

SOME PROPERTIES OF THE SAMPLE COEFFICIENT OF VARIATION

by

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CHAPTER I

INTRODUCTION

The sample coefficient of variation (S.C.V.) is defined, in the statistical literature, in a variety of ways. Among the most common are:

$$v = \frac{s}{\bar{x}} = \frac{\left(\frac{\sum (x_i - \bar{x})^2}{n-1} \right)^{1/2}}{\bar{x}}, \quad v_n = \frac{s_n}{\bar{x}} = \frac{\left(\frac{\sum (x_i - \bar{x})^2}{n} \right)^{1/2}}{\bar{x}},$$

$u = |v|$ and $u_n = |v_n|$. In this dissertation v is used as the defining statistic of the S.C.V.

The S.C.V. has been used in many areas of statistics. Despite this, there seems to be both a lack of knowledge of some of its properties and a deficiency of certain probability tables that could facilitate its use.

A. Some Applications of the S.C.V.

Four situations where the S.C.V. is applicable are exemplified below.

In some applications, see for example Enrick [6], the standard deviation increases in direct proportion to increases in the mean. In such situations the S.C.V. is practically independent of changes in the mean. The S.C.V. may be directly used to compare the variabilities of populations having the above invariance property in conjunction with the possibility of unequal means.

Another area of application is as follows: Past experience has established that the population coefficient of variation (C.V.) is equal

to a value V . A random sample is taken and the S.C.V. is calculated. One may wish to know whether there is significant evidence to reject the hypothesis that the sample came from a population having a C.V. equal to V . An example of such a situation is given by Hald [12], p. 303.

In many theoretical results the C.V. is used. For example; upper bounds for the bias of certain ratio estimators are given in terms of the C.V. of $\bar{X}$, see e.g. Cochran [3], p. 162. In such situations the S.C.V. may be used either as an estimator or in obtaining confidence intervals for the C.V.

It can be easily noticed that the C.V. is a measure of the concentration of the observations around the population mean. If the parent population is normal then the probability that an observation X lies in the interval $((1-g)\mu, (1+q)\mu)$ is equal to

$$\frac{1}{\sqrt{2\pi} \sigma} \int_{(1-q)\mu}^{(1+q)\mu} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\} dx = \frac{1}{\sqrt{2\pi}} \int_{-g/V}^{q/V} e^{-y^2/2} dy.$$

Thus, as would be expected, small values of V imply a high probability that the observations concentrate in a small neighborhood of the true mean. Since the S.C.V. is an estimator of the C.V., it can be used as a measure of such concentration in the sample.

In certain quality control problems, see e.g. Johnson and Welch [14], one may be interested in the proportion defective in a lot. In such situations one may specify a lower boundary L and that the proportion

of objects falling below L be less than α . If a random sample $\{X_i\}_{i=1}^n$ is taken then it may be more meaningful, in this situation, to consider the transformed variables $\{Y_i = X_i - L\}_{i=1}^n$. A small value of the statistic $v_y = \frac{S}{\bar{Y}} = \frac{S}{\bar{X} - L}$ would be a good indication that the above specifications have been met. It will be shown in Chapter IX that v_y can be used as a test statistic for the above criterion. In a similar manner one could consider the situation where an upper boundary, or both an upper and lower boundary, has been specified.

Several numerical examples that illustrate such applications are given in Chapter IX.

B. Purpose

In order to proceed with a further study of the S.C.V. one must be able to show that such a study is necessary. For this purpose some of the inadequacies of previous research in this subject will be illustrated.

In order to be able to apply the non-central t statistic, in the study of the S.C.V., Johnson and Welch [14] correctly state that the S.C.V., in applied problems, is usually less than $1/3$. They then proceed to illustrate their method by considering a situation where, from a sample of size 25, a S.C.V. of 2.6 was observed. A study of their table will reveal their inability to find the probabilities of the S.C.V. for the most useful range of parameters, namely when the C.V. is small. One can, furthermore, notice that even if 2.6 were a practical value for the S.C.V., their neglect of the probability of negative values of the S.C.V. will, in this range, introduce a significant error. This was pointed out by Owen [22].

Koopmans et al [17] show that no fixed sample procedure exists for obtaining a finite length confidence interval for the C.V. whose underlying population is normal. In many practical situations the S.C.V. may, nevertheless, be used to obtain a finite length confidence interval having an insignificant amount of error from the theoretical length.

Hald [12] derived an approximate formula for obtaining percentage points of the S.C.V. He then illustrates the use of his method by considering a situation having sample size 5. It will be shown that such approximate formulas should be used with caution for small sample sizes. In order that one may avoid the use of such approximations the exact percentage points of the S.C.V. will be derived in this dissertation.

It is hoped that the preceeding arguments have indicated the need for further study in this subject. The following paragraphs outline the main purpose of this dissertation.

The main purpose of this dissertation is to extend our knowledge of five areas of the S.C.V. They are: (a) Give a table of percentage points of the S.C.V. derived from a normal parent distribution. Restrict this table to values of the C.V. less than .5. (b) Study the robustness and the rate of convergence to normality, of the S.C.V. (c) Compare several different approximate methods for finding the percentage points of the S.C.V. The interest in these methods lies not only in comparing the accuracy of past results, but also in applying them to those cases where the parameters lie outside the range of applicability of the probability tables. (d) Generalize the S.C.V. in order to be able to handle problems

that occur in the analysis of variance. (e) Suggest ways of using the above results in applied problems. Illustrate the applicability of such methods by solving some examples found in the literature.

CHAPTER II

PARTIAL LITERATURE REVIEW

A significant early piece of work on the mathematical theory of the S.C.V. was done by McKay [19], [20]. He assumed a normal parent distribution and a small C.V. in order to obtain the following approximate distribution of $\frac{v^2}{n}$:

$$n[1 + 1/V^2][v_n^2/(1 + v_n^2)] \cap \chi_{n-1}^2(0), \text{ where } \chi_{n-1}^2(0)$$

refers to the central chi square distribution with $n-1$ degrees of freedom.

A mathematical discussion of McKay's second paper was given by Fieller [8]. E. Pearson [23] used a Monte Carlo sampling plan to verify the validity of McKay's approximation.

Another approximate distribution was given by Hald [12]. He assumed a normal parent distribution and in addition assumed the insignificance of the probability of negative values of the S.C.V. He then used the statistic $Z = S - t_h \bar{X}$, t_h being given by $\Pr\{\frac{S}{X} \leq t_h | V\} = h$, to obtain the following approximation for the values of t_h :

$$t_h = V \left[\frac{1 + u_h \left[\frac{1}{2(n-1)} \left(1 - \frac{v^2 u_h^2}{n} \right) + \frac{v^2}{n} \right]^{1/2}}{1 - \frac{v^2 u_h^2}{n}} \right].$$

Here u_h refers to the h 'th percentile of the standard normal distribution.

Several other articles, dealing with the distribution of the S.C.V. derived from a normal parent, have been published. Among these is one by Hendricks and Robey [13], where a polar transformation is used to obtain an approximate frequency function of v_n . Also of interest is the article by Johnson and Welch [14] who apply their table of the non-central t distribution to find percentage points of the S.C.V.

F. N. David [5] seems to have been the first researcher to consider the S.C.V. for non normal parent distributions. She derived the approximate moments of the S.C.V. for a general parent population, and used these moments to discuss the distributional properties of the S.C.V.

Several papers have appeared which deal with confidence intervals of the population coefficient of variation (C.V.). Notable among these are the following three papers: (i) Johnson and Welch [14] used their table of the non-central t distribution to obtain such confidence limits. It should be noted that they are in general restricted to cases where the S.C.V. is large. (ii) As previously mentioned, Koopmans et al [17] showed that only approximations to the desired length of such confidence intervals can be obtained. They then proceeded to show that certain sequential methods exist for obtaining fixed length confidence intervals of the C.V. whose underlying parent population is normal. (iii) Linhart [18] considered the case of gamma parent distributions. He used the method of maximum likelihood to obtain a method for finding confidence intervals of the C.V.

An alternative method, will in this dissertation be used for obtaining, with relative ease, such confidence intervals. The procedure is based on the exact percentage points of the distribution of the S.C.V. These percentage points are found in Appendix II of this thesis.

Other Commonly Used Scalar Invariant Statistics

Several statistics, having similar properties to the S.C.V., have been advanced and used by other researchers. Notable among these are Gini's sample coefficient of concentration, $G = \frac{\Delta}{2\bar{X}}$ where

$$\Delta = \frac{\sum_{i,j=1}^n |X_i - X_j|}{n(n-1)}, \frac{\text{Range}}{\bar{X}}, \text{ and } \frac{\sum |X_i - \bar{X}|}{n\bar{X}}.$$

Such statistics will have many of the properties given in Chapter IV; see Glasser [11]. However, their distributional properties usually depend on order statistics and, as a result, are more difficult to obtain than those of the S.C.V. In general, one seems to discriminate between these statistics on the basis of degree of difficulty encountered in working with the numerator.

CHAPTER III

Properties of the S.C.V. for a Normal Parent Distribution

A statistic, such as the sample coefficient of variation, can have one of several functions; it might be primarily useful in the estimation of parameters or it can have its chief application in the testing of statistical hypotheses. If the latter is of prime importance then it becomes advantageous to have some knowledge of the distributional properties of this statistic.

Such distributional properties are not always easy to obtain. If the statistic is somewhat complicated then exact results can only be derived by making some particular assumptions on the parent distribution. Such assumptions are at best approximations to reality and they are made because they seem reasonable in practice and because they provide a solution to the problem.

The validity of making such assumptions should be further investigated: This can possibly be done by investigating the effects, on the final result, of slight deviations from the parent distribution assumptions. If such deviations incur only minute changes on the results of the method used then one says that the distribution is robust. Robustness clearly enhances the confidence with which one is willing to use the previously obtained results in an applied situation.

If the parent population is assumed to be normal then the sample mean " $\bar{X}$ " and standard deviation " S " are independent. This result makes possible the derivation of the exact distribution of the S.C.V. Since

in many applications the parent population is approximately normal, the verification of robustness will, in such cases, give reasonable grounds for the use of the normality assumption.

A. Exact Distribution of the S.C.V.

The following notation will be used throughout this chapter: $f(t)$ will refer to the frequency function of the random variable t . A clear distinction will also be made between v and v_n , both of which are defined in Chapter I of this dissertation.

Let $\{X_i\}_{i=1}^n$ be a sequence of independent and identically distributed random variables whose common normal distribution is:

$$N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left[-\frac{(x - \mu)^2}{2\sigma^2} \right].$$

Since it will be slightly easier to work with v_n rather than v , the frequency function of the former statistic will be derived.

Since v_n is scalar invariant one could equivalently consider the sequence

$$\{Y_i = \frac{\sqrt{n}}{\sigma} X_i\}_{i=1}^n.$$

Letting $V = \frac{\sigma}{\mu}$ and $a = \frac{\sqrt{n}}{V}$ one notices that $\bar{Y} \cap N(a,1)$ and

$$S_n^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n} \cap \chi_{n-1}^2.$$

The statistical independence of S_n^2 and $\bar{Y}$ leads to the equation

$$f(s_n, \bar{y}) = \frac{s_n^{(n-2)} \exp\left(-\frac{1}{2}[(\bar{y} - a)^2 + s_n^2]\right)}{\sqrt{2\pi} \Gamma\left(\frac{n-1}{2}\right) 2^{\left(\frac{n-3}{2}\right)}}$$

where $0 \leq s_n < \infty$ and $-\infty < \bar{y} < \infty$.

In what follows, several transformations of the variables will be made. It should be remembered that these will be transformations on continuous frequency function and, therefore, the Jacobian needs to be calculated and appropriately applied.

The transformation

$$v_n = \frac{s_n}{\bar{y}} \text{ and } \bar{y} = \bar{y},$$

in the above equation results in the following frequency function:

$$f(v_n, \bar{y}) = \frac{(v_n \bar{y})^{n-2} |\bar{y}| \exp\left(-\frac{1}{2}[(\bar{y} - a)^2 + (v_n \bar{y})^2]\right)}{\sqrt{2\pi} \Gamma\left(\frac{n-1}{2}\right) 2^{\left(\frac{n-3}{2}\right)}}.$$

For the purpose of simplifications in future equations, let

$$A = \frac{v_n^{n-2} \exp\left[-\frac{a^2 v_n^2}{2(1+v_n^2)}\right]}{\sqrt{2\pi} \Gamma\left(\frac{n-1}{2}\right) 2^{\left(\frac{n-3}{2}\right)}} .$$

It can easily be verified that

$$(\bar{y} - a)^2 + v_n^2 \bar{y}^2 = \left[\bar{y}(1 + v_n^2)^{\frac{1}{2}} - \frac{a}{(1 + v_n^2)^{\frac{1}{2}}} \right]^2 + \frac{a^2 v_n^2}{1 + v_n^2} .$$

Inserting the above equation into the previously given frequency function one obtains

$$f(v_n) = A \int_{-\infty}^{\infty} \bar{y}^{n-2} |\bar{y}| \exp\left[-\frac{1}{2} \left[\bar{y}(1 + v_n^2)^{1/2} - \frac{a}{(1 + v_n^2)^{1/2}} \right]^2\right] d\bar{y} .$$

The transformation $z = \bar{y} (1 + v_n^2)^{1/2}$, $v_n = v_n$, in the above equation, finally results in

$$f(v_n) = \begin{cases} f_1(v_n) & \text{if } v_n \geq 0 \\ f_2(v_n) & \text{if } v_n < 0 \end{cases} \quad (3.1)$$

$$\text{where } f_1(v_n) = \frac{A}{(1 + v_n^2)^{n/2}} I_{n-1} \left(\frac{a}{(1 + v_n^2)^{1/2}} \right)$$

$$\text{and } f_2(v_n) = \frac{(-1)^n A}{(1 + v_n^2)^{n/2}} I_{n-1} \left(- \frac{a}{(1 + v_n^2)^{1/2}} \right).$$

The function $I_{n-1}(b) =$

$$\int_0^\infty z^{n-1} \exp\left[-\frac{1}{2}(z - b)^2\right] dz$$

called the Airy Function, plays an important role in many functional developments in statistics. For a detailed discussion of this function see R. A. Fisher [9].

For the special case $a = 0$, equation (3.1) can be greatly simplified.

For if $a = 0$,

$$I_{n-1} \left(\frac{a}{(1 + v_n^2)^{1/2}} \right) = \int_0^\infty z^{n-1} e^{-z^2/2} dz = \Gamma\left(\frac{n}{2}\right) 2^{(n/2)-1}.$$

Without making restrictions on the parameters, simplifications of equation (3.1) are, in the general case, not possible.

It is interesting to observe that Fieller [8] stated the distribution of $u_n = |v_n|$. If in equation (3.1) the transformation $u_n = |v_n|$ is made, then the frequency function of u_n agrees with Fieller's result.

Our interest lies not in obtaining the frequency function of v_n , but in finding the percentage points of the S.C.V. The following equation shows that the frequency function of v_n can be used for this purpose:

$$\Pr \{v_n \leq v_o \mid V\} = \Pr \left\{v \leq \left(\frac{n}{n-1}\right)^{1/2} v_o \mid V\right\}.$$

B. Moments of the S.C.V.

After deriving the distribution of v_n , and thus of v , the next logical step is to investigate the moments of the S.C.V. The following is an intuitive argument. Whatever the values of u and σ , the moments of v cannot exist. Since $v = \frac{S}{\bar{X}}$, there is always a non zero probability that $\bar{X}$ lies in some neighborhood of zero. The contribution of this neighborhood will cause the moments of v to diverge. A more formal proof for the above statement is as follows:

The independence of S and $\bar{X}$ implies that

$$E \left(\frac{S}{\bar{X}} \right)^r = E(S)^r E \left(\frac{1}{\bar{X}} \right)^r$$

where r is some natural number. Since $E(S^r)$ is a Gamma Integral it has clearly a finite value. One needs now only to investigate

$$E \left(\frac{1}{\bar{X}} \right)^r = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} \left(\frac{1}{\bar{x}} \right)^r \exp \left(-\frac{n(\bar{x} - \mu)^2}{2\sigma^2} \right) d\bar{x} \quad .$$

The above integral, over the subinterval $(0, |\mu|)$, becomes

$$g(\mu) = \int_0^{|\mu|} \left(\frac{1}{\bar{x}} \right)^r \exp \left(- \frac{n(\bar{x} - \mu)^2}{2\sigma^2} \right) d\bar{x} \quad .$$

Since in the interval $(0, |\mu|)$ $(\bar{x} - \mu)^2 \leq 4\mu^2$, it follows that

$$g(\mu) \geq \exp \left(- \frac{2\mu^2 n}{\sigma^2} \right) \int_0^{|\mu|} \left(\frac{1}{\bar{x}} \right)^r d\bar{x} \quad .$$

The divergence of the integral

$$\int_0^{|\mu|} \left(\frac{1}{\bar{x}} \right)^r d\bar{x}$$

implies the non-existence of the moments of v .

It should be realized that the non-existence of the moments of v is a mathematical property which was arrived at on the basis of the normality assumption on the parent distribution. It is possible that for some almost normal parent distributions the moments of v do exist. In such situations the normality assumption may, from an applied standpoint, still be valid. The above situation should be further investigated.

C. Probability that the S.C.V. is Negative

Equation (3.1) will be used to obtain the exact percentage points of the S.C.V. It would, therefore, be helpful to show that, for a particular range of values of the parameters, the probability of negative values of v is insignificant. If this were true then $f_2(v_n)$ could be removed from consideration and in this way equation (3.1) would be considerably

simplified. The following discussion will indicate that the range of parameters for which the probability of negative v is insignificant corresponds to the range of parameters that is most often encountered in practice.

Even though most observations are restricted to the non-negative range, a normal density may still give a good approximation to the data. Such a normality assumption causes a formerly non-existent probability of negative values of the observations. If the normality assumption is valid, this probability should be insignificant. Since

$$\Pr\{X < 0 | V\} = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^0 e^{-(x-\mu)^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1/V} e^{-y^2/2} dy$$

such an infinitesimal probability can only be achieved when V is small, say less than $1/3$.

It was mentioned in Chapter I that our interest centers in the parametric range $V \leq .5$ and $n \geq 5$. For this range it can be shown that $\Pr\{v < 0 | V\} \approx 0$. Since

$$\Pr\{v < 0 | V\} = \Pr\left\{\frac{S}{\bar{X}} < 0 | V\right\}$$

and $s \geq 0$ one obtains

$$\Pr\left\{\frac{S}{\bar{X}} < 0 | V\right\} = \Pr\{\bar{X} < 0 | V\} = N\left(-\frac{\sqrt{n}}{V}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{n}/V} e^{-y^2/2} dy.$$

For the parametric range given above $\Pr\{v < 0 \mid V\}$ is now clearly insignificant.

An easier way to illustrate the above is that the C.V. of $\bar{X}$ is equal to $V/\sqrt{n} = \frac{1}{a}$. This is clearly, in most applications, less than 1/3.

D. A Simplified Formula of the Frequency Function for the Use on the Computer

Equation (3.1) is a somewhat cumbersome statement of the frequency function of v_n . In order to use it in the computation of percentage points on the IBM 7040 at the computing center of the Virginia Polytechnic Institute, certain modifications will have to be made. These modifications will be pointed out at this stage.

The results of the previous section can be used to remove $\Pr\{v < 0 \mid V\}$ from consideration. This leads to the removal of $f_2(v_n)$ from equation (3.1) and, thus, greatly simplifies the numerical work.

The integration of $f_1(v_n)$, on a computer, presents certain problems. Since V is taken to be small, while n has moderate values, the parameter $a = \frac{\sqrt{n}}{V}$ could be quite large. This forces terms such as

$$\exp \left(- \frac{a^2 v_n^2}{2(1 + v_n^2)} \right)$$

to be exceedingly small while causing

$$I_{n-1} \left(\frac{a}{(1 + v_n^2)^{1/2}} \right)$$

to be extremely large. In order to remedy this situation the equation

$$f_1(v_n) = \frac{v_n^{n-2} \exp \left(\frac{-a^2 v_n^2}{2(1 + v_n^2)} \right) \int_0^\infty z^{n-1} e^{-\frac{1}{2} \left(z - \frac{a}{(1 + v_n^2)^{1/2}} \right)^2} dz}{\sqrt{2\pi} \Gamma\left(\frac{n-1}{2}\right) 2^{(n-3)/2} (1 + v_n^2)^{n/2}}$$

will be modified.

For our range of parameters the following approximation, to the Airy Function, gives excellent results:

$$\int_0^{\infty} z^q e^{-\frac{1}{2}(z-b)^2} dz = t^q e^{-\frac{1}{2}(t-b)^2} \left(\frac{2\pi t^2}{t^2 + q} \right)^{1/2} \left[1 - \frac{3q}{4(q + t^2)^2} + \right.$$

$$\left. \frac{5q^2}{6(q + t^2)^3} \right], \text{ where}$$

$$t = \frac{b + (b^2 + 4q)^{1/2}}{2}.$$

This result was obtained by Rosnikoff and Lieberman [25].

The above approximation for the Airy Function reduces the double integral of $f_1(v_n)$, into a single integral. This reduction, to a single integral, will considerably simplify the numerical work necessary to obtain the percentage points of the S.C.V.

One still has to factor $f_1(v_n)$ into a form suitable for computer use. For this purpose the terms have to be balanced so that no overflow occurs. At the same time underflow is allowable only when $f_1(v_n) \approx 0$.

The above factors have been kept in consideration when deriving equation (3.2). Combining $f_1(v_n)$ with the approximation of the Airy Function and simplifying one obtains the following modified form of $f_1(v_n)$:

$$f_1(v_n) \approx \left[\frac{2B \left(\frac{t^2}{t^2 + n - 1} \right)^{1/2}}{C v_n} \right] \left(\frac{v_n t}{\sqrt{2} \Gamma \left(\frac{n-1}{2} \right)^{1/n-1} \exp \left(\frac{(t-b)^2}{2(n-1)} \right) C} \right)^{n-1} \quad (3.2)$$

Here $b = \frac{a}{(1 + v_n^2)^{1/2}}$, $t = \frac{b + [b^2 + 4(n-1)]^{1/2}}{2}$

$$B = 1 - \frac{3(n-1)}{4(n-1 + t^2)^2} + \frac{5(n-1)^2}{6(n-1 + t^2)^3}, \text{ and}$$

$$C = \exp \left(\frac{v_n^2}{2V^2(1 + v_n^2)} \right) (1 + v_n^2)^{1/2}.$$

The symbol $\approx$, instead of an equal sign, is due to the approximation to the Airy Function.

For large values of n a Stirling approximation for the gamma function is necessary. For this purpose equation 6.1.37, of Abramowitz and Stegun [1] p. 257, may be used.

E. Computer Method for Obtaining the Percentage Points of the S.C.V.

Equation (3.2) can be directly used to obtain the percentage points of the S.C.V. Since this function is somewhat complicated a non-standard numerical method had to be used. Since such an approach may find similar uses elsewhere, it is discussed in more complete detail below.

The author observed that most of the probability, for the distribution of the S.C.V., is concentrated in a small interval around V . For

this purpose the total probability interval of, the S.C.V., was subdivided into the following three subintervals:

(a) $0 < v_n < .7 V$, (b) $.7 V < v_n < 1.3 V$, (c) $1.3 V < v_n < bV$.

Here b is suitably chosen in order that $\Pr\{0 < v_n < bV \mid V\} \approx 1$. The author noted that for $V \leq .3$ a value of $b = 3$ was found satisfactory, while for $V > .3$, $b = 3.5$ could be used. When n was small and $V > .4$ a value of $b = 4.5$ gave slightly better approximations to the desired probability value, namely one.

For purposes of illustration the method of finding the percentage point $v_{.005}$, satisfying the equation $\Pr\{v_n < v_{.005} \mid V\} = .005$, is described. First, the interval $0 < v_n < .7 V$ is subdivided into an appropriate number of subintervals required for the application of Simpson's rule. Simpson's rule is then applied on the first two subintervals, i.e. $\text{Prob} = f(v_{n_0}) + 4f(v_{n_1}) + f(v_{n_2})$, and it is determined whether this gives a probability value greater than .005. If not then one more double subinterval is taken and the newly determined area is added to the previous result. This is continued until the probability exceeds .005, at one stage, and is less than .005 at the previous stage. A linear interpolation within this subinterval may be used to obtain $v_{.005}$.

The integration is now continued, from the upper end of the last mentioned subinterval, until the probability .01 is exceeded. This process is continued until v_n is chosen to equal bV . At this stage all the required percentage points have been obtained and, in addition,

the last obtained probability value can be compared with "one" in order to measure the accuracy of the approximation. This method is continued for all combinations of V and n to give the table of Appendix II.

F. Accuracy of the Values of the Table of Appendix II

The previously described method, for obtaining the percentage points of the S.C.V. has several possible sources of error. These sources of error can be subdivided into two classes. They are: (a) The error caused by the linear interpolation in the final subinterval in order to obtain the required percentage point, and (b) All other sources of error.

Errors due to (b) can be controlled by comparing the last obtained value with one. It was found that appropriate subdivisions of the interval of v_n would control this value to less than one unit, in the last stated decimal place of the table of Appendix II, of error.

Errors due to source (a) are slightly more difficult to control. In the interval $.7 V < v_n < 1.3 V$ the slope of $f_1(v_n)$ has its largest values and a linear interpolation might have caused some error. This is precisely the interval that required a large number of subdivisions per unit length and the length of such a subinterval is of the order of the maximum allowable error. The slope of $f_1(v_n)$, on the other two intervals of v_n , is in general much smaller and a linear interpolation seems reasonable. It is still interesting to note that several rows of our table have been rerun after changing the length of the subintervals

and the agreement between such sets of values was found to be exceptionally good.

Several other methods were used to verify the accuracy of our table. For very small values of V a comparison was made between our table and McKay's approximation and the agreement was excellent. For example, when $V = .02$ the difference between these two methods was at most 10^{-5} . A "spot-check" was also made between several values of our table and the results of Resnikoff and Lieberman [25]. The agreement was again exceptionally good. An additional check was made by taking several of our values, say v_0 and integrating equation (3.2) over the interval

$$0 < v_n < \left(\frac{n-1}{n} \right)^{1/2} v_0$$

to see if the desired probability would be obtained. Again the agreement was excellent.

CHAPTER IV

Properties of the S.C.V. Whose Parent Population is Restricted
to the Non-Negative Range

In Chapter III the distribution of the S.C.V. under the normality assumption was discussed. In many situations where the S.C.V. may be useful, the parent distribution from which the data came is clearly not normal. This occurs when the observations must lie in some bounded interval or when there is a clear dependence between S and $\bar{X}$. Even in such situations the normality assumption is often made. The justification for making such an assumption is that the S.C.V. is closely related to the non-central t statistic, i.e. when the parent distribution is normal $\frac{\sqrt{n}}{v}$ is distributed as non-central t with $n-1$ degrees of freedom, and the latter statistic is reasonably robust with respect to changes in parent population. One of the reasons for the robustness of the non-central t is that if the parent population is almost normal then S and $\bar{X}$ are almost independent (actually almost uncorrelated). This statement carries over to the S.C.V. Despite this argument, such justifications can only be considered after a careful scrutiny of the situation.

Throughout this chapter it will be assumed that the observations are restricted to the non-negative range. Such a restriction is clearly not serious or severe since most applications concern themselves with non-negative observations.

A. Existence of Moments of the S.C.V.

A well known upper bound for the S.C.V. will be presently given.

If $\Pr\{X_1 < 0 \mid V\} = 0$ then $v = \frac{S}{\bar{X}}$ is maximized when one of the observations $X_j = C > 0$ while all the other observations are zero.

Then $\bar{X} = \frac{C}{n}$ which implies that $v = \sqrt{n}$. In the discrete case $\Pr\{\bar{X} = 0 \mid V\} = D$, where D may be greater than zero. If this occurs then $S = 0$ and v becomes meaningless. Since, in such a situation, there is no variability it seems natural to let $v = 0$. With this assumption the above inequality also holds for the discrete case.

For Gini's sample coefficient of concentration and several other statistics Glasser [11] developed inequalities similar to the previously given one.

Since $v^r \leq n^{\frac{r}{2}}$ one can obtain

$$\begin{aligned} E(v^r) &= \int_0^{\sqrt{n}} v^r dF(v) \\ &\leq n^{r/2} \int_0^{\sqrt{n}} dF(v) = n^{r/2}. \end{aligned}$$

This establishes the existence of the moments of the S.C.V. whose parent population is restricted to the non-negative range. It is interesting to note that in practice the upper bound $v = \sqrt{n}$ is seldom approached and its practical use is, therefore, very limited.

B. Distributional Properties of the S.C.V. for the Case of Non-Negative Range

The existence of the upper bounds for the moments of the S.C.V. leads to several important results. One of these is that the moments of the S.C.V. uniquely determine their distribution. This follows directly from the result given in Kendall and Stuart [16], p. 110.

This is of particular importance since the moments may be used to approximate the distribution of the S.C.V. While this result does not imply that such moment methods will give good approximations to reality, it does give some indication in this direction.

Of even greater importance is the asymptotic normality of the distribution of the S.C.V. (For a proof see e.g. Cramer [4], p. 366). From Cramer's proof, in conjunction with the moment results of F. N. David [5], the asymptotic distribution of the S.C.V. is normal with mean V and variance

$$\mu_2(v) = \frac{V^2}{n} \left(\frac{1}{2} + V^2 - \frac{w_3}{w_1 w_2} + \frac{w_4}{4w_2^2} \right).$$

Here $w_i = i$ 'th population cumulant. This asymptotic result can, for large sample sizes, be used as an approximate method for finding the percentage points of the S.C.V.

The asymptotic distribution of the S.C.V., from a normal parent, can also be shown to approach normality. This is true despite the fact (see Chapter III) that the moments of the S.C.V., for the normal parent case, do not exist. Such an asymptotic property may, when properly interpreted, find useful applications.

C. Remark Concerning the Exact Distribution of the S.C.V. from Non-Negative Parents

The fact that S and $\bar{X}$ are independent allows one to find the exact distribution of v in the case of the normal parent assumption. Since this independence is, under very general conditions, a unique property of the normal distribution (see Kendall and Stuart [16]),

it is apparent that for other parent distributions such results will be much more difficult to obtain.

Despite this difficulty several people have attempted to solve similar problems. Truksa [28] developed an inductive method for finding the joint frequency function $f(s, \bar{x})$. This method was generalized by Springer [27], who considered the case of observations restricted to the non-negative interval. The difficulty of applying their results limited them to consider only cases of very small sample sizes. This led the author to believe that another approach to the non-normal parent case should be more fruitful.

CHAPTER V

The Approximate Moments of the S.C.V. and Other Pertinent Statistics

A brief discussion of the distributional properties of the S.C.V., for certain non normal parent distributions, has been given in Chapter IV. It was there shown that if the observations cannot be negative then all the moments of v exist. The existence of the moments, in the above mentioned case, can be used to approximate the distribution of the S.C.V.

Most of the methods for approximating the distribution of a random variable are based on the moments of this variable. If, as in our case, the exact values for the moments cannot be found then approximate methods, for finding these moments, become desirable. In general such approximations consist of expanding the moments into the Laurent series

$$\sum_{i=0}^{\infty} \frac{b_i}{n^i}.$$

It is then assumed that, for large enough n , the first few terms of the above mentioned series will give a sufficiently accurate approximation to the true values of these moments.

There are two commonly used methods for obtaining such Laurent series expansions. One of these is based on the properties of orthogonal polynomials while the other is based on Fisher's k statistics. A modified form of the latter method will be exclusively used in this dissertation.

R. A. Fisher in [10] developed the properties of the k statistics. These statistics are linear combinations of augmented symmetric functions of the observations, such that the r 'th k statistic " k_r " is defined as

$E(k_r) = w_r$, w_r being the r 'th population cumulant. One unfortunately encounters, in practice, terms of the form $E(k_{r_1}^{i_1} k_{r_2}^{i_2}) = \mu'(r_1^{i_1} r_2^{i_2})$, which can make the use of the k statistics a lengthy task.

R. A. Fisher in [10] developed a table of cumulants of terms of the form $k_{r_1}^{i_1} k_{r_2}^{i_2}$. The user of this table must still convert such product cumulants into μ 's; a task which may in practice take a considerable amount of time. An excellent discussion of the Fisher k statistics, which includes an extended table of these statistics, is given by Kendall and Stuart [16].

The S.C.V. is a function of only the first two k statistics, namely $k_1 = \bar{X}$ and $k_2 = S^2$. This reduces the algebra to terms of the form $k_1^h k_2^l$, h and l being non-negative integers. For such a situation F. N. David [5] gives a method which greatly simplifies the labor of obtaining the μ 's.

Instead of working with $\mu'(1^h 2^l)$ F. N. David considered the central moments

$$\mu(1^h 2^l) = E(k_1 - w_1)^h (k_2 - w_2)^l = (1^h 2^l).$$

It can be noticed that for $h + l = 1$ the central moment $(1^h 2^l) = 0$, while for $h + l = 2, 3$ $(1^h 2^l)$ equals the equivalent Fisher's product cumulants. For $h + l > 3$ the situation becomes more complicated. In order to remove this complication F. N. David develops a table of corrective terms, correct to $O(\frac{1}{n^3})$, which in conjunction with Fisher's k statistics, gives directly the values of the μ 's. Since many readers may be unfamiliar with the properties of these tables, Appendix I

states the expansions of $(1^{h_1} 2^{h_2})$ correct through terms of $\frac{1}{n^3}$.

A. The Approximate Moments of the S.C.V.

The mean and the next three central moments of the S.C.V. will be considered in this dissertation. F. N. David gives these moments expansions through terms involving $\frac{1}{n^2}$. However, it is possible that, for large V and moderate n , the contribution of the $\frac{1}{n^3}$ terms may be significant. The method of expanding the moments of the S.C.V. will be illustrated by giving a part of the algebra necessary to obtain the first moment. It should, at this point be emphasized that w_r has been defined as the r 'th population cumulant while $V = \frac{\sigma}{\mu}$ is the population C.V.

$$\begin{aligned}
 E\left(\frac{S}{\bar{X}}\right) &= E\left(\frac{k_2^{1/2}}{k_1}\right) = \frac{w_2^{1/2}}{w_1} E\left(\frac{w_2 + k_2 - w_2}{w_2}\right)^{1/2} \left(\frac{w_1 + k_1 - w_1}{w_1}\right)^{-1} \\
 &= VE\left(1 + \frac{k_2 - w_2}{w_2}\right)^{1/2} \left(1 + \frac{k_1 - w_1}{w_1}\right)^{-1} = VE\left[1 + \frac{k_2 - w_2}{2w_2} - \frac{(k_2 - w_2)^2}{8w_2^2}\right. \\
 &\quad + \frac{(k_2 - w_2)^3}{16w_2^3} - \frac{5(k_2 - w_2)^4}{128w_2^4} + \frac{7(k_2 - w_2)^5}{256w_2^5} - \frac{21(k_2 - w_2)^6}{1024w_2^6} \\
 &\quad - \frac{k_1 - w_1}{w_1} - \frac{(k_1 - w_1)(k_2 - w_2)}{w_1 w_2} + \frac{(k_1 - w_1)(k_2 - w_2)^2}{8w_1 w_2^2} \\
 &\quad \left. - \frac{(k_1 - w_1)(k_2 - w_2)^3}{16w_1 w_2^3} + \frac{5(k_1 - w_1)(k_2 - w_2)^4}{128w_1 w_2^4} - \frac{7(k_1 - w_1)(k_2 - w_2)^5}{256w_1 w_2^5}\right]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{(k_1 - w_1)^2}{w_1^2} + \frac{(k_1 - w_1)^2(k_2 - w_2)}{2w_1^2w_2} - \frac{(k_1 - w_1)^2(k_2 - w_2)^2}{8w_1^2w_2^2} \\
& + \frac{(k_1 - w_1)^2(k_2 - w_2)^3}{16w_1^2w_2^3} - \frac{5(k_1 - w_1)^2(k_2 - w_2)^4}{128w_1^2w_2^4} - \frac{(k_1 - w_1)^3}{w_1^3} \\
& - \frac{(k_1 - w_1)^3(k_2 - w_2)}{2w_1^3w_2} - \frac{(k_1 - w_1)^3(k_2 - w_2)^2}{8w_1^3w_2^2} - \frac{(k_1 - w_1)^3(k_2 - w_2)^3}{16w_1^3w_2^3} \\
& + \frac{(k_1 - w_1)^4}{w_1^4} + \frac{(k_1 - w_1)^4(k_2 - w_2)}{2w_1^4w_2} - \frac{(k_1 - w_1)^4(k_2 - w_2)^2}{8w_1^4w_2^2} \\
& - \frac{(k_1 - w_1)^5}{w_1^5} - \frac{(k_1 - w_1)^5(k_2 - w_2)}{w_1^5w_2} + \frac{(k_1 - w_1)^6}{w_1^6} \Bigg) = v \left(1 + \frac{(2)}{2w_2} \right. \\
& - \frac{(2^2)}{8w_2^2} + \frac{(2^3)}{16w_2^3} - \frac{5(2^4)}{128w_2^4} + \frac{7(2^5)}{256w_2^5} - \frac{21(2^6)}{1024w_2^6} - \frac{(1)}{w_1} - \frac{(12)}{w_1w_2} + \frac{(12^2)}{8w_1w_2^2} \\
& - \frac{(12^3)}{16w_1w_2^3} + \frac{5(12^4)}{128w_1w_2^4} - \frac{7(12^5)}{256w_1w_2^5} + \frac{(1^2)}{w_1^2} + \frac{(1^22)}{2w_1^2w_2} - \frac{(1^22^2)}{8w_1^2w_2^2} + \frac{(1^22^3)}{16w_1^2w_2^3} \\
& - \frac{5(1^22^4)}{128w_1^2w_2^4} - \frac{(1^3)}{w_1^3} - \frac{(1^32)}{2w_1^3w_2} + \frac{(1^32^2)}{8w_1^3w_2^2} - \frac{(1^32^3)}{16w_1^3w_2^3} + \frac{(1^4)}{w_1^4} + \frac{(1^42)}{2w_1^4w_2} \\
& \left. - \frac{(1^42^2)}{8w_1^4w_2^2} - \frac{(1^5)}{w_1^5} - \frac{(1^52)}{w_1^5w_2} + \frac{(1^6)}{w_1^6} \right) .
\end{aligned}$$

Appendix I gives us directly the values of $(1^h 2^l)$. Similar expansions can be made for the higher order moments of v . After a considerable amount of simplification the following equations are obtained.

$$\begin{aligned}
 \mu_1'(v) = v & \left[1 + \frac{1}{n} \left(\frac{w_2}{w_1^2} - \frac{w_3}{2w_1w_2} - \frac{w_4}{8w_2^2} \right) - \frac{1}{4(n-1)} + \frac{1}{n^2} \left(-\frac{5w_3}{2w_1^3} + \frac{3w_2^2}{w_1^4} \right. \right. \\
 & + \frac{3w_4}{8w_1^2w_2} + \frac{w_5}{8w_1w_2^2} - \frac{w_3^2}{4w_1^2w_2^2} + \frac{w_6}{16w_2^3} - \frac{3w_3w_4}{16w_1w_2^3} - \frac{15w_4^2}{128w_2^4} \left. \right) + \frac{1}{n(n-1)} \left(\frac{9w_4}{32w_2^2} \right. \\
 & + \frac{w_3}{8w_1w_2} - \frac{w_2}{4w_1^2} \left. \right) + \frac{1}{(n-1)^2} \left(\frac{w_3^2}{4w_2^3} + \frac{1}{32} \right) + \frac{1}{n^3} \left(\frac{29w_4}{8w_1^4} - \frac{35w_2w_3}{2w_1^5} + \frac{15w_3^2}{w_1^6} \right. \\
 & - \frac{w_5}{8w_1^3w_2} + \frac{w_3^2}{2w_1^4w_2} - \frac{w_6}{16w_1^2w_2^2} + \frac{5w_3w_4}{16w_1^3w_2^2} - \frac{w_7}{16w_1w_2^3} + \frac{3w_3w_5}{8w_1^2w_2^3} + \frac{9w_4^2}{128w_1^2w_2^3} \\
 & - \frac{3w_3^3}{8w_1^3w_2^3} - \frac{5w_8}{128w_2^4} - \frac{15w_2^2w_4}{32w_1^2w_2^4} + \frac{5w_3w_6}{32w_1w_2^4} + \frac{15w_4w_5}{64w_1w_2^4} + \frac{35w_4w_6}{128w_2^5} - \frac{105w_3w_4^2}{256w_1w_2^5} \\
 & - \frac{315w_4^3}{1024w_2^6} + \frac{5w_3^2}{16w_1^2w_2^2} + \frac{5w_4}{32w_1^2w_2} + \frac{5w_3}{8w_1^3} - \frac{3w_2^2}{4w_1^4} - \frac{9w_5}{32w_1w_2^2} - \frac{5w_3w_4}{64w_1w_2^3} \\
 & - \frac{25w_6}{64w_2^3} + \frac{95w_4^2}{512w_2^4} - \frac{w_3}{64w_1w_2} - \frac{33w_2^2}{16w_2^3} + \frac{w_2}{32w_1^2} - \frac{5w_3w_5}{4w_2^4} - \frac{145w_4}{256w_2^2} + \frac{5w_3^3}{8w_1w_2^4} \\
 & \left. + \frac{35w_3^2w_4}{32w_2^5} + \frac{5}{128} \right] \quad (5.1)
 \end{aligned}$$

$$\begin{aligned}
\mu_2(v) = E[v - E(v)]^2 = v^2 & \left[\frac{1}{n} \left(\frac{w_2}{w_1^2} - \frac{w_3}{w_1 w_2} + \frac{w_4}{4w_2^2} \right) + \frac{1}{2(n-1)} \right. \\
& + \frac{1}{n^2} \left(\frac{7w_4^2}{32w_2^4} + \frac{w_3 w_4}{4w_1 w_2^3} - \frac{w_6}{8w_2^3} + \frac{w_3^2}{4w_1^2 w_2^2} - \frac{w_5}{4w_1 w_2^2} + \frac{5w_4}{2w_1^2 w_2} - \frac{10w_3}{w_1^3} \right. \\
& \left. \left. + \frac{8w_2^2}{w_1^4} \right) + \frac{1}{n(n-1)} \left(\frac{w_2}{w_1^2} - \frac{w_3}{2w_1 w_2} - \frac{5w_4}{8w_2^2} \right) + \frac{1}{(n-1)^2} \left(-\frac{1}{8} \right. \right. \\
& \left. \left. - \frac{w_3^2}{2w_2^3} \right) + \frac{1}{n^3} \left(-\frac{3w_3}{w_1^3} + \frac{7w_2^2}{2w_1^4} - \frac{3w_4}{4w_1^2 w_2} + \frac{5w_5}{8w_1 w_2^2} - \frac{9w_3^2}{8w_1^2 w_2^2} + \frac{13w_6}{16w_2^3} \right. \right. \\
& + \frac{3w_3 w_4}{8w_1 w_2^3} - \frac{23w_4^2}{64w_2^4} + \frac{41w_4}{32w_2^2} + \frac{w_3}{8w_1 w_2} - \frac{w_2}{4w_1^2} + \frac{17w_3^2}{4w_2^3} - \frac{1}{16} - \frac{107w_2 w_3}{w_1^5} \\
& + \frac{69w_3^3}{w_1^6} + \frac{17w_2^2}{w_1^4 w_2} - \frac{w_3 w_4}{2w_1^3 w_2^2} + \frac{3w_4^2}{16w_2^3 w_1^2} - \frac{5w_3 w_5}{8w_1^2 w_2^3} + \frac{w_3^3}{2w_1^3 w_2^3} + \frac{21w_3 w_4^2}{32w_1 w_2^5} \\
& - \frac{w_3^3}{w_1 w_2^4} - \frac{7w_4 w_5}{16w_1 w_2^4} - \frac{17w_4 w_6}{32w_2^5} + \frac{75w_4^3}{128w_2^6} - \frac{17w_2^2 w_4}{8w_2^5} + \frac{111w_4}{4w_1^4} + \frac{w_7}{8w_1 w_2^3} \\
& \left. \left. + \frac{5w_8}{64w_2^4} - \frac{w_3 w_6}{4w_1 w_2^4} + \frac{5w_3 w_5}{2w_2^4} + \frac{11w_3^2 w_4}{16w_1^2 w_2^4} - \frac{4w_5}{w_2 w_1^3} \right) \right] \quad (5.2)
\end{aligned}$$

$$\begin{aligned}
\mu_3(v) = v^3 & \left[\frac{1}{n^2} \left(-\frac{3w_4^2}{16w_2^4} + \frac{w_6}{8w_2^3} + \frac{9w_3^2}{4w_1^2 w_2^2} - \frac{3w_5}{4w_1 w_2^2} + \frac{3w_4}{w_1^2 w_2} - \frac{10w_3}{w_1^3} \right. \right. \\
& \left. \left. + \frac{6w_2^2}{w_1^4} \right) + \frac{1}{n(n-1)} \left(\frac{3w_4}{4w_2^2} - \frac{3w_3}{w_1 w_2} + \frac{3w_2}{w_1^2} \right) + \frac{1}{(n-1)^2} \left(\frac{w_3^2}{2w_2^3} + \frac{1}{4} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{n^3} \left[-\frac{41w_3^2}{8w_2^3} - \frac{3w_8}{32w_2^4} - \frac{33w_6}{32w_2^3} - \frac{3w_3w_5}{w_2^4} + \frac{27w_4^2}{64w_2^4} - \frac{57w_4}{32w_2^2} + \frac{1}{16} + \frac{39w_4w_6}{64w_2^5} \right. \\
& + \frac{39w_3^2w_4}{16w_2^5} - \frac{83w_4^3}{128w_2^6} + \frac{3w_5}{16w_1w_2^2} + \frac{3w_3w_4}{4w_1w_2^3} + \frac{3w_3}{8w_1w_2} + \frac{3w_3w_6}{16w_1w_2^4} + \frac{15w_6}{8w_1^2w_2^2} + \frac{3w_4w_5}{32w_1w_2^4} \\
& + \frac{3w_3^3}{4w_1w_2^4} - \frac{9w_3w_5}{8w_1^2w_2^3} - \frac{3w_4^2}{16w_1^2w_2^3} + \frac{57w_4}{8w_1^2w_2} + \frac{27w_3^2}{16w_1^2w_2^2} + \frac{15w_4w_3^3}{32w_1^2w_2^4} - \frac{57w_5}{4w_1^3w_2} - \frac{69w_3w_4}{4w_1^3w_2^2} \\
& \left. - \frac{42w_3}{w_1^3} + \frac{19w_3^3}{8w_1^3w_2^3} + \frac{261w_4}{4w_1^4} + \frac{303w_3^2}{4w_1^4w_2} + \frac{63w_2^2}{2w_1^4} - \frac{225w_2w_3}{w_1^5} + \frac{116w_3^3}{w_1^6} - \frac{15w_3w_4^2}{32w_1w_2^5} \right] \quad (5.3)
\end{aligned}$$

$$\begin{aligned}
\mu_4(v) &= v^4 \left[\frac{1}{n^2} \left(\frac{3w_4^2}{16w_2^4} - \frac{3w_3w_4}{2w_1w_2^3} + \frac{3w_3^2}{w_1^2w_2^2} + \frac{3w_4}{2w_1^2w_2} + \frac{3w_2^2}{w_1^4} - \frac{6w_3}{w_1^3} \right) \right. \\
& + \frac{1}{n(n-1)} \left(\frac{3w_4}{4w_2^2} - \frac{3w_3}{w_1w_2} + \frac{3w_2}{w_1^2} \right) + \frac{1}{(n-1)^2} \left(\frac{3}{4} \right) + \frac{1}{n^3} \left(\frac{79w_4}{w_1^4} - \frac{276w_2w_3}{w_1^5} \right. \\
& + \frac{120w_3^3}{w_1^6} - \frac{31w_5}{2w_1^3w_2} + \frac{303w_3^2}{2w_1w_2} + \frac{9w_6}{4w_1^2w_2^2} - \frac{117w_3w_4}{2w_1^3w_2^2} + \frac{105w_3^2}{4w_1^2w_2^2} + \frac{105w_4}{4w_1^2w_2} - \frac{111w_3}{w_1^3} \\
& - \frac{72w_2^2}{w_1^4} - \frac{w_7}{2w_1w_2^3} + \frac{6w_3w_5}{w_1^2w_2^3} + \frac{81w_4^2}{16w_1^2w_2^3} - \frac{15w_3^3}{2w_1^3w_2^3} + \frac{w_8}{16w_2^4} - \frac{27w_3^2w_4}{8w_1^2w_2^4} + \frac{3w_3w_6}{4w_1w_2^4} \\
& - \frac{9w_4w_6}{16w_2^5} - \frac{15w_3w_4^2}{16w_1w_2^5} + \frac{45w_4^3}{64w_2^6} - \frac{21w_2^5}{4w_1w_2^2} - \frac{13w_3w_4}{4w_1w_2^3} - \frac{17w_4^2}{32w_2^4} - \frac{27w_3}{4w_1w_2} + \frac{33w_2}{4w_1^2} \\
& \left. + \frac{2w_3w_5}{w_2^4} - \frac{9w_4}{16w_2^2} + \frac{3w_3^3}{w_1w_2^4} - \frac{9w_3^2w_4}{4w_2^5} - \frac{3}{8} + \frac{3w_6}{8w_2^3} + \frac{3w_4w_5}{8w_1w_2^4} + \frac{3w_3^2}{2w_2^3} \right] \quad (5.4)
\end{aligned}$$

Comparing equations (5.1 - 5.4) with F. N. David's results [5] one notes that they agree through order $\frac{1}{n^2}$. This agreement may be misleading as we both used F. N. David's table of corrective terms. Myers in [21] compares F. N. David's results with the ones he obtained by the use of orthogonal statistics and obtains agreement. This gives an indication that the terms in Appendix I are likely to be error free.

If one considers as significant only expansions through order $\frac{1}{n^3}$, then the following terms have, from a mathematical standpoint, equivalent values:

$$\frac{1}{n-1} \Leftrightarrow \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}$$

$$\frac{1}{n(n-1)} \Leftrightarrow \frac{1}{n^2} + \frac{1}{n^3}$$

$$\frac{1}{(n-1)^2} \Leftrightarrow \frac{1}{n^2} + \frac{2}{n^3}.$$

Using the above results in (5.1 - 5.4) one obtains the following expansions:

$$\begin{aligned} \mu'_1(v) = & V \left[1 + \frac{1}{n} \left(\frac{w_2}{w_1^2} - \frac{w_3}{2w_1w_2} - \frac{w_4}{8w_2^2} - \frac{1}{4} \right) + \frac{1}{n^2} \left(-\frac{5w_3}{2w_1^3} + \frac{3w_2^2}{w_1^4} + \frac{3w_4}{8w_1^2w_2} \right. \right. \\ & + \frac{w_5}{8w_1w_2^2} - \frac{w_3^2}{4w_1^2w_2^2} + \frac{w_6}{16w_2^3} - \frac{3w_3w_4}{16w_1w_2^3} - \frac{15w_4^2}{128w_2^4} + \frac{9w_4}{32w_2^2} + \frac{w_3}{8w_1w_2} - \frac{w_2}{4w_1^2} + \frac{w_3^2}{4w_2^3} \\ & \left. \left. - \frac{7}{32} \right) + \frac{1}{n^3} \left(\frac{29w_4}{8w_1^4} - \frac{35w_2w_3}{2w_1^5} + \frac{15w_3^2}{w_1^6} - \frac{w_5}{8w_1^3w_2} + \frac{w_3^2}{2w_1^4w_2} - \frac{w_6}{16w_1^2w_2^2} + \frac{5w_3w_4}{16w_1^3w_2^2} \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{w_7}{16w_1w_2^3} + \frac{3w_3w_5}{8w_1^2w_2^3} + \frac{9w_4^2}{128w_1^2w_2^3} - \frac{3w_3^3}{8w_1^3w_2^3} - \frac{5w_8}{128w_2^4} - \frac{15w_4w_3^2}{32w_1^2w_2^4} + \frac{5w_3w_6}{32w_1w_2^4} \\
& + \frac{15w_4w_5}{64w_1w_2^4} + \frac{35w_4w_6}{128w_2^5} - \frac{105w_3w_4^2}{256w_1w_2^5} - \frac{315w_4^3}{1024w_2^6} + \frac{5w_3^2}{16w_1^2w_2^2} + \frac{5w_4}{32w_1^2w_2} + \frac{5w_3}{8w_1^3} \\
& - \frac{3w_2^2}{4w_1^4} - \frac{9w_5}{32w_1w_2^2} - \frac{5w_3w_4}{64w_1w_2^3} - \frac{25w_6}{64w_2^3} + \frac{95w_4^2}{512w_2^4} + \frac{7w_3}{64w_1w_2} - \frac{25w_3^2}{16w_2^3} - \frac{7w_2}{32w_1^2} \\
& - \left[\frac{5w_3w_5}{4w_2^4} - \frac{73w_4}{256w_2^2} + \frac{5w_3^3}{8w_1w_2^4} + \frac{35w_3^2w_4}{32w_2^5} - \frac{19}{128} \right] \quad (5.5)
\end{aligned}$$

$$\begin{aligned}
\mu_2(v) = v^2 & \left[\frac{1}{n} \left(\frac{w_2}{w_1^2} - \frac{w_3}{w_1w_2} + \frac{w_4}{4w_2^2} + \frac{1}{2} \right) + \frac{1}{n^2} \left(\frac{7w_4^2}{32w_2^4} + \frac{w_3w_4}{4w_1w_2^3} - \frac{w_6}{8w_2^3} \right. \right. \\
& + \left. \frac{w_3^2}{4w_1^2w_2^2} - \frac{w_5}{4w_1w_2^2} + \frac{5w_4}{2w_1^2w_2} - \frac{10w_3}{w_1^3} + \frac{8w_2^2}{w_1^4} + \frac{3}{8} + \frac{w_2}{w_1^2} - \frac{w_3}{2w_1w_2} - \frac{5w_4}{8w_2^2} - \frac{w_3^2}{2w_2^3} \right) \\
& + \frac{1}{n^3} \left(\frac{3}{16} + \frac{3w_2}{4w_1^2} - \frac{3w_3}{8w_1w_2} + \frac{21w_4}{32w_2^2} + \frac{13w_3^2}{4w_2^3} - \frac{3w_3}{w_1^3} + \frac{7w_2^2}{2w_1^4} - \frac{3w_4}{4w_1^2w_2} + \frac{5w_5}{8w_1w_2^2} \right. \\
& - \frac{9w_3^2}{8w_1^2w_2^2} + \frac{13w_6}{16w_2^3} + \frac{3w_3w_4}{8w_1w_2^3} - \frac{23w_4^2}{64w_2^4} - \frac{107w_2w_3}{w_1^5} + \frac{69w_3^3}{w_1^6} + \frac{17w_2^2}{w_1^4w_2} - \frac{w_3w_4}{2w_1^3w_2^3} \\
& + \frac{3w_4^2}{16w_2^3w_1} - \frac{5w_3w_5}{8w_1^2w_2^3} + \frac{w_3^3}{2w_1^3w_2^3} + \frac{21w_3w_4^2}{32w_1w_2^5} - \frac{w_3^3}{w_1w_2^4} - \frac{7w_4w_5}{16w_1w_2^4} - \frac{17w_4w_6}{32w_2^5} + \frac{75w_4^3}{128w_2^6} \\
& \left. - \frac{17w_3^2w_4}{8w_2^5} + \frac{111w_4}{4w_1^4} + \frac{w_7}{8w_1w_2^3} + \frac{5w_8}{64w_2^4} - \frac{w_3w_6}{4w_1w_2^4} + \frac{5w_3w_5}{2w_2^4} + \frac{11w_3^2w_4}{16w_1^2w_2^4} - \frac{4w_5}{w_1^3w_2} \right] \quad (5.6)
\end{aligned}$$

$$\begin{aligned}
\mu_3(v) = v^3 & \left[\frac{1}{n^2} \left(-\frac{3w_4^2}{16w_2^4} + \frac{w_6}{8w_2^3} + \frac{9w_3^2}{4w_1^2w_2^2} - \frac{3w_5}{4w_1w_2^2} + \frac{3w_4}{w_1^2w_2} - \frac{10w_3}{w_1^3} \right. \right. \\
& + \frac{6w_2^2}{w_1^4} + \frac{3w_4}{4w_2^2} - \frac{3w_3}{w_1w_2} + \frac{3w_2}{w_1^2} + \frac{w_3^2}{2w_2^3} + \frac{1}{4} \left. \right) + \frac{1}{n^3} \left(-\frac{33w_4}{32w_2^2} - \frac{21w_3}{8w_1w_2} \right. \\
& + \frac{3w_2}{w_1^2} - \frac{33w_3^2}{8w_2^3} + \frac{9}{16} - \frac{3w_8}{32w_2^4} - \frac{33w_6}{32w_2^3} - \frac{3w_3w_5}{w_2^4} + \frac{27w_4^2}{64w_2^4} + \frac{39w_4w_6}{64w_2^5} \\
& + \frac{39w_3^2w_4}{16w_2^5} - \frac{83w_4^3}{128w_2^6} + \frac{3w_5}{16w_1w_2^2} + \frac{3w_3w_4}{4w_1w_2^3} + \frac{3w_3w_6}{16w_1w_2^4} + \frac{15w_6}{8w_1^2w_2^2} + \frac{3w_4w_5}{32w_1w_2^4} \\
& + \frac{3w_3^3}{4w_1w_2^4} - \frac{9w_3w_5}{8w_1^2w_2^3} - \frac{3w_4^2}{16w_1^2w_2^3} + \frac{27w_3^2}{16w_1^2w_2^2} + \frac{15w_3^2w_4}{32w_1^2w_2^4} - \frac{57w_5}{4w_1^3w_2} - \frac{69w_3w_4}{4w_1^3w_2^2} \\
& - \frac{42w_3}{w_1^3} + \frac{19w_3^3}{8w_1^3w_2^3} + \frac{261w_4}{4w_1^4} + \frac{303w_3^2}{4w_1^4w_2} + \frac{63w_2^2}{2w_1^4} - \frac{225w_2w_3}{w_1^5} + \frac{116w_3^3}{w_1^6} - \frac{15w_3w_4^2}{32w_1w_2^5} \\
& \left. \left. + \frac{57w_4}{8w_1^2w_2} \right) \right] \quad (5.7)
\end{aligned}$$

$$\begin{aligned}
\mu_4(v) = v^4 & \left[\frac{1}{n^2} \left(\frac{3w_4^2}{16w_2^4} - \frac{3w_3w_4}{2w_1w_2^3} + \frac{3w_3^2}{w_1^2w_2^2} + \frac{3w_4}{2w_1^2w_2} + \frac{3w_2^2}{w_1^4} - \frac{6w_3}{w_1^3} + \frac{3w_4}{4w_2^2} \right. \right. \\
& - \frac{3w_3}{w_1w_2} + \frac{3w_2}{w_1^2} + \frac{3}{4} \left. \right) + \frac{1}{n^3} \left(\frac{3w_4}{16w_2^2} - \frac{39w_3}{4w_1w_2} + \frac{45w_2}{4w_1^2} + \frac{9}{8} + \frac{79w_4}{w_1^4} - \frac{276w_2w_3}{w_1^5} \right. \\
& + \frac{120w_3^3}{w_1^6} - \frac{31w_5}{2w_1^3w_2} + \frac{303w_3^2}{2w_1^4w_2} + \frac{9w_6}{4w_1^2w_2^2} - \frac{117w_3w_4}{2w_1^3w_2^2} + \frac{105w_3^2}{4w_1^2w_2^2} + \frac{105w_4}{4w_1^2w_2} - \frac{111w_3}{w_1^3} \\
& \left. \left. + \frac{57w_4}{8w_1^2w_2} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{72w_2^2}{w_1^4} - \frac{w_7}{2w_1w_2^3} + \frac{6w_3w_5}{w_1^2w_2^3} + \frac{81w_4^2}{16w_1^2w_2^3} - \frac{15w_3^3}{2w_1^3w_2^3} + \frac{w_8}{16w_2^4} - \frac{27w_3^2w_4}{8w_1^2w_2^4} + \frac{3w_3w_6}{4w_1w_2^4} \\
& - \frac{9w_4w_6}{16w_2^5} - \frac{15w_3w_4^2}{16w_1w_2^5} + \frac{45w_4^3}{64w_2^6} - \frac{21w_5}{4w_1w_2^2} - \frac{13w_3w_4}{4w_1w_2^3} - \frac{17w_4^2}{32w_2^4} + \frac{2w_3w_5}{w_2^4} + \frac{3w_3^3}{w_1w_2^4} \\
& - \frac{9w_3^2w_4}{4w_2^5} + \frac{3w_6}{8w_2^3} + \frac{3w_4w_5}{8w_1w_2^4} + \frac{3w_3^2}{2w_2^3} \Bigg] \quad . \quad (5.8)
\end{aligned}$$

Having obtained the approximate moments, it would be appropriate to apply them to some of the more useful parent distributions. For the S.C.V. the two most often used parent distributions are the normal and the two parameter gamma. For some situations where the gamma distribution may be applicable see e.g. Linhart [18].

The gamma frequency function is stated as

$$f(x) = \frac{1}{\Gamma(p)\beta^p} x^{p-1} e^{-x/\beta}, \quad 0 < x < \infty.$$

Since v is scalar invariant one could equivalently consider $Y = X/\beta$, having the frequency

$$f(y) = \frac{1}{\Gamma(p)} y^{p-1} e^{-y}, \quad 0 < y < \infty.$$

For the random variable Y , $w_r = p\Gamma(r)$ and, thus, $V = p^{-1/2}$. Using the above results in equations (5.5 - 5.8) one obtains the following approximate moments for the gamma case:

$$\mu_1'(v) = \frac{1}{p^{1/2}} \left[1 + \frac{1}{4n} \left(-1 - \frac{3}{p} \right) + \frac{1}{32n^2} \left(-7 + \frac{86}{p} + \frac{105}{p^2} \right) \right]$$

$$+ \frac{1}{128n^3} \left[-19 - \frac{1019}{p} - \frac{9465}{p^2} - \frac{8505}{p^3} \right]. \quad (5.9)$$

$$\begin{aligned} \mu_2(v) = & \frac{1}{p} \left[\frac{1}{2n} \left(1 + \frac{1}{p} \right) + \frac{1}{8n^2} \left(3 - \frac{46}{p} - \frac{49}{p^2} \right) + \frac{1}{16n} \left(3 + \frac{271}{p} \right. \right. \\ & \left. \left. + \frac{1}{16n} \left(3 + \frac{271}{p} + \frac{2457}{p} + \frac{2189}{p} \right) \right] \right]. \quad (5.10) \end{aligned}$$

$$\begin{aligned} \mu_3(v) = & \frac{1}{p^3} \left[\frac{1}{4n^2} \left(1 + \frac{14}{p} + \frac{13}{p^2} \right) + \frac{1}{16n^3} \left(9 - \frac{399}{p} - \frac{2841}{p^2} \right. \right. \\ & \left. \left. - \frac{2289}{p^3} \right) \right]. \quad (5.11) \end{aligned}$$

$$\mu_4(v) = \frac{1}{p^2} \left[\frac{3}{4n^2} \left(1 + \frac{2}{p} + \frac{1}{p^2} \right) + \frac{9}{8n^3} \left(1 - \frac{1}{p} + \frac{35}{p^2} + \frac{37}{p^3} \right) \right]. \quad (5.12)$$

The above expansion will be used in the discussion of the robustness of the normality assumption of the parent. A complete discussion of robustness will be given in Chapter VII.

Before considering the approximate moments from a normal parent, it would be appropriate to say something about their validity and then give a brief discussion of their usefulness. It was shown, in Chapter II, that for a normal parent none of the moments, of the S.C.V., exist. We have seen, in Chapter III, that if the observations cannot be negative then all the moments of v exist. In most applications the observations are non-negative and even in such situations the normal frequency

$N(\mu, \sigma^2)$ may fit the data extremely well. Under the above mentioned conditions the truncated normal distribution

$$g(x) = \frac{N(\mu, \sigma^2)}{1 - N(0)} \cdot x \geq 0 \text{ and } N(0) = \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^0 e^{-(x-\mu)^2 / 2\sigma^2} dx$$

would also give a good fit and, thus, could be used as the parent distribution. If the truncated normal is assumed for the parent distribution then the moments of v would exist. At the same time, the moments of the truncated parent distribution (i.e. the parent random variables, not v) should be very close to those of the normal distribution. Thus, one might consider the appropriate moments of v as being those obtained from a truncated normal parent distribution, in order that their values be meaningful.

We have seen, in Chapter III, that for a normal parent the exact distribution of the S.C.V. is known. For this reason it seems appropriate to give a brief discussion of the purpose for the moment expansions, given in this chapter, when a normal parent distribution is assumed. One reason for these expansions is to obtain the first few moments, especially the mean and variance, of the S.C.V. These moments were also used to approximate the distribution of the S.C.V. and it would be appropriate to study the quality of such distributional approximations. For some values of the parameters these approximations will be shown to be exceptionally good and, hence, no tables, in such a parametric region, will be required.

For a normal distribution $w_i = 0$ for $i > 2$ and for this reason the approximate moments of v follow trivially from equations (5.5 - 5.8). They are:

$$\mu_1'(v) = v \left[1 + \frac{1}{n} \left(v^2 - \frac{1}{4} \right) + \frac{1}{n^2} \left(3v^4 - \frac{v^2}{4} - \frac{7}{32} \right) + \frac{1}{n^3} \left(15v^6 - \frac{3v^4}{4} - \frac{7v^2}{32} - \frac{19}{128} \right) \right]. \quad (5.13)$$

$$\mu_2(v) = v^2 \left[\frac{1}{n} \left(v^2 + \frac{1}{2} \right) + \frac{1}{n^2} \left(8v^4 + v^2 + \frac{3}{8} \right) + \frac{1}{n^3} \left(69v^6 + \frac{7}{2} v^4 + \frac{3}{4} v^2 + \frac{3}{16} \right) \right]. \quad (5.14)$$

$$\mu_3(v) = v^3 \left[\frac{1}{n^2} \left(6v^4 + 3v^2 + \frac{1}{4} \right) + \frac{1}{n^3} \left(116v^6 + \frac{63v^4}{2} + 3v^2 + \frac{9}{16} \right) \right]. \quad (5.15)$$

$$\mu_4(v) = v^4 \left[\frac{1}{n^2} \left(3v^4 + 3v^2 + \frac{3}{4} \right) + \frac{1}{n^3} \left(120v^6 + 72v^4 + \frac{45}{4} v^2 + \frac{9}{8} \right) \right]. \quad (5.16)$$

These approximate moment results will find application in a later chapter.

B. The Moments of the Statistic $Z = S - t_0 \bar{X}$.

It was noticed by several workers, e.g. Hald [12], that the statistic $Z = S - t_0 \bar{X}$ can be beneficially applied to the study of the distributional properties of the S.C.V. Z has two important advantages over the S.C.V. If the parent distribution is almost normal then Z becomes a linear combination of two nearly normal variates and, consequently, is

itself almost normal. Therefore, if one knows its mean and variance, one could use Z to obtain approximate percentage points of v . Another advantage of Z is that it is a linear combination of two statistics while v is a ratio of the same two statistics.

If $\Pr(\bar{X} \geq 0 \mid V) = 1$ then $\Pr\left(\frac{S}{\bar{X}} \leq t_o \mid V\right) = \Pr(S - t_o \bar{X} \leq 0 \mid V)$ and it follows that a knowledge of the distribution of Z could be beneficially applied in the study of the S.C.V. Emphasis will be given here to the moments of Z . These moments are given below:

$$\mu'_1(Z) = E(S) - t_o w_1 \quad (5.17)$$

$$\begin{aligned} \mu'_2(Z) &= E(S - t_o \bar{X})^2 - [E(S) - t_o w_1]^2 = w_2 - 2t_o w_1 E\left[S \left(\frac{k_1 - w_1}{w_1}\right)\right] \\ &+ t_o^2 \frac{w_2}{n} - [E(S)]^2 \end{aligned} \quad (5.18)$$

$$\begin{aligned} \mu'_3(Z) &= E(S - t_o \bar{X})^3 = E(S^3) - 3t_o \left(w_1 w_2 + \frac{w_3}{n}\right) + 3t_o w_1 E(S) + 2E\left[S \left(\frac{k_1 - w_1}{w_1}\right)\right] \\ &+ E\left[S \left(\frac{k_1 - w_1}{w_1}\right)^2\right] - t_o^3 \left(w_1^3 + \frac{3w_1 w_2}{n} + \frac{w_3}{n^2}\right) \end{aligned} \quad (5.19)$$

$$\begin{aligned} \mu'_4(Z) &= E(S - t_o \bar{X})^4 = \left[w_2^2 + \frac{1}{n} \left(w_4 + 2w_2^2\right) + \frac{2w_2^2}{n^2} - \frac{2w_2^2}{n^3}\right] - 4t_o w_1 E(S^3) \\ &+ E\left\{S^3 \left(\frac{k_1 - w_1}{w_1}\right)\right\} + 6t_o^2 w_1^2 w_2 \left[1 + \frac{1}{n} \left(\frac{w_2}{w_1^2} + \frac{2w_3}{w_1 w_2}\right) + \frac{1}{n^2} \left(\frac{w_4}{w_1^2 w_2}\right)\right] \end{aligned}$$

$$\begin{aligned}
& - 4t_0 w_1 \left[E(S) + 3E \left\{ S \left(\frac{k_1 - w_1}{w_1} \right) \right\} + 3E \left\{ S \left(\frac{k_1 - w_1}{w_1} \right)^2 \right\} + E \left\{ S \left(\frac{k_1 - w_1}{w_1} \right)^3 \right\} \right] \\
& + t_0^4 \left[w_1^4 + \frac{6w_1^2 w_2}{n} + \frac{1}{n^2} (4w_1 w_3 + 3w_2^2) + \frac{w_4}{n^3} \right], \quad (5.20)
\end{aligned}$$

where

$$\begin{aligned}
E(S) &= w_2^{1/2} \left[1 + \frac{1}{n} \left(-\frac{w_4}{8w_2^2} - \frac{1}{4} \right) + \frac{1}{n^2} \left(\frac{w_6}{16w_2^3} - \frac{15w_4^2}{128w_2^4} + \frac{9w_4}{32w_2^2} + \frac{w_3^2}{4w_2^3} \right. \right. \\
& \left. \left. - \frac{7}{32} \right) + \frac{1}{n^3} \left(-\frac{19}{128} - \frac{73w_4}{256w_2^2} - \frac{25w_3^2}{16w_2^3} - \frac{5w_8}{128w_2^4} - \frac{25w_6}{64w_2^3} - \frac{5w_3 w_5}{4w_2^4} + \frac{95w_4^2}{512w_2^4} \right. \right. \\
& \left. \left. + \frac{35w_4 w_6}{128w_2^5} + \frac{35w_3^2 w_4}{32w_2^5} - \frac{315w_4^3}{1024w_2^6} \right) \right] \\
E \left[S \left(\frac{k_1 - w_1}{w_1} \right) \right] &= w_2^{1/2} \left[\frac{1}{n} \left(\frac{w_3}{2w_1 w_2} \right) + \frac{1}{n^2} \left(-\frac{w_5}{8w_1 w_2^2} + \frac{3w_3 w_4}{16w_1 w_2^3} \right. \right. \\
& \left. \left. - \frac{w_3}{8w_1 w_2} \right) + \frac{1}{n^3} \left(-\frac{7w_3}{64w_1 w_2} + \frac{w_7}{16w_1 w_2^3} + \frac{9w_5}{32w_1 w_2^2} + \frac{5w_3 w_4}{64w_1 w_2^3} - \frac{5w_3^3}{8w_1 w_2^4} \right. \right. \\
& \left. \left. - \frac{5w_3 w_6}{32w_1 w_2^4} - \frac{15w_4 w_6}{64w_1 w_2^4} + \frac{105w_3 w_4^2}{256w_1 w_2^5} \right) \right] \\
E \left[S \left(\frac{k_1 - w_1}{w_1} \right)^2 \right] &= w_2^{1/2} \left[\frac{w_2}{w_1 n} + \frac{1}{n^2} \left(\frac{3w_4}{8w_1^2 w_2} - \frac{w_3^2}{4w_1^2 w_2^2} - \frac{w_2}{4w_1^2} \right) \right]
\end{aligned}$$

$$+ \frac{1}{n^3} \left[-\frac{w_6}{16w_1^2w_2^2} + \frac{5w_3^2}{16w_1^2w_2^2} + \frac{5w_4}{32w_1^2w_2} + \frac{3w_3w_5}{8w_1^2w_2^3} + \frac{9w_4^2}{128w_1^2w_2^3} - \frac{7w_2}{32w_1^2} - \frac{15w_3^2w_4}{32w_1^2w_2^4} \right]$$

$$E \left[S \left(\frac{k_1 - w_1}{w_1} \right)^3 \right] = w_2^{1/2} \left[\frac{1}{n^2} \left(\frac{5w_3}{2w_1} \right) + \frac{1}{n^3} \left(\frac{w_5}{8w_1^3w_2} - \frac{5w_3w_4}{16w_1^3w_2^2} - \frac{5w_3}{8w_1} - \frac{3w_3}{8w_1w_2} \right) \right]$$

$$E(S^3) = w_2^{\frac{3}{2}} \left[1 + \frac{1}{n} \left(\frac{3w_4}{8w_2^2} + \frac{3}{4} \right) + \frac{1}{n^2} \left(-\frac{w_6}{16w_2^3} + \frac{9w_4^2}{128w_2^4} - \frac{15w_4}{32w_2^2} - \frac{w_3^2}{4w_2^3} + \frac{17}{32} \right) + \frac{1}{n^3} \left(\frac{21w_3^2}{16w_2^3} + \frac{3w_8}{128w_2^4} + \frac{21w_6}{64w_2^3} + \frac{3w_3w_5}{4w_2^4} - \frac{21w_4^2}{512w_2^4} + \frac{99w_4}{256w_2^2} + \frac{49}{128} - \frac{15w_4w_6}{128w_2^5} - \frac{15w_3^2w_4}{32w_2^5} + \frac{105w_4^3}{1024w_2^6} \right) \right]$$

$$E \left[S^3 \left(\frac{k_1 - w_1}{w_1} \right) \right] = w_2^{\frac{3}{2}} \left[\frac{1}{n} \left(\frac{3w_3}{2w_1w_2} \right) + \frac{1}{n^2} \left(\frac{3w_5}{8w_1w_2^2} - \frac{3w_3w_4}{16w_1w_2^3} + \frac{9w_3}{8w_1w_2} \right) + \frac{1}{n^3} \left(-\frac{w_7}{16w_1w_2^3} - \frac{15w_5}{32w_1w_2^2} - \frac{17w_3w_4}{64w_1w_2^3} + \frac{51w_3}{64w_1w_2} + \frac{3w_3w_6}{32w_1w_2^4} + \frac{9w_4w_5}{64w_1w_2^4} + \frac{3w_3^3}{8w_1w_2^4} - \frac{45w_3^2w_4}{256w_1w_2^5} \right) \right]$$

Using the above equations one obtains

$$\mu_2(z) = \frac{t_o w_2}{n} - 2t_o w_1 E \left[S \left(\frac{k_1 - w_1}{w_1} \right) \right] + A, \quad (4.21)$$

where

$$\begin{aligned} A = & -w_2 \left[\frac{1}{n} \left(-\frac{w_4}{4w_2} - \frac{1}{2} \right) + \frac{1}{n} \left(\frac{w_6}{8w_2} - \frac{7w_4}{32w_2} + \frac{5w_4}{8w_2} + \frac{w_3}{2w_2} - \frac{3}{8} \right) \right. \\ & + \frac{1}{n^3} \left(\frac{17w_4 w_6}{32w_2^5} - \frac{75w_4^3}{128w_2^6} + \frac{23w_4^2}{64w_2^4} + \frac{17w_3^2 w_4}{8w_2^5} - \frac{21w_4}{32w_2^2} - \frac{13w_6}{16w_2^3} - \frac{13w_3^2}{4w_2^3} - \frac{3}{16} \right. \\ & \left. \left. - \frac{5w_8}{64w_2^4} - \frac{5w_3 w_5}{2w_2^4} \right) \right]. \end{aligned}$$

We will now, as before, apply the above approximate moments to obtain results for particular parent distributions. It can be observed that because the distribution of the S.C.V. only depends on the parameters V and n , one may use any combination of w_1 and w_2 that gives the required V . Using the same argument as given in the previous section the validity of the use of the normal parent distribution, in this situation, becomes apparent. Assuming the parent distribution to be normal and letting $w_1 = 1$ and $w_2^{1/2} = V$ one obtains the approximate moments of Z .

$$\mu_1'(Z) = V \left(1 - \frac{1}{4n} - \frac{7}{32n^2} - \frac{19}{128n^3} \right) - t_o$$

$$\mu_2(Z) = V^2 \left[\frac{1}{2n} + \frac{3}{8n^2} + \frac{3}{16n^3} + \frac{t_o^2}{n} \right]$$

$$E(S) = \mu_1' + t_o$$

$$E(S^3) = V^3 \left[1 + \frac{3}{4n} + \frac{17}{32n^2} + \frac{49}{128n^3} \right]$$

$$\mu_3'(Z) = E(S^3) - 3V^2 t_o + 3 t_o^2 E(S) \left[1 + \frac{V^2}{n} \right] - t_o^3 (1 + 3V^2/n)$$

$$\mu_4'(Z) = V^4 \left[1 + 2 \left(\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} \right) \right] - 4 t_o E(S^3) + 6 t_o^2 V^2 (1 + V^2/n)$$

$$- 4 t_o^3 (1 + 3V^2/n) E(S) + t_o^4 [(1 + 6V^2/n) + 3V^4/n^2] .$$

In a similar manner one obtains the approximate moments of Z for a gamma parent distribution. Letting, as before, $p^{-1/2} = V$ one obtains

$$E(S) = \frac{1}{V} \left[1 - \frac{1}{4n} (1 + 3V^2) - \frac{1}{32n^2} (7 - 86V^2 - 105V^4) - \frac{1}{128n^3} (19 + 1019V^2 \right.$$

$$\left. + 9465V^4 + 8505V^6) \right]$$

$$E \left[S \left[\frac{k_1 - w_1}{w_1} \right] \right] = V \left[\frac{1}{n} - \frac{1}{4n^2} (1 + 3V^2) - \frac{1}{32n^3} (7 - 86V^2 - 105V^4) \right]$$

$$E \left[S \left(\frac{k_1 - w_1}{w_1} \right)^2 \right] = v \left[\frac{1}{n} - \frac{1}{4n^2} (1 - 5v^2) - \frac{1}{32n^3} (7 - 70v^2 - 57v^4) \right]$$

$$E \left[S \left(\frac{k_1 - w_1}{w_1} \right)^3 \right] = v^3 \left[\frac{5}{n^2} - \frac{1}{4n^3} (5 - 9v^2) \right]$$

$$E(S^3) = \frac{1}{v^3} \left[1 + \frac{1}{4n} (3 + 9v^2) + \frac{1}{32n^2} (17 - 122v^2 - 159v^4) + \frac{1}{128n^3} (49 + 969v^2 + 8019v^4 + 7155v^6) \right]$$

$$E \left[S^3 \left(\frac{k_1 - w_1}{w_1} \right) \right] = \frac{1}{v} \left[\frac{3}{4} + \frac{1}{4n^2} (9 + 27v^2) + \frac{1}{32n^3} (51 - 366v^2 - 477v^4) \right]$$

$$A = \frac{1}{v^2} \left[\frac{1}{2n} (1 + 3v^2) + \frac{1}{8n^2} (3 - 46v^2 - 57v^4) - \frac{1}{16n^3} (3 + 271v^2 + 2457v^4 + 2205v^6) \right].$$

The moments of Z , for the normal parent distribution, will be used in Chapter VI in the study of approximate methods for the distribution of the S.C.V. The moment results, for the gamma case can be used in the study of robustness, although, they were found to add little to the accuracy over the use of the moments of the S.C.V. and were for this reason, not used.

C. Some Monte Carlo Results

Moment results, such as those given in this chapter, can have their

accuracy independently verified by the use of Monte Carlo procedures. These have been done for several combinations of n and V and the results agreed favorably with the ones obtained from the approximate moments.

Table (5.1) gives the results of one such combination. This result is for a small value of n and closer agreement can be expected when the sample size is increased.

Table (5.1) consists of $N = 1000$ and then 5000 samples of the S.C.V., each having sample size 15. The population sampled is normal with C.V. = .2. The notation, of Table (5.1), consists of the mean = $E(v)$ and $STD. = [\text{var}(v)]^{1/2}$. β_1 and β_2 are the two Pearson Beta's.

TABLE 5.1

Comparison Between Monte Carlo Results and Approximate Moments
For $V = .2$ and $n = 15$

N	Mean	Std.	β_1	β_2
1000	.1936	.0367	.2865	2.957
5000	.1936	.0369	.2467	2.958
From Appr. Moment Formulas	.1961	.0377	.2294	3.113

CHAPTER VI

Approximate Methods for Obtaining the Percentage Points of the S.C.V.

Several approximate methods have previously been used to obtain the percentage points of the S.C.V. derived from a normal parent distribution. The development of an exact table, for the percentage points of the S.C.V., should diminish their applicability but, at the same time, it cannot exhaust their serviceability.

The main application of the approximate methods have been reserved to the case where the population parameters fall outside the range of the exact table. In this situation the exact table can be used as a guide to the accuracy of the different approximate methods and, thus, determine which of these approximate methods meet a specified degree of accuracy.

In the following sections the different approximate methods will be discussed and then a comparison between them will be made in order to determine the best approximate method for different ranges of the C.V. It should be emphasized that the work of this chapter deals only with cases having the normal parent assumption.

A. Approximate Methods(a) McKay's Approximation

McKay [20] gave the following approximate distribution of v_n^2 :

$$B[v_n^2/(1 + v_n^2)] \cap \chi_{n-1}^2(0), \quad (6.1)$$

where $B = n(1 + 1/v^2)$.

One may be interested in finding t_0 such that

$$\Pr\{B(v_n^2/[1 + v_n^2]) < t_0 \mid V\} = h.$$

Since the case under consideration is $0 < V \leq .5$ and $n \geq 5$, it becomes apparent that

$$\Pr\{v_n < 0 \mid V\} \approx 0. \quad \frac{v_n^2}{1+v_n^2} < 1$$

further implies that $B > t_0$. Applying the above results to equation (6.1) results in

$$\Pr\{v_n < \left(\frac{t_0}{B - t_0} \right)^{1/2} \mid V\} = h,$$

or equivalently

$$\Pr\{v < \left[\frac{\left(\frac{n}{n-1} \right) \left(\frac{t_0}{B - t_0} \right)^{1/2}}{1} \right] \mid V\} = h \quad (6.2)$$

McKay's approximate method will be denoted by M.

(b) Hald's Approximation

The moments of the statistic $Z = S - t_0 \bar{X}$ have been derived in Chapter V. If Z can be assumed to be approximately normal then the following approximation can be used:

$$\frac{Z - E(Z)}{\sigma(Z)} \sim N(0,1).$$

The Z statistic can, therefore, be used as an approximate method for finding the percentage points of the S.C.V.

The moments of Z were found to the order of accuracy of $O(\frac{1}{n^3})$ and their use, in this situation, involves the computation of the roots of a somewhat complicated equation. Using a computer to obtain the approximate percentage points of the S.C.V., it was found that the, at times,

slight increase in accuracy of the results did not warrant the extra algebra needed in obtaining them.

A simple version, for this case, can still be found if one restricts the variance to an accuracy of $O(\frac{1}{n})$. Such a situation was considered by Hald [12] whose results are illustrated below.

To the order of accuracy stated above one can assume that

$$Z|N\left(\frac{t_o}{V} - 1, \frac{1}{2(n-1)} + \frac{t_o^2}{n}\right),$$

or equivalently

$$\frac{Z - \frac{t_o}{V} + 1}{\left(\frac{1}{2(n-1)} + \frac{t_o^2}{n}\right)^{1/2}} | N(0,1).$$

Let $t_o = t_h$ be the h 'th percentile point of v , while u_h denotes the h 'th standard normal percentile point. Then the following equation holds:

$$\Pr\left\{\frac{Z - t_h/V + 1}{\left(\frac{1}{2(n-1)} + \frac{t_h^2}{n}\right)^{1/2}} < u_h | V\right\} = h.$$

One also has, from the definition of Z , that $\Pr\{Z < 0 | V\} = h$.

Combining the latter two equations one obtains

$$u_h = - \frac{\left(1 - \frac{t_h}{V}\right)}{\left(\frac{1}{2(n-1)} + \frac{t_h^2}{n}\right)^{1/2}}.$$

Solving this equation for t_h results in

$$t_h = V \left[\frac{1 + u_h \left(\frac{1}{2(n-1)} C + \frac{V^2}{n} \right)^{1/2}}{C} \right]$$

where $C = 1 - \frac{V^2 u_h^2}{n}$.

This approximate method will be denoted by Z.

(c) Pearson's Moment Approximation

Approximations for the mean and the next three central moments of the S.C.V. were obtained in Chapter V. These moments can be used to fit a Pearson distribution to v and this distribution can be used to obtain approximate percentage points of the S.C.V. Once β_1 and β_2 are calculated the table of Johnson et al [15] can be used as an aid in the calculation of the required percentage points.

This approximate method will be denoted by P.

(d) S.C.V. Assumed Normal

In Chapter IV it was mentioned that the S.C.V. is asymptotically normally distributed. One can, consequently, assume that the S.C.V. is approximately normal with mean and variance given by equations (5.13 and 5.14). For this approximation three cases will be distinguished. They are: (i) The mean and variance formulae are given by (5.13 and 5.14): This case will be denoted by N1. (ii) By N2 will be denoted the case where the mean and variance of equations (5.13 and 5.14), are given with an accuracy of $O(\frac{1}{n})$. (iii) By N3 will be denoted the case where the mean is given by V while the variance is given with an accuracy of $O(\frac{1}{n})$.

B. Comparisons of the Approximate and Exact Results

For each of the approximate methods mentioned in Part A a table, similar to the one of Appendix II, was constructed. The inclusion of an auxiliary table, 6.1, was deemed necessary for the purpose of presenting some idea of the types of values obtained from the above mentioned tables.

A glance at Table 6.1 reveals the high quality of McKay's approximation for small values of V . McKay's approximation also gives reasonable results for intermediate values of V . The Pearson fit approximation behaves as expected.

The real surprise was the superiority of N_3 over N_1 and N_2 . N_3 is regarded as superior to the other two cases because it generally showed a better fit at the "tails" of the distribution, while N_1 usually showed a better fit at the center. It is interesting to note that F. N. David [5] used McKay's approximation to hypothesize the superiority of N_3 .

Another approximation that gave reasonably good results was Z . One can note that Z does not seem to be much better than N_3 and, in fact, it gives a worse fit for some parametric regions. If the complete moment expansions of Z , i.e. the ones given in Chapter V, were used then the approximations, at the "tails", would have in general been worse.

It should at this point be remarked that we are dealing with a large amount of data. Any reduction of this data into a small table must of necessity lose a great deal of information and can, therefore, only serve as a rough guide to the quality of the approximations.

TABLE 6.1

Comparison Between the Exact Percentage Points and
Those Obtained by the Various Approximate Methods

	PERCENTAGE POINTS						
	1%	5%	10%	50%	90%	95%	99%
<u>n = 10, V = .04</u>							
Exact	.01926	.02430	.02721	.03851	.05112	.05489	.06213
M	.01925	.02430	.02721	.03851	.05111	.05487	.06212
Z	.01806	.02448	.02791	.04000	.05211	.05555	.06201
P	.01933	.02431	.02720	.03851	.05116	.05490	.06199
N1	.01727	.02361	.02699	.03891	.05084	.05422	.06056
N2	.01817	.02427	.02753	.03901	.05049	.05374	.05985
N3	.01916	.02526	.02852	.04000	.05148	.05474	.06084
<u>n = 10, V = .10</u>							
Exact	.04802	.06064	.06792	.09630	.12816	.13773	.15623
M	.04798	.06058	.06786	.09620	.12803	.13759	.15607
Z	.04507	.06110	.06966	.10000	.13067	.13944	.15602
P	.04822	.06066	.06788	.09628	.12827	.13778	.15588
N1	.04278	.05877	.06729	.09736	.12743	.13596	.15196
N2	.04506	.06045	.06866	.09760	.12654	.13475	.15014
N3	.04746	.06285	.07106	.10000	.12894	.13715	.15254

TABLE 6.1(continued)

	<u>PERCENTAGE POINTS</u>						
	<u>1%</u>	<u>5%</u>	<u>10%</u>	<u>50%</u>	<u>90%</u>	<u>95%</u>	<u>99%</u>
<u>n = 10, V = .10</u>							
Exact	.06322	.07282	.07818	.09825	.11993	.12635	.13865
M	.06319	.07279	.07814	.09820	.11987	.12628	.13858
Z	.06212	.07318	.07909	.10000	.12108	.12709	.13842
P	.06327	.07284	.07818	.09824	.11996	.12635	.13858
N1	.06088	.07197	.07788	.09874	.11960	.12551	.13660
N2	.06165	.07253	.07833	.09880	.11927	.12507	.13595
N3	.06285	.07373	.07953	.10000	.12047	.12627	.13715
<u>n = 20, V = .20</u>							
Exact	.12546	.14481	.15564	.19654	.24150	.25498	.28110
M	.12522	.14452	.15532	.19616	.24106	.25451	.28061
Z	.12344	.14557	.15745	.20000	.24387	.25661	.28094
P	.12566	.14485	.15563	.19652	.24157	.25501	.28092
N1	.11980	.14264	.15482	.19778	.24075	.25293	.27577
N2	.12145	.14384	.15578	.19790	.24002	.25196	.27435
N3	.12355	.14594	.15788	.20000	.24212	.25406	.27645

TABLE 6.1(continued)

	<u>PERCENTAGE POINTS</u>						
	1%	5%	10%	50%	90%	95%	99%
n = 40, V = .20							
<u>Exact</u>	<u>.14702</u>	<u>.16134</u>	<u>.16924</u>	<u>.19832</u>	<u>.22937</u>	<u>.23853</u>	<u>.25615</u>
M	.14680	.16119	.16907	.19813	.22915	.23832	.25597
Z	.14623	.16181	.17017	.20000	.23049	.23927	.25594
P	.14711	.16138	.16924	.19831	.22939	.23854	.25607
N1	.14432	.16032	.16884	.19892	.22900	.23753	.25352
N2	.14489	.16073	.16917	.19895	.22873	.23717	.25301
N3	.14594	.16178	.17022	.20000	.22978	.23822	.25406
n = 100, V = .20							
<u>Exact</u>	<u>.16631</u>	<u>.17571</u>	<u>.18082</u>	<u>.19934</u>	<u>.21863</u>	<u>.22424</u>	<u>.23439</u>
M	.16621	.17563	.18074	.19926	.21856	.22417	.23488
Z	.16604	.17591	.18120	.20000	.21906	.22452	.23482
P	.16632	.17572	.18082	.19934	.21864	.22425	.23493
N1	.16525	.17530	.18067	.19958	.21849	.22385	.23390
N2	.16539	.17541	.18074	.19958	.21842	.22375	.23377
N3	.16581	.17583	.18116	.20000	.21884	.22417	.23419

TABLE 6.1(continued)

	<u>PERCENTAGE POINTS</u>						
	<u>1%</u>	<u>5%</u>	<u>10%</u>	<u>50%</u>	<u>90%</u>	<u>95%</u>	<u>99%</u>
n = 50, V = .3							
<u>Exact</u>	<u>.22678</u>	<u>.24665</u>	<u>.25756</u>	<u>.29803</u>	<u>.34172</u>	<u>.35471</u>	<u>.37988</u>
M	.22622	.24610	.25703	.29752	.34125	.35430	.37962
Z	.22606	.24725	.25869	.30000	.34309	.35569	.37985
P	.22694	.24667	.25755	.29802	.34175	.35473	.37979
N1	.22252	.24492	.25687	.29901	.34116	.35310	.37551
N2	.22323	.24544	.25727	.29904	.34081	.35264	.37485
<u>N3</u>	<u>.22419</u>	<u>.24640</u>	<u>.25823</u>	<u>.30000</u>	<u>.34177</u>	<u>.35360</u>	<u>.37581</u>
n = 100, V = .3							
<u>Exact</u>	<u>.24781</u>	<u>.26229</u>	<u>.27018</u>	<u>.29903</u>	<u>.32946</u>	<u>.33839</u>	<u>.35550</u>
M	.24744	.26196	.26987	.29877	.32928	.33825	.35546
Z	.24748	.26261	.27076	.30000	.33013	.33885	.35545
P	.24785	.26229	.27018	.29902	.32948	.33840	.35549
N1	.24567	.26144	.26985	.29951	.32918	.33759	.35336
N2	.24591	.26162	.26999	.29952	.32905	.33742	.35313
<u>N3</u>	<u>.24636</u>	<u>.26210</u>	<u>.27047</u>	<u>.30000</u>	<u>.32953</u>	<u>.33790</u>	<u>.35361</u>

TABLE 6.1(continued)

	<u>PERCENTAGE POINTS</u>						
	<u>1%</u>	<u>5%</u>	<u>10%</u>	<u>50%</u>	<u>90%</u>	<u>95%</u>	<u>99%</u>
n = 50, v = .4							
Exact	.2990	.3260	.3411	.3974	.4598	.4786	.5156
M	.2974	.3246	.3397	.3962	.4590	.4781	.5157
Z	.2981	.3269	.3426	.4000	.4617	.4801	.5160
P	.2992	.3261	.3410	.3974	.4598	.4787	.5154
N1	.2911	.3228	.3397	.3992	.4588	.4757	.5074
N2	.2924	.3237	.3404	.3993	.4582	.4749	.5062
N3	.2931	.3244	.3411	.4000	.4589	.4756	.5069
n = 100, v = .4							
Exact	.3275	.3475	.3584	.3987	.4420	.4549	.4797
M	.3265	.3466	.3576	.3981	.4418	.4548	.4800
Z	.3272	.3479	.3592	.4000	.4429	.4555	.4798
P	.3276	.3475	.3584	.3987	.4420	.4549	.4796
N1	.3236	.3459	.3578	.3996	.4415	.4534	.4756
N2	.3240	.3462	.3580	.3996	.4413	.4531	.4752
N3	.3244	.3466	.3584	.4000	.4416	.4534	.4756

TABLE 6.1(continued)

	<u>PERCENTAGE POINTS</u>						
	<u>1%</u>	<u>5%</u>	<u>10%</u>	<u>50%</u>	<u>90%</u>	<u>95%</u>	<u>99%</u>
n = 50, V = .5							
<u>Exact</u>	<u>.3686</u>	<u>.4034</u>	<u>.4228</u>	<u>.4969</u>	<u>.5811</u>	<u>.6072</u>	<u>.6591</u>
M	.3653	.4003	.4198	.4946	.5803	.6071	.6609
Z	.3678	.4045	.4247	.5000	.5836	.6092	.6600
P	.3691	.4034	.4227	.4968	.5813	.6073	.6587
N1	.3554	.3977	.4203	.5000	.5796	.6022	.6446
N2	.3575	.3993	.4215	.5000	.5785	.6007	.6425
N3	.3575	.3993	.4215	.5000	.5785	.6007	.6425
n = 100, V = .5							
<u>Exact</u>	<u>.4051</u>	<u>.4310</u>	<u>.4452</u>	<u>.4985</u>	<u>.5567</u>	<u>.5743</u>	<u>.6085</u>
M	.4029	.4290	.4434	.4973	.5567	.5748	.6102
Z	.4049	.4316	.4462	.5000	.5579	.5752	.6089
P	.4053	.4310	.4452	.4984	.5568	.5743	.6084
N1	.3985	.4282	.4441	.5000	.5559	.5718	.6015
N2	.3993	.4288	.4445	.5000	.5555	.5712	.6007
N3	.3993	.4288	.4445	.5000	.5555	.5712	.6007

From Table 6.1 one observed that as n increases, in general, the accuracy of the approximation increases. One can for the above reason establish a standard of accuracy and then find the smallest n that meets this accuracy. Such an n will have to be found for each value of V . This standard must be flexible as an error in the percentage points when V is small is more significant than for large V .

Taking all the above mentioned factors into consideration Table 6.2 is given. The entries in this table, are the smallest n 's necessary for the approximations to be of high quality. It was felt that such a high quality approximation can be achieved when the maximum allowable error ranges from no deviation, from the true results, in the first four decimal places for $V = .02$ to a slight deviation in the third decimal place for $V = .5$. Values outside the closed interval $(5, 100)$ were not considered: a dash indicates that the minimum sample size, n , is greater than 100 while an entry 5 indicates that the required value for n is less than or equal to 5.

C. Recommendations for Use of the Approximate Methods

From the discussions of the previous section it seems obvious that the only approximations one need consider are M , P , Z , and $N3$. Despite the fact that P is probably the best of the above approximations, it will in general, due to its complexity, not be used. This leaves two approximations based on moment properties, i.e. Z and $N3$, and one obtained from a mathematical derivation.

Of the above three approximations $N3$ is the simplest one to use.

TABLE 6.2

Guide to the Accuracy of the Approximate Methods

<u>y =</u>	<u>M</u>	<u>Z</u>	<u>P</u>	<u>N1</u>	<u>N2</u>	<u>N3</u>
.02	5	60	15	70	70	60
.04	5	60	15	70	70	60
.06	5	60	20	80	80	70
.08	8	70	20	80	80	70
.1	8	70	20	80	80	70
.12	10	70	20	90	90	80
.14	10	80	20	100	90	90
.16	10	90	25	100	100	90
.18	15	90	25	100	100	90
.2	20	100	30	-	-	100
.22	25	100	30	-	-	100
.24	30	100	30	-	-	-
.26	40	100	35	-	-	-
.28	50	100	35	-	-	-
.3	50	100	40	-	-	-
.32	60	100	40	-	-	-
.34	70	100	45	-	-	-
.36	80	-	45	-	-	-
.38	90	-	50	-	-	-
.4	100	-	50	-	-	-
.42	-	-	50	-	-	-
.44	-	-	50	-	-	-
.46	-	-	50	-	-	-
.48	-	-	50	-	-	-
.5	-	-	60	-	-	-

It also seems to give, throughout our range of parameters, reasonable results. If one considers the sample size to be large, say greater than 100, then N_3 can be recommended for use.

M seems to give better results than its two competitors. For small values of V the results of McKay's approximation are exceptionally good. The quality of McKay's approximation decreases, as V increases, but even for $V = .5$ it still gives reasonable results. This is especially true for large sample sizes.

Z seems to lack both the simplicity of N_3 and the accuracy of M . Although it, at times, gives quite accurate results, its use seems to be limited.

One should at this point discuss the uses of such approximations. The important use seems to be to obtain the approximate percentage points, of the S.C.V., for $n > 100$. Another possible use is as an alternative to interpolation, in n , of the exact percentage points. If the four point Lagrange interpolation formula, see e.g. Abramowitz and Stegun [1] p. 878, seems to be too laborious to use, then a linear interpolation, in n , might be preferable to the use of an approximate method. It would seem then, to the author, that the use of the approximations should be restricted to the case when $n > 100$. This argument can be modified if the user either has ready access to a high speed computer or is interested in $\Pr\{v < v_0 \mid V\} = \alpha$ for non-tabulated values of α . For $n > 100$, N_3 gives reasonably good results and can, therefore, be recommended for general use. One could possibly

still prefer to use, for smaller values of V , the slightly more complicated McKay's result in order to obtain better approximation. This choice is left to the reader.

CHAPTER VII

Rates of Convergence to Normality and Robustness of the S.C.V.A. Rates of Convergence to Normality of the S.C.V.

In the previous chapter several approximate formulae, for the distribution of the S.C.V., were discussed. Approximations dealing with asymptotic normality have traditionally played a fundamental role in statistics and it would, therefore, seem appropriate to give a more complete discussion of those approximations that are directly related to the asymptotic normality of the S.C.V.

If one allows

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2}$$

then it is well known for a normal distribution β_1 and β_2 are equal to 0 and 3.0 respectively. It would, therefore, be reasonable to expect that as the distribution of the S.C.V. approaches normality the above conditions would be approximated. For this purpose the approximate moments, developed in Chapter V, can be used.

Equations (5.6) and (5.7 - 5.8), which are correct to $O(\frac{1}{n})$ and $O(\frac{1}{n^2})$ respectively, become:

$$\mu_2(v) \approx A_2 = \frac{v^2}{n} \left(v^2 + \frac{1}{2} - \frac{w_3}{w_1 w_2} + \frac{w_4}{4w_2^2} \right) \quad (7.1)$$

$$\begin{aligned} \mu_3(v) \approx A_3 = \frac{v^3}{n^2} & \left(\frac{1}{4} + 3v^2 + 6v^4 - \frac{3w_4^2}{16w_2^4} + \frac{w_6}{8w_2^3} + \frac{9w_3^2}{4w_1^2 w_2^2} - \frac{3w_5}{4w_1 w_2^2} \right. \\ & \left. + \frac{3w_4}{w_1^2 w_2} - \frac{10w_3}{w_1^3} + \frac{3w_4}{4w_2^2} - \frac{3w_3}{w_1 w_2} + \frac{w_3^2}{2w_2^3} \right) \end{aligned} \quad (7.2)$$

$$\begin{aligned} \mu_4(v) \approx A_4 = \frac{3v^4}{n^2} & \left(\frac{1}{4} + v^2 + v^4 + \frac{w_4^2}{16w_2^4} - \frac{w_3 w_4}{2w_1 w_2^3} + \frac{w_3^2}{w_1^2 w_2^2} + \frac{w_4}{2w_1^2 w_2} \right. \\ & \left. - \frac{2w_3^2}{w_2^3} + \frac{w_4}{4w_2^2} - \frac{w_3}{w_1 w_2} \right). \end{aligned} \quad (7.3)$$

After some appreciable algebra one can obtain

$$\beta_2 \approx \frac{A_4}{A_2^2} = 3. \quad (7.4)$$

Similarly

$$\beta_1 \approx \frac{A_3^2}{A_2^3} = \frac{1}{2n} [g(w_1, w_2, \dots)], \quad (7.5)$$

where g is a function independent of n . The above equations show that, in general, the value of n which is necessary to make

$$\frac{A_3^2}{A_2^3}$$

approximately zero is large enough for β_2 to approximate 3. Consequently, stronger consideration should usually be given to the skewness parameter " β_1 " than to β_2 , as once the requirements for $\beta_1 = 0$ are met, it can be seen that the sample size would usually be large enough to give a satisfactory approximation to $\beta_2 = 3$.

One can now study the values of β_1 for different parent distributions. For this purpose let $\beta_i(N)$, $\beta_i(G)$ $i = 1, 2$ be the Pearson β 's, of the S.C.V., for the normal and gamma parent distributions respectively. Let $A_i(N)$ and $A_i(G)$ $i = 1, 2, 3$ have a similar correspondence to the A_i 's mentioned above. Then after a moderate amount of algebra the following results are obtained:

$$\beta_1(N) = \frac{\sim A_3^2(N)}{A_2^3(N)} = \frac{1}{2n} (1 + 18V^2 + 72V^4 - 80V^6 \dots), \quad (7.6)$$

and

$$\beta_1(G) = \frac{\sim A_3^2(G)}{A_2^3(G)} = \frac{1}{2n} (1 + 25V^2 + 144V^4 - 144V^6 \dots). \quad (7.7)$$

One can again be reminded that

$$\frac{A_4(N)}{A_2^2(N)} = \frac{A_4(G)}{A_2^2(G)} = 3.$$

For example, if $V = .3$ and $n = 100$ one obtains

$$\frac{A_3(N)}{A_2^{3/2}(N)} = .125 \text{ while } \frac{A_3(G)}{A_2^{3/2}(G)} = .147.$$

Had we used equations (5.9 - 5.16) the results would have been .1263 and .1396 respectively.

It may be instructive to give a short table of values of β_1 for different combinations of n and V . Using equations (5.9 - 5.16), extensive tables, of the β_1 's, have been developed and they can readily be used for the above mentioned purpose. Several such values are given in Table 7.1.

Such values of the β_1 's give an indication of the accuracy of the assumption that the S.C.V. is normally distributed. Table 7.1 again indicates that the skewness parameter " β_1 " is usually a more serious cause of anormality than is β_2 . One is at this stage still not sure about the correspondence between the value of β_1 and the nearness of the distribution to normality. An idea of the nearness of the distribution to normality can be obtained by comparing the values of P with N_1 , of Table 6.1. Near agreement between the corresponding values of N_1 and P would clearly indicate that the S.C.V. is an almost normally distributed random variable. At the same time the corresponding values of β_1 can be observed in order possibly to obtain an idea of the correspondence between β_1 and the normality of the S.C.V.

Table 7.1 indicates that the distribution of the S.C.V. from a normal parent distribution is generally nearer to normality than the corresponding distribution from a gamma parent. This agrees with the results of equations (7.7) and (7.8). A comparison between Table 6.1 and 7.1 indicates that for the distribution of the S.C.V.

TABLE 7.1

Values of the Pearson β 's for Different Values of the Parameters

n	v	<u>Normal</u>		<u>Gamma</u>	
		$\sqrt{\beta_1}$	β_2	$\sqrt{\beta_1}$	β_2
10	.04	.2475	2.968	.2456	2.9707
10	.10	.2633	2.985	.2506	3.0024
20	.10	.1800	3.0019	.1780	3.0155
20	.20	.2211	3.0390	.2082	3.0969
40	.20	.1540	3.0230	.1571	3.0578
100	.20	.0964	3.0101	.1030	3.0253
50	.30	.1798	3.0494	.1865	3.1224
100	.30	.1263	3.0257	.1396	3.0644
50	.40	.2371	3.0984	.2380	3.2492
100	.40	.1663	3.0513	.1858	3.1305
50	.50	.3072	3.1679	.2866	3.4455
100	.50	.2146	3.0886	.2376	3.2327
300	.50	.1228	3.0306	.1494	3.0793
200	.30	.0890	3.0131	.1013	3.0300
300	.30	.0726	3.0088	.0835	3.0221
200	.35	.1023	3.0190	.1184	3.0478
300	.40	.0954	3.0176	.1137	3.0447

to be regarded as reasonably close to normal, $\sqrt{\beta_1}$ should be less than or equal to say 0.1. Therefore, when $V \leq .3$ and $n > 100$ the normality assumption, of the S.C.V., may be used. It should be remembered that this was derived from a visual inspection of the tables and the reader is free to use the tables and choose his own criterion for nearness to normality. The use of the normality assumption, even for $V = .5$ and $n > 100$, should not cause too serious an error. When V is restricted to very small values, one could clearly use smaller values of n .

B. Robustness of the Distribution of the S.C.V.

Unfortunately, no practical exact results for the percentage points, of the S.C.V. from a non-normal parent distribution, seem to be known. In order to study robustness, of the distribution of the S.C.V., one is led to consider approximate results.

All cumulants above the second, of a normal distribution, are equal to zero. One may, therefore, study the effects that the variation of the higher population cumulants may have on the distribution of the S.C.V. By substituting these cumulants into the approximate moment results of Chapter V, and then using these moments in a Gram - Charlier series, the effects of non-normality may be studied.

Another approach is to consider the asymptotic normality of the S.C.V. From the discussions of Chapters IV and V the S.C.V. is, under very general conditions, asymptotically distributed as:

$$N \left(v, \frac{v^2}{n} \left[\frac{1}{2} + v^2 - \frac{w_3}{w_1 w_2} + \frac{w_4}{4w_2^2} \right] \right) .$$

Therefore, the distributions of all S.C.V.'s whose population cumulants have the property $\frac{1}{2} + v^2 >> \frac{w_3}{w_1 w_2} + \frac{w_4}{4w_2^2}$ are asymptotically

identical. One can, consequently, reason that if the parent distribution has the above property then, for large enough n , the distribution of the S.C.V. is robust, since the parent population plays no significant role in the asymptotic result.

One would still be interested in knowing the robustness of the distribution of the S.C.V. from moderate sample sizes. From the results of Chapter VI, it becomes evident that the Pearson approximation may be used, with some caution, to investigate such robustness. The reason for the necessity of caution is that P is an approximation and one cannot be sure, when comparing such results, whether the differences are due to lack of fit or robustness.

The gamma distribution is taken as a possible alternative parent. Since the approximate moments, for the gamma case, have been derived and given in Chapter V they may be used to find the approximate percentage points of the S.C.V. Using the Pearson approximation a complete table of such percentage points was derived. Before discussing the results of this table it may be interesting to see some numerical illustrations. For this purpose table 7.2 is included. It contains the percentage points obtained from a Pearson fit approximation for the gamma parent case.

TABLE 7.2

Selected Percentage Points of the S.C.V. Obtained From a Pearson Fit From a Gamma Parent Distribution

<u>n</u>	<u>v</u>	<u>1%</u>	<u>5%</u>	<u>10%</u>	<u>50%</u>	<u>90%</u>	<u>95%</u>	<u>99%</u>
10	.04	.01933	.02432	.02720	.03850	.05113	.05486	.06194
10	.10	.04810	.06074	.06799	.09620	.12775	.13714	.15510
20	.10	.06337	.07294	.07826	.09817	.11969	.12604	.13822
20	.20	.12622	.14561	.15628	.19601	.23942	.25248	.27793
40	.20	.14792	.16199	.16968	.19798	.22825	.23722	.25453
100	.20	.16691	.17611	.18110	.19919	.21809	.22361	.23415
50	.30	.22922	.24842	.25886	.29714	.33855	.35101	.37541
100	.30	.24974	.26358	.27110	.29853	.32765	.33626	.35287
50	.40	.3037	.3299	.3440	.3955	.4522	.4697	.5049
100	.40	.3317	.3504	.3605	.3976	.4377	.4498	.4735
50	.50	.3755	.4100	.4282	.4934	.5662	.5894	.6374
100	.50	.4123	.4362	.4491	.4964	.5484	.5645	.5966

Table 7.2, in conjunction with the corresponding values of the exact results and P of Table 6.1, can be used to study robustness.

For purposes of simplification let P_1 and E stand for the Pearson approximation and exact results, of Table 6.1, respectively. Let P_2 signify the values that are given in Table 7.2. It can be noticed that for small values of V and n , say $V < .1$ $n \geq 10$, there was good agreement between the values of P_1 , P_2 and E . It may, therefore, be assumed that for small values of V it makes little difference which, of these two, parent distributions is used. As V increases, while n is held fixed, the difference between P_1 and P_2 become noticeable. It is difficult to explain such differences as due to the approximations since the results of P_1 had far better agreement with the true results than did those of P_2 . This indicates that as V increases to .5 the robustness, of the percentage points, becomes worse and a relatively large sample size is necessary to give reasonable robustness. One might have expected the above results from the discussions of the previous section.

One may notice that all the tables, dealing with approximations, consider the same parametric cases. This was done, not because of lack of data, but in order to give the reader an opportunity to compare the values of these tables.

CHAPTER VIII

Some Statistics Related to the S.C.V.

Although the previously obtained results dealt with the properties of the S.C.V., they could possibly be also used for other purposes. This chapter is devoted to the discussion of several such uses.

A. Relationship Between the Non-Central t and S.C.V.

Given two independently distributed variables U_1 and U_2 such that $U_1 \sim N(0,1)$ while $U_2 \sim \chi_r^2(0)$ then

$$\frac{U_1 + a}{\sqrt{U_2/r}}$$

follows a non-central t distribution with a non-centrality parameter "a". This implies that if the parent population is normal, then

$$\frac{\sqrt{n} \bar{X}}{S} = \frac{\sqrt{n}}{v} \sim t'_{n-1} \left(a = \frac{\sqrt{n}}{v} \right).$$

If v is the S.C.V. then

$$\Pr\{v \leq v_0 \mid V\} = \Pr\{v \leq 0 \mid V\} + \Pr\{0 \leq v \leq v_0 \mid V\}.$$

It was shown in Chapter III that

$$\Pr\{v \leq 0 \mid V\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a} e^{-y^2/2} dy = N(-a).$$

From the above discussion one clearly has

$$\Pr\{0 \leq v \leq v_0 \mid V\} = \Pr\{t'_{n-1}(a) \geq \frac{\sqrt{n}}{v_0} \mid V\}.$$

Making the same restrictions on the parameters as those of Chapter

III results in $N(-a) \approx 0$. This work could, therefore, have equivalently been considered as dealing with some properties of the non-central t distribution. One, unfortunately, seems to find little discussion of the properties of the non-central t distribution for small values of V , or equivalently for large values of the non-centrality parameter. This author was unable to find a tabulation of the exact percentage points of the non-central t , for small values of V .

For large values of V extensive tabulation of the percentage points of the non-central t distribution have been made. Two good sources, for such tabulation and also for discussions of the properties and applications of the non-central t distribution, are Resnikoff and Lieberman [25] and Owen [22]. Their tables could also be useful for the study of the S.C.V. if one is concerned with a situation where $V > .5$.

It should now be obvious that all the results, of this dissertation, are also valid for the non-central t distribution, when $V \leq .5$ and $n \geq 5$. These include the discussions on the approximate distributions, asymptotic normality, and robustness. It also includes the tabulation of the exact percentage points that are given in Appendix II.

B. The Generalized Sample Coefficient of Variation

In many applied problems, especially those dealing with analysis of variance, the S.C.V., as defined here, is not directly used. In its place are substituted certain similar statistics which may be regarded

as generalizations of the S.C.V.

One may, for example, consider a single factor completely randomized experiment consisting of N random variables

$$\{x_{ij}\}_{i=1}^{n_j} \quad j=1, \dots, k$$

which are independently distributed on $N(\mu_j, \sigma^2)$.

Letting

$$\bar{x}_{.j} = \sum_{i=1}^{n_j} x_{ij} / n_j$$

one can obtain the following analysis of variance table.

Source of Variation	d.f.	Sum of Squares	Mean Squares
Treatments	$k - 1$	$SSR = \sum_{j=1}^k n_j (\bar{x}_{.j} - \bar{x})^2$	$\frac{SSR}{k - 1}$
Error	$N - k = e$	$SSE = \sum_{j=1}^k \sum_{i=1}^{n_j} (x_{ij} - \bar{x}_{.j})^2$	$MSE = SSE/e$
Total	$N - 1$	$\sum_{j=1}^k \sum_{i=1}^{n_j} (x_{ij} - \bar{x})^2 = SST$	

$$\text{Since } \bar{X} = \sum_{j=1}^k \sum_{i=1}^{n_j} x_{ij} / N$$

$$\text{is an unbiased estimator of } \mu = \sum_{j=1}^k n_j \mu_j / N$$

while MSE is an unbiased estimator for σ^2 , it would be reasonable to replace the S.C.V. by

$$G = \frac{\sqrt{\text{MSE}}}{\bar{X}}$$

Let us call this the Generalized Sample Coefficient of Variation (G.S.C.V.). It can be noted that the S.C.V. is the special case of the G.S.C.V. where the degrees of freedom due to error are equal to $N-1$. The S.C.V. is, therefore, the special case where only one sample is being considered.

Several illustrations of the use of the G.S.C.V. can be found in Federer [7].

It is fortunate that the table of Appendix II, developed for the case of the S.C.V., can be used to treat more general situations, such as the G.S.C.V. This will be illustrated below.

In general one is given the model $\underline{y} = X' \underline{\beta} + \underline{\epsilon}$ where

$$\underline{y} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{bmatrix}$$

$$\underline{\beta} = \begin{bmatrix} \beta_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \beta_p \end{bmatrix}$$

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_N \end{bmatrix}$$

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x_{1N} \\ x_{21} & x_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x_{2N} \\ \cdot & \cdot & & & & & & & & & & & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & & & & & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & & & & & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & & & & & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & & & & & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & & & & & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & & & & & & & & & & & \cdot \\ x_{p1} & x_{p2} & & & & & & & & & & & & & & & & & x_{pN} \end{bmatrix}$$

Here X is a matrix of constants, $\underline{\beta}$ is the vector of parameters, while $\underline{t}$ is a random vector distributed as $N(\underline{0}, \sigma^2 I)$. The notation used here is due to Scheffe' [26].

From Scheffe' [26], p. 12, $SSE = \underline{y}' \underline{y} - \hat{\underline{\beta}}' X \underline{y}$, where $XX' \hat{\underline{\beta}} = X \underline{y}$. Scheffe' further shows that $SSE/\sigma^2 \cap \chi^2_{n-r}(0)$ or $SSE/\sigma^2 \cap \chi^2_{\theta}(0)$, where r is the rank of X .

It is of interest to show that SSE and $\bar{Y} = \sum_{i=1}^n y_i / N$

are independent. This follows directly from a theorem of Scheffe' [26], p. 26 that the least square estimator (linear unbiased minimum variance estimator) of an estimable function is independent of SSE .

A parametric function $\psi = \underline{c}' \underline{\beta}$ is said to be estimable if there exists a constant vector $\underline{a}$ such that $E(\underline{a}' \underline{y}) = \psi$, identically in $\underline{\beta}$.

Equivalently, ψ is said to be estimable if, and only if, there exists a vector $\underline{a}$ such that $\underline{c}' = \underline{a}' X'$.

Choosing $\underline{a}' = (\frac{1}{N}, \frac{1}{N}, \dots, \frac{1}{N})$ results in $E(\bar{Y}) = E(\underline{a}'\underline{y}) = \underline{a}'X'\underline{\beta}$. Therefore $E(\bar{Y})$ is estimable. From the results of Scheffe' [26], pg. 14-15, $\bar{Y}$ is the least square estimator of $E(\bar{Y})$. This establishes the independence of $\bar{Y}$ and SSE.

Call $G = \frac{\sqrt{MSE}}{\bar{Y}}$ the G.S.C.V.

Since G is scalar invariant one could equivalently consider the transformation, of the y 's, to a new set of variables $\{U\}$ by multiplying each of the y 's by $\frac{\sqrt{N}}{\sigma}$. Then

$$\bar{U} \sim N(a = \frac{\sqrt{N}}{V}, 1) \quad (8.1)$$

while

$$\frac{SSE(U)}{N} \sim \chi_e^2(0). \quad (8.2)$$

Thus, $\left(\frac{e}{N}\right)^{1/2} G$ is distributed as v_n , see Chapter III, with $a = \frac{\sqrt{N}}{V}$ and $n = e + 1$.

Appendix II can be thought of as a tabulation for the special case $e = N-1$, that is, when all the degrees of freedom are due to error. It will presently be shown that by a simple transformation it is possible to use Appendix II to obtain the percentage points of G .

To obtain the percentage points of G the value of the non-centrality parameter, $a = \frac{\sqrt{N}}{V}$, must be known. In order to use our table the non-centrality parameter must be of the form $a = \frac{\sqrt{n}}{V}, = \frac{\sqrt{e+1}}{V'}$. Let V' be an

auxiliary variable satisfying the equation

$$a = \frac{\sqrt{N}}{V} = \frac{\sqrt{e+1}}{V'}.$$

Or equivalently

$$V' = \left(\frac{e+1}{N} \right)^{1/2} V.$$

Appendix II can now be used by letting $u = e+1$ and the value of V , to be used in the table, be V' . This gives the appropriate percentage points of

$$v = \left(\frac{e+1}{e} \right)^{1/2} ; v_n = \left(\frac{e+1}{e} \right)^{1/2} \left(\frac{e}{N} \right)^{1/2} G.$$

Or equivalently

$$\Pr\{v < v_0 \mid V, e, N\} = \Pr\left\{\left(\frac{e+1}{N}\right)^{1/2} G < v_0 \mid V, e, N\right\} =$$

$$\Pr\left\{G < \left(\frac{N}{e+1}\right)^{1/2} v_0 \mid V, e, N\right\}.$$

The above result could have also been obtained by the use of the properties of the non-central t distribution. This can be seen from the argument given below. From equations (8.1 and 8.2) one obtains that

$$\frac{\bar{U}}{\left(\frac{SSE(U)}{Ne}\right)^{1/2}} \sim t'_e(\sqrt{N}/V).$$

It was also seen that

$$t'_e = \frac{\sqrt{e+1}}{v}.$$

This implies that

$$v = \frac{\sqrt{e+1}}{t'_e} = \left(\frac{e+1}{N} \right)^{1/2} G.$$

This result, as expected, is equivalent to the one obtained before.

Different statistics, related to the non-central t distribution, can be treated in a similar manner.

CHAPTER IX

Numerical Examples

In the previous chapters the properties of the S.C.V. have been investigated. The results obtained from the previous investigations will be illustrated by several examples.

A. Hypothesis Testing

Example 1 (Hald's Example)

The following example was given by Hald [12], p. 303:

"On the basis of experience for a long time a laboratory has found an estimate, 2.4%, of the population coefficient of variation for the compressive strength of samples of concrete made and cured under standard conditions. Accepting this as a standard, what upper limit for future sample coefficients of variation should be used for $n = 5$?"

$\alpha = .025.$

One is, thus, interested in finding v_0 such that $\Pr\{v > v_0 \mid V = .024\} = .025$. From the table in Appendix II one obtains $v_0 = .04008$. It is interesting to note that Hald, by using his approximate method (see Chapter VI), obtains the value $v_0 = .041$.

The table of Appendix II can be used to obtain the power curve for a test, such as the one illustrated here. The calculation of such a power function table is quite easy, as one can see from the example given below.

Power is defined as $\Pr\{\text{reject null hypothesis} \mid V\}$ which in Hald's

example becomes $\Pr\{v > .04008 \mid V\}$. Let us illustrate the calculations of such power by finding the value of V that corresponds to a power of .9.

From Appendix II one finds for $n = 5$ and probability = .1 the required value lies between $V = .06$ and .08. Using linear interpolation one obtains

$$V = .06 + .02 \left(\frac{.04008 - .03092}{.04120 - .03092} \right) = .07782.$$

Continuing in this fashion one can, if one wishes, obtain 13 points of the power curve. For Hald's example they are

<u>Power</u>	<u>V</u>	<u>Power</u>	<u>V</u>	<u>Power</u>	<u>V</u>
.005	.02078	.25	.03453	.975	.11557
.01	.02199	.5	.04375	.99	.14795
.025	.02400	.75	.05783	.995	.17775
.05	.02601	.90	.07782		
.10	.02872	.95	.09528		

It may be interesting to see how power varies with n . For this purpose one can use Hald's example where instead of $n = 5$, one can choose different values of n . Maintaining power at .9 the results given below show, for varying n , the corresponding values of the rejection point v_0 and the required value of the parameter V .

n	5	10	20	50	100
V	.07782	.05134	.04033	.03324	.03014
v_0	.04008	.03491	.03157	.02873	.02734

The above calculations, for Hald's example, indicate the increase in sensitivity of the power of the test for increasing n.

B. Confidence Intervals for C.V.

Example 2(calculations of the confidence intervals by the use of Appendix II)

Assume that $n = 20$ and the S.C.V. has been calculated to be .05. Interest centers on finding a 95% confidence interval on V.

One can obtain such a confidence interval by solving the following two equations:

$$(i) \quad \Pr\{v \leq .05 \mid V\} = .025$$

$$(ii) \quad \Pr\{v \geq .05 \mid V\} = .025 .$$

From Appendix II one notices that the U.L. lies between $V = .06$ and .08. Thus, by linear interpolation one obtains $U.L. = .07313$. Similarly the lower limit $L.L. = .03800$.

In discussing such confidence intervals it was assumed that the $\Pr\{v \leq v_0 \mid V\}$ is a monotonic function of V. This, as pointed out by Owens [22], is not exactly correct. In general as V increases this probability decreases, but as V becomes very large μ becomes small compared to σ . The probability, consequently, increases that $\bar{X}$ will

be negative, a situation which destroys the above mentioned monotonicity. Since we restricted ourselves to cases where $\Pr\{\bar{X} < 0 \mid V\} \approx 0$ the above mentioned monotonicity can be assumed to hold.

Example 3 (Linhart's example - illustrates use of the approximation for large n)

Linhart [18] uses a somewhat complicated method to obtain approximate confidence intervals for the C.V. derived from a gamma parent. Linhart's example involves $n = 326$, $v = .2113$ and a 95% confidence interval is desired. Therefore, $u_h = \pm 1.96$. For so large a sample size approximation N3 can be used. One, thus, only has to solve for V in

$$\left(\frac{v - V}{\sqrt{1/(2n)}} \right)^2 = u_h^2$$

or equivalently

$$V = v \left(\frac{1 + u_h \sqrt{1/(2n)}}{1 - u_h^2/(2n)} \right) = .2113 \left(\frac{1 \pm 1.96 \sqrt{1/[2(326)]}}{1 - \frac{(1.96)^2}{2(326)}} \right).$$

Therefore, the approximate 95% confidence interval is given by
 $.1951 < V < .2279$.

By comparison, Linhart obtained the following confidence interval.

$$.192 < V < .224$$

It is interesting to note that had one inserted in the above approximate formula, the mean of Linhart's confidence interval, namely .208

as the S.C.V., one would have obtained his confidence interval. This led the author to believe that Linhart might have made a slight round-off error.

C. Applications to Quality Control

In certain quality control problems one is interested in the proportion defective in the population. In many applications an object is said to be defective if the value of some measurement falls below a constant L . Let us assume that the value of any random number, of this population, $X \cap N(\mu, \sigma^2)$. If σ is the largest allowable proportion defective, $Y = X - L$ and $V_y = \frac{\sigma}{\mu - L}$ then the following equation must hold:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1/V_y} e^{-z^2/2} dz = \alpha.$$

Therefore, given a value of α one can find, from a table of the normal integral, the corresponding V_y . For any $V < V_y$, the proportion defective would be less than α .

Let $v_y = \frac{S}{\bar{X} - L} = \frac{S_y}{\bar{Y}}$. The test now becomes similar to the one used for Hald's example, namely, reject the hypothesis when $v_y > v_o$, where v_o is determined from $\Pr(v_y > v_o \mid V_y) = \text{significance level of the test}$. One can also obtain the power of such a test for different alternative values of V_y .

Such quality control problems have been discussed by several people, see for example Owen [22]. The table of Appendix II can be considered as an extension of certain other tables, such as those of Owen [22], for solving such quality control problems. Our table is especially useful for very small values of α ; this being precisely the region where little previous tabulation seems to have been done.

Example 4 (Testing hypothesis on proportion defective)

In a certain plant relatively inexpensive objects, but ones performing a crucial function, are produced. The following inspection is to be made: A sample of size 5 will be taken and the products will be rejected if the sample indicates that $\alpha > 10^{-6}$. Develop a test having a significance level equal to .005.

In order to develop such a test one first has to solve

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1/v_y} e^{-z^2/2} dz = 10^{-6} .$$

The solution for the above equation is

$$\frac{1}{v_y} = 4.75342 \text{ or } v_y = .21037 .$$

One needs now to find v_0 such that $\Pr\{v_y > v_0 \mid V_y\} = .005$.

From Appendix II one obtains $v_0 = .42670$. One would, thus, reject the hypothesis if $v_y > .42670$. One could in addition, using the same method as in Example 1, obtain a power curve for the above test.

Example 5 (Confidence intervals for the proportion defective)

Suppose that in the previous example one obtains a $v_y = 0.3$.

Find a one sided 95% confidence interval for the proportion defective.

First let us find the value of V_y such that $\Pr\{v_y < .3 \mid V_y\} = .05$.

From the table of Appendix II one obtains $V_y = .4247$. Since

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1/V_y} e^{-z^2/2} dz = \alpha$$

one obtains the one sided 95% confidence interval for α as $\alpha \leq .00926$.

If, instead of a lower limit L , an upper limit U is specified, the problem could be equivalently solved by considering the test statistic $v_y = \frac{S}{U - \bar{X}}$. One can also consider the case where there is both an upper and lower bound. In the latter situation the following three decision procedure may be considered:

H_{-1} : A proportion of at least α_1 of the population lies below L .

H_1 : A proportion of at least α_2 of the population lies above U .

H_0 : Is true if both H_{-1} and H_1 are false.

If $\alpha_1 = \alpha_2$ one could find V_y such that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1/V_y} e^{-z^2/2} dz = \alpha_1$$

Then find v_0 satisfying $\Pr\{v_y > v_0 \mid V_y\} = \frac{\alpha^*}{2}$ where α^* is the significance level of the test. Then one rejects H_0 if either $\frac{S}{U - \bar{X}}$ or $\frac{S}{\bar{X} - L}$ is greater

than v_0 . The power curve for such a test is slightly more complicated as the alternatives for H_{-1} and H_1 need not be the same.

Example 6 (Calculation of percentage points from the non-central t distribution)

Assume that $t'_{15}(70.8)$ follows a non-central t distribution with 15 degrees of freedom and a non-centrality parameter equal to 70.8.

It is desired to solve for t_0 in $\Pr\{t'_{15}(70.8) > t_0\} = .05$.

Since

$$\frac{\sqrt{n}\bar{X}}{S} = \frac{\sqrt{n}}{v} \left(t'_{n-1} \left(\frac{\sqrt{n}}{V} \right) \right)$$

one can use the S.C.V. to solve the above problem.

$\frac{\sqrt{n}}{V} = 70.8$ implies that $V = \frac{4}{70.8} = .05650$. Therefore,

$\Pr\{t' > t_0 \mid V\} = \Pr\{v < \frac{\sqrt{n}}{t_0} \mid V\}$ or $v_0 = \frac{\sqrt{n}}{t_0}$ which

implies that $t_0 = \frac{\sqrt{n}}{v_0}$.

From Appendix II $v_0 = .03929$ and, thus, $t_0 = \frac{\sqrt{n}}{v_0} = \frac{4}{.03929} = 101.80708$.

Example 7 (Kramer's example of the G.S.C.V.)

Example 7 will be discussed in greater than usual detail because it involves a real situation with real data and, at the same time, illustrates a possible area of application of the G.S.C.V. Dr. Clyde Y. Kramer, Consultant for the Agricultural Experiment Station at the Virginia Polytechnic Institute, in analyzing agricultural data, has noticed that

a high value of the G.S.C.V., for certain kinds of situations, indicates that something has gone wrong with the experiment. The G.S.C.V. is, therefore, used as a preliminary check on the quality of the experiment. If it is found to be quite large, a search for the cause is undertaken.

It was also observed that for different crops the observed G.S.C.V.'s, of different experiments usually congregate about a certain value, say V . Since the G.S.C.V. is an estimator of the C.V., the above V value may be considered as the C.V. One can, therefore, use the above information to decide when to question the results.

For experiments dealing with corn, the values of the G.S.C.V. often congregate about the value .06. One can, therefore, assume this to be the value of the C.V.: this information can be used in the analysis of the following problem, the notation being the same as that of Chapter VIII.

Given $N = 80$, $e = 57$ and $V = .06$. Interest centers on finding a g_0 such that $\Pr\{G > g_0 \mid V = .06\} = \alpha = .01$. For this purpose one needs the following results:

$$C = \left(\frac{N}{e+1} \right)^{1/2} = 1.17445$$

$$D = \frac{1}{C} = .851462$$

$$V' = DV = .06(.851462) = .05109.$$

Using linear interpolation in the table of Appendix II, with V replaced by V' and n by $e+1$, one can find the value v_0 such that $\Pr\{v > v_0 \mid V', e+1\} = .01$. v_0 is found to be equal to .06236.

Therefore, $g_0 = v_0 C = .07324$.

One, therefore, would suspect that something happened if $G > .07324$.

It may be interesting to consider some points on the power curve for the hypothesis of the above problem. Several such points are given below, where $\text{Power} = \Pr\{G > .07324 \mid V\}$.

<u>Power</u>	<u>V</u>	<u>Power</u>	<u>V</u>	<u>Power</u>	<u>V</u>
.10	.06556	.50	.07368	.90	.08357
.25	.06920	.75	.07862		

At this point Kramer mentioned that the experiment cannot possibly be suspected for G even as large as .08. It was then thought that maybe the estimate for V was possibly inaccurate. Taking the same data, but with $V = .09$, a value of $V' = .07663$ results and this in turn yields $v_0 = .09360$ and finally $g_0 = .10993$.

The experiment could not be suspected even for $G = .11$ and, thus, a new approach appears to be necessary.

If G is considered as a measure of the quality of the experiment then populations having different values of V may be admissible. If this is the case then one is only interested in deciding on the largest allowable V and then performing a test based on this value of V being the C.V. Acceptance of the experiment for C.V. = V would clearly imply acceptance for any C.V. $< V$, therefore, a test based on V would cover all the admissible cases.

Suppose one decided that the largest possible $V = .15$. Then $V' =$

.15D = .12772. This implies that $v_0 = .15809$ and, thus, $g_0 = .18567$.

Therefore, experiments having $G > .18567$ will be subject to scrutiny.

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A P P E N D I C E S

APPENDIX I

EXPECTED CENTRAL PRODUCT CUMULANTS

$$(1^h 2^l) = E[k_1 - w_1]^h (k_2 - w_2)^l]$$

$$(1^2) = \frac{w_2}{n}$$

$$(1^3) = \frac{w_3}{n^2}$$

$$(1^4) = \frac{w_4}{n^3} + \frac{3w_2^2}{n^2}$$

$$(1^5) = \frac{10w_2w_3}{n^3}$$

$$(1^6) = \frac{15w_2^3}{n^3}$$

$$(12) = \frac{w_3}{n}$$

$$(1^22) = \frac{w_4}{n^2}$$

$$(1^32) = \frac{w_5}{n^3} + \frac{3w_2w_3}{n^2}$$

$$(1^42) = \frac{6w_2w_4}{n^3} + \frac{4w_3^2}{n^3}$$

$$(1^52) = \frac{15w_3w_2^2}{n^3}$$

$$(2^2) = \frac{w_4}{n} + \frac{2w_2^2}{n-1}$$

$$(12^2) = \frac{w_5}{n^2} + \frac{4w_2w_3}{n(n-1)}$$

$$(1^2 2^2) = \frac{1}{n^3} (w_6 + 4w_3^2 + 4w_2 w_4) + \frac{1}{n^2} (w_2 w_4 + 2w_3^2) + \frac{2w_2^3}{n(n-1)}$$

$$(1^3 2^2) = \frac{1}{n^3} (3w_2 w_5 + 7w_3 w_4 + 14w_2^2 w_3)$$

$$(1^4 2^2) = \frac{1}{n^3} (12w_2 w_3^2 + 3w_2^2 w_4 + 6w_2^4)$$

$$(2^3) = \frac{w_6}{n^2} + \frac{12w_2 w_4}{n(n-1)} + \frac{1}{(n-1)^2} (4w_3^2 + 8w_2^3) - \frac{8w_3^2}{n^3}$$

$$(12^3) = \frac{3w_3 w_4}{n^2} + \frac{6w_3 w_2^2}{n(n-1)} + \frac{1}{n^3} (w_7 + 12w_2 w_5 + 12w_3 w_4 + 8w_3 w_4 + 24w_2^2 w_3)$$

$$(1^2 2^3) = \frac{1}{n^3} (w_2 w_6 + 6w_3 w_5 + 3w_4^2 + 18w_2^2 w_4 + 28w_2 w_3^2 + 8w_2^4)$$

$$(1^3 2^3) = \frac{1}{n^3} (9w_2 w_3 w_4 + 6w_3^3 + 18w_2^3 w_3)$$

$$(2^4) = \frac{3w_4^2}{n^2} + \frac{12w_2^2 w_4}{n(n-1)} + \frac{12w_2^4}{(n-1)^2} + \frac{1}{n^3} (w_8 + 24w_2 w_6 + 32w_3 w_5 + 32w_4^2$$

$$+ 144w_2^2 w_4 + 96w_2 w_3^2 + 48w_2^4)$$

$$(12^4) = \frac{1}{n^3} (4w_3 w_6 + 6w_4 w_5 + 12w_2^2 w_5 + 72w_2 w_3 w_4 + 16w_3^3 + 80w_2^3 w_3)$$

$$(1^2 2^4) = \frac{1}{n^3} (12w_3^2 w_4 + 24w_2^2 w_3^2 + 3w_2 w_4^2 + 12w_2^3 w_4 + 12w_2^5)$$

$$(2^5) = \frac{1}{n^3} (10w_{46} w + 20w_{26}^2 w + 120w_{24} w^2 + 320w_{24}^3 w + 40w_{34}^2 w + 80w_{32}^2 w + 160w_2^5)$$

$$(12^5) = \frac{1}{n^3} (15w_{34} w^2 + 60w_{234}^2 w + 60w_{23}^4 w)$$

$$(2^6) = \frac{1}{n^3} (15w_4^3 + 90w_{24}^2 w^2 + 180w_{24}^4 w + 120w_2^6)$$

APPENDIX II

EXACT PERCENTAGE POINTS OF THE S.C.V.

The Entries are the v_o 's Such That

$$\Pr\{v \leq v_o \mid V\} = \alpha.$$

Here 100α is the Appropriate Percentage Value.

APPENDIX II

This table is expected to be exact to within two units of the last recorded decimal place. If the C.V. is close to .5 and the sample size, n , is in the neighborhood of 5, then this error may be slightly exceeded in the "tails".

Linear interpolations for the values of the C.V. are exceptionally good: this has been verified by determining additional exact intermediate values and these were used to confirm the accuracy of linear interpolation in V . No error exceeding one unit in the last recorded place was found. Linear interpolation for n does not give results of similar accuracy. For most purposes the results obtained would be sufficiently exact. If further accuracy is desired, one of the approximate methods discussed in this dissertation can be used, or a four point Lagrange formula with two pairs of equidistant tabular values, e.g. suppose the value at $n = 54$ is required, suitable interpolating values would be $n = 40, 50, 60$ and 70 .

n = 5

<u>V</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.00455	.00545	.00696	.00843	.01031	.01386	.01832
.04	.00909	.01090	.01391	.01685	.02062	.02773	.03664
.06	.01363	.01634	.02086	.02527	.03092	.04158	.05497
.08	.01817	.02176	.02779	.03367	.04120	.05542	.07330
.10	.02268	.02718	.03471	.04206	.05147	.06925	.09164
.12	.02719	.03258	.04161	.05042	.06171	.08306	.10998
.14	.03167	.03795	.04848	.05875	.07192	.09685	.12834
.16	.03614	.04331	.05532	.06705	.08210	.11062	.14670
.18	.04058	.04863	.06214	.07532	.09224	.12436	.16508
.20	.04500	.05393	.06891	.08355	.10235	.13806	.18346
.22	.04938	.05919	.07565	.09173	.11240	.15173	.20186
.24	.05374	.06442	.08234	.09987	.12241	.16537	.22028
.26	.05807	.06961	.08899	.10795	.13237	.17896	.23871
.28	.06236	.07476	.09559	.11599	.14226	.19250	.25715
.30	.06661	.07987	.10215	.12396	.15210	.20600	.27562
.32	.0708	.0849	.1086	.1319	.1619	.2195	.2941
.34	.0750	.0900	.1151	.1397	.1716	.2328	.3126
.36	.0791	.0949	.1215	.1475	.1812	.2462	.3311
.38	.0832	.0998	.1278	.1552	.1908	.2595	.3496
.40	.0873	.1047	.1341	.1629	.2003	.2727	.3682
.42	.0913	.1095	.1402	.1705	.2097	.2858	.3867
.44	.0952	.1143	.1464	.1780	.2190	.2989	.4053
.46	.0991	.1190	.1524	.1854	.2283	.3119	.4239
.48	.1030	.1236	.1584	.1928	.2375	.3249	.4425
.50	.1068	.1282	.1644	.2000	.2466	.3377	.4611

n = 5

<u>v</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02321	.02790	.03081	.03339	.03645	.03857
.04	.04643	.05583	.06167	.06685	.07300	.07725
.06	.06968	.08382	.09262	.10044	.10972	.11615
.08	.09297	.11191	.12372	.13423	.14672	.15538
.10	.11631	.14013	.15502	.16828	.18409	.19508
.12	.13971	.16851	.18656	.20267	.22192	.23534
.14	.16320	.19708	.21839	.23747	.26033	.27631
.16	.18678	.22589	.25058	.27275	.29941	.31812
.18	.21047	.25496	.28317	.30860	.33930	.36094
.20	.23427	.28432	.31622	.34508	.38011	.40490
.22	.25820	.31402	.34979	.38230	.42196	.45018
.24	.28228	.34410	.38394	.42035	.46503	.49700
.26	.30652	.37458	.41874	.45933	.50944	.54553
.28	.33092	.40551	.45426	.49932	.55538	.59604
.30	.35551	.43693	.49056	.54047	.60304	.64878
.32	.3803	.4689	.5277	.5829	.6526	.7041
.34	.4053	.5014	.5658	.6267	.7044	.7622
.36	.4305	.5345	.6049	.6721	.7586	.8237
.38	.4559	.5683	.6452	.7191	.8155	.8888
.40	.4816	.6028	.6866	.7680	.8754	.9580
.42	.5075	.6381	.7294	.8191	.9389	1.0324
.44	.5337	.6742	.7736	.8723	1.0061	1.1121
.46	.5602	.7111	.8193	.9281	1.0778	1.1983
.48	.5869	.7490	.8668	.9867	1.1545	1.2919
.50	.6140	.7878	.9161	1.0485	1.2373	1.3953

n = 8

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.00752	.00841	.00983	.01113	.01272	.01559	.01904
.04	.01503	.01682	.01964	.02225	.02544	.03118	.03809
.06	.02253	.02522	.02945	.03336	.03814	.04676	.05713
.08	.03001	.03359	.03924	.04445	.05083	.06233	.07618
.10	.03747	.04195	.04900	.05551	.06349	.07788	.09523
.12	.04491	.05027	.05873	.06654	.07613	.09341	.11429
.14	.05231	.05856	.06843	.07754	.08872	.10892	.13335
.16	.05967	.06681	.07808	.08849	.10128	.12440	.15242
.18	.06699	.07501	.08768	.09939	.11379	.13984	.17150
.20	.07427	.08317	.09724	.11025	.12625	.15526	.19058
.22	.08150	.09127	.10673	.12104	.13865	.17063	.20967
.24	.08866	.09931	.11616	.13176	.15100	.18596	.22877
.26	.09577	.10729	.12552	.14242	.16327	.20124	.24788
.28	.10282	.11520	.13481	.15301	.17548	.21648	.26700
.30	.10980	.12304	.14403	.16352	.18761	.23166	.28613
.32	.1167	.1308	.1532	.1740	.1997	.2468	.3053
.34	.1236	.1385	.1622	.1843	.2116	.2619	.3244
.36	.1303	.1461	.1712	.1946	.2235	.2769	.3436
.38	.1370	.1537	.1801	.2047	.2353	.2918	.3627
.40	.1436	.1611	.1889	.2148	.2471	.3067	.3819
.42	.1502	.1685	.1976	.2248	.2587	.3215	.4011
.44	.1566	.1757	.2062	.2347	.2702	.3362	.4203
.46	.1630	.1829	.2147	.2445	.2816	.3509	.4395
.48	.1693	.1900	.2231	.2542	.2930	.3655	.4587
.50	.1755	.1971	.2315	.2638	.3042	.3800	.4779

n = 8

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02273	.02621	.02836	.03026	.03250	.03405
.04	.04546	.05244	.05675	.06056	.06507	.06818
.06	.06822	.07872	.08521	.09095	.09775	.10246
.08	.09101	.10507	.11378	.12149	.13062	.13694
.10	.11384	.13152	.14248	.15220	.16374	.17174
.12	.13672	.15808	.17136	.18315	.19717	.20691
.14	.15966	.18479	.20044	.21437	.23097	.24253
.16	.18267	.21166	.22977	.24593	.26523	.27870
.18	.20577	.23873	.25939	.27786	.30001	.31550
.20	.22896	.26601	.28932	.31023	.33538	.35303
.22	.25224	.29353	.31961	.34308	.37141	.39137
.24	.27564	.32132	.35030	.37648	.40822	.43066
.26	.29916	.34939	.38143	.41048	.44585	.47098
.28	.32280	.37778	.41303	.44513	.48441	.51245
.30	.34658	.40652	.44516	.48052	.52402	.55523
.32	.3705	.4356	.4779	.5167	.5648	.5995
.34	.3946	.4651	.5112	.5538	.6068	.6453
.36	.4188	.4950	.5451	.5917	.6502	.6929
.38	.4433	.5254	.5798	.6307	.6950	.7424
.40	.4679	.5562	.6152	.6708	.7416	.7940
.42	.4926	.5876	.6515	.7121	.7899	.8482
.44	.5176	.6195	.6886	.7547	.8403	.9049
.46	.5428	.6519	.7267	.7987	.8928	.9646
.48	.5682	.6849	.7657	.8441	.9477	1.0274
.50	.5938	.7186	.8057	.8912	1.0051	1.0938

n = 10

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.00878	.00963	.01095	.01216	.01361	.01619	.01926
.04	.01755	.01926	.02190	.02430	.02721	.03238	.03851
.06	.02631	.02887	.03283	.03644	.04080	.04856	.05777
.08	.03505	.03846	.04374	.04855	.05437	.06472	.07703
.10	.04376	.04802	.05463	.06064	.06792	.08087	.09630
.12	.05244	.05755	.06547	.07269	.08143	.09700	.11556
.14	.06108	.06704	.07628	.08470	.09491	.11311	.13484
.16	.06968	.07648	.08704	.09667	.10835	.12919	.15411
.18	.07823	.08587	.09775	.10859	.12174	.14523	.17340
.20	.08672	.09520	.10841	.12045	.13508	.16124	.19268
.22	.09516	.10448	.11900	.13224	.14837	.17721	.21198
.24	.10354	.11370	.12953	.14398	.16158	.19314	.23128
.26	.11186	.12285	.13997	.15564	.17472	.20903	.25059
.28	.12007	.13190	.15033	.16721	.18780	.22486	.26991
.30	.12822	.14087	.16062	.17871	.20080	.24065	.28923
.32	.1363	.1498	.1709	.1901	.2137	.2564	.3086
.34	.1443	.1586	.1810	.2015	.2266	.2721	.3279
.36	.1522	.1673	.1910	.2127	.2393	.2877	.3473
.38	.1601	.1760	.2009	.2238	.2520	.3032	.3666
.40	.1678	.1845	.2107	.2349	.2646	.3187	.3860
.42	.1754	.1930	.2205	.2458	.2770	.3341	.4053
.44	.1830	.2013	.2301	.2567	.2894	.3495	.4247
.46	.1904	.2096	.2396	.2674	.3017	.3647	.4441
.48	.1978	.2177	.2491	.2780	.3139	.3799	.4635
.50	.2051	.2258	.2584	.2886	.3260	.3951	.4829

n = 10

<u>v</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02250	.02555	.02743	.02908	.03104	.03239
.04	.04501	.05112	.05489	.05821	.06213	.06485
.06	.06754	.07673	.08240	.08741	.09333	.09742
.08	.09009	.10241	.11001	.11673	.12467	.13017
.10	.11268	.12816	.13773	.14619	.15623	.16318
.12	.13532	.15402	.16559	.17586	.18804	.19649
.14	.15801	.18000	.19365	.20576	.22017	.23017
.16	.18076	.20613	.22190	.23593	.25267	.26432
.18	.20359	.23242	.25040	.26644	.28559	.29897
.20	.22649	.25889	.27916	.29729	.31901	.33421
.22	.24948	.28557	.30823	.32854	.35297	.37010
.24	.27257	.31247	.33762	.36025	.38755	.40678
.26	.29577	.33964	.36741	.39247	.42281	.44427
.28	.31907	.36705	.39758	.42522	.45883	.48268
.30	.34250	.39477	.42817	.45856	.49567	.52212
.32	.3661	.4228	.4592	.4925	.5334	.5627
.34	.3898	.4511	.4908	.5272	.5722	.6045
.36	.4136	.4799	.5229	.5627	.6120	.6477
.38	.4376	.5089	.5556	.5989	.6530	.6924
.40	.4617	.5384	.5890	.6361	.6953	.7386
.42	.4860	.5683	.6229	.6741	.7389	.7867
.44	.5105	.5987	.6576	.7132	.7841	.8367
.46	.5351	.6295	.6930	.7534	.8309	.8889
.48	.5599	.6608	.7292	.7947	.8794	.9433
.50	.5849	.6926	.7663	.8372	.9298	1.0004

n = 12

<u>V</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.00973	.01054	.01178	.01290	.01424	.01661	.01939
.04	.01945	.02106	.02355	.02579	.02847	.03321	.03878
.06	.02915	.03158	.03530	.03866	.04270	.04980	.05818
.08	.03884	.04207	.04703	.05152	.05690	.06638	.07757
.10	.04850	.05253	.05874	.06434	.07108	.08295	.09697
.12	.05812	.06295	.07041	.07714	.08523	.09950	.11637
.14	.06769	.07334	.08204	.08989	.09933	.11602	.13578
.16	.07722	.08367	.09362	.10259	.11340	.13251	.15519
.18	.08669	.09395	.10514	.11524	.12742	.14898	.17460
.20	.09611	.10417	.11660	.12783	.14138	.16541	.19402
.22	.10546	.11433	.12799	.14036	.15529	.18180	.21344
.24	.11475	.12443	.13932	.15282	.16914	.19815	.23287
.26	.12398	.13443	.15057	.16522	.18292	.21445	.25231
.28	.13310	.14434	.16175	.17752	.19663	.23071	.27175
.30	.14213	.15418	.17284	.18974	.21026	.24692	.29120
.32	.1511	.1640	.1838	.2019	.2238	.2631	.3107
.34	.1600	.1736	.1947	.2139	.2373	.2792	.3301
.36	.1688	.1832	.2055	.2259	.2507	.2952	.3496
.38	.1775	.1927	.2163	.2378	.2640	.3112	.3691
.40	.1861	.2020	.2269	.2495	.2772	.3271	.3885
.42	.1946	.2113	.2374	.2612	.2903	.3430	.4080
.44	.2030	.2205	.2478	.2727	.3033	.3587	.4275
.46	.2112	.2295	.2581	.2842	.3162	.3744	.4470
.48	.2194	.2385	.2682	.2955	.3290	.3901	.4665
.50	.2275	.2474	.2783	.3067	.3417	.4057	.4860

n = 12

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02232	.02507	.02675	.02824	.02999	.03120
.04	.04465	.05015	.05353	.05651	.06003	.06246
.06	.06700	.07527	.08036	.08485	.09016	.09381
.08	.08937	.10045	.10727	.11330	.12042	.12534
.10	.11177	.12570	.13429	.14188	.15085	.15707
.12	.13422	.15105	.16143	.17062	.18152	.18907
.14	.15671	.17649	.18872	.19957	.21244	.22137
.16	.17926	.20206	.21621	.22877	.24371	.25409
.18	.20188	.22780	.24390	.25824	.27533	.28723
.20	.22457	.25368	.27183	.28801	.30736	.32087
.22	.24733	.27974	.30002	.31815	.33986	.35508
.24	.27019	.30602	.32851	.34866	.37289	.38992
.26	.29313	.33251	.35730	.37960	.40650	.42546
.28	.31618	.35922	.38646	.41101	.44074	.46175
.30	.33934	.38621	.41598	.44294	.47571	.49895
.32	.3626	.4134	.4459	.4754	.5114	.5370
.34	.3860	.4410	.4763	.5085	.5479	.5762
.36	.4095	.4688	.5071	.5422	.5854	.6164
.38	.4332	.4970	.5384	.5765	.6238	.6579
.40	.4570	.5255	.5703	.6117	.6632	.7006
.42	.4809	.5544	.6027	.6476	.7038	.7448
.44	.5050	.5837	.6357	.6843	.7456	.7906
.46	.5292	.6133	.6693	.7219	.7887	.8381
.48	.5536	.6434	.7036	.7605	.8331	.8872
.50	.5782	.6739	.7386	.8001	.8791	.9385

n = 15

<u>V</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01079	.01154	.01268	.01370	.01492	.01704	.01952
.04	.02157	.02306	.02535	.02739	.02983	.03408	.03905
.06	.03233	.03458	.03801	.04107	.04473	.05111	.05857
.08	.04308	.04607	.05064	.05473	.05961	.06813	.07810
.10	.05379	.05752	.06324	.06835	.07446	.08513	.09762
.12	.06446	.06895	.07581	.08195	.08928	.10212	.11715
.14	.07508	.08033	.08834	.09550	.10407	.11908	.13669
.16	.08565	.09166	.10082	.10901	.11882	.13601	.15622
.18	.09617	.10294	.11324	.12247	.13351	.15292	.17576
.20	.10663	.11414	.12559	.13587	.14816	.16979	.19531
.22	.11702	.12527	.13788	.14920	.16275	.18662	.21485
.24	.12735	.13632	.15009	.16245	.17728	.20342	.23441
.26	.13758	.14730	.16223	.17563	.19175	.22017	.25396
.28	.14771	.15821	.17429	.18874	.20614	.23688	.27353
.30	.15776	.16904	.18627	.20177	.22044	.25354	.29309
.32	.1678	.1798	.1982	.2147	.2347	.2702	.3127
.34	.1777	.1904	.2099	.2276	.2489	.2867	.3322
.36	.1874	.2009	.2216	.2403	.2629	.3032	.3518
.38	.1971	.2113	.2332	.2530	.2769	.3196	.3714
.40	.2067	.2216	.2447	.2655	.2908	.3360	.3910
.42	.2161	.2318	.2560	.2780	.3046	.3523	.4106
.44	.2255	.2419	.2673	.2903	.3183	.3686	.4302
.46	.2348	.2519	.2785	.3026	.3319	.3848	.4498
.48	.2439	.2618	.2895	.3147	.3454	.4009	.4694
.50	.2529	.2716	.3004	.3267	.3588	.4169	.4890

n = 15

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02212	.02453	.02602	.02732	.02886	.02992
.04	.04424	.04909	.05206	.05468	.05776	.05989
.06	.06638	.07367	.07815	.08209	.08674	.08995
.08	.08854	.09830	.10430	.10958	.11582	.12013
.10	.11072	.12299	.13054	.13719	.14507	.15050
.12	.13295	.14776	.15688	.16495	.17449	.18109
.14	.15521	.17263	.18338	.19288	.20414	.21193
.16	.17753	.19761	.21002	.22100	.23407	.24311
.18	.19991	.22271	.23684	.24938	.26429	.27466
.20	.22234	.24795	.26387	.27800	.29485	.30660
.22	.24485	.27335	.29111	.30693	.32583	.33900
.24	.26743	.29894	.31860	.33616	.35723	.37193
.26	.29010	.32470	.34637	.36578	.38910	.40544
.28	.31285	.35065	.37442	.39578	.42149	.43958
.30	.33570	.37684	.40280	.42618	.45446	.47442
.32	.3586	.4032	.4315	.4570	.4880	.5100
.34	.3817	.4299	.4606	.4884	.5223	.5464
.36	.4049	.4568	.4900	.5203	.5573	.5837
.38	.4282	.4840	.5199	.5527	.5930	.6219
.40	.4515	.5115	.5502	.5858	.6296	.6612
.42	.4751	.5393	.5810	.6194	.6671	.7015
.44	.4987	.5674	.6122	.6538	.7055	.7431
.46	.5225	.5959	.6440	.6888	.7449	.7860
.48	.5464	.6247	.6763	.7246	.7855	.8302
.50	.5705	.6538	.7092	.7612	.8272	.8759

n = 16

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01108	.01181	.01292	.01391	.01509	.01715	.01955
.04	.02214	.02360	.02583	.02782	.03018	.03431	.03911
.06	.03319	.03538	.03873	.04171	.04526	.05145	.05867
.08	.04422	.04714	.05161	.05558	.06032	.06858	.07822
.10	.05521	.05887	.06446	.06943	.07535	.08570	.09778
.12	.06617	.07057	.07726	.08324	.09036	.10280	.11734
.14	.07708	.08222	.09003	.09700	.10533	.11987	.13691
.16	.08794	.09382	.10274	.11072	.12025	.13692	.15647
.18	.09874	.10535	.11539	.12438	.13513	.15394	.17605
.20	.10948	.11681	.12799	.13799	.14995	.17093	.19562
.22	.12016	.12821	.14052	.15154	.16472	.18788	.21520
.24	.13076	.13954	.15298	.16502	.17943	.20479	.23478
.26	.14126	.15079	.16537	.17843	.19407	.22166	.25437
.28	.15168	.16197	.17765	.19174	.20865	.23849	.27396
.30	.16201	.17303	.18985	.20497	.22315	.25527	.29356
.32	.1723	.1840	.2020	.2181	.2376	.2720	.3132
.34	.1824	.1949	.2140	.2312	.2519	.2887	.3328
.36	.1925	.2057	.2259	.2442	.2662	.3053	.3524
.38	.2024	.2164	.2378	.2570	.2803	.3218	.3720
.40	.2123	.2269	.2495	.2698	.2944	.3383	.3916
.42	.2220	.2374	.2611	.2825	.3084	.3548	.4112
.44	.2316	.2478	.2726	.2950	.3223	.3712	.4309
.46	.2412	.2580	.2840	.3075	.3361	.3875	.4505
.48	.2506	.2681	.2952	.3199	.3498	.4037	.4701
.50	.2599	.2782	.3064	.3321	.3634	.4199	.4898

n = 16

<u>v</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02206	.02439	.02582	.02708	.02856	.02958
.04	.04412	.04880	.05167	.05419	.05716	.05921
.06	.06620	.07324	.07755	.08135	.08583	.08892
.08	.08830	.09772	.10350	.10860	.11461	.11876
.10	.11043	.12226	.12953	.13595	.14352	.14876
.12	.13260	.14688	.15568	.16344	.17263	.17897
.14	.15480	.17159	.18195	.19111	.20194	.20944
.16	.17705	.19641	.20837	.21895	.23152	.24022
.18	.19936	.22135	.23496	.24703	.26137	.27133
.20	.22173	.24642	.26175	.27535	.29157	.30285
.22	.24417	.27164	.28874	.30396	.32212	.33480
.24	.26667	.29703	.31598	.33286	.35308	.36723
.26	.28926	.32260	.34346	.36213	.38452	.40021
.28	.31193	.34837	.37123	.39174	.41643	.43378
.30	.33470	.37433	.39930	.42176	.44889	.46801
.32	.3576	.4005	.4277	.4522	.4819	.5029
.34	.3805	.4269	.4564	.4831	.5156	.5386
.36	.4036	.4536	.4855	.5145	.5499	.5752
.38	.4268	.4806	.5150	.5465	.5850	.6126
.40	.4501	.5078	.5449	.5790	.6209	.6510
.42	.4735	.5353	.5753	.6121	.6575	.6904
.44	.4970	.5631	.6061	.6458	.6951	.7309
.46	.5207	.5912	.6374	.6802	.7337	.7726
.48	.5445	.6197	.6692	.7153	.7732	.8156
.50	.5684	.6485	.7015	.7512	.8138	.8600

n = 18

<u>V</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01157	.01227	.01334	.01428	.01540	.01735	.01961
.04	.02314	.02454	.02667	.02856	.03080	.03469	.03921
.06	.03469	.03679	.03999	.04282	.04618	.05203	.05882
.08	.04622	.04902	.05328	.05706	.06155	.06936	.07843
.10	.05772	.06122	.06654	.07127	.07689	.08667	.09804
.12	.06918	.07338	.07977	.08545	.09220	.10396	.11766
.14	.08059	.08550	.09295	.09958	.10748	.12123	.13727
.16	.09195	.09757	.10608	.11368	.12272	.13848	.15689
.18	.10324	.10958	.11916	.12772	.13791	.15570	.17651
.20	.11447	.12150	.13218	.14171	.15305	.17288	.19614
.22	.12563	.13336	.14513	.15562	.16813	.19003	.21577
.24	.13673	.14515	.15802	.16946	.18315	.20714	.23540
.26	.14775	.15687	.17080	.18323	.19811	.22422	.25503
.28	.15865	.16851	.18350	.19693	.21299	.24124	.27467
.30	.16946	.18004	.19612	.21054	.22781	.25823	.29432
.32	.1802	.1915	.2087	.2241	.2426	.2752	.3140
.34	.1909	.2028	.2211	.2375	.2572	.2920	.3336
.36	.2014	.2141	.2335	.2509	.2718	.3089	.3533
.38	.2118	.2252	.2457	.2641	.2863	.3256	.3729
.40	.2221	.2362	.2578	.2773	.3007	.3424	.3926
.42	.2324	.2472	.2699	.2903	.3150	.3590	.4123
.44	.2425	.2580	.2818	.3033	.3292	.3756	.4319
.46	.2524	.2687	.2936	.3161	.3433	.3921	.4516
.48	.2623	.2792	.3053	.3288	.3574	.4086	.4713
.50	.2721	.2897	.3169	.3414	.3713	.4250	.4910

n = 18

<u>v</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02196	.02414	.02548	.02666	.02804	.02900
.04	.04392	.04830	.05098	.05334	.05612	.05804
.06	.06590	.07249	.07653	.08008	.08426	.08715
.08	.08790	.09672	.10213	.10688	.11251	.11638
.10	.10992	.12100	.12781	.13380	.14088	.14575
.12	.13197	.14535	.15358	.16084	.16940	.17534
.14	.15407	.16979	.17947	.18803	.19815	.20513
.16	.17621	.19433	.20551	.21538	.22710	.23523
.18	.19840	.21898	.23171	.24297	.25634	.26560
.20	.22064	.24375	.25806	.27076	.28585	.29637
.22	.24295	.26867	.28464	.29883	.31574	.32752
.24	.26533	.29373	.31141	.32716	.34598	.35910
.26	.28778	.31896	.33843	.35582	.37661	.39118
.28	.31031	.34437	.36571	.38478	.40771	.42378
.30	.33293	.36997	.39325	.41415	.43930	.45699
.32	.3556	.3958	.4211	.4439	.4714	.4908
.34	.3784	.4218	.4492	.4740	.5041	.5254
.36	.4013	.4480	.4777	.5046	.5374	.5606
.38	.4243	.4746	.5066	.5357	.5713	.5967
.40	.4474	.5013	.5358	.5673	.6059	.6336
.42	.4707	.5283	.5654	.5994	.6413	.6714
.44	.4940	.5556	.5955	.6321	.6775	.7102
.46	.5175	.5832	.6260	.6654	.7145	.7501
.48	.5411	.6111	.6569	.6994	.7524	.7910
.50	.5648	.6393	.6883	.7340	.7912	.8332

n = 20

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01200	.01267	.01369	.01459	.01566	.01751	.01965
.04	.02399	.02534	.02737	.02918	.03131	.03501	.03930
.06	.03597	.03799	.04104	.04375	.04695	.05251	.05895
.08	.04793	.05062	.05469	.05830	.06258	.07000	.07860
.10	.05985	.06322	.06830	.07282	.07818	.08747	.09825
.12	.07172	.07577	.08189	.08731	.09375	.10493	.11790
.14	.08355	.08828	.09543	.10176	.10929	.12236	.13756
.16	.09533	.10073	.10892	.11616	.12479	.13977	.15722
.18	.10705	.11313	.12234	.13051	.14024	.15715	.17688
.20	.11871	.12546	.13570	.14481	.15564	.17450	.19654
.22	.13030	.13773	.14900	.15905	.17098	.19182	.21621
.24	.14183	.14993	.16223	.17321	.18626	.20909	.23588
.26	.15323	.16204	.17539	.18729	.20148	.22633	.25556
.28	.16455	.17404	.18847	.20130	.21664	.24353	.27524
.30	.17579	.18596	.20143	.21522	.23173	.26068	.29492
.32	.1870	.1978	.2143	.2291	.2468	.2778	.3146
.34	.1980	.2096	.2271	.2429	.2617	.2948	.3343
.36	.2090	.2212	.2398	.2565	.2765	.3118	.3540
.38	.2198	.2327	.2524	.2701	.2913	.3288	.3737
.40	.2305	.2441	.2649	.2836	.3060	.3457	.3934
.42	.2412	.2555	.2773	.2969	.3206	.3625	.4131
.44	.2517	.2667	.2896	.3102	.3350	.3793	.4328
.46	.2621	.2777	.3017	.3234	.3494	.3960	.4525
.48	.2724	.2887	.3138	.3364	.3638	.4127	.4722
.50	.2825	.2996	.3257	.3494	.3780	.4292	.4919

n = 20

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02187	.02393	.02520	.02630	.02761	.02851
.04	.04375	.04788	.05041	.05263	.05525	.05705
.06	.06564	.07185	.07566	.07901	.08295	.08567
.08	.08754	.09587	.10097	.10545	.11074	.11439
.10	.10948	.11993	.12635	.13199	.13865	.14325
.12	.13144	.14406	.15182	.15865	.16671	.17228
.14	.15344	.16827	.17739	.18544	.19496	.20154
.16	.17548	.19257	.20310	.21241	.22342	.23103
.18	.19757	.21698	.22897	.23956	.25213	.26085
.20	.21971	.24150	.25498	.26692	.28110	.29096
.22	.24191	.26615	.28118	.29453	.31041	.32144
.24	.26418	.29095	.30760	.32238	.34002	.35234
.26	.28651	.31590	.33421	.35054	.37004	.38369
.28	.30892	.34101	.36107	.37899	.40047	.41550
.30	.33141	.36631	.38819	.40777	.43134	.44786
.32	.3540	.3918	.4156	.4369	.4626	.4808
.34	.3766	.4175	.4432	.4665	.4945	.5144
.36	.3994	.4434	.4712	.4964	.5269	.5486
.38	.4223	.4695	.4995	.5268	.5599	.5835
.40	.4452	.4959	.5282	.5576	.5936	.6192
.42	.4683	.5225	.5572	.5890	.6279	.6558
.44	.4914	.5493	.5866	.6208	.6629	.6932
.46	.5147	.5765	.6165	.6532	.6987	.7315
.48	.5381	.6039	.6467	.6862	.7353	.7709
.50	.5616	.6316	.6774	.7198	.7727	.8113

n = 25

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01283	.01345	.01438	.01519	.01615	.01781	.01972
.04	.02566	.02689	.02874	.03038	.03230	.03562	.03944
.06	.03846	.04031	.04310	.04555	.04844	.05342	.05917
.08	.05125	.05371	.05743	.06070	.06456	.07121	.07889
.10	.06400	.06709	.07173	.07582	.08065	.08899	.09862
.12	.07671	.08042	.08599	.09091	.09672	.10675	.11834
.14	.08938	.09372	.10021	.10596	.11275	.12450	.13807
.16	.10201	.10694	.11438	.12096	.12875	.14221	.15780
.18	.11456	.12011	.12850	.13592	.14470	.15991	.17753
.20	.12704	.13322	.14256	.15082	.16061	.17757	.19727
.22	.13944	.14626	.15656	.16567	.17646	.19520	.21700
.24	.15177	.15923	.17050	.18045	.19227	.21280	.23674
.26	.16403	.17213	.18432	.19516	.20801	.23036	.25649
.28	.17622	.18492	.19807	.20977	.22367	.24788	.27623
.30	.18828	.19760	.21175	.22431	.23927	.26536	.29598
.32	.2003	.2103	.2254	.2388	.2548	.2828	.3157
.34	.2121	.2228	.2389	.2532	.2703	.3002	.3355
.36	.2239	.2352	.2523	.2675	.2857	.3175	.3552
.38	.2356	.2475	.2656	.2817	.3009	.3348	.3750
.40	.2471	.2597	.2788	.2958	.3162	.3520	.3948
.42	.2586	.2718	.2918	.3098	.3313	.3692	.4145
.44	.2699	.2837	.3048	.3237	.3463	.3863	.4343
.46	.2811	.2956	.3177	.3375	.3613	.4034	.4541
.48	.2922	.3074	.3305	.3512	.3761	.4204	.4738
.50	.3032	.3190	.3431	.3648	.3909	.4374	.4936

n = 25

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02170	.02352	.02464	.02562	.02677	.02756
.04	.04340	.04706	.04930	.05126	.05357	.05516
.06	.06511	.07062	.07398	.07694	.08042	.08280
.08	.08684	.09421	.09871	.10268	.10734	.11055
.10	.10859	.11785	.12351	.12850	.13436	.13842
.12	.13037	.14154	.14840	.15442	.16152	.16643
.14	.15218	.16531	.17337	.18045	.18884	.19460
.16	.17402	.18915	.19845	.20664	.21632	.22303
.18	.19591	.21309	.22365	.23299	.24403	.25166
.20	.21785	.23712	.24901	.25950	.27197	.28060
.22	.23984	.26127	.27452	.28624	.30015	.30980
.24	.26188	.28554	.30019	.31319	.32864	.33939
.26	.28399	.30994	.32606	.34037	.35744	.36933
.28	.30616	.33449	.35213	.36783	.38655	.39963
.30	.32840	.35919	.37839	.39555	.41609	.43043
.32	.3507	.3841	.4049	.4236	.4460	.4617
.34	.3731	.4091	.4317	.4519	.4763	.4935
.36	.3956	.4343	.4587	.4806	.5071	.5258
.38	.4181	.4597	.4860	.5097	.5384	.5587
.40	.4408	.4854	.5136	.5392	.5702	.5922
.42	.4635	.5112	.5415	.5690	.6025	.6264
.44	.4864	.5372	.5697	.5993	.6354	.6612
.46	.5093	.5635	.5983	.6300	.6690	.6969
.48	.5323	.5900	.6272	.6612	.7032	.7333
.50	.5555	.6168	.6565	.6929	.7380	.7705

n = 30

<u>V</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01345	.01402	.01487	.01563	.01651	.01803	.01977
.04	.02689	.02803	.02974	.03125	.03302	.03605	.03954
.06	.04031	.04203	.04459	.04686	.04951	.05407	.05931
.08	.05371	.05601	.05943	.06244	.06599	.07208	.07908
.10	.06708	.06995	.07423	.07800	.08245	.09008	.09885
.12	.08041	.08385	.08900	.09353	.09888	.10806	.11863
.14	.09370	.09771	.10374	.10902	.11528	.12602	.13840
.16	.10694	.11151	.11842	.12447	.13164	.14396	.15818
.18	.12013	.12526	.13304	.13987	.14796	.16188	.17796
.20	.13322	.13895	.14760	.15522	.16424	.17977	.19774
.22	.14624	.15258	.16210	.17051	.18046	.19763	.21752
.24	.15919	.16614	.17653	.18575	.19664	.21545	.23731
.26	.17206	.17960	.19090	.20091	.21275	.23325	.25709
.28	.18487	.19295	.20519	.21598	.22880	.25100	.27688
.30	.19757	.20623	.21938	.23098	.24478	.26871	.29667
.32	.2102	.2195	.2335	.2459	.2607	.2864	.3165
.34	.2227	.2326	.2475	.2608	.2765	.3040	.3363
.36	.2351	.2456	.2615	.2756	.2923	.3216	.3561
.38	.2474	.2585	.2753	.2902	.3080	.3391	.3759
.40	.2595	.2713	.2890	.3048	.3236	.3566	.3957
.42	.2716	.2839	.3026	.3193	.3392	.3740	.4155
.44	.2836	.2965	.3162	.3337	.3546	.3914	.4353
.46	.2954	.3090	.3296	.3480	.3700	.4088	.4551
.48	.3071	.3213	.3429	.3621	.3852	.4260	.4749
.50	.3187	.3335	.3561	.3762	.4004	.4432	.4947

n = 30

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02156	.02322	.02423	.02512	.02616	.02688
.04	.04313	.04645	.04847	.05025	.05234	.05378
.06	.06471	.06970	.07275	.07542	.07857	.08073
.08	.08631	.09299	.09706	.10065	.10486	.10776
.10	.10792	.11631	.12143	.12595	.13124	.13489
.12	.12956	.13969	.14587	.15133	.15775	.16217
.14	.15122	.16312	.17040	.17682	.18438	.18959
.16	.17292	.18663	.19502	.20243	.21117	.21719
.18	.19466	.21022	.21976	.22819	.23814	.24503
.20	.21644	.23389	.24464	.25412	.26533	.27308
.22	.23827	.25767	.26964	.28020	.29270	.30141
.24	.26015	.28156	.29478	.30648	.32039	.33001
.26	.28208	.30557	.32010	.33300	.34829	.35898
.28	.30407	.32970	.34561	.35969	.37655	.38823
.30	.32613	.35397	.37128	.38670	.40508	.41791
.32	.3482	.3784	.3972	.4139	.4340	.4480
.34	.3704	.4030	.4233	.4414	.4632	.4785
.36	.3927	.4277	.4496	.4692	.4929	.5095
.38	.4150	.4526	.4762	.4974	.5229	.5410
.40	.4375	.4777	.5030	.5258	.5534	.5729
.42	.4600	.5029	.5301	.5547	.5844	.6055
.44	.4826	.5284	.5575	.5838	.6159	.6386
.46	.5052	.5541	.5852	.6134	.6479	.6724
.48	.5280	.5799	.6131	.6434	.6804	.7069
.50	.5509	.6060	.6414	.6738	.7135	.7420

n = 35

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01393	.01446	.01526	.01596	.01679	.01819	.01980
.04	.02785	.02892	.03052	.03192	.03357	.03638	.03961
.06	.04176	.04336	.04576	.04786	.05034	.05457	.05941
.08	.05564	.05779	.06098	.06379	.06709	.07274	.07922
.10	.06948	.07217	.07617	.07969	.08383	.09091	.09902
.12	.08330	.08652	.09133	.09556	.10053	.10905	.11883
.14	.09706	.10082	.10645	.11140	.11721	.12718	.13864
.16	.11078	.11508	.12152	.12719	.13386	.14529	.15845
.18	.12444	.12927	.13654	.14293	.15046	.16338	.17826
.20	.13805	.14341	.15150	.15862	.16702	.18144	.19807
.22	.15156	.15749	.16641	.17425	.18353	.19947	.21789
.24	.16499	.17151	.18126	.18983	.19999	.21747	.23770
.26	.17834	.18542	.19601	.20534	.21639	.23544	.25752
.28	.19162	.19923	.21068	.22079	.23274	.25337	.22734
.30	.20483	.21297	.22528	.23617	.24902	.27127	.29716
.32	.2179	.2267	.2398	.2515	.2652	.2891	.3170
.34	.2309	.2402	.2543	.2667	.2814	.3069	.3368
.36	.2438	.2537	.2686	.2818	.2975	.3247	.3566
.38	.2566	.2671	.2829	.2968	.3135	.3424	.3765
.40	.2693	.2803	.2970	.3118	.3294	.3601	.3963
.42	.2819	.2935	.3110	.3266	.3452	.3777	.4161
.44	.2943	.3065	.3250	.3414	.3610	.3953	.4360
.46	.3066	.3194	.3388	.3561	.3767	.4128	.4558
.48	.3189	.3323	.3526	.3707	.3923	.4303	.4757
.50	.3310	.3450	.3662	.3851	.4078	.4477	.4955

n = 35

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02146	.02299	.02391	.02473	.02569	.02634
.04	.04292	.04598	.04784	.04948	.05140	.05271
.06	.06440	.06899	.07179	.07426	.07715	.07912
.08	.08588	.09204	.09579	.09909	.10296	.10561
.10	.10739	.11512	.11983	.12398	.12884	.13220
.12	.12891	.13824	.14393	.14895	.15482	.15889
.14	.15047	.16142	.16811	.17401	.18094	.18571
.16	.17205	.18467	.19238	.19918	.20721	.21273
.18	.19367	.20799	.21676	.22450	.23362	.23993
.20	.21533	.23139	.24124	.24996	.26022	.26730
.22	.23703	.25488	.26585	.27555	.28704	.29498
.24	.25878	.27847	.29060	.30134	.31403	.32284
.26	.28057	.30217	.31549	.32733	.34134	.35106
.28	.30242	.32599	.34055	.35347	.36884	.37951
.30	.32433	.34993	.36580	.37990	.39669	.40837
.32	.3463	.3740	.3912	.4065	.4248	.4375
.34	.3683	.3982	.4168	.4334	.4532	.4671
.36	.3904	.4226	.4426	.4605	.4820	.4971
.38	.4126	.4471	.4686	.4880	.5112	.5275
.40	.4348	.4717	.4949	.5157	.5408	.5584
.42	.4572	.4966	.5214	.5437	.5707	.5898
.44	.4796	.5216	.5481	.5721	.6011	.6216
.46	.5020	.5468	.5752	.6008	.6320	.6541
.48	.5246	.5722	.6024	.6298	.6633	.6871
.50	.5472	.5977	.6300	.6593	.6951	.7207

n = 40

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01432	.01482	.01557	.01623	.01700	.01832	.01983
.04	.02862	.02963	.03114	.03246	.03400	.03664	.03966
.06	.04292	.04443	.04670	.04868	.05099	.05496	.05949
.08	.05719	.05920	.06223	.06487	.06797	.07326	.07932
.10	.07143	.07395	.07774	.08105	.08492	.09156	.09915
.12	.08564	.08867	.09321	.09719	.10185	.10984	.11898
.14	.09981	.10334	.10864	.11329	.11876	.12810	.13881
.16	.11391	.11798	.12402	.12936	.13562	.14635	.15865
.18	.12794	.13253	.13936	.14537	.15245	.16457	.17848
.20	.14192	.14702	.15464	.16134	.16924	.18277	.19832
.22	.15583	.16145	.16987	.17726	.18598	.20094	.21816
.24	.16968	.17581	.18504	.19312	.20267	.21908	.23800
.26	.18346	.19011	.20014	.20892	.21931	.23718	.25784
.28	.19713	.20434	.21513	.22466	.23589	.25526	.27768
.30	.21069	.21845	.23005	.24031	.25241	.27330	.29753
.32	.2242	.2325	.2449	.2559	.2689	.2913	.3174
.34	.2376	.2464	.2597	.2714	.2853	.3093	.3372
.36	.2509	.2603	.2744	.2868	.3016	.3272	.3571
.38	.2641	.2740	.2890	.3022	.3178	.3451	.3769
.40	.2772	.2877	.3035	.3174	.3340	.3629	.3968
.42	.2902	.3012	.3178	.3326	.3501	.3807	.4166
.44	.3030	.3146	.3321	.3477	.3661	.3984	.4365
.46	.3158	.3279	.3463	.3626	.3821	.4161	.4564
.48	.3284	.3412	.3604	.3775	.3979	.4337	.4762
.50	.3409	.3542	.3744	.3923	.4137	.4513	.4961

n = 40

<u>v</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02137	.02280	.02366	.02442	.02531	.02592
.04	.04275	.04560	.04733	.04886	.05064	.05186
.06	.06414	.06842	.07103	.07332	.07601	.07784
.08	.08554	.09127	.09476	.09783	.10143	.10389
.10	.10695	.11415	.11854	.12239	.12691	.13003
.12	.12839	.13708	.14237	.14703	.15250	.15628
.14	.14985	.16005	.16628	.17176	.17821	.18263
.16	.17134	.18309	.19027	.19660	.20404	.20915
.18	.19286	.20619	.21435	.22155	.23000	.23586
.20	.21442	.22937	.23853	.24660	.25615	.26273
.22	.23602	.25263	.26283	.27184	.28249	.28982
.24	.25766	.27599	.28725	.29724	.30899	.31717
.26	.27935	.29944	.31181	.32277	.33577	.34474
.28	.30108	.32300	.33652	.34852	.36274	.37262
.30	.32287	.34667	.36138	.37448	.39000	.40079
.32	.3447	.3705	.3864	.4006	.4175	.4292
.34	.3666	.3944	.4116	.4270	.4453	.4581
.36	.3886	.4184	.4370	.4536	.4734	.4873
.38	.4106	.4426	.4626	.4805	.5019	.5169
.40	.4327	.4670	.4884	.5076	.5307	.5469
.42	.4549	.4915	.5144	.5350	.5599	.5774
.44	.4771	.5161	.5407	.5628	.5894	.6082
.46	.4994	.5410	.5672	.5908	.6194	.6397
.48	.5218	.5660	.5939	.6191	.6498	.6715
.50	.5443	.5911	.6209	.6478	.6806	.7040

n = 45

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01464	.01512	.01583	.01645	.01718	.01843	.01985
.04	.02927	.03022	.03165	.03290	.03436	.03686	.03970
.06	.04388	.04532	.04746	.04934	.05153	.05528	.05955
.08	.05847	.06039	.06325	.06576	.06869	.07369	.07940
.10	.07304	.07544	.07902	.08216	.08583	.09209	.09925
.12	.08757	.09045	.09475	.09853	.10294	.11048	.11910
.14	.10206	.10543	.11044	.11486	.12003	.12885	.13895
.16	.11650	.12036	.12609	.13116	.13708	.14721	.15880
.18	.13086	.13521	.14170	.14741	.15409	.16554	.17866
.20	.14516	.15001	.15725	.16362	.17107	.18385	.19851
.22	.15940	.16474	.17275	.17977	.18800	.20213	.21837
.24	.17357	.17941	.18818	.19586	.20488	.22038	.23823
.26	.18767	.19401	.20352	.21189	.22171	.23861	.25809
.28	.20171	.20855	.21879	.22786	.23849	.25680	.27795
.30	.21561	.22298	.23400	.24375	.25521	.27496	.29781
.32	.2295	.2374	.2492	.2596	.2719	.2931	.3177
.34	.2432	.2516	.2642	.2753	.2885	.3112	.3375
.36	.2568	.2658	.2792	.2910	.3050	.3292	.3574
.38	.2704	.2798	.2940	.3066	.3214	.3472	.3773
.40	.2838	.2938	.3088	.3221	.3378	.3652	.3972
.42	.2971	.3076	.3235	.3375	.3541	.3831	.4170
.44	.3103	.3214	.3381	.3528	.3704	.4010	.4369
.46	.3234	.3350	.3526	.3681	.3865	.4188	.4568
.48	.3364	.3486	.3669	.3832	.4026	.4365	.4766
.50	.3493	.3620	.3812	.3983	.4186	.4543	.4965

n = 45

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02130	.02264	.02345	.02416	.02500	.02557
.04	.04261	.04529	.04691	.04834	.05001	.05116
.06	.06392	.06795	.07039	.07254	.07506	.07679
.08	.08525	.09063	.09391	.09678	.10016	.10249
.10	.10659	.11335	.11747	.12108	.12534	.12826
.12	.12795	.13611	.14108	.14544	.15060	.15412
.14	.14933	.15892	.16476	.16989	.17594	.18007
.16	.17074	.18178	.18851	.19445	.20140	.20623
.18	.19218	.20471	.21235	.21911	.22704	.23252
.20	.21366	.22770	.23629	.24388	.25281	.25895
.22	.23517	.25077	.26033	.26877	.27873	.28562
.24	.25672	.27393	.28449	.29383	.30486	.31249
.26	.27832	.29718	.30878	.31906	.33120	.33958
.28	.29996	.32053	.33320	.34443	.35769	.36695
.30	.32165	.34399	.35776	.37000	.38453	.39457
.32	.3434	.3676	.3825	.3957	.4115	.4225
.34	.3652	.3912	.4074	.4217	.4388	.4508
.36	.3870	.4150	.4324	.4479	.4664	.4793
.38	.4090	.4390	.4577	.4743	.4943	.5082
.40	.4310	.4631	.4831	.5010	.5225	.5376
.42	.4530	.4873	.5087	.5280	.5510	.5673
.44	.4751	.5116	.5346	.5552	.5799	.5974
.46	.4973	.5362	.5606	.5826	.6092	.6280
.48	.5196	.5608	.5869	.6104	.6388	.6589
.50	.5419	.5857	.6134	.6385	.6689	.6904

n = 50

<u>V</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01491	.01537	.01605	.01664	.01734	.01852	.01986
.04	.02981	.03073	.03209	.03328	.03467	.03703	.03973
.06	.04470	.04608	.04811	.04990	.05199	.05554	.05959
.08	.05957	.06141	.06412	.06651	.06930	.07405	.07946
.10	.07441	.07670	.08011	.08310	.08659	.09254	.09932
.12	.08922	.09196	.09607	.09966	.10385	.11102	.11919
.14	.10397	.10718	.11199	.11618	.12109	.12948	.13906
.16	.11867	.12234	.12786	.13267	.13830	.14793	.15892
.18	.13331	.13746	.14367	.14912	.15547	.16635	.17879
.20	.14790	.15253	.15944	.16551	.17261	.18475	.19866
.22	.16242	.16754	.17515	.18186	.18970	.20313	.21853
.24	.17689	.18249	.19080	.19815	.20674	.22148	.23841
.26	.19129	.19735	.20640	.21438	.22374	.23980	.25828
.28	.20556	.21210	.22193	.23055	.24068	.25809	.27816
.30	.21974	.22678	.23737	.24665	.25756	.27634	.29803
.32	.2339	.2415	.2528	.2627	.2744	.2946	.3179
.34	.2479	.2560	.2680	.2786	.2912	.3128	.3378
.36	.2619	.2704	.2832	.2945	.3079	.3309	.3577
.38	.2757	.2847	.2983	.3103	.3245	.3490	.3776
.40	.2894	.2990	.3134	.3260	.3411	.3671	.3974
.42	.3030	.3131	.3283	.3417	.3575	.3851	.4173
.44	.3165	.3271	.3431	.3572	.3740	.4031	.4372
.46	.3299	.3411	.3578	.3727	.3903	.4210	.4571
.48	.3432	.3549	.3725	.3881	.4066	.4389	.4770
.50	.3564	.3686	.3870	.4034	.4228	.4567	.4969

n = 50

<u>v</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02124	.02251	.02327	.02394	.02474	.02528
.04	.04249	.04502	.04656	.04790	.04949	.05057
.06	.06374	.06755	.06986	.07188	.07427	.07590
.08	.08500	.09010	.09319	.09591	.09910	.10128
.10	.10628	.11268	.11657	.11998	.12399	.12674
.12	.12757	.13530	.13999	.14411	.14897	.15230
.14	.14889	.15796	.16348	.16832	.17405	.17797
.16	.17023	.18067	.18704	.19263	.19923	.20376
.18	.19161	.20345	.21067	.21703	.22453	.22966
.20	.21301	.22629	.23440	.24154	.24998	.25580
.22	.23445	.24920	.25823	.26618	.27561	.28208
.24	.25592	.27219	.28216	.29095	.30137	.30853
.26	.27744	.29527	.30622	.31592	.32733	.33527
.28	.29900	.31844	.33040	.34100	.35353	.36215
.30	.32061	.34172	.35471	.36623	.37988	.38940
.32	.3423	.3651	.3792	.3917	.4065	.4168
.34	.3640	.3886	.4038	.4173	.4334	.4446
.36	.3857	.4122	.4286	.4431	.4605	.4726
.38	.4076	.4359	.4535	.4692	.4879	.5010
.40	.4294	.4598	.4786	.4955	.5156	.5298
.42	.4514	.4837	.5039	.5220	.5437	.5588
.44	.4734	.5079	.5294	.5488	.5720	.5883
.46	.4955	.5321	.5551	.5758	.6007	.6182
.48	.5176	.5565	.5810	.6031	.6297	.6485
.50	.5398	.5811	.6072	.6307	.6591	.6792

n = 60

<u>V</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01535	.01577	.01640	.01694	.01758	.01866	.01989
.04	.03070	.03154	.03279	.03388	.03515	.03732	.03977
.06	.04603	.04729	.04917	.05080	.05272	.05597	.05966
.08	.06134	.06302	.06553	.06771	.07027	.07461	.07955
.10	.07662	.07873	.08187	.08460	.08780	.09325	.09944
.12	.09187	.09440	.09817	.10147	.10531	.11187	.11933
.14	.10707	.11004	.11445	.11830	.12280	.13048	.13922
.16	.12223	.12562	.13068	.13509	.14026	.14907	.15911
.18	.13734	.14116	.14687	.15185	.15768	.16764	.17900
.20	.15238	.15665	.16300	.16856	.17507	.18619	.19889
.22	.16736	.17207	.17909	.18522	.19242	.20472	.21878
.24	.18227	.18742	.19511	.20183	.20972	.22322	.23868
.26	.19711	.20271	.21107	.21839	.22698	.24169	.25857
.28	.21186	.21792	.22696	.23488	.24419	.26014	.27847
.30	.22654	.23305	.24278	.25131	.26134	.27856	.29837
.32	.2411	.2481	.2585	.2677	.2784	.2969	.3183
.34	.2556	.2631	.2742	.2840	.2955	.3153	.3382
.36	.2700	.2779	.2898	.3002	.3125	.3336	.3581
.38	.2843	.2927	.3053	.3163	.3294	.3519	.3780
.40	.2985	.3074	.3207	.3324	.3462	.3701	.3979
.42	.3126	.3220	.3360	.3484	.3630	.3883	.4178
.44	.3266	.3365	.3513	.3643	.3797	.4065	.4377
.46	.3405	.3509	.3664	.3801	.3964	.4246	.4576
.48	.3543	.3652	.3815	.3959	.4130	.4427	.4775
.50	.3680	.3793	.3964	.4115	.4295	.4607	.4974

n = 60

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02114	.02229	.02299	.02360	.02431	.02480
.04	.04228	.04459	.04598	.04720	.04864	.04962
.06	.06344	.06690	.06900	.07084	.07299	.07447
.08	.08460	.08923	.09204	.09450	.09739	.09936
.10	.10577	.11159	.11512	.11821	.12184	.12433
.12	.12696	.13398	.13824	.14198	.14637	.14938
.14	.14817	.15641	.16142	.16581	.17097	.17452
.16	.16940	.17889	.18466	.18972	.19568	.19976
.18	.19066	.20142	.20798	.21373	.22050	.22516
.20	.21195	.22401	.23137	.23783	.24544	.25069
.22	.23327	.24667	.25485	.26205	.27053	.27637
.24	.25462	.26940	.27843	.28638	.29577	.30225
.26	.27601	.29220	.30211	.31085	.32117	.32831
.28	.29744	.31509	.32591	.33546	.34675	.35456
.30	.31892	.33806	.34983	.36022	.37252	.38104
.32	.3404	.3611	.3739	.3851	.3985	.4078
.34	.3620	.3843	.3981	.4102	.4247	.4348
.36	.3836	.4076	.4224	.4355	.4511	.4620
.38	.4053	.4310	.4469	.4610	.4778	.4896
.40	.4270	.4544	.4715	.4866	.5047	.5174
.42	.4488	.4781	.4963	.5125	.5319	.5455
.44	.4706	.5018	.5212	.5386	.5594	.5740
.46	.4925	.5257	.5464	.5649	.5872	.6028
.48	.5144	.5497	.5717	.5915	.6153	.6320
.50	.5364	.5738	.5972	.6183	.6436	.6615

n = 70

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01569	.01609	.01667	.01717	.01776	.01877	.01990
.04	.03138	.03216	.03333	.03434	.03552	.03753	.03981
.06	.04705	.04823	.04998	.05150	.05328	.05629	.05971
.08	.06271	.06428	.06661	.06864	.07101	.07504	.07961
.10	.07833	.08030	.08322	.08577	.08874	.09379	.09952
.12	.09393	.09629	.09980	.10286	.10644	.11252	.11942
.14	.10948	.11224	.11635	.11993	.12412	.13124	.13933
.16	.12499	.12815	.13286	.13697	.14177	.14994	.15924
.18	.14044	.14401	.14933	.15397	.15939	.16863	.17914
.20	.15584	.15982	.16575	.17092	.17697	.18729	.19905
.22	.17118	.17557	.18211	.18783	.19452	.20593	.21896
.24	.18644	.19125	.19842	.20469	.21202	.22455	.23887
.26	.20164	.20687	.21467	.22149	.22948	.24315	.25878
.28	.21676	.22242	.23086	.23824	.24690	.26171	.27869
.30	.23180	.23789	.24697	.25493	.26426	.28025	.29860
.32	.2468	.2533	.2630	.2716	.2816	.2988	.3185
.34	.2616	.2686	.2790	.2881	.2988	.3172	.3384
.36	.2764	.2838	.2949	.3046	.3160	.3357	.3584
.38	.2911	.2989	.3107	.3210	.3332	.3541	.3783
.40	.3057	.3140	.3264	.3374	.3502	.3725	.3982
.42	.3202	.3289	.3421	.3536	.3673	.3908	.4181
.44	.3346	.3438	.3576	.3698	.3842	.4091	.4380
.46	.3488	.3585	.3731	.3859	.4011	.4273	.4579
.48	.3630	.3732	.3885	.4020	.4179	.4456	.4779
.50	.3771	.3877	.4038	.4179	.4347	.4637	.4978

n = 70

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02106	.02212	.02277	.02333	.02399	.02444
.04	.04213	.04425	.04554	.04666	.04798	.04889
.06	.06320	.06639	.06833	.07002	.07201	.07337
.08	.08428	.08855	.09114	.09341	.09607	.09789
.10	.10537	.11074	.11399	.11684	.12019	.12248
.12	.12648	.13295	.13688	.14033	.14436	.14714
.14	.14760	.15521	.15982	.16387	.16862	.17188
.16	.16875	.17750	.18282	.18748	.19295	.19671
.18	.18992	.19985	.20588	.21118	.21741	.22169
.20	.21112	.22224	.22902	.23496	.24196	.24678
.22	.23234	.24470	.25223	.25885	.26664	.27201
.24	.25360	.26722	.27554	.28285	.29147	.29741
.26	.27489	.28982	.29894	.30697	.31644	.32298
.28	.29622	.31248	.32244	.33121	.34157	.34873
.30	.31759	.33523	.34605	.35559	.36687	.37466
.32	.3390	.3581	.3698	.3801	.3924	.4008
.34	.3604	.3810	.3936	.4048	.4181	.4273
.36	.3819	.4040	.4176	.4296	.4440	.4539
.38	.4035	.4271	.4417	.4547	.4701	.4808
.40	.4251	.4503	.4660	.4799	.4964	.5079
.42	.4467	.4737	.4904	.5052	.5230	.5353
.44	.4684	.4971	.5149	.5308	.5498	.5630
.46	.4902	.5207	.5397	.5566	.5769	.5911
.48	.5120	.5443	.5645	.5826	.6042	.6194
.50	.5338	.5681	.5896	.6088	.6319	.6481

n = 80

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01597	.01634	.01688	.01736	.01791	.01885	.01992
.04	.03193	.03267	.03376	.03471	.03582	.03770	.03983
.06	.04788	.04899	.05063	.05206	.05372	.05655	.05975
.08	.06381	.06529	.06748	.06939	.07161	.07539	.07966
.10	.07972	.08157	.08431	.08670	.08949	.09422	.09958
.12	.09559	.09781	.10111	.10399	.10734	.11304	.11950
.14	.11142	.11402	.11789	.12125	.12518	.13185	.13941
.16	.12721	.13019	.13462	.13848	.14298	.15064	.15933
.18	.14295	.14631	.15131	.15567	.16076	.16941	.17925
.20	.15864	.16238	.16796	.17282	.17850	.18817	.19917
.22	.17426	.17840	.18455	.18993	.19621	.20690	.21909
.24	.18982	.19435	.20110	.20698	.21387	.22562	.23901
.26	.20531	.21024	.21758	.22399	.23150	.24431	.25893
.28	.22072	.22606	.23400	.24094	.24908	.26297	.27886
.30	.23606	.24180	.25036	.25784	.26661	.28160	.29878
.32	.2513	.2575	.2666	.2747	.2841	.3002	.3187
.34	.2665	.2731	.2829	.2914	.3015	.3188	.3386
.36	.2816	.2886	.2990	.3082	.3189	.3373	.3586
.38	.2966	.3040	.3151	.3248	.3362	.3559	.3785
.40	.3115	.3193	.3311	.3414	.3535	.3743	.3984
.42	.3263	.3346	.3470	.3578	.3707	.3928	.4183
.44	.3410	.3497	.3628	.3743	.3878	.4112	.4383
.46	.3556	.3648	.3785	.3906	.4049	.4295	.4582
.48	.3701	.3797	.3942	.4069	.4219	.4479	.4781
.50	.3845	.3946	.4097	.4231	.4388	.4662	.4981

n = 80

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02100	.02199	.02259	.02311	.02372	.02414
.04	.04200	.04398	.04518	.04623	.04746	.04830
.06	.06301	.06599	.06779	.06937	.07122	.07249
.08	.08402	.08801	.09042	.09254	.09501	.09671
.10	.10505	.11005	.11309	.11574	.11886	.12099
.12	.12609	.13213	.13579	.13900	.14276	.14534
.14	.14714	.15423	.15854	.16231	.16673	.16976
.16	.16822	.17638	.18134	.18568	.19077	.19427
.18	.18932	.19857	.20420	.20913	.21492	.21891
.20	.21044	.22082	.22713	.23266	.23917	.24365
.22	.23159	.24311	.25013	.25629	.26353	.26852
.24	.25277	.26547	.27321	.28002	.28803	.29355
.26	.27398	.28789	.29638	.30385	.31266	.31873
.28	.29523	.31038	.31965	.32780	.33743	.34407
.30	.31651	.33295	.34301	.35188	.36235	.36958
.32	.3378	.3556	.3665	.3761	.3874	.3953
.34	.3592	.3783	.3901	.4005	.4128	.4213
.36	.3806	.4011	.4138	.4250	.4382	.4474
.38	.4020	.4240	.4376	.4496	.4639	.4738
.40	.4235	.4470	.4616	.4744	.4897	.5004
.42	.4451	.4701	.4857	.4994	.5158	.5273
.44	.4666	.4934	.5099	.5246	.5422	.5544
.46	.4883	.5166	.5343	.5500	.5687	.5818
.48	.5100	.5401	.5588	.5755	.5955	.6095
.50	.5317	.5636	.5835	.6013	.6226	.6375

n = 90

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01620	.01655	.01706	.01751	.01804	.01892	.01993
.04	.03239	.03309	.03412	.03502	.03607	.03784	.03985
.06	.04857	.04962	.05117	.05252	.05409	.05676	.05978
.08	.06473	.06613	.06820	.07001	.07211	.07567	.07970
.10	.08086	.08262	.08521	.08747	.09011	.09457	.09963
.12	.09697	.09907	.10220	.10492	.10809	.11346	.11955
.14	.11304	.11550	.11915	.12234	.12605	.13234	.13948
.16	.12906	.13188	.13607	.13972	.14398	.15121	.15941
.18	.14503	.14822	.15295	.15708	.16189	.17006	.17934
.20	.16096	.16451	.16979	.17439	.17976	.18889	.19926
.22	.17682	.18074	.18658	.19166	.19760	.20770	.21919
.24	.19262	.19692	.20331	.20888	.21540	.22649	.23912
.26	.20836	.21303	.21999	.22606	.23316	.24526	.25905
.28	.22402	.22908	.23661	.24318	.25087	.26400	.27899
.30	.23960	.24505	.25316	.26025	.26855	.28272	.29892
.32	.2551	.2610	.2697	.2773	.2862	.3014	.3189
.34	.2705	.2768	.2861	.2942	.3037	.3201	.3388
.36	.2859	.2925	.3024	.3111	.3213	.3387	.3587
.38	.3011	.3082	.3187	.3279	.3387	.3573	.3787
.40	.3163	.3238	.3349	.3447	.3561	.3759	.3986
.42	.3314	.3392	.3510	.3613	.3735	.3944	.4185
.44	.3463	.3546	.3671	.3780	.3908	.4129	.4385
.46	.3612	.3699	.3830	.3945	.4080	.4314	.4584
.48	.3760	.3852	.3989	.4110	.4252	.4498	.4783
.50	.3907	.4003	.4147	.4273	.4423	.4682	.4983

n = 90

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02094	.02188	.02244	.02293	.02351	.02390
.04	.04189	.04376	.04489	.04587	.04702	.04781
.06	.06284	.06565	.06735	.06883	.07057	.07176
.08	.08380	.08756	.08983	.09181	.09414	.09573
.10	.10478	.10949	.11234	.11483	.11776	.11976
.12	.12576	.13144	.13489	.13790	.14143	.14385
.14	.14676	.15343	.15747	.16101	.16516	.16801
.16	.16778	.17545	.18011	.18419	.18897	.19225
.18	.18882	.19752	.20281	.20744	.21288	.21661
.20	.20988	.21963	.22556	.23076	.23687	.24106
.22	.23096	.24180	.24839	.25417	.26097	.26564
.24	.25208	.26402	.27129	.27768	.28519	.29036
.26	.27322	.28630	.29427	.30128	.30953	.31522
.28	.29440	.30865	.31734	.32499	.33401	.34023
.30	.31561	.33106	.34051	.34882	.35863	.36540
.32	.3368	.3536	.3638	.3728	.3834	.3908
.34	.3581	.3761	.3871	.3969	.4084	.4163
.36	.3795	.3988	.4106	.4211	.4335	.4421
.38	.4008	.4215	.4342	.4455	.4588	.4680
.40	.4222	.4443	.4579	.4700	.4843	.4942
.42	.4437	.4672	.4818	.4947	.5100	.5206
.44	.4652	.4902	.5057	.5195	.5359	.5473
.46	.4867	.5133	.5298	.5445	.5620	.5742
.48	.5083	.5366	.5541	.5697	.5884	.6014
.50	.5299	.5599	.5785	.5951	.6149	.6288

n = 100

<u>v</u>	<u>1/2%</u>	<u>1%</u>	<u>2-1/2%</u>	<u>5%</u>	<u>10%</u>	<u>25%</u>	<u>50%</u>
.02	.01639	.01672	.01722	.01764	.01814	.01898	.01993
.04	.03278	.03344	.03443	.03528	.03628	.03796	.03987
.06	.04915	.05015	.05163	.05291	.05441	.05694	.05980
.08	.06551	.06684	.06881	.07053	.07252	.07591	.07973
.10	.08184	.08350	.08598	.08813	.09063	.09487	.09966
.12	.09814	.10014	.10312	.10570	.10872	.11382	.11960
.14	.11440	.11675	.12023	.12325	.12678	.13276	.13953
.16	.13062	.13331	.13730	.14078	.14482	.15169	.15947
.18	.14680	.14984	.15434	.15826	.16284	.17060	.17940
.20	.16292	.16631	.17134	.17571	.18082	.18949	.19934
.22	.17899	.18273	.18829	.19312	.19877	.20837	.21927
.24	.19500	.19910	.20518	.21049	.21668	.22723	.23921
.26	.21094	.21540	.22203	.22781	.23456	.24606	.25915
.28	.22681	.23164	.23881	.24507	.25239	.26487	.27909
.30	.24261	.24781	.25554	.26229	.27018	.28365	.29903
.32	.2583	.2639	.2722	.2794	.2879	.3024	.3190
.34	.2740	.2799	.2888	.2965	.3056	.3211	.3389
.36	.2895	.2959	.3053	.3136	.3233	.3398	.3589
.38	.3050	.3117	.3218	.3306	.3409	.3585	.3788
.40	.3204	.3275	.3382	.3475	.3584	.3772	.3987
.42	.3357	.3432	.3545	.3643	.3759	.3958	.4187
.44	.3509	.3588	.3707	.3811	.3933	.4143	.4386
.46	.3660	.3744	.3868	.3978	.4107	.4329	.4586
.48	.3810	.3898	.4029	.4144	.4280	.4514	.4785
.50	.3959	.4051	.4189	.4310	.4452	.4699	.4985

n = 100

<u>V</u>	<u>75%</u>	<u>90%</u>	<u>95%</u>	<u>97-1/2%</u>	<u>99%</u>	<u>99-1/2%</u>
.02	.02090	.02178	.02231	.02278	.02332	.02370
.04	.04180	.04357	.04463	.04557	.04666	.04741
.06	.06271	.06536	.06697	.06837	.07002	.07114
.08	.08362	.08717	.08932	.09120	.09340	.09491
.10	.10454	.10901	.11171	.11407	.11683	.11872
.12	.12548	.13086	.13412	.13697	.14031	.14260
.14	.14643	.15275	.15658	.15992	.16385	.16654
.16	.16740	.17467	.17908	.18293	.18745	.19055
.18	.18839	.19663	.20163	.20601	.21115	.21468
.20	.20940	.21863	.22424	.22916	.23493	.23889
.22	.23043	.24068	.24692	.25239	.25881	.26322
.24	.25149	.26279	.26967	.27570	.28280	.28769
.26	.27258	.28495	.29249	.29912	.30691	.31228
.28	.29369	.30718	.31540	.32263	.33114	.33700
.30	.31484	.32946	.33839	.34625	.35550	.36188
.32	.3360	.3518	.3615	.3700	.3800	.3869
.34	.3572	.3742	.3847	.3938	.4047	.4122
.36	.3785	.3968	.4080	.4178	.4295	.4376
.38	.3998	.4194	.4314	.4420	.4545	.4632
.40	.4211	.4420	.4549	.4662	.4797	.4890
.42	.4425	.4648	.4785	.4906	.5051	.5151
.44	.4639	.4876	.5022	.5152	.5306	.5413
.46	.4854	.5106	.5261	.5399	.5564	.5678
.48	.5069	.5336	.5501	.5648	.5824	.5946
.50	.5284	.5567	.5743	.5899	.6085	.6216

SOME PROPERTIES OF THE SAMPLE COEFFICIENT OF VARIATION

by

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ABSTRACT

In this dissertation some properties of the sample coefficient of variation are investigated and the results are summarized below.

The exact frequency function of the S.C.V., whose parent distribution is assumed to be normal, is derived. The probability of negative values of the S.C.V. is discussed and the moments of the S.C.V. are shown not to exist. The above mentioned frequency function is then factored into a form suitable for computer calculations and an exact table of percentage points is derived and given in the Appendix.

All moments of the S.C.V., whose parent population is restricted to the non-negative range, are shown to exist. Upper bounds for such moments are given and these upper bounds are used to show that the moments of the S.C.V. uniquely determine their distribution. Cramer also shows that this distribution is asymptotically normal.

Using Fisher's k statistics, approximations to the mean and next three central moments of the S.C.V., are obtained through the $\frac{1}{n^3}$ terms. Investigations for the normal and gamma parent cases are then undertaken. The approximate moments of the Statistic $Z = S - t_0 \bar{X}$ are also derived and several of its properties are then studied.

Several approximations to the distribution of the S.C.V. are investigated. Tables of their percentage points are then derived and compared with the previously obtained exact results. It is found that for small values of the population coefficient of variation (C.V.), McKay's approximation gives excellent results. This approximation becomes worse as the

C.V. increases. For intermediate sample sizes a Pearson fit to the approximate moments of the S.C.V. gives very good results. For large sample sizes, say greater than 100, the asymptotic normality approximation gives adequately accurate percentage points. Surprisingly, the latter approximation is more accurate than the one which assumes normality of the S.C.V. and mean and variance given to $O(\frac{1}{n^3})$. Also included are several tables which summarize the degrees of accuracy of the above approximations.

A more thorough investigation into the asymptotic normality, of the S.C.V., is undertaken. Also, there is investigated the robustness, with respect to changes of the parent populations, of the percentage points of the S.C.V. For this purpose the effects, on the percentage points, of replacing the normal parent assumption by a gamma distribution is investigated. For small values of the C.V. the robustness is found to be exceptionally good. As the C.V. increased, larger sample sizes are found to be necessary to achieve a robustness of high quality. This situation is again illustrated by a table.

For values of the C.V. less than .5 the above properties, of the S.C.V., also hold for the non-central t distribution. In fact, the exact tables derived in this dissertation can be used to find the exact percentage points for small values of the C.V. of the non-central t.

The S.C.V. is then generalized to a form suitable for handling certain problems in the analysis of variance. The exact percentage points of this generalized sample coefficient of variation (G.S.C.V.)

can be found by the use of the previously derived percentage points for the S.C.V.

The above results are illustrated by several numerical examples. These examples describe the application of the S.C.V. to testing statistical hypothesis; finding the power curves of such tests; obtaining confidence intervals for the C.V. It is also used to study certain quality control problems and in the application of the G.S.C.V. to the analysis of experimental data.