables are referred to the current configuration.

By using a geometrically nonlinear formulation, the finite element method has been used in analyzing arch and shell instability problems [34-37]. A special numerical technique must be adopted to trace the path of the load-deflection curve near the limit point (critical buckling load) and in the post-buckling region because the stiffness matrix at the vicinity of the limit point is nearly singular and the descending branch of the load-deflection curve in the post-buckling region is characterized by a negative-definite stiffness matrix. This means that the structure can withstand only a decreasing load after buckling. Many methods have been proposed to overcome this problem. Among these are the simple method of suppressing equilibrium iterations [39,40], the introduction of artificial spring [34], the displacement control method [41,42], and the "constant-arc-length method" of Riks [43] and Wempner [46]. Reviews of these most commonly used techniques are contained in References [44,45]. Among these methods the modified Riks-Wempner method appears to be the most effective in conjunction with the finite element method. Many investigators [43-48] have already used this method in its original

or modified versions to determine the pre- and post-buckling behaviors of various types of structures, such as arch, shell and dome. In most of these works only isotropic material is considered. Very few works of nonlinear buckling analysis for laminated composite structures [65] are reported in the literature.

In many industrial applications the structures are composed of three-dimensional solid continuum with the thin shell-like portions connected to them. In modelling such structures 3-D solid elements and shell elements are employed, for three-dimensional solid and shell-like portions. Since the nodal degrees-of-freedom for these two types of elements are incompatible with each other, the connections of shell elements to the solid elements present considerable difficulty. Three possible approaches to model these practical structures exist: (1) Discretize shell-like portions of the structure with solid elements also. This approach may be computationally expensive and impractical. Also it may lead to erroneous results for thin sheet metal structures due to shear locking phenomenon or the stiffness matrix becoming ill-conditioned. (2) Use multipoint constraint equations for connecting three-dimensional solid elements and shell elements. (3) Develop transition elements which can provide proper connections between the two portions of the structure modelled with three-dimensional solids and the curved shell elements. The transition elements for three-dimensional stress analysis can be found in the works by Bathe and his colleagues [15,16,49] and Surana [50,51]. The transition element possesses the properties of both three-dimensional solid element and degenerated 3-D shell element which are all derived using three-dimensional elasticity equations. Hence the compatibility between transition element and solid element (or shell element) is preserved. These transition elements can be employed to model: (1) solid-to-shell transition regions, in which the transition element provides connections between the two portions of the structure being modelled, namely, the 3-D solid elements and shell ele-

ments, (2) the intersections of different shell surfaces. The major advantage of employing transition element is to eliminate the constraint equations which otherwise need be included at these transition regions. Although the transition element has been used for linear and geometrically nonlinear static analysis of isotropic material, the application for linear and nonlinear static and transient analyses of laminated anisotropic shell-type structures has not yet been reported.

The degenerated 3-D shell element has been investigated for geometrically nonlinear analysis of isotropic and laminated anisotropic shells [14,16,17,33]. From the review of the literature it appears that the geometrically nonlinear analyses of isotropic and laminated anisotropic stiffened shells and shell-type structures with discontinuous geometries have not received the attention they deserve. Most practical structures involve stiffened panels and pressure vessels with geometric discontinuities (e.g., windows, doors, holes, etc.), and experience geometric nonlinearities. The problem of calculating stresses, frequencies and buckling loads in composite structures are magnified by the anisotropy and bending-stretching coupling. Motivated by these observations the following program of research was undertaken.

1.3 Present Study

The present study is directed to deal with the following topics:

- Development of incremental total Lagrangian formulations for shell, beam and three-dimensional solid elements. Also, the development of the transition elements used to model solid-to-shell transition regions and shell intersections.
- Development of a numerical algorithm based on the incremental total Lagrangian formulation and the Newton-Raphson method for geometrically nonlinear static and

transient analyses of isotropic and layered anisotropic curved beams, arches, shells, stiffened shells and the other practical shell-type structures.

- Buckling and post-buckling analysis of isotropic and layered anisotropic arches, shells and stiffened shells by using the modified Riks-Wempner method.

The incremental formulation for nonlinear analysis by using total Lagrangian description is presented in Chapter 2 and the associated finite element model is described in Chapter 3. Chapter 4 contains the iterative solution procedures, Newton-Raphson method and modified Riks-Wempner method, which are used to solve the nonlinear finite element equations. The developments of degenerated 3-D shell element, degenerated curved beam element, 3-D solid element and solid-shell transition element incorporated with the incremental formulation are presented in Chapter 5. Chapter 6 contains the discussions of numerical results of linear and geometrically nonlinear static and transient analyses of isotropic and laminated composite beams, shells, stiffened shells and some general shell-type structures. Comparisons of the present results with the results available in the literature show very good agreements. Many numerical results of stiffened shells and some general shell-type structures could serve as references for future investigations. In Chapter 7 a summary, conclusions and recommendations of the present study are included.

CHAPTER 2

FORMULATION OF THE INCREMENTAL EQUATIONS OF MOTION BY THE TOTAL LAGRANGIAN DESCRIPTION

2.1 Introduction

In the linear description of the motion of solid bodies it is assumed that the displacements are infinitely small and that the material is linearly elastic. In addition we assume the configuration and the nature of the boundary conditions remain unchanged during the entire deformation process. These assumptions imply that the displacement vector $\{U\}$ is a linear function of the applied load vector $\{R\}$. The evaluation of element matrices and load vectors are performed over the original volume of the finite elements and are assumed to be constant and independent of the element displacements.

The nonlinearity in solid mechanics arises from two distinct sources. One is due to the kinematics of deformation of the body and the other from the constitutive behavior (i.e., stress-strain relations). The analyses in which the first type of nonlinearity is considered are called geometrically nonlinear analyses, and those in which the second type is considered are called materially nonlinear analyses. The geometrically nonlinear analysis can be subclassified into two cases : (i) large displacements, large rotations and small strains (ii) large displacements, large rotations and large strains. In the first case it is assumed that the rotations of line elements are large, but their extensions and changes of angles between two line elements are small. In the second case the extension of a line element and angle changes between two line elements are large, and rotations of a line element are also large.

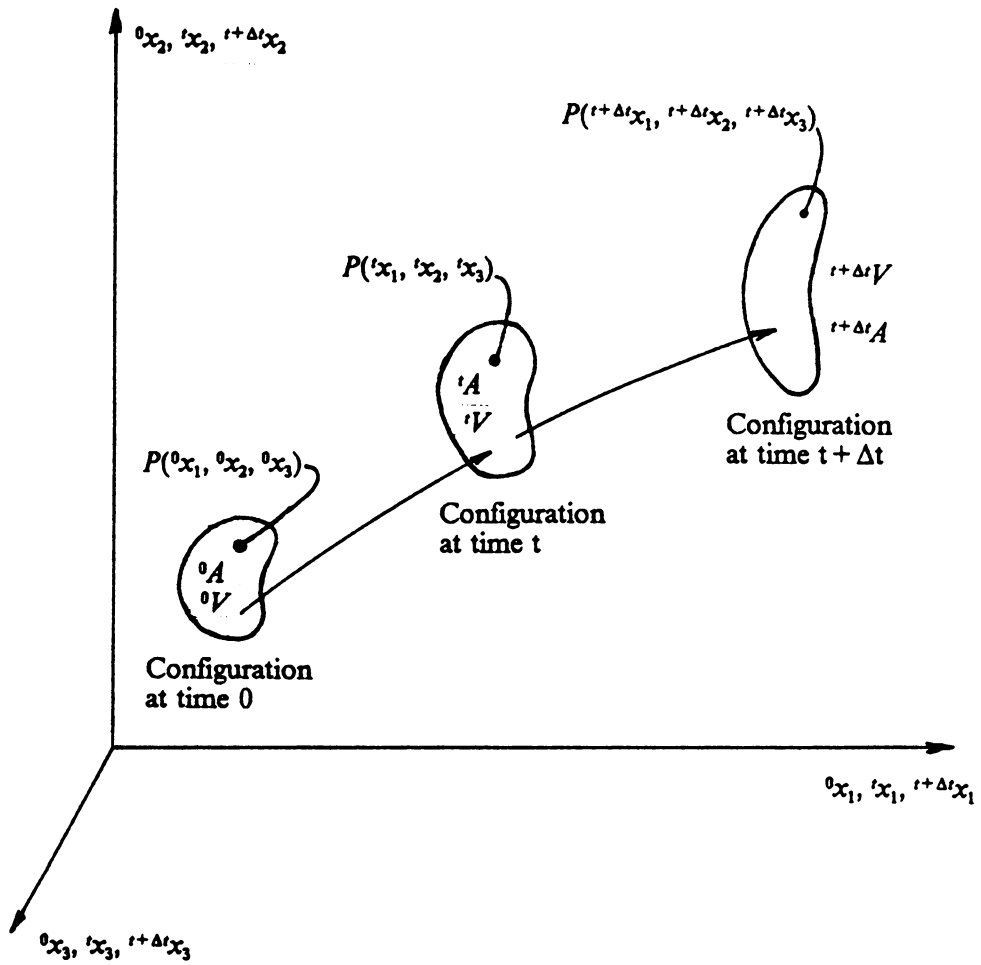
In the present study the first type of geometrical nonlinearity is considered, and it is assumed that the material is linear elastic.

Consider the motion of a body in a fixed Cartesian coordinate system as shown in Fig. 2.1 and assume that the body can experience large displacements and rotations. We wish to determine the configuration of the body for different times and loads. The formulation to be described assumes that the solutions for the kinematic and static variables for all time steps from time 0 to time t , inclusive, have been obtained, and that the solution for time $t + \Delta t$ is pursued next. Hence in the present formulation we follow all particles of the body as it deforms from the initial configuration to the final configuration. This type of description is called the Lagrangian description, which differs from the Eulerian description usually used in the analysis of fluid mechanics problems. There are various methods used to describe motion of a continuum in Newtonian mechanics as described in References [38,52].

In the Lagrangian description of motion all variables are referred to a reference configuration, which can be the initial configuration or any other convenient configuration. The description in which all variables are referred to the current configuration is called the updated Lagrangian description [31,32] and the one in which all variables are referred to the initial configuration is called the total Lagrangian description [17,33].

2.2 Principle of Virtual Displacements

Since the displacement based finite element procedure will be employed for numerical solution, the principle of virtual displacements is used to express the equilibrium of the body in the configuration at time $t + \Delta t$. The principle of virtual displacements requires that



$$1+\Delta t1x_i = 0x_i + 1+\Delta t1u_i$$

$$1x_i = 0x_i + 1u_i \quad i = 1,2,3$$

$$1+\Delta t1x_i = 1x_i + 1u_i$$

Figure 2.1 Motion of a continuum in Cartesian coordinate systems

$$\int_{t+\Delta t V} {}^{t+\Delta t}\tau_{ij} \delta_{t+\Delta t} e_{ij} {}^{t+\Delta t}dV = {}^{t+\Delta t}R \quad (2.1)$$

where ${}^{t+\Delta t}\tau_{ij}$ = the Cartesian components of the Cauchy stress tensor at time $t + \Delta t$.

(The Cauchy stresses are always referred to the configuration in which they occur ${}^{t+\Delta t}\tau_{ij} \equiv {}^{t+\Delta t}_{t+\Delta t}\tau_{ij}$),

${}_{t+\Delta t}e_{ij}$ = the Cartesian components of the infinitesimal strain tensor associated with the displacements u_i in going from the configuration at time t to the configuration at time $t + \Delta t$, i.e. ${}_{t+\Delta t}e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial {}^{t+\Delta t}x_j} + \frac{\partial u_j}{\partial {}^{t+\Delta t}x_i} \right)$ which are also referred to this unknown configuration at time $t + \Delta t$,

${}^{t+\Delta t}x_i$ = Cartesian components of a point in configuration at time $t + \Delta t$, the left superscript refers to the configuration of the body,

and

$${}^{t+\Delta t}R = \int_{t+\Delta t A} {}^{t+\Delta t}_{t+\Delta t}t_k \delta u_k {}^{t+\Delta t}dA + \int_{t+\Delta t V} {}^{t+\Delta t}\rho {}^{t+\Delta t}_{t+\Delta t}f_k \delta u_k {}^{t+\Delta t}dV \quad (2.2)$$

${}^{t+\Delta t}_{t+\Delta t}t_k$ and ${}^{t+\Delta t}_{t+\Delta t}f_k$ are the components of the externally applied surface and body force vectors respectively, δu_k is a virtual variation in the current displacement components ${}^{t+\Delta t}u_k$, and $\delta_{t+\Delta t}e_{ij}$ are the virtual variations in strains.

Equation (2.1) cannot be solved directly since the configuration at time $t + \Delta t$ is unknown. This is an important difference compared with the linear analysis in which we assume that the displacements are infinitesimally small so that the configuration of the body does not change. In a large deformation analysis special attention must be given to the fact that the configuration of the body is changing continuously. This change in configuration can be dealt with by defining auxiliary stress and strain meas-

ures. The objective in their definition is to express the internal work in eqn. (2.1) in terms of an integral over a volume that is known. The stress and strain measures that we shall use are the 2nd Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor.

An approximate solution of eqn. (2.1) can be obtained by referring all variables to a previously calculated known equilibrium configuration and linearizing the resulting equation. This solution is then improved by iteration. In principle, any one of the already calculated equilibrium configurations could be used. In practice total Lagrangian (T. L.) and updated Lagrangian (U. L.) formulations are choosed. In the T. L. formulation all static and kinematic variables are referred to the initial configuration at time 0. The U. L. formulation is based on the same procedures that are used in the T. L. formulation, but all static and kinematic variables are referred to the configuration at time t. Both the T. L. and U. L. formulations include all kinematic nonlinear effects due to large displacements, large rotations and large strains. In the present study the total Lagrangian formulation is adopted.

2.3 Total Lagrangian Formulation

In the formulation all variables in eqns. (2.1) and (2.2) are referred to the initial configuration at time 0 of the body. The applied forces in eqn. (2.2) are evaluated using

$$\begin{aligned} {}^{t+\Delta t}_{t+\Delta t} \rho \quad {}^{t+\Delta t}_{t+\Delta t} dA &= {}^{t+\Delta t}_0 \rho \quad {}^0 dA \\ {}^{t+\Delta t}_{t+\Delta t} \rho \quad {}^{t+\Delta t}_{t+\Delta t} dV &= {}^0 \rho \quad {}^{t+\Delta t}_0 dV \end{aligned} \quad (2.3)$$

when the loading is deformation-independent and can be specified prior to the incremental analysis.

The volume integral of Cauchy stresses times variations in infinitesimal strains in eqn. (2.1) can be transformed to give [38]

$$\int_{t+\Delta t V} {}^{t+\Delta t}\tau_{ij} \delta_{t+\Delta t} e_{ij} {}^{t+\Delta t}dV = \int_{0V} {}^{t+\Delta t}S_{ij} \delta {}^{t+\Delta t}_0 \varepsilon_{ij} {}^0dV \quad (2.4)$$

where ${}^{t+\Delta t}S_{ij}$ = Cartesian components of the 2nd Piola-Kirchhoff stress tensor corresponding to configuration at time $t + \Delta t$ but measured in the configuration at time 0.

${}^{t+\Delta t}_0 \varepsilon_{ij}$ = Cartesian components of the Green-Lagrange strain tensor in the configuration at time $t + \Delta t$, referred to the configuration at time 0, and ${}^{t+\Delta t}_0 \varepsilon_{ij}$ are defined as ${}^{t+\Delta t}_0 \varepsilon_{ij} = \frac{1}{2}({}^{t+\Delta t}_0 u_{i,j} + {}^{t+\Delta t}_0 u_{j,i} + {}^{t+\Delta t}_0 u_{k,i} {}^{t+\Delta t}_0 u_{k,j})$,
 ${}^{t+\Delta t}_0 u_{i,j} = \frac{\partial {}^{t+\Delta t}u_i}{\partial x_j}$

${}^{t+\Delta t}u_i$ = components of displacement vector from initial position at time 0 to configuration at time $t + \Delta t$, ${}^{t+\Delta t}u_i = {}^{t+\Delta t}x_i - {}^0x_i$

The 2nd Piola-Kirchhoff stress tensor referred to the configuration at time 0 is defined as

$${}^{t+\Delta t}_0 S_{ij} = \frac{{}^0\rho}{{}^{t+\Delta t}_0 \rho} {}^{t+\Delta t}x_{i,s} {}^{t+\Delta t}\tau_{sr} {}^{t+\Delta t}_0 x_{r,j}$$

and is energetically conjugate to the Green-Lagrange strain tensor. Also the Cauchy stress tensor is energetically conjugate to the infinitesimal strain tensor. Hence, the total internal virtual work can be calculated using as stress measures either the Cauchy or the 2nd Piola-Kirchhoff stress tensors provided the conjugate strain tensors are employed and the integrations are performed over the current and original volumes, respectively. The relation of eqn. (2.4) explains that the 2nd Piola-Kirchhoff stress and Green-Lagrange strain tensors are energetically conjugate.

Substituting the relations in eqn. (2.3) and (2.4) into eqns. (2.1) and (2.2), the following equilibrium equation for the body in the configuration at time $t + \Delta t$ but referred to the configuration at time 0 is obtained

$$\int_{0_V} {}^{t+\Delta t}{}_0S_{ij} \delta {}^{t+\Delta t}{}_0\varepsilon_{ij} {}^0dV = {}^{t+\Delta t}R \quad (2.5)$$

where ${}^{t+\Delta t}R$ is calculated using

$${}^{t+\Delta t}R = \int_{0_A} {}^{t+\Delta t}{}_0t_k \delta u_k {}^0dA + \int_{0_V} {}^0\rho {}^{t+\Delta t}{}_0f_k \delta u_k {}^0dV \quad (2.6)$$

Since the stresses ${}^{t+\Delta t}{}_0S_{ij}$ and strains ${}^{t+\Delta t}{}_0\varepsilon_{ij}$ are unknown, the following incremental decompositions are used

$${}^{t+\Delta t}{}_0S_{ij} = {}^t{}_0S_{ij} + {}_0S_{ij} \quad (2.7)$$

$${}^{t+\Delta t}{}_0\varepsilon_{ij} = {}^t{}_0\varepsilon_{ij} + {}_0\varepsilon_{ij} \quad (2.8)$$

where ${}^t{}_0S_{ij}$ and ${}^t{}_0\varepsilon_{ij}$ are the known 2nd Piola-Kirchhoff stresses and Green-Lagrange strains in the configuration at time t . Using the definition of the Green-Lagrange strain tensor and ${}^{t+\Delta t}{}_0u_i = {}^t{}_0u_i + u_i$, where u_i = the increment in displacement components, it follows that

$${}_0\varepsilon_{ij} = {}_0e_{ij} + {}_0\eta_{ij} \quad (2.9)$$

where

$${}_0e_{ij} = \frac{1}{2}({}_0u_{i,j} + {}_0u_{j,i} + {}^t{}_0u_{k,i} {}_0u_{k,j} + {}^t{}_0u_{k,i} {}^t{}_0u_{k,j}) \quad (2.10)$$

= linear part of strain increment ${}_0\varepsilon_{ij}$

$${}_0\eta_{ij} = \frac{1}{2} {}_0u_{k,i} {}_0u_{k,j} \quad (2.11)$$

= nonlinear part of strain increment ${}_0\varepsilon_{ij}$

and $\delta^{t+\Delta t}{}_0\varepsilon_{ij} = \delta^t{}_0\varepsilon_{ij} + \delta_0\varepsilon_{ij} = \delta_0\varepsilon_{ij}$ since $\delta^t{}_0\varepsilon_{ij}$ are already known.

The incremental 2nd Piola-Kirchhoff stress components ${}_0S_{ij}$ are related to the incremental Green-Lagrange strain components ${}_0\varepsilon_{ij}$ using the constitutive tensor ${}_0C_{ijrs}$, i.e.

$${}_0S_{ij} = {}_0C_{ijrs} {}_0\varepsilon_{rs} \quad (2.12)$$

Using eqns. (2.7) - (2.12), eqn. (2.5) can be written as

$$\int_{0_V} {}_0C_{ijrs} {}_0\varepsilon_{rs} \delta_0\varepsilon_{ij} {}^0dV + \int_{0_V} {}^tS_{ij} \delta_0\eta_{ij} {}^0dV = {}^{t+\Delta t}R - \int_{0_V} {}^tS_{ij} \delta_0e_{ij} {}^0dV \quad (2.13)$$

which represents a nonlinear equilibrium equation for the incremental displacements u_i .

2.4 Linearization of Incremental Equations of Motion

The solution of eqn. (2.13) cannot be calculated directly, since they are nonlinear in the displacement increments. Approximate solution can be obtained by assuming that ${}_0\varepsilon_{ij} = {}_0e_{ij}$ in eqn. (2.13). This means that, in addition to using $\delta_0\varepsilon_{ij} = \delta_0e_{ij}$, the incremental constitutive relation employed is

$${}_0S_{ij} = {}_0C_{ijrs} {}_0e_{rs}$$

Hence, in the T. L. formulation the approximate equation to be solved is

$$\int_{0_V} {}_0C_{ijrs} {}_0e_{rs} \delta_0e_{ij} {}^0dV + \int_{0_V} {}^tS_{ij} \delta_0\eta_{ij} {}^0dV = {}^{t+\Delta t}R - \int_{0_V} {}^tS_{ij} \delta_0e_{ij} {}^0dV \quad (2.14)$$

In dynamic analysis, the applied body forces include inertial forces. In this case we have

$$\int_{t+\Delta t_V} {}^{t+\Delta t}\rho \quad {}^{t+\Delta t}\ddot{u}_i \quad \delta {}^{t+\Delta t}u_i \quad {}^{t+\Delta t}dV = \int_{0_V} {}^0\rho \quad {}^{t+\Delta t}\ddot{u}_i \quad \delta {}^{t+\Delta t}u_i \quad {}^0dV \quad (2.15)$$

and hence the mass matrix can be evaluated using the initial configuration of the body. Using Hamilton's principle we obtain the equations of motion of the moving body at time $t + \Delta t$ in the variational form as

$$\int_{0_V} {}^0\rho \quad {}^{t+\Delta t}\ddot{u}_i \quad \delta {}^{t+\Delta t}u_i \quad {}^0dV + \int_{0_V} {}^0C_{ijrs} \quad {}^0e_{rs} \quad \delta {}^0e_{ij} \quad {}^0dV + \int_{0_V} {}^0S_{ij} \quad \delta {}^0\eta_{ij} \quad {}^0dV = {}^{t+\Delta t}R - \int_{0_V} {}^0S_{ij} \quad \delta {}^0e_{ij} \quad {}^0dV \quad (2.16)$$

where $\delta {}^{t+\Delta t}u_i = \delta u_i$

CHAPTER 3

DISPLACEMENT FINITE ELEMENT MODEL

3.1 Introduction

Based on the principle of virtual displacements the incremental equations of motion, eqn. (2.16) presented in Chapter 2, can be used to develop general nonlinear displacement finite element model. The generalized displacements are the primary variables in the governing finite element equations.

The basic steps in deriving the governing finite element equations include: (1) The selection of proper interpolation functions. (2) The interpolating of the element displacements and coordinates with these functions. (3) Substituting the displacement and geometry fields into the governing equations of motion. Then invoking the principle of virtual displacements for each of the nodal point displacements in turn and the governing finite element equation are obtained. Only a single element of a specific type is considered in the above derivation. The final algebraic equations of motion for an assemblage of elements are obtained by assembling the governing equations of motion of each element.

3.2 Finite-Element Discretization

It is important that the coordinates and displacements are interpolated using the same interpolation functions so that the displacement compatibility across element boundaries can be preserved in all configurations. Hence

$$\begin{aligned} {}^0x_i &= \sum_{k=1}^n \varphi_k {}^0x_i^k, & {}^tx_i &= \sum_{k=1}^n \varphi_k {}^tx_i^k \\ {}^{t+\Delta t}x_i &= \sum_{k=1}^n \varphi_k {}^{t+\Delta t}x_i^k & i &= 1, 2, 3 \end{aligned} \quad (3.1)$$

$${}^t u_i = \sum_{k=1}^n \varphi_k {}^t u_i^k, \quad u_i = \sum_{k=1}^n \varphi_k u_i^k \quad i = 1, 2, 3 \quad (3.2)$$

where the right superscript k indicates the quantity at nodal point k, φ_k is the interpolation function corresponding to nodal point k, and n is the number of element nodal points.

3.3 Finite-Element Model

Using eqns. (3.1) and (3.2) to evaluate the displacement derivatives required in the integrals, eqn. (2.16) becomes,

$${}_0^t[M]^{t+\Delta t}\{\ddot{\Delta}^e\} + ({}_0^t[K_L] + {}_0^t[K_{NL}])\{\Delta^e\} = {}^{t+\Delta t}\{R\} - {}_0^t\{F\} \quad (3.3)$$

where $\{\Delta^e\}$ is the vector of nodal incremental displacements from time t to time t + Δt in one element, and ${}_0^t[M]^{t+\Delta t}\{\ddot{\Delta}^e\}$, ${}_0^t[K_L]\{\Delta^e\}$, ${}_0^t[K_{NL}]\{\Delta^e\}$, and ${}_0^t\{F\}$ are obtained by evaluating the integrals

$$\int_{0_V} {}_0\rho {}^{t+\Delta t}\ddot{u}_i {}^{\delta t+\Delta t}u_i {}^0dV, \int_{0_V} {}_0C_{ijrs} {}_0e_{rs} {}^{\delta 0}e_{ij} {}^0dV, \int_{0_V} {}_0^tS_{ij} {}^{\delta 0}\eta_{ij} {}^0dV$$

and $\int_{0_V} {}_0^tS_{ij} {}^{\delta 0}e_{ij} {}^0dV$ respectively, i.e.

$${}_0^t[K_L] = \int_{0_V} {}_0^t[B_L]^T {}_0[C] {}_0^t[B_L] {}^0dV \quad (3.4)$$

$${}_0^t[K_{NL}] = \int_{0_V} {}_0^t[B_{NL}]^T {}_0^t[S] {}_0^t[B_{NL}] {}^0dV \quad (3.5)$$

$${}_0^t[M] = \int_{0_V} {}_0\rho {}^t[H]^T {}^t[H] {}^0dV \quad (3.6)$$

$${}_0^t\{F\} = \int_{0_V} {}_0^t[B_L]^T {}_0^t\{\hat{S}\} {}^0dV \quad (3.7)$$

In the above equations, ${}_0[B_L]$ and ${}_0[B_{NL}]$ are linear and nonlinear strain-displacement transformation matrices, ${}_0[C]$ is the incremental stress-strain material property matrix, ${}_0[S]$ is a matrix of 2nd Piola-Kirchhoff stress components, ${}_0\{\hat{S}\}$ is a vector of these stresses, and ${}^t[H]$ is the incremental displacement interpolation matrix. All matrix elements correspond to the configuration at time t and are defined with respect to the configuration at time 0.

It is important to note that eqn. (3.3) is only approximation to the actual solution to be solved in each time step, i.e. eqn. (2.13). Therefore it may be necessary to iterate in each time step until eqn. (2.13) is satisfied to a required tolerance.

Note that the finite element equations (3.3) are 2nd order differential equations in time. In order to obtain numerical solutions at each time step, eqn. (3.3) needs to be converted to algebraic equations.

3.4 Newmark Scheme for Time Discretization

In this study the Newmark integration scheme is used to convert the ordinary differential equations in time to algebraic equations. In the Newmark scheme, displacements and velocities are approximated by

$$\begin{aligned} {}^{t+\Delta t}\{\Delta\} &= {}^t\{\Delta\} + {}^t\{\dot{\Delta}\} \Delta t + \left[\left(\frac{1}{2} - \beta\right) {}^t\{\ddot{\Delta}\} + \beta {}^{t+\Delta t}\{\ddot{\Delta}\}\right] (\Delta t)^2 \\ {}^{t+\Delta t}\{\dot{\Delta}\} &= {}^t\{\dot{\Delta}\} + [(1 - \alpha) {}^t\{\ddot{\Delta}\} + \alpha {}^{t+\Delta t}\{\ddot{\Delta}\}] \Delta t \end{aligned} \quad (3.8)$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$ for the constant average acceleration method and Δt is the time step. Rearranging eqns. (3.3) and (3.8), we obtain

$${}_0^t[K] \{\Delta\} = {}^{t+\Delta t}\{\hat{R}\} \quad (3.9)$$

where $\{\Delta\}$ = vector of nodal incremental displacements at time $t = {}^{t+\Delta t}\{\Delta\} - {}^t\{\Delta\}$, and

$$\begin{aligned}\delta[\hat{K}] &= a_0 \delta[M] + \delta[K_L] + \delta[K_{NL}] \\ {}^{t+\Delta t}\{\hat{R}\} &= {}^{t+\Delta t}\{R\} - {}^t\{F\} + \delta[M] (a_1 {}^t\{\dot{\Delta}\} + a_2 {}^t\{\ddot{\Delta}\}) \\ a_0 &= \frac{1}{\beta(\Delta t)^2}, \quad a_1 = a_0 \Delta t, \quad a_2 = \frac{1}{2\beta} - 1\end{aligned}$$

Once eqn. (3.9) is solved for $\{\Delta\}$ at time $t + \Delta t$, the acceleration and velocity vectors are obtained by

$$\begin{aligned}{}^{t+\Delta t}\{\ddot{\Delta}\} &= a_0\{\Delta\} - a_1 {}^t\{\dot{\Delta}\} - a_2 {}^t\{\ddot{\Delta}\} \\ {}^{t+\Delta t}\{\dot{\Delta}\} &= {}^t\{\dot{\Delta}\} + a_3 {}^t\{\ddot{\Delta}\} + a_4 {}^{t+\Delta t}\{\ddot{\Delta}\}\end{aligned}\tag{3.10}$$

where $a_3 = (1 - \alpha) \Delta t$, $a_4 = \alpha \Delta t$

CHAPTER 4

SOLUTION PROCEDURES

4.1 Introduction

In Chapter 3 we obtained the final governing finite element equations to be solved at each time step t , i.e. eqn. (3.9), for each element. For an assemblage of elements the governing equations are constructed by assembling eqn. (3.9) of each element together. We noted in Chapter 3 that eqn. (3.3) is only linearized approximation to the actual governing equation of motion, eqn. (2.13), and so is eqn. (3.9). Therefore it is necessary to solve eqn. (3.9) iteratively at each time step until eqn. (2.13) is satisfied to a required tolerance. Two iterative procedures are often used in the finite element analysis of non-linear problems. One is the direct iteration, also known as the Picard method. The other is the Newton-Raphson method. Here the Newton-Raphson method is adopted.

4.2 Newton-Raphson Method

The right-hand side of eqn. (3.9) corresponds to the out-of-balance load vector which is not yet balanced by element stresses at time t when the continuum moves from configuration at time t to one at time $t + \Delta t$. Hence an increment in the nodal point displacements is required. This updating of the nodal displacements in the iteration is continued until the out-of-balance loads are small.

Assume that in the iterative solution we have evaluated ${}^{t+\Delta t}\{\Delta\}^{(i-1)}$. Then by the Taylor series expansion and neglecting the 2nd and higher-order terms we obtain the equation to be solved at the i -th iteration as

$$\begin{aligned}
& {}^{t+\Delta t}[\hat{K}]^{(i-1)} (\{\Delta\}^{(i)} - \{\Delta\}^{(i-1)}) \\
& = - {}^t_0[\hat{K}] \{\Delta\}^{(i-1)} + {}^{t+\Delta t}\{\hat{R}\} \\
& = - a_0 {}^t_0[M] \{\Delta\}^{(i-1)} - ({}^t_0[K_L] + {}^t_0[K_{NL}]) \{\Delta\}^{(i-1)} + {}^{t+\Delta t}\{R\} - {}^t_0\{F\} + \\
& \quad a_1 {}^t_0[M] {}^t\{\dot{\Delta}\} + a_2 {}^t_0[M] {}^t\{\ddot{\Delta}\}
\end{aligned} \tag{4.1}$$

where ${}^t_0\{F\} + ({}^t_0[K_L] + {}^t_0[K_{NL}]) \{\Delta\}^{(i-1)}$ is the nodal point forces that correspond to the element stresses in the configuration after the $(i-1)$ -th iteration, which can also be expressed as ${}^{t+\Delta t}{}_0\{F\}^{(i-1)}$, and from eqn. (3.7) it is given by

$${}^{t+\Delta t}{}_0\{F\}^{(i-1)} = \int_{0V} {}^{t+\Delta t}{}_0[B_L^{(i-1)}]^T {}^{t+\Delta t}{}_0\{\hat{S}\}^{(i-1)} {}^0dV \tag{4.2}$$

Eqn. (4.1) can now be written as

$$\begin{aligned}
& (a_0 {}^{t+\Delta t}{}_0[M] + {}^{t+\Delta t}{}_0[K_L] + {}^{t+\Delta t}{}_0[K_{NL}])^{(i-1)} (\{\Delta\}^{(i)} - \{\Delta\}^{(i-1)}) = \\
& {}^{t+\Delta t}\{R\} - {}^t_0[M] (a_0\{\Delta\}^{(i-1)} - a_1 {}^t\{\dot{\Delta}\} - a_2 {}^t\{\ddot{\Delta}\}) - {}^{t+\Delta t}{}_0\{F\}^{(i-1)}
\end{aligned} \tag{4.3}$$

which is the equation to be solved at the i -th iteration. The solution at $t + \Delta t$ is given by

$${}^{t+\Delta t}\{\Delta\}^{(i)} = {}^{t+\Delta t}\{\Delta\}^{(i-1)} + (\{\Delta\}^{(i)} - \{\Delta\}^{(i-1)}) \tag{4.4}$$

in which $\{\Delta\}^{(0)} = \{0\}$.

A geometrical explanation of the Newton-Raphson iteration is given in Fig. 4.1 for a one-dimensional problem.

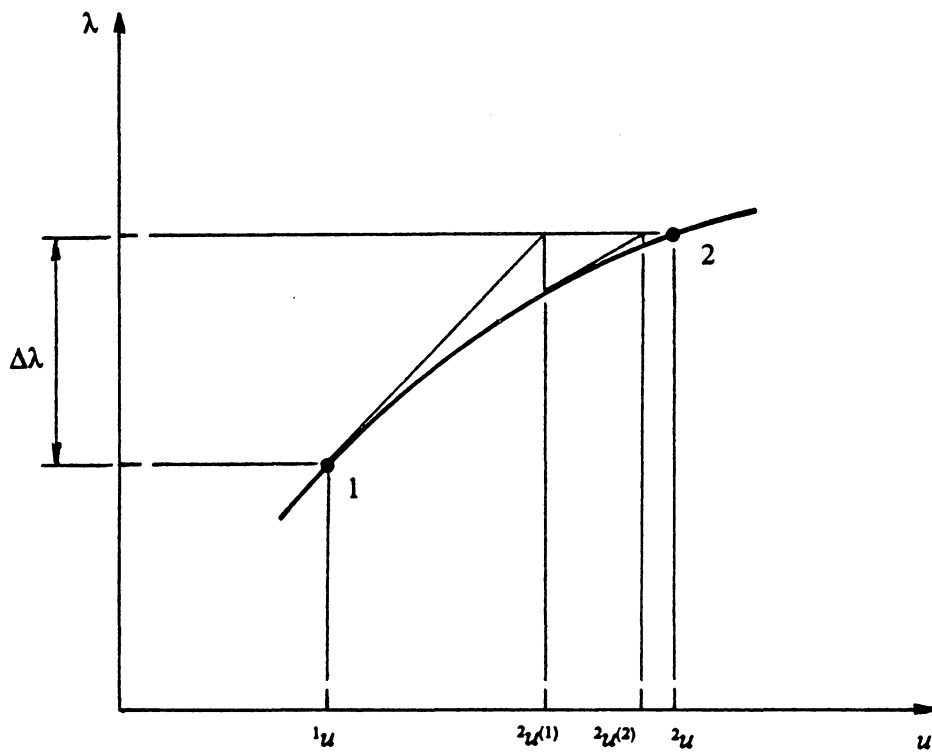


Figure 4.1 Geometric interpretation of the Newton-Raphson method for a one-dimensional problem with tangent stiffness matrix updated at each iteration

The procedure of the Newton-Raphson method is summarized as follows. For each iteration in a fixed time step (or a load step for static analysis) the following computations are carried out:

1. Establish the system matrices ${}_0[M]$, ${}_0[K_L]$, ${}_0[K_{NL}]$, and ${}^{t+\Delta t}_0\{F\}^{(i-1)}$ in eqn. (4.3) by using the approximate nodal displacements, strains and stresses from the last iteration.
2. Evaluate the right-hand side of eqn. (4.3).
3. Solve eqn. (4.3) for $\{\delta\Delta\}^{(i)} = \{\Delta\}^{(i)} - \{\Delta\}^{(i-1)}$.
4. Update the nodal displacements by eqn. (4.4).
5. Test for convergence.
6. If the process has not converged, return to step 1. Otherwise go to the next time step (or load step).

In order to reduce the amount of computations per iteration, the Newton-Raphson method is modified by using the same system matrices ${}_0[M]$, ${}_0[K_L]$, ${}_0[K_{NL}]$ for several iterations. These system matrices are updated only at the beginning of each time step (or load step) or only when the convergence rate becomes poor. This modified method may require more iterations to reach a new equilibrium point.

4.3 Modified Riks/Wempner Method

As stated in Chapter 1, since the Newton-Raphson method fails to trace the response beyond the limit point, the modified Riks/Wempner method is adopted in the present study for the nonlinear post-buckling analysis. The theoretical development of this

method can be found in References [44,47]. In Riks/Wempner method the load increment for each load step is considered to be an unknown and solved as part of the solution. The iteration is performed on the normal plane to the tangent of the first iteration (or on a circle or sphere) and the new equilibrium point will be the intersection of the normal plane (or circle, sphere) with the equilibrium path. In contrast, for Newton-Raphson method the iterations are performed on the constant load level.

Assume that the loading is proportional, then the iterative equations corresponding to eqn. (4.3) become (consider the static case only)

$$\begin{aligned}
 & ({}^{t+\Delta t}_0[K_L] + {}^{t+\Delta t}_0[K_{NL}])^{(i-1)} \{\delta\Delta\}^{(i)} \\
 & = ({}^{t+\Delta t}\lambda^{(i-1)} + \Delta\lambda^{(i)})\{\bar{Q}\} - {}^{t+\Delta t}_0\{F\}^{(i-1)} \\
 & = \Delta\lambda^{(i)}\{\bar{Q}\} + {}^{t+\Delta t}_0\{Re\}^{(i-1)}
 \end{aligned} \tag{4.5}$$

where $\{\bar{Q}\}$ = constant load distribution vector,

${}^{t+\Delta t}_0\{Re\}^{(i-1)}$ = unbalanced force vector at iteration (i-1)

$$= {}^{t+\Delta t}\lambda^{(i-1)} \{\bar{Q}\} - {}^{t+\Delta t}_0\{F\}^{(i-1)}$$

${}^{t+\Delta t}_0\{F\}^{(i-1)}$ = vector of nodal point forces equivalent to the element stresses at time $t + \Delta t$ and iteration (i-1)

$$\{\delta\Delta\}^{(i)} = {}^{t+\Delta t}\{\Delta\}^{(i)} - {}^{t+\Delta t}_0\{\Delta\}^{(i-1)}$$

= vector of increments in the nodal displacements at iteration i

$\Delta\lambda^{(i)}$ = load increment

$$= {}^{t+\Delta t}\lambda^{(i)} - {}^{t+\Delta t}\lambda^{(i-1)}$$

In order to solve $N + 1$ variables, $\{\delta\Delta\}^{(i)}$ and $\Delta\lambda^{(i)}$, an additional equation is used to constrain the length of the load step

$$f(\Delta\lambda^{(i)}, \{\delta\Delta\}^{(i)}) = 0 \tag{4.6}$$

Several constraint equations have been presented such as the tangent constant arc-length [43] and the spherical constant arc-length [44,45]. For the tangent constant arc-length the iterations are performed on the "normal plane" while the iterations are performed on the "sphere" for the spherical constant arc-length.

The algorithm of the modified Riks/Wempner method is briefly summarized in the following (also see Figures 4.2, 4.3 and 4.4). The detailed theoretical development can be found in References [43-48].

The first iteration

1. For the first load step (i.e. $t=0$), choose a load increment $\Delta\lambda^{(1)}$.
2. Calculate the system stiffness matrices ${}_0[K_L]$, ${}_0[K_{NL}]$ in the current configuration $\{ \Delta \}$.

3. Solve the equation

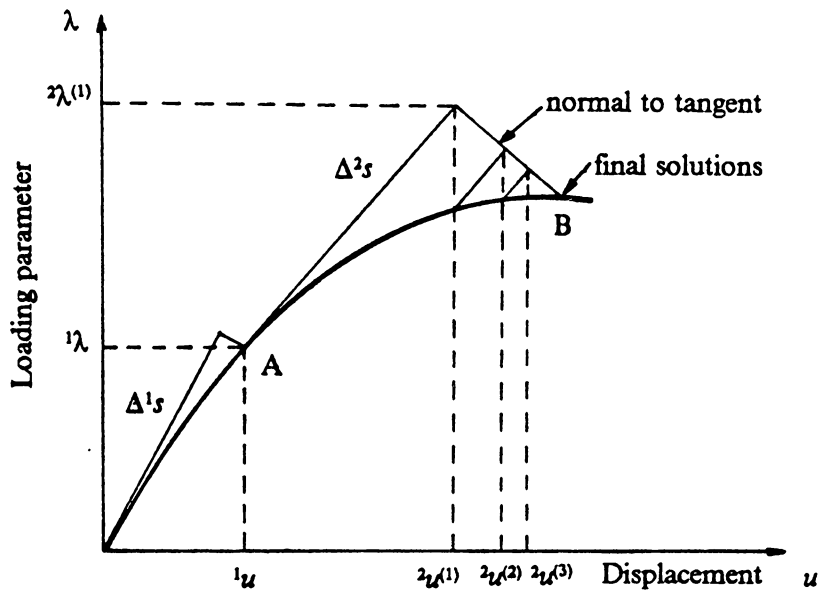
$$({}_0[K_L] + {}_0[K_{NL}]) \{ \delta\Delta \}^{(1)T} = \{ \bar{Q} \}$$

where $\{ \bar{Q} \}$ is the constant load distribution vector.

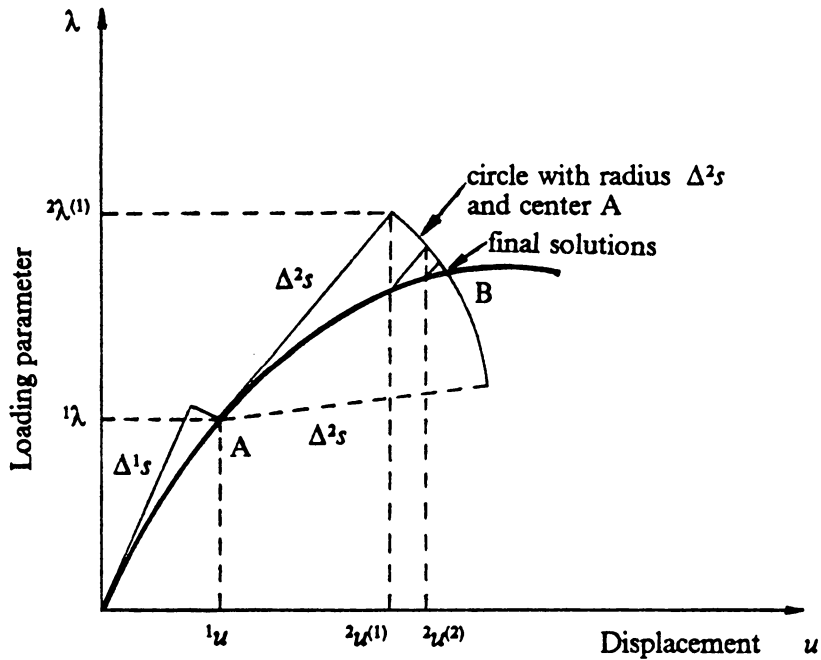
4.
 - a. For the first load step, compute the generalized arc length Δs from the constraint equation

$$\Delta s = \Delta\lambda^{(1)} (\{ \delta\Delta \}^{(1)T} \cdot \{ \delta\Delta \}^{(1)T} + 1)^{1/2} \quad (\text{by Ramm [44]})$$

$$\text{or } \Delta s = \Delta\lambda^{(1)} (\{ \delta\Delta \}^{(1)T} \cdot \{ \delta\Delta \}^{(1)T})^{1/2} \quad (\text{by Crisfield [45]})$$



(a) Iteration along the normal plane



(b) Iteration along the circle

Figure 4.2 Riks/Wempner method of a one-dimensional problem with tangent stiffness matrix updated at the beginning of each load step only

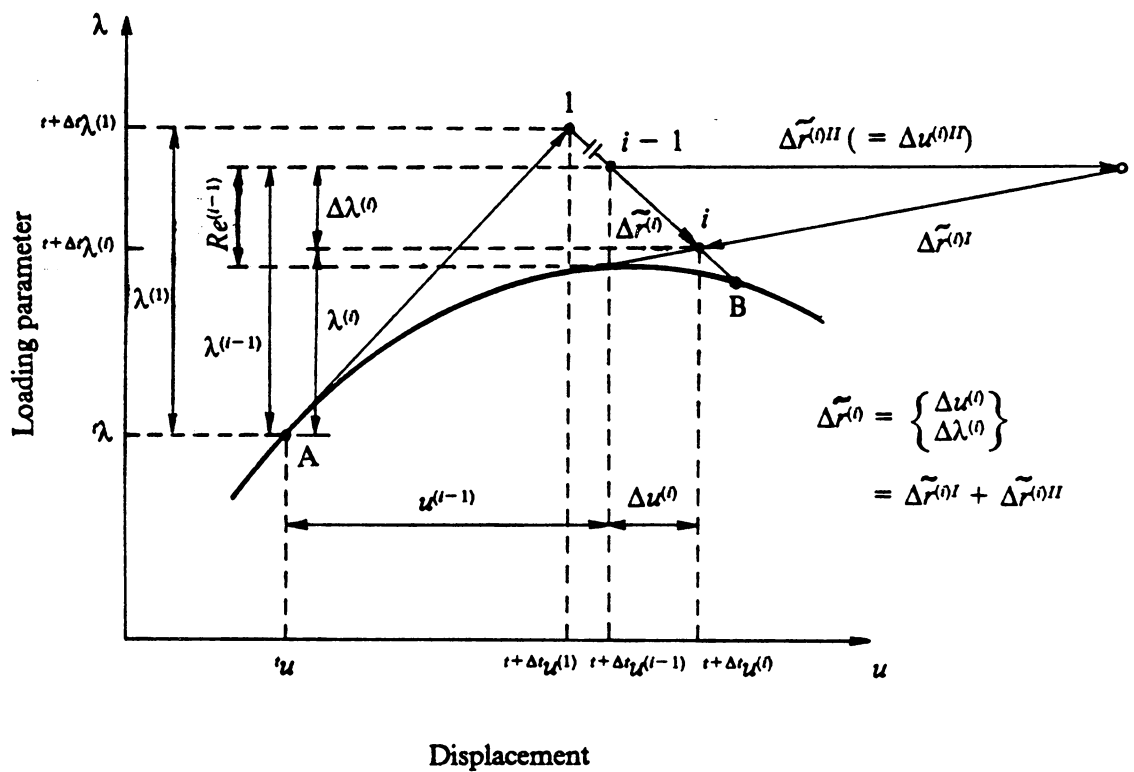


Figure 4.3 Iteration procedure along normal plane with stiffness matrix updated in each iteration

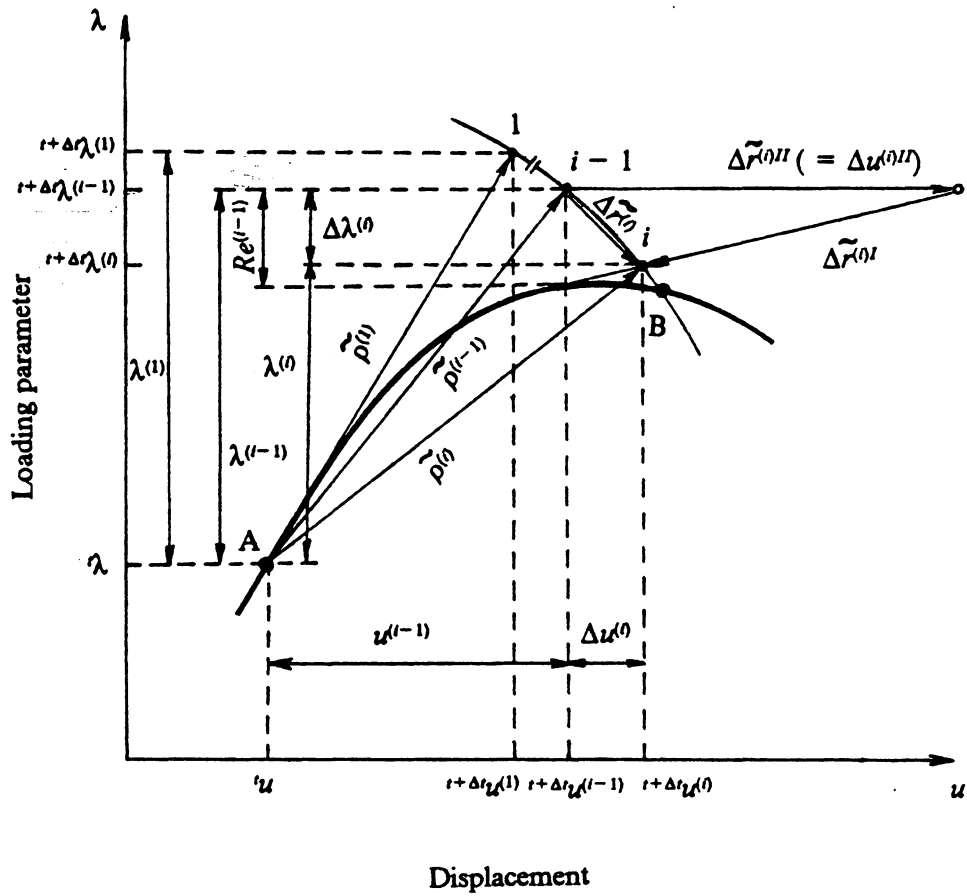


Figure 4.4 Iteration procedure along a circle with stiffness matrix updated in each iteration

- b. For the second and higher load steps, compute the initial incremental loading parameter $\Delta\lambda^{(1)}$ by

$$\Delta\lambda^{(1)} = \pm \hat{\Delta s} / (\{\delta\Delta\}^{(1)T} \cdot \{\delta\Delta\}^{(1)T} + 1)^{1/2} \quad (\text{by Ramm [44]})$$

$$\text{or } \Delta\lambda^{(1)} = \pm \hat{\Delta s} / (\{\delta\Delta\}^{(1)T} \cdot \{\delta\Delta\}^{(1)T})^{1/2} \quad (\text{by Crisfield [45]})$$

Plus indicates loading and minus indicates unloading. The sign of $\Delta\lambda^{(1)}$ is chosen so that the dot product of the vectors

$$\Delta\lambda^{(1)} \begin{Bmatrix} \{\delta\Delta\}^{(1)T} \\ 1 \end{Bmatrix} \text{ and } \begin{Bmatrix} {}^t\{\Delta\} - {}^{t-\Delta t}\{\Delta\} \\ {}^t\lambda - {}^{t-\Delta t}\lambda \end{Bmatrix}$$

is positive.

5. Compute the incremental nodal displacements

$$\{\delta\Delta\}^{(1)} = \Delta\lambda^{(1)} \{\delta\Delta\}^{(1)T}$$

and update the total nodal displacements and the total load parameter

$${}^{t+\Delta t}\{\Delta\}^{(1)} = {}^t\{\Delta\} + \{\delta\Delta\}^{(1)}, \quad {}^{t+\Delta t}\lambda^{(1)} = {}^t\lambda + \Delta\lambda^{(1)}$$

6. Check convergence. If convergence is achieved, go to step 14. If not, continue with the 2nd iteration by going to step 7.

The i-th iteration, $i = 2, 3, \dots$

7. Compute the nodal force vector ${}^{t+\Delta t}\{F\}^{(i-1)}$ corresponds to the previous iteration ${}^{t+\Delta t}\{\Delta\}^{(i-1)}$.

8. Update the external load vector

$${}^{t+\Delta t}\{Q\}^{(i-1)} = {}^{t+\Delta t}\lambda^{(i-1)} \{\bar{Q}\}$$

9. Update the system stiffness matrices ${}^{t+\Delta t}{}_0[K_L]^{(i-1)}$ and ${}^{t+\Delta t}{}_0[K_{NL}]^{(i-1)}$ if desirable.

10. Solve for $\{\delta\Delta\}^{(n)I}$ and $\{\delta\Delta\}^{(n)II}$ from the two sets of equations

$$({}_0[K_L] + {}_0[K_{NL}])^{(i-1)} \{\delta\Delta\}^{(n)I} = \{\bar{Q}\}$$

$$({}_0[K_L] + {}_0[K_{NL}])^{(i-1)} \{\delta\Delta\}^{(n)II} = {}^{t+\Delta t}\lambda^{(i-1)} \{\bar{Q}\} - {}^{t+\Delta t}\{F\}^{(i-1)}$$

11. Compute the incremental load parameter $\Delta\lambda^{(n)}$

a. If iteration on the normal plane

$$\Delta\lambda^{(n)} = -(\{\delta\Delta\}^{(n)I} \cdot \{\delta\Delta\}^{(n)II}) / (\{\delta\Delta\}^{(n)I} \cdot \{\delta\Delta\}^{(n)I} + 1)$$

b. If iteration on the sphere, $\Delta\lambda^{(n)}$ is obtained from the following quadratic equation

$$a(\Delta\lambda^{(n)})^2 + 2b\Delta\lambda^{(n)} + c = 0$$

where by Ramm [44]

$$a = \{\delta\Delta\}^{(n)I} \cdot \{\delta\Delta\}^{(n)I} + 1$$

$$b = \lambda^{(i-1)} + \{\delta\Delta\}^{(n)I} \cdot (\{\delta\Delta\}^{(n)II} + \{\Delta\}^{(i-1)})$$

$$c = \{\delta\Delta\}^{(n)II} \cdot (\{\delta\Delta\}^{(n)II} + 2\{\Delta\}^{(i-1)})$$

Alternatively, by Crisfield [45]

$$a = \{\delta\Delta\}^{(n)I} \cdot \{\delta\Delta\}^{(n)I}$$

$$b = \{\delta\Delta\}^{(n)I} \cdot (\{\delta\Delta\}^{(n)II} + \{\Delta\}^{(i-1)})$$

$$c = (\{\delta\Delta\}^{(n)II} + \{\Delta\}^{(i-1)}) \cdot (\{\delta\Delta\}^{(n)II} + \{\Delta\}^{(i-1)}) - (\Delta\hat{s})^2$$

and $\Delta\hat{s}$ is the arc-length of the current load step.

Two solutions $\Delta\lambda_1^{(n)}$ for this quadratic equation and two corresponding vectors

$$\{\delta r\}^{(n)} = \begin{Bmatrix} \{\delta\Delta\}^{(n)} \\ \Delta\lambda_1^{(n)} \end{Bmatrix}, \quad \{\delta r\}_2^{(n)} = \begin{Bmatrix} \{\delta\Delta\}_2^{(n)} \\ \Delta\lambda_2^{(n)} \end{Bmatrix}$$

the improved solution corresponds to the smaller of $|\{\delta r\}^{(i)}|$ and $|\{\delta r\}_2^{(i)}|$

12. Compute the incremental nodal displacements

$$\{\delta \Delta\}^{(i)} = \Delta \lambda^{(i)} \{\delta \Delta\}^{(i)I} + \{\delta \Delta\}^{(i)II}$$

and update the total nodal displacements and the total load parameter

$${}^{t+\Delta t}\{\Delta\}^{(i)} = {}^{t+\Delta t}\{\Delta\}^{(i-1)} + \{\delta \Delta\}^{(i)}, \quad {}^{t+\Delta t}\lambda^{(i)} = {}^{t+\Delta t}\lambda^{(i-1)} + \Delta \lambda^{(i)}$$

13. Repeat steps 7 to 12 until the process has satisfied the convergence criteria.

14. Adjust the arc length for the subsequent load step by

$$\Delta \hat{s} = \Delta s (\hat{I}/I)^{1/2}$$

to control the number of future iterations, where

Δs = the arc length in the current load step,

$\hat{I}$ = no. of desired iterations (4 or 5 in the present study),

I = no. of iterations required in the current load step.

15. Start a new load step by returning to step 2.

4.4 Convergence Criteria

The incremental solution at the end of each iteration should be checked to see whether it has converged within the preassigned tolerance. Here the displacement criterion is adopted, i.e.

$$\left[\sum_{j=1}^{NEQ} (\Delta_j^{(i)} - \Delta_j^{(i-1)})^2 / \sum_{j=1}^{NEQ} ({}^{t+\Delta t}\Delta_j^{(i)})^2 \right]^{1/2} \leq \varepsilon$$

where ε is the displacement convergence tolerance and is set to 0.001 in the present study.

Another convergence criteria are also used, such as measuring the out-of-balance load vector or measuring the work done by the out-of-balance loads. Some experiences with these criteria can be found in Reference [53].

CHAPTER 5

ELEMENT DEVELOPMENT

5.1 Introduction

In this Chapter, we develop expressions of ${}_0[B_L]$, ${}_0[B_{NL}]$ and $[H]$ matrices for the degenerated 3-D shell, degenerated 3-D curved beam, three-dimensional solid and solid-shell transition elements, in order to find the stiffness matrices ${}_0[K_L]$, ${}_0[K_{NL}]$ and mass matrix ${}_0[M]$ for those elements.

5.2 Degenerated 3-D Shell Element

Degenerated 3-D shell element is obtained by imposing two constraints on the three-dimensional isoparametric solid element. These two constraints are: (1) straight line normal to midsurface before deformation remains straight but not normal after deformation, (2) the transverse normal components of strain and hence stress are ignored in the development. Therefore the nonlinear formulation admits arbitrarily large displacements and rotations of the shell element but small strains since the thickness does not change and the normal does not distort.

Consider the solid 3-D element shown in Fig. 5.1. Let ξ , η be the curvilinear coordinates in the middle surface of the shell and ζ be the coordinate in the thickness direction. Here ξ , η and ζ are normalized such that they vary between -1 and +1. The coordinates of a typical point in the element can be written as

$$x_i = \sum_{k=1}^n \varphi_k(\xi, \eta) \left[\frac{1+\zeta}{2} (x_i^k)_{top} + \frac{1-\zeta}{2} (x_i^k)_{bottom} \right] \quad (5.1)$$

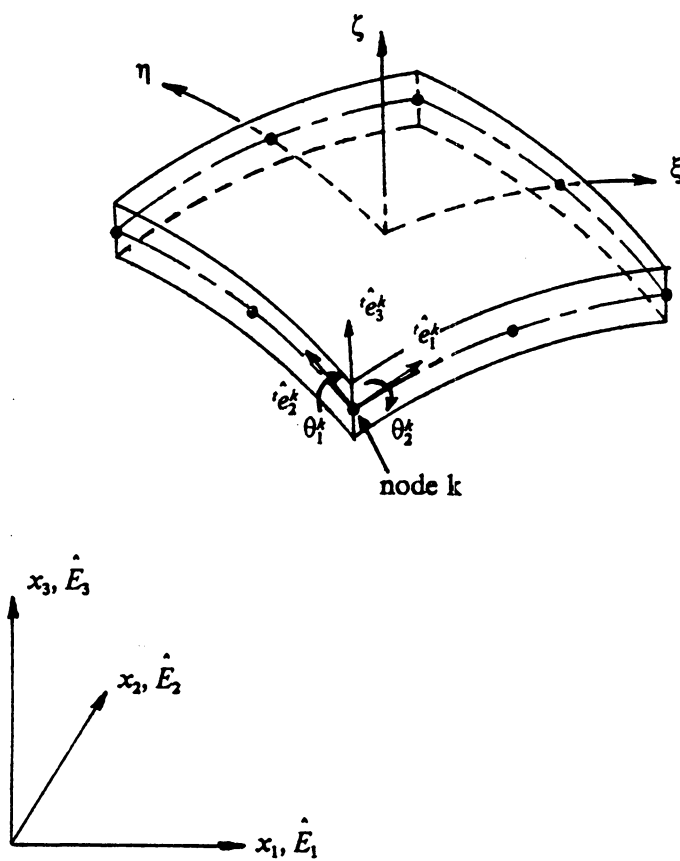


Figure 5.1 Geometry of general curved shell element

where n is the number of nodes in the element, and $\varphi_k(\xi, \eta)$ is the finite element interpolation function associated with node k . If $\varphi_k(\xi, \eta)$ are derived as interpolation functions of a parent element, square or triangular in plane, then compatibility is achieved at the interfaces of curved space shell elements. Define

$$V_{3i}^k = (x_i^k)_{top} - (x_i^k)_{bottom}$$

$$\hat{e}_{3i}^k = \vec{V}_3^k / |\vec{V}_3^k|$$

where $\vec{V}_3^k$ is the vector connecting the upper and lower points of the normal at node k .

Eqn. (5.1) can be rewritten as

$$x_i = \sum_{k=1}^n \varphi_k(\xi, \eta) [(x_i^k)_{mid} + \frac{\zeta}{2} h_k e_{3i}^k] \quad (5.2)$$

where $h_k = |\vec{V}_3^k|$ is the thickness of the shell element at node k . Hence, the coordinates of any point in the element at time t are interpolated by the expression

$${}^t x_i = \sum_{k=1}^n \varphi_k(\xi, \eta) [{}^t x_i^k + \frac{\zeta}{2} h_k {}^t e_{3i}^k] \quad (5.3)$$

and the displacements by

$${}^t u_i = {}^t x_i - {}^0 x_i = \sum_{k=1}^n \varphi_k(\xi, \eta) [{}^t u_i^k + \frac{\zeta}{2} h_k ({}^t e_{3i}^k - {}^0 e_{3i}^k)] \quad (5.4)$$

$$u_i = {}^{t+\Delta t} u_i - {}^t u_i = \sum_{k=1}^n \varphi_k(\xi, \eta) [u_i^k + \frac{\zeta}{2} h_k ({}^{t+\Delta t} e_{3i}^k - {}^t e_{3i}^k)] \quad (5.5)$$

Here ${}^t u_i^k$ and u_i^k denote, respectively, the displacement and incremental displacement components in the x_i -direction at the k -th node and time t . For small rotation $d\vec{\Omega}$ at each node

$$d\vec{\Omega} = \theta_2^k {}^t \hat{e}_1^k + \theta_1^k {}^t \hat{e}_2^k + \theta_3^k {}^t \hat{e}_3^k$$

the increment of vector ${}^t \hat{e}_3^k$ can be written as

$$\Delta \hat{e}_3^k = {}^{t+\Delta t}\hat{e}_3^k - {}^t\hat{e}_3^k = d\vec{\Omega} \times {}^t\hat{e}_3^k = \theta_1^k {}^t\hat{e}_1^k - \theta_2^k {}^t\hat{e}_2^k \quad (5.6)$$

Then eqn. (5.5) becomes

$$u_i = \sum_{k=1}^n \varphi_k(\xi, \eta) \left[u_i^k + \frac{\zeta}{2} h_k (\theta_1^k {}^t\hat{e}_{1i}^k - \theta_2^k {}^t\hat{e}_{2i}^k) \right] \quad i = 1, 2, 3 \quad (5.7)$$

The unit vectors ${}^t\hat{e}_1^k$ and ${}^t\hat{e}_2^k$ at node k can be obtained from the relations

$$\begin{aligned} {}^t\hat{e}_1^k &= \frac{\hat{E}_2 \times {}^t\hat{e}_3^k}{|\hat{E}_2 \times {}^t\hat{e}_3^k|} \\ {}^t\hat{e}_2^k &= {}^t\hat{e}_3^k \times {}^t\hat{e}_1^k \end{aligned} \quad (5.8)$$

where $\hat{E}_i$ are the unit vectors of the stationary global coordinate system. Eqn. (5.7) can be written in matrix form as

$$\{u\} = \{u_1 \ u_2 \ u_3\}^T = [H]_{3 \times 5n} \{\Delta^*\}_{5n \times 1}$$

where $\{\Delta^*\}$ = the vector of nodal incremental displacements (five per node).

$$= \{u_i^k \ \theta_1^k \ \theta_2^k\}^T \quad i = 1, 2, 3 \quad k = 1 \text{ to } n \quad (n \text{ is the number of nodes})$$

$[H]_{3 \times 5n}$ = the incremental displacement interpolation matrix

$$= \begin{bmatrix} \varphi_k & 0 & 0 & \frac{\varphi_k}{2} \zeta h_k {}^t\hat{e}_{11}^k & -\frac{\varphi_k}{2} \zeta h_k {}^t\hat{e}_{21}^k \\ 0 & \varphi_k & 0 & \frac{\varphi_k}{2} \zeta h_k {}^t\hat{e}_{12}^k & -\frac{\varphi_k}{2} \zeta h_k {}^t\hat{e}_{22}^k \\ 0 & 0 & \varphi_k & \frac{\varphi_k}{2} \zeta h_k {}^t\hat{e}_{13}^k & -\frac{\varphi_k}{2} \zeta h_k {}^t\hat{e}_{23}^k \end{bmatrix}$$

For each time step or iteration step one can find 3 unit vectors at each node from eqns. (5.6) and (5.8).

From eqn. (2.10) the linear strain increments $\{_0e\} = \{_0e_{11} \ _0e_{22} \ _0e_{33} \ 2_0e_{12} \ 2_0e_{13} \ 2_0e_{23}\}^T$ can be expressed as

$$\{_0e\} = {}^t[A] \{_0u\} \quad (5.9)$$

where

$\{_0u\}$ = the vector of derivatives of incremental displacements

$$= \{_0u_{1,1} \ _0u_{1,2} \ _0u_{1,3} \ _0u_{2,1} \ _0u_{2,2} \ _0u_{2,3} \ _0u_{3,1} \ _0u_{3,2} \ _0u_{3,3}\}^T$$

$$_0u_{i,j} = \frac{\partial u_i}{\partial {}^0x_j}$$

= derivative of displacement increment u_i w.r.t. coordinate ${}^0x_j$

and

$${}^t[A]_{6 \times 9} = \begin{bmatrix} 1 + {}_0u_{1,1} & 0 & 0 & {}_0u_{2,1} & 0 & 0 & {}_0u_{3,1} & 0 & 0 \\ 0 & {}_0u_{1,2} & 0 & 0 & 1 + {}_0u_{2,2} & 0 & 0 & {}_0u_{3,2} & 0 \\ 0 & 0 & {}_0u_{1,3} & 0 & 0 & {}_0u_{2,3} & 0 & 0 & 1 + {}_0u_{3,3} \\ {}_0u_{1,2} & 1 + {}_0u_{1,1} & 0 & 1 + {}_0u_{2,2} & {}_0u_{2,1} & 0 & {}_0u_{3,2} & {}_0u_{3,1} & 0 \\ {}_0u_{1,3} & 0 & 1 + {}_0u_{1,1} & {}_0u_{2,3} & 0 & {}_0u_{2,1} & 1 + {}_0u_{3,3} & 0 & {}_0u_{3,1} \\ 0 & {}_0u_{1,3} & {}_0u_{1,2} & 0 & {}_0u_{2,3} & 1 + {}_0u_{2,2} & 0 & 1 + {}_0u_{3,3} & {}_0u_{3,2} \end{bmatrix}$$

The vectors $\{_0u\}$ and $\{_0e\}$ are related to the displacement increments at nodes by

$$\{_0u\} = [N] \{u\} = [N] {}^t[H] \{\Delta^*\}$$

$$\{_0e\} = {}^t[A] \{_0u\} = {}^t[A] [N] {}^t[H] \{\Delta^*\} \equiv {}_0[B_L] \{\Delta^*\}$$

where

$[N]^T$ = the operator of differentials

$$= \begin{bmatrix} \frac{\partial}{\partial^0 x_1} & \frac{\partial}{\partial^0 x_2} & \frac{\partial}{\partial^0 x_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial^0 x_1} & \frac{\partial}{\partial^0 x_2} & \frac{\partial}{\partial^0 x_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial^0 x_1} & \frac{\partial}{\partial^0 x_2} & \frac{\partial}{\partial^0 x_3} \end{bmatrix}$$

and ${}^0[B_L] = {}^t[A] [N] {}^t[H] =$ the linear incremental strain-displacement matrix.

The components of ${}^t[A]$ contains ${}^i u_{i,j}$. From eqn. (5.4) the global displacements are related to the natural curvilinear coordinates ξ, η and linear coordinate ζ . Hence the derivatives of these displacements ${}^i u_{i,j}$ with respect to the global coordinates ${}^0 x_1, {}^0 x_2$ and ${}^0 x_3$ are obtained by a matrix relation

$$[{}^i u_{i,j}] = \begin{bmatrix} \frac{\partial^i u_1}{\partial^0 x_1} & \frac{\partial^i u_2}{\partial^0 x_1} & \frac{\partial^i u_3}{\partial^0 x_1} \\ \frac{\partial^i u_1}{\partial^0 x_2} & \frac{\partial^i u_2}{\partial^0 x_2} & \frac{\partial^i u_3}{\partial^0 x_2} \\ \frac{\partial^i u_1}{\partial^0 x_3} & \frac{\partial^i u_2}{\partial^0 x_3} & \frac{\partial^i u_3}{\partial^0 x_3} \end{bmatrix} = {}^0[J]^{-1} \begin{bmatrix} \frac{\partial^i u_1}{\partial \xi} & \frac{\partial^i u_2}{\partial \xi} & \frac{\partial^i u_3}{\partial \xi} \\ \frac{\partial^i u_1}{\partial \eta} & \frac{\partial^i u_2}{\partial \eta} & \frac{\partial^i u_3}{\partial \eta} \\ \frac{\partial^i u_1}{\partial \zeta} & \frac{\partial^i u_2}{\partial \zeta} & \frac{\partial^i u_3}{\partial \zeta} \end{bmatrix} \quad (5.10)$$

The Jacobian matrix ${}^0[J]$ is defined as

$${}^0[J] = \begin{bmatrix} \frac{\partial^0 x_1}{\partial \xi} & \frac{\partial^0 x_2}{\partial \xi} & \frac{\partial^0 x_3}{\partial \xi} \\ \frac{\partial^0 x_1}{\partial \eta} & \frac{\partial^0 x_2}{\partial \eta} & \frac{\partial^0 x_3}{\partial \eta} \\ \frac{\partial^0 x_1}{\partial \zeta} & \frac{\partial^0 x_2}{\partial \zeta} & \frac{\partial^0 x_3}{\partial \zeta} \end{bmatrix} \quad (5.11)$$

and is computed from the coordinate definition of eqn. (5.3). The derivatives of displacements $'u_i$ with respect to the coordinates ξ, η and ζ can be computed from eqn. (5.4).

In the evaluations of element matrices in eqns. (3.4) to (3.7), the integrands, (i.e. ${}_0[B_L], {}_0[C], {}_0[B_{NL}], {}_0[S], {}_0[H]$ and ${}_0\{\hat{S}\}$) should be expressed in the same coordinate system, namely the global coordinate system (${}^0x_1, {}^0x_2, {}^0x_3$) or the local curvilinear system (x'_1, x'_2, x'_3) which is aligned with the shell element midsurface. In this study we express the matrices and vector of the integrands in the local coordinate system (x'_1, x'_2, x'_3).

The number of stress and strain components are reduced to five since we neglect the transverse normal components of stress and strain. Hence, the global derivatives of displacements, $[\frac{\partial u_{i,j}}{\partial x_k}]$ which are obtained in eqn. (5.10), are transformed to the local derivatives of the local displacements along the orthogonal coordinates by the following relation (also see Ref. [11])

$$\begin{bmatrix} \frac{\partial u'_1}{\partial x'_1} & \frac{\partial u'_2}{\partial x'_1} & \frac{\partial u'_3}{\partial x'_1} \\ \frac{\partial u'_1}{\partial x'_2} & \frac{\partial u'_2}{\partial x'_2} & \frac{\partial u'_3}{\partial x'_2} \\ \frac{\partial u'_1}{\partial x'_3} & \frac{\partial u'_2}{\partial x'_3} & \frac{\partial u'_3}{\partial x'_3} \end{bmatrix} = [\theta]_{3 \times 3}^T [{}_0u_{i,j}] [\theta]_{3 \times 3} \quad (5.12)$$

where $[\theta]^T$ is the transformation matrix between the local coordinate system (x'_1, x'_2, x'_3) at the integration point and the global coordinate system (${}^0x_1, {}^0x_2, {}^0x_3$). The transformation matrix $[\theta]$ is obtained by interpolating the three orthogonal unit vectors ($\hat{e}_1, \hat{e}_2, \hat{e}_3$) at each node

$$[\theta] = \begin{bmatrix} \sum_{k=1}^n \varphi_k {}^t e_{11}^k & \sum_{k=1}^n \varphi_k {}^t e_{21}^k & \sum_{k=1}^n \varphi_k {}^t e_{31}^k \\ \sum_{k=1}^n \varphi_k {}^t e_{12}^k & \sum_{k=1}^n \varphi_k {}^t e_{22}^k & \sum_{k=1}^n \varphi_k {}^t e_{32}^k \\ \sum_{k=1}^n \varphi_k {}^t e_{13}^k & \sum_{k=1}^n \varphi_k {}^t e_{23}^k & \sum_{k=1}^n \varphi_k {}^t e_{33}^k \end{bmatrix}$$

Since the element matrices are evaluated using numerical integration, the transformation must be performed at each integration point during the numerical integration.

In order to obtain ${}_0[B_L]$, the vector of derivatives of incremental displacements $\{u_0\}$ needs to be evaluated. Equations (5.10) and (5.12) can be used again except that ${}^t u_i$ are replaced by u_i and the interpolation equation for u_i , eqn. (5.5), is applied.

The constitutive relation, ${}_0[C']$, for the k-th lamina of a laminated composite shell in the local coordinate system (x'_1, x'_2, x'_3) can be expressed as

$${}_0[C']_{(k)} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & 0 & 0 \\ C'_{12} & C'_{22} & C'_{23} & 0 & 0 \\ C'_{13} & C'_{23} & C'_{33} & 0 & 0 \\ 0 & 0 & 0 & C'_{44} & C'_{45} \\ 0 & 0 & 0 & C'_{45} & C'_{55} \end{bmatrix} \quad (5.13)$$

where

$$C'_{11} = m^4 Q_{11} + 2m^2 n^2 (Q_{12} + 2Q_{33}) + n^4 Q_{22}$$

$$C'_{12} = m^2 n^2 (Q_{11} + Q_{22} - 4Q_{33}) + (m^4 + n^4) Q_{12}$$

$$C'_{13} = mn[m^2 Q_{11} - n^2 Q_{22} - (m^2 - n^2)(Q_{12} + 2Q_{33})]$$

$$\begin{aligned}
C'_{22} &= n^4 Q_{11} + 2m^2 n^2 (Q_{12} + 2Q_{33}) + m^4 Q_{22} \\
C'_{23} &= mn[n^2 Q_{11} - m^2 Q_{22} + (m^2 - n^2)(Q_{12} + 2Q_{33})] \\
C'_{33} &= m^2 n^2 (Q_{11} + Q_{22} - 2Q_{12}) + (m^2 - n^2)^2 Q_{33} \\
C'_{44} &= m^2 Q_{44} + n^2 Q_{55}, \quad C'_{45} = mn(Q_{44} - Q_{55}) \\
C'_{55} &= m^2 Q_{55} + n^2 Q_{44} \\
m &= \cos \theta_{(k)}, \quad n = \sin \theta_{(k)}
\end{aligned}$$

Q_{ij} , which are the plane stress-reduced stiffnesses of an orthotropic lamina in the material coordinate system, are reduced from the constitutive relations for three-dimensional orthotropic body by neglecting the normal stress in the thickness direction. The Q_{ij} can be expressed in terms of engineering constants of a lamina

$$\begin{aligned}
Q_{11} &= \frac{E_1}{1 - \nu_{12} \nu_{21}}, \quad Q_{12} = \frac{\nu_{12} E_2}{1 - \nu_{12} \nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}} \\
Q_{44} &= (G_{13}) AK, \quad Q_{55} = (G_{23}) AK, \quad Q_{33} = G_{12}
\end{aligned} \tag{5.14}$$

where "AK" is the shear correction coefficient which is taken to be equal to 5/6.

To evaluate element matrices in eqns. (3.4) - (3.7), we employ the Gaussian quadrature to perform the integrations. Since we are dealing with laminated composite structures, the integration through the thickness involves individual lamina. One way is to use Gaussian quadrature through the thickness direction. Since the constitutive relation ${}_0[C]$ is different from layer to layer and is not a continuous function in the thickness direction, the integration should be performed separately for each layer (see [14,54]). This increases the computation time as the number of layers is increased. An alternative way is to perform explicit integration through the thickness and reduce the problem to a 2-D one. The Jacobian matrix, in general, is a function of ξ , η and ζ . The

terms in ζ may be neglected provided the thickness to curvature ratios are small. Thus the Jacobian matrix ${}^0[J]$ becomes independent of ζ and explicit integration can be employed. If ζ terms are retained in ${}^0[J]$, Gaussian points through the thickness should be added. In the present study we assume that the Jacobian matrix is independent of ζ in the evaluation of element matrices and the internal nodal force vector.

Since the explicit integration is performed through the thickness, the expression for $[\frac{\partial u'_i}{\partial x'_j}]$, ${}_0[A']$, $\{{}_0u'\}$, ${}^t[H]$, ${}_0[B'_L]$ and $\{{}_0\varepsilon'_{ij}\}$ are now expressed in an explicit form in terms of ζ ,

$$\begin{aligned}
[\frac{\partial u'_i}{\partial x'_j}]_{3 \times 3} &= [GU] + \zeta [GS] \\
{}_0[A']_{5 \times 9} &= [SD] + \zeta [TD] \\
\{{}_0u'\}_{9 \times 1} &= ([DH1] + \zeta [DH2])_{9 \times 5n} \{\Delta^*\}_{5n \times 1} \\
{}^t[H]_{3 \times 5n} &= [DK1] + \zeta [DK2] \\
{}_0[B'_L]_{5 \times 5n} &= ([SD] + \zeta [TD])([DH1] + \zeta [DH2]) \\
&= [SD][DH1] + \zeta([TD][DH1] + [SD][DH2]) \\
&\quad + \zeta^2 [TD][DH2] \\
\{{}_0\varepsilon'_{ij}\}_{5 \times 1} &= \{S1\} + \zeta \{S2\} + \zeta^2 \{S3\}
\end{aligned} \tag{5.15}$$

where prime on the variables indicates that they are expressed in the local coordinate system (x'_1, x'_2, x'_3) , which is aligned with the shell element midsurface, and the matrices and vectors on the right-hand side are functions of ξ and η . The integrands of element matrices and internal nodal force vector are now explicit functions of ζ and we can use explicit integration through the thickness and use the Gaussian quadrature to perform numerical integration on the midsurface of the shell element.

For thin shell structures, in order to avoid 'element locking' we use reduced integration scheme to evaluate the stiffness coefficients associated with the transverse shear deformation. Hence we split the constitutive matrix ${}^0[C']$ into two parts, one without transverse shear moduli ${}^0[C']_B$, and the other with only transverse shear moduli ${}^0[C']_S$. Full integration is used to evaluate the stiffness coefficients containing ${}^0[C']_B$, and reduced integration is used for those containing ${}^0[C']_S$.

If a shell element is subjected to a distributed load (such as the weight or pressure), the corresponding load vector ${}^{t+\Delta t}\{R\}$ from eqn. (2.6) is given by

$${}^{t+\Delta t}\{R\}_{5n \times 1} = \int_{{}^0A} {}^t[H]^T \begin{bmatrix} {}^{t+\Delta t}P_1 \\ {}^{t+\Delta t}P_2 \\ {}^{t+\Delta t}P_3 \end{bmatrix} {}^0dA \quad (5.16)$$

where

${}^{t+\Delta t}P_i$ = the component of distributed load in the ${}^0x_i$ direction at time $t + \Delta t$.

${}^0A$ = the area of upper or middle or bottom surface of the shell element
depending on the position of the loading.

and the loading is assumed deformation-independent.

Substituting ${}^t[H]$ into eqn. (5.16), eqn. (5.16) becomes

$$\begin{aligned}
{}^{t+\Delta t}\{R\}_{5n \times 1} &= \int_{0_A} \begin{bmatrix} \varphi_k & 0 & 0 \\ 0 & \varphi_k & 0 \\ 0 & 0 & \varphi_k \\ -\frac{\zeta}{2}\varphi_k h_k {}^te_{11}^k & -\frac{\zeta}{2}\varphi_k h_k {}^te_{12}^k & -\frac{\zeta}{2}\varphi_k h_k {}^te_{13}^k \\ -\frac{\zeta}{2}\varphi_k h_k {}^te_{21}^k & -\frac{\zeta}{2}\varphi_k h_k {}^te_{22}^k & -\frac{\zeta}{2}\varphi_k h_k {}^te_{23}^k \end{bmatrix} \begin{bmatrix} {}^{t+\Delta t}P_1 \\ {}^{t+\Delta t}P_2 \\ {}^{t+\Delta t}P_3 \end{bmatrix} {}^0dA \\
&= \sum_{r=1}^{NGP} \sum_{s=1}^{NGP} \begin{bmatrix} \varphi_k {}^{t+\Delta t}P_1 \\ \varphi_k {}^{t+\Delta t}P_2 \\ \varphi_k {}^{t+\Delta t}P_3 \\ \sum_{l=1}^3 \frac{\zeta}{2}\varphi_k h_k {}^{t+\Delta t}P_l {}^te_{1l}^k \\ \sum_{l=1}^3 -\frac{\zeta}{2}\varphi_k h_k {}^{t+\Delta t}P_l {}^te_{2l}^k \end{bmatrix} W_{\xi_r} W_{\eta_s} |{}^0J|_{(\xi_r, \eta_s)} 2/h \quad (5.17)
\end{aligned}$$

where

$h = \sum_{k=1}^{NPE} \varphi_k(\xi, \eta) h_k$ = the shell thickness at each Gaussian point,

W = the weight at each Gaussian point,

and $|{}^0J|$ is the determinant of the Jacobian matrix in eqn. (5.11) at each Gaussian point.

Here the ζ terms are retained in Jacobian matrix and let ζ equal to 1 or -1 or 0 respectively when the distributed loading is on upper or bottom or middle surface.

If the loading is deformation-dependent, the load vector ${}^{t+\Delta}\{R\}$ cannot be evaluated using eqn. (5.16). If the load (or time) step is small enough, the loading can be approximated by using the intensity of loading corresponding to time $t + \Delta t$, but integrating over the area calculated in the last iteration. Hence ${}^{t+\Delta}\{R\}$ is calculated by

$${}^{t+\Delta t}\{R\}_{5n \times 1} = \int_{{}^{t+\Delta t}A^{(t-1)}} {}^t[H]^T \begin{bmatrix} {}^{t+\Delta t}P_1 \\ {}^{t+\Delta t}P_2 \\ {}^{t+\Delta t}P_3 \end{bmatrix} {}^{t+\Delta t}dA \quad (5.18)$$

and the Jacobian matrix ${}^0[J]$ is replaced by ${}^{t+\Delta t}[J]^{(t-1)}$ in eqn. (5.17).

5.3 Degenerated Curved Beam Element

Here we consider the formulation of the more general three dimensional beam elements with rectangular cross-section. The basic kinematic and static assumptions used are: (1) plane sections originally normal to the neutral axis remain plane and undistorted under deformation but not necessarily normal to this axis, (2) only the longitudinal stress and two transverse shear stresses are nonzero for isotropic material but for laminated composite beam only the transverse normal stress perpendicular to the layers is neglected. The displacements and rotations of the element can be arbitrarily large but the element strains are assumed to be small, which means that the cross-sectional area of the beam does not change. Using the T. L. formulation of a continuous medium for nonlinear analysis presented in Chapter 2, the general 3-D beam element for nonlinear analysis is formulated here. Similar to the shell element developed in the previous section, the general beam element presented here is a doubly degenerated element from a full isoparametric continuum element by imposing the two constraints. This beam element exhibits the displacement compatibility with the shell element and has the transverse shear deformation and variable thickness properties. Hence, this beam element can be used to model stiffeners of a stiffened shell structure. Also, with the introduction of offsets of the beam neutral axis from the reference axis in the element formulation, 3-D stiffeners and curved beams of general cross-section may be modelled [15,26].

The basic geometry of the beam element developed here results from the appropriate degeneration of a 3-D isoparametric continuum element as shown in Fig. 5.2. The process may be applied to either the cubic, parabolic or linear isoparametric 3-D hexahedra. Let ξ be a curvilinear coordinate line along the length (neutral axis) of the beam while η and ζ are linear coordinate lines in cross-section plane. ξ, η, ζ vary between -1 and 1 in the usual way. Assume the cross-section at any point to be rectangular, two well defined thickness directions appear and an expression can be written to relate the Cartesian coordinates of any point within the beam to its curvilinear coordinates and the Cartesian coordinates of the node points along the reference axis. If the neutral and reference axes are coincident (no offsets), the node points are located at the intersection of the η, ζ planes with the coordinate line ξ . The section properties of the beam are specified by a pair of thicknesses a_k and b_k , and a set of orthogonal unit vectors $\hat{e}_1^k, \hat{e}_2^k$ and $\hat{e}_3^k$ at each node as shown in Fig. 5.3.

The $\hat{e}_2^k$ and $\hat{e}_3^k$ are perpendicular to the neutral axis and aligned with two thickness directions (see Fig. 5.3). The relationship between the Cartesian and curvilinear coordinates which defines the basic geometry for an n noded element at time t is given by

$${}^t x_i = \sum_{k=1}^n \varphi_k(\xi) [{}^t x_i^k + (\frac{\zeta}{2} a_k + \Delta_k^\zeta) {}^t e_{3i}^k + (\frac{\eta}{2} b_k + \Delta_k^\eta) {}^t e_{2i}^k] \quad (5.19)$$

where Δ_k^ξ and Δ_k^η are the offsets in the directions $\hat{e}_3^k$ and $\hat{e}_2^k$ respectively at node k, $\varphi_k(\xi)$ are the interpolation functions of a line element and ${}^t x_i^k$ are the Cartesian coordinates of node k. The offsets exist when the neutral axis of the beam may be displaced from the reference axis which is passing through the nodal points. The introduction of offset makes it possible that the stiffening elements use the nodes of the shell elements so that the node number of a stiffened shell structure will not increase.

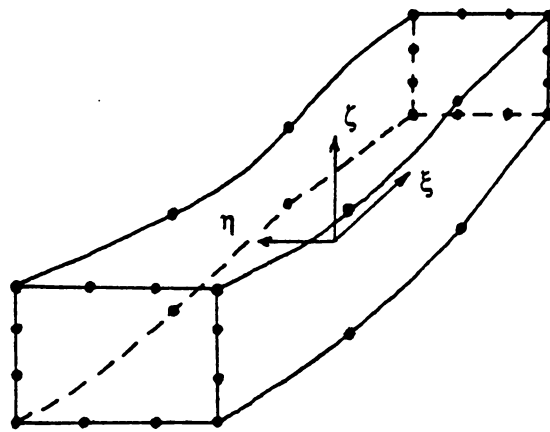


Figure 5.2 An isoparametric cubic element representing a curved beam

Coordinate system
at node 1

At time t

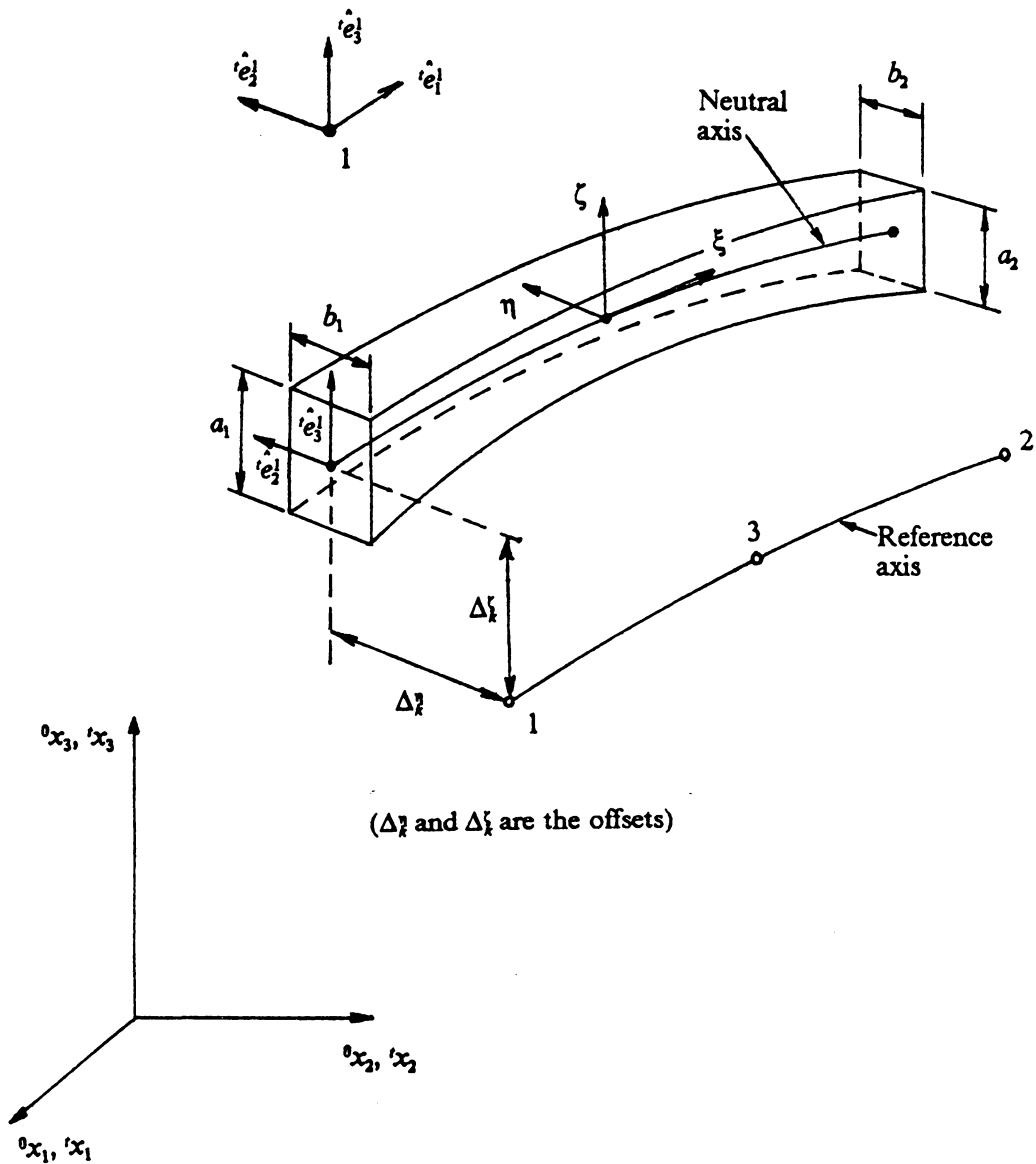


Figure 5.3 Geometry of a parabolic curved beam element

The three components of displacement of each point within the element can be expressed in terms of the three Cartesian components of displacement u_i^k at the reference line nodes and three rotations θ_1^k, θ_2^k and θ_3^k which trace the change in orientations of $\hat{e}_1^k, \hat{e}_2^k, \hat{e}_3^k$ local coordinate system at each node. So the local orthogonal vectors $\hat{e}_1^k, \hat{e}_2^k, \hat{e}_3^k$ form a coordinate system for the corresponding scalar rotations θ_2^k, θ_1^k and θ_3^k as shown in Fig. 5.4. It is noted that the nodal displacements u_i^k are global while the rotations θ_i^k are local. From eqn. (5.19) the components of displacement increment can be written as

$$\begin{aligned} u_i &= {}^{t+\Delta t}u_i - {}^tu_i \\ &= {}^{t+\Delta t}x_i - {}^tx_i \\ u_i &= \sum_{k=1}^n \varphi_k(\xi) \left[u_i^k + \left(\frac{\zeta}{2}a_k + \Delta_k^\zeta \right) e_{3i}^k + \left(\frac{\eta}{2}b_k + \Delta_k^\eta \right) e_{2i}^k \right] \end{aligned} \quad (5.20)$$

where

$$\begin{aligned} \hat{e}_3^k &= \text{increment of unit vector } \hat{e}_3^k \\ &= (\text{small rotation vector}) \times \hat{e}_3^k \\ &= (\theta_2^k \hat{e}_1^k + \theta_1^k \hat{e}_2^k + \theta_3^k \hat{e}_3^k) \times \hat{e}_3^k \\ &= -\theta_2^k \hat{e}_2^k + \theta_1^k \hat{e}_1^k \\ \hat{e}_2^k &= \text{increment of unit vector } \hat{e}_2^k \\ &= (\text{small rotation vector}) \times \hat{e}_2^k \\ &= (\theta_2^k \hat{e}_1^k + \theta_1^k \hat{e}_2^k + \theta_3^k \hat{e}_3^k) \times \hat{e}_2^k \\ &= \theta_2^k \hat{e}_3^k - \theta_3^k \hat{e}_1^k \end{aligned}$$

$$\text{Hence } u_i = \sum_{k=1}^n \varphi_k u_i^k + \varphi_k \left[\left(\frac{\eta}{2}b_k + \Delta_k^\eta \right) \hat{e}_{3i}^k - \left(\frac{\zeta}{2}a_k + \Delta_k^\zeta \right) \hat{e}_{2i}^k \right] \theta_1^k +$$

$$\varphi_k \left(\frac{\zeta}{2}a_k + \Delta_k^\zeta \right) \hat{e}_{1i}^k \theta_1^k - \varphi_k \left(\frac{\eta}{2}b_k + \Delta_k^\eta \right) \hat{e}_{1i}^k \theta_3^k \quad i = 1, 2, 3 \quad (5.21)$$

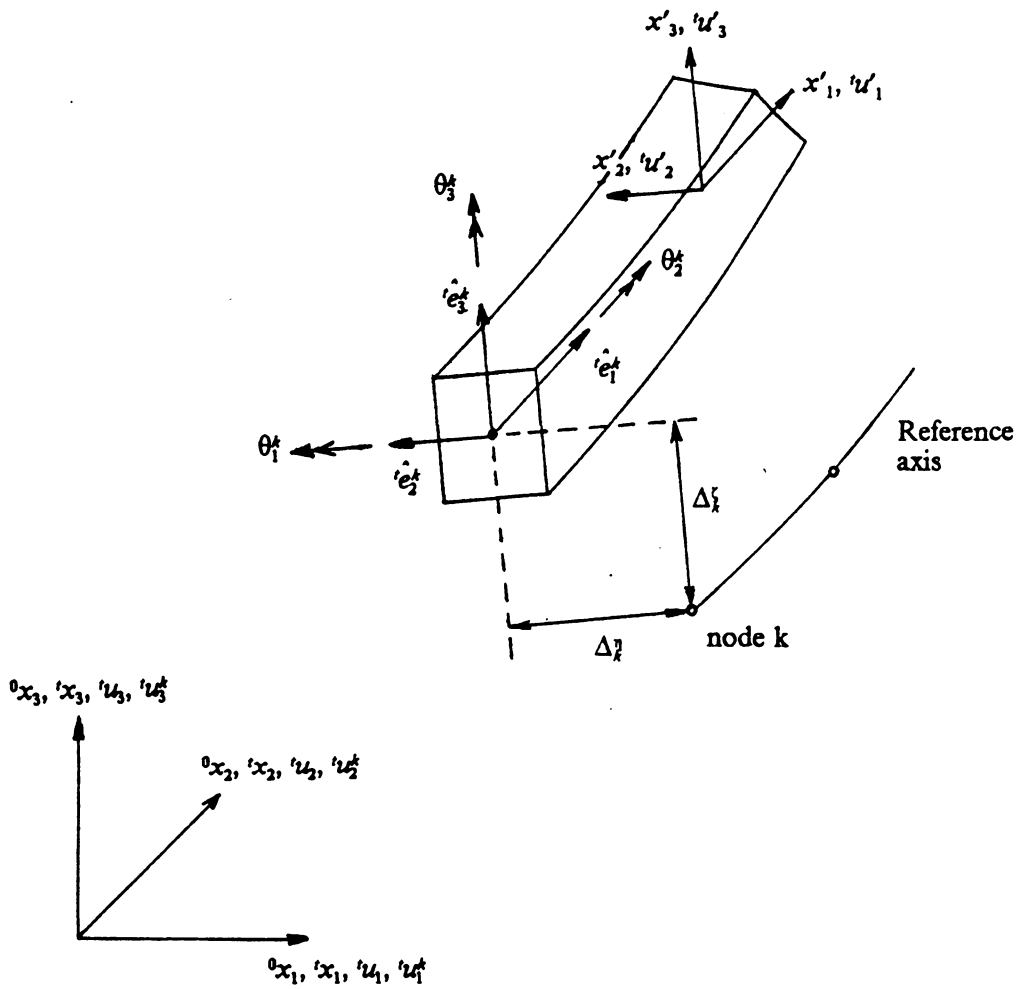


Figure 5.4 Local and global coordinate systems for kinematic and static variables of a curved beam element

From this expression the incremental displacement interpolation matrix ${}^t[H]$ for the beam element can be formulated as:

$${}^t[H]_{3 \times 6n} = \quad (5.22)$$

$$\begin{bmatrix} \varphi_k & 0 & 0 & \varphi_k(\frac{\zeta}{2}a_k + \Delta\zeta_k)'e_{11}^k & \varphi_k[(\frac{\eta}{2}b_k + \Delta\eta_k)'e_{31}^k - (\frac{\zeta}{2}a_k + \Delta\zeta_k)'e_{21}^k] & -\varphi_k(\frac{\eta}{2}b_k + \Delta\eta_k)'e_{11}^k \\ 0 & \varphi_k & 0 & \varphi_k(\frac{\zeta}{2}a_k + \Delta\zeta_k)'e_{12}^k & \varphi_k[(\frac{\eta}{2}b_k + \Delta\eta_k)'e_{32}^k - (\frac{\zeta}{2}a_k + \Delta\zeta_k)'e_{22}^k] & -\varphi_k(\frac{\eta}{2}b_k + \Delta\eta_k)'e_{12}^k \\ 0 & 0 & \varphi_k & \varphi_k(\frac{\zeta}{2}a_k + \Delta\zeta_k)'e_{13}^k & \varphi_k[(\frac{\eta}{2}b_k + \Delta\eta_k)'e_{33}^k - (\frac{\zeta}{2}a_k + \Delta\zeta_k)'e_{23}^k] & -\varphi_k(\frac{\eta}{2}b_k + \Delta\eta_k)'e_{13}^k \end{bmatrix}$$

Using eqns. (3.4) to (3.7) and this ${}^t[H]$ for beam element, the beam element matrices and internal force vector needed in the eqn. (4.3) can be obtained. The processes to evaluate element matrices and internal force vector are the same as that for the shell element in the last section. The main difference is that in carrying out the numerical integration of element matrices, we perform explicit intergration through the two thickness directions and reduce the problem to a 1-D one. The Jacobian matrix ${}^t[J]$ becomes independent of η and ζ if we neglect the terms in η and ζ , and then explicit integration can be employed to evaluate the element matrices and the internal nodal force vector.

Since the explicit integration is performed through two thickness directions, η and ζ , the expressions for $[\frac{\partial {}^t u'_i}{\partial x'_j}]$, ${}_0[A']$, $\{{}_0 u'\}$, ${}^t[H]$, ${}_0[B'_L]$ and $\{{}_0 \epsilon'_{ij}\}$ are now expressed in an explicit form of η and ζ , i.e.

$$[\frac{\partial {}^t u'_i}{\partial x'_j}]_{3 \times 3} = [GU] + \zeta [GS] + \eta [GQ]$$

$${}_0[A']_{5 \times 9} = [SD] + \zeta [TD] + \eta [UD]$$

$$\{{}_0 u'\}_{9 \times 1} = ([DH1] + \zeta [DH2] + \eta [DH3])_{9 \times 6n} \{\Delta^s\}_{6n \times 1}$$

$${}^t[H]_{3 \times 6n} = [DK1] + \zeta [DK2] + \eta [DK3]$$

$$\begin{aligned} {}^t[B'_L]_{5 \times 6n} &= ([SD] + \zeta [TD] + \eta [UD])([DH1] + \zeta [DH2] + \eta [DH3]) \\ &= [SD][DH1] + \zeta ([TD][DH1] + [SD][DH2]) \\ &\quad + \zeta^2 [TD][DH2] + \eta ([UD][DH1] + [SD][DH3]) \\ &\quad + \eta^2 [UD][DH3] + \eta \zeta ([UD][DH2] + [TD][DH3]) \end{aligned}$$

$$\{{}^t\epsilon'_{ij}\}_{5 \times 1} = \{S1\} + \zeta \{S2\} + \zeta^2 \{S3\} + \eta \{S4\} + \zeta \eta \{S5\} + \eta^2 \{S6\}$$

where prime indicates that the quantities are expressed in the local coordinate system (x'_1, x'_2, x'_3) , which is aligned with the neutral axis of the beam element, and the matrices and vectors on the right-hand side are functions of ξ only. The integrands of element matrices and internal nodal force vector are now explicit functions of η and ζ . Therefore explicit integration is employed through the two thickness directions, and the Gaussian quadrature is used to perform numerical integration in the longitudinal direction. The reduced/selective integration scheme is used to avoid "element locking".

For laminated composite beam the same constitutive relations as that for laminated composite shell is used and five components of stress and strain are retained.

The external force vector ${}^{t+\Delta t}\{R\}$ for a beam element subjected to distributed line load is evaluated by letting $\eta = 0$, then similar to eqn. (2.6) we have

$${}^{t+\Delta t}\{R\}_{6n \times 1} = \int_{{}^0L} {}^t[H]^T \begin{bmatrix} {}^{t+\Delta t}P_1 \\ {}^{t+\Delta t}P_2 \\ {}^{t+\Delta t}P_3 \end{bmatrix} {}^0dL \quad (5.23)$$

where

${}^{t+\Delta t}P_i$ = the component of distributed load in the ${}^0x_i$ direction at time $t + \Delta t$.

${}^0L$ = the length of line which is the intersection of $\eta = 0$ plane with

upper or lower surface of the beam.

and the loading is assumed deformation-independent.

Substituting ${}^t[H]$ from eqn. (5.22) into eqn. (5.23), we obtain

$${}^{t+\Delta t}\{R\}_{6n \times 1} = \sum_{r=1}^{NGP} \begin{bmatrix} \varphi_k {}^{t+\Delta t}P_1 \\ \varphi_k {}^{t+\Delta t}P_2 \\ \varphi_k {}^{t+\Delta t}P_3 \\ \sum_{i=1}^3 \varphi_k \left(\frac{\zeta}{2} a_k + \Delta_k^\zeta \right) {}^t e_{1i}^k {}^{t+\Delta t}P_i \\ \sum_{i=1}^3 \varphi_k \left[\Delta_k^\eta {}^t e_{3i}^k - \left(\frac{\zeta}{2} a_k + \Delta_k^\zeta \right) {}^t e_{2i}^k \right] {}^{t+\Delta t}P_i \\ \sum_{i=1}^3 - \varphi_k \Delta_k^\eta {}^t e_{1i}^k {}^{t+\Delta t}P_i \end{bmatrix} W_{\xi_r} |{}^0J|_{\xi_r} 4/(ab) \quad (5.24)$$

where

$a = \sum_{k=1}^{NPE} \varphi_k(\xi, \eta) a_k$ = the beam thickness in ζ direction at each Gaussian point,

$b = \sum_{k=1}^{NPE} \varphi_k(\xi, \eta) b_k$ = the beam thickness in η direction at each Gaussian point,

W = the weight at each Gaussian point,

and $|{}^0J|$ is the determinant of the Jacobian matrix at each Gaussian point. Here the η and ζ terms are retained in Jacobian matrix and let $\eta = 0$ and $\zeta = 1$ or -1 or 0 respectively when the distributed line loading is on upper or bottom or middle surface. If the loading is deformation-dependent, similar modifications as discussed for shell element should be made.

In analyzing stiffened plate or shell, the external load vector is evaluated for shell elements only by using eqn. (5.17) and we do not compute eqn. (5.24) since the stiffener elements have common nodes with the shell elements.

The system matrices of the stiffened shell are obtained by adding the element matrices of the stiffeners to that of the shell elements. Since the nodal rotation degrees-of-freedom are around the local coordinate axes at the node, we should note the misorientation of the two coordinate systems attached on the shell element and beam element respectively. In the formulations of stiffener element, suitable transformations of nodal variables and axes system to that of a shell element are necessary if there is a misorientation between the shell and the stiffener axes systems. In this study two kinds of stiffener orientations relative to shell are considered. The first one is that the neutral axis of stiffener is parallel to the ξ axis of the shell element. In this case there is no misorientation between the shell and the stiffener axes systems (see Fig. 5.5(a)). It means the nodal variables of the stiffener element $\{u_i^t, \theta_1^t, \theta_2^t, \theta_3^t\}_B^T$ are the same as that of the shell element $\{u_i^t, \theta_1^t, \theta_2^t, \theta_3^t\}_S^T$. Also the local coordinate systems for beam and shell elements at the node in common are the same. The second one is that when the neutral axis of stiffener is placed parallel to the η axis of the shell element, there is a misorientation between the shell and the stiffener axes systems as can be seen in Fig. 5.5(b). The relations of nodal variables and axes systems of these two elements are also shown in Fig. 5.5(b).

The total number of nodes for a shell structure with stiffeners is the same as that without stiffeners. The nodes of a stiffening element are part of a shell element and the stiffening element matrices are added to the shell element matrices directly to obtain the stiffened shell matrices.

5.4 Three-Dimensional Continuum Element

The geometric description and displacement representation for a 3-D continuum element are much easier than that of degenerated 3-D shell element and beam element.

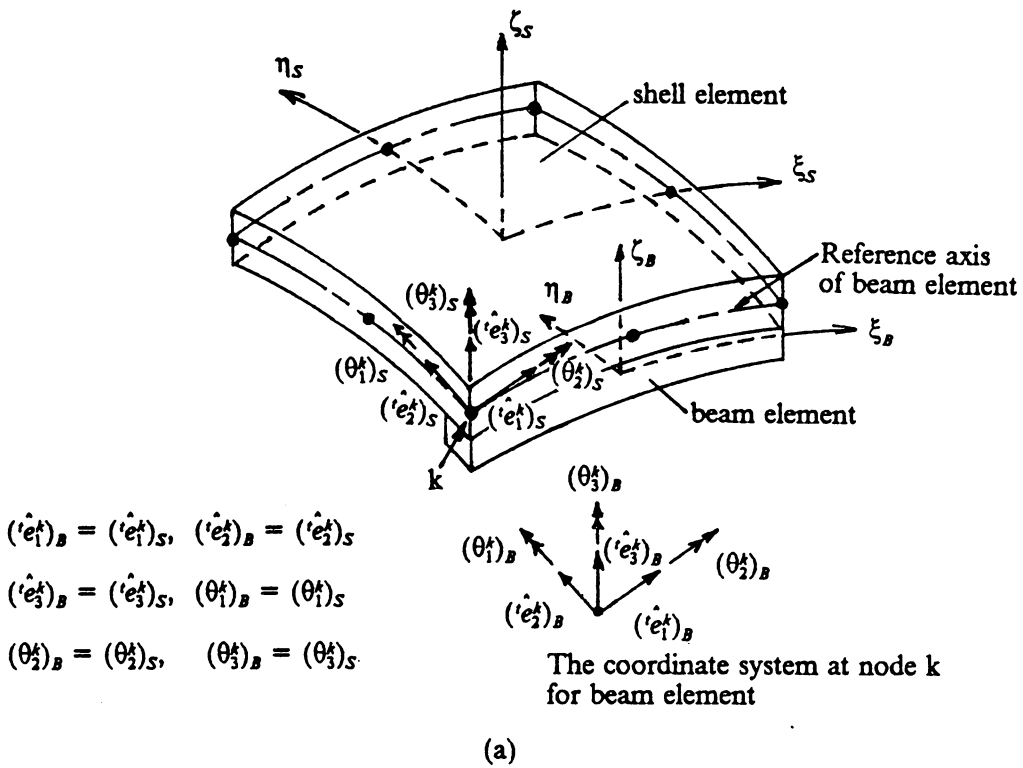
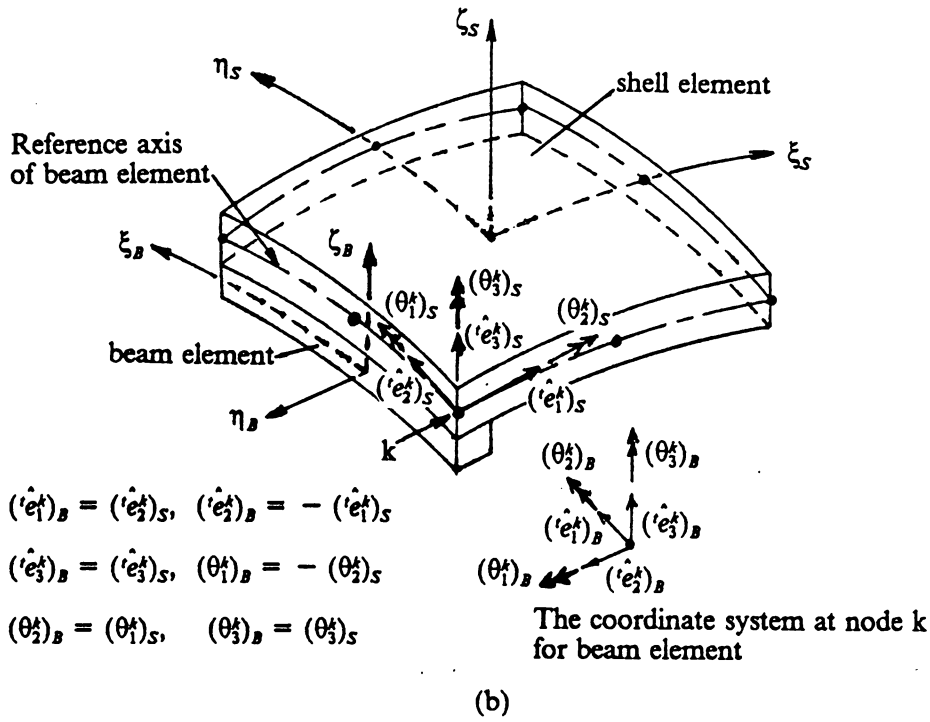


Figure 5.5 Geometric relations of stiffener axes system to shell axes system

The relationship between the Cartesian and curvilinear coordinates for an n node element at time t is given by

$${}^t x_i = \sum_{k=1}^n \varphi_k(\xi, \eta, \zeta) {}^t x_i^k \quad (5.26)$$

The displacements at time t is ${}^t u_i = {}^t x_i - {}^0 x_i$. Hence, the incremental displacements from time t to $t + \Delta t$ are obtained as

$$\begin{aligned} u_i &= {}^{t+\Delta t} x_i - {}^t x_i \\ &= \sum_{k=1}^n \varphi_k(\xi, \eta, \zeta) ({}^{t+\Delta t} x_i^k - {}^t x_i^k) \\ &= \sum_{k=1}^n \varphi_k(\xi, \eta, \zeta) u_i^k \end{aligned} \quad (5.27)$$

It is noted that now there are only 3 d.o.f. at each node, i.e. three incremental translation displacements. From eqn. (5.27) the incremental displacement interpolation matrix $[H]$ for the 3-D continuum element can be formulated as follows:

$$[H]_{3 \times 3n} = \begin{bmatrix} \varphi_k & 0 & 0 \\ 0 & \varphi_k & 0 \\ 0 & 0 & \varphi_k \end{bmatrix} \quad (5.28)$$

Using eqns. (3.4) to (3.7) again and $[H]$ in eqn. (5.28), the element matrices and internal nodal force vector for 3-D continuum element can be obtained. The procedure is similar to that in the previous sections for shell and beam elements, except that no kinematic and static assumptions are imposed. Hence there are six components for stresses and strains respectively in local coordinate system (x'_1, x'_2, x'_3) aligned with the element. That is

$$\{ {}^0 e' \}_{6 \times 1} = \{ {}^0 e'_{11} \quad {}^0 e'_{22} \quad {}^0 e'_{33} \quad 2 {}^0 e'_{23} \quad 2 {}^0 e'_{13} \quad 2 {}^0 e'_{12} \}^T$$

$$= '[A']_{6 \times 9} \{u'\}_{9 \times 1} \quad (5.29)$$

Also the transformation matrix $[\theta]$ at each integration point is formulated in a different way. Here $[\theta]$ is given by

$$[\theta] = [\hat{q}_1, \hat{q}_2, \hat{q}_3] \quad (5.30)$$

where

$$\bar{Q}_3 = \begin{bmatrix} \frac{\partial' x_1}{\partial \xi} \\ \frac{\partial' x_2}{\partial \xi} \\ \frac{\partial' x_3}{\partial \xi} \end{bmatrix} \times \begin{bmatrix} \frac{\partial' x_1}{\partial \eta} \\ \frac{\partial' x_2}{\partial \eta} \\ \frac{\partial' x_3}{\partial \eta} \end{bmatrix}, \quad \hat{q}_3 = \frac{\bar{Q}_3}{|\bar{Q}_3|}$$

$$\bar{Q}_1 = \hat{E}_2 \times \bar{Q}_3, \quad \hat{q}_1 = \frac{\bar{Q}_1}{|\bar{Q}_1|}$$

$$\bar{Q}_2 = \bar{Q}_3 \times \bar{Q}_1, \quad \hat{q}_2 = \frac{\bar{Q}_2}{|\bar{Q}_2|}$$

The constitutive matrix ${}_0[C']_{(k)}$ for the k-th lamina of laminated composite structure in the local coordinate system (x'_1, x'_2, x'_3) can be expressed as

$${}_0[C']_{(k)} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & 0 & 0 & C'_{16} \\ C'_{12} & C'_{22} & C'_{23} & 0 & 0 & C'_{26} \\ C'_{13} & C'_{23} & C'_{33} & 0 & 0 & C'_{36} \\ 0 & 0 & 0 & C'_{44} & C'_{45} & 0 \\ 0 & 0 & 0 & C'_{45} & C'_{55} & 0 \\ C'_{16} & C'_{26} & C'_{36} & 0 & 0 & C'_{66} \end{bmatrix} \quad (5.31)$$

where

$$\begin{aligned} C'_{11} &= m^4 Q_{11} + 2m^2 n^2 (Q_{12} + 2Q_{66}) + n^4 Q_{22} \\ C'_{12} &= m^2 n^2 (Q_{11} + Q_{22} - 4Q_{66}) + (m^4 + n^4) Q_{12} \\ C'_{13} &= m^2 Q_{13} + n^2 Q_{23} \\ C'_{16} &= mn[m^2 Q_{11} - n^2 Q_{22} - (m^2 - n^2)(Q_{12} + 2Q_{66})] \\ C'_{22} &= n^4 Q_{11} + 2m^2 n^2 (Q_{12} + 2Q_{66}) + m^4 Q_{22} \\ C'_{23} &= n^2 Q_{13} + m^2 Q_{23} \\ C'_{26} &= mn[n^2 Q_{11} - m^2 Q_{22} + (m^2 - n^2)(Q_{12} + 2Q_{66})] \\ C'_{33} &= Q_{33} \\ C'_{36} &= mn(Q_{13} - Q_{23}) \\ C'_{44} &= m^2 Q_{44} + n^2 Q_{55}, \quad C'_{45} = mn(Q_{55} - Q_{44}) \\ C'_{55} &= m^2 Q_{55} + n^2 Q_{44} \\ C'_{66} &= m^2 n^2 (Q_{11} + Q_{22} - 2Q_{12}) + (m^2 - n^2)^2 Q_{66} \\ m &= \cos \theta_{(k)}, \quad n = \sin \theta_{(k)} \end{aligned}$$

Q_{ij} are the constitutive relations for a three-dimensional orthotropic lamina, which can be expressed in terms of the engineering constants of a lamina

$$Q_{11} = \frac{E_1(1 - \nu_{23} \nu_{32})}{\Delta}, \quad Q_{12} = E_1 \frac{\nu_{21} + \nu_{31} \nu_{23}}{\Delta} = E_2 \frac{\nu_{12} + \nu_{32} \nu_{13}}{\Delta}$$

$$\begin{aligned}
Q_{13} &= E_3 \frac{v_{13} + v_{12} v_{23}}{\Delta}, & Q_{22} &= \frac{E_2(1 - v_{13} v_{31})}{\Delta} \\
Q_{23} &= E_3 \frac{v_{23} + v_{21} v_{13}}{\Delta}, & Q_{33} &= \frac{E_3(1 - v_{12} v_{21})}{\Delta} \\
Q_{44} &= G_{23}, & Q_{55} &= G_{13}, & Q_{66} &= G_{12}
\end{aligned} \tag{5.32}$$

where $\Delta = 1 - v_{12} v_{21} - v_{23} v_{32} - v_{31} v_{13} - 2 v_{21} v_{32} v_{13}$

In carrying out the integration of element matrices internal nodal force vector, the Gaussian quadrature is used in all ξ , η and ζ directions to perform the numerical integrations. For thin structures modelled with 3-D continuum elements, the reduced integration scheme is applied to evaluate the stiffness coefficients associated with the transverse shear deformation to avoid the locking phenomenon.

If the 3-D continuum element is subjected to surface distributed loading which is deformation-independent, the external load vector in eqn. (2.6) becomes

$$\begin{aligned}
{}^{t+\Delta t}\{R\}_{3n \times 1} &= \int_{0_A} {}^t[H]^T \begin{bmatrix} {}^{t+\Delta t}P_1 \\ {}^{t+\Delta t}P_2 \\ {}^{t+\Delta t}P_3 \end{bmatrix} {}^0dA \\
&= \sum_{r=1}^{NGP} \sum_{s=1}^{NGP} \begin{bmatrix} \varphi_k {}^{t+\Delta t}P_1 \\ \varphi_k {}^{t+\Delta t}P_2 \\ \varphi_k {}^{t+\Delta t}P_3 \end{bmatrix} W_{\xi_r} W_{\eta_s} |{}^0J|_{(\xi_r, \eta_s)} 2/TH
\end{aligned} \tag{5.33}$$

where

${}^{t+\Delta t}P_i$ = the component of distributed load in the ${}^0x_i$ direction at time $t + \Delta t$.

${}^0A$ = the area of upper or middle or bottom surface of the 3-D solid element depending on the position of the loading,

$TH = \sum_{k=1}^{NPEH} \varphi_k(\xi, \eta, \zeta) h_k$ = the thickness of 3-D solid element at each Gaussian point,

W = the weight at each Gaussian point,

and ζ is equal to 1 or -1 or 0 respectively when the distributed loading is on upper or bottom or middle surface. Eqn. (5.33) is similar to eqn. (5.17) for the shell element.

The 3-D continuum element developed here is used in conjunction with shell element and solid-shell transition element to model general shell-type structures.

5.5 Solid-to-Shell Transition Element

In the analysis of actual shell structures, e.g. arch dams and foundations, turbine blades mounted on a shaft which have relatively thick root and thin tip, it may be necessary to model shell-to-solid transition regions. Also for another kind of practical shell structure like folded plate, the intersection of two shell surfaces present some difficulties in modelling. These transition regions can be modelled by using transition elements while use 3-D continuum element and shell element to model three-dimensional solid and shell-like portions respectively. The transition elements developed here possess the properties of both shell element in Section 5.1 and solid element in Section 5.3, and there are nodes on the top and bottom surfaces in addition to that on the mid-surface. The interpolation functions, corresponding to the nodes which are common with the adjacent 3-D continuum element, are the usual functions used in the three-dimensional continuum elements, and the interpolation functions associated with the other mid-

surface nodes are those of the shell elements. Also the degrees of freedom of the nodes which are common with the adjacent 3-D continuum element is three and five degrees of freedom for the other nodes.

An example of the transition element developed here is shown in Figure 5.6. Fig. 5.6(a) shows a 16-node three-dimensional isoparametric solid element where eight-node parabolic faces are connected by linear edges. Figure 5.6(c) is the transition element which will provides proper connections between the elements shown in Fig. 5.6(a) and Fig. 5.6(b). Its $\zeta = -1$ face is compatible with $\xi = \pm 1$ and $\eta = \pm 1$ faces of the 3-D solid element in Fig. 5.6(a), whereas its $\xi = +1$ face is compatible with $\xi = \pm 1$ and $\eta = \pm 1$ faces of the curved shell element in Fig. 5.6(b). More transition elements for three-dimensional stress analysis can be found in Ref. [50].

Referring to eqns. (5.3) and (5.26) the coordinates of any point in the transition element at time t are interpolated by the expression

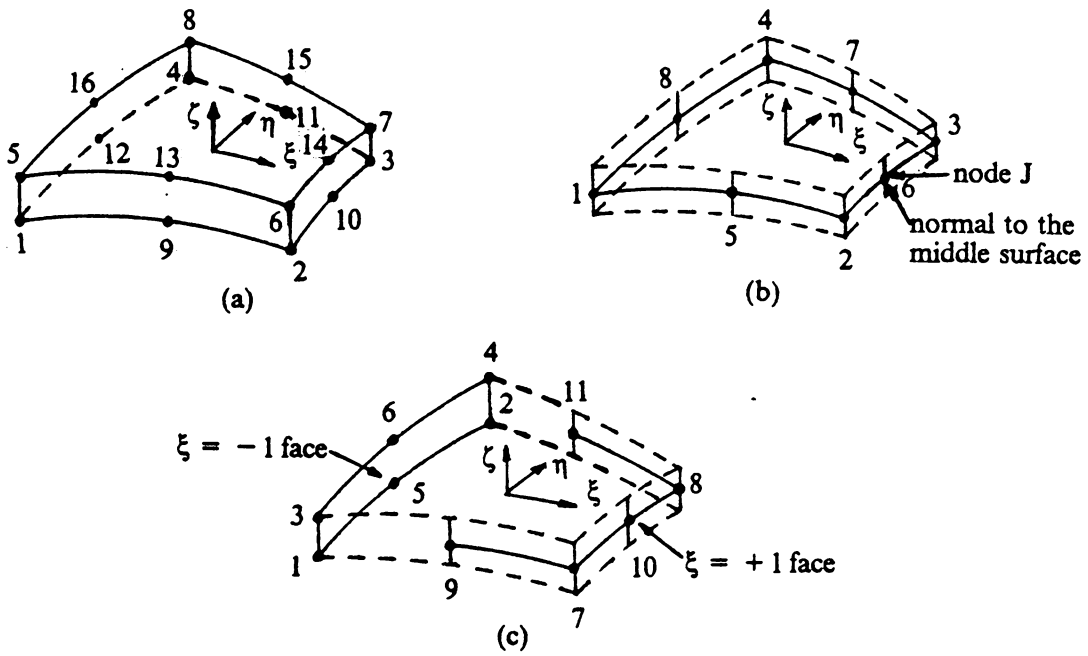
$${}^t x_i = \sum_{k=1}^m \varphi_k(\xi, \eta, \zeta) {}^t x_i^k + \sum_{k=m+1}^n \bar{\varphi}_k(\xi, \eta) \left[{}^t x_i^k + \frac{\zeta}{2} h_k {}^t e_{3i}^k \right] \quad (5.34)$$

The displacements at time t are given by

$$\begin{aligned} {}^t u_i &= {}^t x_i - {}^0 x_i \\ &= \sum_{k=1}^m \varphi_k(\xi, \eta, \zeta) {}^t u_i^k + \sum_{k=m+1}^n \bar{\varphi}_k(\xi, \eta) \left[{}^t u_i^k + \frac{\zeta}{2} h_k ({}^t e_{3i}^k - {}^0 e_{3i}^k) \right] \end{aligned} \quad (5.35)$$

And from eqns. (5.6) and (5.27) the incremental displacements from time t to time $t + \Delta t$ can be written as

$$u_i = {}^{t+\Delta t} x_i - {}^t x_i$$



interpolation functions for
this transition element:

$$\varphi_k = \frac{1}{8}(1 + \xi\xi_k)(1 + \eta\eta_k)(1 + \zeta\zeta_k)(\xi\xi_k + \eta\eta_k - 1) \quad k = 1, 2, 3, 4$$

$$\varphi_k = \frac{1}{4}(1 - \eta^2)(1 + \xi\xi_k)(1 + \zeta\zeta_k) \quad k = 5, 6$$

$$\varphi_k = \frac{1}{4}(1 + \xi\xi_k)(1 + \eta\eta_k)(\xi\xi_k + \eta\eta_k - 1) \quad k = 7, 8$$

$$\varphi_k = \frac{1}{2}(1 - \xi^2)(1 + \eta\eta_k) \quad k = 9, 11$$

$$\varphi_k = \frac{1}{2}(1 - \eta^2)(1 + \xi\xi_k) \quad k = 10$$

Figure 5.6 (a) 16-node three-dimensional solid element; (b) 8-node curved shell element; (c) transition element

$$= \sum_{k=1}^m \varphi_k(\xi, \eta, \zeta) u_i^k + \sum_{k=m+1}^n \bar{\varphi}_k(\xi, \eta) [u_i^k + \frac{\zeta}{2} h_k (\theta_1^k {}^t e_{1i}^k - \theta_2^k {}^t e_{2i}^k)] \quad i = 1, 2, 3 \quad (5.36)$$

where n is the total number of nodes of each transition element, m is the number of nodes which are common with the adjacent 3-D continuum element and $n-m$ is the number of the other mid-surface nodes. From eqn. (5.36) the incremental displacement interpolation matrix ${}^t[H]$ for this transition element can be formulated as follows:

$${}^t[H_k] = \begin{bmatrix} \varphi_k & 0 & 0 \\ 0 & \varphi_k & 0 \\ 0 & 0 & \varphi_k \end{bmatrix} \quad \text{if } 1 \leq k \leq m$$

$${}^t[H_k] = \begin{bmatrix} \bar{\varphi}_k & 0 & 0 & \frac{\bar{\varphi}_k \zeta}{2} h_k {}^t e_{11}^k & -\frac{\bar{\varphi}_k \zeta}{2} h_k {}^t e_{21}^k \\ 0 & \bar{\varphi}_k & 0 & \frac{\bar{\varphi}_k \zeta}{2} h_k {}^t e_{12}^k & -\frac{\bar{\varphi}_k \zeta}{2} h_k {}^t e_{22}^k \\ 0 & 0 & \bar{\varphi}_k & \frac{\bar{\varphi}_k \zeta}{2} h_k {}^t e_{13}^k & -\frac{\bar{\varphi}_k \zeta}{2} h_k {}^t e_{23}^k \end{bmatrix} \quad \text{if } m+1 \leq k \leq n \quad (5.37)$$

Applying eqns. (3.4) to (3.7) again and ${}^t[H]$ in eqn. (5.37) for solid-shell transition element, the element matrices and internal nodal force vector can be obtained by following the similar procedures as used in Sections 5.2 to 5.4. In Section 5.2 for shell ele-

ment the normal stress and strain in the thickness direction are neglected to comply with shell assumptions while in Section 5.4 for three-dimensional continuum element none of the stress and strain components are neglected. Since the transition element is located between the shell and solid elements in the practical application, the validity of including this transverse normal stress or neglecting it is worth of investigation. Here the transition elements with and without the transverse normal stress (throughout the whole element) are developed. For each case the corresponding stresses, strains and constitutive relations should be employed. It means if neglect the transverse normal strain and stress components,

$$\{ {}_0e' \}_{5 \times 1} = \{ {}_0e'_{11} \quad {}_0e'_{22} \quad 2{}_0e'_{12} \quad 2{}_0e'_{13} \quad 2{}_0e'_{23} \}^T$$

and

$$\{ {}_0S' \}_{5 \times 1} = \{ {}_0S'_{11} \quad {}_0S'_{22} \quad {}_0S'_{12} \quad {}_0S'_{13} \quad {}_0S'_{23} \}^T$$

and use the constitutive relations in eqn. (5.13). For the other case

$$\{ {}_0e' \}_{6 \times 1} = \{ {}_0e'_{11} \quad {}_0e'_{22} \quad {}_0e'_{33} \quad 2{}_0e'_{23} \quad 2{}_0e'_{13} \quad 2{}_0e'_{12} \}^T,$$

$$\{ {}_0S' \}_{6 \times 1} = \{ {}_0S'_{11} \quad {}_0S'_{22} \quad {}_0S'_{33} \quad {}_0S'_{23} \quad {}_0S'_{13} \quad {}_0S'_{12} \}^T$$

and use the constitutive relations in eqn. (5.31).

The transformation matrix $[\theta]$ is also evaluated using eqn. (5.30). In the evaluations of the element matrices and internal nodal force vector we cannot use explicit integration through the thickness direction since some of the interpolation functions are functions of ξ , η and ζ . Therefore the Gaussian quadrature is used in all ξ , η and ζ directions to perform the numerical integration. For laminated composite structures each transition element can contain more than one layer. Hence the expressions for ${}_0[K_L]$, ${}_0[K_{NL}]$ and ${}_0\{F\}$ become

$${}_0[K_L] = \int_{0_V} {}_0[B'_L]^T {}_0[C'] {}_0[B'_L] {}_0dV$$

$$\begin{aligned}
&= \sum_{k=1}^p \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 {}_0[B'_L]^T {}_0[C']_{(k)} {}_0[B'_L] \frac{h_{(k)}}{h} |{}^0J| d\xi d\eta d\zeta_k \\
&= \sum_{k=1}^p \left(\sum_{r=1}^{NGP} \sum_{s=1}^{NGP} \sum_{t=1}^{NGP} {}_0[B'_L]^T {}_0[C']_{(k)} {}_0[B'_L] W_{\xi_r} W_{\eta_s} W_{\zeta_t} |{}^0J| \frac{h_{(k)}}{h} \right) \quad (5.38)
\end{aligned}$$

where

ζ_k varies from -1 to 1 in that layer and

$$\zeta = -1 + \frac{1}{h} \left[-h_{(k)}(1 - \zeta_k) + 2 \sum_{j=1}^k h_{(j)} \right]$$

$$d\zeta = \frac{h_{(k)}}{h} d\zeta_k$$

p= Total number of layers

$h_{(k)}$ = The thickness of the k-th layer

h= The total thickness of the composite

Similarly

$$\begin{aligned}
{}_0[K_{NL}] &= \int_{0_V} {}_0[B'_{NL}]^T {}_0[S'] {}_0[B'_{NL}] {}^0dV \\
&= \sum_{k=1}^p \left(\sum_{r=1}^{NGP} \sum_{s=1}^{NGP} \sum_{t=1}^{NGP} {}_0[B'_{NL}]^T {}_0[S']_{(k)} {}_0[B'_{NL}] W_{\xi_r} W_{\eta_s} W_{\zeta_t} |{}^0J| \frac{h_{(k)}}{h} \right) \quad (5.39)
\end{aligned}$$

and

$$\begin{aligned}
{}_0\{F\} &= \int_{0_V} {}_0[B'_L]^T {}_0\{S'\} {}^0dV \\
&= \sum_{k=1}^p \left(\sum_{r=1}^{NGP} \sum_{s=1}^{NGP} \sum_{t=1}^{NGP} {}_0[B'_L]^T {}_0\{S'\}_{(k)} W_{\xi_r} W_{\eta_s} W_{\zeta_t} |{}^0J| \frac{h_{(k)}}{h} \right) \quad (5.40)
\end{aligned}$$

For thin structure modelled with this transition element, the reduced integration scheme is applied in eqns. (5.38) to (5.40) to evaluate the stiffness coefficients associated with the transverse shear deformation to avoid the locking phenomenon.

The external load vector for transition element subjected to distributed surface loading, which is deformation-independent, can be obtained by

$${}^{t+\Delta t}\{R\} = \int_{0A} {}^t[H]^T \begin{bmatrix} {}^{t+\Delta t}P_1 \\ {}^{t+\Delta t}P_2 \\ {}^{t+\Delta t}P_3 \end{bmatrix} {}^0dA \quad (5.41)$$

$$= \sum_{r=1}^{NGP} \sum_{s=1}^{NGP} \begin{bmatrix} \varphi_k {}^{t+\Delta t}P_1 \\ \varphi_k {}^{t+\Delta t}P_2 \\ \varphi_k {}^{t+\Delta t}P_3 \\ \cdot \\ \cdot \\ \cdot \\ \bar{\varphi}_k {}^{t+\Delta t}P_1 \\ \bar{\varphi}_k {}^{t+\Delta t}P_2 \\ \bar{\varphi}_k {}^{t+\Delta t}P_3 \\ \frac{\zeta}{2} \bar{\varphi}_k h_k ({}^{t+\Delta t}P_1 {}^te_{11}^k + {}^{t+\Delta t}P_2 {}^te_{12}^k + {}^{t+\Delta t}P_3 {}^te_{13}^k) \\ - \frac{\zeta}{2} \bar{\varphi}_k h_k ({}^{t+\Delta t}P_1 {}^te_{21}^k + {}^{t+\Delta t}P_2 {}^te_{22}^k + {}^{t+\Delta t}P_3 {}^te_{23}^k) \end{bmatrix} W_{\xi_r} W_{\eta_s} |{}^0J|_{(\xi_r, \eta_s)} 2/h$$

where h is the thickness of the transition element at each Gaussian point and the order of $r^{+\Delta r}\{R\}$ is $[3m + 5(n-m)]$ by 1.

CHAPTER 6

SAMPLE ANALYSIS

6.1 Introduction

A number of representative problems were analyzed using the elements and solution procedures developed in the present study. Some of the problems have the analytic solutions or have been analyzed in the literature by using different finite-element models. Comparisons of the present results with the solutions available in the literature can show the accuracy and applicability of the formulations and elements in the present study. For those problems which have not been analyzed yet the present results can be used as references for future investigations.

Static as well as transient analyses were performed for each kind of element developed here. Newton-Raphson method was used for nonlinear transient analysis while modified Riks/Wempner method was employed for post-buckling analysis. For most of the problems the reduced/selective integration scheme was used. Although many problems were analyzed during this investigation, the results of only a few sample problems are presented in this Chapter.

6.2 Static Analysis

6.2.1 Plate and Shell Structures

1. A cylindrical shell roof under the action of self-weight

This problem, as shown in Fig. 6.1(a), has been used frequently for assesment of shell finite element performance [55]. In Reference [55], the authors showed that with the all-round reduction of integration order the parabolic and cubic shell elements show a more

rapid convergence and better accuracy than with the reduced integration order applied to transverse shear terms only. The results obtained in the present study with 9-node parabolic shell element and all-round reduction of integration order are shown in Fig. 6.1(b) and 6.1(c). The exact solution is solved by Scordelis [56]. Even with one element the results have the tendency of exact solutions and for further subdivision the results are close to the exact ones.

2. Simply-supported spherical shell subjected to a point load

The isotropic shell shown in Fig. 6.2(a) was analyzed for its large displacement response with four 9-node elements and sixteen 4-node elements, modelling one-quarter of the shell. Fig. 6.2(b) shows the responses calculated, including the post-buckling range, with the modified Riks-Wempner method which automatically determines the load increments. Fig. 6.2(b) also includes the results by Bathe [57].

3. Shallow cylindrical shell with a center point load

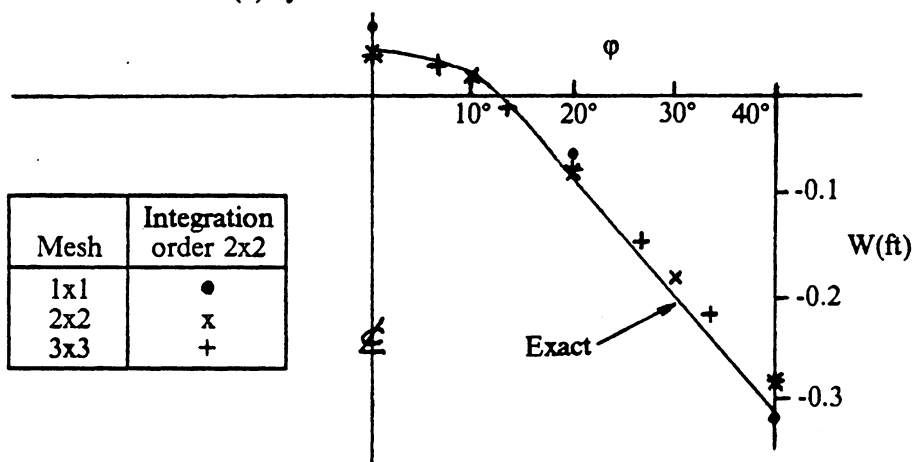
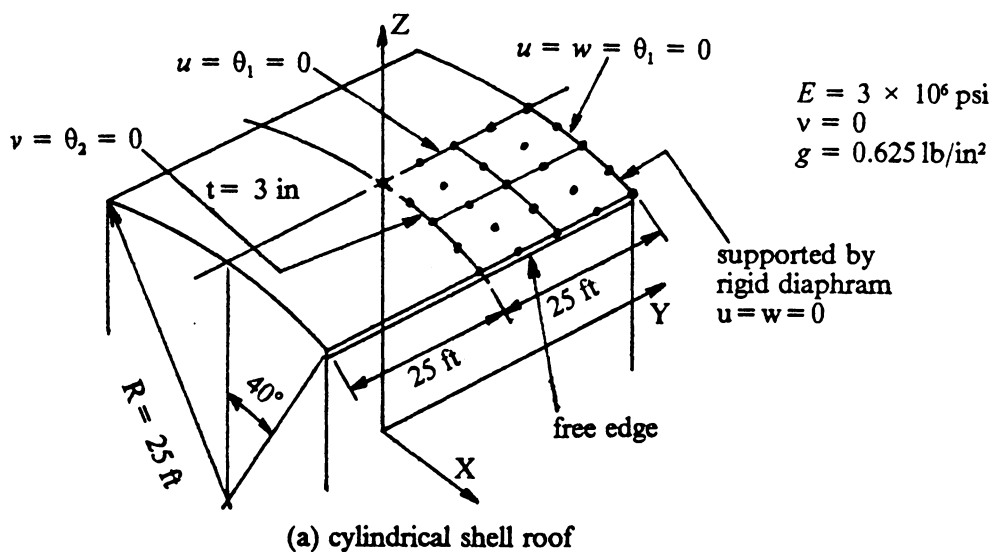
The isotropic shallow cylindrical shell under one concentrated load described in Fig. 6.3 is hinged at the longitudinal edges and free at the curved boundaries. The structure exhibits snap-through as well as snap-back phenomena. One-quarter of the shell was analyzed with four 9-node degenerated shell elements. The solution obtained by Crisfield [45] is also shown in Fig. 6.3 to be compared with the present result.

4. 2-layer cross-ply(0/90) and angle-ply(-45/45) simply-supported plates with uniform loading

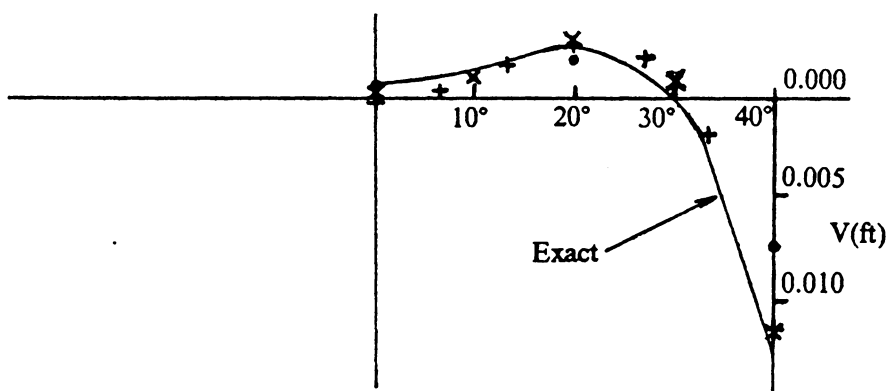
The material properties of the lamina are:

$$E_1/E_2 = 25, E_2 = 7.031 \times 10^5 \text{ N/cm}^2, G_{12}/E_2 = 0.5, G_{13} = G_{12}, G_{23}/E_2 = 0.2, \nu_{12} = 0.25$$

The laminated plates in this problem do not buckle under the lateral uniform loading. The modified Riks/Wempner method is adopted for nonlinear analyses of these two plates. This method eliminates the difficulty to prescribe the load levels without an approximate knowledge of the load carrying capacity of the structure. Laminated



(b) Vertical displacement on mid-section



(c) Axial displacement at support

Figure 6.1 A cylindrical shell roof under self-weight

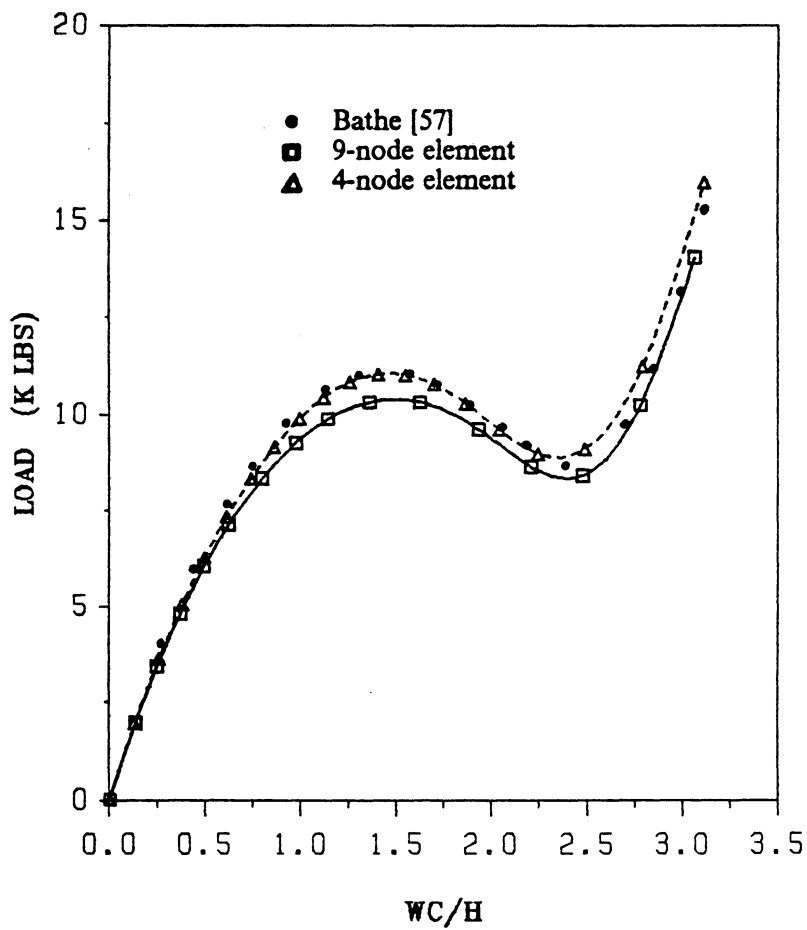
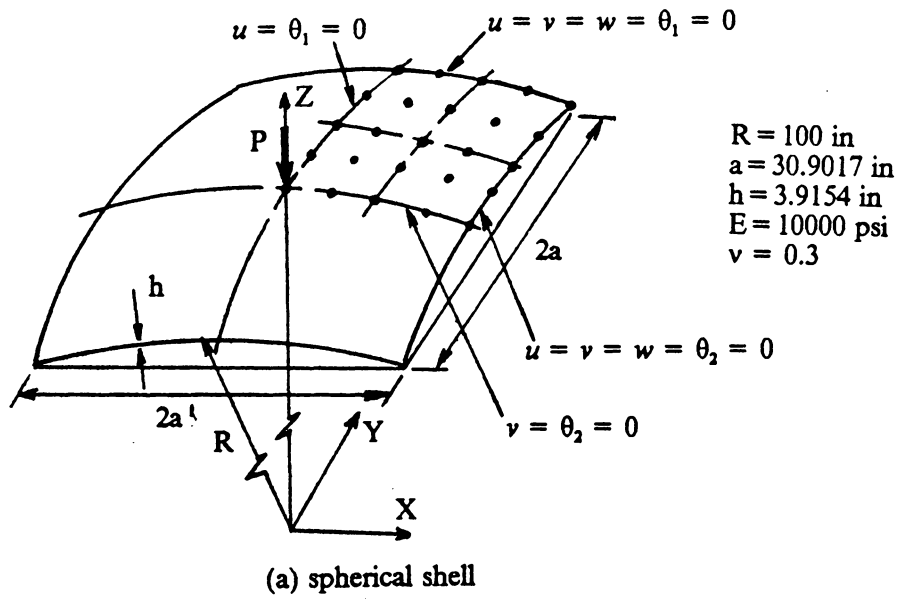


Figure 6.2 Geometric nonlinear response of a spherical shell

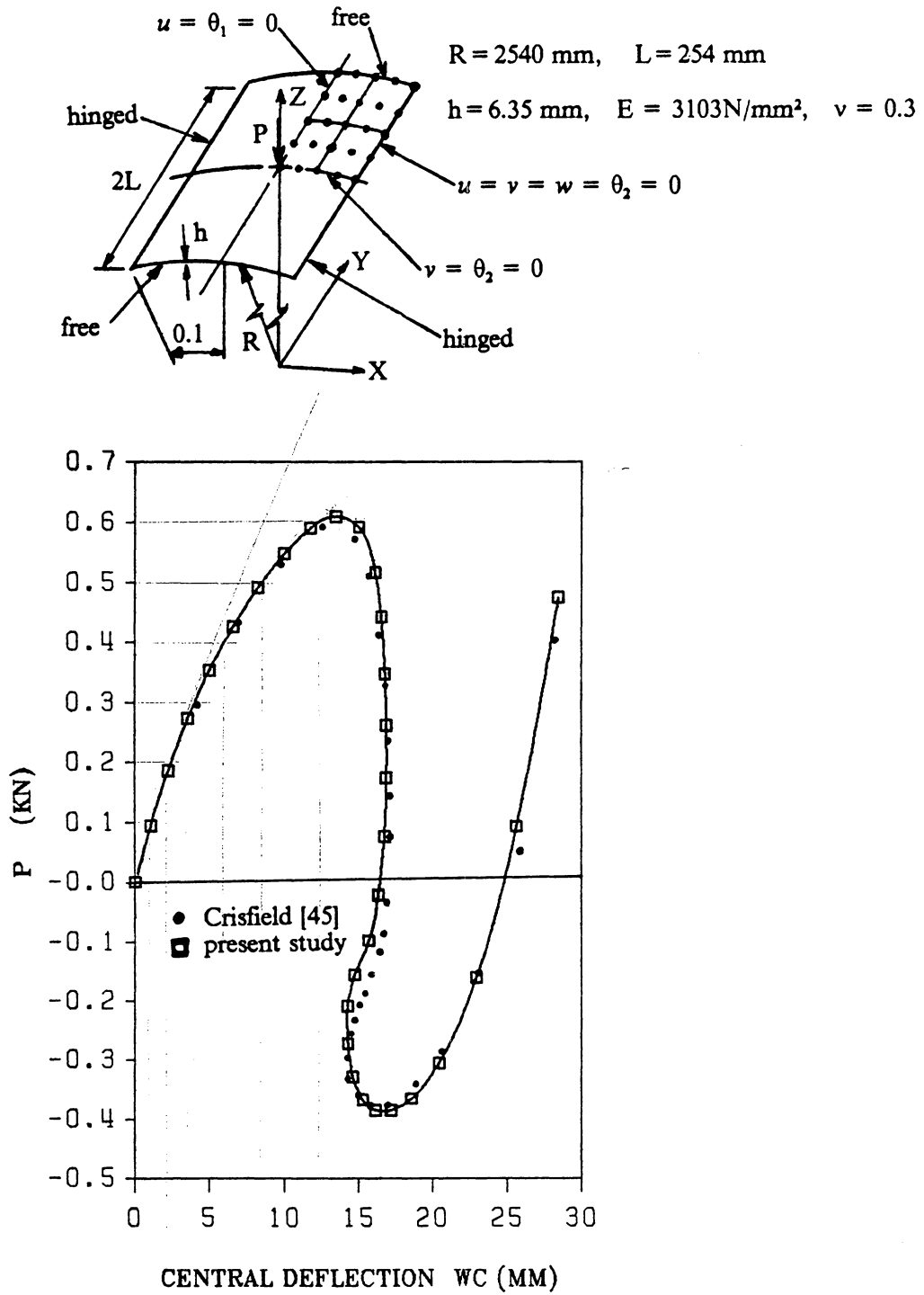


Figure 6.3 Shallow cylindrical shell with a center point load

anisotropic plates exhibit, in general, bending-stretching coupling (i.e. coupling between the inplane displacements and the bending displacements). If bending-stretching coupling exists, a quarter-plate model usually cannot be used to analyze plates with biaxial symmetry w.r.t. geometry and loading (see [58,59] for details). Three types of simply-supported boundary conditions (BC1, BC2 and BC3) for a quarter-plate model are defined in [58]. For the cross-ply(0/90) rectangular plate quarter-plate model of BC1 or BC3 gives the same results as that of the full-plate model for corresponding edge conditions. For the 2-layer angle-ply(-45/45) rectangular plate the boundary and symmetry conditions of BC2 are correct if quarter-plate model is used. The effect of edge boundary conditions and symmetry conditions on the nonlinear responses of laminated rectangular plates is obvious as shown in Fig. 6.4.

5. Orthotropic simply-supported(BC1) plate under uniform loading

Fig. 6.5 shows the plate and material properties analyzed. A quarter of the plate with BC1 boundary and symmetry conditions is modelled with four 9-node shell elements. The Present results shown in Fig. 6.5 are in good agreement with the experimental data in [60].

6. 4-layer(0/90/90/0) clamped plate under uniform loading

Fig. 6.6 shows the plate analyzed and its material properties. A quarter of the plate is modelled with four 9-node elements. The present results, the experimental data in [60] and the finite-element solutions of [61] are shown in Fig. 6.6. The present results are in good agreement with that obtained with mixed finite element model in [61].

7. 2-layer cross-ply(0/90) and angle-ply(-45/45) simply-supported spherical shell with uniform loading

The shell described in Fig. 6.7(a) was analyzed for its large displacement response with four 9-node degenerated shell elements, modelling one-quarter of the shell. Similar to the 2-layer laminated plates discussed above, the effect of edge boundary conditions

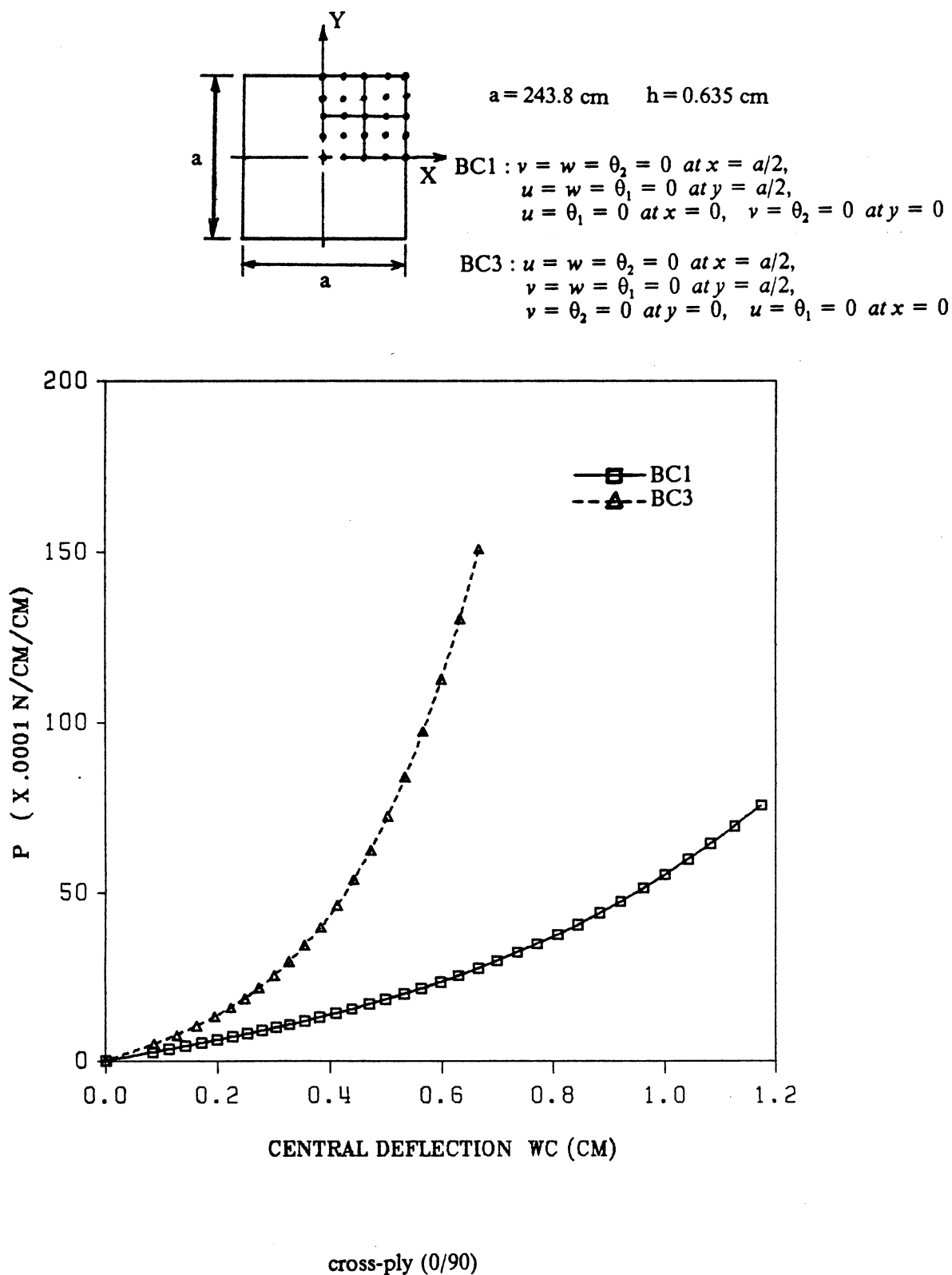
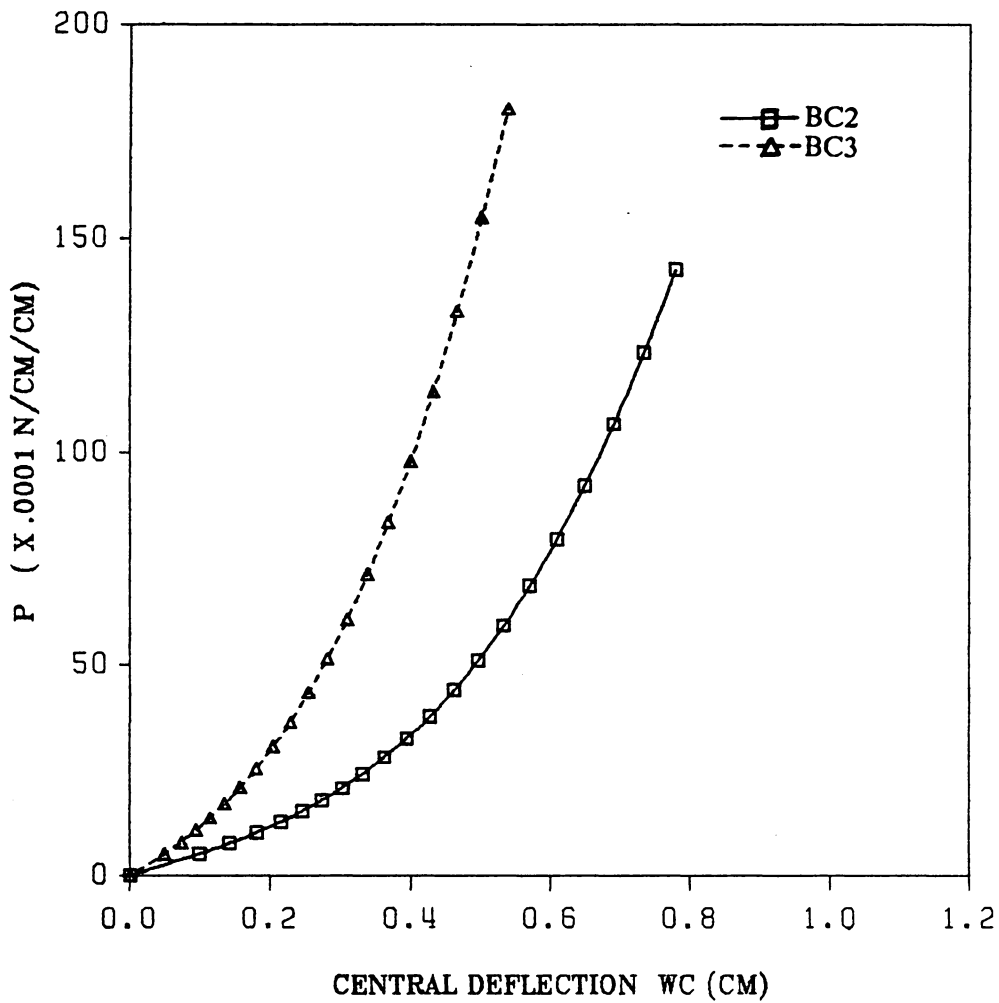


Figure 6.4 Bending of 2-layer simply-supported plates under uniform loading

$$\begin{aligned}
 \text{BC2 : } & u = w = \theta_2 = 0 \text{ at } x = a/2, \\
 & v = w = \theta_1 = 0 \text{ at } y = a/2, \\
 & u = \theta_2 = 0 \text{ at } y = 0, \quad v = \theta_1 = 0 \text{ at } x = 0
 \end{aligned}$$



angle-ply (-45/45)

Figure 6.4 Bending of 2-layer simply-supported plates under uniform loading

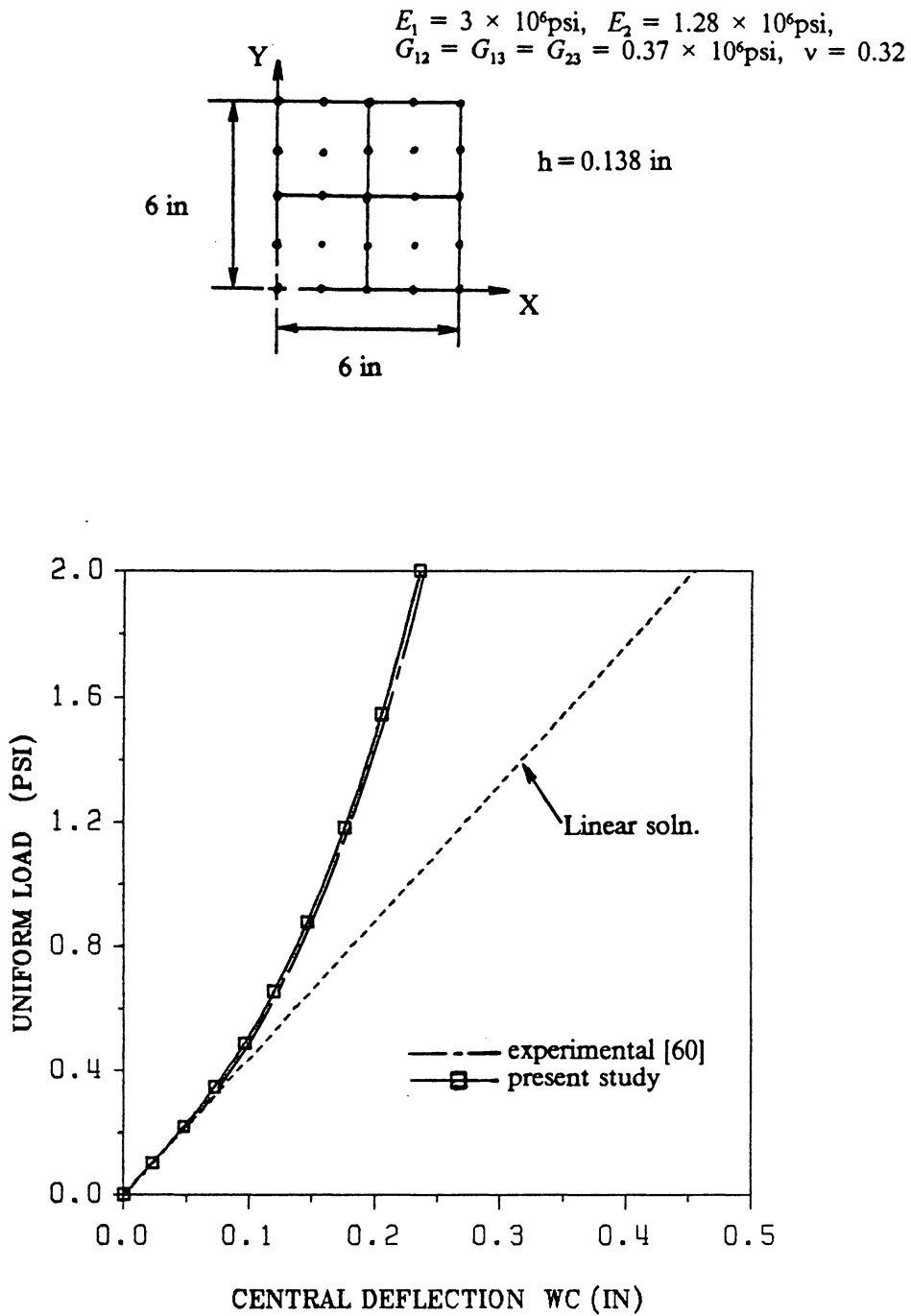


Figure 6.5 A simply-supported(BC1) orthotropic plate under uniform loading

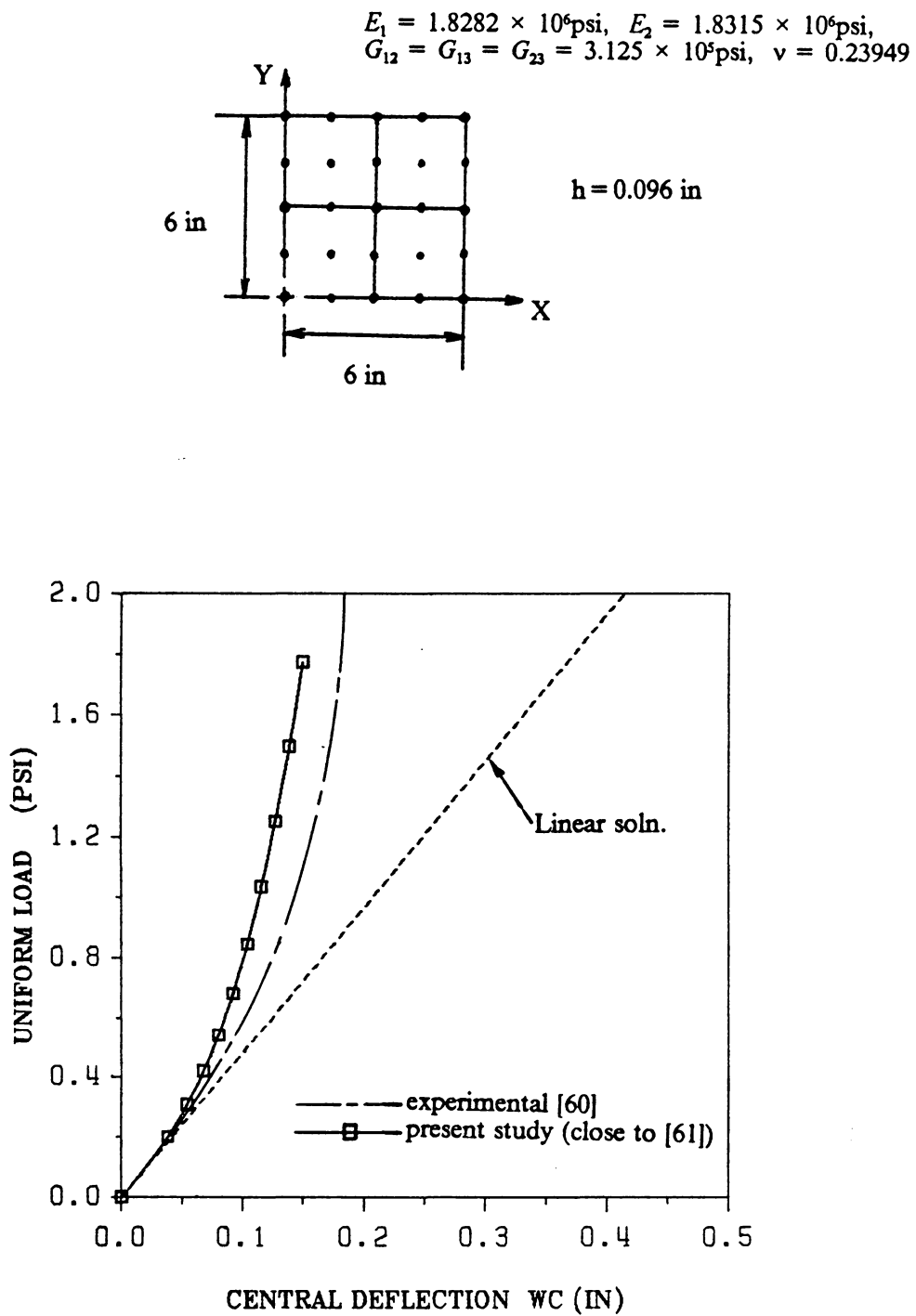


Figure 6.6 4-layer(0/90/90/0) clamped plate under uniform loading

and symmetry conditions on the nonlinear response is investigated. The effect is very prominent as shown in Fig 6.7. The same shell structure of 2-layer cross-ply(0/90) has been analyzed in [62], in which the Newton-Raphson method was used and only pre-buckling response was obtained.

8. 9-layer cross-ply(0/90/0/90....) simply-supported spherical shell

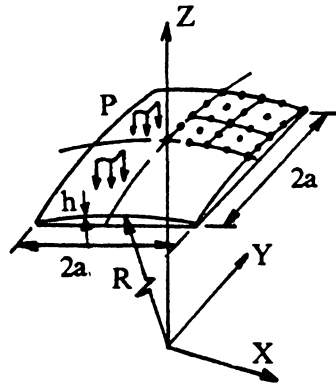
A laminated spherical shell of nine-layers of graphite-epoxy material ($E_1/E_2 = 40$, $E_2 = 10^6$ psi, $G_{12}/E_2 = 0.6$, $G_{13} = G_{12}$, $G_{23}/E_2 = 0.5$, $\nu_{12} = 0.25$) with the same geometry as that in Fig. 6.7(a) is analyzed for large-deflection bending. Only one-quarter of the shell was modelled with four 9-node degenerated shell elements, and BC1 simply-supported boundary conditions and uniform loading were used here. The load-deflection curve obtained with the modified Riks/Wempner method is compared with that obtained by Noor[5] in Fig. 6.8. The present results are in very good agreement with Noor's results. Note that the laminated shell exhibits softening first and then stiffening and does not have a limit point. This situation is similar to that in Fig. 6.7(b) with the same boundary conditions.

6.2.2 Beam Structures

1. Linear displacement of a deep straight beam

This problem is considered (see Fig. 6.9) to illustrate the combined effects of bending and transverse shear. The tip deflections obtained in the present study are ($E = 3 \times 10^6$ psi, $\nu = 0$):

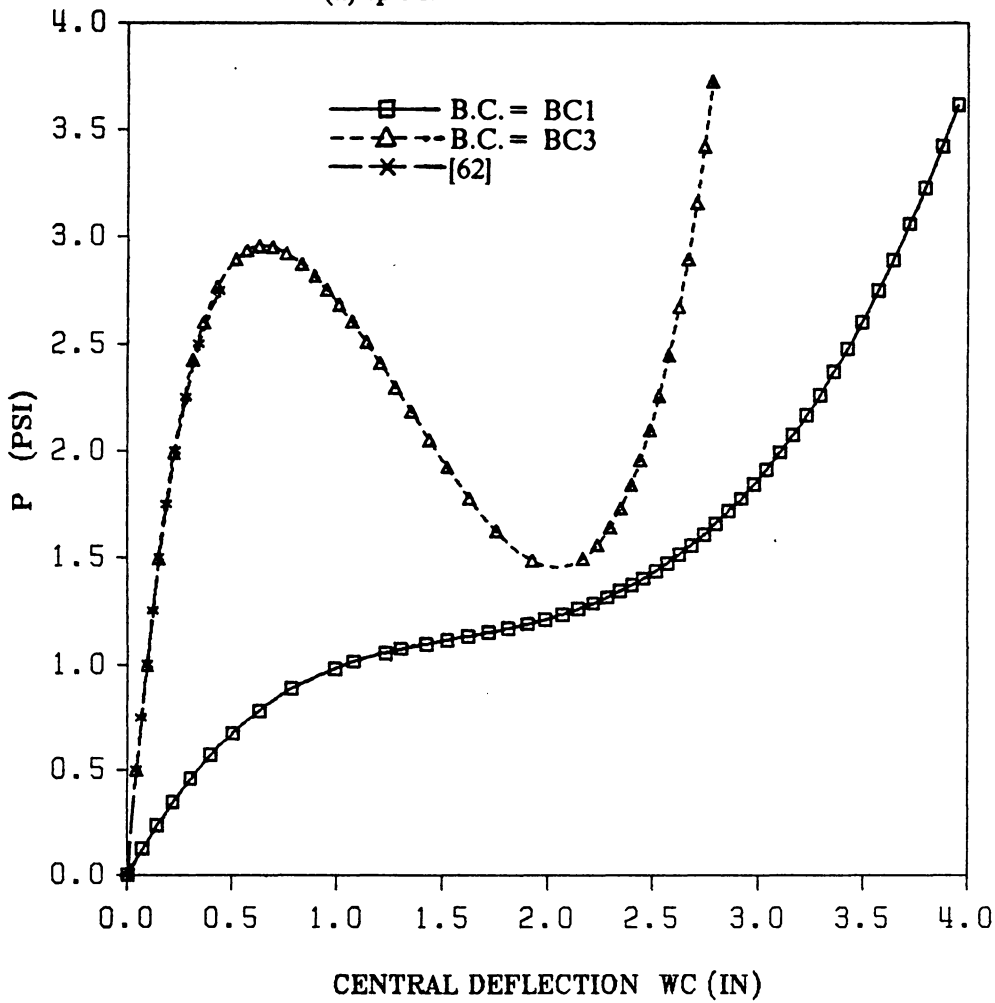
Mesh	Tip deflection (in)
One 3-node beam element	-0.42667 E-05
Two 3-node beam element	-0.42667 E-05
One 9-node shell element	-0.42667 E-05
Classical beam theory	-0.26667 E-05



$R = 1000$ in
 $a = 50$ in
 $h = 1$ in

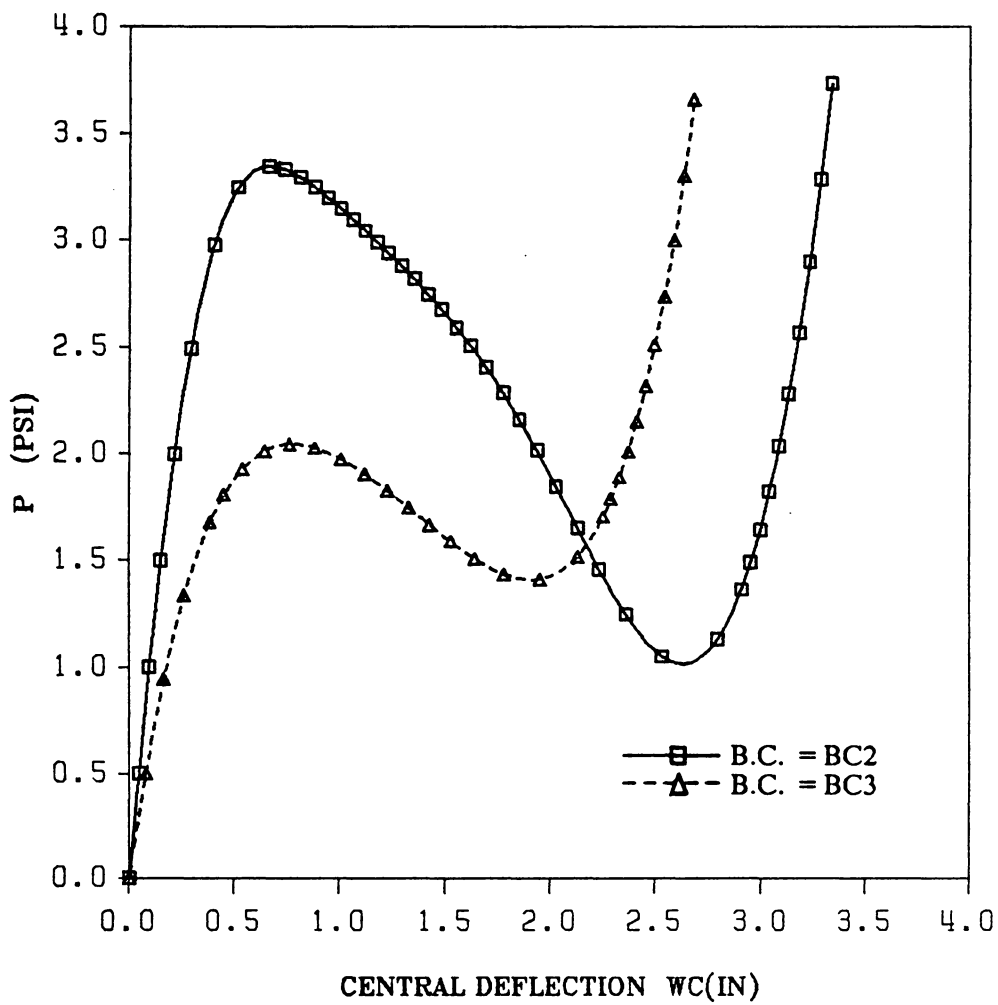
$E_1/E_2 = 25$, $E_2 = 10^6$ psi, $G_{12}/E_2 = 0.5$,
 $G_{12} = G_{13}$, $G_{23}/E_2 = 0.2$, $\nu_{12} = 0.25$

(a) spherical shell



(b) cross-ply(0/90)

Figure 6.7 Bending of 2-layer simply-supported shells under uniform loading



(c) angle-ply(-45/45)

Figure 6.7 Bending of 2-layer simply-supported shells under uniform loading

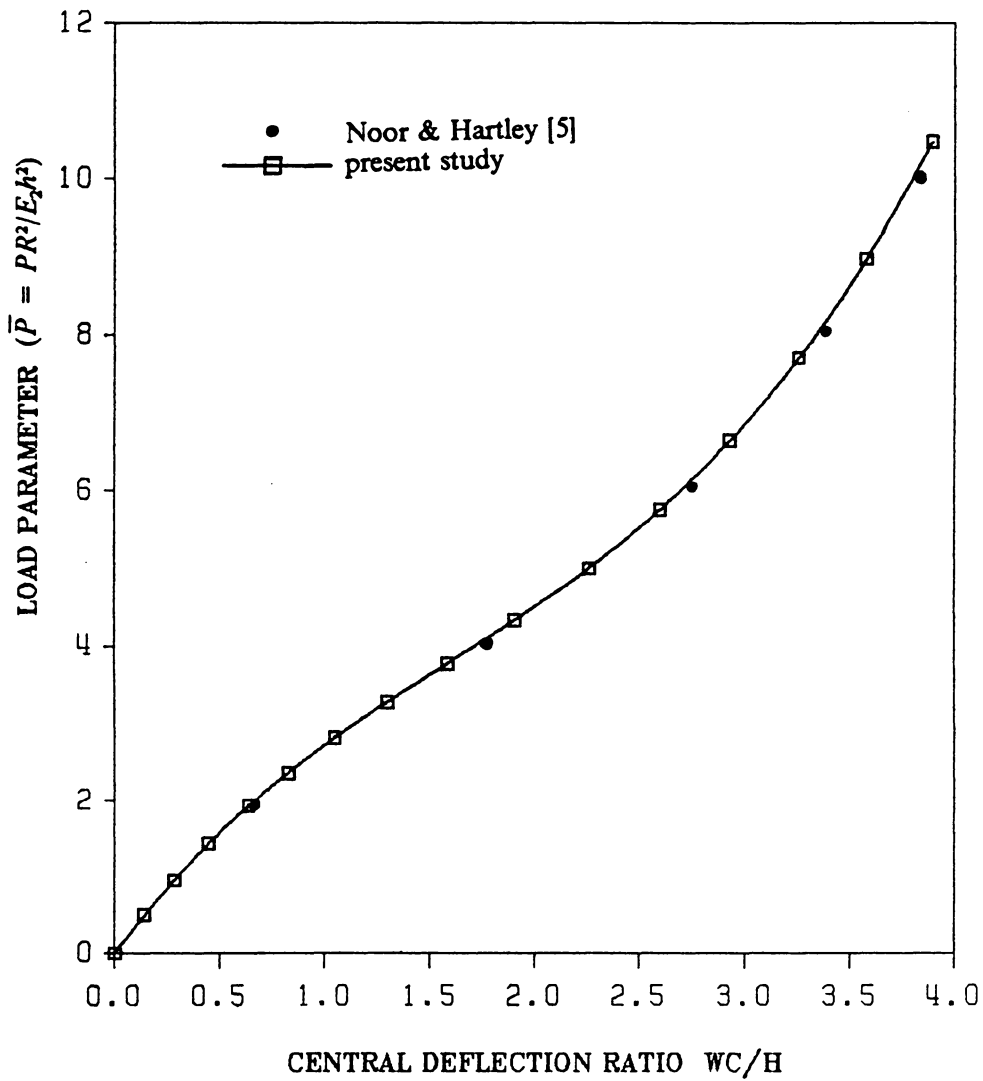


Figure 6.8 Bending of 9-layer cross-ply simply-supported shell under uniform loading

The classical beam theory neglects the transverse shear deformation. Therefore the deflection it predicts is smaller than the results obtained in the present study.

2. Linear displacement of a shallow straight beam

The same beam as that in the last problem except that the depth is $h = 1$ in and the end load is 3 lbs. The results obtained in the present study are:

Mesh	Tip deflection (in)
Two 3-node beam element	-0.4024 E-02
One 9-node shell element	-0.4024 E-02
Classical beam theory	-0.4000 E-02

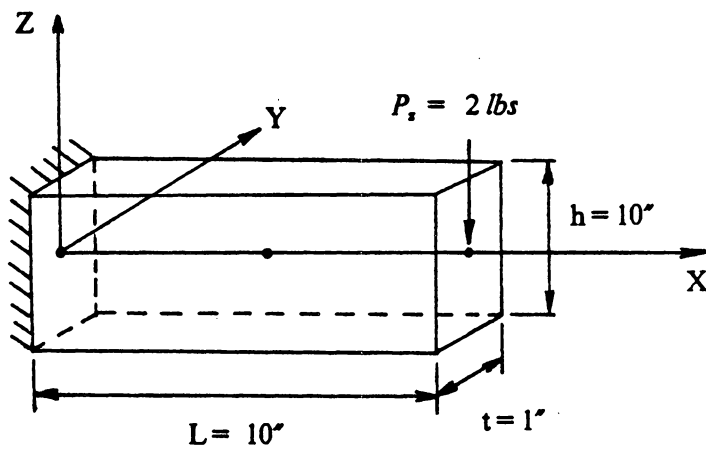
This beam is shallower than that in the last problem. Hence the transverse shear effect is not so obvious and the results in the present study are quite close to that obtained by the classical beam theory.

3. Linear analysis of a straight beam with multi-loads

This cantilever beam (length = 25 in, depth = 1 in and width = 5 in) is subjected to three concentrated forces and one torsion, where $P_z = -0.1$ lb, $P_y = 0.1$ lb, $P_x = 0.1$ lb and $M_x = -0.1$ lb-in at the free end. The Young's modulus $E = 10^7$ psi, Poisson ratio $\nu = 0.3$. The results obtained in the present study are:

Deflections at loaded end (in)	One 3-node element	Two 3-node elements	Three 3-node elements	Three 2-node elements [19]
u	0.4550 E-07	0.4550 E-07	0.4550 E-07	0.4550 E-07
v	0.3956 E-05	0.4594 E-05	0.4659 E-05	0.4549 E-05
w	-0.1139 E-03	-0.1139 E-03	-0.1139 E-03	-0.1137 E-03
θ_x	0.6825 E-05	0.6825 E-05	0.6825 E-05	0.6824 E-05
θ_y	0.7145 E-07	0.7145 E-07	0.7145 E-07	0.5773 E-07
θ_z	0.2730 E-06	0.2730 E-06	0.2730 E-06	0.2730 E-06

From the above table, u , w , θ_x , θ_y , and θ_z have converged at one element level. Also the present results are quite close to that in Ref. [19]. In [19] the authors used the classical beam theory to develop a 2-node curved beam element with 8 d.o.f. at each node.



classical beam theory:

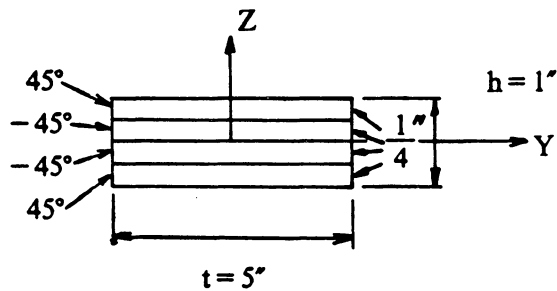
$$\delta_{T_p} = -\frac{PL^3}{3EI} = -0.26667 \times 10^{-5} \text{ in}$$

Figure 6.9 Deep cantilever beam under end load

4. Linear analysis of a straight composite beam with multi-loads

The same cantilever beam and loads used in the last problem except that it is laminated with 4 layers with the following material properties and lamination scheme used:

$$E_L = 3 \times 10^7 \text{ psi}, \quad E_T = 3 \times 10^6 \text{ psi}, \quad G_{LT} = 10^6 \text{ psi}, \quad \nu_{LT} = 0.3$$



The results obtained in the present study are:

Deflections at loaded end (in)	One 3-node element	Two 3-node elements	Three 3-node elements	Three 2-node elements [19]
u	0.5113 E-07	0.5113 E-07	0.5113 E-07	0.5113 E-07
v	0.4153 E-05	0.5016 E-05	0.5133 E-05	0.5112 E-05
w	-0.1430 E-03	-0.1430 E-03	-0.1430 E-03	-0.1294 E-03
θ_x	0.8576 E-05	0.8576 E-05	0.8576 E-05	0.7777 E-05
θ_y	0.1735 E-05	0.1735 E-05	0.1735 E-05	0.2071 E-06
θ_z	0.3068 E-06	0.3068 E-06	0.3068 E-06	0.3067 E-06

v , w , θ_x and θ_y in the present study are different from that in Ref. [19]. Therefore we can see that the effect of transverse shear deformation is more significant for composite material than for isotropic material.

5. A curved isotropic beam

The cantilever curved beam (an arc of a circle) shown in Fig. 6.10 is subjected to an end load. The Young's modulus $E = 10^6$ psi, Poisson ratio $\nu = 0$. The effect on convergence rate of the integration order of parabolic element developed here is studied for this structure. The transverse deflection (in inches) at loaded end are:

NGP = 3, LGP = 2				
Two 3-node elements	Three 3-node elements	Four 3-node elements	Six 3-node elements	Eight 3-node elements
-0.27849	-0.40151	-0.45716	-0.47455	-0.47616

NGP = 2, LGP = 2		
Two 3-node elements	Four 3-node elements	Eight 3-node elements
-0.45770	-0.47399	-0.47635

The corresponding analytical solution from Timoshenko [63] is 0.4655 in. From the present results we see that the convergence is slow when reduced/selective integration (i.e. NGP = 3, LGP = 2) is used, and the use of reduced integration for all terms shows a faster convergence.

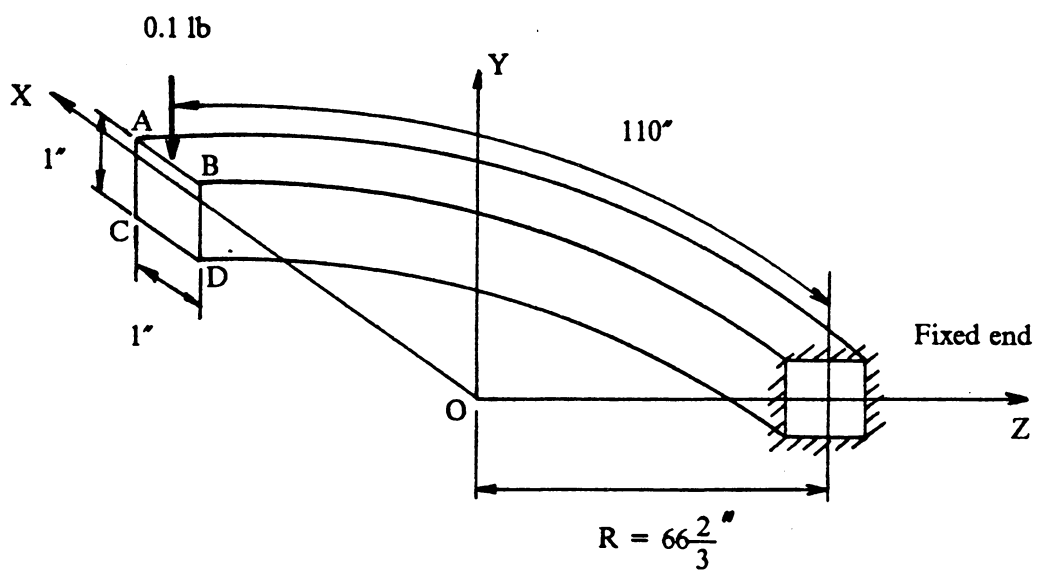
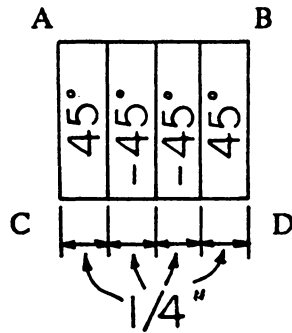


Figure 6.10 Circularly curved isotropic beam with end load

6. Laminated curved beam

The graphite fiber reinforced epoxy laminated curved beam has the same geometry as that analyzed in the last problem, and the lay-up details are shown below:

$$E_L = 3 \times 10^7 \text{ psi}, \quad E_T = 3 \times 10^6 \text{ psi}, \quad G_{LT} = 10^6 \text{ psi}, \quad \nu_{LT} = 0.3$$



The transverse deflection (in inches) at loaded end are:

NGP = 3, LGP = 2				
Two 3-node elements	Three 3-node elements	Four 3-node elements	Six 3-node elements	Eight 3-node elements
-0.038428	-0.060863	-0.066630	-0.069460	-0.071365

NGP = 2, LGP = 2		
Two 3-node elements	Four 3-node elements	Eight 3-node elements
-0.071205	-0.073812	-0.074086

The result given in Ref. [19] is 0.0604 in.

7. Large deflection of a clamped beam with a center point load

The beam is described in Fig. 6.11. Due to symmetry only one-half of the beam was modelled with two 3-node beam elements. Fig. 6.11 shows the results in the present study with reduced/selective integration scheme, which are quite close to the solutions by Noor and Peters [37] using the mixed model.

8. Large deflection of a cantilever beam under uniform loading

The Fig. 6.12 shows the geometry and material properties of the beam. The results obtained in the present study by modelling the beam with two 3-node beam elements are also shown in Fig. 6.12 and compared with the solutions by Reddy and Heyliger [64] using 8-node quadratic plane elasticity element.

9. Large deflection of clamped shallow circular arch subjected to a center point load

The geometry of this arch is described in Fig. 6.13. The material properties are:

$$E = 6.895 \times 10^6 \text{ N/cm}^2, \quad \nu = 0.25, \quad \rho = 2.5393 \times 10^{-4} \frac{\text{lb} \cdot \text{f} - \text{sec}^2}{\text{in}^4}$$

Due to symmetry of geometry and loading only one-half of the arch was modelled with four 3-node beam element. The results in the present study with reduced/selective integration scheme are compared with the solutions by Sharifi and Popov [34] using ten 2-node planar beam elements.

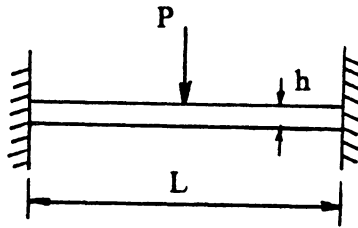
10. Symmetrical buckling analyses of isotropic shallow arches

Two shallow arches examples investigated by Sharifi and Popov [34] were analyzed by using the present nonlinear finite element model. The modified Riks/Wempner method is used as the solution procedure to obtain the pre-buckling and post-buckling responses. The geometries and loadings of both arches are shown in Figures 6.14 and 6.15 respectively. The material properties used are:

$$E = 10^7 \text{ psi}, \quad \nu = 0.25, \quad \rho = 2.5393 \times 10^{-4} \frac{\text{lb} \cdot \text{f} - \text{sec}^2}{\text{in}^4}$$

Arch 1 is subjected to a center point load while arch 2 is under a uniform loading. In each case, due to symmetry, only one-half of the arch was modelled with five 3-node beam elements. The results obtained in the present study and that in Ref. [34] are shown in Figures 6.14 and 6.15. In Ref. [34] updated Lagrangian description was adopted and a planar 2-node beam element which included one transverse shear strain was developed. Also in order to obtain post-buckling response the method of fictitious elastic springs was used there.

11. Symmetrical buckling analyses of 2-layer cross-ply(0/90) and



$L = 50.8 \text{ cm}$, $h = 0.3175 \text{ cm}$, unit width
 $E = 20.684 \times 10^6 \text{ N/cm}^2$,
 $G = 10.342 \times 10^6 \text{ N/cm}^2$, $\nu = 0$

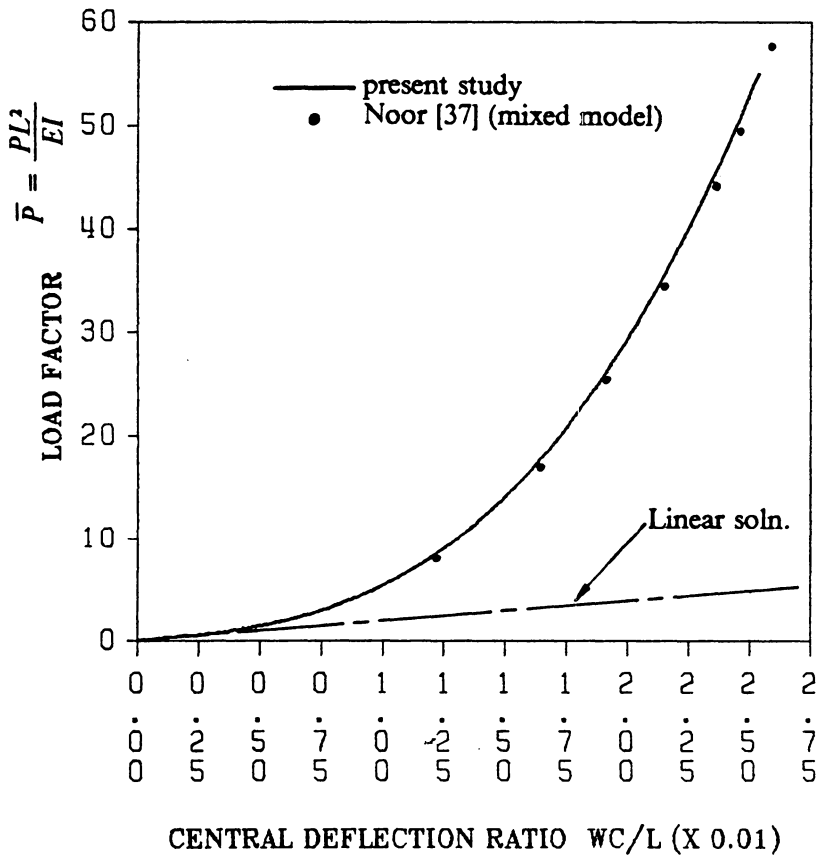


Figure 6.11 Large deflection of a clamped beam with a center point load

$E = 1.2 \times 10^4 \text{ psi}$
 $\nu = 0.2$
 The uniform load is conservative.

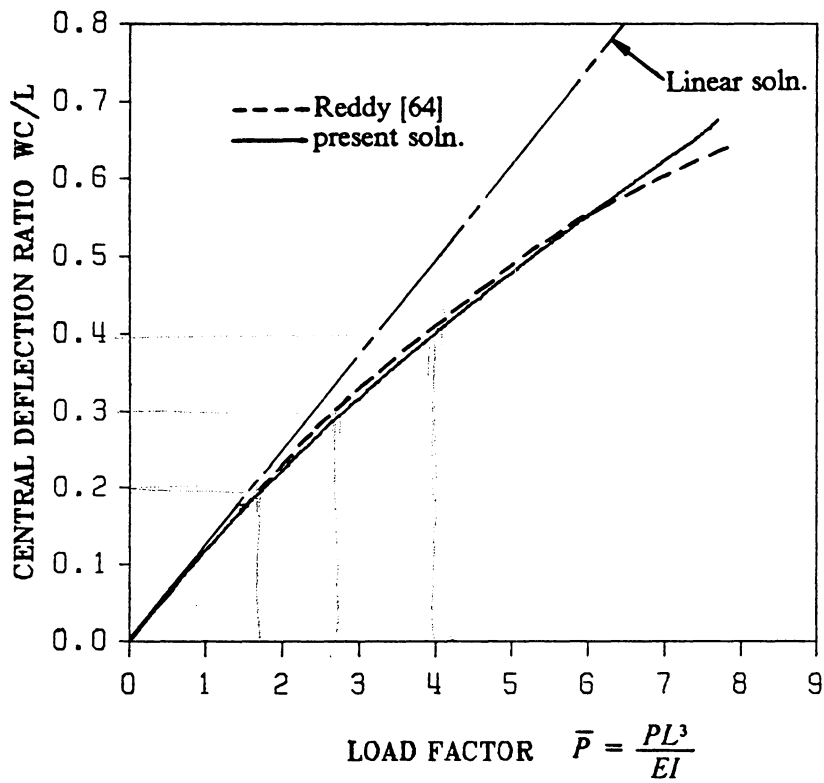
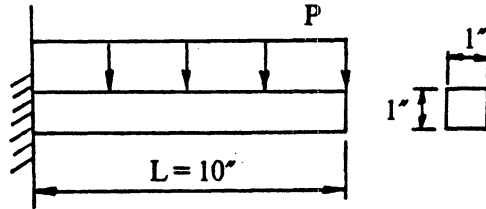


Figure 6.12 Large deflection of a cantilever beam under uniform load

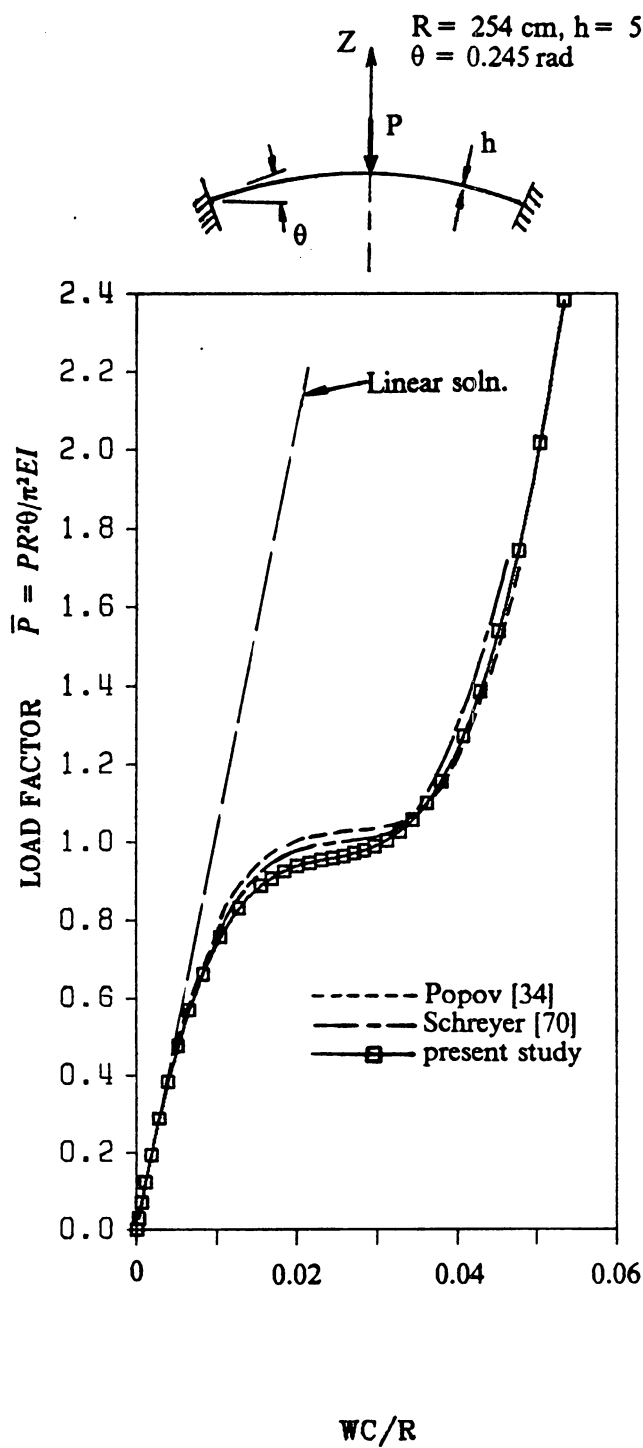


Figure 6.13 Large deflection of a clamped shallow arch under a center point load

angle-ply(-45/45) shallow arches

The geometries and loading of these two 2-layer composite arches are the same as that of arch 1 in the last problem. The material properties are:

$$E_1/E_2 = 25, \quad E_2 = 10^6 \text{ psi}, \quad G_{12}/E_2 = 0.5, \quad G_{13} = G_{12}, \\ G_{23}/E_2 = 0.2, \quad \nu_{12} = 0.25, \quad \rho = 2.3763 \times 10^{-5} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}^4}$$

Since the symmetrical deformation modes exist for both arches, only one-half of the arch was modelled by five 3-node beam elements for each beam. The pre- and post-buckling responses obtained in the present study with the modified Riks/Wempner method are shown in Fig. 6.16.

6.2.3 Stiffened Plates and Shells

1. Analysis of a stiffened plate with eccentric stiffeners

This problem shown in Fig. 6.17 illustrates the use of the degenerated shell element in conjunction with the degenerated beam element developed in the present study. The nodes are located on the midplane of the plate and the stiffness of stiffeners are added to that of the shell elements without any additional degrees-of-freedom. The reduced/selective integration scheme was used to evaluate the stiffness matrices of shell and beam elements. The present results are compared with the solution calculated using the classical beam theory. They are shown below (where NGP = the number of integration point with the bending part, LGP = the number of integration point with the transverse shear part):

Mesh	NGP	LGP	Transverse deflection (in)
One 9-node shell element and Two 3-node beam elements	3	2	-0.25447 E-03
Two 9-node shell elements and Four 3-node beam elements	3	2	-0.25533 E-03
Classical beam theory			-0.25195 E-03

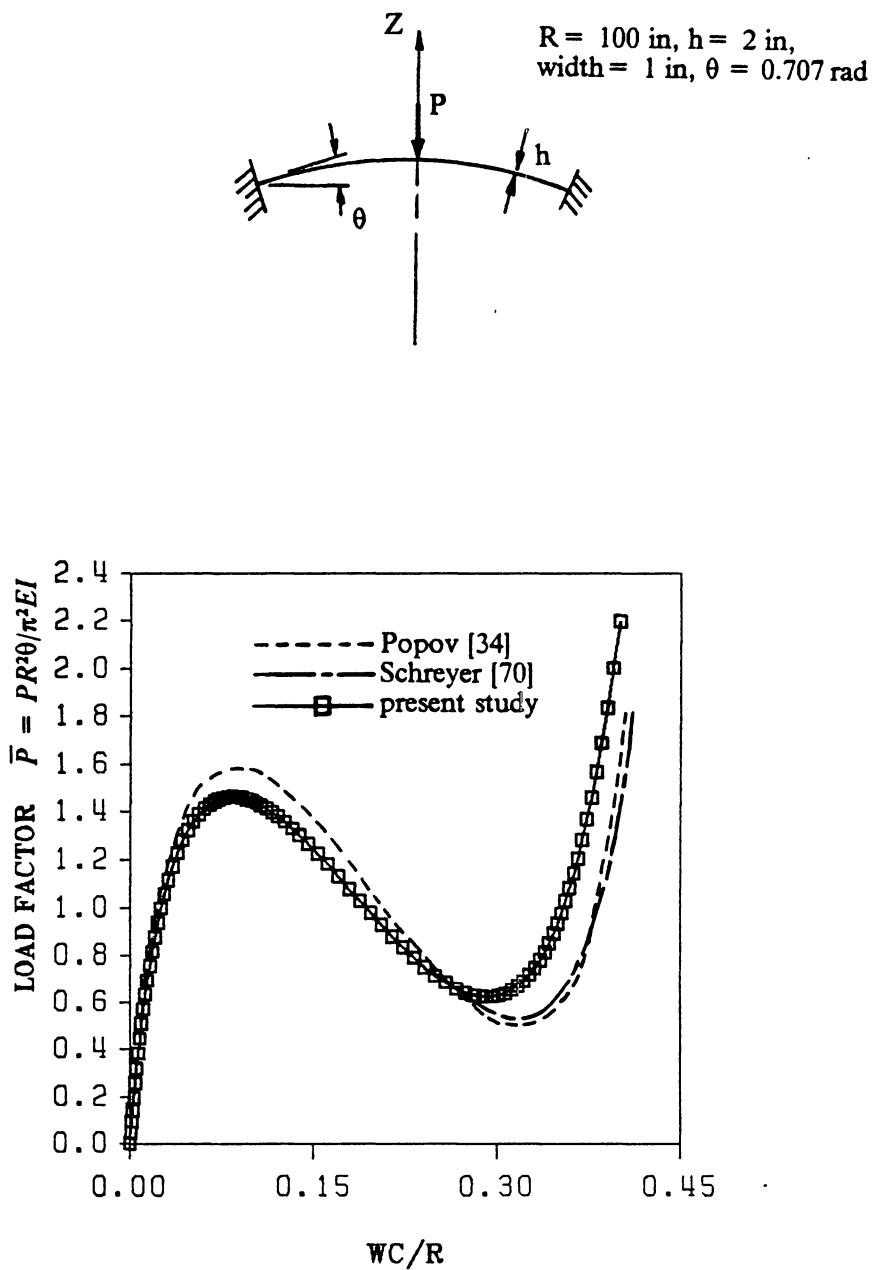


Figure 6.14 Symmetrical buckling of shallow arch (arch 1)

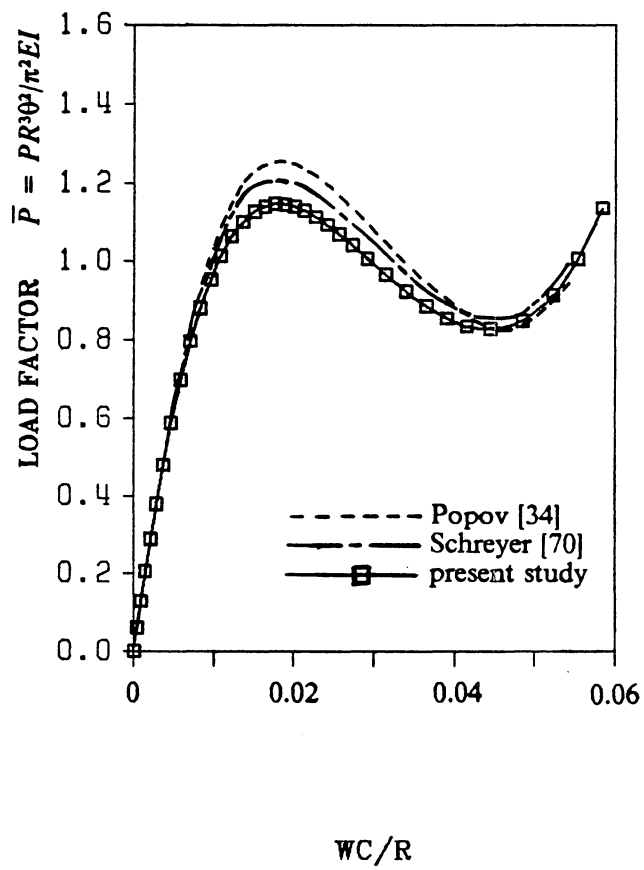
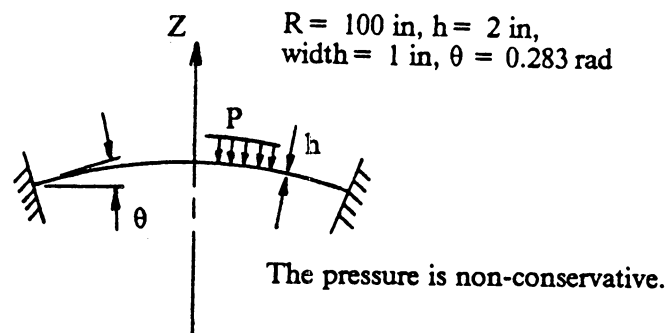


Figure 6.15 Symmetrical buckling of shallow arch (arch 2)

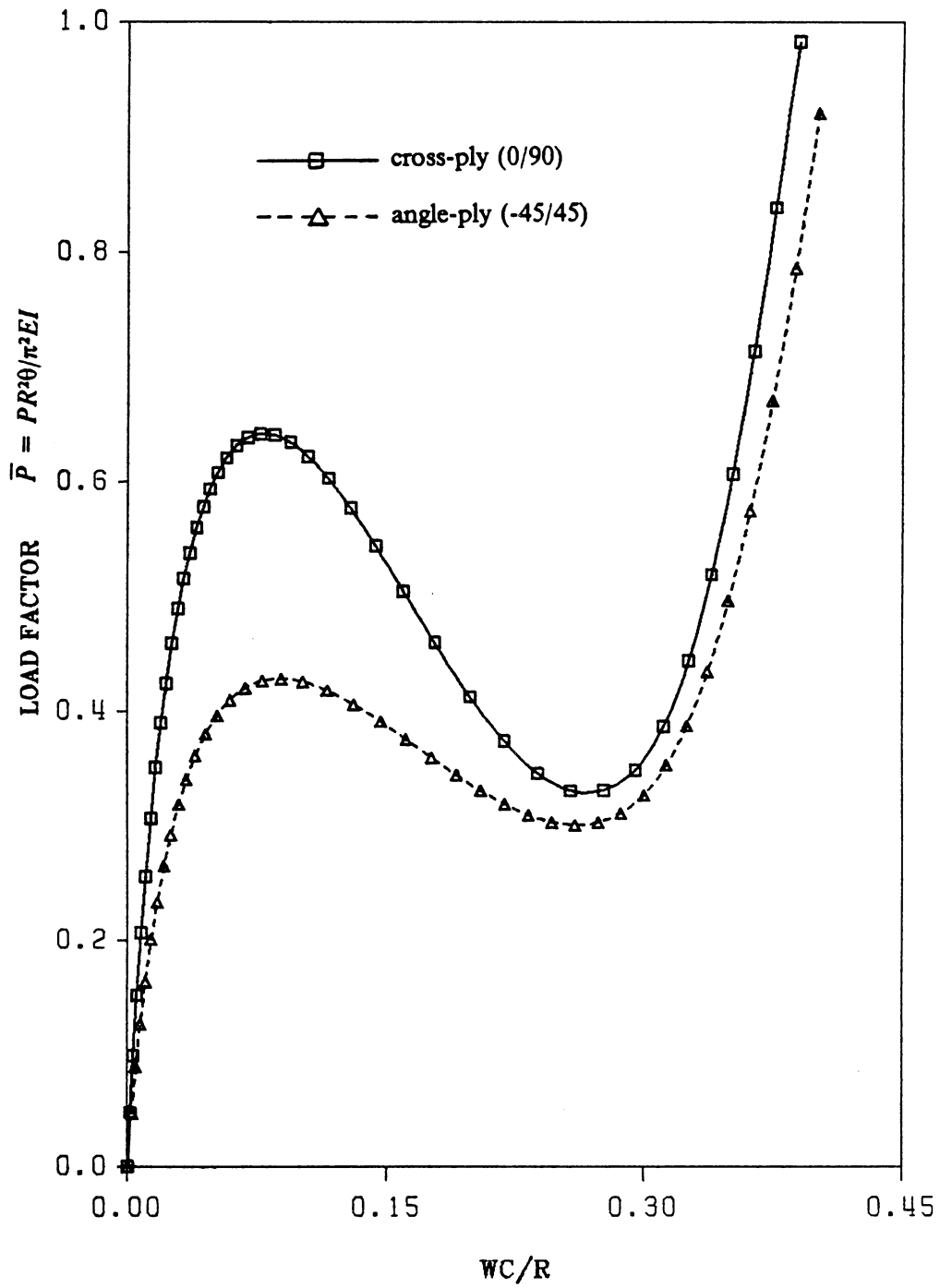


Figure 6.16 Symmetrical buckling analyses of 2-layer shallow arches under center point load

The load-deflection curves for plate with and without stiffeners are shown in Fig. 6.17. From the results it is clear that by adding stiffeners to the plate, the geometric linear range is extended considerably (i.e. plate with stiffeners exhibits almost linear response).

2. A square plate resting on elastic edge beams and supported at corners

Fig. 6.18 shows the plate geometry and material properties. Due to symmetry we only consider one-quarter of the structure. The same problem was solved by Timoshenko [66] who assumed the elastic edge beams are of zero torsional rigidity. The results obtained with 9-node shell element and 3-node beam element are shown below:

Mesh	NGP	LGP	Vertical displacement at the center (in)
2 x 2	3	2	-0.095922
2 x 2	2	2	-0.095957
3 x 3	3	2	-0.095790
3 x 3	2	2	-0.095800
Timoshenko [66]			-0.120290

3. A cantilever plate with symmetric stiffeners

Fig. 6.19 shows the geometry and material properties of the stiffened plate. The results obtained in the present study are shown below:

Mesh	NGP	LGP	Transverse deflection at loaded point (cm)
Four 9-node shell elements and Six 3-node beam elements	3	2	-0.18103
Four 9-node shell elements and Six 3-node beam elements	2	2	-0.18106
Nine 9-node shell elements and nine 3-node beam elements	3	2	-0.18137
Nine 9-node shell elements and nine 3-node beam elements	2	2	-0.18139

4. A cantilever cylindrical shell with symmetric stiffeners

Fig. 6.20 describes the geometry and material properties of this stiffened shell. The results obtained in the present study are compared with solutions given by Venkatesh and Rao [21]. They are presented in the table below. From the results it can be seen that with the reduced integration on all terms the 9-node parabolic shell element and 3-node parabolic beam element show a more rapid convergence and better accuracy than with the selective integration. This situation does not exist for the linear 4-node shell element and 2-node beam element.

Mesh	NGP	LGP	Radial displ. at loaded corner (cm) x 1000	Tangential displ. at loaded corner (cm) x 1000
Four 9-node shell elements and 12 3-node beam elements	3	2	-66.919	13.700
Four 9-node shell elements and 12 3-node beam elements	2	2	-165.83	32.362
Nine 9-node shell elements and 18 3-node beam elements	3	2	-113.92	22.389
Nine 9-node shell elements and 18 3-node beam elements	2	2	-166.13	32.420
16 4-node shell elements and 24 2-node beam elements	2	1	-164.14	33.033
16 4-node shell elements and 24 2-node beam elements	1	1	-164.45	32.486
36 4-node shell elements and 36 2-node beam elements	2	1	-165.43	32.725
36 4-node shell elements and 36 2-node beam elements	1	1	-165.57	32.486
9 4-node shell elements (Ref. [21])			-166.90	32.45

5. A cantilever cylindrical shell with eccentric stiffeners

The geometry and material properties are shown in Fig. 6.21. The results obtained in the present study are compared with solutions given by Venkatesh and Rao [21]. They are presented in the following:

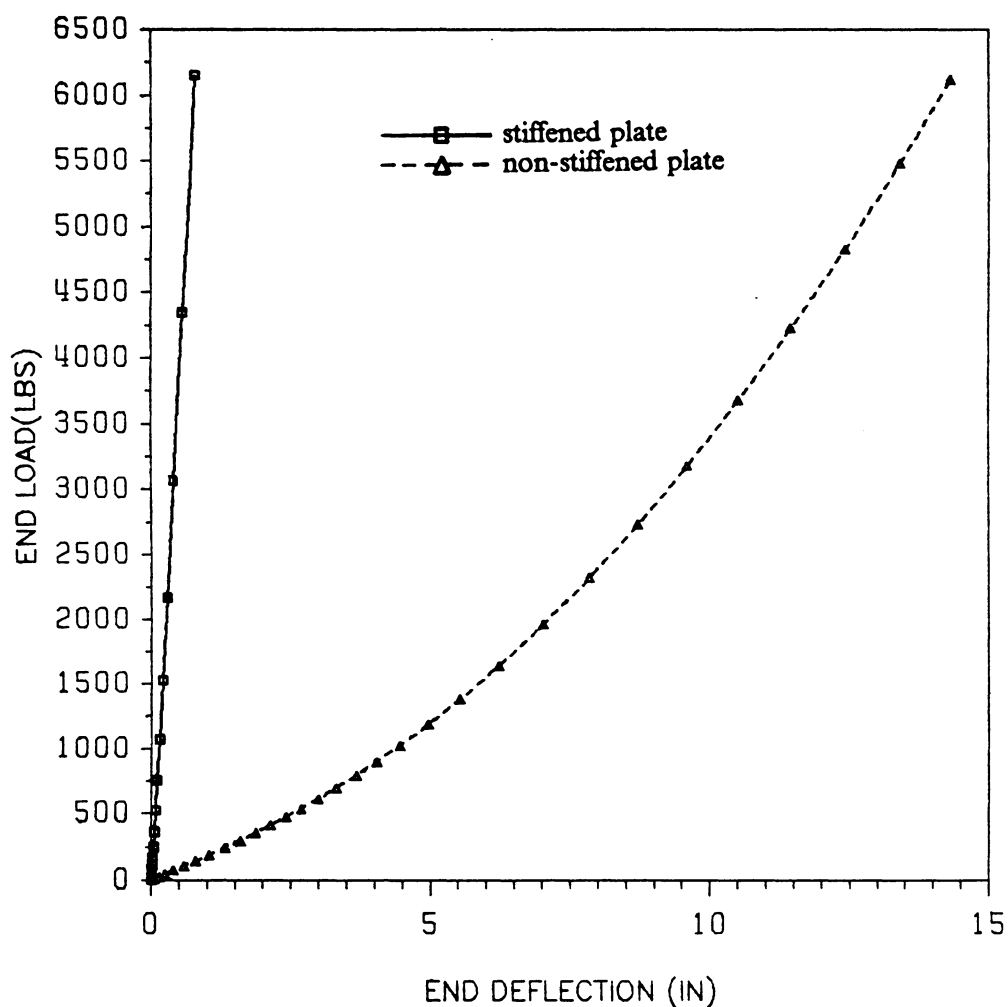
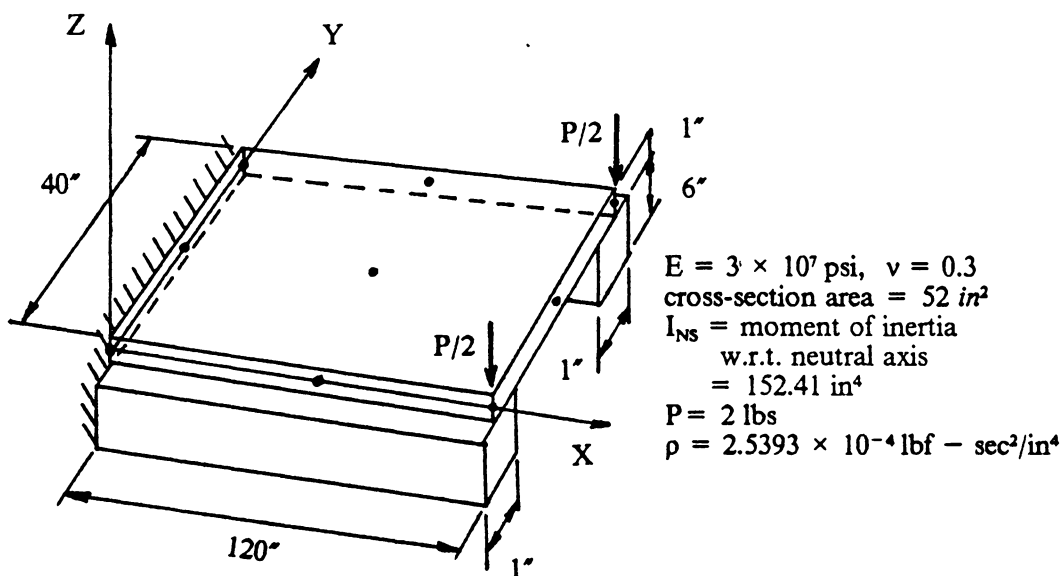


Figure 6.17 Cantilever stiffened plate subjected end load

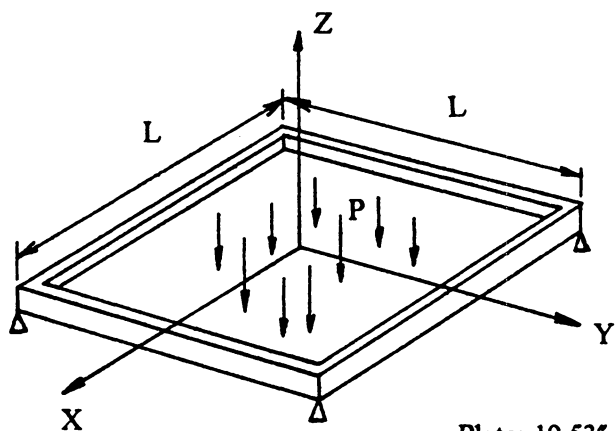


Plate: $19.53'' \times 19.53'' \times 0.2''$
 Edge beams: $0.5''$ width, $1''$ depth
 $E = 10^7$ psi, $\nu = 0.25$
 $P = 1$ psi
 G of edge beams $= 4 \times 10^6$ psi

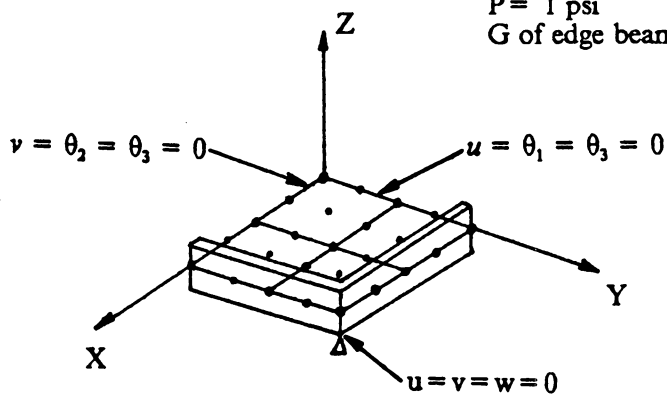


Figure 6.18 A square plate resting on elastic edge beam

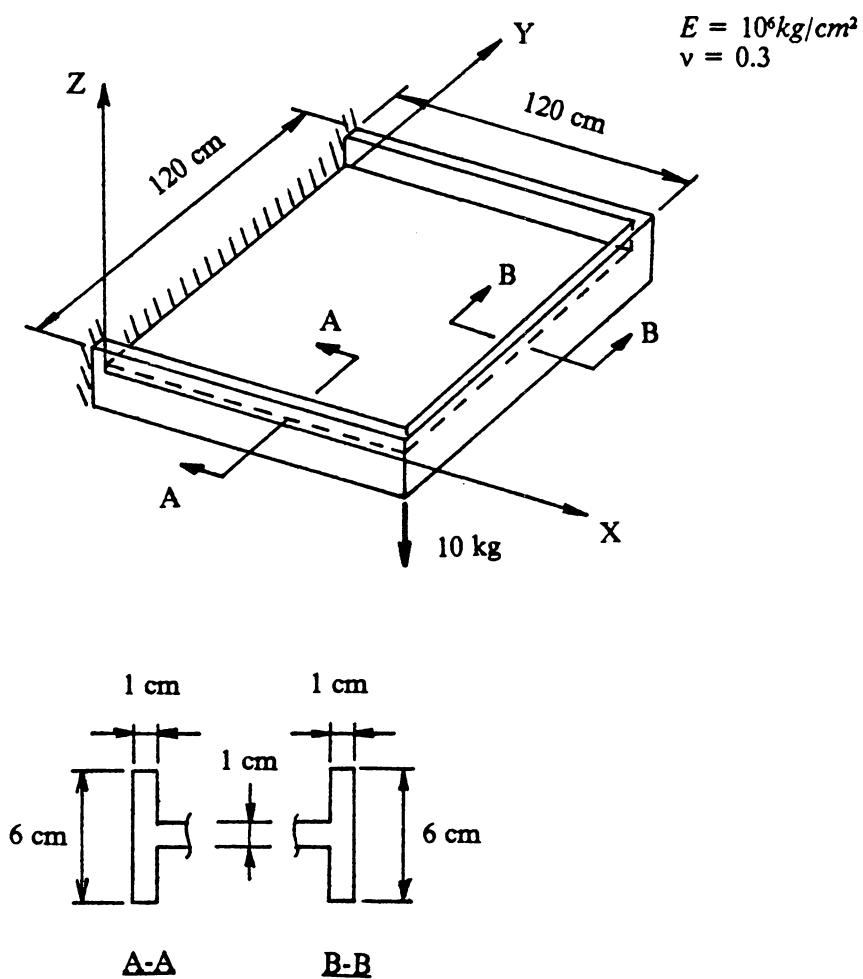


Figure 6.19 A cantilever plate with symmetric stiffeners

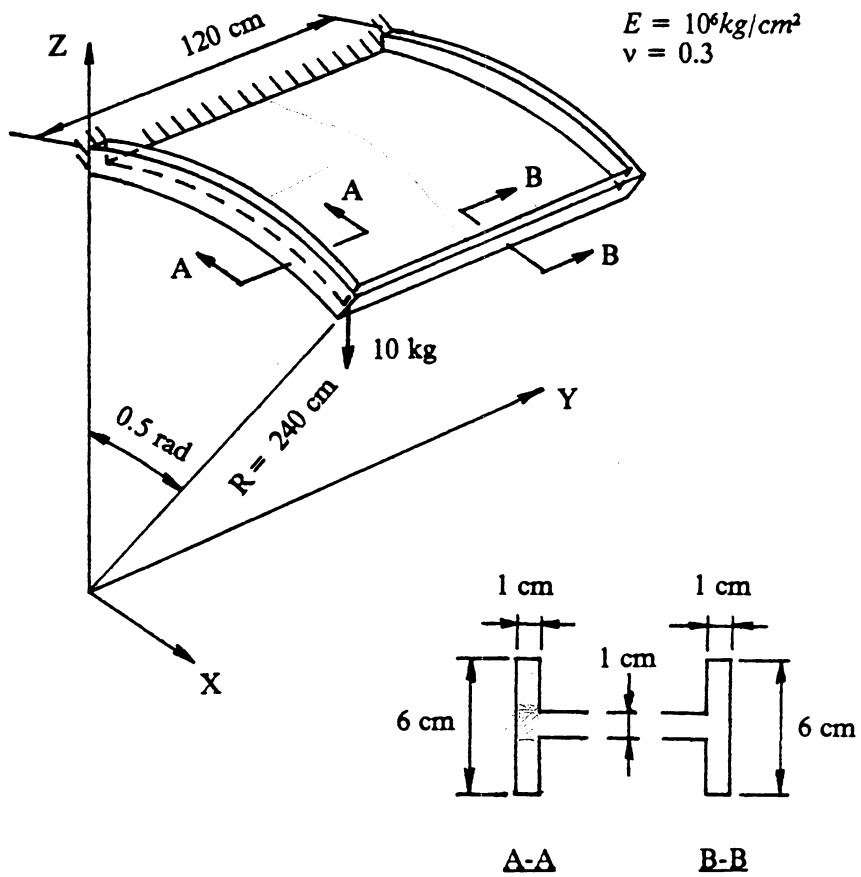


Figure 6.20 A cantilever cylindrical shell with symmetric stiffeners

Mesh	NGP	LGP	Radial displ. at loaded corner (cm) x 1000	Tangential displ. at loaded corner (cm) x 1000
Four 9-node shell elements and 8 3-node beam elements	2	2	-53.766	11.283
Nine 9-node shell elements and 12 3-node beam elements	2	2	-54.440	11.415
16 4-node shell elements and 16 2-node beam elements	2	1	-48.462	10.366
36 4-node shell elements and 24 2-node beam elements	2	1	-50.857	10.763
9 4-node shell elements (Ref. [21])			-55.30	11.51
16 4-node shell elements (Ref. [21])			-55.79	11.62

The radial deflections of hoop stiffeners and the axial stiffener obtained in the present study are also shown in Fig. 6.21. If the radius approaches infinity, i.e. the structure becomes a cantilever plate with eccentric stiffeners, and the other dimensions and loading are the same, the results obtained are shown below:

Mesh	NGP	LGP	Transverse displ. at loaded corner (cm) x 1000
Four 9-node shell elements and six 3-node beam elements	3	2	-50.574
Four 9-node shell elements and six 3-node beam elements	2	2	-51.035
Nine 9-node shell elements and nine 3-node beam elements	3	2	-50.845
Nine 9-node shell elements and nine 3-node beam elements	2	2	-51.245

The geometric nonlinear response of this stiffened shell is shown in Fig. 6.22. Since the curve is almost straight, this stiffened shell does not have notable geometric nonlinearity. The response of the same cylindrical shell without stiffeners is also included in Fig. 6.22 to show the effect of the stiffeners on the cylindrical shell.

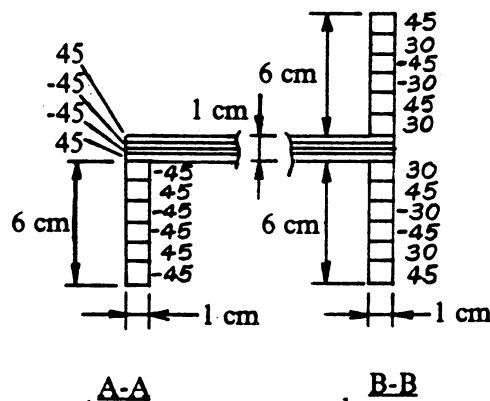
6. A composite eccentrically stiffened cantilever cylindrical shell

A glass-epoxy layered cantilever cylindrical shell reinforced with eccentrically mounted layered stiffeners was analyzed in the present study. The geometry and loading are the same as in the last problem. The material properties and lamination details are shown below:

Material: glass epoxy

$$E_L = 0.516 \times 10^6 \text{ kg/cm}^2, \quad E_T = 0.137 \times 10^6 \text{ kg/cm}^2,$$

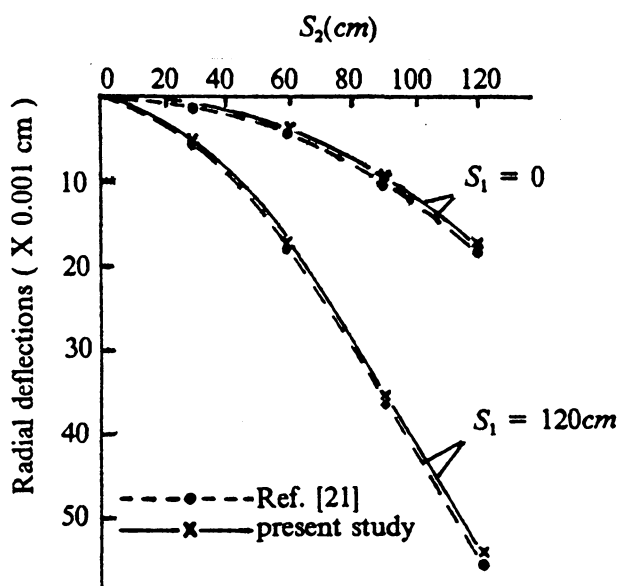
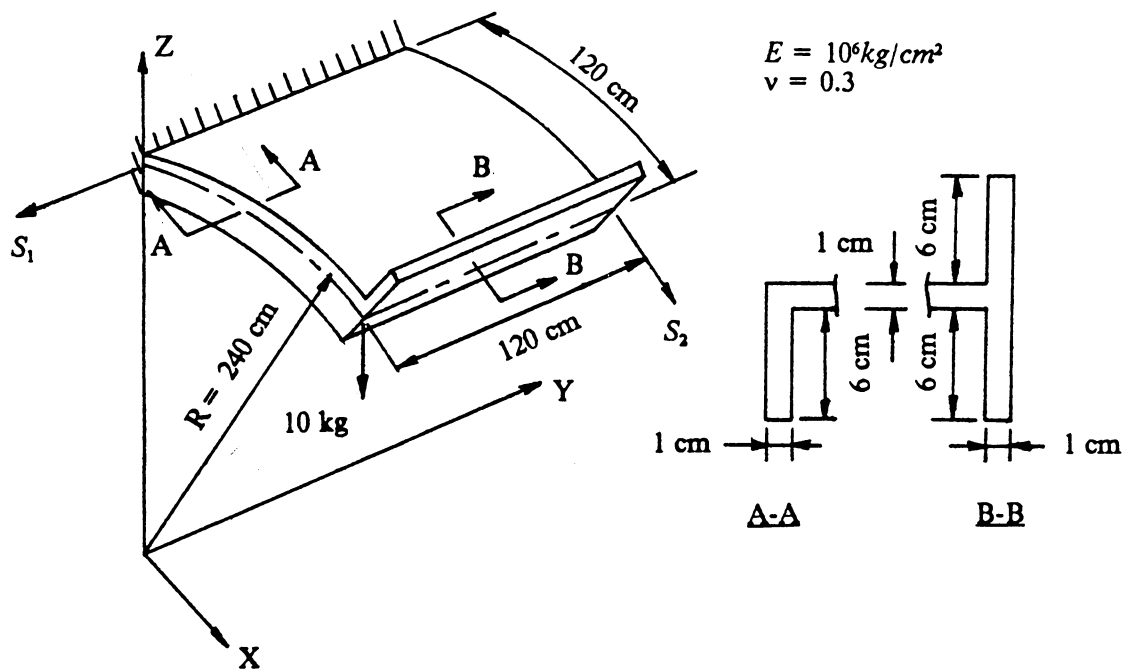
$$G_{LT} = 0.861 \times 10^5 \text{ kg/cm}^2, \quad \nu_{LT} = 0.25$$



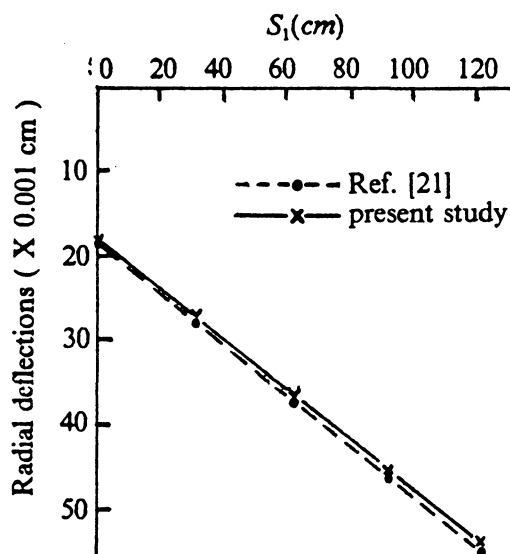
The results obtained in the present study are compared with those given by Venkatesh and Rao [21]. They are presented in the following and Fig. 6.23.

Mesh	NGP	LGP	Radial displ. at loaded corner (cm) x 1000	Tangential displ. at loaded corner (cm) x 1000
Four 9-node shell elements and 8 3-node beam elements	2	2	-194.87	41.205
Nine 9-node shell elements and 12 3-node beam elements	2	2	-197.33	41.695
Ref. [21]			-203.0 (app.)	

7. Static analysis of a simply-supported square plate with symmetric stiffeners and subjected to uniform pressure



Hoop stiffener



Axial stiffener

Figure 6.21 A cantilever cylindrical shell with eccentric stiffeners

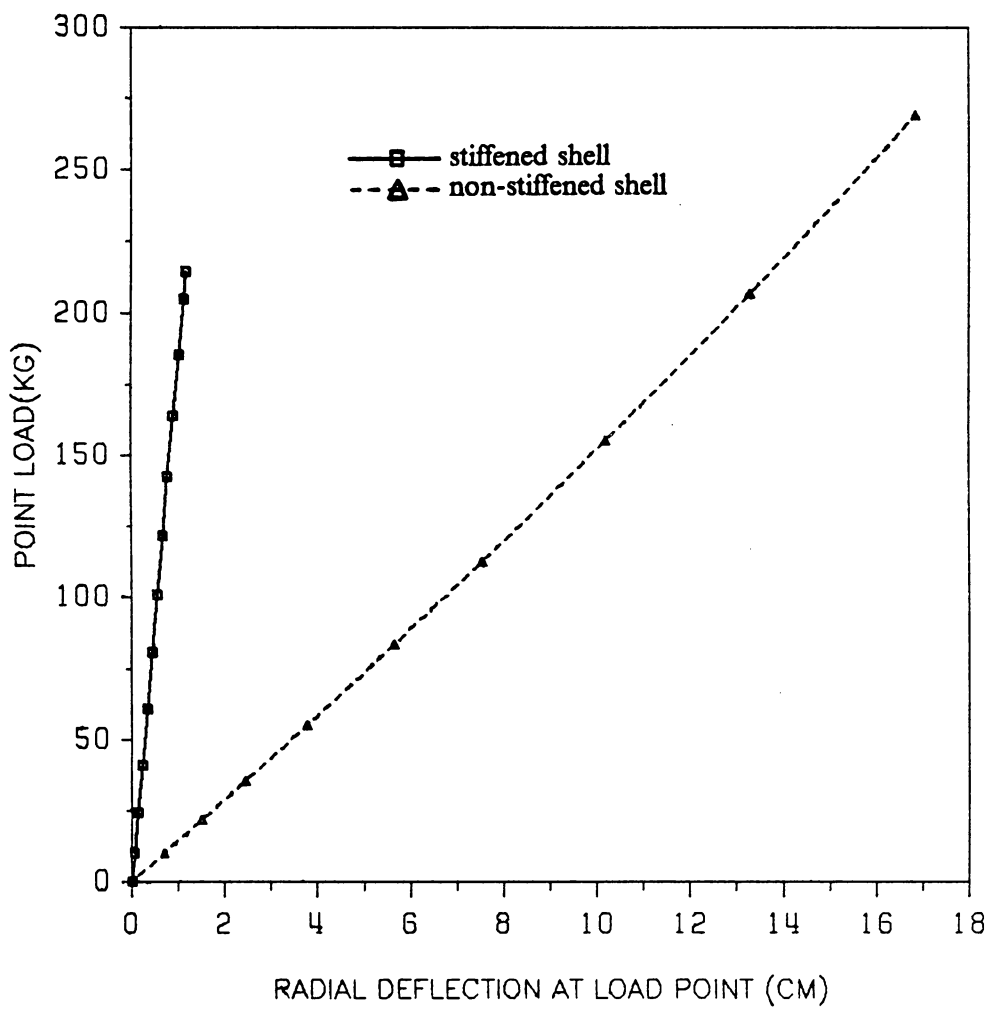
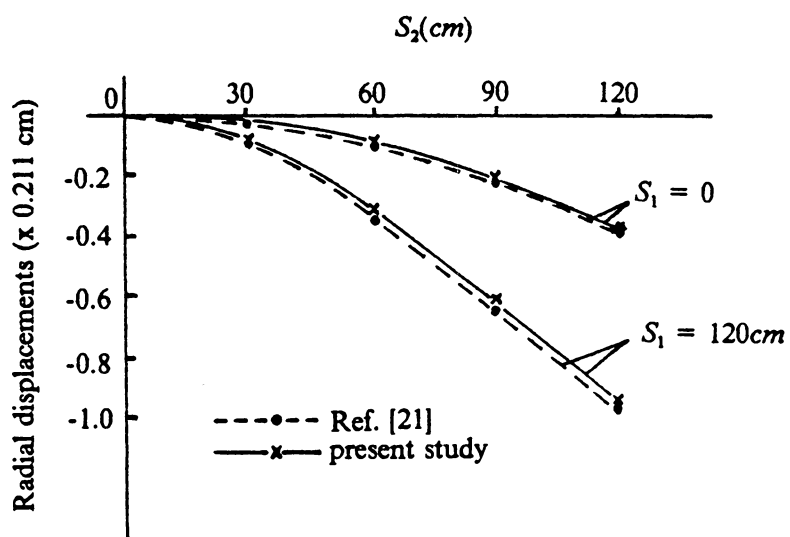


Figure 6.22 Static nonlinear analysis of cylindrical shell with eccentric stiffeners



Radial displacement of the hoop stiffeners

Figure 6.23 A cantilever composite cylindrical shell with eccentric stiffeners

The geometry of the stiffened plate is shown in Fig. 6.24. The material properties are:

$$E = 30 \times 10^6 \text{ psi}, \quad \nu = 0.3, \quad \rho = 2.5393 \times 10^{-4} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}^4}$$

Due to symmetry, only one quarter of the stiffened plate was modelled with four 9-node shell elements and eight 3-node beam elements. The geometric nonlinear responses of this stiffened plate and the same plate without stiffeners are shown in the Fig. 6.24. The same non-stiffened plate was analyzed also in Ref. [14] and the present results are close to that in Ref. [14]. The effect of stiffeners on the response of the plate can be seen from Fig. 6.24. The linear solutions for this stiffened plate are also included in the figure and we can see that the stiffened plate in this case exhibits high geometric nonlinearity.

8. Static analysis of a simply-supported spherical shell with symmetric stiffeners

The geometry of the stiffened spherical shell is shown in Fig. 6.25. The material properties are:

$$E = 10^4 \text{ psi}, \quad \nu = 0.3, \quad \rho = 2.3763 \times 10^{-5} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}^4}$$

Due to symmetry, only one quarter of the stiffened shell was modelled with sixteen 4-node shell elements and sixteen 2-node beam elements. The geometric nonlinear responses of this stiffened shell and the same shell without stiffeners (and subjected to a center point load) are shown in Fig. 6.25. The same non-stiffened shell was analyzed by Bathe and Ho [57] who used flat 3-node triangular shell element which includes transverse shear deformation. The stiffened shell buckles under a center point load. The effect of stiffeners on the the response of the shell can be seen from the figure.

9. Static analyses of 2-layer cross-ply and angle-ply simply-supported stiffened laminated plates

The geometries of the two stiffened laminated plates and the associated stacking sequences are shown in Figures 6.26 and 6.27. The material properties used are:

$$\begin{aligned} E_x/E_y &= 25, \quad E_y = 7.031 \times 10^5 \text{ N/cm}^2, \quad \nu_{xy} = 0.25, \quad G_{xy}/E_y = 0.5, \\ G_{xz} &= G_{xy}, \quad G_{yz}/E_y = 0.2, \quad \rho = 2.547 \times 10^{-6} \frac{\text{N sec}^2}{\text{cm}^4} \end{aligned}$$

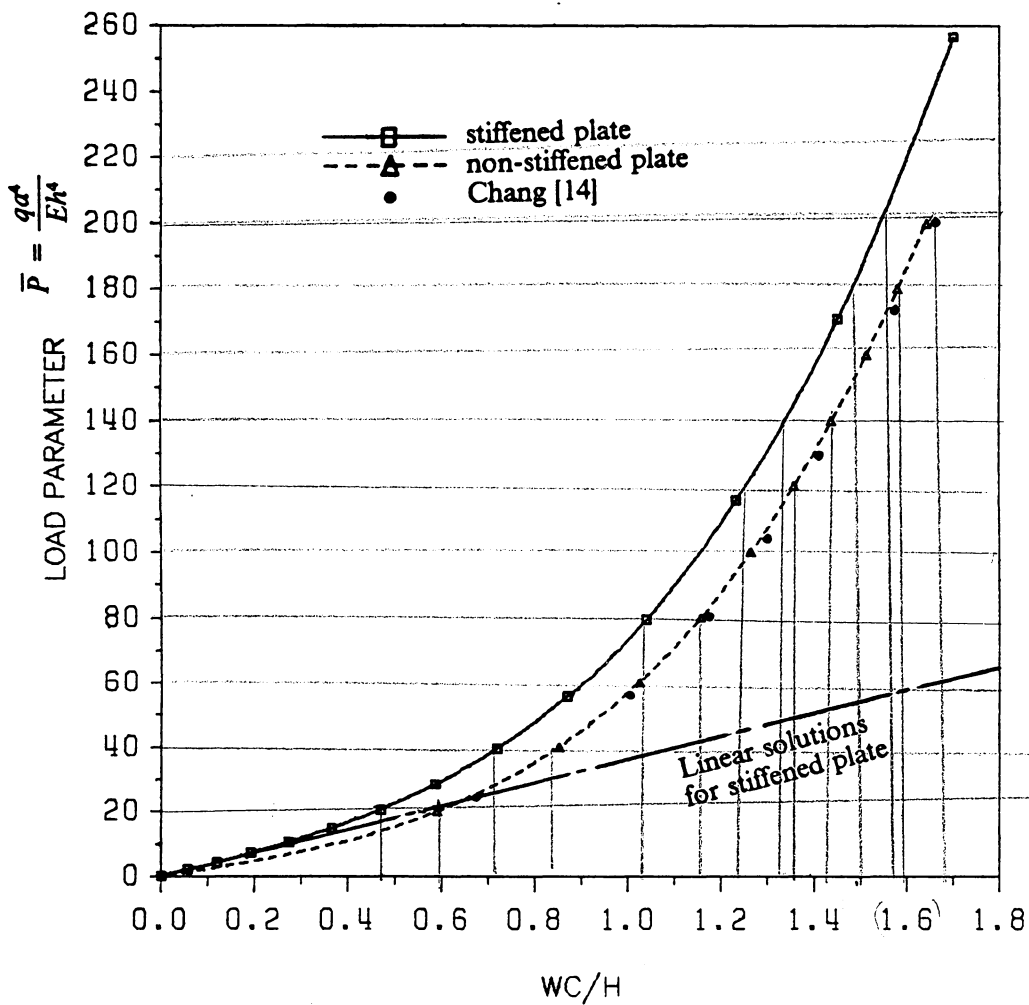
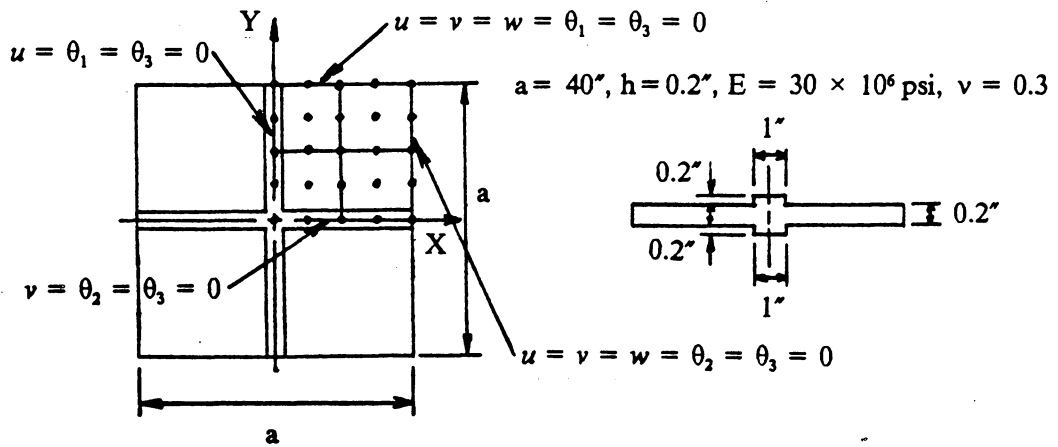


Figure 6.24 Static analysis of a simply-supported square plate with symmetric stiffeners and under uniform pressure

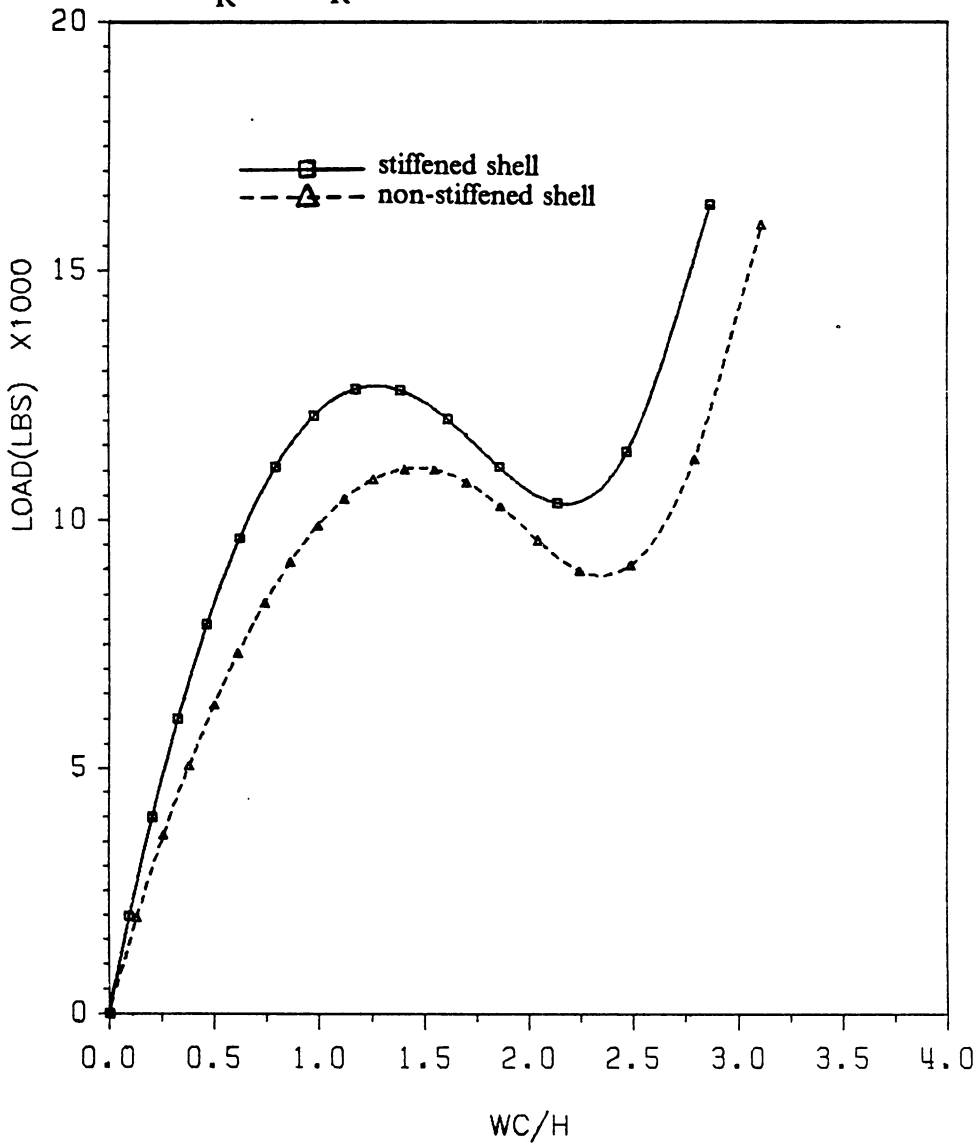
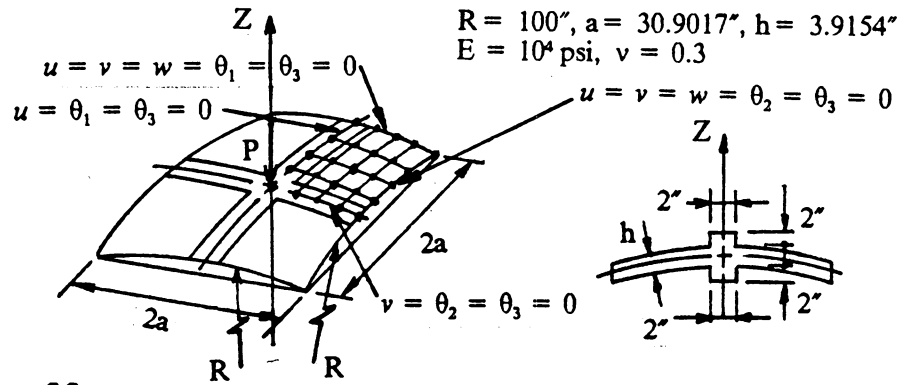


Figure 6.25 Static analysis of a simply-supported spherical shell with symmetric stiffeners and under a center point load

For unsymmetrically laminated plates the word "simply-supported" does not have a unique meaning because of the bending-stretching coupling. Also, usually we cannot use quarter-plate models to analyze plates with biaxial symmetry with respect to the geometry and loading. For the edge boundary conditions shown in the figures, we can use quarter-plate models only if the the symmetry conditions are BC2 for angle-ply and BC3 for cross-ply. The geometric nonlinear responses of these two plates with and without stiffeners are shown in Figures 6.26 and 6.27, respectively. Only one quarter of the plate was modelled with four 9-node shell elements and eight 3-node beam elements.

10. Static analyses of 2-layer cross-ply and angle-ply simply-supported stiffened laminated spherical shells subjected to uniform pressure

The geometries of the two stiffened laminated spherical shells and the stacking sequences are shown in Figures 6.28 and 6.29. The material properties are:

$$E_x/E_y = 25, \quad E_y = 10^6 \text{ psi}, \quad \nu_{xy} = 0.25, \quad G_{xy}/E_y = 0.5, \\ G_{xz} = G_{xy}, \quad G_{yz}/E_y = 0.2, \quad \rho = 2.3763 \times 10^{-5} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}^4}$$

The edge boundary conditions are the same as that in the last problem. For cross-ply and angle-ply stiffened shells the symmetry conditions are BC3 and BC2 respectively. One quarter of each stiffened shell was modelled with four 9-node shell elements and eight 3-node beam elements. The geometric nonlinear responses of these two shells with and without stiffeners are shown in Figures 6.28 and 6.29, respectively.

6.2.4 Applications of Solid-Shell Transition Element

In this section examples are presented to illustrate the use of the solid-shell transition element developed in the present study. In order to demonstrate the accuracy of the transition element, the same structures were also analyzed using different finite elements developed here, e.g. the degenerated shell element or 3-D continuum element. Because

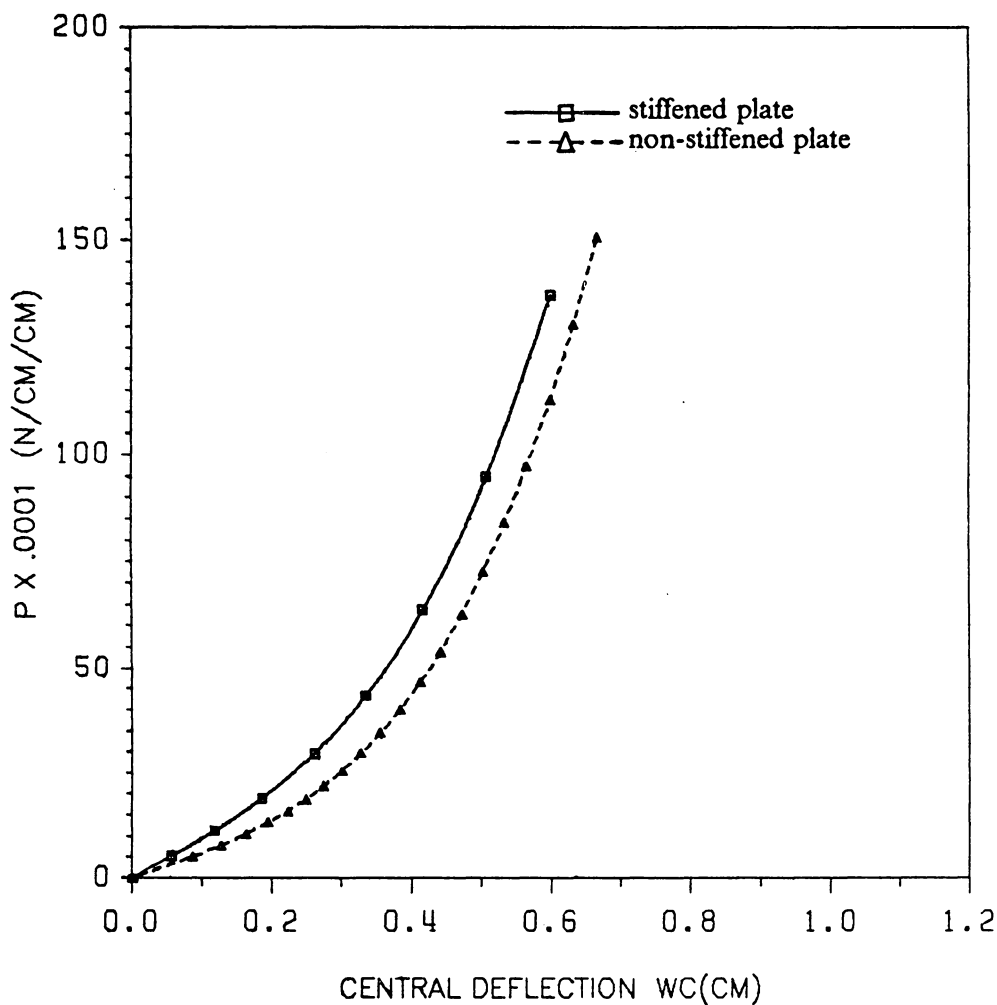
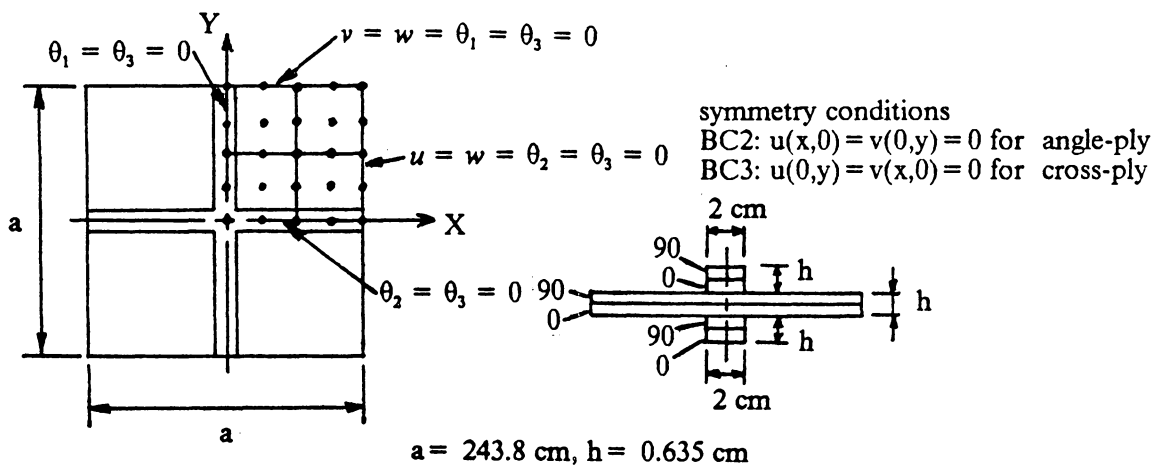


Figure 6.26 Static analysis of 2-layer cross-ply(0/90) simply-supported stiffened plate under uniform pressure

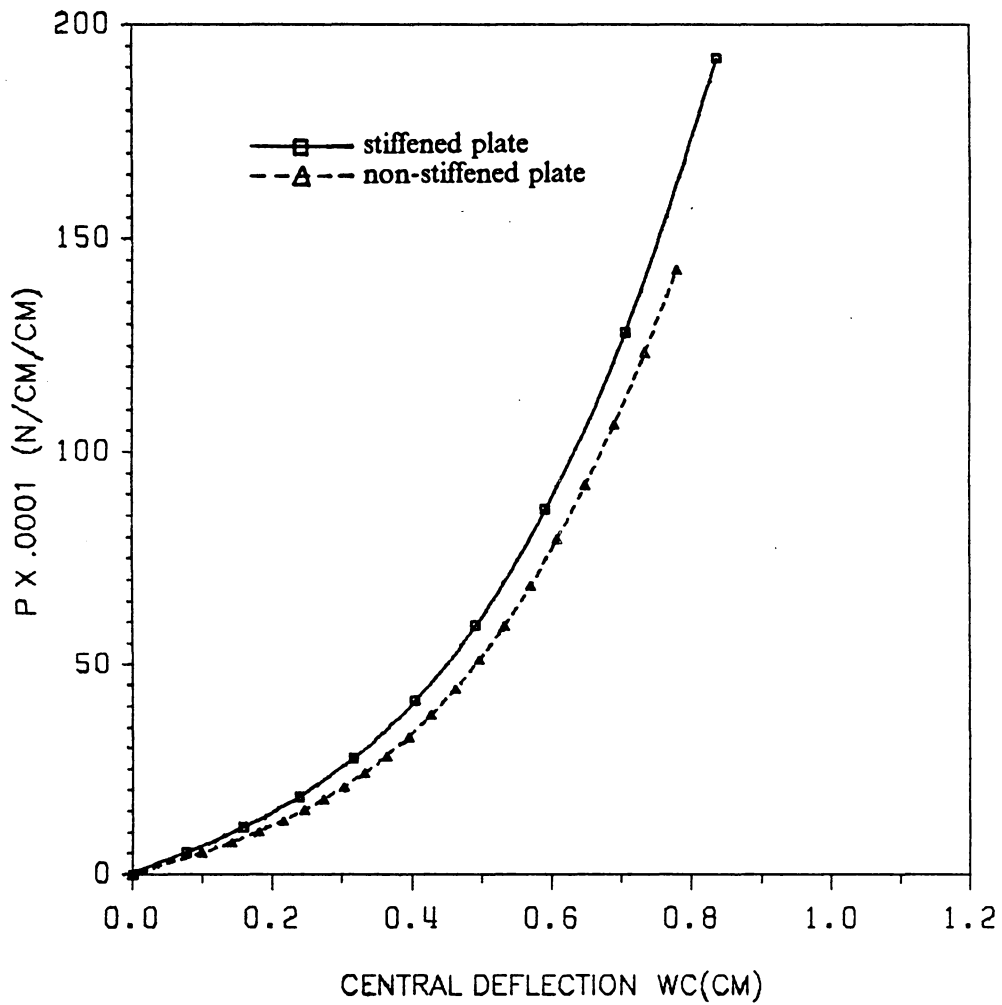
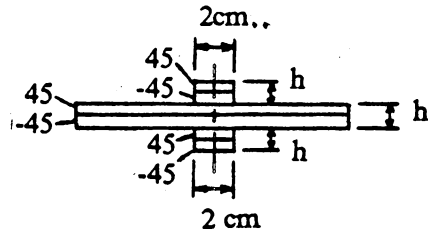


Figure 6.27 Static analysis of 2-layer angle-ply(-45/45) simply-supported stiffened plate under uniform pressure

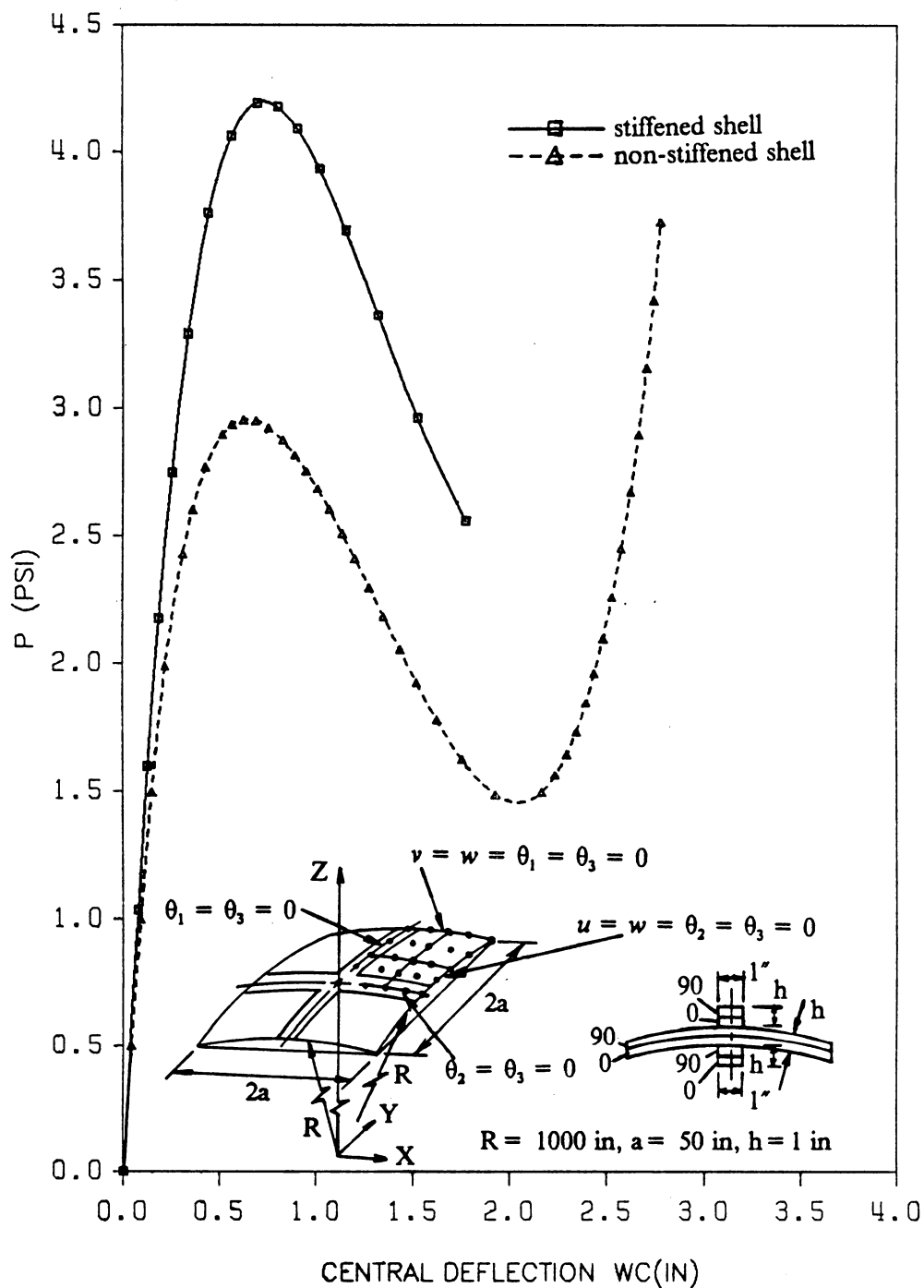


Figure 6.28 Static analysis of 2-layer cross-ply(0/90) simply-supported stiffened spherical shell under uniform pressure

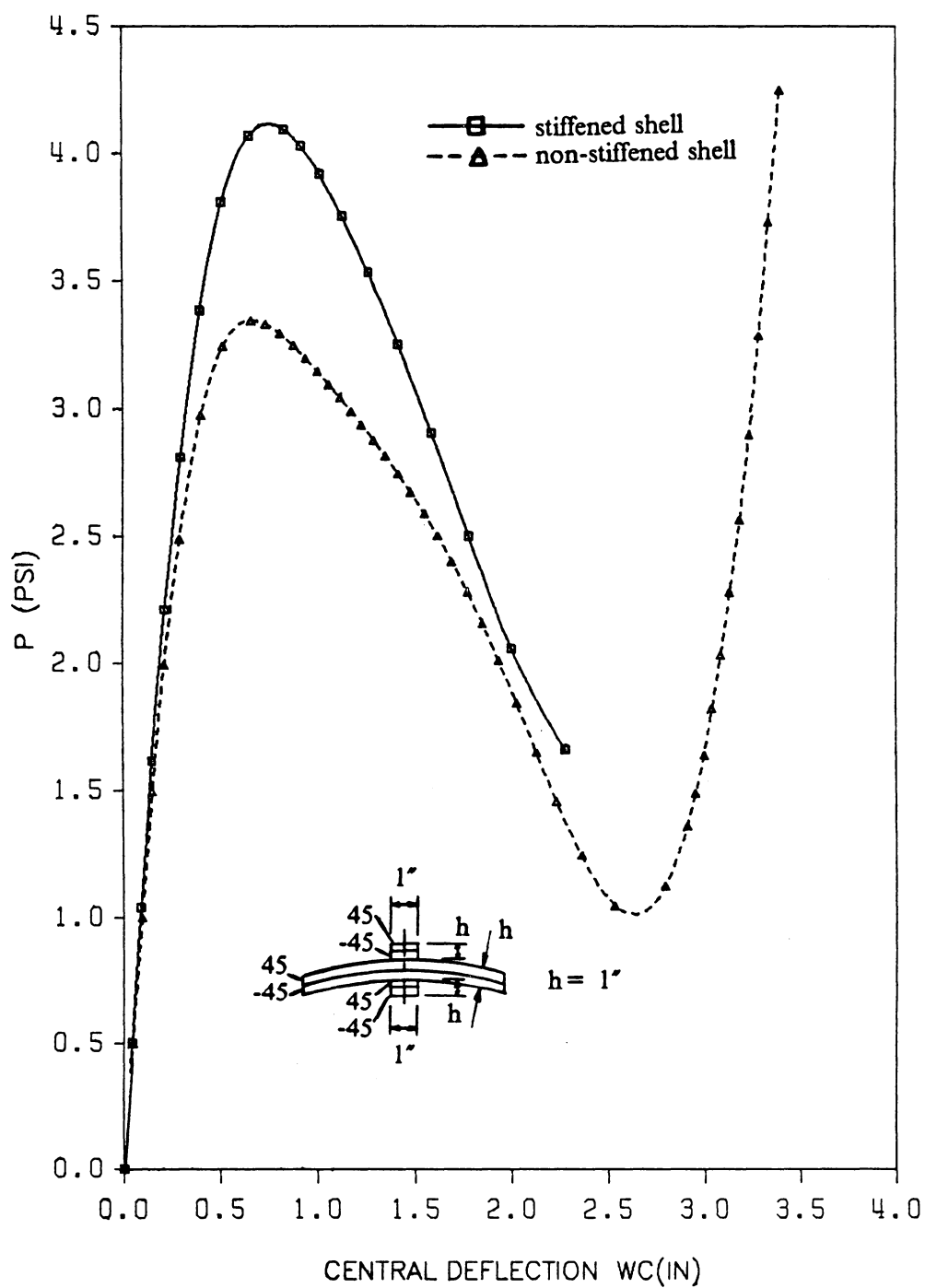


Figure 6.29 Static analysis of 2-layer angle-ply(-45/45) simply-supported stiffened spherical shell under uniform pressure

the transition element accounts for the transverse shear deformations, reduced integration was used to evaluate the transverse shear terms.

1. A cantilever plate

The plate and the four finite element models used are shown in Fig. 6.30. For the bending loading of $P = 30$ lb/in at the free end, the lateral deflections w at $x = 2$ in, 4 in, 6 in and 8 in are:

0.011280 in, 0.040992 in, 0.082992 in, 0.13114 in for model A

0.011283 in, 0.040998 in, 0.083002 in, 0.13115 in for model B

0.011280 in, 0.040992 in, 0.082995 in, 0.13114 in for model C

0.011283 in, 0.040998 in, 0.083002 in, 0.13115 in for model D

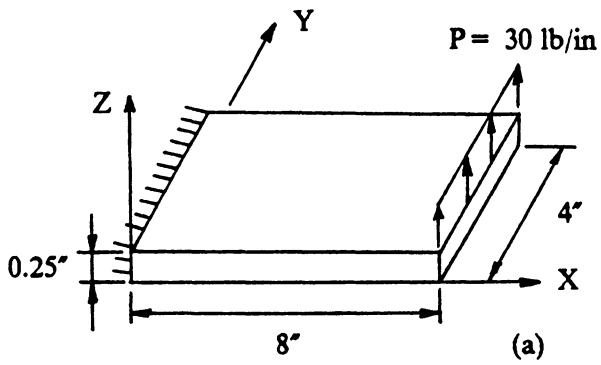
In this example problem we assume $\sigma_z = 0$ and $\epsilon_z = 0$ through the whole transition element. The integration order for parabolic 3-D solid element and parabolic transition element is $3 \times 3 \times 2$ for the bending part and $2 \times 2 \times 2$ for the transverse shear part. For models B and D, since the normal stress σ_z and the normal strain ϵ_z are neglected through the whole plate, i.e. the stiffness in the transverse normal direction becomes zero, the deflections are expected to be larger as compared with models A and C. This example demonstrates the accuracy of the transition element in the linear analysis and it illustrates the manner in which the transition elements can be connected to 3-D solid elements or with themselves. This example was also analyzed by Surana [50].

2. A simply-supported isotropic beam subjected to a center point load

This example is used to evaluate the performance of the solid-shell transition elements in geometric nonlinear analysis. The material properties are:

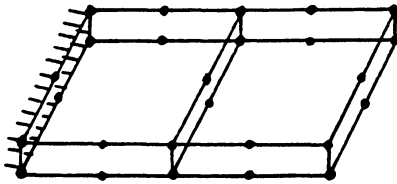
$$E = 3 \times 10^7 \text{ psi}, \quad \nu = 0, \quad \rho = 2.5393 \times 10^{-4} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}^4}$$

Fig. 6.31 shows the simply-supported beam and the finite element models. In Fig 6.31(b) the beam is modelled with three 11-node transition elements (back to back). Another

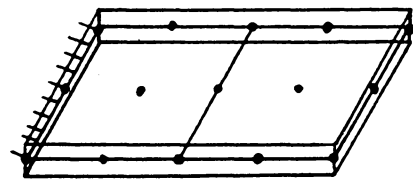


$$E = 3 \times 10^7 \text{ psi}$$

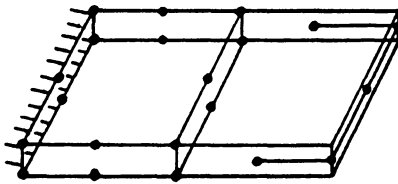
$$\nu = 0$$



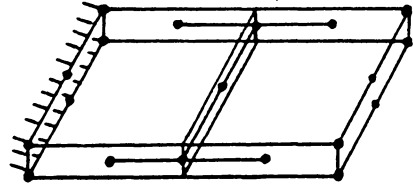
(b) model A



(c) model B



(d) model C



(e) model D

Figure 6.30 Cantilever plate and its finite element models

finite element model using three 8-node degenerated shell elements is shown in Fig. 6.31(c). For linear analysis, when $p = -0.6$ lb, the center deflections are:

-0.0204815 in for the model with transition elements

-0.02048 in for classical beam theory

Fig. 6.32 shows plots of the load factor vs the central deflection ratio for these two models and the results are almost the same. For the transition element model the deflections with $\sigma_x \neq 0$ are less but very close to that with $\sigma_x = 0$. The effect of σ_x on the performance of the transition element were investigated in details in Ref. [51].

3. A simply-supported 2-layer (0/90) beam subjected to a center point load

The same simply-supported beam as that in the last problem was used except that the beam is composed of 2-layer (0/90) T300/5208 graphite-epoxy material. The material properties are:

$$E_1 = 19.2 \times 10^6 \text{ psi}, \quad E_2 = E_3 = 1.56 \times 10^6 \text{ psi}, \quad \nu_{12} = \nu_{13} = 0.24, \quad \nu_{23} = 0.49,$$

$$G_{12} = G_{13} = 0.82 \times 10^6 \text{ psi}, \quad G_{23} = 0.523 \times 10^6 \text{ psi}, \quad \rho = 2.3763 \times 10^{-5} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}^4}$$

The two finite element models used in the last problem were used for the present problem. This example demonstrates the performance of solid-shell transition element used in the composite plate analysis. Fig. 6.33 shows good agreement of the results of both models in the geometric nonlinear analysis.

6.2.5 General Shell-Type Structures

1. A nonuniform cantilever plate

The plate considered consists of a short thick slab (2 in $\times$ 1 in $\times$ 0.7 in) and a very thin long plate (10 in $\times$ 1 in $\times$ 0.1 in) joined together at the mid-plane as shown in Fig. 6.34(a). Two loading are applied respectively at the free end of the plate. Loading I is a bending load of $p = 3$ lb/in and loading II is a inplane load of $p = 120000$ lb/in. Two different models are used to model this structure. Model A in Fig. 6.34(b) contains ten

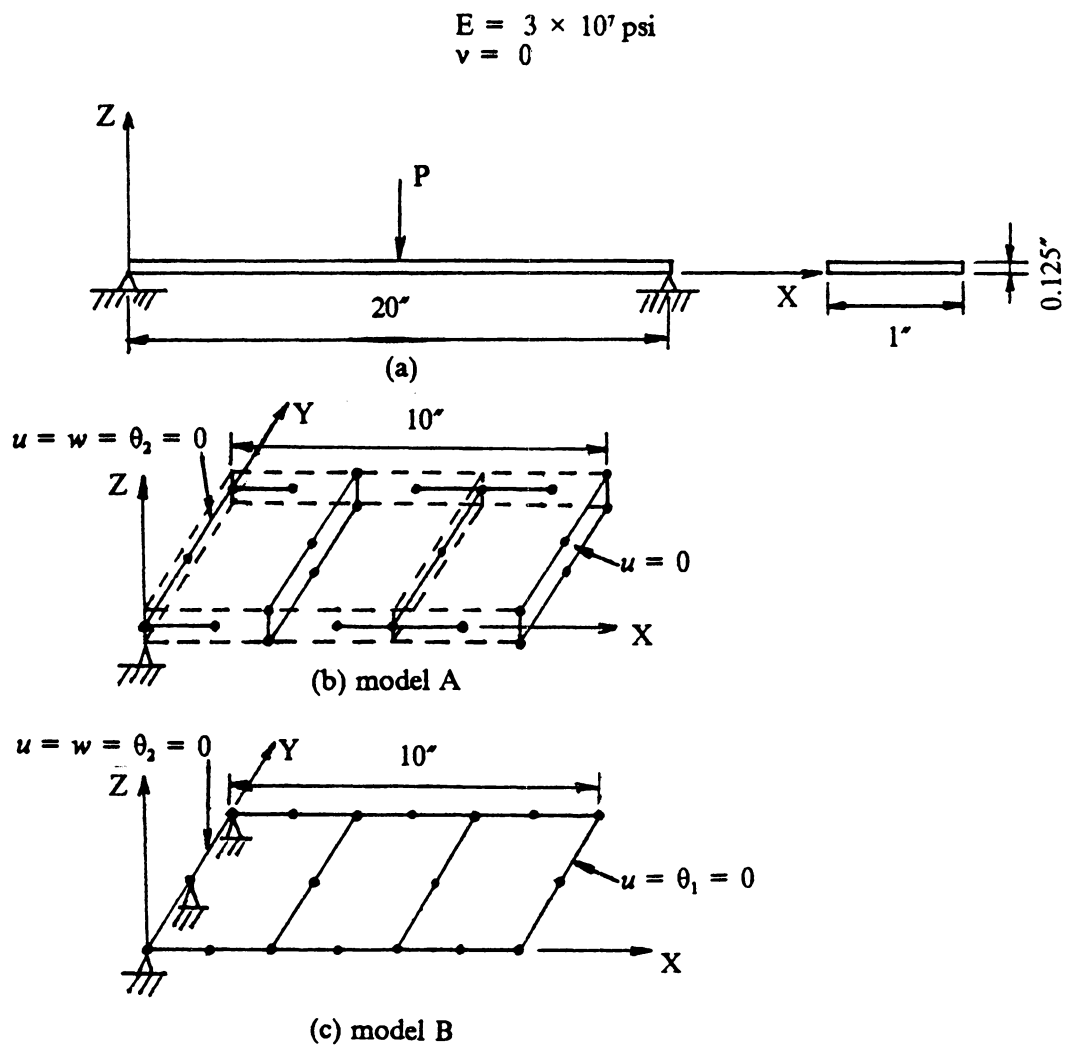


Figure 6.31 Simply-supported beam and its finite element models

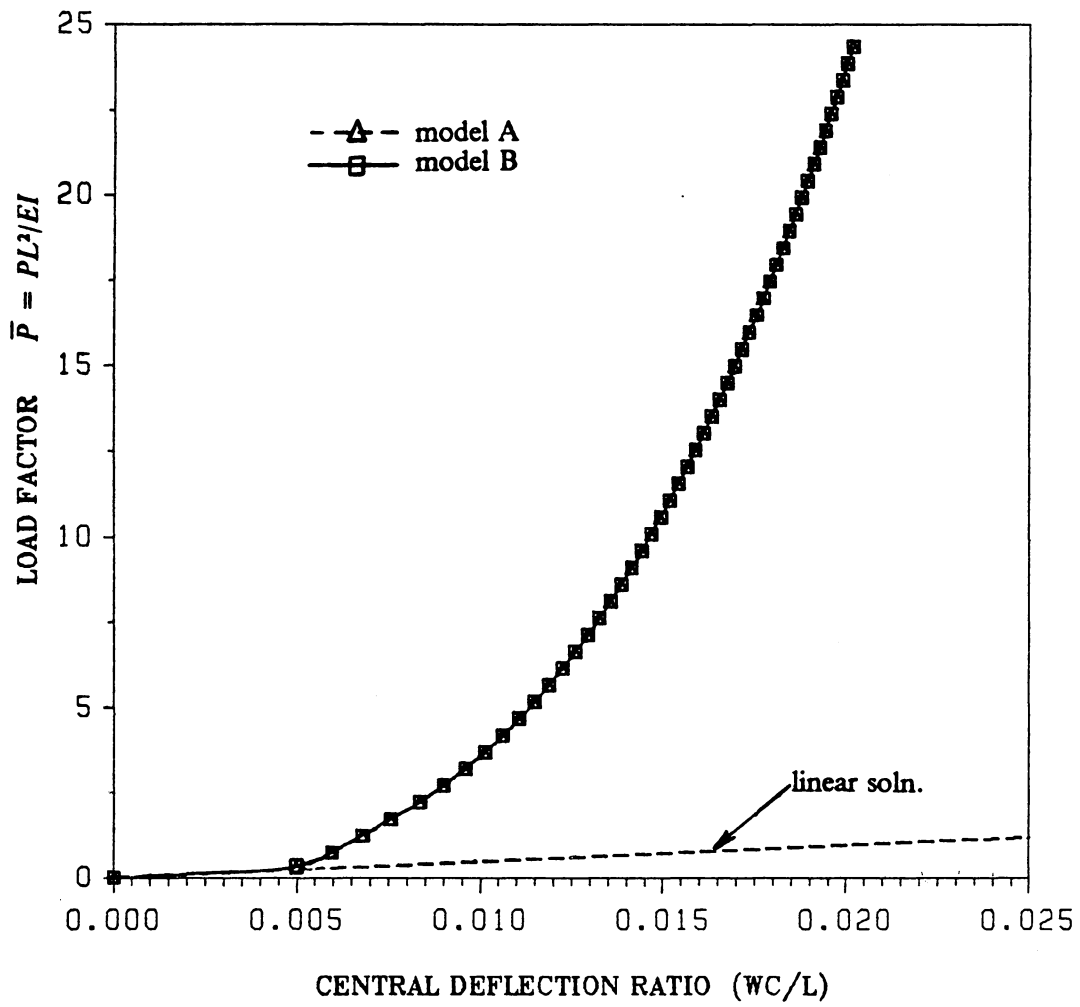


Figure 6.32 Geometric nonlinear response of simply-supported beam under center point load

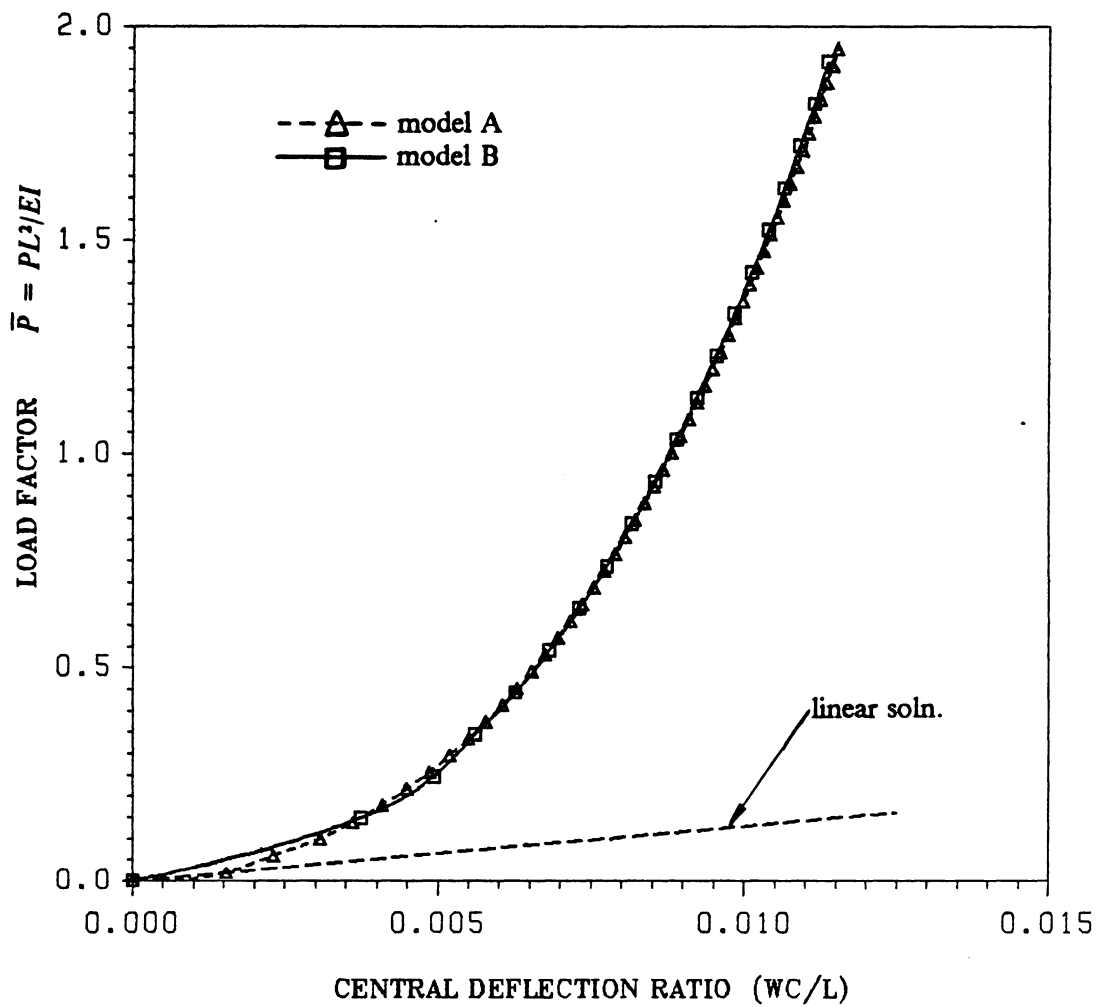


Figure 6.33 Geometric nonlinear response of 2-layer(0/90) simply-supported beam under center point load

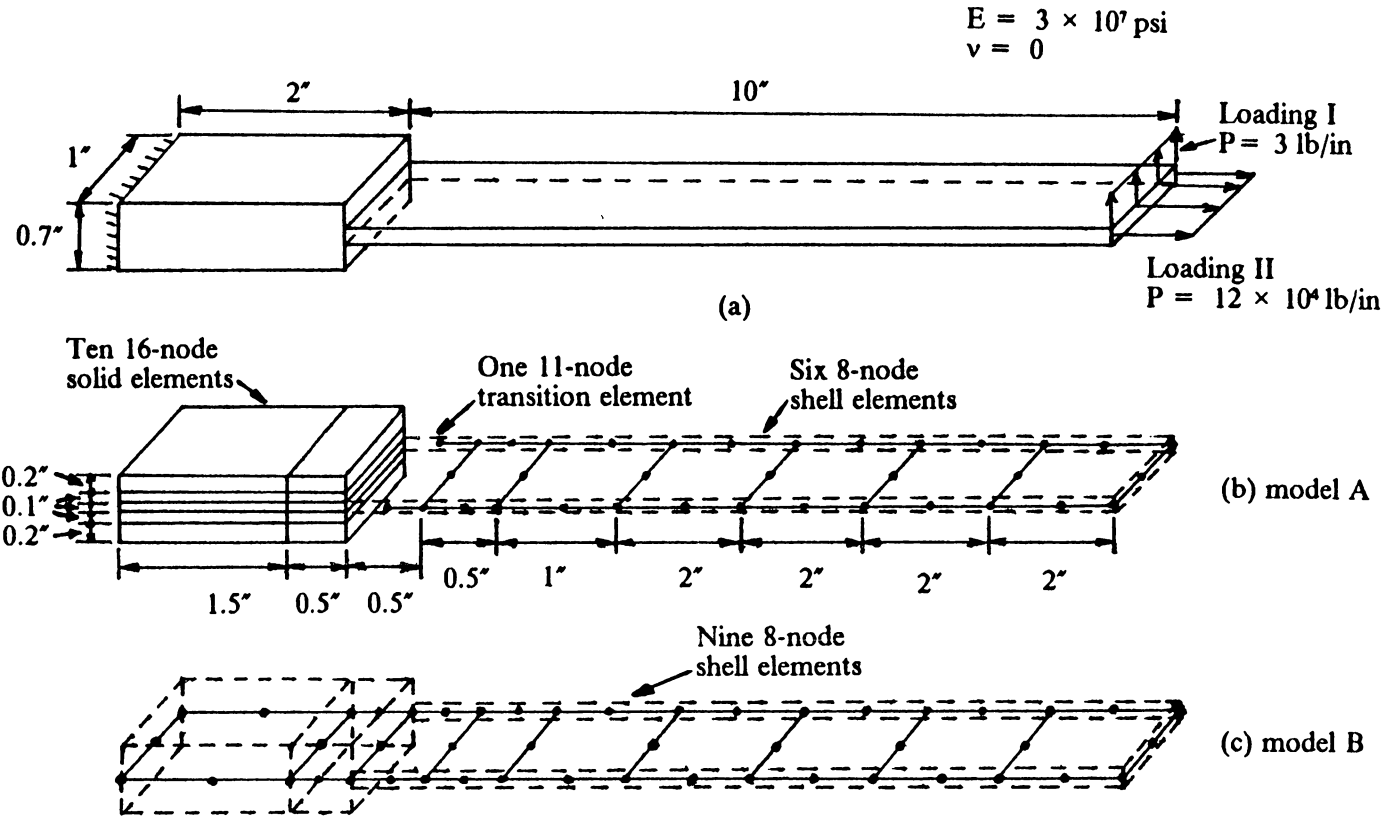
16-node 3-D solid elements, one 11-node transition element and six 8-node degenerated shell elements. In model B of Fig. 6.34(c) the entire plate was modelled with nine 8-node degenerated shell elements. The deflections of both models for loading I are shown in Figures 6.35 to 6.40. Fig. 6.35 shows a plot of the deflection w along the length of the plate. Fig. 6.36 shows the sharp change in the deflection w from $x = 2$ in to $x = 2.25$ in. This is due to the thickness change of the plate at $x = 2$ in. Figures 6.37 to 6.40 show plots of deflection u along the thickness of the plate at $x = 2$ in, $x = 1.75$ in, $x = 1.5$ in and $x = 0.75$ in respectively. From Figures 6.37 to 6.40 we can see model B fails to predict the correct deformation. The results obtained in the present study with model A is quite close to that by Surana [50] using the same model. Also the results of model B in the present study are close to that in [50] using the same model. The deflections w of model B are almost the same as that of model A. For loading II, Fig. 6.41 shows a plot of the deflection u along the length of the plate. Fig. 6.42 shows a sharp change in the deflection u from $x = 2$ in to $x = 2.5$ in. Fig. 6.43 shows plots of deflection u through the thickness of the plate at $x = 2$ in, $x = 1.75$ in and $x = 1.5$ in for models A and B. Again model B can not predict the correct deformation. The differences are most significant at $x = 2$ in. For loading II, the results of models A and B are close to that in [50] with the same models. This example demonstrates the use of transition elements in conjunction with solid elements and degenerated shell elements.

2. A cantilever folded roof structure

This example shows the application of transition elements to model the intersections of different shell surfaces. The use of transition element can eliminate the constraint equations at the transition regions. The folded plate and the finite element model used are shown in Fig. 6.44. The material properties are:

$$E = 3 \times 10^7 \text{ psi}, \quad \nu = 0, \quad \rho = 2.5393 \times 10^{-4} \frac{\text{lb} \cdot \text{f} - \text{sec}^2}{\text{in}^4}$$

Figure 6.34 Cantilever nonuniform plate and its finite element models



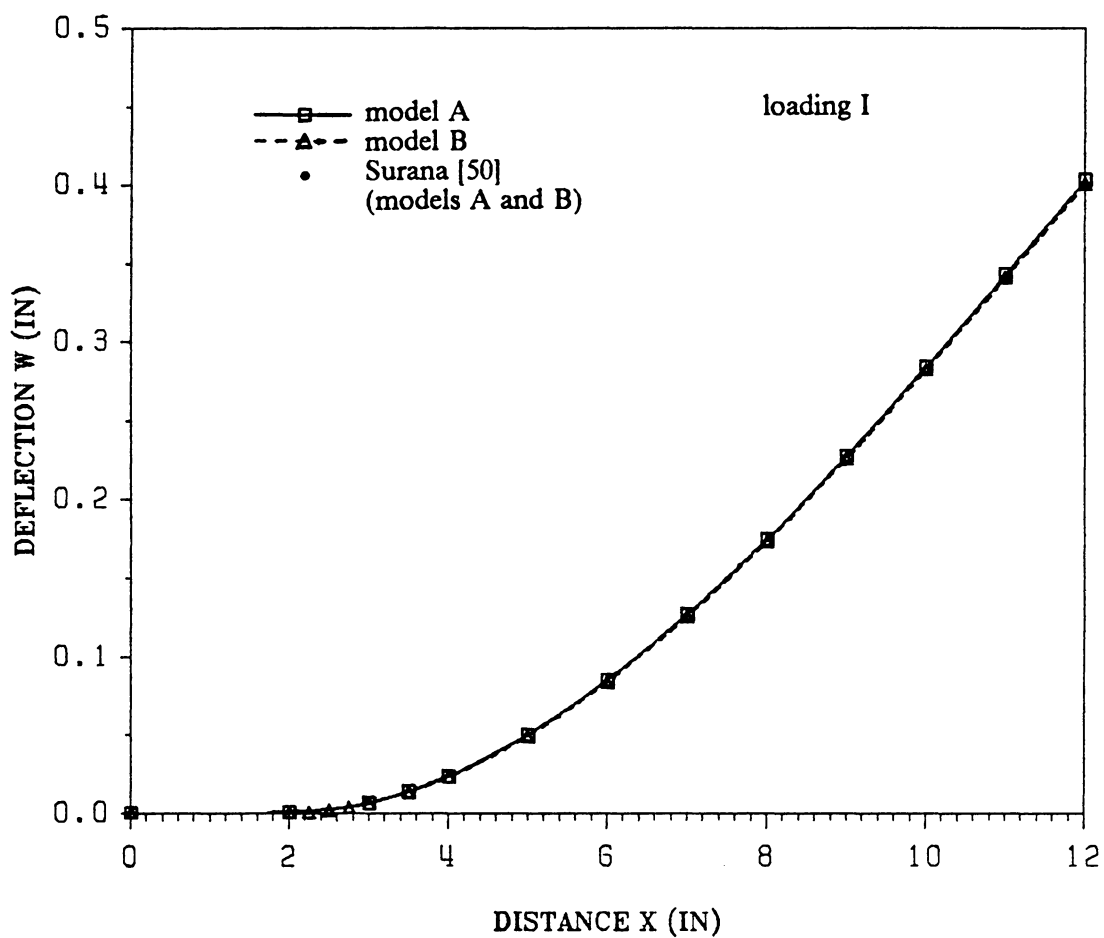


Figure 6.35 Deflection w along the length of the plate

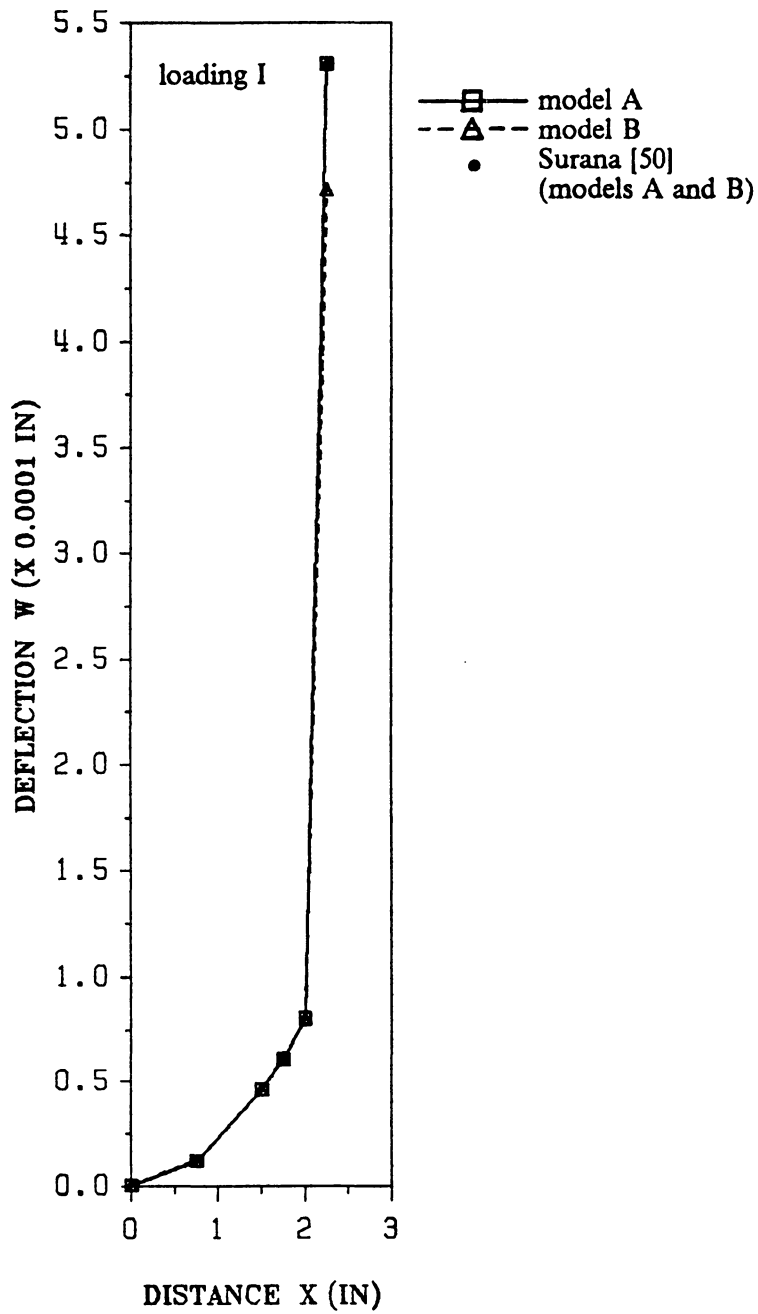


Figure 6.36 Deflection w along the length of the plate from $x = 0$ in to $x = 2.25$ in

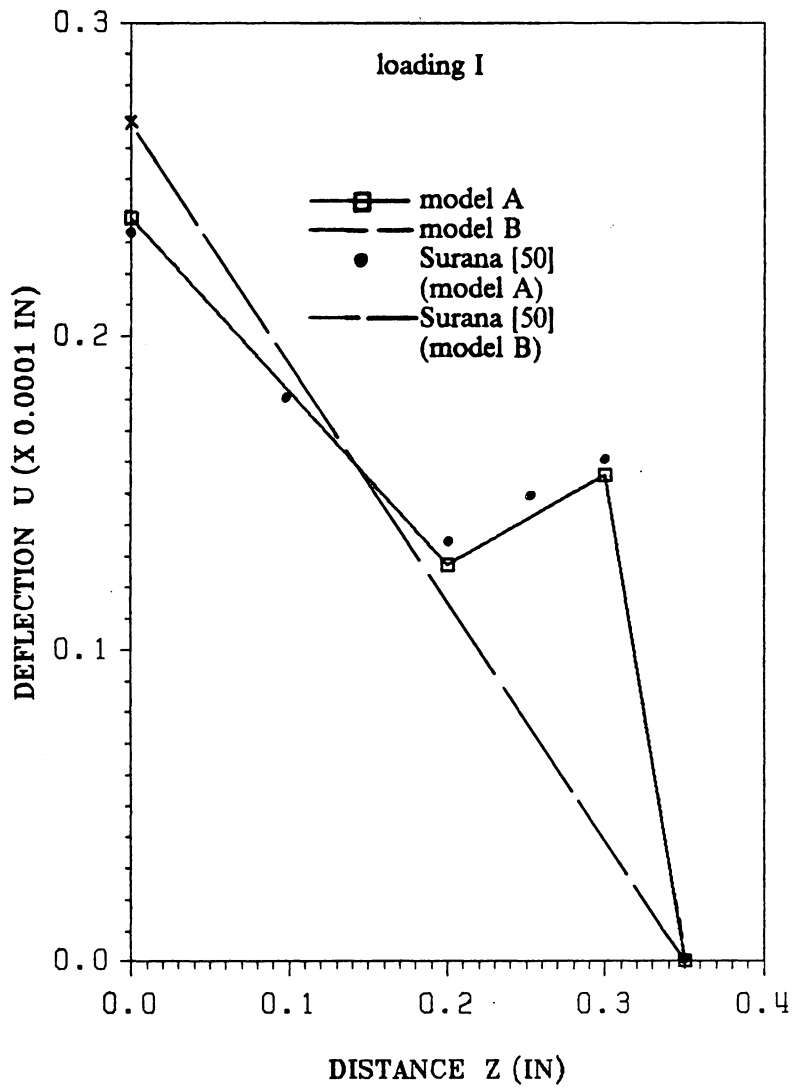


Figure 6.37 Deflection u along the depth of the plate at $x = 2$ in

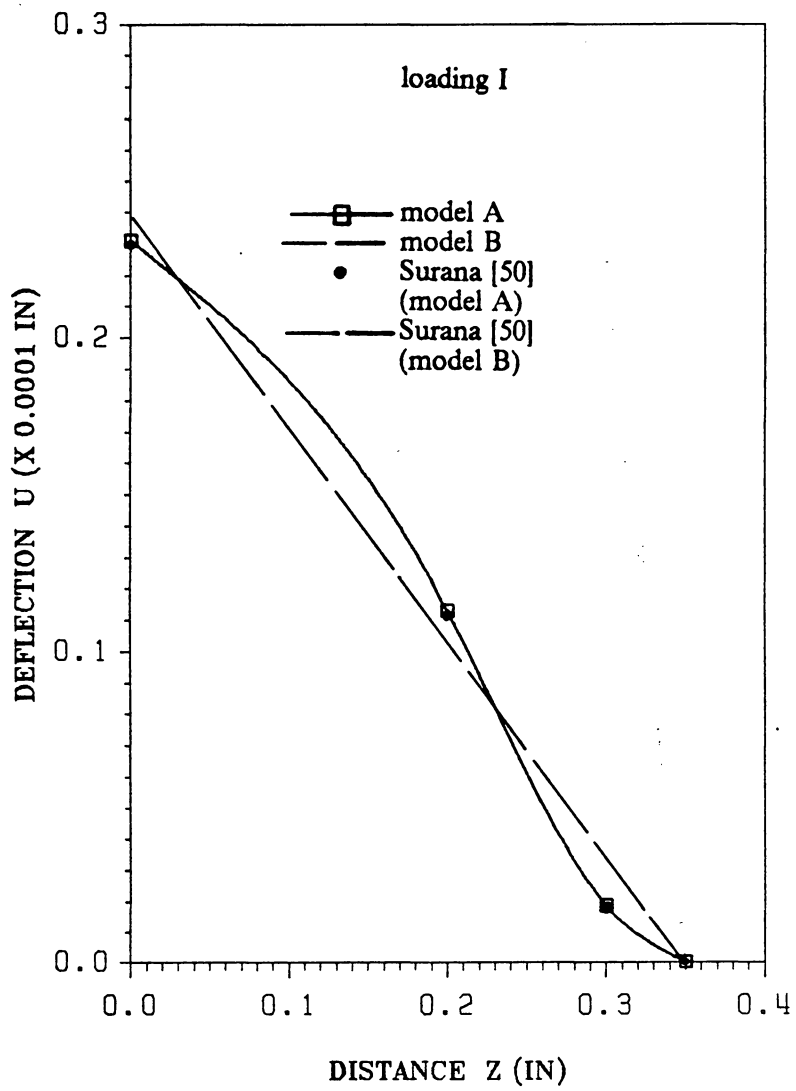


Figure 6.38 Deflection u along the depth of the plate at $x = 1.75$ in

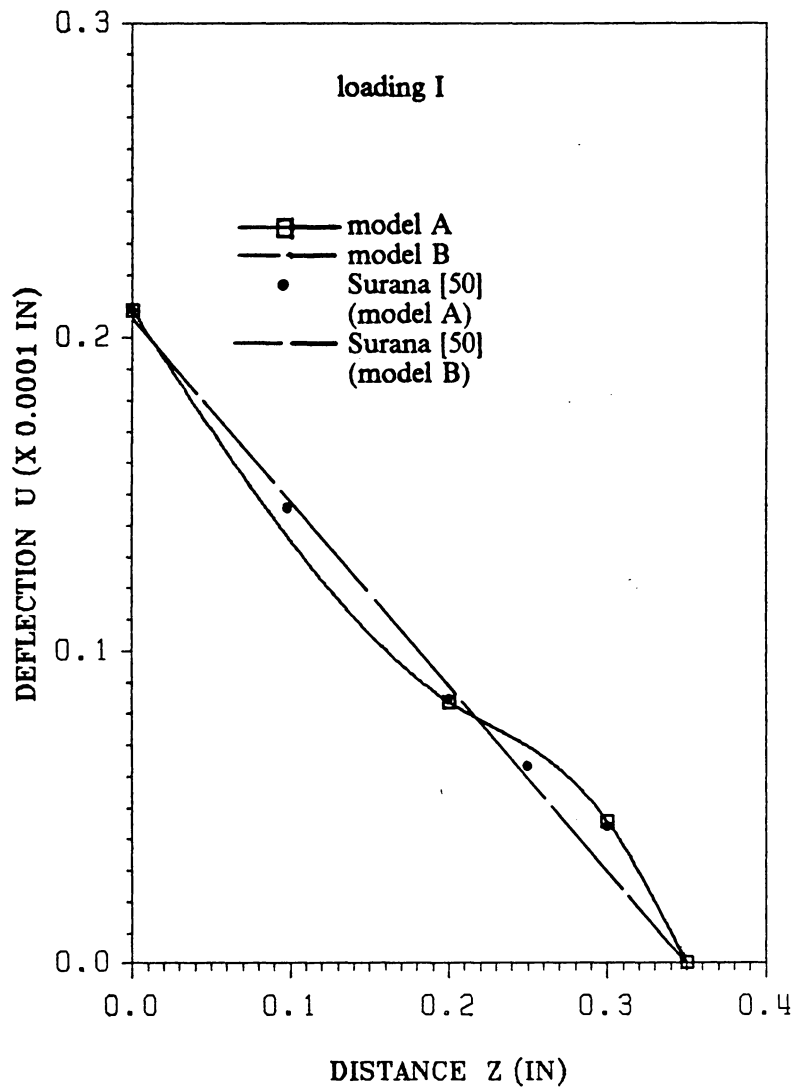


Figure 6.39 Deflection u along the depth of the plate at $x = 1.5$ in

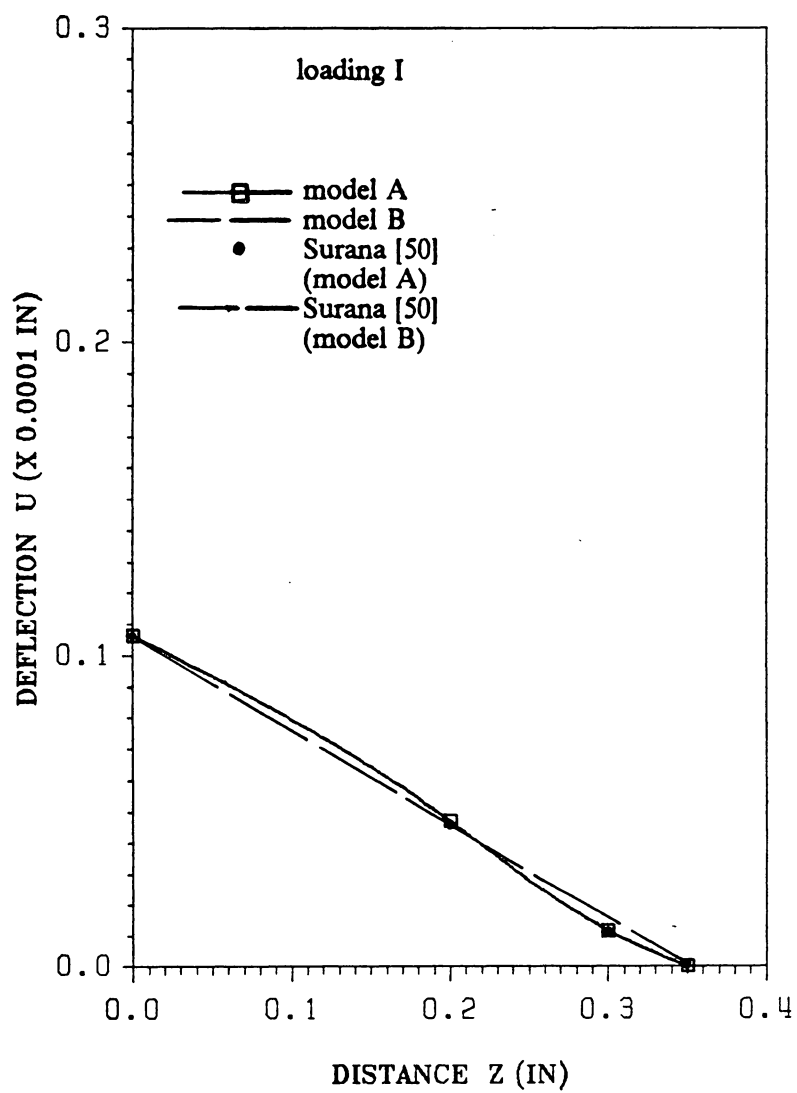


Figure 6.40 Deflection u along the depth of the plate at $x = 0.75$ in

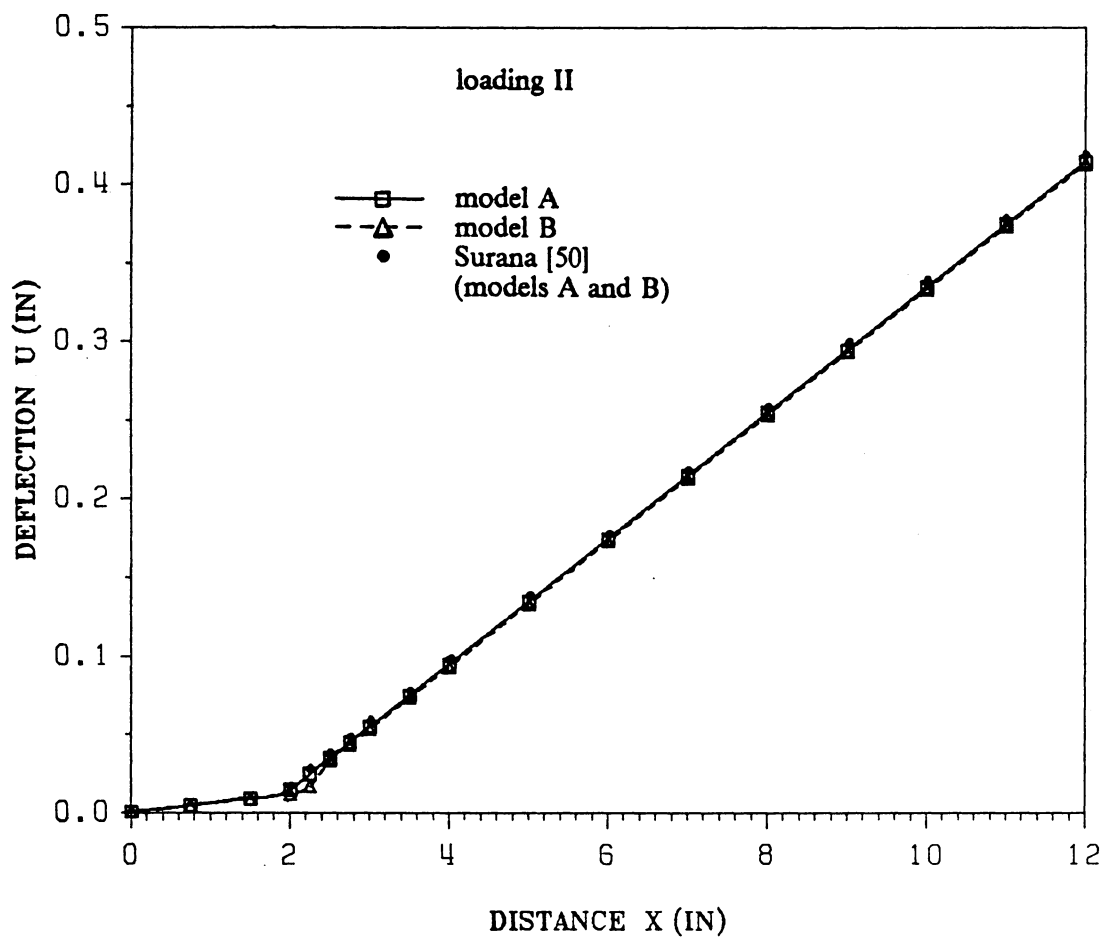


Figure 6.41 Deflection u along the length of the plate

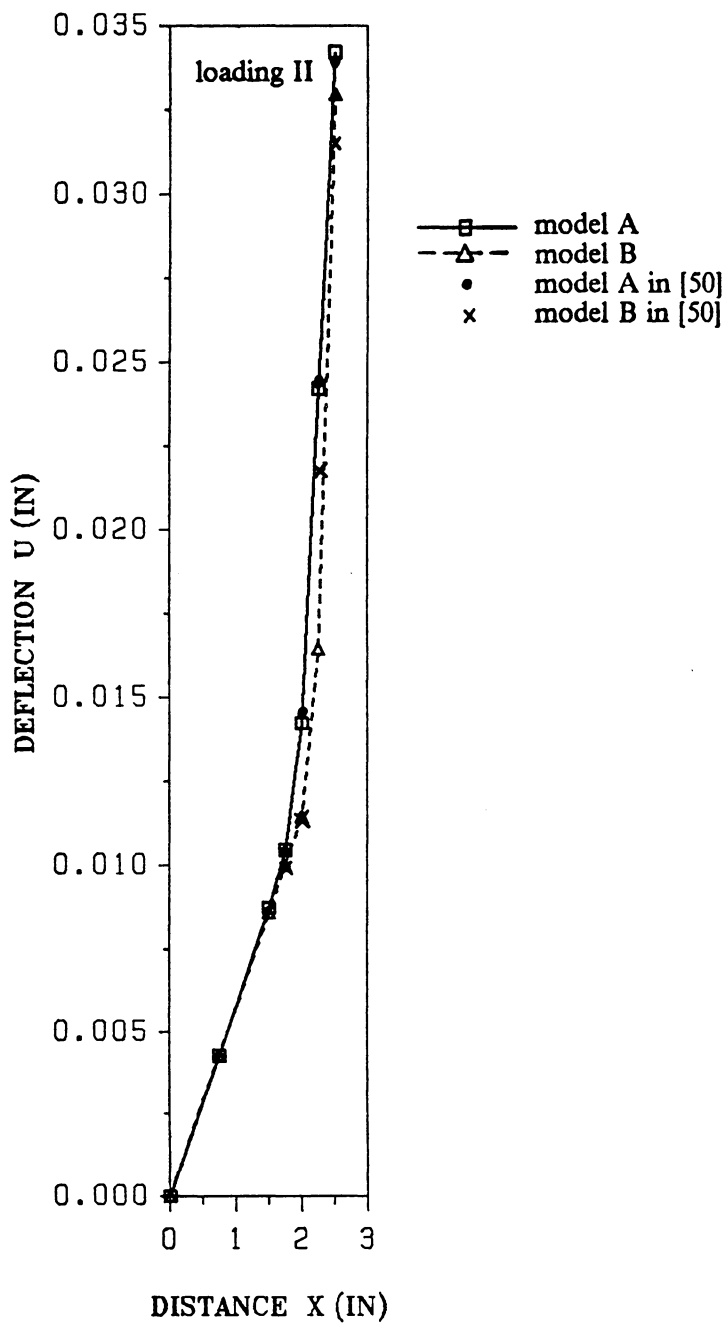


Figure 6.42 Deflection u along the length of the plate from $x = 0$ in to $x = 2.5$ in

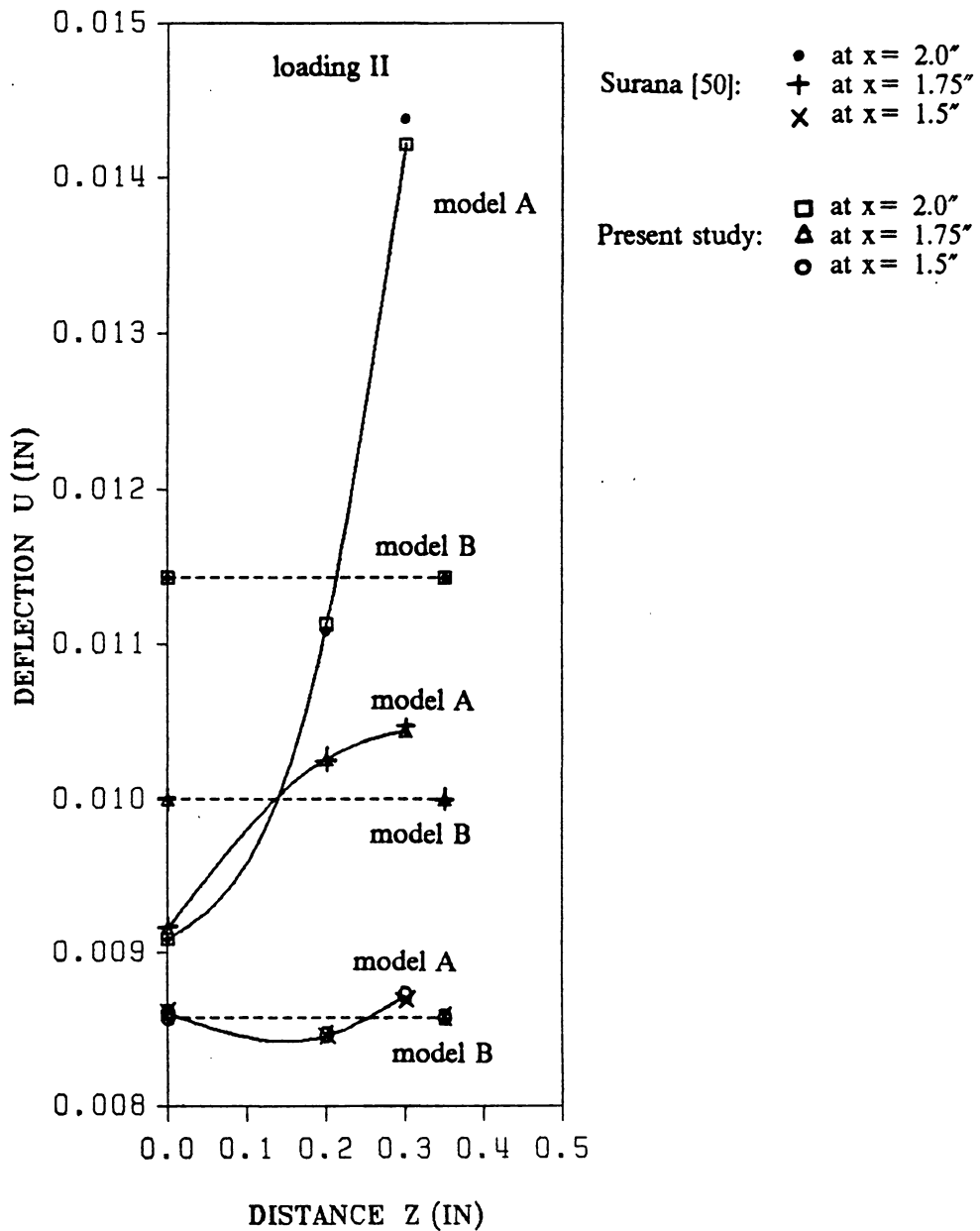


Figure 6.43 Deflection u along the depth of the plate at $x = 2$ in, 1.75 in, 1.5 in

The predicted bending stresses of the folded roof in the present study are shown in Fig. 6.45, which are in good agreement with the results obtained with 2-D fine mesh by Bathe and Ho [49]. With the same accuracy the model employing transition element has much less degrees-of-freedom in analysis as compared with the 2-D model and does not need the constraint equations at the transition regions. Large deformation analysis of this folded plate was also performed here. The loading is the same. Fig. 6.46 shows the results of geometric nonlinear analysis compared with the linear solution. In the load range shown the roof does not exhibit high geometric nonlinearity.

3. A simply-supported beam with nonuniform thickness under a center point load

Figure 6.47 shows the geometry of the nonuniform beam and its finite element models. The material properties are:

$$E = 3 \times 10^7 \text{ psi}, \quad \nu = 0, \quad \rho = 2.5393 \times 10^{-4} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}^4}$$

Static response of this beam is studied by using two finite element models. Model A in Fig. 6.47(b) combines three 16-node solid elements, one 11-node solid-shell transition element and one 8-node degenerated shell element. In model B of Fig. 6.47(c) we use three 8-node shell elements. For linear analysis, when $p = -0.6 \text{ lb}$ the center deflections are:

$$-0.010882 \text{ in} \quad \text{for model A}$$

$$-0.011791 \text{ in} \quad \text{for model B}$$

$$-0.011171 \text{ in} \quad \text{for classical beam theory}$$

The geometric nonlinear solutions of these two models are shown in Fig. 6.48. The load-deflection curves of these two models are close to each other. From the results we can see that this nonuniform simply-supported beam becomes stiffer as the loading increases.

4. A layered composite simply-supported beam with nonuniform thickness under a center point load

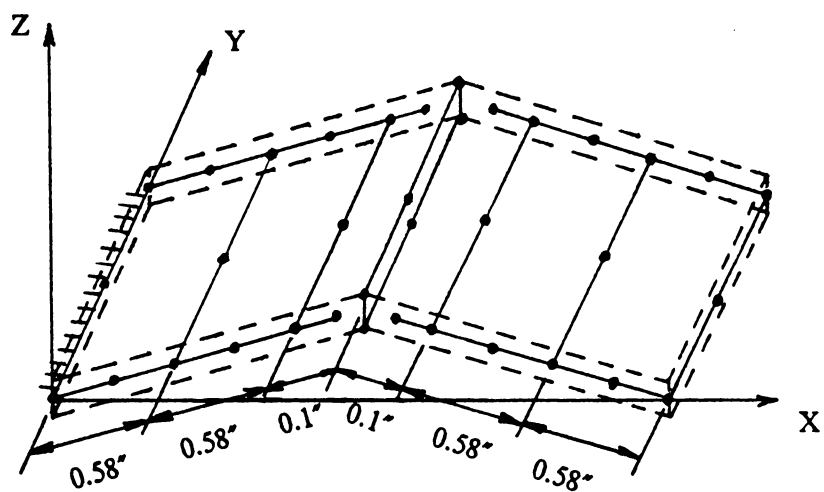
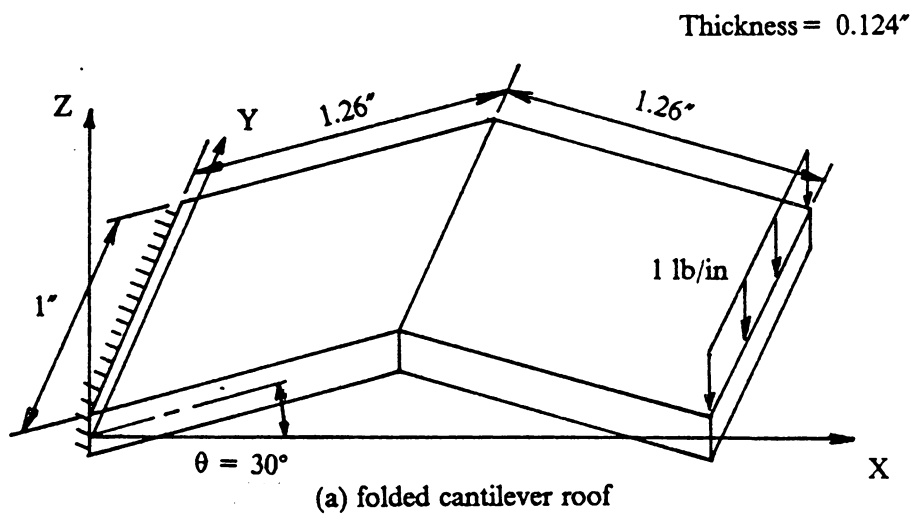


Figure 6.44 Folded cantilever roof under a line loading and its finite element model

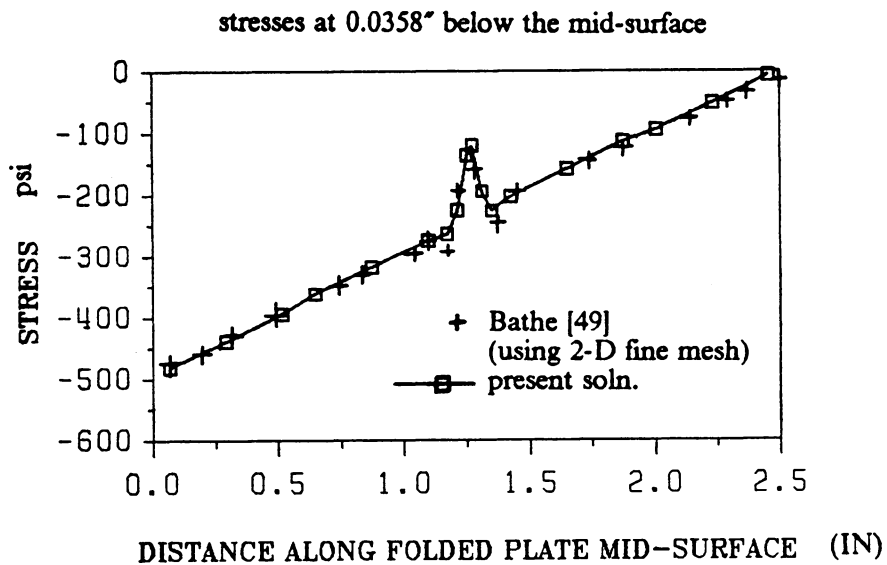
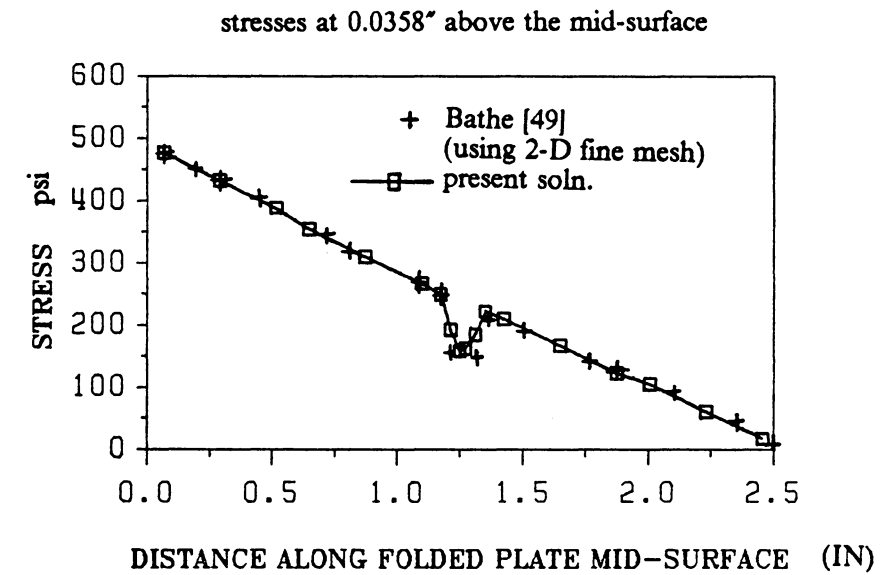


Figure 6.45 Bending stresses of folded roof along its length

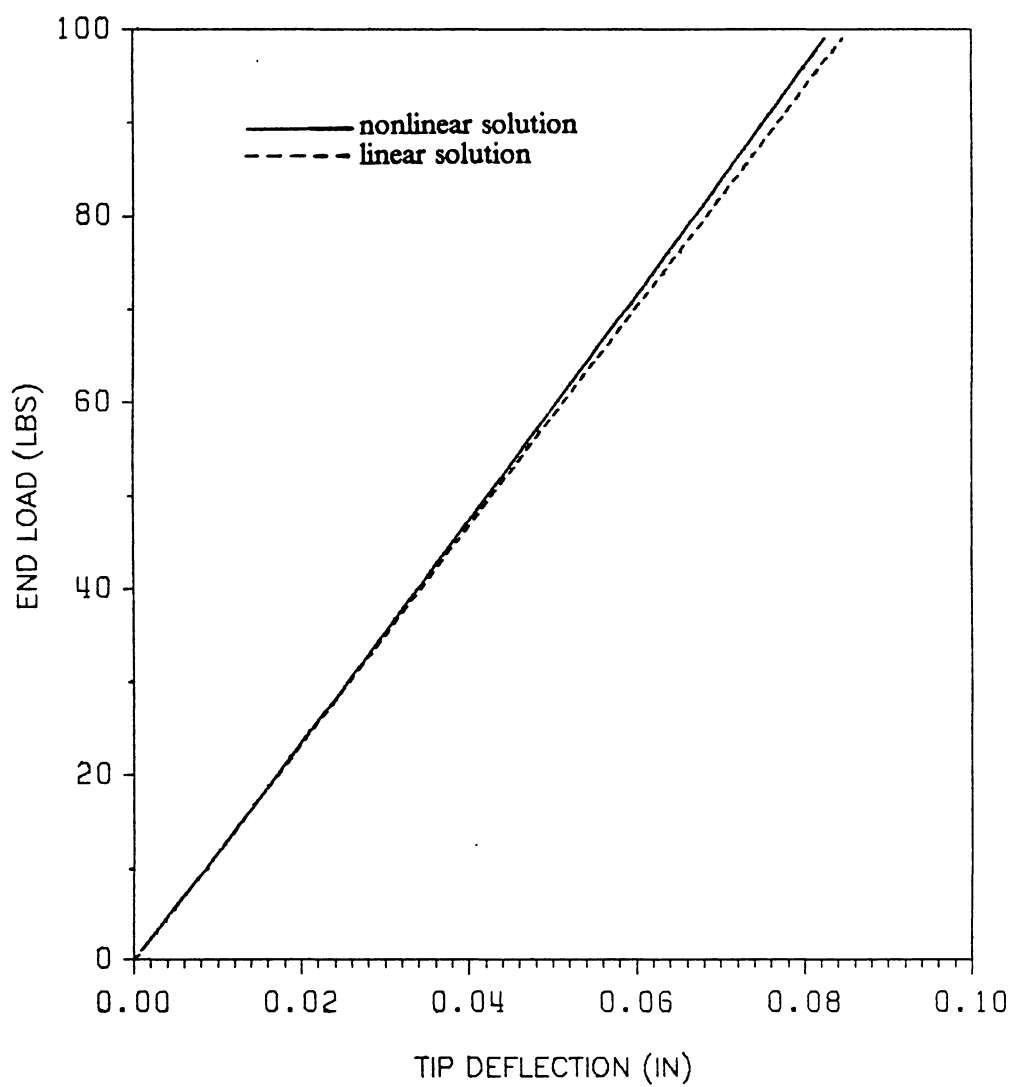
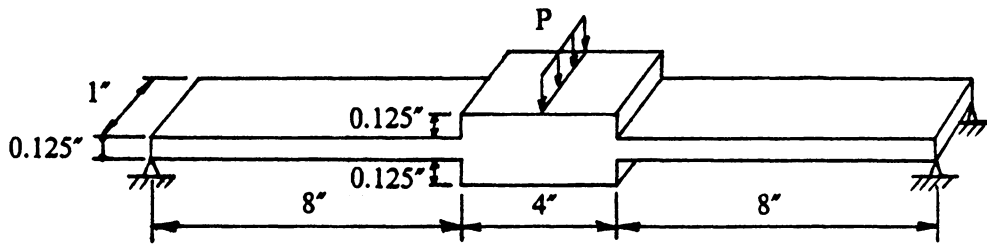
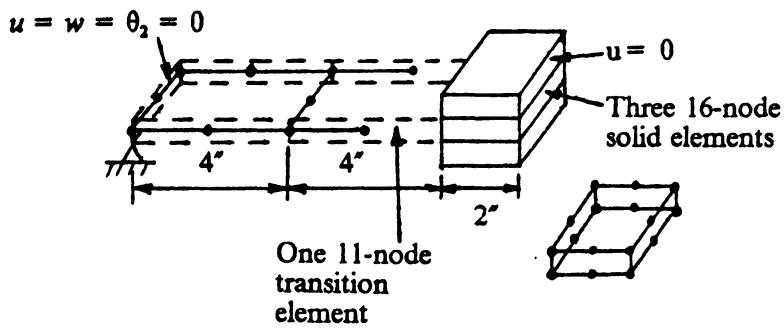


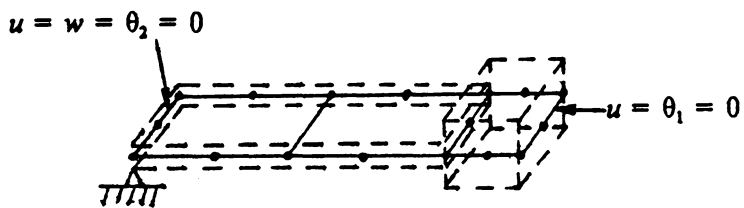
Figure 6.46 Bending of folded roof under end loading



(a)



(b) model A



(c) model B

Figure 6.47 Simply-supported beam with nonuniform thickness and its finite element models

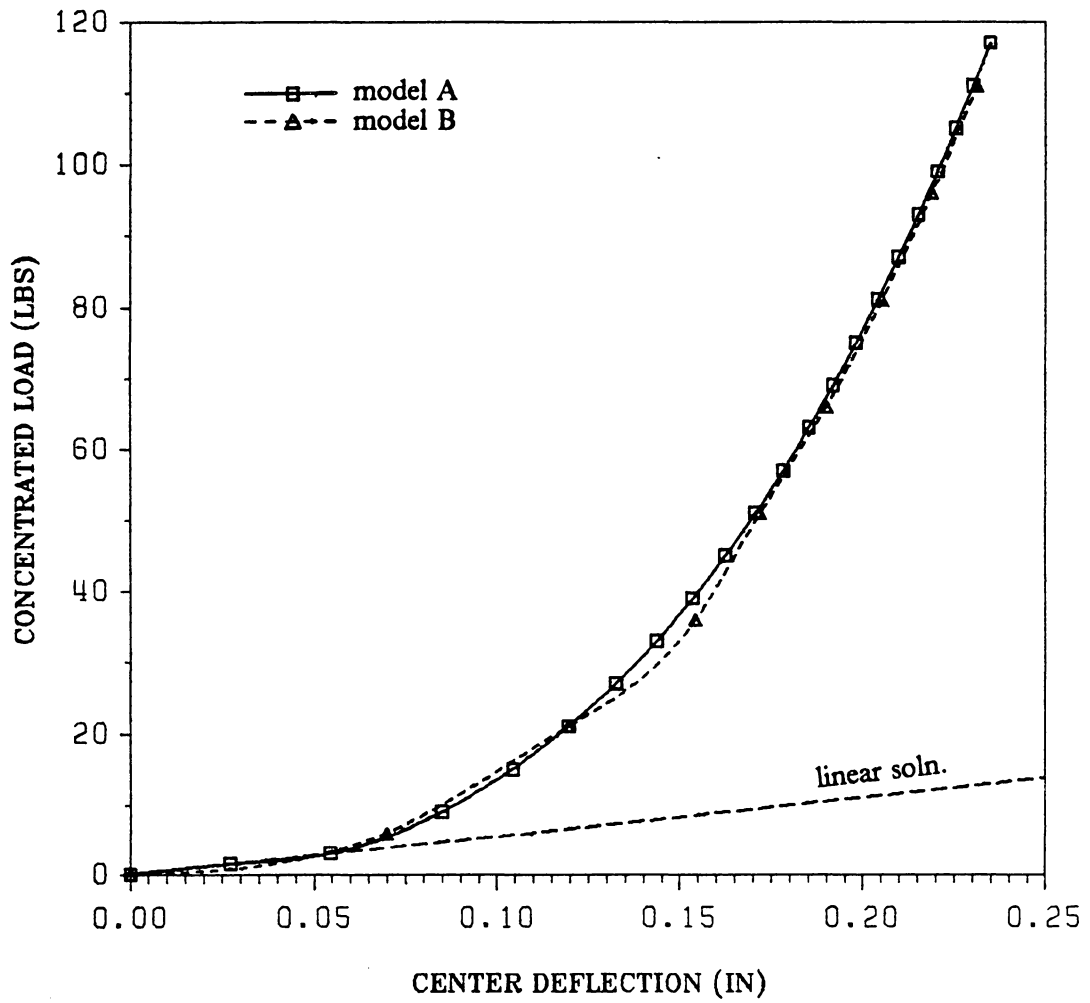
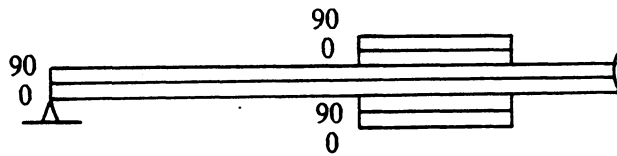


Figure 6.48 Bending of nonuniform beam under center point load

The geometry of the beam is the same as that in the last problem except that it is composed of 2-layer(0/90) and 6-layer (0/90/0/90/0/90) T300/5208 Graphite-epoxy material. The stacking sequence is shown below:



The material properties used are the same as those in problem 3 of Section 6.2.4. The finite element model shown in Fig. 6.47(b) was used to perform the geometric nonlinear analysis. Fig. 6.49 shows the loads versus the central deflection for the problem. From the results we see that the structure becomes softer in the small load range and then stiffer as the load increases.

6.3 Transient analysis

In all example problems considered in this section, zero initial conditions are assumed and no damping is included. Since the natural frequency of nonlinear vibration depends on the amplitude and no estimate on the time increment for the nonlinear dynamic analysis is available in the literature, a convergence study was conducted for each problem to select a time increment that yielded a stable and accurate solution while keeping the computational effort to a minimum. The following estimate, which is used

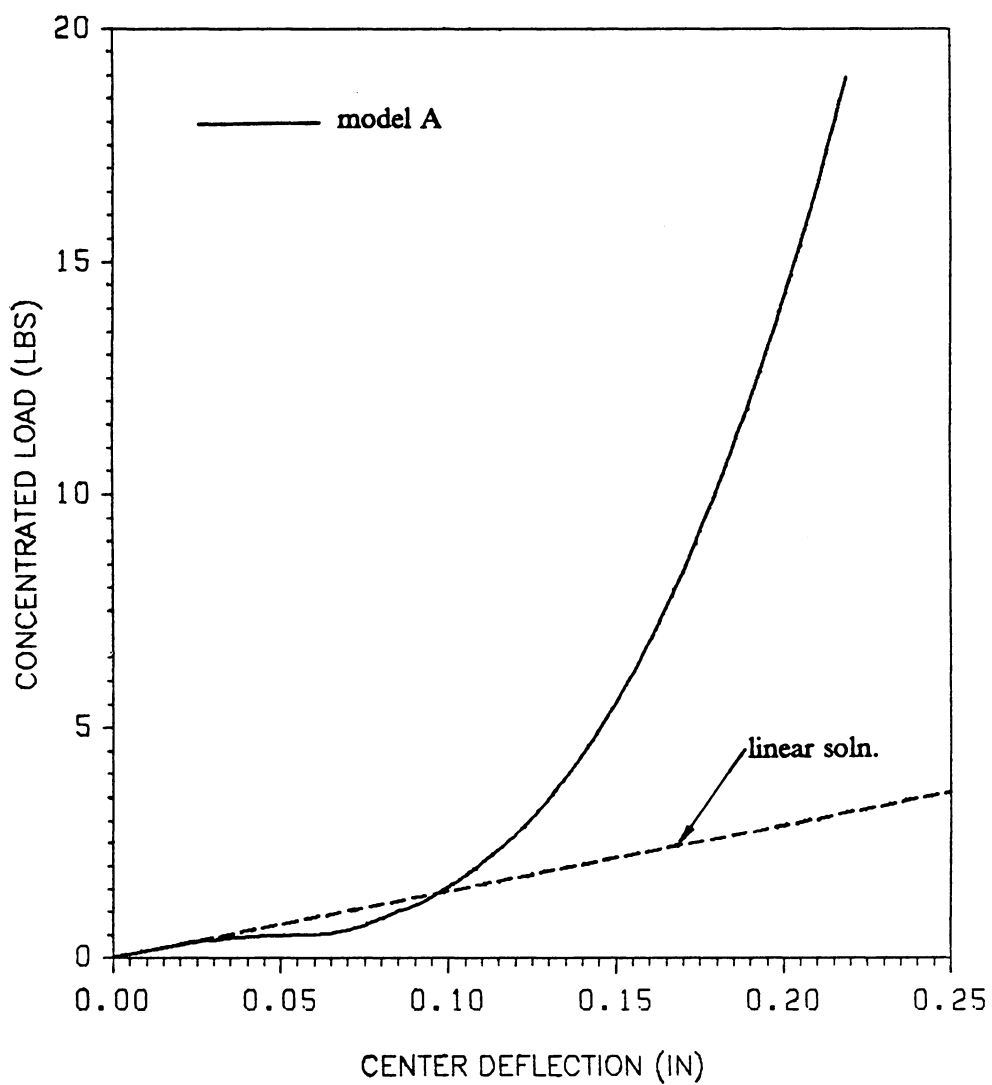


Figure 6.49 Bending of nonuniform composite beam under center point load

for linear analysis, was used as a starting time increment in the present study for shell-type structures

$$\Delta t \leq 0.25(\rho h/D)^{1/2} (\Delta x)^2$$

where ρ = density, h = thickness, D = smaller of D_{11} and D_{22} ,

$$(\text{for isotropic material } D = \frac{Eh^3}{12(1 - \nu^2)})$$

Δx = the minimum distance between any two global nodes of the mesh.

6.3.1 Plate and Shell Structures

1. Isotropic simply-supported plate under uniform loading

The geometry of the plate is shown in Fig. 6.50. The material properties used are:

$$E = 2.1 \times 10^6 \text{ N/cm}^2, \nu = 0.25, \rho = 8 \times 10^{-6} \frac{\text{N} - \text{sec}^2}{\text{cm}^4}$$

The time step is taken to be $\Delta t = 10 \mu\text{sec}$. The intensity of the step loading is $q = 10 \text{ N/cm}^2$. Due to symmetry, only one quarter of the plate is modelled with four 9-node shell elements. The results of the present nonlinear transient analysis are shown in Fig. 6.50, which are very close to that reported by Reddy [67]. In Fig. 6.50 the static nonlinear deflection at center for this loading is also shown.

2. 2-layer cross-ply(0/90) and angle-ply(45/-45) square plate under uniform loading

The geometry of the plate is shown in Fig. 6.51. The material properties used are:

$$E_x/E_y = 25, E_y = 7.031 \times 10^5 \text{ N/cm}^2, G_{xy}/E_y = 0.5, G_{xy} = G_{xz},$$

$$G_{yz}/E_y = 0.2, \nu_{xy} = 0.25, \rho = 2.547 \times 10^{-6} \text{ N s}^2/\text{cm}^4$$

The time step used is $\Delta t = 0.002 \text{ sec}$. The intensity of the step loading is $q = -50 \times 10^{-4} \text{ N/cm}^2$. The edge boundary conditions and symmetry conditions used for cross-ply and angle-ply plates are BC3 and BC2, respectively. Hence quarter-plate model can be used for both cases. Four 9-node shell elements are contained in the mesh. The results of the present nonlinear transient analyses for both plates are shown in Fig. 6.51, which are very close to those reported by Reddy [68]. In Fig. 6.51 the static nonlinear deflections at center for this loading are also shown.

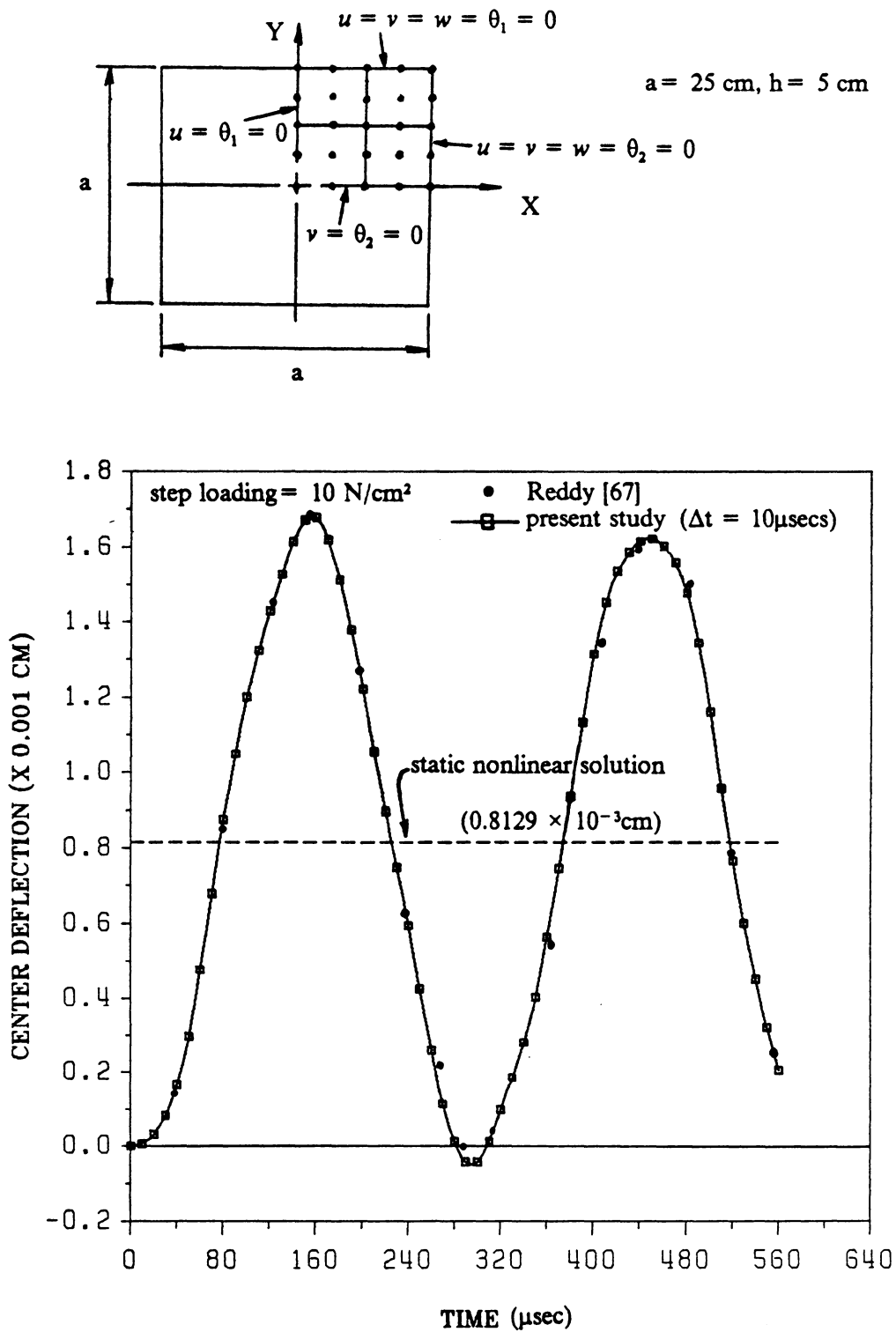


Figure 6.50 Nonlinear transient response of simply-supported plate under uniform loading

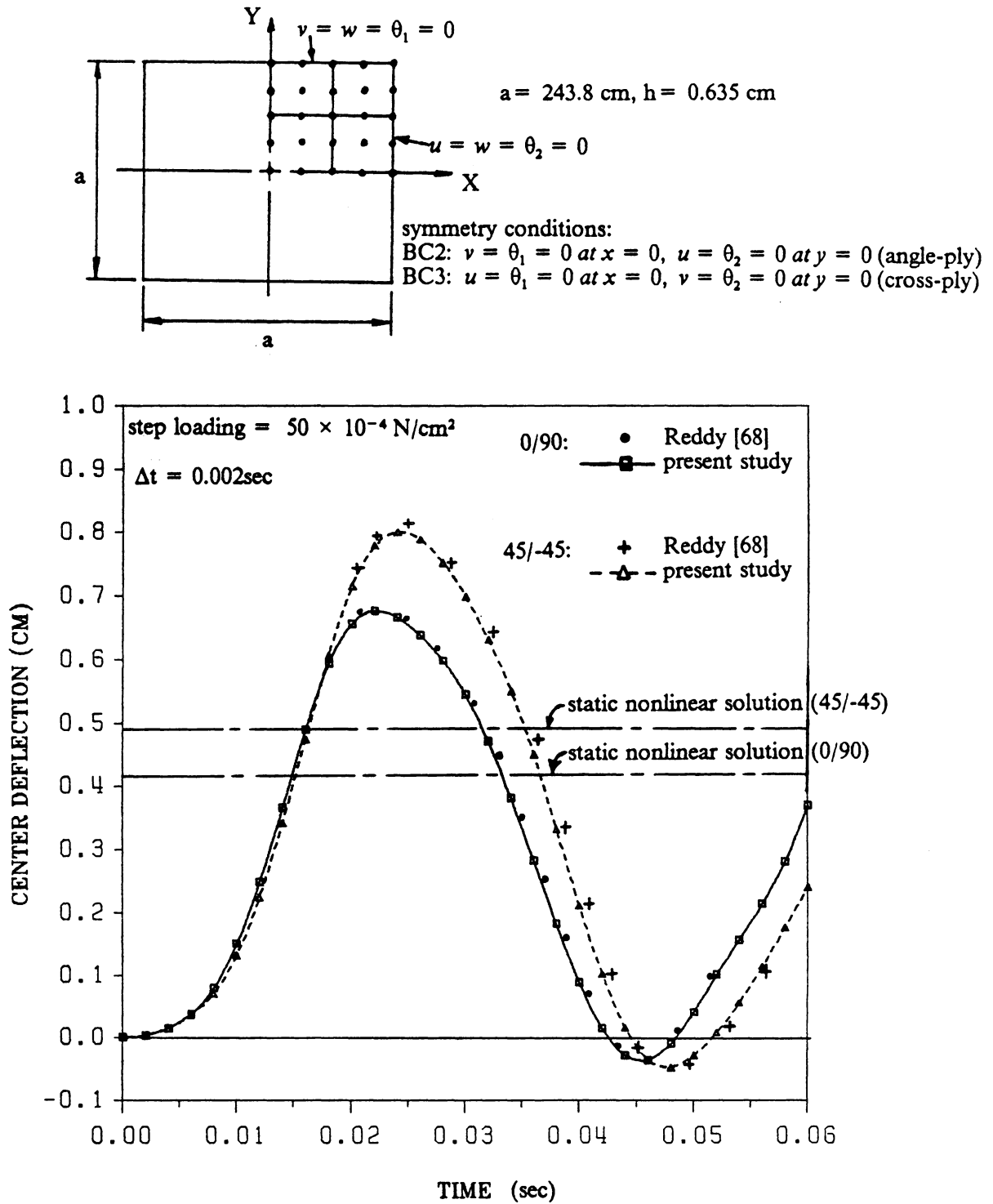


Figure 6.51 Nonlinear transient responses of 2-layer cross-ply and angle-ply simply-supported plates under uniform loading

6.3.2 Beam Structures

1. Nonlinear transient response of a clamped isotropic beam

The geometry of the beam is shown in Fig. 6.52. The material properties used are:

$$E = 3 \times 10^7 \text{psi}, \quad \nu = 0.0, \quad \rho = 2.5393 \times 10^{-4} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}^4}$$

The intensity of the step load is $q = 640$ lbs, and the time step is $\Delta t = 50 \mu\text{sec}$. Due to symmetry, only one-half of the beam was modelled with two 3-node beam elements. The results of the present nonlinear transient analysis are shown in Fig. 6.52, which are compared to the results of Mondkar and Powell [69], who used 8-node plane stress elements. In Fig. 6.52 the static nonlinear deflection at center for this loading is also shown to be compared with the transient response.

2. Nonlinear transient response of a shallow circular arch

The arch has the same material properties and geometry as that of problem 9 in Section 6.2.2 and are subjected to a center concentrated force. Also the arch was modelled with the same mesh. The step load is 37600 lbs. The time step is $50 \mu\text{sec}$. Fig. 6.53 shows the nonlinear transient response and the nonlinear static solution at this step load.

3. Nonlinear transient analysis of arch 1 of problem 10 in Section 6.2.2

Same finite element model as that in static analysis was used. Since this arch will buckle under center point load, two step loadings were applied respectively. One is 7500 lbs which is below buckling load. The other is 15000 lbs which is larger than buckling load. The time step is 0.0002 sec. The nonlinear transient responses of these two step loadings and their corresponding nonlinear static solutions are shown in Fig. 6.54.

4. Nonlinear transient analysis of 2-layer cross-ply(0/90) arch of problem 11 in Section 6.2.2

Same finite element model as that in static analysis was used. Two step loadings were applied respectively. One is 3500 lbs. The other is 4500 lbs. The time step is 0.0002 sec. The nonlinear transient responses of these two step loadings and their corresponding

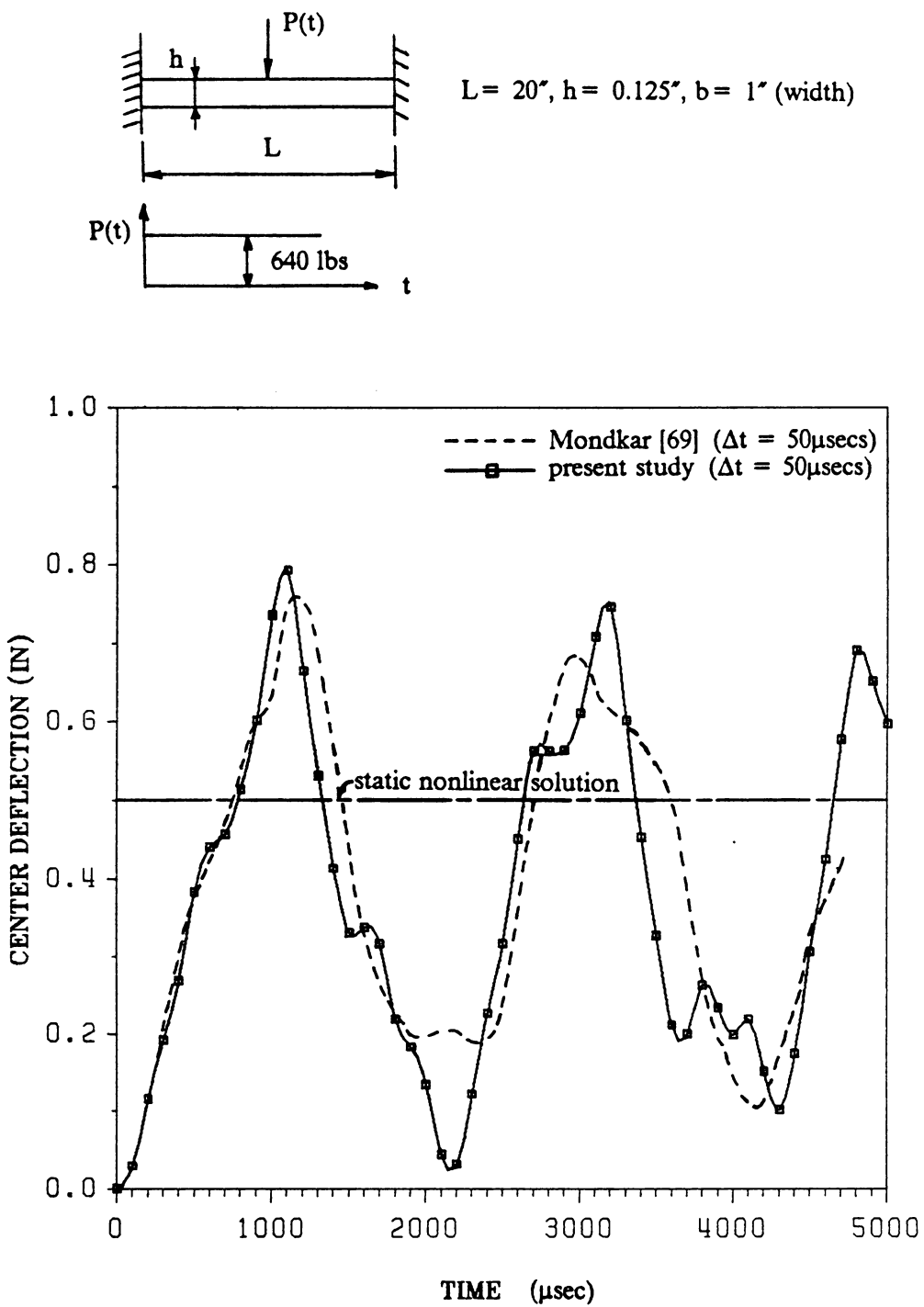


Figure 6.52 Nonlinear transient response of clamped beam under a center point load

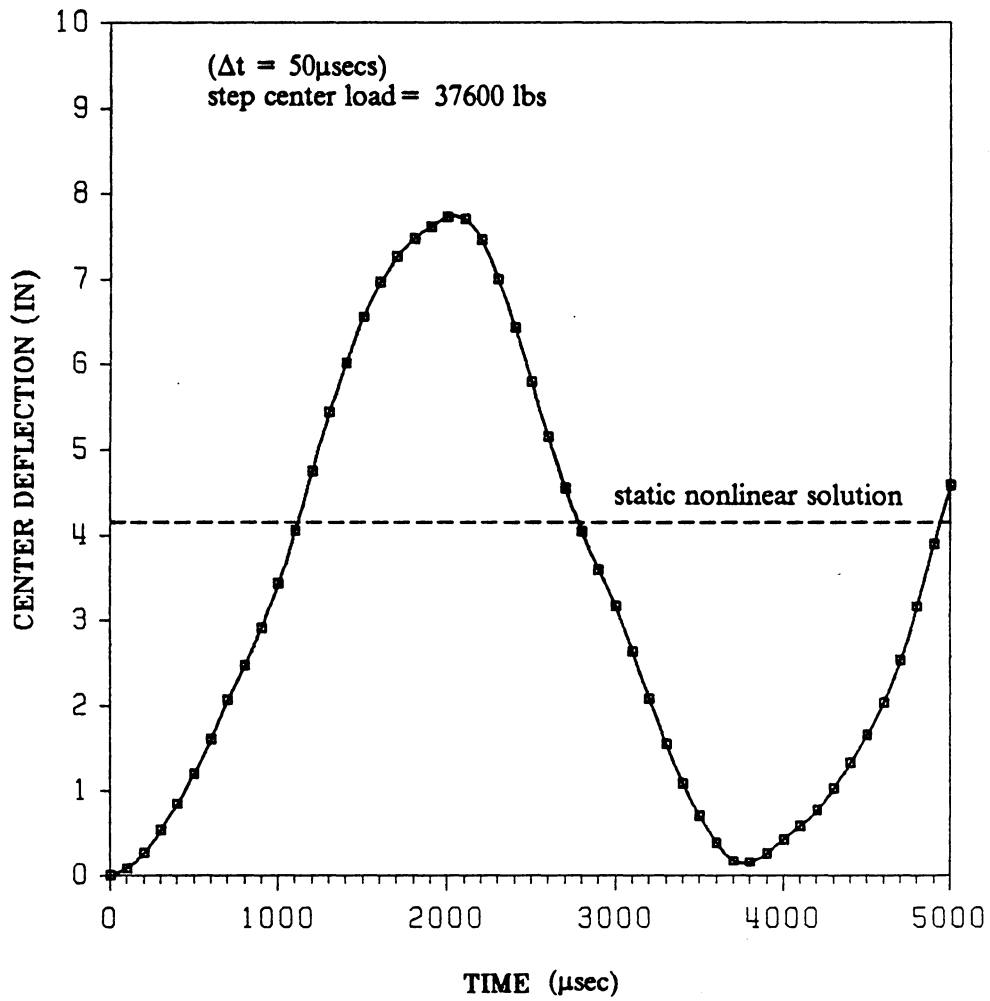


Figure 6.53 Nonlinear transient response of shallow circular arch under a center point load

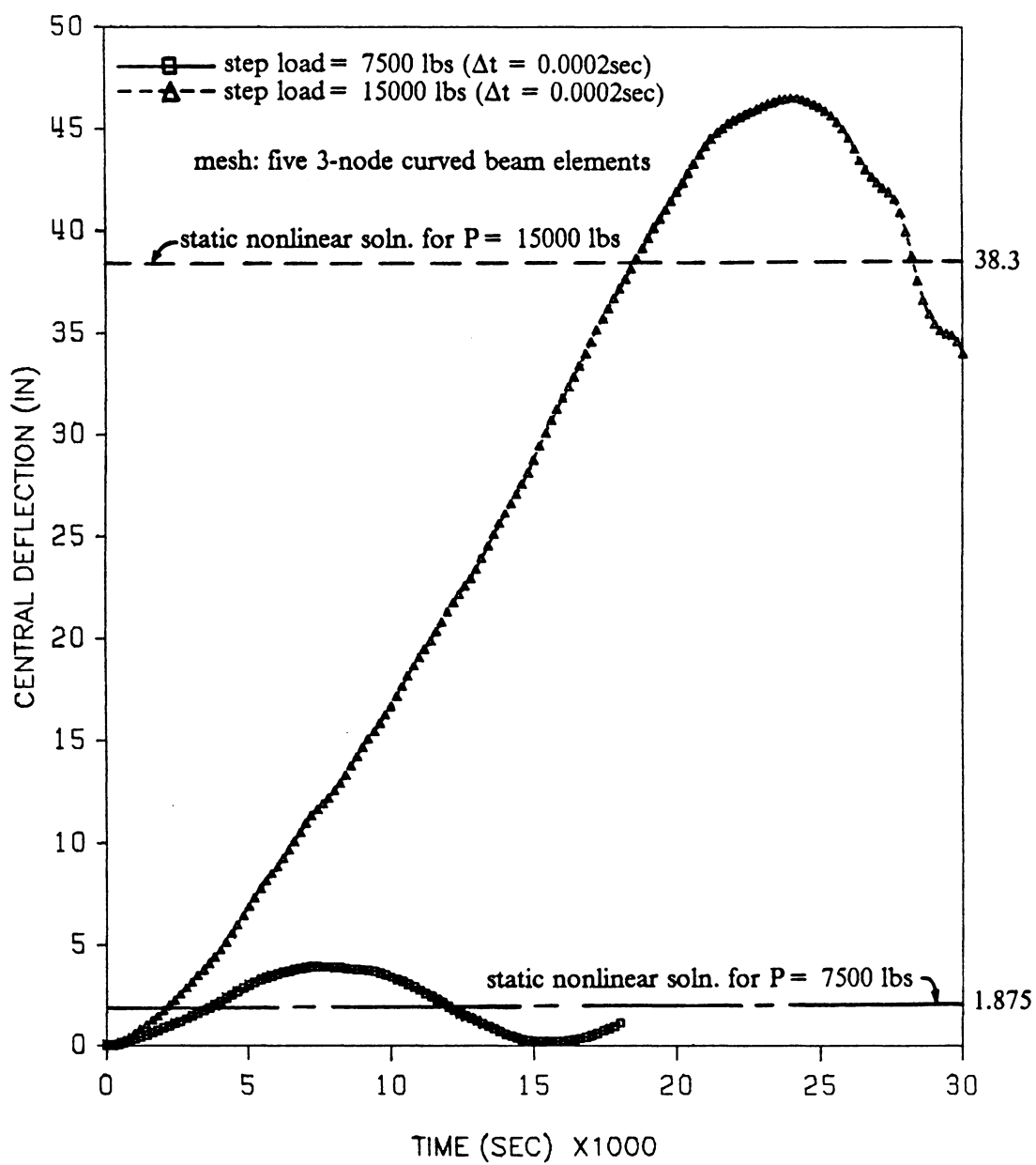


Figure 6.54 Nonlinear transient response of arch 1 under a center point load

nonlinear static solutions are shown in Fig. 6.55. The period and amplitude of transient response increase when the step load increases.

6.3.3 Stiffened Plates and Shells

1. Nonlinear transient analysis of problem 7 in Section 6.2.3

The same mesh as that for static analysis was used. The step uniform pressure is -3.75 psi. The time step $\Delta t = 0.0002$ sec. The nonlinear transient responses of the stiffened and non-stiffened plates and the corresponding static nonlinear solutions are shown in Fig. 6.56.

2. Nonlinear transient analysis of problem 8 in Section 6.2.3

The same mesh was used. The nonlinear transient responses of three step loadings, 5000 lbs, 8000 lbs and 10000 lbs, and the corresponding static nonlinear solutions are shown in Fig. 6.57. The time step $\Delta t = 0.0007$ sec. Note that the period and amplitude of transient response increase when the step load increases.

3. Nonlinear transient analysis of problem 9 in Section 6.2.3

The same mesh and boundary conditions as that for static analysis were used. The step uniform pressure is $-50 \times 10^{-4} N/cm^2$. The time step $\Delta t = 0.002$ sec. The nonlinear transient responses of the stiffened and non-stiffened plates and the corresponding static nonlinear solutions are shown in Figures 6.58 and 6.59.

4. Nonlinear transient analysis of problem 10 in Section 6.2.3

The same 2-layer cross-ply spherical shell was analyzed for the transient response. The same mesh and boundary conditions were used. The step uniform load is -3.25 psi. The time step is $\Delta t = 0.0005$ sec. The nonlinear transient responses of the stiffened and non-stiffened plates and the corresponding static nonlinear solutions at this step loading are shown in Figures 6.60. Note that this step loading already exceeds the critical load

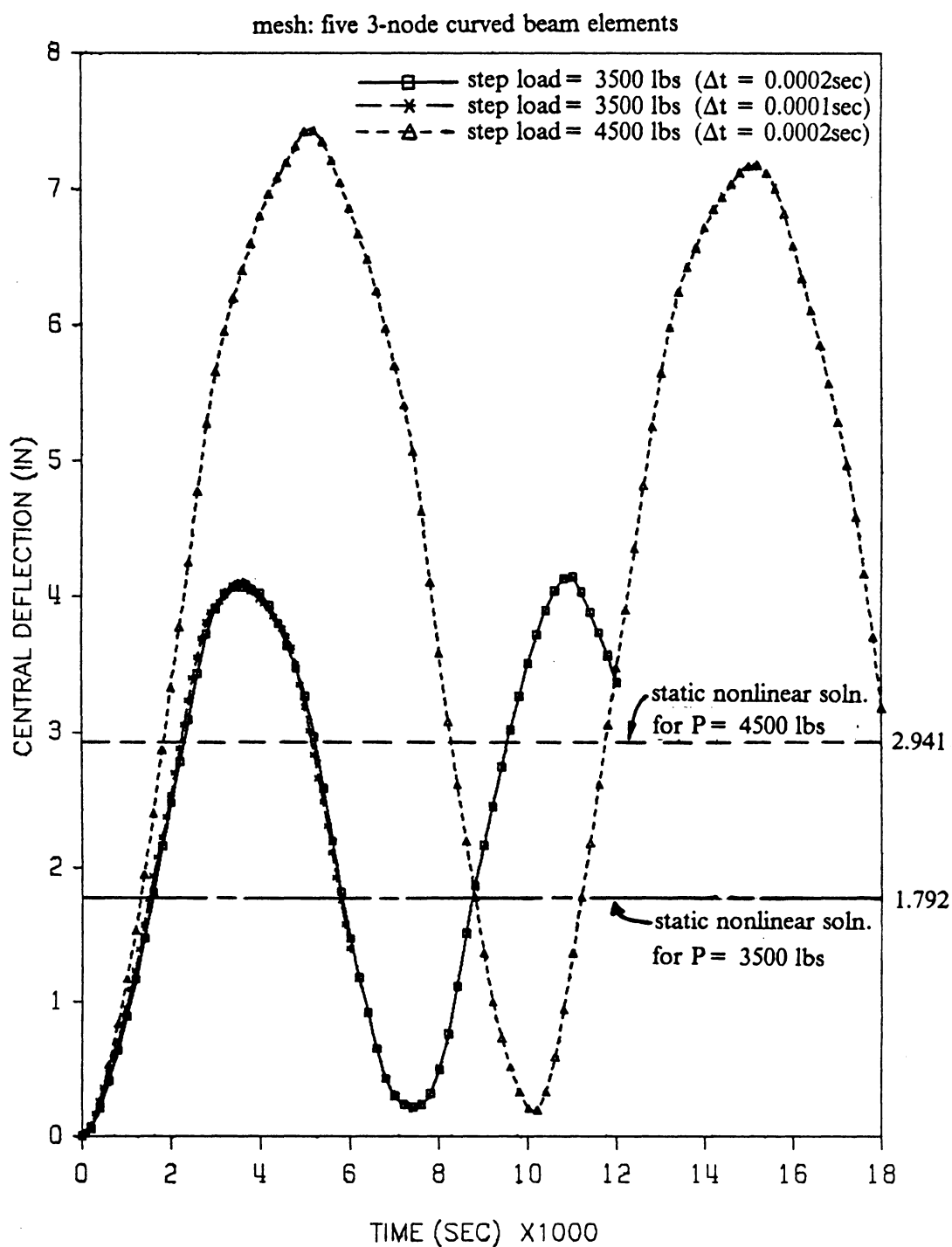


Figure 6.55 Nonlinear transient response of 2-layer cross-ply shallow arch under a center point step load

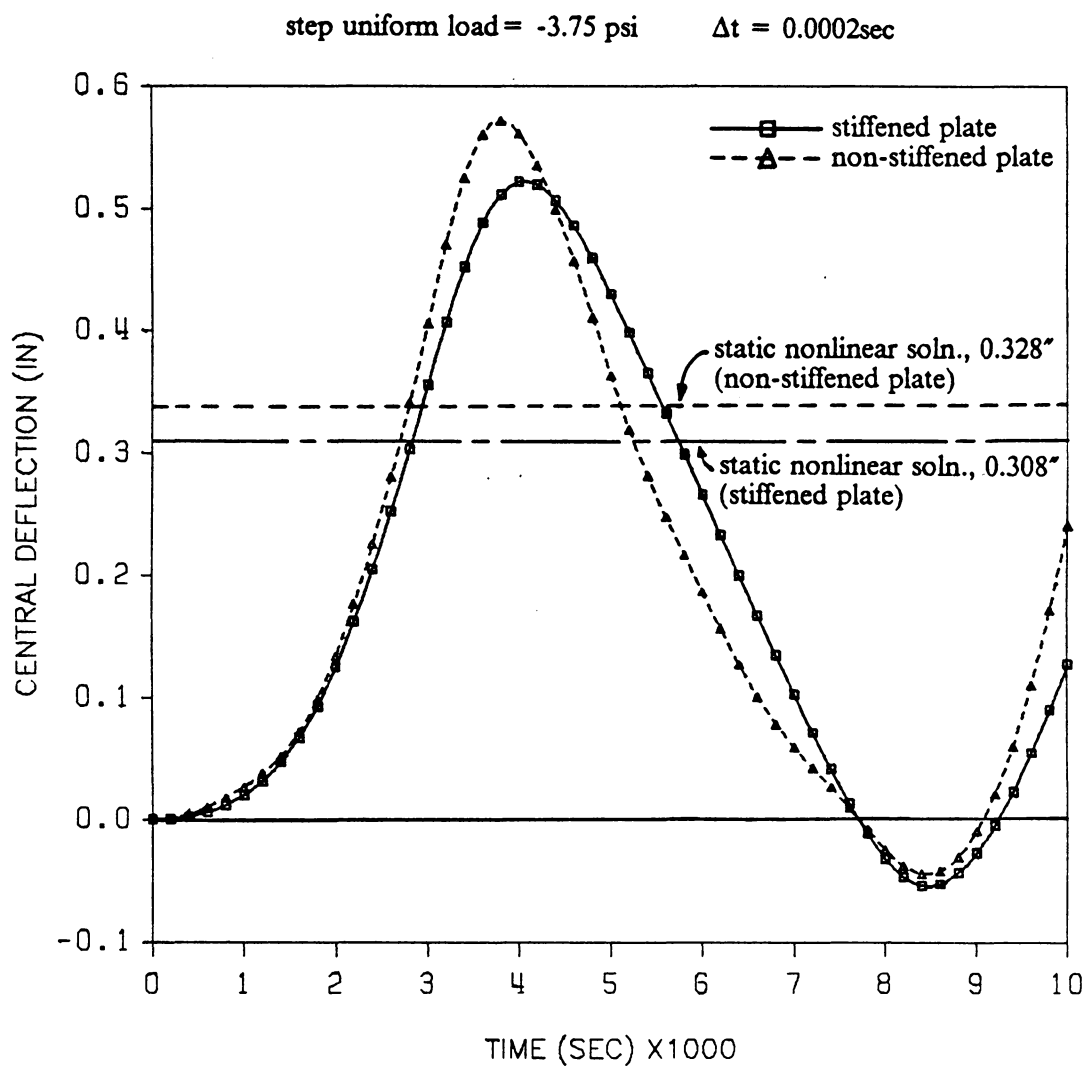


Figure 6.56 Nonlinear transient response of simply-supported stiffened plate under a step uniform pressure

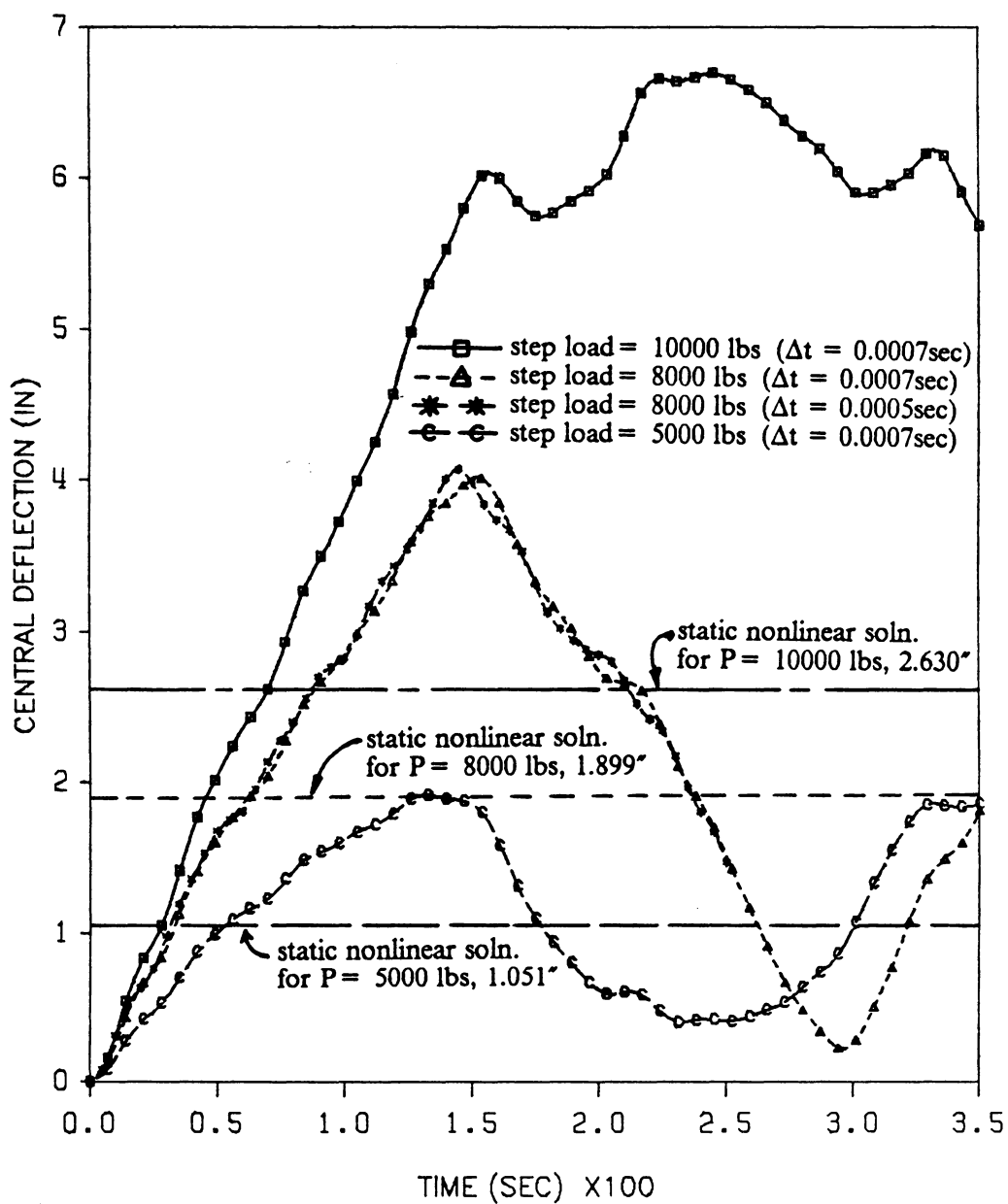


Figure 6.57 Nonlinear transient responses of simply-supported stiffened spherical shell under a step point load

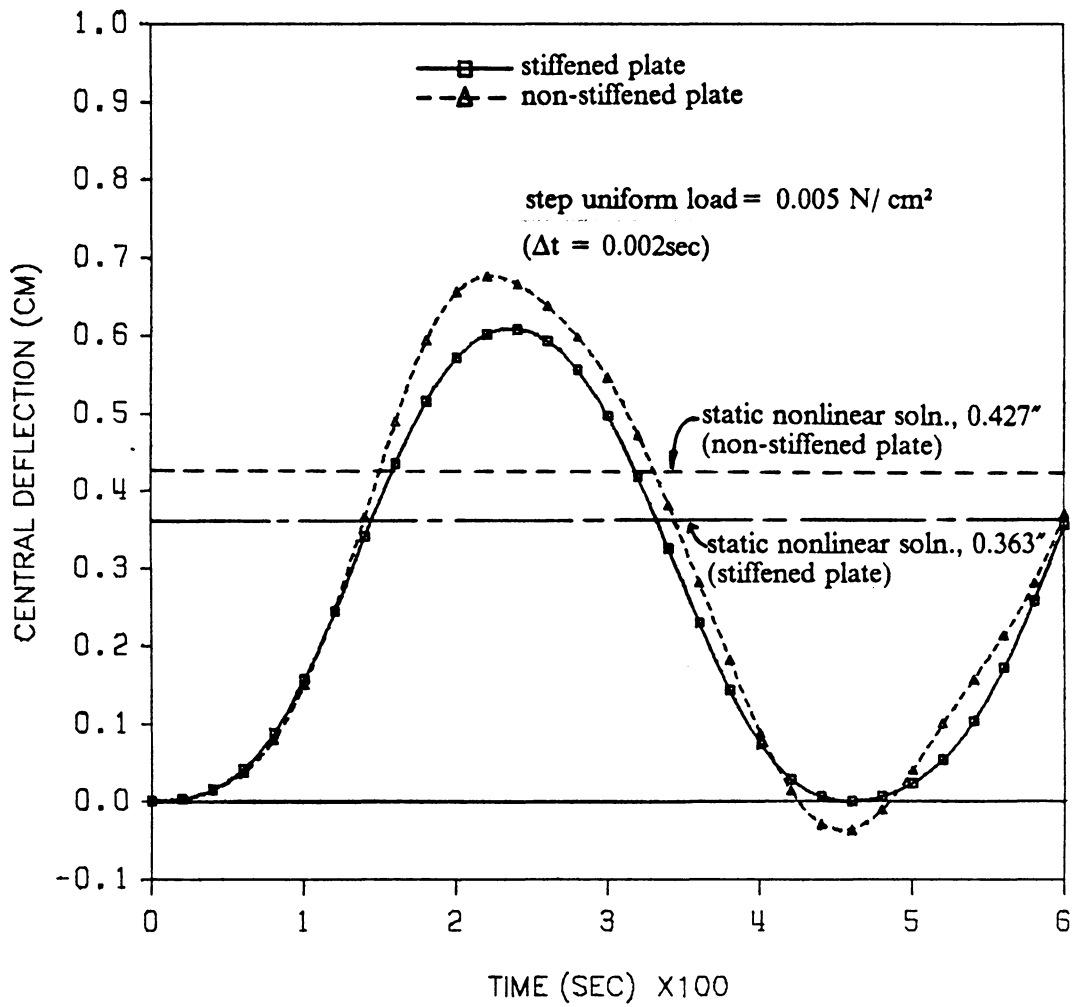


Figure 6.58 Nonlinear transient response of simply-supported 2-layer cross-ply stiffened plate under a step uniform pressure

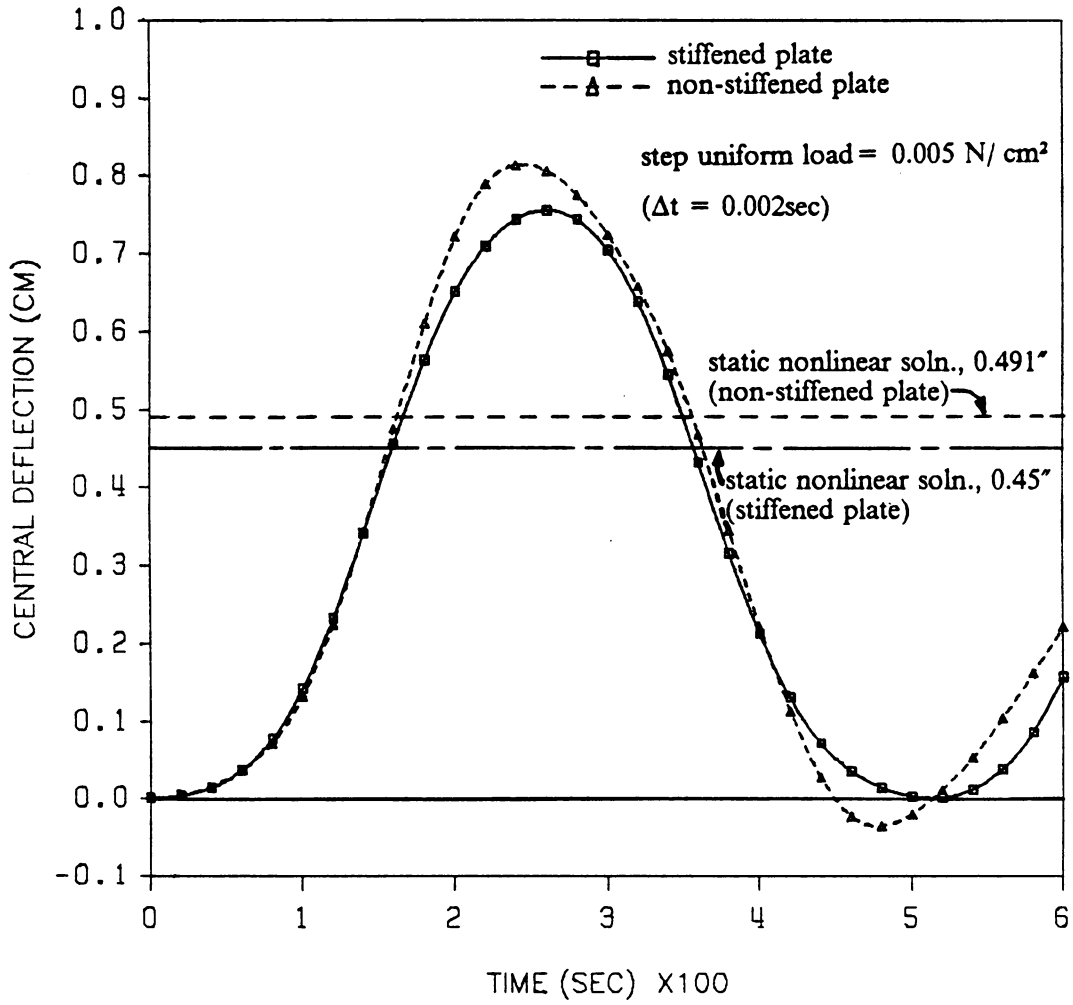


Figure 6.59 Nonlinear transient response of simply-supported 2-layer angle-ply stiffened plate under a step uniform pressure

of the non-stiffened shell and that the static and dynamic deflections of non-stiffened shell are much larger than that for stiffened shell.

6.3.4 Applications of Solid-shell Transition Element

1. Nonlinear transient analysis of problem 2 in Section 6.2.4

Two finite element models of static analysis were used again for transient analysis. The step point load is 201 lbs. The time step $\Delta t = 0.00005$ sec for the model with 8-node degenerated shell element, $\Delta t = 0.00004$ sec for the model with 11-node solid-shell transition element. The transient responses of both models are shown in Fig. 6.61 and close to each other. The corresponding static nonlinear solution at this load is also shown in the figure.

2. Nonlinear transient analysis of problem 3 in Section 6.2.4

Two finite element models of static analysis were used again for transient analysis. The step point load is 15 lbs. The time step $\Delta t = 0.00005$ sec for the model with 8-node degenerated shell element, $\Delta t = 0.00004$ sec for the model with 11-node solid-shell transition element. The transient responses of both models are shown in Fig. 6.62 and close to each other. The corresponding static nonlinear solution at this load is also shown in the figure.

6.3.5 General Shell-Type Structures

1. Nonlinear transient analysis of problem 2 in Section 6.2.5

The same finite element model of static analysis was used again for transient analysis. The step load is 65 lbs. The time step $\Delta t = 0.00001$ sec. The nonlinear transient response and the corresponding static nonlinear solution at this load step are shown in Fig. 6.63.

2. Nonlinear transient analysis of problem 3 in Section 6.2.5

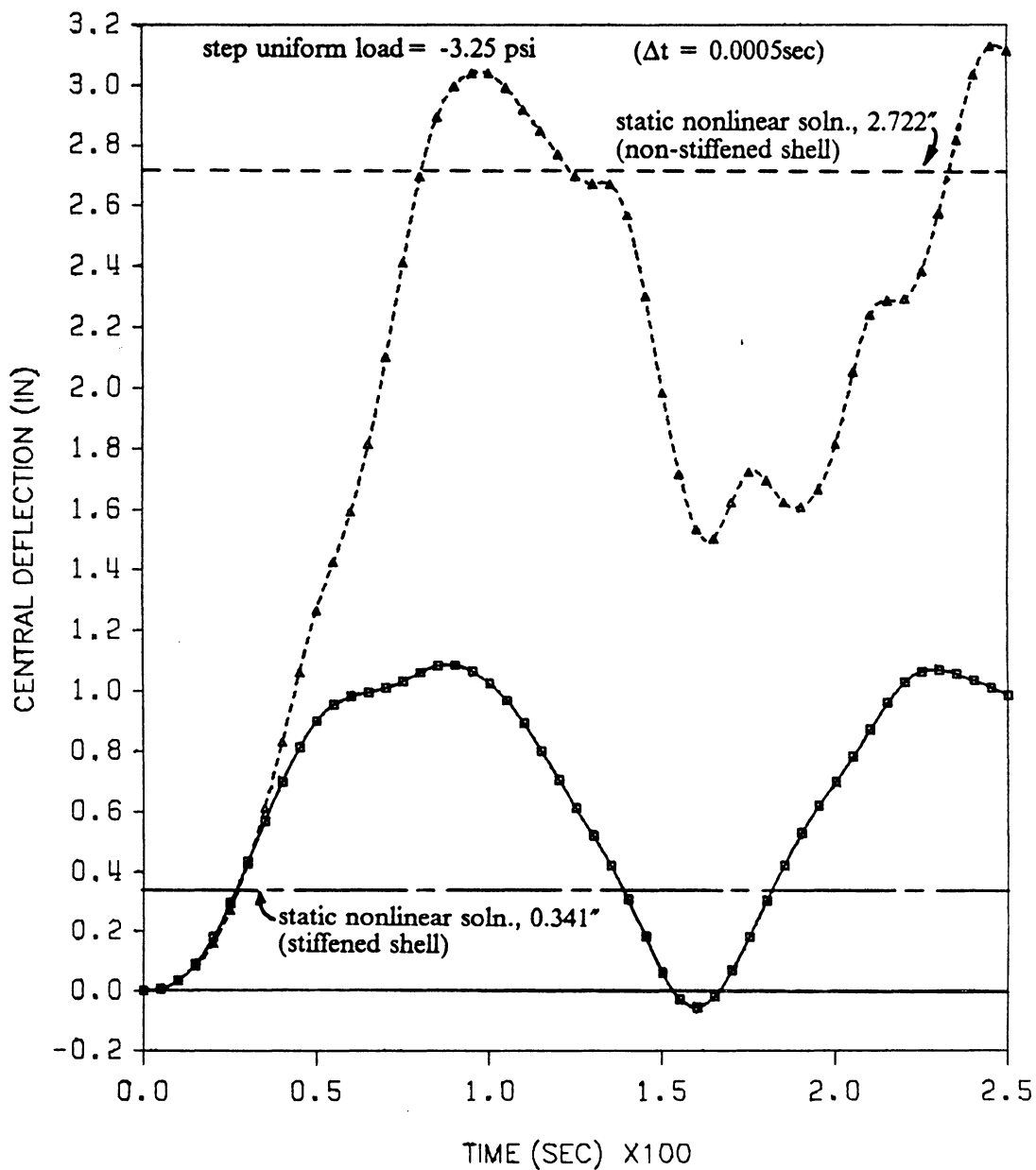


Figure 6.60 Nonlinear transient response of simply-supported 2-layer cross-ply(0/90) stiffened shell under a step uniform pressure

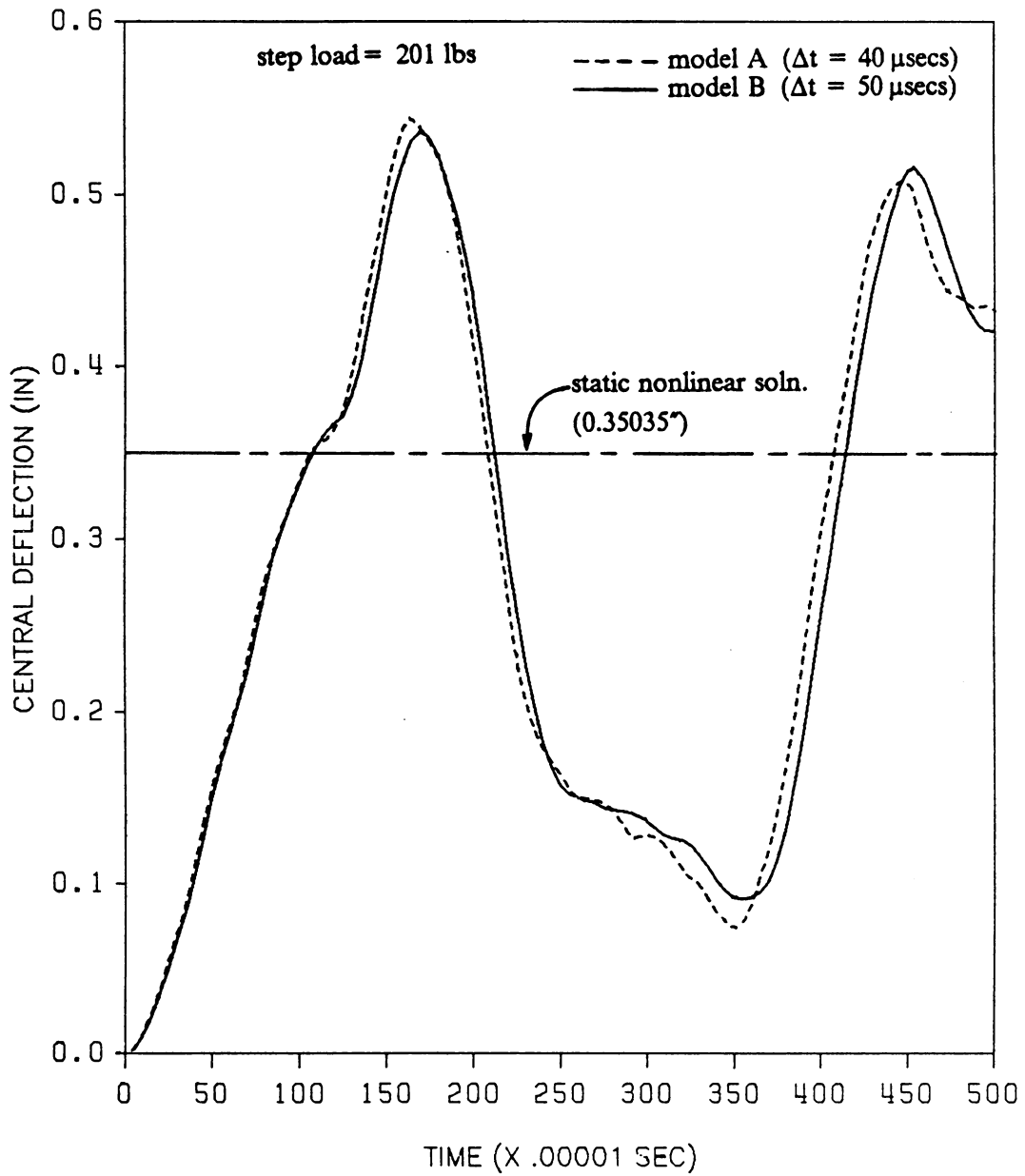


Figure 6.61 Nonlinear transient response of simply-supported beam under a step point load

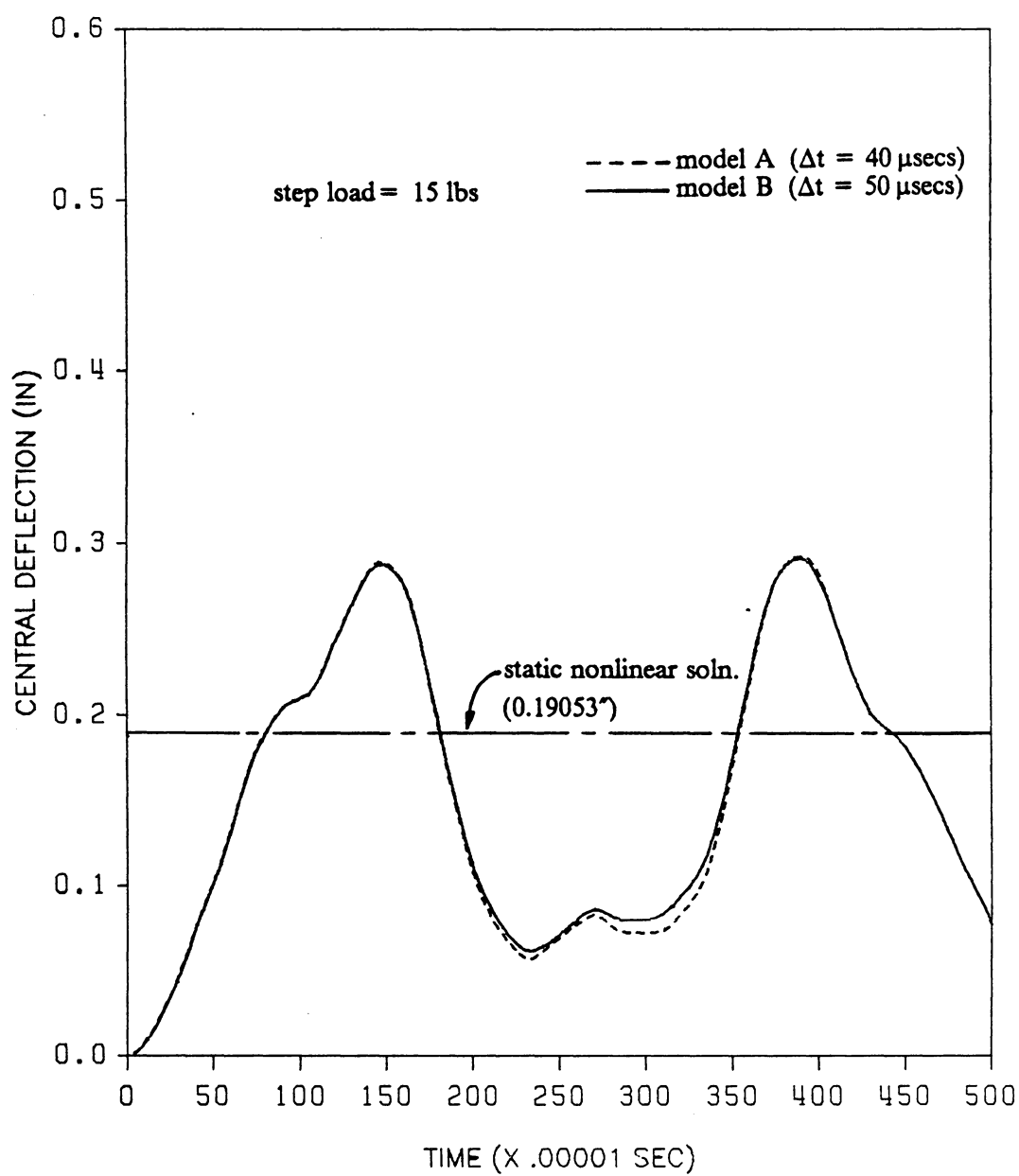


Figure 6.62 Nonlinear transient response of simply-supported 2-layer cross-ply(0/90) beam under a step point load

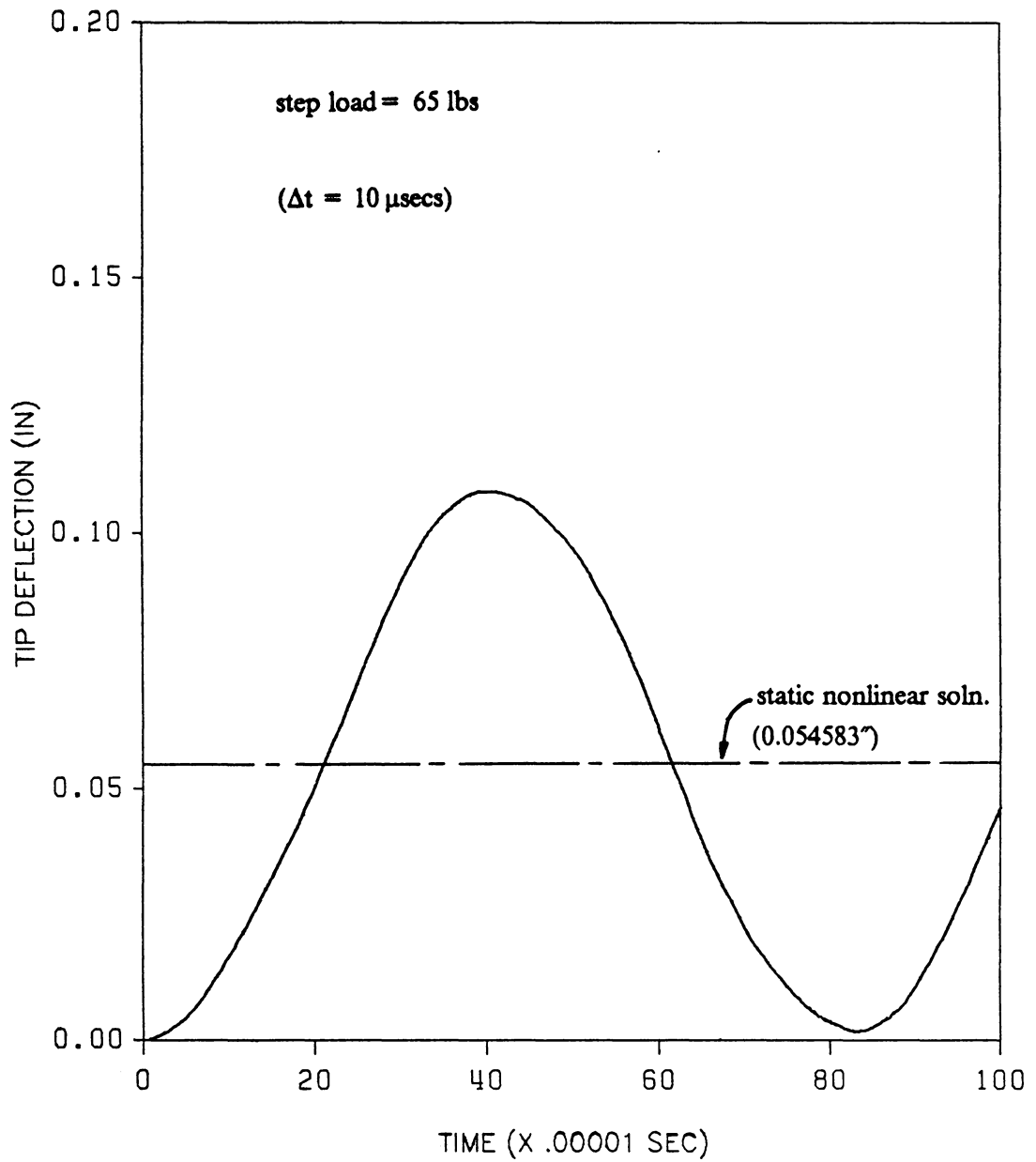


Figure 6.63 Nonlinear transient response of a cantilever folded roof under a step point load

The same finite element models of static analysis were used again for transient analysis. The step load is 63 lbs. The time step $\Delta t = 0.00008$ sec. The nonlinear transient responses of these two finite element models and the corresponding static nonlinear solutions at this load step are shown in Fig. 6.64.

3. Nonlinear transient analysis of problem 4 in Section 6.2.5

The same finite element model as that of static analysis was used again for transient analysis. The step load is 18 lbs. Fig. 6.65 shows the nonlinear transient responses for two different time step, 0.00005 sec and 0.0001 sec, which are close to each other. The static nonlinear deflection at this load step is also shown in the figure.

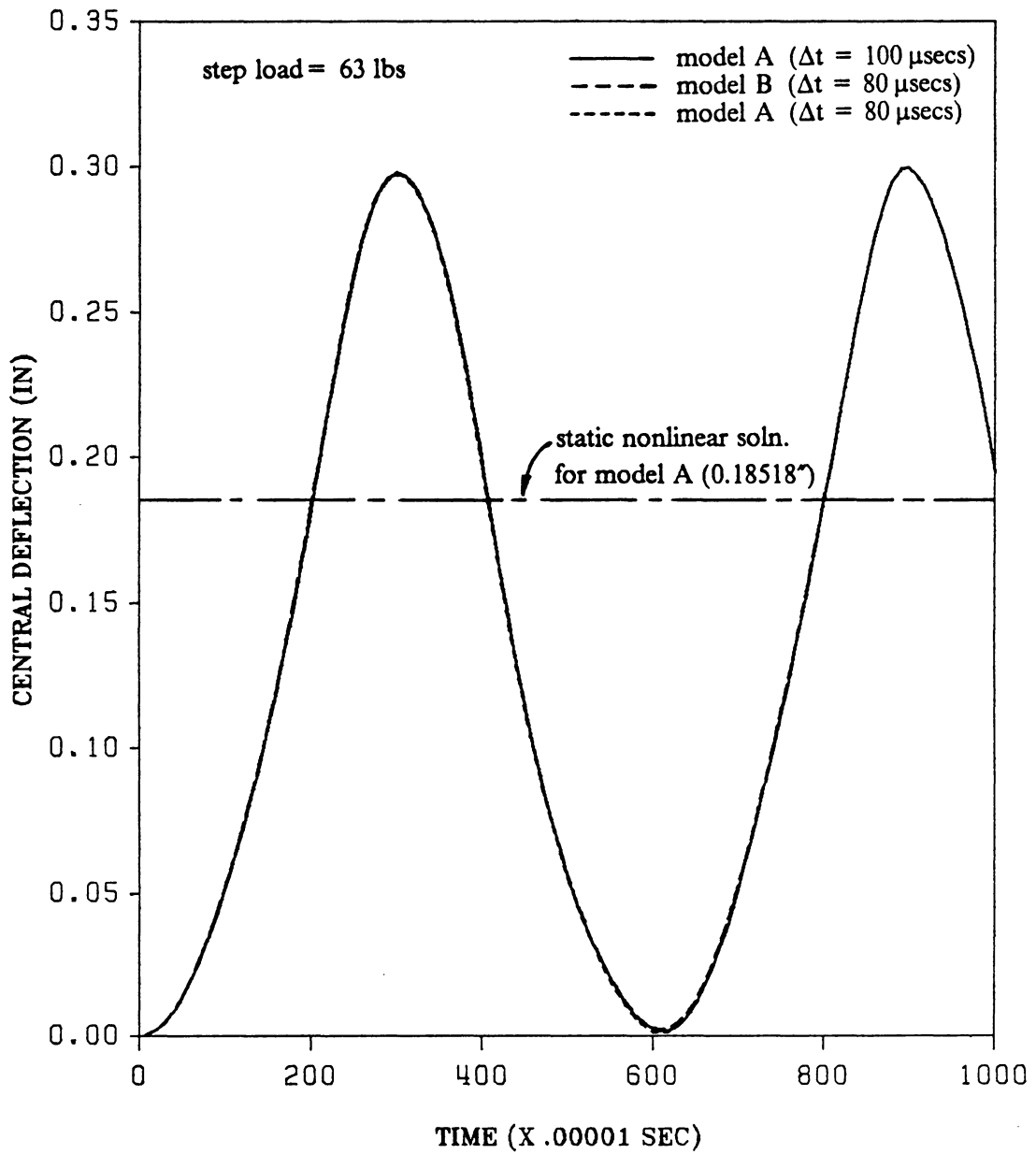


Figure 6.64 Nonlinear transient response of a simply-supported beam with nonuniform thickness under a step point load

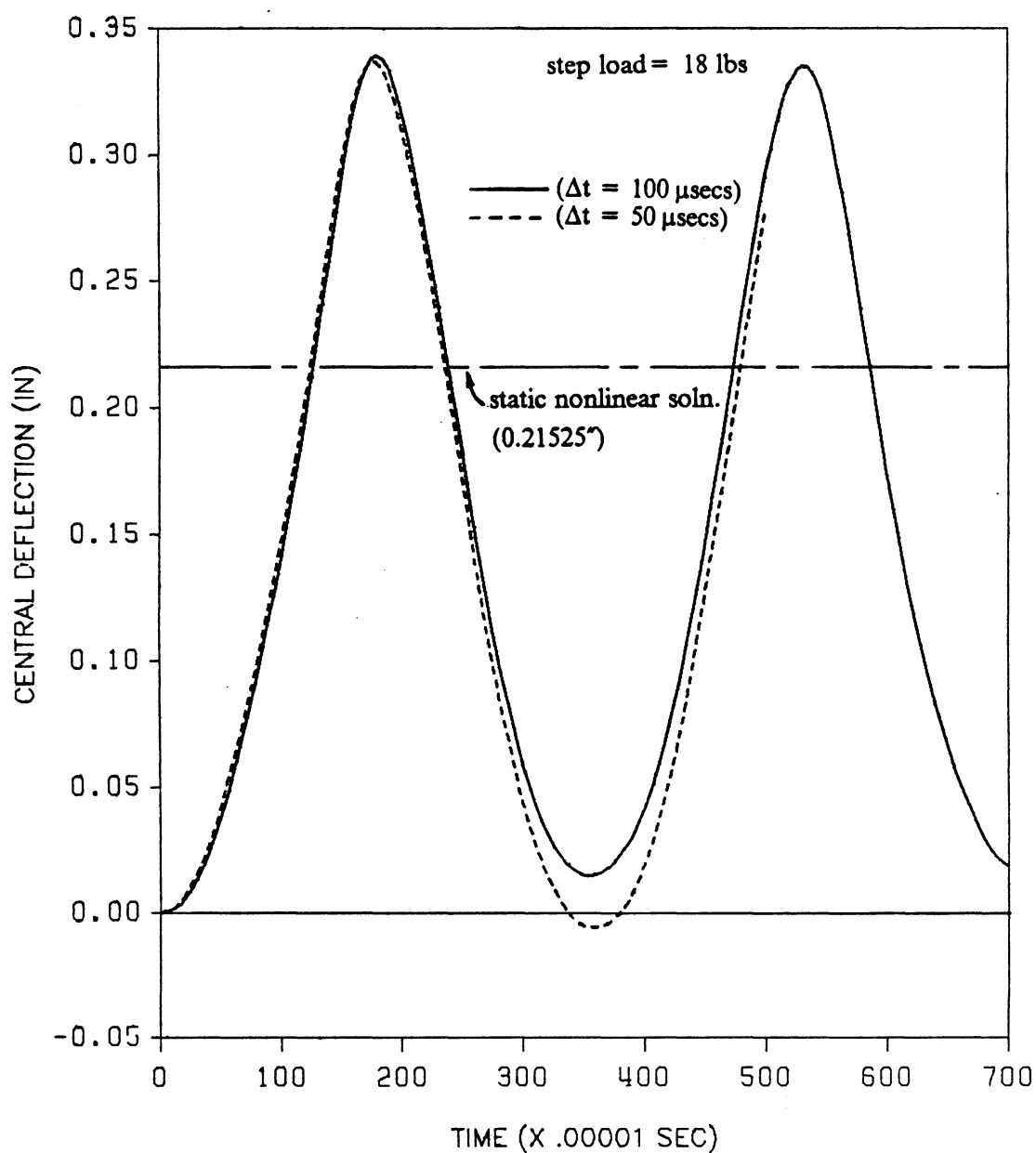


Figure 6.65 Nonlinear transient response of a simply-supported layered composite beam with nonuniform thickness under a step point load

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Summary and Conclusions

The present study dealt with the following major topics:

1. The formulation of incremental equations of motion of a continuum by the total Lagrangian description based on the principle of virtual displacements.
2. The development of displacement finite element model from the linearized incremental equations of motion by the continuum mechanics strain-displacement relations and interpolations of coordinates and displacements.
3. Time discretization of the equations of motion and construction of solution procedures to solve the system equations iteratively for nonlinear analysis.
4. Imposing some assumptions from shell and beam theories on the 3-dimensional isoparametric continuum element to develop continuum-based shell, beam and solid-shell transition elements for anisotropic as well as isotropic materials.

Numerical examples, for both static and transient analyses, demonstrate the validity and efficiency of the present formulations and solution procedures. The present results are in good agreement with those available in the literature. Many of the numerical results of general shell-type structures included here can serve as references for future investigations. The present work is the one which can analyze isotropic and laminated general shell-type structures accurately and efficiently.

7.2 Recommendations

The formulations presented here can be extended to contain more considerations in practical structural analysis. The inclusion of thermal load in the present formulations is straight forward since the temperature field within the element can be interpolated in the similar way as displacement field. Also if the membrane strains are large, the thickness is updated by evaluating the normal strain increments from the constitutive equations via the zero normal stress assumption.

The present formulations can be extended to incorporate with nonlinear material models, for example plasticity and nonlinear viscoelasticity, in case the improved anisotropic material models are available. The materially nonlinear analysis can be performed either separately or in combination with geometric nonlinearity.

Another natural extension of the present research is to incorporate with failure criteria to deal with the failure analysis of laminated composite structures.

One area which needs more study is the cost reduction and accuracy of nonlinear analysis for complex practical structures, e.g. the structures with cut-out or discontinuous stiffeners which need more detailed stress analysis near the discontinuities. The development of global/local algorithms may greatly improve efficiency. The combination of present formulations with some global/local analysis strategies is therefore worth future investigation.

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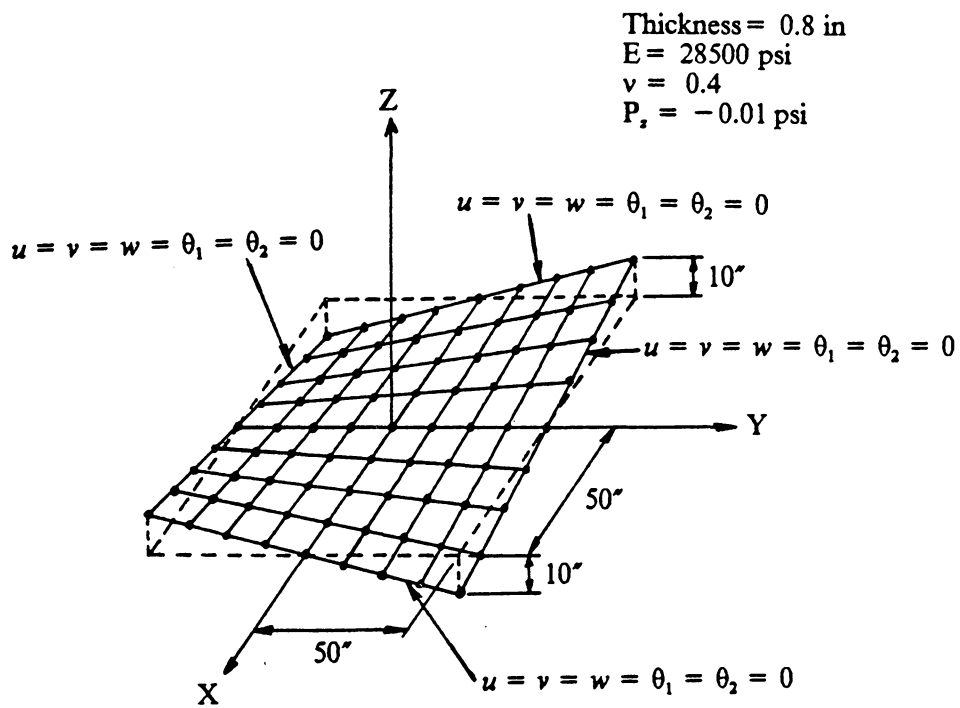
APPENDIX

A.1 A clamped hypar shell

The geometry and boundary conditions of a clamped hypar shell are shown in Fig. A.1. Full shell was modelled with sixteen 9-node shell elements because no symmetric bending exists for this problem. The clamped shell is subjected to a uniform lateral pressure $P_z = -0.01$ psi. The linear deflection at the center obtained in the present study is -0.024442 in as compared with -0.024082 in obtained by Rao [20] who used sixteen 4-node shell elements based on a classical thin shell theory.

A.2 Stress variations of problem 10 in Section 6.2.3

The stress variations of 2-layer cross-ply simply-supported stiffened spherical shell with respect to the loading are plotted in Figures A.2 and A.3. The stresses were measured at the apex of the stiffened shell for each load step. Fig. A.2 shows the stresses within the shell while Fig. A.3 shows the stresses within the stiffeners.



Mesh: sixteen 9-node shell elements

Figure A.1 A clamped hyper shell

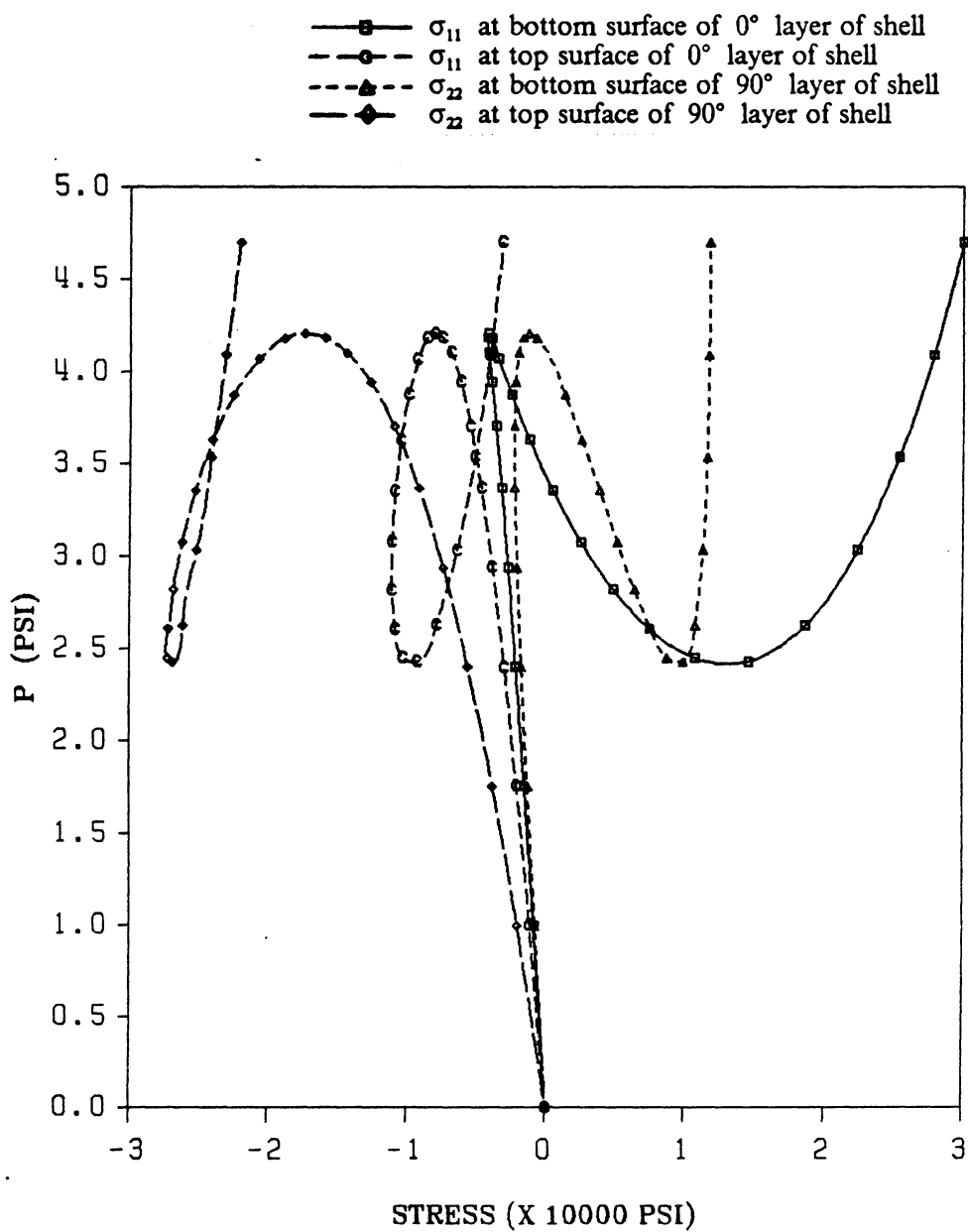


Figure A.2 Stress variations of problem 10 in Section 6.2.3 (shell portion)

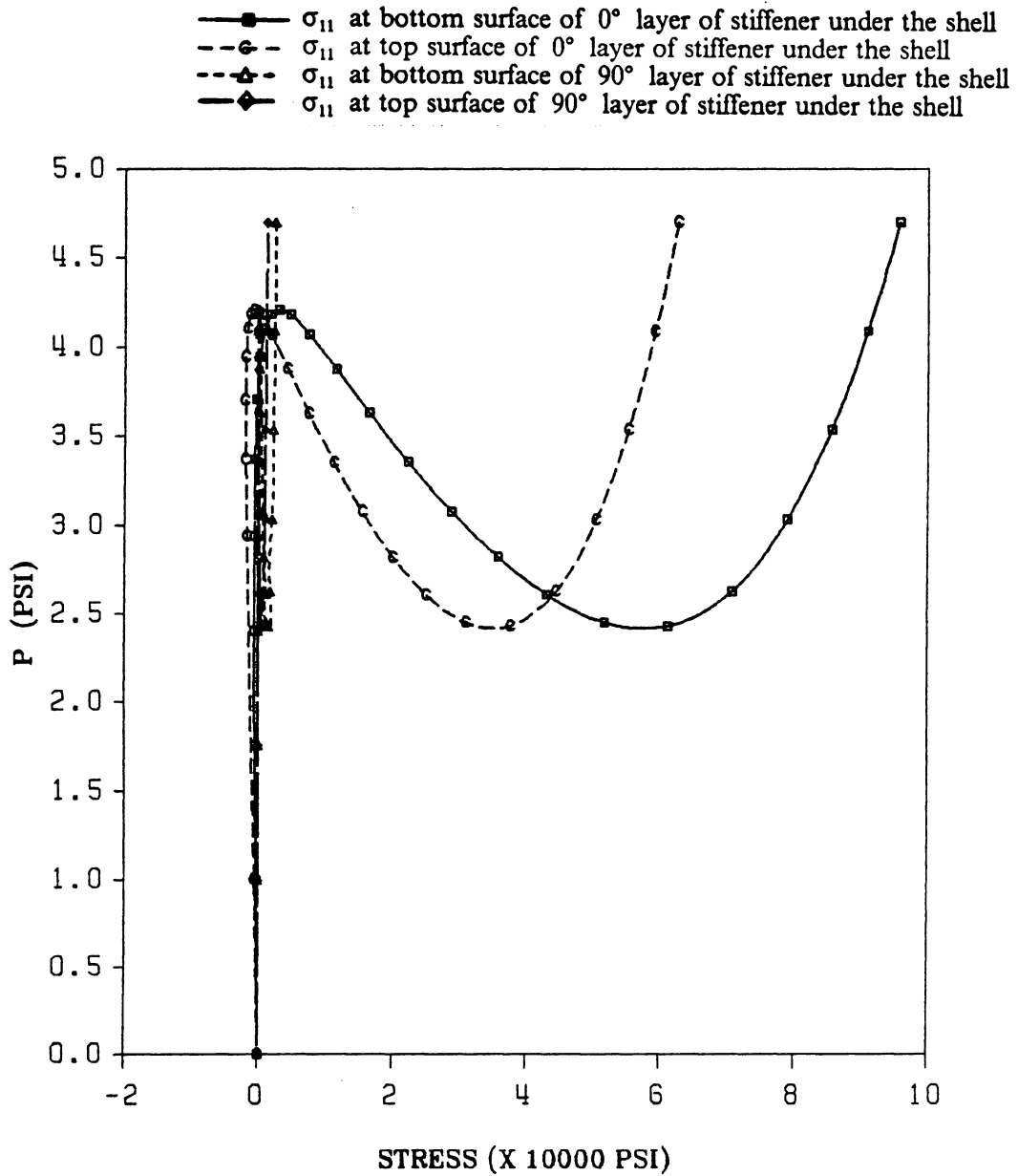


Figure A.3 Stress variations of problem 10 in Section 6.2.3 (stiffener portion)

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