

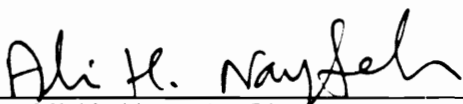
**A Theoretical and Experimental Study of Modal Interactions in Resonantly
Forced Structures**

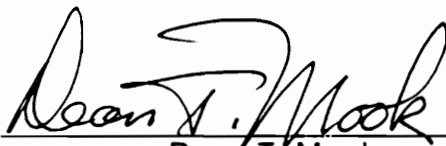
by

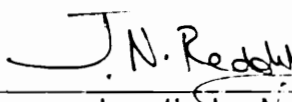
Balakumar Balachandran

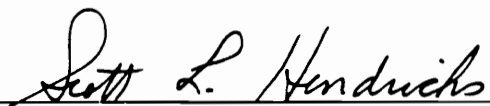
Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in
Engineering Mechanics

APPROVED:


Ali H. Nayfeh, Chairman


Dean T. Mook


Junuthula N. Reddy


Scott L. Hendricks


Rakesh K. Kapania

October, 1990

Blacksburg, Virginia

**A Theoretical and Experimental Study of Modal Interactions in Resonantly
Forced Structures**

by

Balakumar Balachandran

Ali H. Nayfeh, Chairman

Engineering Mechanics

(ABSTRACT)

The influence of modal interactions on the response of harmonically excited flexible L-shaped metallic and composite structures has been investigated analytically and experimentally. Each metallic structure possesses a two-to-one internal resonance, while each composite structure possesses a three-to-one internal resonance and either a two-to-one or a one-to-one internal resonance. For the metallic structures, a weakly nonlinear analysis is used to derive the autonomous system of equations which describe the evolution of the amplitudes and phases of the internally resonant modes. These equations are obtained for primary- and secondary-resonant excitations. The excitation frequency or amplitude is used as a control parameter and the resulting bifurcations (saddle-node, pitchfork, and Hopf bifurcations) are studied. Theoretical analyses for internally resonant systems are used to predict and explain the responses of the composite structures.

In the experiments, during primary-resonant excitations, low excitation (milli g) levels are used, while during secondary-resonant excitations, high

excitation (one to two g) levels are used. During primary-resonant excitations of the metallic structures, the saturation phenomenon and its breakdown, two-period quasiperiodic motions, and chaotically modulated motions are observed. The experimentally observed locations of jumps and transitions to quasiperiodic motions are in good agreement, respectively, with the analytically predicted pitchfork (saddle-node) and Hopf-bifurcation points. During subharmonic excitations of the higher mode, the metallic structure exhibits modal saturation and its breakdown, leading from periodic to modulated motions. The metallic structure also exhibits nonlinear periodic, quasiperiodic, and chaotically modulated responses to combination-resonant excitations. The experimental observations made during secondary-resonant excitations are in good agreement with the analytical predictions. All the observed responses of the metallic structures are planar. During primary-resonant excitations of a composite structure, nonlinear planar and nonplanar motions, nonplanar periodically and chaotically modulated motions, and modal saturation are observed. Nonlinear responses of a composite structure could not be excited during secondary-resonant excitations even at high excitation levels. In all cases, the transition from periodic to chaotically modulated motions occurred via quasiperiodic motions.

Analytical approximations for representative third-order and fourth-order autonomous systems are derived to study motions near the Hopf-bifurcation points of each of these systems.

Acknowledgements

First, I would like to thank my advisor, Professor Ali H. Nayfeh, for providing an opportunity to work with him and for all his teaching, advice, and support. Next, I would like to thank Professor Dean T. Mook for his invaluable advice, help, and numerous suggestions.

Further, I am thankful to Professors J. N. Reddy and S. L. Hendricks for their teaching and advice. I am also thankful to Professors R. K. Kapania and W. L. Neu for their help. I would also like to thank all the professors who have taught me at Virginia Polytechnic Institute and State University (VPI&SU), Blacksburg, VA, U.S.A. and Indian Institute of Technology, Madras, India.

I would like to express my appreciation to Nestor Sanchez, Larry Zavodney, and my friends at VPI&SU for their friendship. I would also like to express my thanks to cheerful Sally Shrader, for being infinitely helpful.

This work was supported by the Center for Innovative Technology under Grant No. MAT-90-006; the Army Research Office under Grant No. DAAL03-89-K-0180; and the Office of Naval Research under Contract No. N00014-90-J-1149.

To my parents and brother,

for their love and support.

Table of Contents

- 1. Introduction 1**
 - 1.1. Earlier studies 3
 - 1.2. Present study 15

- 2. Some Relevant Concepts and Notions 21**
 - 2.1. Attractors 23
 - 2.2. Types of attractors 25
 - 2.2.1. Point attractors 26
 - 2.2.2. Periodic attractors or limit cycles 28
 - 2.2.3. Quasiperiodic attractors 31
 - 2.2.4. Strange attractors 32
 - 2.3. Local bifurcations 34
 - 2.3.1. Bifurcations of point attractors 35
 - 2.3.2. Bifurcations of periodic attractors 41
 - 2.4. Transitions to chaotic motions 44
 - 2.5. Tools for characterizing attractors 46
 - 2.5.1. Poincaré sections 47
 - 2.5.2. Frequency spectra 48

2.5.3. Lyapunov exponents	49
2.5.4. Dimension	51
3. Analysis for the Metallic Structures	54
3.1. Linear undamped free vibrations	59
3.2. Primary resonance of the lower (first) mode	68
3.3. Primary resonance of the higher (second) mode	75
3.4. Subharmonic resonance of the higher mode	77
3.5. Combination resonance of additive type	81
3.6. Stability analysis	84
3.7. Analytical Predictions	85
4. Experiments with the Metallic Structures and Comparison with Theory	103
4.1. Primary resonance of the second (higher) mode	108
4.2. Primary resonance of the first (lower) mode	121
4.3. Subharmonic resonance of order one-half of the second mode	128
4.4. Combination resonance of the additive type	131
5. Experiments with the Composite Structures	177
5.1. Brief Analysis	180
5.2. Experiments	183
6. Motion near a Hopf Bifurcation of Three- and Four-Dimensional Systems	205
6.1. Three-dimensional system	208
6.1.1. Perturbation Analysis	210
6.1.2. Comparison	214
6.1.3. Stability Analysis	215
6.2. Four-dimensional system	219

6.2.1. Perturbation Analysis	221
6.2.2. Numerical Results	231
7. Conclusions and Recommendations	241
7.1. Conclusions	241
7.2. Recommendations	243
References	245
Table 1. Interaction Coefficients	255
Appendix A. Metallic structures: Equations of motion	256
Appendix B. Perturbation analysis to determine frequency correction	260
Appendix C. Metallic structures analysis: Integrals	266
Appendix D. Dimensions and properties of structures	273
D.1. Dimensions and properties of metallic structures	273
D.2. Dimensions of composite structures	277
Appendix E. Experimental procedures	280
E.1. Determination of linear frequencies	280
E.2. Determination of modal-damping coefficients	282
E.3. Pseudo-phase plane and Poincaré section	283
E.4. Time-dependent modal decompositions	285
E.5. Pointwise dimension calculations	288

Appendix F. Coefficients of Section 6.2. 294

Vita 298

1. Introduction

An understanding of the dynamical characteristics of a structural system is essential for its design and control. The nonlinearities in a structural system can have important influences on these characteristics. One such influence leads to an interaction between the different modes of oscillation. When the frequencies of a system are commensurable or nearly commensurable, internal (autoparametric) resonances may occur in the system (Tondl, 1965, Nayfeh and Mook, 1979, Schmidt and Tondl, 1986). These internal resonances lead to modal interactions and consequently to an energy exchange between the directly excited and indirectly excited modes of oscillation. As a result, several interesting nonlinear phenomena appear in the responses to harmonic excitations.

The extent of the modal interactions and the conditions under which they occur depend on the linear natural frequencies ω_i and nonlinearities of a system (Nayfeh and Mook, 1979). When the nonlinearity is cubic, internal

resonances can occur if $\omega_n \simeq \omega_m$, $\omega_n \simeq | \pm 2\omega_m \pm \omega_k |$, or $\omega_n \simeq | \pm \omega_m \pm \omega_k \pm \omega_l |$. When the nonlinearity is quadratic, besides some of the above resonances, internal resonances can occur if $\omega_n \simeq 2\omega_m$ or $\omega_n \simeq \omega_m + \omega_k$. Weakly nonlinear analyses of vibrations of many finite-degree-of-freedom and continuous systems lead to sets of nonlinear differential equations with the nonlinearities primarily being quadratic and/or cubic. In these equations, when the excitation terms appear as multiplying terms they are called parametric, while when the excitation terms appear as additive terms they are called external. There are different types of external and parametric excitations. Their classification depends on the relationship between the excitation frequency and natural frequencies of a system (Nayfeh and Mook, 1979). From the governing equations of motion, one can obtain the equations which describe the evolution of the amplitudes and phases of the internally resonant modes of a system. These equations are normally referred to as modulation, evolution, or envelope equations because they describe motions that occur on a slow time scale. An interesting feature is that these equations are qualitatively similar in form for different physical systems. Next, some of the earlier studies on internally resonant systems are briefly reviewed before the present work is discussed.

1.1. Earlier studies

One of the first observations of the influence of an internal resonance on the response of a system was made by Froude (1863). He remarked that a ship whose frequency in heave is twice its frequency in roll has undesirable seakeeping characteristics. Since then, several different physical systems with internal resonances have been studied. These systems can be classified, based on the type of internal resonance they possess, as systems with a one-to-one, two-to-one, three-to-one, or combination internal resonance. Also, some systems may possess more than one type of internal resonance.

One-to-one internal resonance

One-to-one internal resonances ($\omega_m \simeq \omega_n$) occur in several physical systems with quadratic or cubic nonlinearities, such as spherical pendulums, surface waves in closed basins, elastic shells and rings, strings, and beams.

Miles (1962, 1984a) examined the response of a spherical pendulum to planar harmonic excitations and found that the system can exhibit periodic nonplanar and modulated responses for certain excitation frequencies. He used the damping/excitation frequency as a control parameter and showed that the modulated motions are either quasiperiodic or chaotic and that there is a period-doubling sequence leading to chaotic motions. Miles (1984b) studied surface waves in a vertical, circular cylinder subjected to harmonic horizontal

excitations and concluded that nonlinear periodic, periodically modulated, and chaotically modulated responses are possible in that system. Further, he observed that the equations governing the amplitudes and phases of motion have a form similar to those derived for spherical pendulums and strings (Miles, 1984a, 1984c). In their recent review, Miles and Henderson (1990) examined and discussed studies of parametrically forced surface waves. Kambe and Umeki (1990) also studied modal interactions in parametrically forced surface waves. Miles used the time-averaged Lagrangian (e.g., Nayfeh, 1973) to derive the equations governing the amplitudes and phases of the internally resonant modes of oscillation in all of his studies.

Evensen (1966) conducted experiments with a thin elastic ring and observed coupling between in-plane and out-of-plane modes. Chen and Babcock (1975) observed interactions between two flexural modes of a finite length cylindrical shell in their experiments. Maewal (1986a) numerically studied the modulation equations governing the response of a circular cylindrical shell and a circular ring and showed that chaotically modulated responses are possible in these systems. He points out that the equations governing the amplitudes and phases of the motion of the shell are similar in form to those derived by Miles (1984a, 1984b) for other physical systems. Recently, Raouf and Nayfeh (1990a) presented a bifurcation analysis (control parameter: excitation frequency) for harmonically excited cylindrical shells. They used the method of multiple scales (Nayfeh, 1973,1981) to derive the modulation equations and showed that

a shell can display nonlinear periodic, quasiperiodic, and chaotic responses for certain parameter values.

Miles (1965), Anand (1966), Narasimha (1968), and Nayfeh and Mook (1979) analyzed motions of strings subjected to harmonic in-plane excitations and found that nonplanar responses (whirling or ballooning motions) are possible for some excitation frequencies. Miles (1984c) conducted a bifurcation analysis (control parameter: excitation frequency) and discussed possible bifurcations in the system. He did not find any periodically or chaotically modulated motions in his numerical investigations. Johnson and Bajaj (1989) investigated the same problem and showed that periodically and chaotically modulated responses of a string are possible for some parameter values. They used the method of averaging (e.g., Nayfeh, 1973) in their analysis. In their study a detailed picture of possible bifurcations (control parameter: damping/excitation frequency) which can occur for certain parameter values is given.

Ho, Scott, and Easley (1975) found whirling motions in their analysis of the forced nonlinear response of a simply supported beam, with two perpendicular axes of symmetry and fixed length. The nonplanar responses were a consequence of the coupling between in-plane and out-of-plane modes, as in the case of the string. Maewal (1986b) showed that a simply supported beam with a symmetric cross-section subjected to a harmonic in-plane excitation can exhibit chaotically modulated motions due to interaction between in-plane and

out-of-plane flexural modes. Haight and King (1972) examined the response of a cantilever beam subjected to a harmonic in-plane base excitation and found that nonplanar responses occur in certain conditions. Crespo da Silva and Glynn (1978) and Crespo da Silva (1980) found that an isotropic cantilever beam subjected to a harmonic transverse excitation at the base can exhibit nonplanar periodic and aperiodic responses due to interaction between flexural modes in two principal directions. Pai and Nayfeh (1990) considered the same problem and showed that nonplanar periodic (steady whirling), periodically modulated (unsteady whirling), and chaotically modulated motions are possible for certain values of the damping and excitation frequency. Nayfeh and Pai (1989) demonstrated the existence of similar motions for a parametrically excited isotropic cantilever beam.

Two-to-one internal resonance

Two-to-one internal resonances ($\omega_n \simeq 2\omega_m$) can occur in systems with quadratic nonlinearities. These systems include spring pendulums, ships, surface waves in closed basins, shells, arches, frames, and beam structures. Most of the material in this dissertation is to do with structures possessing a two-to-one internal resonance. Hence, this dissertation is closely related to the studies discussed in this section.

Gorelik and Witt (1933) experimentally observed modal coupling in a spring pendulum. Sethna (1965) was one of the first to analytically study the

response of two-degree-of-freedom systems with quadratic nonlinearities to primary-resonant excitations. He used the method of averaging for his asymptotic analysis to study a pendulum consisting of two masses connected by a spring. He showed that nonlinear periodic and amplitude- and phase-modulated motions can occur when the lower mode is excited and nonlinear periodic motions can occur when the higher mode is excited.

As mentioned earlier, Froude (1863) remarked that a ship whose frequency in heave is twice its frequency in roll has undesirable seakeeping characteristics. Nayfeh, Mook, and Marshall (1973) considered the response of a ship whose frequency in pitch is approximately twice its frequency in roll to primary-resonant excitations. They were the first to find the saturation phenomenon, which in the present system occurs when the second (pitch) mode is excited by a primary-resonant excitation. They found nonlinear periodic and modulated responses when the first (roll) mode was excited by a primary-resonant excitation. Nayfeh (1988) presented a detailed study of the nonlinear response of an internally resonant ship to primary-resonant excitations.

Yamamoto and Yasuda (1977) and Yamamoto et al. (1977) conducted analog-computer simulations of resonantly forced oscillations of a two-degree-of-freedom system with quadratic and cubic nonlinearities. They observed amplitude- and phase-modulated motions during primary-resonant excitations of the first and second modes. Miles (1984d) considered the

response of two quadratically coupled oscillators to a primary-resonant excitation of the lower mode. He used the excitation frequency as a control parameter and demonstrated that quasiperiodic and chaotic motions are possible in this system. Hatwal, Mallik, and Ghosh (1983a, 1983b) presented numerical and experimental results for the response of a two-degree-of-freedom system (a spring-mass-damper system with an attached pendulum) with quadratic nonlinearities to a harmonic excitation. They showed that chaotic responses are possible during primary-resonant excitations of the second mode. However, moderately high excitation levels were used in their study.

Haddow, Barr, and Mook (1984) conducted a theoretical and experimental investigation of the response of a two-degree-of-freedom beam structure to a harmonic excitation. They were the first to experimentally verify the saturation phenomenon. The paper followed from Haddow's (1983) work. He derived the equations of motion of the structure by treating a discrete system obtained by lumping the inertias of the beams with those of the concentrated masses. The method of multiple scales was used to determine approximate solutions of the resulting ordinary-differential equations for primary- and secondary-resonant excitations. In his study, the predicted linear and nonlinear periodic responses agreed qualitatively with the experimental observations. Hopf bifurcations (Guckenheimer and Holmes, 1983) were not considered in his study. His experimental observations include an extensive collection of force- and frequency-response curves for primary-resonant excitations. Nayfeh and

Zavodney (1988) experimentally observed amplitude- and phase-modulated motions in a similar structure subjected to a harmonic excitation of the lower mode. The structures treated in this dissertation are similar in form to those first used by Haddow et al. (1984) and later used by Nayfeh and Zavodney (1988).

Miles (1985) analyzed the response of a double pendulum subjected to a principal parametric excitation of the lower mode. He did not find any Hopf bifurcations in the study. Nayfeh (1983a, 1987a) treated principal parametric resonances of the lower and higher modes of a general two-degree-of-freedom system with quadratic nonlinearities and showed that Hopf bifurcations can occur in both cases. He mentioned that the equations of motion governing the response of double pendulums and ships constrained to pitch and roll are a special form of the equations considered in his study. He used the detuning between the frequencies of the two modes and the excitation frequency as control parameters for his study of principal parametric resonances of the lower mode and found regions in the parameter space where amplitude- and phase-modulated responses can occur. He also found a period-doubling sequence leading from quasiperiodic to chaotically modulated motions.

Nayfeh and Zavodney (1986) studied the response of a two-degree-of-freedom system with quadratic nonlinearities to a combination parametric resonance and found Hopf bifurcations and period-doubling bifurcations of quasiperiodic solutions. They used the detuning between the frequencies of the two modes,

excitation frequency, and excitation amplitude as control parameters. Streit, Bajaj, and Krousgrill (1988) treated a system that models motions of a robotic manipulator (governing equations similar in form to those treated by Nayfeh and Zavodney, 1986) and found periodically and chaotically modulated responses and a sequence of period-doubling bifurcations leading to chaotically modulated responses. They varied the nonlinear coefficients in the governing equations and numerically studied the resulting qualitative changes in the response.

Goodier and McIvor (1964), and McIvor and Lovell (1968) studied motions of cylindrical shells and found nonlinear responses due to coupling between a breathing and a flexural mode of the shell. McIvor and Sonstegard (1966) studied the axisymmetric response of spherical shells, taking into account the coupling of two flexural modes. Yasuda and Kushida (1984) investigated theoretically and experimentally the response of a shallow spherical shell to a harmonic excitation. Nayfeh and Raouf (1987a, 1987b) and Raouf and Nayfeh (1990b) analyzed the motions of infinitely long circular cylindrical and spherical shells subjected to primary-resonant excitations of the low- and high-frequency modes. They showed that the saturation phenomenon, periodically modulated motions, chaotically modulated motions, and period-doubling cascades leading from quasiperiodic to chaotic motions are possible for certain parameter values. Nayfeh, Raouf, and Nayfeh (1990) treated the response of cylindrical shells to a subharmonic radial excitation of the breathing mode.

Mook and co-workers (1985, 1986) studied the response of arches to subharmonic and combination resonances. They showed that the saturation phenomenon and periodically modulated responses are possible during subharmonic excitation of the higher mode and that combination resonances can also lead to modulated motions. They also discuss the influence of the presence/absence of the internal resonance on the response of the system in detail.

Miles (1976) considered resonantly forced oscillations of surface waves in a cylindrical basin. A transverse base excitation at a frequency close to the frequency of the first mode was considered and nonlinear periodic and modulated responses were found for some parameter values. Ibrahim and Barr (1975a) studied the response of a container partially filled with liquid and supported by an elastic structure to a primary-resonant excitation of the high-frequency mode. Miles (1984e) examined fluid motions in a circular cylinder subjected to principal parametric excitations of the first mode. He considered the case of perfect tuning (i. e., $\omega_n = 2\omega_m$) and found nonlinear periodic responses for some parameter values but did not find any Hopf bifurcations. As a consequence, he did not find any modulated motions. Holmes (1986) found transverse homoclinic orbits (Guckenheimer and Holmes, 1983) in parametrically excited surface waves and demonstrated that chaotic motions (irregular sloshing) are possible as a consequence. Nayfeh (1987b) studied surface waves in a cylindrical container subjected to a principal parametric excitation of the lower mode. He relaxed Miles' assumption of

perfectly tuned internal resonance and found nonlinear periodic, quasiperiodic, and chaotically modulated motions for certain parameter values. The detuning between the frequencies of the modes and the excitation frequency were used as control parameters for his bifurcation analysis. Gu and Sethna (1987) investigated parametrically excited surface waves in a rectangular container. They discussed in detail possible bifurcations and showed that nonlinear periodic, quasiperiodic, and chaotically modulated responses are possible for certain parameter values. Miles and Henderson (1990) reviewed the studies mentioned in this paragraph and a few others in their review article on parametrically excited surface waves.

Two-to-one internal resonances are also of interest in other problems. For example, Nayfeh (1971) studied the case of a perfectly tuned internal resonance in the context of a two-dimensional restricted problem of three bodies. However, unlike the previous studies, this study involves a Hamiltonian system.

Three-to-one internal resonance

Three-to-one internal resonances ($\omega_n \simeq 3\omega_m$) can occur in systems with cubic nonlinearities, such as beams. Sethna (1963) examined a two-degree-of-freedom system with internal and external resonances. He mentioned that modulated motions are possible, if the internal resonance is

not perfectly tuned, for some excitation frequencies close to the frequency of the lower mode.

Nayfeh and co-workers (1974a, 1974b) and Sridhar, Nayfeh, and Mook (1975) studied the response of uniform and non-uniform hinged-clamped beams subjected to harmonic excitations. They considered different primary and secondary resonances and found that nonlinear periodic responses are possible due to the interaction between in-plane flexural modes. Nayfeh, Mook, and Nayfeh (1987) considered uniform beams with a static axial load and a restraining spring at one end and a hinge at other end. They showed that nonlinear periodic and modulated responses are possible during primary-resonant excitations of the first or second mode. Nayfeh, Nayfeh, and Mook (1988) treated the response of a uniform hinged-clamped beam to a primary-resonant excitation of the higher mode and showed that quasiperiodic responses and period-doubling cascades leading from quasiperiodic to chaotic motions are possible. Crespo da Silva and Zaretzky (1988) studied the response of isotropic cantilever and clamped-pinned/sliding beams of nearly square and rectangular cross sections to primary-resonant excitations of the higher mode. For beams of nearly square cross-section they showed that interaction between in-plane flexural modes can lead to nonlinear periodic planar responses. They considered interaction between the in-plane flexural modes (three-to-one internal resonance) and in-plane and out-of-plane flexural modes (one-to-one internal resonance) for beams of rectangular cross-section and showed that nonlinear periodic nonplanar responses are possible.

Tso and Asmis (1974) analyzed the response of a two-degree-of-freedom system with cubic nonlinearities subjected to a principal parametric resonance of the first mode. Tezak, Mook, and Nayfeh (1978) treated the nonlinear responses of a uniform hinged-clamped beam subjected to principal and combination parametric resonances.

Tousi and Bajaj (1985) studied bifurcations in a two-degree-of-freedom system with cubic nonlinearities subjected to a primary-resonant excitation of the lower mode. They used the excitation frequency as a control parameter and showed that the system can exhibit quasiperiodic motions, chaotically modulated motions, and period-doubling sequences culminating in chaos.

Combination internal resonance

As mentioned before, this type of resonance can occur in systems with quadratic and/or cubic nonlinearities. These systems include circular membranes, beam frames, and plates. Some of the relevant references which treat this type of resonance in the presence of an external resonance include Ibrahim and Barr (1975b), Nayfeh and Mook (1978), Sridhar, Mook, and Nayfeh (1975, 1978), Lobitz, Nayfeh, and Mook (1977), Yasuda and Uno (1983), and Nayfeh, Nayfeh, and Mook (1990). In the last-mentioned reference, a T-shaped beam-mass structure was subjected to a harmonic excitation of the third mode. The first three natural frequencies of the structure were such that the third natural frequency was approximately the sum of the first and second natural

frequencies (additive combination resonance). The experiments and associated analysis show the existence of modal saturation, quasiperiodic, and phase-locked responses. Nayfeh (1983b) examined the nonlinear responses of a system to a parametric excitation.

1.2. *Present study*

In all the studies discussed in the earlier section, periodically and chaotically modulated motions occur in the region enclosed by the Hopf-bifurcation points. So, an analytical prediction of the Hopf-bifurcation points for an internally resonant system, such as the structures treated in the current work, can help in identifying the control-parameter values for which periodically and chaotically modulated motions are likely to occur. Also, the results of the different studies suggest that the chaotic motions can occur in internally resonant multi-degree-of-freedom systems at small levels of primary-resonant excitations (weak forcing). However, all the previous studies have by far been theoretical, and experimental confirmation, analysis of such modulated motions, and observations of transitions from periodic to chaotic motions in a structural system have been lacking. Furthermore, all the previous studies of the nonlinear response of internally resonant structural systems have been limited to structures made of isotropic materials. So far, the commonly studied nonlinear phenomenon in composite structures (specifically,

composite plates) has been the dependence of the frequency on the amplitude of oscillation during free oscillations (e.g., Chia, 1980). The nonlinear response of a composite structure in the presence of an internal resonance has not received much attention.

In this dissertation, the issues discussed in the previous paragraph are addressed. To this end, the influence of modal interactions on the response of flexible L-shaped beam-mass structures subjected to resonant excitations has been investigated. A typical beam-mass structure is shown in Fig. 1.1. It consists of two light-weight beams and two concentrated masses. The beams are made of steel in the case of the metallic structures and either a glass-epoxy or a graphite-epoxy composite material in the case of the composite structures. In each metallic (composite) structure, the first two (three) natural frequencies occur in a frequency range less than 20.0 Hz while the higher frequencies are far separated from these frequencies. The first two frequencies of each metallic structure are such that $\omega_2 \simeq 2\omega_1$, where ω_j is the frequency of the j th flexural mode. Let the first three frequencies of a composite structure be ω_1 , ω_t , and ω_2 , in that order. In each composite structure, ω_t represents the frequency of the first torsional mode, while ω_1 and ω_2 represent the frequencies of the first and second flexural modes, respectively. For a glass-epoxy composite structure, they are such that $\omega_2 \simeq 2\omega_t$ and $\omega_2 \simeq 3\omega_1$ while for a graphite-epoxy composite structure, they are such that $\omega_1 \simeq \omega_t$ and $\omega_2 \simeq 3\omega_1$. These frequency relationships can lead to internal resonances. Essentially, each metallic (composite) structure behaves

like a two-degree-of-freedom (three-degree-of-freedom) system. The equations which govern these structures are qualitatively similar in form to those which govern other physical systems (examples: ships, shells, surface waves). Hence, the structures treated in the present study can be regarded as analogues of other nonlinearly coupled oscillators or internally resonant physical systems. The organization of this dissertation is described in the following paragraphs.

In Chapter 2, a brief overview is provided of some concepts of nonlinear dynamics and notions relevant to this dissertation. In Chapter 3, an analytical model is derived to predict the qualitative and quantitative behavior of a metallic structure for primary- and secondary-resonant excitations. During primary-resonant excitations, the excitation frequency Ω is close to a natural frequency ω_j of a structure, while during secondary-resonant excitations, the excitation frequency is far away from the natural frequencies of a structure. Here, we consider the secondary-resonant excitations $\Omega \simeq 2\omega_2$ and $\Omega \simeq \omega_1 + \omega_2$. First, the Lagrangian for weakly nonlinear motions of an undamped structure is formulated and time averaged over the period of the primary oscillation (fast time scale). Subsequently, requiring the time-averaged Lagrangian to be stationary with respect to the variables which vary on a slow time scale, we obtain the corresponding Euler-Lagrange equations. The resulting autonomous system of equations describe the evolution of the amplitudes and phases of the internally resonant modes of a

structure. Later, modal damping is assumed and modal-damping coefficients, determined from experiments, are included in the analytical model.

The experiments conducted with the metallic structures and comparisons between experimental and analytical results are presented in Chapter 4. Fourier spectra, Poincaré sections, time-dependent modal decompositions, pseudo-phase planes, and dimension calculations are used to analyze the response of a structure. For primary-resonant excitations, the analytically predicted excitation-parameter values for modal saturation and its breakdown, jumps, and Hopf bifurcations are in good agreement with those observed in the experiments. The qualitative nature of the analytical and experimental results are compared for secondary-resonant excitations. Experimental observations of transitions from periodic to chaotic motions are also presented in Chapter 4. In Chapter 5, results from experiments with the composite structures are presented in the form of frequency spectra, Poincaré sections, and pseudo-phase planes. The composite structures are subjected to primary- and secondary-resonant excitations. A brief analysis is also provided to explain the responses of the composite structures. The present experimental study is one of the first to study modal interactions in periodically forced composite structures.

In Chapter 6, the analysis of motions near Hopf-bifurcation points of representative three- and four-dimensional systems is presented. The method of multiple scales is used to derive asymptotic expansions for the amplitude

and frequency of the limit cycle near a Hopf-bifurcation point of a system. Numerical studies carried out to ascertain the range of validity of the analytical approximations are also presented. Finally, we close the dissertation with concluding remarks and suggestions for future work in Chapter 7.

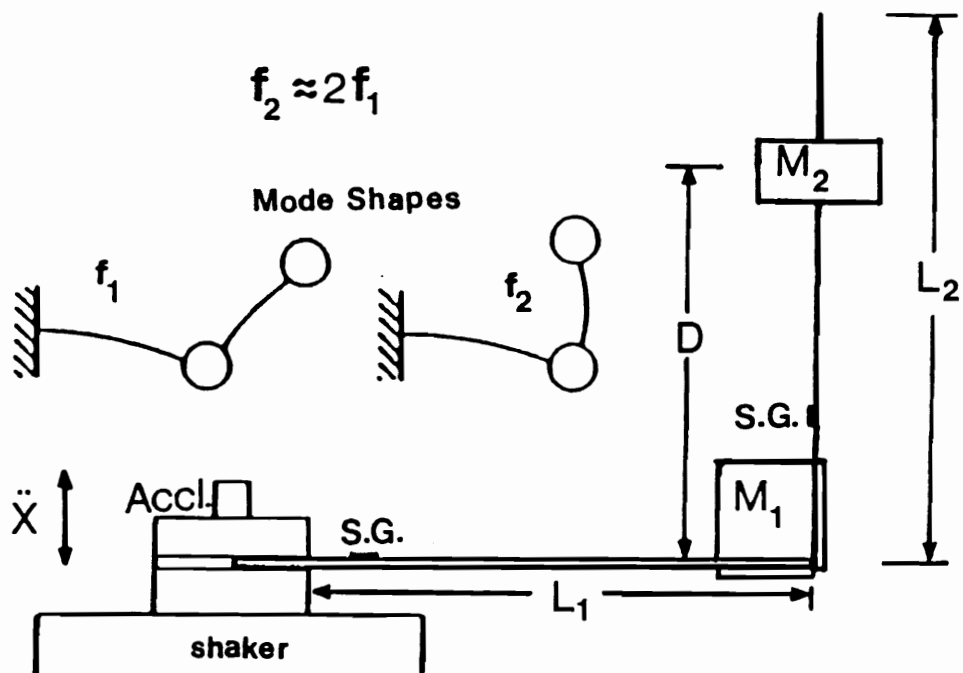


Figure 1.1. A typical beam-mass structure used in the study.

2. Some Relevant Concepts and Notions

In this chapter, we present some concepts and notions which are relevant to the current study. Most of the material is to do with the theory of nonlinear dynamical systems, which had its origins in Poincaré's work in the late nineteenth century (Guckenheimer and Holmes, 1983). A historical outline of concepts dating from Poincaré's initial work to recent times can be found in Jackson (1989). The material of this chapter in one form or the other can be traced to Hale (1963), Nayfeh and Mook (1979), Carr (1981), Eckmann (1981), Guckenheimer and Holmes (1983), Haken (1983), Farmer, Ott, and Yorke (1983), Bergé, Pomeau, and Vidal (1984), Eckmann and Ruelle (1985), Seydel (1988), Ruelle (1989), and Jackson (1989). There are several topics and references which have not been covered here. However, this does not mean that they are not relevant to the current work.

In general, a physical system is classified as either a conservative (Hamiltonian) or a nonconservative system. As the structures treated in the

present study dissipate energy due to damping, they fall under the class of nonconservative systems. Further, in the analysis we use systems with deterministic parameters to model these structures. Hence here, we will be concerned with deterministic nonconservative systems. In these systems, the asymptotic state of motion (time $t \rightarrow \infty$) is attracted to a subset of the phase space, defined by the coordinates of motion. Loosely speaking, this subset is called an attractor.

We consider systems whose time evolution is governed by an autonomous system of equations of the form:

$$\frac{dx(t)}{dt} = f[x(t); \gamma] , \quad x \in \mathbb{R}^n, \quad \gamma \in \mathbb{R}^k \quad (2.1)$$

where $\mathbb{R}^n$ stands for the n -dimensional real Euclidean space, the vector x is the set of n variables that describe the state of the system at time t , the m -differentiable vector function f determines the nonlinear time evolution of the state variables, and the vector γ is the set of k control parameters. Systems governed by equation (2.1) are also referred to as continuous-time dynamical systems (i.e., flows) as opposed to discrete-time dynamical systems whose evolution is governed by algebraic equations. In the context of this dissertation, equation (2.1) describes the evolution of the slowly varying amplitudes and phases (i.e., the vector x) of the internally resonant modes of a structure. Equation (2.1) is referred to as the modulation or averaged equation in the later chapters of this dissertation. In some studies (e.g.,

Sethna and Bajaj, 1978), the method of averaging is used to obtain the modulation equations. In these cases, one can use integral-manifold theorems (Hale, 1963) to relate the solutions of the averaged equations and their stability to the solutions of the original unaveraged system of equations.

The n -dimensional space whose coordinates are the state variables is also known as the phase space. One can distinguish between conservative and dissipative systems by examining how an arbitrary set of initial conditions evolves in time in this finite dimensional space. The change in the volume of the arbitrary domain is described by the divergence of the vector field f , that is, $\nabla \cdot f$. For conservative systems $\nabla \cdot f = 0$, while for dissipative systems $\nabla \cdot f < 0$. This implies that an arbitrary set of initial conditions has the same volume as time evolves in a conservative system while its volume shrinks to zero as time unfolds in a dissipative system (generalized Liouville's theorem, Jackson, 1989). As dissipation causes contractions of volumes in phase space, it takes the asymptotic state of motion of the system to a subset of $\mathbb{R}^n$, loosely called an attractor.

2.1. *Attractors*

There are several definitions for an attractor (e.g., Eckmann and Ruelle, 1985; Guckenheimer and Holmes, 1983). Here, we are interested in attractors of systems governed by equation (2.1). This equation can be thought of as a

nonlinear evolution operator T' that acts on a set of initial conditions $y \in V$ (a finite volume in phase space) such that $T' y = x(y,t)$. Also, we let $x(y,t)$ belong to the volume V . Repeated applications of T' may take one to an attractor A , which is defined to have the following properties (Eckmann, 1981):

1. **Invariance** : $T'A = A$
2. **Attractivity** : Attractor A has a shrinking neighborhood: that is, there is an open neighborhood U of A , $U \supset A$ such that $T'U \subset U$ for $t > 0$ and as t becomes large the evolution leads to A .
3. **Recurrence** : If S is an open set in V such that $S \cap A$ is not a null set, then there are arbitrarily large values of t such that $T^t x \in S \cap A$ when $x \in S \cap A$. In other words, trajectories starting from a state in an open subset of A repeatedly come to this state, time and again for arbitrarily large values of t .
4. **Indecomposability** : Attractor A cannot be split up into two nontrivial invariant pieces.

In the above statements and hereafter, the symbol $\supset$ denotes a superset, the symbol $\subset$ denotes a subset, and the symbol $\cap$ stands for the intersection of two sets. Property 2 rules out unstable equilibria while Property 3 implies that the flow T' is nowhere transient on A . Property 4 is also referred to as irreducibility and implies that an attractor cannot be decomposed into two distinct smaller attractors. A more mathematically precise definition can be found in Eckmann and Ruelle (1985). For an operational definition, one could say that an attractor is a set on which experimental points ($T^t x$) accumulate for

large t (Ruelle, 1989). If A is an attractor, its basin of attraction is defined as the set of initial points Y such that $T^t Y$ approaches A as $t \rightarrow \infty$. Next, the different types of attractors are discussed.

2.2. *Types of attractors*

Based on their geometric structures attractors can be classified as point attractors (fixed points), periodic attractors (limit cycles), quasiperiodic attractors (tori), and strange (chaotic) attractors. In experiments, they are typically identified by examining their frequency spectra and Poincaré sections (Bergé et al., 1984; Eckmann and Ruelle, 1985). The procedure used to construct Poincaré sections in the present study is discussed in Appendix E. As the dimension of a system increases or the dynamics become complicated, studying the geometrical structure or analyzing the frequency spectrum of an attractor becomes difficult, and one has to use dimension calculations (Ruelle, 1989; Farmer et al., 1983) and invariant measures, such as the Lyapunov exponents (e.g., Haken, 1983; Bergé et al, 1984), to identify the type of an attractor. Operational definitions of Poincaré sections and Lyapunov exponents are provided in Sections 2.5.1 and 2.5.3, respectively.

2.2.1. Point attractors

The solutions of equation (2.1) are called fixed points or constant solutions (equilibrium points of the dynamical system) if they satisfy the condition $T'P = P$ or $f(P, \gamma) = 0$ for all t . A point attractor is a fixed point of equation (2.1). However, not all fixed points are attractors. For a fixed point to be an attractor, the attractivity condition must be satisfied. To determine if this condition is satisfied, we perturb the fixed point with a disturbance $\xi(t)$, introduce $x = P + \xi$ into equation (2.1), expand f in a Taylor series about P , linearize the resulting equation in ξ , and obtain the so-called variational equation:

$$\frac{d\xi(t)}{dt} = D_P f(P; \gamma)\xi(t) + O(|\xi|^2) \quad (2.2)$$

where $D_P f$ is an $n \times n$ Jacobian matrix, the matrix of first partial derivatives of f . It should be noted that equation (2.2) is a set of n linear equations with real constant coefficients. Let λ_n denote the eigenvalues of $D_P f$. Because equation (2.2) has real constant coefficients, the complex conjugate of any its eigenvalues is also an eigenvalue. The attractivity condition demands that the real part of each eigenvalue λ_n should be negative, that is, the fixed point P should be asymptotically stable. If any eigenvalue of $D_P f$ has a positive real part, then the corresponding fixed point is unstable and it does not qualify as an attractor. To give an example, we consider point attractors of a two-dimensional system. Let the eigenvalues of $D_P f$ be λ_1 and λ_2 . These

eigenvalues are either both real or form a complex conjugate pair. When the eigenvalues are complex (real), the corresponding fixed-point attractor is called a focus (node).

If the real parts of all the eigenvalues λ_n of a fixed point are nonzero, the fixed point is called hyperbolic or nondegenerate fixed point (e.g., Seydel, 1988) and linearization is sufficient to determine the stability of such a fixed point. A hyperbolic fixed point for which all the λ_n have positive real parts is called an unstable fixed point (in this study, we do not have any such fixed points). If a hyperbolic fixed point has some eigenvalues with positive real parts and some eigenvalues with negative real parts, it is called a nonstable fixed point or a saddle point (e.g., Parker and Chua, 1987). An unstable fixed point is unstable in forward time and stable in reverse time, while a nonstable fixed point is unstable in both forward and reverse times. If any λ_n has a zero real part, then linearization is insufficient to determine the stability of the fixed point. The corresponding fixed point is called a nonhyperbolic or degenerate fixed point (e.g., Holmes, 1990). One needs to use a nonlinear analysis to determine its stability. In the nonlinear analysis, one may begin by using the center manifold theory (Carr, 1981), to reduce the order of the system before carrying out the stability analysis. In this approach, one obtains the equation on the center manifold and uses a perturbation analysis to study solutions of this equation.

In the context of the current work, point attractors of the averaged or modulation equations (equation 2.1) imply constant amplitudes and phases for the internally resonant modes of a structure. This in turn means that the motion of the structure is periodic (period being a multiple of the period of forcing) and that, its corresponding frequency spectrum has one basic frequency.

2.2.2. Periodic attractors or limit cycles

These attractors are of fundamental importance in many physical systems (Haken, 1983). They correspond to stable closed trajectories or orbits in the phase space and stable periodic solutions of equation (2.1). Let a be a point on the limit cycle, having the period T , which is finite. Then $T^T a = a$ and $T^t a \neq a$ for $0 < t < T$. The point a is a periodic point with period T and lies on a periodic or closed orbit $X = \{T^t a : 0 \leq t \leq T\}$. Alternatively $X(t) = X(t + T)$, where $X(t)$ is a solution of equation (2.1). Again to determine the attractivity condition for the closed orbit, we perturb the solution $X(t)$ with a disturbance $\xi(t)$, introduce $x = X + \xi$ into equation (2.1), expand f in a Taylor series, linearize the resulting equation, and obtain

$$\frac{d\xi(t)}{dt} = D_X f(X; \gamma; t) \xi(t) \quad (2.3)$$

where $D_X f$ is an $n \times n$ matrix of periodic coefficients with period T .

Equation (2.3) is studied by using Floquet theory (Nayfeh and Mook, 1979), which gives us information on the growth or decay of $\xi(t)$. There are n fundamental sets of solutions, namely, $\xi_1, \xi_2, \dots, \xi_n$ for equation (2.3), each of which is an n -dimensional vector. These vectors can be used as columns to form an $n \times n$ matrix Φ (fundamental matrix), which satisfies

$$\Phi(t + T) = \Phi(t) C \quad (2.4)$$

where C is an $n \times n$ constant matrix. If the initial conditions are chosen such that $\Phi(0) = I$, then the matrix $C = \Phi(T)$ is called the monodromy matrix. Its eigenvalues are known as the Floquet multipliers λ . The quantities $\frac{1}{T} \ln \lambda$ are called characteristic exponents. Here, for the sake of simplicity we will assume that the matrix $\Phi(T)$ has distinct eigenvalues. Besides this case, Nayfeh and Mook (1979) also deal with the case where $\Phi(T)$ has multiple eigenvalues. In the case of distinct eigenvalues, one can introduce a transformation $W = \Phi P$ into equation (2.4) and obtain

$$W(t + T) = W(t) P^{-1} C P = W(t) B \quad (2.5)$$

where B is a diagonal matrix and P^{-1} is the inverse of the matrix P . This means that w_j , the j th column of matrix W , satisfies

$$w_j(t + T) = \lambda_j w_j(t) \quad (2.6)$$

which in turn implies that $w_j(t + KT) = \lambda_j^K w_j(t)$. Therefore if $|\lambda_j| < 1$, then $w_j(t + KT) \rightarrow 0$ as $K \rightarrow \infty$, and if $|\lambda_j| > 1$, then $w_j(t + KT) \rightarrow \infty$ as $K \rightarrow \infty$.

Consequently, the solution X is asymptotically stable if all the Floquet multipliers lie within the unit circle in the complex plane. However, it is known that one of the Floquet multipliers λ_k of the monodromy matrix for an autonomous system is ± 1 (e.g., Seydel 1988). This multiplier corresponds to a disturbance ξ , which is tangent to the limit cycle or periodic orbit. Hence, here the attractivity condition for X to be an attractor is that $|\lambda_j| < 1$ for all $j \neq k$. When this condition is satisfied, the limit cycle X is said to be orbitally stable. If the magnitude of any of the Floquet multipliers is greater than 1, then the corresponding limit cycle is unstable and does not qualify as an attractor. There are several studies that examine periodic motions in the Poincaré section rather than in the phase plane (e.g., Parker and Chua, 1987; Seydel, 1988). Periodic attractors in the phase space correspond to point attractors in the Poincaré section.

There are some orbits which do not qualify as attractors but deserve attention because of their importance in understanding the dynamics of a system. These orbits, which can be considered as limiting cases of periodic orbits, are homoclinic and heteroclinic orbits. Homoclinic orbits have trajectories running to and from the same saddle point while heteroclinic orbits connect distinct saddle points. These orbits appear as boundaries which separate regions of qualitatively different behavior. There are many studies that focus on these orbits because a system can exhibit unusual behavior in their neighborhoods.

In the present study, a periodic attractor of equation (2.1) means that the amplitudes and phases of the internally resonant modes of a structure vary periodically. Their frequency of oscillation is referred to as the modulation frequency. When the modulation and excitation frequencies are incommensurate, the corresponding motion of the structure is called two-period quasiperiodic, and the response spectrum has two basic frequencies (excitation and modulation frequencies). However, if the modulation and excitation frequencies are commensurate, one refers to a phase-locked condition, which here translates to periodic motions of the structure.

2.2.3. Quasiperiodic attractors

As stated at the end of the preceding section, the periodic attractors of equation (2.1) correspond to two-period quasiperiodic motions (quasiperiodic attractors) of the structure. The dependent variables which describe this motion can be written as

$$g(t) = F(f_1 t, f_2 t) \quad (2.7)$$

where t stands for time and the frequencies f_i are incommensurate. Equation (2.7) also describes motions composed of a collection of two oscillators with frequencies f_1 and f_2 .

Now, let us suppose that equation (2.1) describes a motion composed of a collection of k oscillators with k incommensurate frequencies $f_1, f_2, f_3, \dots, f_k$. Then, the corresponding motion lies on a space formed by a product of k circles, which is known as a k -dimensional torus T^k (Eckmann and Ruelle, 1985). One should note that the symbol T^k used here is different from the operator T' used earlier. If the k -torus possesses the properties of an attractor, then the asymptotic motion of the dynamical system will be of the form

$$X(t) = F(f_1 t, f_2 t, \dots, f_k t) \quad (2.8)$$

For a more detailed and refined discussion on the occurrence and stability of a k -torus, one is referred to Grebogi, Ott, and Yorke (1985). In this context, a theorem by Newhouse, Ruelle, and Takens (1978) suggests that three-period quasiperiodic motions are less likely to be realized in physical situations or experiments. However, as Grebogi, Ott, and Yorke (1985) point out, the possibility of realization of k -period quasiperiodic motions depends strongly on the strength of the nonlinearities in a system. In the present study, the modulation equations did not exhibit any quasiperiodic solutions.

2.2.4. Strange attractors

Attractors that are not point, periodic, or quasiperiodic attractors fall under the class of strange attractors. They are of significant interest as they are

normally associated with chaotic motions. In general, the adjective strange is associated with the geometry of an attractor, while the adjective chaotic is associated with the dynamics on the attractor. However, most strange attractors are chaotic, and hence these words are used interchangeably in the literature and in this study (for some more details, see Appendix E). Chaotic motions are known to occur in continuous-time systems whose dimension $n \geq 3$. In discrete-time systems, these motions can occur for $n < 3$ (Ruelle, 1989). Several examples of physical systems in which chaotic motions occur can be found in the literature (e.g., Swinney, 1983; Moon, 1987).

In this paragraph, we discuss the geometrical structure of a strange attractor. First, let us recall that dissipation causes volumes in phase space to contract and, as a consequence an attractor has zero volume. As opposed to point, periodic, and quasiperiodic attractors, the strange attractor has a complex structure. This complex structure is a result of stretching in some directions on the attractor (one should note that at the same time the volume element experiences contractions in other directions so that the change of volume with time is negative). In order to remain confined in a bounded domain, the volume element gets folded, which after some time results in a multilayered structure (Grassberger and Procaccia, 1983). An intersection of a transverse line with this structure would reveal a fractal structure.

A strange attractor can be identified by examining its Lyapunov exponents, dimension, Poincaré section, and frequency spectrum. A strange attractor is

characterized by a positive Lyapunov exponent, a noninteger value for the dimension, and a broadband character in the frequency spectrum. Further, the Poincaré section for a strange attractor does not correspond to any simple geometrical form. Ruelle (1989) also discusses how statistical analysis (ergodic theory) can be used to obtain some invariant probability measures to study chaotic motions. In the present study, the structures are seen to exhibit chaotic motions under certain conditions. In these cases, the motion is ascertained to be chaotic by examining its dimension, Poincaré section, and frequency spectrum.

2.3. *Local bifurcations*

Here, we use the frequency and amplitude of the excitation as control parameters and study qualitative changes that occur in a structure's response as a control parameter is varied. In the course of the analysis, we study motions or solutions near a fixed point or limit cycle of equation (2.1). So, the qualitative changes in the structure of the solutions are local in nature and are called local bifurcations (e.g., Guckenheimer and Holmes, 1983). The parameter values ($\gamma = \gamma_0$ in equation (2.1)) at which these bifurcations take place are called bifurcation values. Bifurcations which deal with changes in the global aspects of a flow in the phase space are called global bifurcations. In studies of these bifurcations, one is normally concerned with homoclinic or

heteroclinic orbits (Guckenheimer and Holmes, 1983). Global-bifurcation methods are often used to analyze the complex behavior (chaotic motions) of a system (e.g., Mees and Sparrow, 1987).

The analysis used to study local bifurcations is called local analysis. In the current work, all the local bifurcations occur as a single control parameter is varied while the other parameters are fixed. Hence, the smallest dimension of the parameter space in which these bifurcations occur is one and the associated bifurcations are called codimension-one bifurcations (Guckenheimer and Holmes, 1983; Bergé et al., 1984). When a fixed point experiences a codimension-one bifurcation, at the bifurcation point, either an eigenvalue of $D_x f$ is zero or a complex conjugate pair of eigenvalues of $D_x f$ are purely imaginary. Next, the different types of local bifurcation are discussed.

2.3.1. Bifurcations of point attractors

As the control parameter γ is varied, the implicit function theorem (e.g., Seydel, 1988) can be used to describe the point attractors or fixed points x as smooth functions of γ when one is away from the places where the Jacobian $D_x f$ is singular. Now, let $D_x f$ be singular at the fixed point $x = x_0$ and the control parameter $\gamma = \gamma_0$. If one constructs a space with coordinates x and γ , then in this space (control-phase space), several branches of solutions may come together at the location (x_0, γ_0) . This point is normally called a bifurcation point

(Guckenheimer and Holmes, 1983). Local bifurcation analysis deals with solutions that emerge from such bifurcation points.

It is common in local analysis, to first reduce the order of a system by using the center manifold theorem. One determines the different eigenvalues of the $n \times n$ matrix $D_x f$ and classifies them based on the real parts of the eigenvalues as, eigenvalues with zero real parts (number of them equal to n_c), eigenvalues with positive real parts, and eigenvalues with negative real parts. After having done so, one determines the respective eigenvectors and the spaces spanned by these eigenvectors. These spaces, known as eigenspaces, are of three types, namely, stable eigenspaces (correspond to eigenvalues with negative real parts), unstable eigenspaces (correspond to eigenvalues with positive real parts), and center eigenspaces (correspond to eigenvalues with zero real parts). These eigenspaces are also referred to as invariant subspaces, as they are invariant under the action of the linearized equations (i.e., if one starts initially from a point $y \in \Gamma$, where Γ is an invariant space, the evolution of this point remains in Γ for all time t). Similar spaces also exist for the nonlinear equations. These spaces, called manifolds (essentially smooth curved surfaces), are classified typically as unstable manifolds, stable manifolds, and center manifolds. Each manifold is tangent to its respective eigenspace at the fixed or equilibrium point (Guckenheimer and Holmes, 1983).

Using the center manifold theory one can reduce the order of a system from n to n_c . In the case of codimension-one bifurcations, it is known that $n_c = 1$ if

one eigenvalue is zero while the other eigenvalues are away from the imaginary axis and $n_c = 2$ when a complex conjugate pair of eigenvalues lie on the imaginary axis while all the other eigenvalues are away from it. One can further simplify this n_c – dimensional system using the method of normal forms to obtain the respective normal form for a particular bifurcation. Holmes (1990) illustrates the procedure for obtaining the normal form for a particular bifurcation in his review article.

Next, the different bifurcations that a point attractor can experience, namely, saddle-node, pitchfork, transcritical, and Hopf bifurcations are discussed. The first three bifurcations occur when an eigenvalue of $D_x f$ crosses from the left half to the right half of the complex plane along the real axis. The fourth one, the Hopf bifurcation, occurs when a pair of complex conjugate eigenvalues crosses transversely from the left half to the right half of the complex plane. The normal forms and accompanying bifurcation diagrams for these bifurcations can be found in many textbooks (e.g., Guckenheimer and Holmes, 1983; Bergé et al., 1984; Seydel, 1988). Typical bifurcation diagrams are shown in Fig. 2.1. In each case, a measure of the response amplitude x is plotted versus the control parameter μ and the point $\mu = 0$ is the bifurcation point. A bifurcation is called supercritical if it locally has stable solutions on either side of the bifurcation point and is called subcritical (reverse) if it locally has stable solutions on only one side of the bifurcation point. One should note that the above and following discussion also apply to periodic attractors when one views them in their Poincaré sections.

Saddle-node bifurcation

Typically, during such a bifurcation, locally two half-branches, one a stable branch (nodes) and another an unstable branch (saddles) emerge or merge at the bifurcation point. This scenario is depicted in Fig. 2.1a and is governed by the following normal form.

$$\frac{dx(t)}{dt} = \mu - x^2 \quad (2.9)$$

In equation (2.9) both x and μ are scalar parameters. This bifurcation is also sometimes referred to as a tangent bifurcation, as one has a vertical tangency at the point of bifurcation in the response curves. In a physical situation, this bifurcation may lead to a jump from one type of attractor to another.

Transcritical bifurcation

Figure 2.1b shows the bifurcation diagram for a transcritical bifurcation. This type of bifurcation is characterized by the presence of one nontrivial half-branch on either side of the bifurcation point. This nontrivial half-branch is stable on one side and unstable on the other side of the bifurcation point. The normal form for this bifurcation is

$$\frac{dx(t)}{dt} = \mu x - x^2 \quad (2.10)$$

where again the x and μ are real scalar parameters. In the present work, we did not find any such bifurcations.

Pitchfork bifurcation

Let x_1, x_2, x_3 , and x_4 represent the state variables of a four-dimensional system governed by equation (2.1). Further, let the form of the equations be such that x_1, x_2, x_3 , and x_4 and $-x_1, -x_2, x_3$, and x_4 are its solutions. When these two solutions are identical, there is a reflection symmetry in the $x_1 - x_2$ plane. A system can possess other types of symmetries too (e.g., Seydel, 1988). The pitchfork bifurcation normally breaks a reflection symmetry of the system. It is governed by the normal form

$$\frac{dx(t)}{dt} = \mu x + \alpha x^3 \quad (2.11)$$

where again the x, μ , and α are real scalar parameters. When α is negative, one has a supercritical bifurcation (Fig. 2.1c), and when it is positive, one has a subcritical bifurcation (Fig. 2.1d). In both cases, the symmetric fixed point loses its stability after the bifurcation. During a supercritical bifurcation, two stable half-branches of unsymmetric fixed points emerge from the bifurcation point, while during a subcritical bifurcation, two unstable half-branches of unsymmetric fixed points merge at the bifurcation point. It is possible in some cases, to recognize these bifurcations in the phase plane and the frequency

spectrum. In the present work, some of these bifurcations are discerned by studying a reflection symmetry of the system.

Hopf bifurcation

The previously discussed bifurcations of point attractors involved only fixed-point solutions of equation (2.1) and are called stationary bifurcations. As opposed to these bifurcations, a Hopf bifurcation involves fixed-point and limit-cycle solutions of equation (2.1) and is called a nonstationary bifurcation. Further, a Hopf bifurcation can be either a subcritical or a supercritical bifurcation. According to a theorem postulated by Hopf (Seydel, 1988) the following conditions have to be satisfied for this bifurcation to occur:

1. $f(x_o, \gamma_o) = 0$.
2. $D_x f$ has a pair of purely imaginary eigenvalues $(\pm i\omega)$ while its other eigenvalues have nonzero real parts.
3. The rate of change of the real part of the eigenvalues, which cross the imaginary axis, with respect to the control parameter γ is not equal to zero at $\gamma = \gamma_o$.

When the above conditions are satisfied a limit cycle of period $2\pi/\omega$ is born at (x_o, γ_o) . This limit cycle is stable or unstable depending on the nature of the bifurcation. In Chapter 6, we develop a third-order expansion for motion near a Hopf-bifurcation point and use it to determine the amplitude and frequency of the limit cycle as well as its stability. The occurrence of a Hopf bifurcation

in a physical process can be recognized typically by the appearance of another incommensurate frequency in the frequency spectrum. In the experiments described in the present work, the transition from periodic to quasiperiodic motions is attributed to a Hopf bifurcation. In the next section bifurcations of periodic attractors are discussed.

2.3.2. Bifurcations of periodic attractors

The bifurcations of periodic attractors are normally discerned by studying the Floquet multipliers of the monodromy matrix $\Phi(T)$ in equation (2.4). At a bifurcation point, one or more of the Floquet multipliers leave the unit circle in the complex plane. The nature of their departure helps in characterizing the resulting bifurcation as a saddle-node (cyclic-fold) bifurcation, pitchfork (symmetry-breaking) bifurcation, period-doubling (flip or subharmonic) bifurcation, or secondary Hopf (Neimark) bifurcation (e.g., Thompson and Stewart, 1986). Some of these bifurcations can be either supercritical or subcritical. Again, when a supercritical bifurcation occurs, one locally has stable solutions on both sides of the bifurcation point and when a subcritical bifurcation occurs one locally has stable solutions on only one side of the bifurcation point. One can also study bifurcations of periodic attractors by examining the bifurcations of their projections in the Poincaré section.

Saddle-node or cyclic-fold bifurcation — Jump bif.

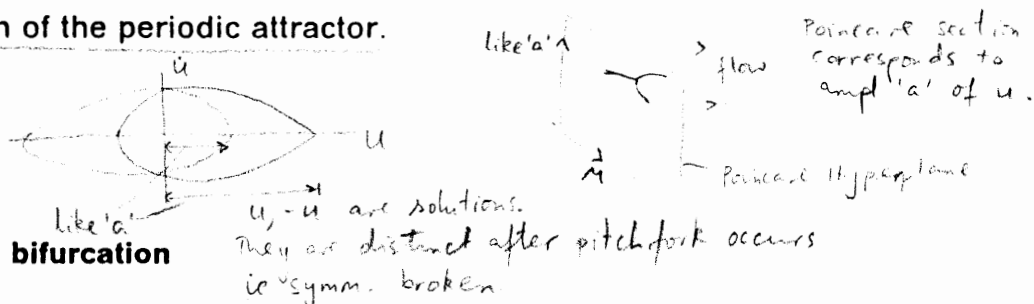
This bifurcation occurs when a Floquet multiplier of $\Phi(T)$ leaves the unit circle through $+1$ in the complex plane and occurs when a system does not possess any natural symmetries. This bifurcation is analogous to the saddle-node bifurcation discussed earlier in the context of fixed-point attractors although here one deals with orbits instead of fixed points.

Pitchfork bifurcation — Symm breaking.

This bifurcation also occurs when a Floquet multiplier leaves the unit circle through $+1$ in the complex plane. When this bifurcation occurs a reflection symmetry of the system is broken. There are also other ways of distinguishing the pitchfork bifurcation (e.g., the analysis used in Section 6.1 of Chapter 6).

This bifurcation corresponds to a pitchfork bifurcation of a fixed point in the

Poincaré section of the periodic attractor.



Period-doubling bifurcation

This bifurcation occurs when a Floquet multiplier leaves the unit circle through -1 in the complex plane. As a consequence, the period (frequency) of the limit cycle doubles (reduces by one-half). Unlike the previous two bifurcations, this bifurcation is not analogous to any of the bifurcations a fixed point

experiences. In the Poincaré section, this bifurcation is characterized by a flipping between two points and hence, it is also called a flip bifurcation.

Secondary-Hopf bifurcation

This bifurcation occurs when a pair of complex conjugate Floquet multipliers leaves the unit circle in the complex plane transversely. In the Poincaré section, this bifurcation corresponds to a Hopf bifurcation of a fixed-point attractor. Again, this bifurcation can introduce a new incommensurate frequency into the motion. As a consequence, one goes from a periodic motion (limit cycle) to a quasiperiodic motion (torus). If the motions are stable, the bifurcation takes one from a periodic attractor to a quasiperiodic attractor. Due to this fact, this bifurcation is also referred to as bifurcation to a torus.

In studying the bifurcations of periodic attractors, one usually needs to determine the stable and unstable limit cycles. Several articles and texts discuss how to determine these unstable limit cycles numerically (e.g., Seydel, 1988). A method for determining these unstable limit cycles was first provided by Aprille and Trick (1972).

For a discussion on bifurcations from quasiperiodic attractors, we refer the reader to Haken (1983). Next, we briefly discuss some typical transitions to chaotic motions.

2.4. *Transitions to chaotic motions*

One of the first proposed transitions to chaotic motions was Landau's scenario (Bergé et al., 1984). However, this scenario lacks experimental proof. Now, there are essentially three widely accepted and observed transitions to chaotic motions (Eckmann, 1981; Bergé et al., 1984). They are the Ruelle-Takens scenario, the Feigenbaum sequence (period-doubling route) and the Pomeau-Mannville scenario (route via intermittency). In all scenarios, the sequence starts from a periodic motion.

Chaos in the Ruelle-Takens scenario (Newhouse, Ruelle, and Takens, 1978) is the culmination of a finite sequence of Hopf bifurcations. If a system goes through three Hopf bifurcations (the second and third being secondary Hopf bifurcations), starting from a rest (point attractor) state, as a control parameter is varied, then it is likely that the system possesses a strange attractor after the third Hopf bifurcation. Each Hopf bifurcation introduces an incommensurate frequency in the spectrum. This route is in marked contrast with Landau's scenario, which suggests that an infinite number of Hopf bifurcations are required before a system reaches a turbulent or chaotic state. The Ruelle-Takens scenario has been observed in some experiments.

In the period-doubling scenario, a periodic orbit undergoes a sequence of period-doubling bifurcations as a control parameter is varied, culminating in a motion whose period is infinite (chaotic motion). A certain ratio of the

control parameters for successive bifurcations tends to a constant (Feigenbaum number) after a large number of bifurcations. This sequence and the corresponding constant have been found to apply to a variety of different systems. One can discern the bifurcations by observing the appearance of subharmonics in the spectrum. The spectrum shows a broadband character at the culmination of the bifurcation sequence. In systems with symmetry, the first period-doubling bifurcation is preceded by a symmetry-breaking bifurcation (e.g., Swift and Wiesenfeld, 1984).

In the Pomeau-Mannville scenario, as a control parameter is varied, first a periodic orbit experiences a saddle-node bifurcation. After this bifurcation occurs, the motion consists of intermittent intervals of periodic and chaotic motions. As the control parameter is varied further, the duration of occurrence of the periodic motion decreases eventually culminating in a state of chaotic motion.

There is also another route (Bergé et al., 1984) that leads to chaotic motions. In this route, as a control parameter is varied, initially after two Hopf bifurcations one has a two-period quasiperiodic motion (torus) or a motion composed of two basic frequencies. As the control parameter is varied further, periodic motion (one basic frequency) follows due to frequency locking and consequently, there is a breakdown of the torus. This frequency-locked motion exists over a range of the control-parameter values and eventually

gives way to a chaotic motion at another critical value of the control parameter.

In the present study, transitions from periodic to chaotic motions are found to occur through two-period quasiperiodic motions. The initial motions are periodic in each case. As the control parameter is varied, these motions give way to two-period quasiperiodic motions. Further on, in some of the transitions, one of the basic periods of the quasiperiodic motions undergoes a sequence of period doublings as a control parameter is varied, culminating in chaotically modulated motions. The other transitions do not follow the period-doubling sequence or the sequences discussed in the two previous paragraphs. Our study could not ascertain if these other transitions followed the Ruelle-Takens scenario or not. Next, we briefly discuss some of the available tools used for characterizing attractors, namely, Lyapunov exponents, dimension calculations, Poincaré sections, and frequency spectra. There are also other tools for characterizing attractors such as correlation functions and multispectral analyses.

2.5. Tools for characterizing attractors

In this section, we discuss some of the tools that are used to characterize attractors.

2.5.1. Poincaré sections

Poincaré sections are obtained differently in autonomous and nonautonomous systems and are typically used to ascertain if a motion is periodic, quasiperiodic, or chaotic. A nonautonomous system can be transformed into a higher-dimensional autonomous system if the explicit time-dependent terms are periodic in the corresponding equations. However, in each case, the Poincaré section is a surface, whose dimension is lower than the dimension of the original system. For an n -dimensional autonomous system governed by equation (2.1), a Poincaré section is an $(n-1)$ -dimensional surface which is transverse to the flow (trajectory) of equation (2.1). The intersections of a trajectory which cross this section in the same direction form a characteristic pattern for each attractor. A convenient Poincaré section is $x_j = \text{constant}$. Further, we can obtain a one-sided section by ensuring that the intersections correspond to crossings in the same direction by imposing the condition that $\dot{x}_j < 0$ or $\dot{x}_j > 0$ at the section. A Poincaré map is a map which relates one intersection to the subsequent intersection on the Poincaré section. A single point or a collection of a finite number of points in the Poincaré section corresponds to a periodic attractor, and a collection of points that form a closed curve corresponds to a two-period quasiperiodic attractor. A collection of points in the Poincaré section that do not lie on any simple geometrical form corresponds to a chaotic attractor.

Let us consider an n -dimensional nonautonomous system in which the explicit time-dependent terms have a period T . This system can be converted into an $(n + 1)$ -dimensional autonomous system (e.g., Parker and Chua, 1987). In this event, the Poincaré section is an n -dimensional surface, which can be constructed from the intersections of the trajectory with this section every T seconds apart. As mentioned before, the intersections form a characteristic pattern for a particular attractor.

In our experimental study, we use the period of excitation to define a Poincaré section. Each of these sections reveals information related to the other incommensurate frequencies of motion.

2.5.2. Frequency spectra

The frequency spectrum (amplitude or power spectrum) helps in distinguishing periodic, quasiperiodic, and chaotic motions from each other. It is determined as a fast Fourier transform of a time series (e.g., time history of one of the state variables in equation (2.1)). The spectrum of a periodic motion has discrete spectral lines at a basic frequency and its multiples. The spectrum of an n -period quasiperiodic motion is composed of n basic frequencies (this implies the presence of n incommensurate frequencies) while the spectrum of a chaotic motion has a broadband character. Several examples of the use of frequency spectra to characterize motions can be found in the literature (e.g., Eckmann and Ruelle, 1985; Moon, 1987).

2.5.3. Lyapunov exponents

The Lyapunov exponents (also called characteristic exponents) are generalizations of eigenvalues of an equilibrium point or characteristic exponents of a periodic orbit (e.g., Parker and Chua, 1987). Essentially, they describe the exponential rate at which a perturbation increases or decreases with time. For an equilibrium point of an n -dimensional system, such as equation (2.1), there are n Lyapunov exponents $\bar{\lambda}_i$, where each of them is defined such that

$$\bar{\lambda}_i = \lim_{t \rightarrow \infty} \frac{1}{t} \text{Re}[\lambda_i t] \quad (2.12)$$

where Re denotes the real part and λ_i is an eigenvalue of the Jacobian $D_x F$. For a stable equilibrium point, the $\bar{\lambda}_i$ are all less than zero.

Similarly, for a periodic orbit of period T , the Lyapunov exponents are defined in terms of the characteristic multipliers or the eigenvalues λ_i of the monodromy matrix in equation (2.4) as

$$\bar{\lambda}_i = \frac{1}{T} \ln |\lambda_i| \quad (2.13)$$

For a periodic orbit of an autonomous system, one of the Lyapunov exponents is always zero and this corresponds to an eigenvalue of the monodromy matrix which is always one. Again for a stable periodic orbit, the other $(n-1)$ Lyapunov exponents should be less than zero. Haken (1983) has shown that

a Lyapunov exponent is always zero for any bounded attractor of an autonomous system except an equilibrium point.

Further, the Lyapunov exponents of a trajectory of a system are a measure of the exponential rate of divergence of trajectories surrounding the trajectory in question. Let $X(t)$ represent a trajectory of a system governed by equation (2.1) and $\xi(t)$ represent a disturbance provided to this trajectory. When the disturbance is small, it is governed by the linearization

$$\frac{d\xi(t)}{dt} = D_X f(X; \gamma; t) \xi(t) \quad (2.14)$$

where $D_X f$ is an $n \times n$ matrix with time dependent coefficients. If we consider an initial deviation $\xi(0)$, its evolution is described by

$$\xi(t) = \Phi(t) \xi(0) \quad (2.15)$$

where $\Phi(t)$ is the fundamental matrix. For an appropriately chosen $\xi(0)$, the rate of exponential expansion in the direction of $\xi(0)$ is given by

$$\bar{\lambda}_i = \lim_{t \rightarrow \infty} \frac{1}{t} \frac{\|\xi(t)\|}{\|\xi(0)\|} \quad (2.16)$$

where the symbol $\|\cdot\|$ means a vector norm. We have n such Lyapunov exponents for the system governed by equation (2.1).

Due to dissipation, contraction should outweigh expansion and this in turn implies that the sum of the Lyapunov exponents (exponential rates of

divergence) should be less than zero. If m Lyapunov exponents are zero, while the others are less than zero, then the corresponding motion lies on an m -torus or an m -period quasiperiodic attractor. For chaotic motions at least one of the Lyapunov exponents is positive. This suggests, that an initial disturbance grows exponentially in the direction corresponding to the positive exponent and can be taken to mean that the motion is sensitive to initial conditions. One should note that unstable motions also have positive Lyapunov exponents. However, in the case of chaotic motions the sum of the Lyapunov exponents will be negative, while in the case of unstable motions the sum of the Lyapunov exponents will be positive. In general, all points in the basin of attraction of an attractor have the same Lyapunov exponents as the attractor. For more details and discussion on these exponents, one is referred to Eckmann and Ruelle (1985), Parker and Chua (1987), Haken (1983), and Seydel (1988).

2.5.4. Dimension

There are several means of obtaining the dimension of an attractor (Farmer et al., 1983). The dimension of a fixed point is zero, the dimension of a periodic orbit is one, and the dimension of an m -torus is m . Unlike these cases where the dimension is an integer, the dimension of a chaotic attractor is nonintegral or fractal. In this study, the pointwise dimension is used to verify the chaotic nature of any motion that is identified to be chaotic from its

spectrum and/or Poincaré section. We discuss the method used to determine this dimension in Appendix E.

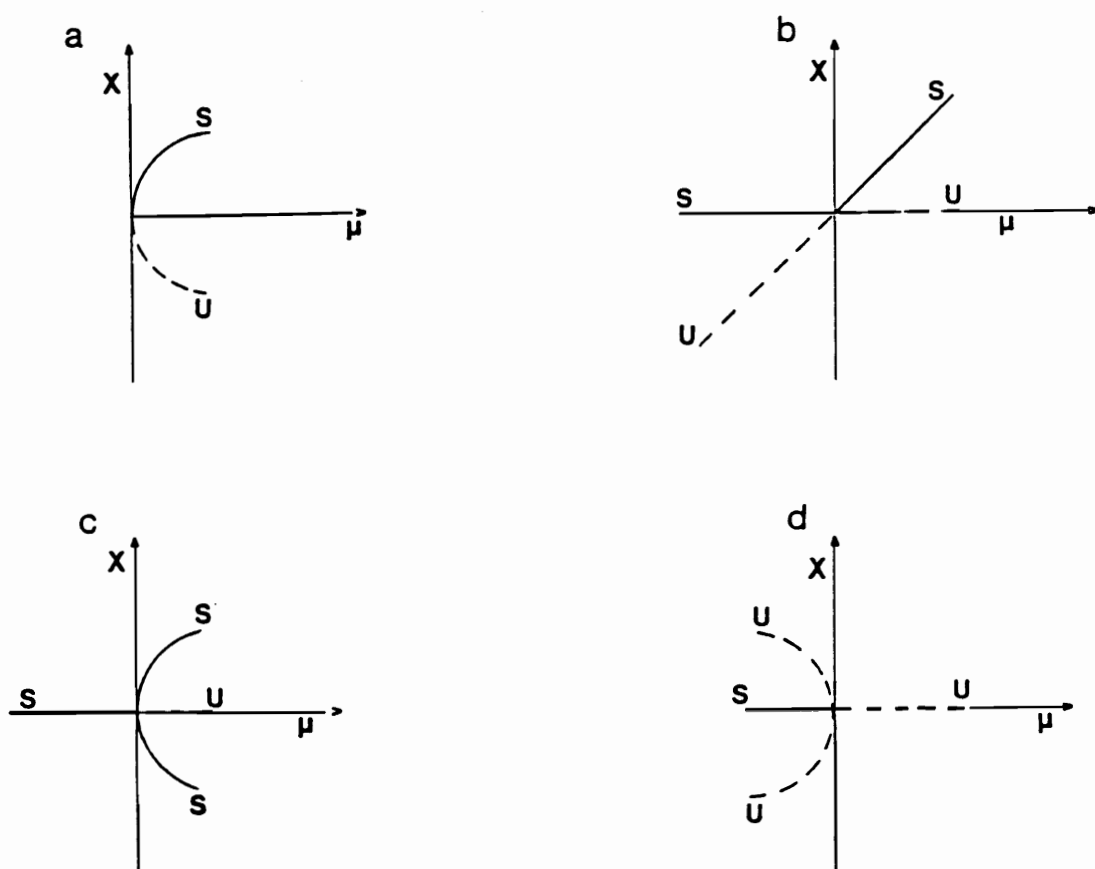


Figure 2.1. Typical bifurcation diagrams: a) saddle-node bifurcation, b) transcritical bifurcation, c) supercritical pitchfork bifurcation, and d) subcritical pitchfork bifurcation. In each bifurcation diagram, the symbols U and S denote unstable and stable fixed points, respectively.

3. Analysis for the Metallic Structures

In this chapter, we derive a first-order approximate solution for weakly nonlinear motions of the metallic structure shown in Fig. 3.1. It is made of two light-weight steel beams and two rigid masses. In the analysis, the beams are assumed to be linear elastic and inextensional. The kinetic and potential energies of the structure are formed taking nonlinearities into account. Next, the resulting Lagrangian for the undamped structure is time averaged over the period of primary oscillation or fast time scale. Then, requiring the time-averaged Lagrangian to be stationary with respect to the variables which vary on a slow time scale leads to the corresponding Euler-Lagrange equations. Subsequently, assuming modal damping and including modal-damping coefficients in these equations, we obtain an autonomous system of equations describing the evolution of the amplitudes and phases of the interacting modes for primary- and secondary-resonant excitations. For these different excitations, we analyze the stability of the solutions of the modulation equations and discuss the corresponding analytical predictions.

The lengths L_1 , L_2 , and D , see Fig. 3.1, define the physical dimensions of the beam-mass structure. As shown in this figure, the rigid mass M_1 is located at the junction of the vertical and horizontal beams while the other rigid mass M_2 is positioned on the vertical beam, such that the structure is tuned for modal interaction. The mass M_1 located at the junction of the horizontal and vertical beams is made up of three pieces. Holes are drilled in the beams and pieces and screws are run through them to hold the three pieces and beams together. The coordinate systems and displacements used in the analysis are shown in Fig. 3.2. The rotary inertias of the rigid masses M_1 and M_2 are taken into account while those of the slender beams are neglected since we are interested only in the first two modes of the structure. For simplicity, the finiteness of the rigid masses is not taken into account in determining the potential and translational kinetic energies of the beam-mass structure and each rigid mass with the beam lengths enclosed within it is treated as one total concentrated mass. The concentrated masses are represented by m_1 and m_2 in Fig. 3.2. Also, the dimensions run to and from the centers of these concentrated masses. The structure is split into three sections of lengths ℓ_1 , d , and ℓ_3 , respectively, and each section is assumed to be inextensional. A Lagrangian description is used in the analysis. The undeformed arc length s_i is used to locate a beam element in beam section i , and its axial and transverse displacements are represented by $u_i(s_i, t)$ and $v_i(s_i, t)$, respectively. The displacements are measured with respect to the coordinate system located at the beginning of the beam section and are functions of the arc length s_i and time t . The rotation $\psi_i(s_i, t)$ is the rigid-body rotation, which a

beam element in section i experiences as it goes from the undeformed to the deformed position. The beam elements are assumed to experience no shear deformation and no warping. The base displacement X due to the excitation is measured from the origin of the inertial reference frame. Further, we neglect the terms which arise due to Earth's rotation. Let ds_i (ds_i') represent the length of an infinitesimal beam element in beam section i in the undeformed (deformed) position. Then, the inextensionality constraint implies that $ds_i = ds_i'$ and can be expressed as

$$\left[1 + \frac{\partial u_i}{\partial s_i}\right]^2 + \left[\frac{\partial v_i}{\partial s_i}\right]^2 = 1 \quad (3.1)$$

Also,

$$\sin \psi_i = \frac{\partial v_i}{\partial s_i} \quad (3.2)$$

The inextensionality constraint relates the axial displacements to the transverse displacements. However, in general, the use of the inextensionality constraint depends on factors such as boundary conditions, material properties (anisotropy), and magnitude of displacements.

The kinetic energy T and potential energy V of the structure are given by

$$\begin{aligned}
T = & \frac{1}{2} \int_0^{\ell_1} \rho_1 [\dot{u}_1^2 + (\dot{v}_1 + \dot{X})^2] ds_1 + \frac{1}{2} m_1 \left[(\dot{X} + \dot{v}_1|_{s_1=\ell_1})^2 + \dot{u}_1^2|_{s_1=\ell_1} \right] \\
& + \frac{1}{2} J_1 \dot{\psi}_1^2|_{s_1=\ell_1} + \frac{1}{2} \int_0^d \rho_2 \left[(\dot{X} + \dot{v}_1|_{s_1=\ell_1} + \dot{u}_2)^2 + (\dot{u}_1|_{s_1=\ell_1} - \dot{v}_2)^2 \right] ds_2 \\
& + \frac{1}{2} m_2 \left[(\dot{X} + \dot{v}_1|_{s_1=\ell_1} + \dot{u}_2|_{s_2=d})^2 + (\dot{u}_1|_{s_1=\ell_1} - \dot{v}_2|_{s_2=d})^2 \right] + \frac{1}{2} J_2 \dot{\psi}_2^2|_{s_2=d} \\
& + \frac{1}{2} \int_0^{\ell_3} \rho_2 \left[(\dot{X} + \dot{v}_1|_{s_1=\ell_1} + \dot{u}_2|_{s_2=d} + \dot{u}_3)^2 + (\dot{u}_1|_{s_1=\ell_1} - \dot{v}_2|_{s_2=d} - \dot{v}_3)^2 \right] ds_3
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
V = & \frac{1}{2} \int_0^{\ell_1} (EI)_1 \left(\frac{\partial \psi_1}{\partial s_1} \right)^2 ds_1 + \frac{1}{2} \int_0^d (EI)_2 \left(\frac{\partial \psi_2}{\partial s_2} \right)^2 ds_2 + \frac{1}{2} \int_0^{\ell_3} (EI)_2 \left(\frac{\partial \psi_3}{\partial s_3} \right)^2 ds_3 \\
& + \int_0^{\ell_1} \rho_1 g (X + v_1) ds_1 + [m_1 + m_2 + \rho_2 (d + \ell_3)] g (X + v_1|_{s_1=\ell_1}) \\
& + \int_0^d \rho_2 g u_2 ds_2 + (m_2 + \rho_2 \ell_3) g u_2|_{s_2=d} + \int_0^{\ell_3} \rho_2 g u_3 ds_3
\end{aligned} \tag{3.4}$$

where ρ_1 and $(EI)_1$ are the mass per unit length and flexural rigidity of the horizontal beam, respectively, and ρ_2 and $(EI)_2$ are the corresponding quantities for the vertical beam. Moreover, J_1 and J_2 are the moments of inertia of masses m_1 and m_2 , respectively, the overdot indicates the derivative with respect to time t , and g is the acceleration due to gravity. If the finiteness of the rigid masses is taken into account in determining the potential and translational kinetic energy, the expressions for the kinetic and potential energies contain additional terms.

The Lagrangian L , which is $T - V$, is augmented by the constraints to obtain the augmented Lagrangian L_{aug} as

$$\begin{aligned}
 L_{aug} = T - V - \frac{1}{2} \int_0^{\ell_1} \lambda_1 \left\{ \left(1 + \frac{\partial u_1}{\partial s_1} \right)^2 + \left(\frac{\partial v_1}{\partial s_1} \right)^2 - 1 \right\} ds_1 \\
 - \frac{1}{2} \int_0^d \lambda_2 \left\{ \left(1 + \frac{\partial u_2}{\partial s_2} \right)^2 + \left(\frac{\partial v_2}{\partial s_2} \right)^2 - 1 \right\} ds_2 \\
 - \frac{1}{2} \int_0^{\ell_3} \lambda_3 \left\{ \left(1 + \frac{\partial u_3}{\partial s_3} \right)^2 + \left(\frac{\partial v_3}{\partial s_3} \right)^2 - 1 \right\} ds_3
 \end{aligned} \tag{3.5}$$

where the λ_i are the Lagrange multipliers. Here, they are the constraint forces that keep each beam section inextensional and, in general, are functions of the position s_i and the displacements u_i and v_i . However, in the linear equations of motion they depend on the position s_i only because their dependence on u_i and v_i corresponds to higher-order terms. When we derive the nonlinear equations, we require the time-averaged Lagrangian to be stationary with respect to the variables that vary on the slow time scale and these multipliers.

At this stage, to study small but finite amplitude oscillations about the static equilibrium position, we introduce a small dimensionless parameter ε , which is the order of the oscillatory part of the transverse displacement v_i , as a bookkeeping device. This parameter serves to establish the different orders of magnitude, the results are independent of its choice, and it is ultimately absorbed in the solution. The axial displacements are assumed to be caused

by the transverse displacements and are $O(\varepsilon^2)$. However, the introduction of the constraint equations enables the axial displacements to be treated as independent variables in the analysis. Next, we present the linear equations of motion and associated boundary conditions for the undamped beam-mass structure.

3.1. *Linear undamped free vibrations*

Introducing equations (3.2)-(3.4) into equation (3.5), retaining terms up to $O(\varepsilon^2)$, and using Hamilton's principle (Meirovitch, 1980), we obtain the following equations of motion and boundary conditions. The steps leading to these equations are briefly discussed in Appendix A.

Equations of motion:

$$\rho_1 \ddot{v}_1 + (EI)_1 \frac{\partial^4 v_1}{\partial s_1^4} + \rho_1 g = 0 \quad (3.6)$$

in the region $0 < s_1 < \ell_1$,

$$\rho_2 \ddot{v}_2 + (EI)_2 \frac{\partial^4 v_2}{\partial s_2^4} + m_2 g \frac{\partial^2 v_2}{\partial s_2^2} + \rho_2 (d + \ell_3 - s_2) g \frac{\partial^2 v_2}{\partial s_2^2} - \rho_2 g \frac{\partial v_2}{\partial s_2} = 0 \quad (3.7)$$

in the region $0 < s_2 < d$, and

$$\rho_2(\ddot{v}_3 + \ddot{v}_2 \Big|_{s_2=d}) + (EI)_2 \frac{\partial^4 v_3}{\partial s_3^4} + \rho_2(\ell_3 - s_3)g \frac{\partial^2 v_3}{\partial s_3^2} - \rho_2 g \frac{\partial v_3}{\partial s_3} = 0 \quad (3.8)$$

in the region $0 < s_3 < \ell_3$.

Boundary conditions:

$$v_1(0, t) = 0 \quad (3.9)$$

$$\frac{\partial v_1}{\partial s_1}(0, t) = 0 \quad (3.10)$$

$$v_2(0, t) = 0 \quad (3.11)$$

$$\frac{\partial v_1}{\partial s_1}(\ell_1, t) = \frac{\partial v_2}{\partial s_2}(0, t) \quad (3.12)$$

$$(EI)_1 \frac{\partial^3 v_1}{\partial s_1^3}(\ell_1, t) = [m_1 + m_2 + \rho_2(d + \ell_3)] [\ddot{v}_1(\ell_1, t) + g] \quad (3.13)$$

$$(EI)_1 \frac{\partial^2 v_1}{\partial s_1^2}(\ell_1, t) + J_1 \frac{\partial \ddot{v}_1}{\partial s_1}(\ell_1, t) = (EI)_2 \frac{\partial^2 v_2}{\partial s_2^2}(0, t) \quad (3.14)$$

$$v_3(0, t) = 0 \quad (3.15)$$

$$\frac{\partial v_2}{\partial s_2}(d, t) = \frac{\partial v_3}{\partial s_3}(0, t) \quad (3.16)$$

$$m_2 g \frac{\partial v_2}{\partial s_2}(d, t) + (EI)_2 \frac{\partial^3 v_2}{\partial s_2^3}(d, t) = (EI)_2 \frac{\partial^3 v_3}{\partial s_3^3}(0, t) + m_2 \ddot{v}_2(d, t) \quad (3.17)$$

$$(EI)_2 \frac{\partial^2 v_2}{\partial s_2^2}(d, t) + J_2 \frac{\partial \ddot{v}_2}{\partial s_2}(d, t) = (EI)_2 \frac{\partial^2 v_3}{\partial s_3^2}(0, t) \quad (3.18)$$

$$(EI)_2 \frac{\partial^3 v_3}{\partial s_3^3}(\ell_3, t) = 0 \quad (3.19)$$

$$(EI)_2 \frac{\partial^2 v_3}{\partial s_3^2}(\ell_3, t) = 0 \quad (3.20)$$

Again, if the finiteness of the rigid masses had been considered in determining the potential and translational kinetic energy of the beam-mass structure, equations (3.7), (3.8), (3.13), (3.14), (3.17), and (3.18) would contain additional terms. Also, equations (3.9) and (3.10) reflect an ideal-clamp assumption. Experiments with the current structure and other structures in the laboratory indicate that these boundary conditions are hard to satisfy in the laboratory. In the analysis, one can account for the nonideal clamp by including regular and torsional springs of "appropriate" stiffnesses at the fixed end. This is mathematically equivalent to satisfying the boundary conditions (3.9) and (3.10) at a "small" distance to the left of the physical boundary or increasing

the length ℓ_1 . Here, we increased the length ℓ_1 to account for the nonideal clamp. This is further discussed, later, in this section.

A closed-form solution of equations (3.6)-(3.8) with the boundary conditions (3.9)-(3.20) is desirable. Assuming displacements about the static equilibrium position, we drop the terms which are not functions of time from equations (3.6) and (3.13). From here on, the displacements v_i represent the transverse displacements measured from the static equilibrium position and equations (3.6) and (3.13) are to be understood as not containing the static terms. For the resulting system of equations, a closed-form solution is not available due to the presence of terms with variable coefficients in equations (3.7) and (3.8). However, one could use a numerical method, such as a finite-difference or finite-element method, to obtain the solution of equations (3.6)-(3.8). Here, we treat the underlined terms in equations (3.7) and (3.8) as a perturbation to the linear eigenvalue problem, formed by the remaining terms, and use the method of strained parameters (Nayfeh, 1973) to determine corrections to the natural frequencies due to the perturbation. At first order, we have the unperturbed linear eigenvalue problem. At the next order, we have a system of inhomogeneous differential equations and inhomogeneous boundary conditions, the inhomogeneity being due to the perturbation. By defining an adjoint problem, we determine the solvability condition (Nayfeh, 1981) and then the corrections to the linear natural frequencies. The details of the perturbation analysis are outlined in Appendix B. The numerical results of the analysis, which will be discussed later, show that the underlined terms in

equations (3.7) and (3.8) are very small in comparison to the other terms in the equations and can be treated as perturbations. Here, we consider the unperturbed eigenvalue problem for the determination of the natural frequencies and associated mode shapes.

The structure is assumed to execute synchronous motion and the displacements v_i are assumed to have the form

$$v_i(s_i, t) = V_i(s_i) G(t) \quad (3.21)$$

where $V_i(s_i)$ depends on the arc length s_i and $G(t)$ is harmonic with frequency ω . Before proceeding to find the solution of equations (3.6)-(3.20), we examine the properties of the system of equations. The system is self-adjoint and consequently, the eigenfunctions corresponding to distinct eigenvalues are orthogonal. In order to show that the system is self-adjoint, we define three differential operators as

$$L_1 = (EI)_1 \frac{d^4}{ds_1^4} \quad (3.22)$$

in the region $0 < s_1 < \ell_1$,

$$L_2 = (EI)_2 \frac{d^4}{ds_2^4} + m_2 g \frac{d^2}{ds_2^2} \quad (3.23)$$

in the region $0 < s_2 < d$, and

$$L_3 = (EI)_2 \frac{d^4}{ds_3^4} \quad (3.24)$$

in the region $0 < s_3 < \ell_3$. We consider two sets of comparison functions (Meirovitch, 1980), namely, U_1 , U_2 , and U_3 and W_1 , W_2 , and W_3 , where the U_i and W_i are functions of s_i , and introduce the following inner products for the respective beam sections:

$$(U_1, L_1 W_1) = \int_0^{\ell_1} U_1 L_1 W_1 ds_1 \quad (3.25)$$

$$(U_2, L_2 W_2) = \int_0^d U_2 L_2 W_2 ds_2 \quad (3.26)$$

$$(U_3, L_3 W_3) = \int_0^{\ell_3} U_3 L_3 W_3 ds_3 \quad (3.27)$$

If the system is self-adjoint (Meirovitch, 1980), then

$$(U_1, L_1 W_1) + (U_2, L_2 W_2) + (U_3, L_3 W_3) = (W_1, L_1 U_1) + (W_2, L_2 U_2) + (W_3, L_3 U_3) \quad (3.28)$$

The above relationship is satisfied by the current problem. So, the eigenfunctions of the system are expected to be orthogonal to each other. One could also show that the system is self-adjoint by defining an adjoint system

for the system of differential equations and boundary conditions (Nayfeh, 1981).

Next, substituting equation (3.21) into equations (3.6)-(3.8) and separating space and time, we find the following solution:

$$V_1 = D_1 \sin \beta_1 s_1 + D_2 \cos \beta_1 s_1 + D_3 \sinh \beta_1 s_1 + D_4 \cosh \beta_1 s_1 \quad (3.29)$$

$$V_2 = D_5 \sin r_3 s_2 + D_6 \cos r_3 s_2 + D_7 \sinh r_2 s_2 + D_8 \cosh r_2 s_2 \quad (3.30)$$

$$V_3 = D_9 \sin \beta_2 s_3 + D_{10} \cos \beta_2 s_3 + D_{11} \sinh \beta_2 s_3 + D_{12} \cosh \beta_2 s_3 - V_2 \Big|_{s_2=d} \quad (3.31)$$

where

$$\beta_i^4 = \frac{\omega^2 \rho_i}{(EI)_i} \quad (3.32)$$

and

$$r_2^2 = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \beta_2^4}, \quad r_3^2 = \frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} + \beta_2^4}, \quad \alpha = \frac{m_2 g}{(EI)_2} \quad (3.33)$$

By using the boundary conditions (3.9) through (3.20), we obtain the characteristic equation whose roots yield the eigenfrequencies ω . Here, the constant D_1 is kept arbitrary. In the following sections, the V_i are referred to as ϕ_{ij} where the i and j denote the beam section and mode, respectively. The mode shape or the eigenfunction is defined by equations (3.29)-(3.33). Thus,

the linear natural frequencies and corresponding mode shapes are analytically determined. Here, our interest lies in the frequencies and mode shapes of the first two modes of vibration of the structure. The frequencies of the higher modes of vibration are far separated from those of the first two modes.

Several cases of the beam-mass structure (referred to as Structures I, II, III, IV, V, and VI) were considered in the experiments. Their properties and dimensions are given in Appendix D. The mode shapes observed in the experiments are also shown in Fig. 3.1. They are in agreement with the analytically predicted mode shapes. The analytically determined frequencies are discussed in the next paragraph. Because the values of the damping coefficients, determined by experiments and discussed in Appendix E, are extremely small, we do not consider the effect of damping in the analytical predictions of the linear frequencies.

For Structure I, whose dimensions are shown in Appendix D, the first two linear natural frequencies were analytically found to be 8.160 Hz and 17.291 Hz. Including the underlined terms in equations (3.7) and (3.8), we obtained frequency corrections of -0.0313 Hz (-0.38%) and -0.0017 Hz (-0.01%) to the first and second natural frequencies, respectively. These corrections are small and so the underlined terms in equations (3.7) and (3.8) are negligible in comparison with the other terms in the equations. The second natural frequency is very sensitive to the length ℓ_1 , whereas the first natural frequency is very sensitive to the length d . The length ℓ_1 , given in Appendix D, was

measured from the physical boundary of the fixed end of the horizontal beam in the L-shaped structure (see Fig. 3.1). Here, this length was increased so that the analytically predicted second natural frequency agreed with the corresponding experimental value. This is justified because the clamp at the fixed end of the structure is not an ideal one. We also found it necessary to decrease the length d so that the analytically predicted first natural frequency agreed with the experimental value. The modified lengths ℓ_1 and d are 156.40 mm and 87.15 mm, respectively. For these dimensions, the analytical model yields 8.131 Hz and 16.443 Hz for the first two linear natural frequencies (corresponding values of experimentally determined damped-resonant frequencies are 8.13 Hz and 16.44 Hz, respectively). These lengths are used for further analyses of Structure I.

For Structure II, whose dimensions are shown in Appendix D, we obtained the frequencies 8.167 Hz and 17.291 Hz for the first two modes of the structure, respectively. Increasing ℓ_1 to 156.05 mm and decreasing d to 86.90 mm fine tuned the analytical model and yielded the frequencies 8.160 Hz and 16.504 Hz (corresponding values of the experimentally determined damped-resonant frequencies are 8.16 Hz and 16.50 Hz, respectively). As in the case of Structure I, the modified values of ℓ_1 and d are used for further analyses. Next, we consider the forced response of the structure to primary- and secondary-resonant excitations.

3.2. Primary resonance of the lower (first) mode

The low excitation levels are expressed by

$$\ddot{X} = -\varepsilon^2 F \cos \Omega t \quad (3.34)$$

where an overdot indicates the derivative with respect to time t , F is $O(1)$, $\varepsilon^2 F$ is the excitation amplitude, and Ω is the excitation frequency. The excitation is ordered at $O(\varepsilon^2)$ so that the influence of the nonlinear terms and primary-resonant excitation is realized at the same order in the analysis.

The frequency relationship between the first two frequencies of the structure is written as

$$\omega_2 = 2 \omega_1 + \varepsilon \sigma_1 \quad (3.35)$$

where ω_i is the i th radian natural frequency of the structure and σ_1 is the detuning of the internal resonance. The excitation frequency Ω is held close to either ω_2 or ω_1 . The first case is called primary resonance of the second mode while the second case is called primary resonance of the first mode.

The response of the structure should contain only the directly excited modes and the modes indirectly excited through the internal resonance because all other modes, which are not directly or indirectly excited, will decay with time due to the presence of damping. Although damping has not been considered

so far in the analysis, as mentioned earlier, modal damping will be included later in the equations. So here, for either case of primary resonance, the structure's response is expected to contain at most two modes, the first two flexural modes, one directly excited through the primary resonance and the other indirectly excited through the two-to-one internal resonance.

The weakly nonlinear responses of the structure can be determined analytically by one of the following approaches. First, a perturbation analysis, such as the method of multiple scales, is applied directly to the governing partial-differential equations and boundary conditions. The partial-differential equations for the beam-mass structure can be obtained from equation (3.5) by using Hamilton's principle (the steps leading to the nonlinear equations of motion are provided in Appendix A). Second, a perturbation analysis is applied to the nonlinear temporal differential equations resulting from using the Galerkin procedure on the governing partial-differential equations and boundary conditions. This approach is popular in studies of nonlinear oscillations of beams and plates. Third, averaging the Lagrangian, equation (3.5) in this case, over the period of the primary oscillation or fast time scale and requiring the averaged Lagrangian to be stationary with respect to the variables which vary on a slow time scale leads to the corresponding Euler-Lagrange equations. Here, the latter approach is used as it is less cumbersome and yields the modulation or envelope equations directly.

In this case, the excitation frequency Ω is held close to the frequency of the first mode and a detuning σ_2 , called the external detuning, is introduced as

$$\Omega = \omega_1 + \varepsilon \sigma_2 \quad (3.36)$$

The transverse and axial displacements of an element in beam section j are approximated in terms of the first two flexural modes of vibration as

$$v_j = \varepsilon [A_1 \phi_{j1} + A_2 \phi_{j2}] \quad (3.37)$$

$$u_j = \varepsilon^2 [A_1^2 \eta_{j1} + A_2^2 \eta_{j2} + 2A_1 A_2 \eta_{j3}] \quad (3.38)$$

where the A_i are functions of time while the ϕ_{ji} and η_{ji} are functions of the spatial coordinate s_j . Here, as mentioned earlier, the ϕ_{ji} is V_j for the i th mode and is determined by equations (3.29)-(3.33). The arbitrary constants in the eigenfunctions have been absorbed into the respective A_i . The form of the axial displacements is chosen by keeping in mind that they are caused by the transverse displacements (see equation (3.1)) and the spatial functions η_{ji} are determined later. The temporal function A_k corresponds to the k th mode and has the form

$$A_k = p_k \cos(k \Omega t) + q_k \sin(k \Omega t) \quad (3.39)$$

where the p_k and q_k are real quantities that vary on the slow time scale τ , defined as $\tau = \varepsilon t$. Later, the Euler-Lagrange equations are written in terms of these p_k and q_k . Also, the time derivative of A_k is

$$\dot{A}_k = k \Omega [-p_k \sin(k\Omega t) + q_k \cos(k\Omega t)] + \varepsilon [p'_k \cos(k\Omega t) + q'_k \sin(k\Omega t)] \quad (3.40)$$

where the prime indicates the derivative with respect to the time scale τ .

Substituting equations (3.2)-(3.4) and (3.34)-(3.40) into equation (3.5), expanding the result, and retaining terms up to $O(\varepsilon^3)$ in the Lagrangian, we obtain

$$L_{aug} = \mathcal{L} + O(\varepsilon^4) \quad (3.41)$$

Only terms up to $O(\varepsilon^3)$ have been retained because the two-to-one internal resonance is activated by quadratic nonlinearities (see Nayfeh and Mook, 1979). The Lagrangian $\mathcal{L}$ has the form

$$\begin{aligned} \mathcal{L} = & \varepsilon \{ \Gamma_1 A_1 + \Gamma_2 A_2 \} + \varepsilon^2 \{ \Gamma_3 A_1^2 + \Gamma_4 A_2^2 + \Gamma_5 A_1 A_2 + \Gamma_6 \dot{A}_1^2 \\ & + \Gamma_{17} \frac{F \cos \Omega t}{\Omega^2} + \Gamma_7 \dot{A}_2^2 + \Gamma_8 \dot{A}_1 \dot{A}_2 \} + \varepsilon^3 \{ \Gamma_{11} A_1 \dot{A}_1^2 \\ & - \Gamma_9 \frac{\dot{A}_1 F \sin \Omega t}{\Omega} - \Gamma_{10} \frac{\dot{A}_2 F \sin \Omega t}{\Omega} + \Gamma_{12} A_1 \dot{A}_1 \dot{A}_2 + \Gamma_{13} A_2 \dot{A}_2^2 \\ & + \Gamma_{14} A_2 \dot{A}_1 \dot{A}_2 + \Gamma_{15} A_2 \dot{A}_1^2 + \Gamma_{16} A_1 \dot{A}_2^2 \} \end{aligned} \quad (3.42)$$

where the Γ_i are as defined in Appendix C. In the next step, $\mathcal{L}$ is time averaged over the period $T = 2\pi/\Omega$ of the fast time scale. Because the p_k and q_k vary on a slow time scale, they are assumed to be constants over the period T . Carrying out the averaging, we obtain

$$\begin{aligned}
\langle \mathcal{L} \rangle &= \frac{1}{T} \int_0^T \mathcal{L} dt \\
&= \frac{\varepsilon^2}{2} \left\{ \Gamma_3(p_1^2 + q_1^2) + \Gamma_4(p_2^2 + q_2^2) + \Gamma_6 \Omega^2(p_1^2 + q_1^2) \right. \\
&\quad + 4\Gamma_7 \Omega^2(p_2^2 + q_2^2) \left. \right\} + \varepsilon^3 \left\{ \Gamma_6 \Omega(-p_1 q'_1 + p'_1 q_1) + \frac{\Gamma_9}{2} (p_1 F) \right. \\
&\quad + 2\Gamma_7 \Omega(-p_2 q'_2 + p'_2 q_2) \\
&\quad \left. + \left(\frac{\Gamma_{15}}{2} - \Gamma_{12} \right) \Omega^2 \left[\frac{p_2}{2} (q_1^2 - p_1^2) - p_1 q_1 q_2 \right] \right\}
\end{aligned} \tag{3.43}$$

Now, requiring the time-averaged Lagrangian $\langle \mathcal{L} \rangle$ to be stationary with respect to the λ_i leads to the following equations:

$$\frac{d}{d\tau} \left(\frac{\partial \langle \mathcal{L} \rangle}{\partial \lambda'_i} \right) - \frac{\partial \langle \mathcal{L} \rangle}{\partial \lambda_i} = 0 \quad \text{for } i = 1, 2, 3 \tag{3.44}$$

which yield the spatial functions η_{ij} as

$$\eta_{i1} = -\frac{1}{2} \int_0^{s_i} \left(\frac{d\phi_{i1}}{ds_i} \right)^2 ds_i \quad \text{for } i = 1, 2, 3 \tag{3.45}$$

$$\eta_{i2} = -\frac{1}{2} \int_0^{s_i} \left(\frac{d\phi_{i2}}{ds_i} \right)^2 ds_i \quad \text{for } i = 1, 2, 3 \tag{3.46}$$

$$\eta_{i3} = -\frac{1}{2} \int_0^{s_i} \left(\frac{d\phi_{i1}}{ds_i} \right) \left(\frac{d\phi_{i2}}{ds_i} \right) ds_i \quad \text{for } i = 1, 2, 3 \tag{3.47}$$

Equations (3.45) and (3.46) follow from equation (3.44) while relation (3.47) can be inferred from the form of equations (3.45), (3.46), and (3.38). One could also obtain all three relations by requiring $\mathcal{L}$ to be stationary with respect to the λ_i . By inspecting the relationship between the stiffness and inertia terms of the linear free-vibration problem, we obtain

$$\Gamma_3 = -\omega_1^2 \Gamma_6, \quad \Gamma_4 = -\omega_2^2 \Gamma_7 \quad (3.48)$$

Using equations (3.35), (3.36), and (3.48) in equation (3.43) leads to

$$\begin{aligned} \langle \mathcal{L} \rangle = \varepsilon^3 \left\{ \Gamma_6 \sigma_2 \omega_1 (p_1^2 + q_1^2) + \Gamma_7 \omega_2 (2\sigma_2 - \sigma_1) (p_2^2 + q_2^2) \right. \\ + \Gamma_6 \omega_1 (-p_1 q'_1 + p'_1 q_1) + 2\Gamma_7 \omega_1 (-p_2 q'_2 + p'_2 q_2) \\ \left. + \left(\frac{\Gamma_{15}}{2} - \Gamma_{12} \right) \omega_1^2 \left[\frac{p_2}{2} (q_1^2 - p_1^2) - p_1 q_1 q_2 \right] + \frac{\Gamma_9}{2} (p_1 F) \right\} \end{aligned} \quad (3.49)$$

Requiring $\langle \mathcal{L} \rangle$ to be stationary with respect to the four variables p_1, p_2, q_1 , and q_2 leads to the following Euler-Lagrange equations:

$$\frac{d}{d\tau} \left(\frac{\partial \langle \mathcal{L} \rangle}{\partial p'_i} \right) - \frac{\partial \langle \mathcal{L} \rangle}{\partial p_i} = 0 \quad \text{for } i = 1, 2 \quad (3.50)$$

$$\frac{d}{d\tau} \left(\frac{\partial \langle \mathcal{L} \rangle}{\partial q'_i} \right) - \frac{\partial \langle \mathcal{L} \rangle}{\partial q_i} = 0 \quad \text{for } i = 1, 2 \quad (3.51)$$

Substituting equation (3.49) into equations (3.50) and (3.51) yields

$$p'_1 + v_1 q_1 + \Lambda_1 (p_2 q_1 - p_1 q_2) = 0 \quad (3.52)$$

$$q'_1 - v_1 p_1 + \Lambda_1 (p_1 p_2 + q_1 q_2) = \Lambda_3 F \quad (3.53)$$

$$p'_2 + v_2 q_2 - 2\Lambda_2 p_1 q_1 = 0 \quad (3.54)$$

$$q'_2 - v_2 p_2 + \Lambda_2 (p_1^2 - q_1^2) = 0 \quad (3.55)$$

where

$$v_1 = \sigma_2, \quad v_2 = (2\sigma_2 - \sigma_1), \quad \Lambda_3 = \Gamma_9/4\Gamma_6\omega_1 \quad (3.56)$$

and

$$\Lambda_1 = \omega_1 \left(\frac{\Gamma_{15}}{2} - \Gamma_{12} \right) / 2\Gamma_6, \quad \Lambda_2 = \omega_1 \left(\frac{\Gamma_{15}}{2} - \Gamma_{12} \right) / 8\Gamma_7 \quad (3.57)$$

Adding modal damping to equations (3.52)-(3.55), we have

$$p'_1 + \mu_1 p_1 + v_1 q_1 + \Lambda_1 (p_2 q_1 - p_1 q_2) = 0 \quad (3.58)$$

$$q'_1 + \mu_1 q_1 - v_1 p_1 + \Lambda_1 (p_1 p_2 + q_1 q_2) = \Lambda_3 F \quad (3.59)$$

$$p'_2 + \mu_2 p_2 + v_2 q_2 - 2\Lambda_2 p_1 q_1 = 0 \quad (3.60)$$

$$q'_2 + \mu_2 q_2 - v_2 p_2 + \Lambda_2 (p_1^2 - q_1^2) = 0 \quad (3.61)$$

where μ_1 and μ_2 are the damping coefficients of the first and second modes, respectively. The form of equations (3.58)-(3.61) is similar to those obtained for the case of primary resonance of the first mode of a pair of nonlinearly coupled oscillators by Sethna (1965), Nayfeh, Mook, and Marshall (1973), Haddow, Barr, and Mook (1984), Miles (1984d), Nayfeh and Raouf (1987a), and

Nayfeh (1989). In these studies, the solutions of equations (3.58)-(3.61) have been investigated in great detail, and the qualitative nature of their analytical results are also applicable to the current study. The analytical predictions for the structure's response to primary resonance of the first mode are compared with the experimental observations in Chapter 4.

3.3. Primary resonance of the higher (second) mode

For this case, the excitation frequency Ω is held close to the frequency of the second mode and their relationship is expressed by introducing an external-detuning parameter σ_2 as

$$\Omega = \omega_2 + \varepsilon \sigma_2 \quad (3.62)$$

while the relationship between the first two frequencies of the structure is expressed in equation (3.35). The axial and transverse displacements of an element in beam section j are approximated as in the earlier case, by equations (3.37) and (3.38), with the temporal function A_k now defined by

$$A_k = p_k \cos\left(\frac{1}{2} k \Omega t\right) + q_k \sin\left(\frac{1}{2} k \Omega t\right) \quad (3.63)$$

Here, we substitute equations (3.2)-(3.4), (3.34), (3.35), (3.37), (3.38), (3.62), and (3.63) into equation (3.5), expand the result, and retain terms up to $O(\varepsilon^3)$. The

resulting $\mathcal{L}$ is given in equation (3.42). Averaging it over the period T of the primary oscillation $4\pi/\Omega$, we obtain the following time-averaged Lagrangian:

$$\begin{aligned} \langle \mathcal{L} \rangle = \varepsilon^3 \left\{ \frac{\Gamma_6}{2} \omega_1 (\sigma_1 + \sigma_2) (p_1^2 + q_1^2) + \Gamma_7 \sigma_2 \omega_2 (p_2^2 + q_2^2) + \frac{\Gamma_{10}}{2} (p_2 F) \right. \\ \left. + \Gamma_6 \omega_1 (-p_1 q'_1 + p'_1 q_1) + 2\Gamma_7 \omega_1 (-p_2 q'_2 + p'_2 q_2) \right. \\ \left. + 4 \left(\frac{\Gamma_{15}}{2} - \Gamma_{12} \right) \omega_1^2 \left[\frac{p_2}{8} (q_1^2 - p_1^2) - \frac{p_1 q_1 q_2}{4} \right] \right\} \end{aligned} \quad (3.64)$$

where again the p_k and q_k are functions of the slow time scale. In this case, the corresponding Euler-Lagrange equations are

$$p'_1 + v_3 q_1 + \Lambda_1 (p_2 q_1 - p_1 q_2) = 0 \quad (3.65)$$

$$q'_1 - v_3 p_1 + \Lambda_1 (p_1 p_2 + q_1 q_2) = 0 \quad (3.66)$$

$$p'_2 + v_4 q_2 - 2\Lambda_2 p_1 q_1 = 0 \quad (3.67)$$

$$q'_2 - v_4 p_2 + \Lambda_2 (p_1^2 - q_1^2) = \Lambda_6 F \quad (3.68)$$

where Λ_1 and Λ_2 are defined in equation (3.57) and

$$v_3 = \frac{1}{2} (\sigma_1 + \sigma_2), \quad v_4 = \sigma_2, \quad \Lambda_6 = \Gamma_{10}/4 \Gamma_7 \omega_2 \quad (3.69)$$

Including modal damping in equations (3.65)-(3.68), we arrive at

$$p'_1 + \mu_1 p_1 + v_3 q_1 + \Lambda_1 (p_2 q_1 - p_1 q_2) = 0 \quad (3.70)$$

$$q'_1 + \mu_1 q_1 - v_3 p_1 + \Lambda_1 (p_1 p_2 + q_1 q_2) = 0 \quad (3.71)$$

$$p_2' + \mu_2 p_2 + \nu_4 q_2 - 2\Lambda_2 p_1 q_1 = 0 \quad (3.72)$$

$$q_2' + \mu_2 q_2 - \nu_4 p_2 + \Lambda_2 (p_1^2 - q_1^2) = \Lambda_6 F \quad (3.73)$$

where, as before, the μ_i are the modal-damping coefficients. Again, the form of equations (3.70)-(3.73) is identical to those obtained for the case of primary resonance of the second mode of a pair of quadratically coupled oscillators by Sethna (1965), Nayfeh et al. (1973), Haddow et al. (1984), Nayfeh and Raouf (1987a,1987b), and Nayfeh (1989). In these studies, the different solutions of equations (3.70)-(3.73) have been examined in great detail, and one can predict the qualitative nature of the structure's behavior from these analytical results. The analytical predictions for the structure's response to primary resonant excitations of the second mode are compared with the experimental observations in Chapter 4.

3.4. Subharmonic resonance of the higher mode

In this case, the relatively high excitation levels are expressed by

$$\ddot{X} = -\varepsilon F \cos \Omega t \quad (3.74)$$

where F is $O(1)$, εF is the excitation amplitude, and Ω is the excitation frequency. The excitation is ordered at $O(\varepsilon)$ so that the influence of the

nonlinear terms and subharmonic-resonant excitation is realized at the same order in the analysis. The excitation frequency Ω is such that

$$\Omega = 2\omega_2 + \varepsilon\sigma_2 \quad (3.75)$$

where σ_2 is the detuning of the external resonance and the two-to-one internal resonance is expressed in equation (3.35). Again here, the structure's response is expected to contain at most two modes, the first two modes.

The transverse and axial displacements of an element in beam section j are approximated as in equations (3.37) and (3.38) and the spatial functions η_{ij} and ϕ_{ij} are defined in Section 3.2. However, here the form of the temporal functions A_k is different and they include the forced response of the linear system. They are given by

$$A_1 = p_1 \cos\left(\frac{1}{4}\Omega t\right) + q_1 \sin\left(\frac{1}{4}\Omega t\right) + \Gamma_{22} \frac{F}{(\omega_1^2 - \Omega^2)} \cos \Omega t \quad (3.76)$$

$$A_2 = p_2 \cos\left(\frac{1}{2}\Omega t\right) + q_2 \sin\left(\frac{1}{2}\Omega t\right) + \Gamma_{23} \frac{F}{(\omega_2^2 - \Omega^2)} \cos \Omega t \quad (3.77)$$

where

$$\Gamma_{22} = \frac{\Gamma_9}{2\Gamma_6} \quad \text{and} \quad \Gamma_{23} = \frac{\Gamma_{10}}{2\Gamma_7} \quad (3.78)$$

The forced response of the linear system is determined from the equations given in Section (3.1) after adding excitation terms to them. Substituting

equations (3.2)-(3.4), (3.35), (3.37), (3.38), and (3.75)-(3.78) into equation (3.5), expanding the result, and retaining terms up to $O(\varepsilon^3)$ in the Lagrangian, we obtain the following equation:

$$\begin{aligned}
 \mathcal{L} = \varepsilon \left\{ \Gamma_1 A_1 + \Gamma_2 A_2 + \Gamma_{17} \frac{F \cos \Omega t}{\Omega^2} \right\} + \varepsilon^2 \left\{ \Gamma_3 A_1^2 + \Gamma_4 A_2^2 + \Gamma_5 A_1 A_2 \right. \\
 + \Gamma_6 \dot{A}_1^2 + \Gamma_7 \dot{A}_2^2 + \Gamma_{18} \left(\frac{F \sin \Omega t}{\Omega} \right)^2 - \Gamma_9 \frac{\dot{A}_1 F \sin \Omega t}{\Omega} - \Gamma_{10} \frac{\dot{A}_2 F \sin \Omega t}{\Omega} \\
 + \Gamma_8 \dot{A}_1 \dot{A}_2 \left. \right\} + \varepsilon^3 \left\{ \Gamma_{11} A_1 \dot{A}_1^2 + \Gamma_{12} A_1 \dot{A}_1 \dot{A}_2 + \Gamma_{13} A_2 \dot{A}_2^2 \right. \\
 + \Gamma_{14} A_2 \dot{A}_1 \dot{A}_2 + \Gamma_{15} A_2 \dot{A}_1^2 + \Gamma_{16} A_1 \dot{A}_2^2 + \Gamma_{19} \frac{A_1 \dot{A}_1 F \sin \Omega t}{\Omega} \\
 + \Gamma_{20} \frac{A_2 \dot{A}_2 F \sin \Omega t}{\Omega} + \Gamma_{21} \frac{A_2 \dot{A}_1 F \sin \Omega t}{\Omega} + \Gamma_{21} \frac{A_1 \dot{A}_2 F \sin \Omega t}{\Omega} \left. \right\} \quad (3.79)
 \end{aligned}$$

where the Γ_i are defined in Appendix C. Next averaging $\mathcal{L}$ over the period of the primary oscillation $T = 8\pi/\Omega$ leads to

$$\begin{aligned}
 \langle \mathcal{L} \rangle = \frac{\varepsilon^2 F^2}{2} \left\{ -\frac{\Gamma_{22}^2 \Gamma_6}{(\omega_1^2 - \Omega^2)} - \frac{\Gamma_{23}^2 \Gamma_7}{(\omega_2^2 - \Omega^2)} + \frac{\Gamma_{22} \Gamma_{23} (\Gamma_5 + \Gamma_8 \Omega^2)}{(\omega_1^2 - \Omega^2)(\omega_2^2 - \Omega^2)} \right. \\
 + \frac{\Gamma_{18}}{\Omega^2} + \frac{\Gamma_9 \Gamma_{22}}{(\omega_1^2 - \Omega^2)} + \frac{\Gamma_{10} \Gamma_{23}}{(\omega_2^2 - \Omega^2)} \left. \right\} \\
 + \varepsilon^3 \left\{ \frac{\Gamma_6}{2} \omega_1 \left(\sigma_1 + \frac{\sigma_2}{2} \right) (p_1^2 + q_1^2) + \frac{\Gamma_7}{2} \sigma_2 \omega_2 (p_2^2 + q_2^2) \right. \\
 + \Gamma_6 \omega_1 (-p_1 q_1' + p_1' q_1) + \Gamma_7 \omega_2 (-p_2 q_2' + p_2' q_2) \\
 + \left(\frac{\Gamma_{15}}{2} - \Gamma_{12} \right) \omega_1^2 \left[\frac{p_2}{2} (q_1^2 - p_1^2) - p_1 q_1 q_2 \right] \\
 + F (q_2^2 - p_2^2) \left[\frac{\Gamma_{13} \Gamma_{23}}{4} + \frac{(2\Gamma_{14} - \Gamma_{16})}{15} \Gamma_{22} + \frac{\Gamma_{20}}{8} \right] \left. \right\} \quad (3.80)
 \end{aligned}$$

where the p_k and q_k are functions of the slow time scale τ and the prime indicates the derivative with respect to τ . Again requiring $\langle \mathcal{L} \rangle$ to be stationary with respect to the four variables p_1, p_2, q_1 , and q_2 leads to the following Euler-Lagrange equations:

$$p_1' + v_5 q_1 + \Lambda_1(p_2 q_1 - p_1 q_2) = 0 \quad (3.81)$$

$$q_1' - v_5 p_1 + \Lambda_1(p_1 p_2 + q_1 q_2) = 0 \quad (3.82)$$

$$p_2' + v_6 q_2 - 2\Lambda_2 p_1 q_1 = -\Lambda_7 F q_2 \quad (3.83)$$

$$q_2' - v_6 p_2 + \Lambda_2(p_1^2 - q_1^2) = -\Lambda_7 F p_2 \quad (3.84)$$

where Λ_1 and Λ_2 are defined in equation (3.57),

$$v_5 = \frac{1}{2} (\sigma_1 + \frac{1}{2} \sigma_2), \quad v_6 = \frac{\sigma_2}{2} \quad (3.85)$$

and

$$\Lambda_7 = \frac{\left\{ \frac{\Gamma_{13}\Gamma_{23}}{4} + \frac{(2\Gamma_{14} - \Gamma_{16})}{15} \Gamma_{22} + \frac{\Gamma_{20}}{8} \right\}}{\Gamma_7 \omega_2} \quad (3.86)$$

Including modal damping in equations (3.81)-(3.84) leads to

$$p_1' + \mu_1 p_1 + v_5 q_1 + \Lambda_1(p_2 q_1 - p_1 q_2) = 0 \quad (3.87)$$

$$q_1' + \mu_1 q_1 - v_5 p_1 + \Lambda_1(p_1 p_2 + q_1 q_2) = 0 \quad (3.88)$$

$$p'_2 + \mu_2 p_2 + \nu_6 q_2 - 2\Lambda_2 p_1 q_1 = -\Lambda_7 F q_2 \quad (3.89)$$

$$q'_2 + \mu_2 q_2 - \nu_6 p_2 + \Lambda_2 (p_1^2 - q_1^2) = -\Lambda_7 F p_2 \quad (3.90)$$

where the μ_i are the modal-damping coefficients. The form of equations (3.87)-(3.90) is similar to those obtained in the study of Nayfeh, Raouf, and Nayfeh (1990) in the context of cylindrical shells and Nayfeh and Nayfeh (1990) in the context of surface waves. Similar equations were also treated by Mook, Plaut, and HaQuang (1985) and Nayfeh (1987a). In these studies the different possible solutions of these equations have been examined in detail, and the qualitative nature of their analytical results are also applicable to the current study.

3.5. *Combination resonance of additive type*

For this case, the excitation frequency Ω is held close to the sum of the frequencies of the first and second modes and their relationship is expressed by introducing an external detuning σ_2 defined as

$$\Omega = \omega_1 + \omega_2 + \varepsilon \sigma_2 \quad (3.91)$$

The excitation level is given by equation (3.74). The form of the axial and transverse displacements are approximated as in equations (3.37) and (3.38)

and the spatial functions η_{ij} and ϕ_{ij} are given in Section 3.2. In this case, the temporal functions A_1 and A_2 are defined as

$$A_1 = \left\{ p_1 \cos\left(\frac{1}{3} \Omega t\right) + q_1 \sin\left(\frac{1}{3} \Omega t\right) + \Gamma_{22} \frac{F}{(\omega_1^2 - \Omega^2)} \cos \Omega t \right\} \quad (3.92)$$

$$A_2 = \left\{ p_2 \cos\left(\frac{2}{3} \Omega t\right) + q_2 \sin\left(\frac{2}{3} \Omega t\right) + \Gamma_{23} \frac{F}{(\omega_2^2 - \Omega^2)} \cos \Omega t \right\} \quad (3.93)$$

where the p_k and q_k are functions of the slow time scale and Γ_{22} and Γ_{23} are defined in equation (3.77). Again here, the forced response of the linear system is included in the approximation for the temporal function.

Substituting equations (3.2)-(3.4), (3.35), (3.37), (3.38), and (3.91)-(3.93) into equation (3.5), expanding the result, and retaining terms up to $O(\varepsilon^3)$ in the Lagrangian leads to an equation similar to equation (3.79). Subsequently, averaging $\mathcal{L}$ over the period of the primary oscillation $6\pi/\Omega$ leads to

$$\begin{aligned} \langle \mathcal{L} \rangle = & \frac{\varepsilon^2 F^2}{2} \left\{ -\frac{\Gamma_{22}^2 \Gamma_6}{(\omega_1^2 - \Omega^2)} - \frac{\Gamma_{23}^2 \Gamma_7}{(\omega_2^2 - \Omega^2)} + \frac{\Gamma_{22} \Gamma_{23} (\Gamma_5 + \Gamma_8 \Omega^2)}{(\omega_1^2 - \Omega^2)(\omega_2^2 - \Omega^2)} \right. \\ & \left. + \frac{\Gamma_{18}}{\Omega^2} + \frac{\Gamma_9 \Gamma_{22}}{(\omega_1^2 - \Omega^2)} + \frac{\Gamma_{10} \Gamma_{23}}{(\omega_2^2 - \Omega^2)} \right\} \\ & + \varepsilon^3 \left\{ \frac{\Gamma_6}{3} \omega_1 (\sigma_1 + \sigma_2) (p_1^2 + q_1^2) + \frac{2\Gamma_7}{3} \omega_2 (\sigma_2 - \frac{\sigma_1}{2}) (p_2^2 + q_2^2) \right. \\ & + \Gamma_6 \omega_1 (-p_1 q'_1 + p'_1 q_1) + \Gamma_7 \omega_2 (-p_2 q'_2 + p'_2 q_2) \\ & \left. + \left(\frac{\Gamma_{15}}{2} - \Gamma_{12} \right) \omega_1^2 \left[\frac{1}{2} p_2 (q_1^2 - p_1^2) - p_1 q_1 q_2 \right] - F (p_1 p_2 - q_1 q_2) \Gamma_{24} \right\} \end{aligned} \quad (3.94)$$

where

$$\Gamma_{24} = \left[\frac{\Gamma_{12}\Gamma_{22}}{8} + \frac{\Gamma_{14}\Gamma_{23}}{20} + \frac{3\Gamma_{15}\Gamma_{22}}{16} + \frac{3\Gamma_{16}\Gamma_{23}}{5} + \frac{\Gamma_{21}}{4} \right] \quad (3.95)$$

and the other Γ_i are given in Appendix C. Determining the Euler-Lagrange equations by requiring $\langle \mathcal{L} \rangle$ to be stationary with respect to p_1, p_2, q_1 , and q_2 and including the modal-damping coefficients μ_i leads to

$$p_1' + \mu_1 p_1 + \nu_7 q_1 + \Lambda_1(p_2 q_1 - p_1 q_2) = \Lambda_8 F q_2 \quad (3.96)$$

$$q_1' + \mu_1 q_1 - \nu_7 p_1 + \Lambda_1(q_1 q_2 + p_1 p_2) = \Lambda_8 F p_2 \quad (3.97)$$

$$p_2' + \mu_2 p_2 + \nu_8 q_2 - 2\Lambda_2 p_1 q_1 = \Lambda_9 F q_1 \quad (3.98)$$

$$q_2' + \mu_2 q_2 - \nu_8 p_2 + \Lambda_2(p_1^2 - q_1^2) = \Lambda_9 F p_1 \quad (3.99)$$

where Λ_1 and Λ_2 are defined in equation (3.57) and

$$\nu_7 = \frac{1}{3}(\sigma_1 + \sigma_2), \quad \nu_8 = \frac{1}{3}(2\sigma_2 - \sigma_1) \quad (3.100)$$

The coefficients of the forcing terms Λ_7 and Λ_8 are given by

$$\Lambda_8 = \frac{-3\Gamma_{24}}{2\Gamma_6(\omega_1 + \omega_2)} \quad (3.101)$$

and

$$\Lambda_9 = \frac{-3\Gamma_{24}}{4\Gamma_7(\omega_1 + \omega_2)} \quad (3.102)$$

The form of equations (3.96)-(3.99) is similar to those studied by Nayfeh and Zavodney (1986), Mook, HaQuang, and Plaut (1986), and Streit, Krousgrill, and Bajaj (1988). The qualitative nature of their analytical results are also applicable to the present study.

3.6. *Stability analysis*

In each of the above cases, fixed points (i.e., $p'_k = q'_k = 0$) and periodic solutions of the autonomous system of equations correspond to periodic and periodically modulated motions of the beam-mass structure, respectively. For primary-resonant excitations, it follows from either equations (3.58)-(3.61) or equations (3.70)-(3.73) that the p_k and q_k depend on eight parameters, namely, the excitation amplitude F , the detuning of the internal resonance σ_1 , the detuning of the external resonance σ_2 , the damping coefficients μ_n , the coefficient of the forcing Λ_3 or Λ_6 , and the nonlinear interaction coefficients Λ_1 and Λ_2 . In the case of subharmonic-resonant excitations, equations (3.87)-(3.90), there are the same eight parameters except that now the coefficient of the forcing is Λ_7 . In the case of combination-resonant excitations, equations (3.96)-(3.99), there are nine parameters because now there are two coefficients of forcing, namely Λ_8 and Λ_9 .

Here, we examine qualitative changes (bifurcations) in the response of the system when either excitation parameter, F or σ_2 , is varied while the other parameters are constant. As described in Chapter 2, the stability of the fixed points is determined by perturbing the autonomous system, obtaining the linearized equations governing the perturbation, and examining the eigenvalues of the Jacobian matrix. In each case, the stability of a fixed point depends on the roots (eigenvalues) of an equation of the following form

$$\lambda^4 + r_1\lambda^3 + r_2\lambda^2 + r_3\lambda + r_4 = 0 \quad (3.103)$$

where r_1 is always $2(\mu_1 + \mu_2)$, while the other r_k depend on the case in question. Noting that the μ_k are positive and that the sum of the eigenvalues is $-r_1$ it is inferred, that we always have some eigenvalues with negative real parts. As a consequence, we have either stable fixed points or nonstable fixed points (see Chapter 2). The saddle-node or pitchfork bifurcations occur when the condition $r_4 = 0$ is satisfied while one of the necessary conditions for Hopf bifurcations to occur is $r_3^2 - r_1r_2r_3 + r_4r_1^2 = 0$. The stability of the periodic solutions of the modulation equations is discussed in Chapters 2 and 6.

3.7. Analytical Predictions

In this section, the analytical results that follow from the modulation equations are summarized. The qualitative nature of the analytical predictions in the

references mentioned in Sections 3.2-3.5 are directly applicable to the present beam-mass structures as well because the governing envelope or modulation equations are the same. So, the results presented in this section are based on the references mentioned in Sections 3.2-3.5. During primary-resonant or secondary-resonant excitations, each structure responds at frequencies which are close to its respective natural frequencies. Throughout this dissertation, the forced response at a frequency close to the natural frequency of a particular mode is referred to as that particular mode's response. If the number of modes which appear in the forced response is n , then the forced response can be called an n -mode response.

The analysis essentially predicts that the beam-mass structure can exhibit four types of responses. The first type of response is called a linear periodic response. In this case, at a nontrivial excitation amplitude, the response spectrum is characterized by a spectral line at the excitation frequency Ω only. This type of response is not possible in the case of primary resonance of the first mode, but it is possible in the case of primary resonance of the second mode. The corresponding fixed point has the form $p_1 = q_1 = 0$ while $p_2 \neq 0$ and $q_2 \neq 0$. In the case of subharmonic- and combination-resonant excitations, the fixed point corresponding to a linear periodic response has the form $p_k = q_k = 0$. This fixed point is sometimes called a trivial fixed point.

The second type of response is a nonlinear periodic response. In the case of primary resonance of the first mode, the response is characterized by spectral

lines at Ω and 2Ω , and in the case of primary resonance of the second mode, the response is characterized by spectral lines at Ω and $\frac{\Omega}{2}$. The response in the case of subharmonic excitation is expected to have spectral lines at $\frac{\Omega}{4}$, $\frac{\Omega}{2}$, and Ω . In the case of combination-resonant excitations, the spectral lines should appear at $\frac{\Omega}{3}$, $2\frac{\Omega}{3}$, and Ω . The corresponding fixed point is sometimes called a nontrivial fixed point. The analysis also predicts modal saturation (Nayfeh and Mook, 1979) for primary-resonant and subharmonic-resonant excitations of the second mode.

As opposed to the first two cases, where the response corresponds to a fixed point of the autonomous system of equations, the third type of response corresponds to a limit-cycle solution of these equations. If the frequency of a limit-cycle solution is commensurate with the excitation frequency, then phase locking occurs and the resulting response is periodic. When the frequencies of the limit-cycle solution and the excitation are incommensurate, the resulting response is periodically modulated. This type of motion normally occurs near a Hopf bifurcation and is characterized by the presence of a uniformly spaced sideband structure around each of the carrier frequencies. The respective carrier frequencies for the different cases of excitation are given in the previous paragraph.

The fourth type of response is a chaotically modulated response. This type of response is normally expected to occur in a region enclosed by the Hopf-bifurcation points, which can be analytically predicted. The spectrum of

a chaotically modulated response is expected to show a broadband character around the carrier frequencies. Again, the carrier frequencies for the different excitation cases are as discussed before.

We compare the qualitative and quantitative nature of the analytical predictions and the corresponding experimental observations for cases of primary-resonant excitations. For each of these cases, the analytical solutions are determined from the equations governing their respective p_k and q_k . In these equations, we use the modal-damping coefficients determined from the experiments and the analytically determined Λ_i . The analytically determined values of Λ_i for Structures I and II are shown in Table I. Fixed points and periodic solutions of the autonomous system of equations correspond to periodic and periodically modulated motions of a metallic structure, respectively.

For periodic motions, the analytical predictions of the modal amplitudes a_k at a location on the structure are given by

$$a_k = C_k \sqrt{(p_k^2 + q_k^2)} \quad (3.104)$$

where the p_k and q_k are the fixed points of the envelope equations and the C_k are appropriate scaling constants. Here, the C_k are chosen to tune the analytical predictions with experimental measurements of the displacements for the free-end of the L-shaped structure.

For secondary-resonant excitations of the metallic structure, only qualitative comparisons are made between theory and experiment. The comparisons are mainly made by examining the frequency content of the response spectrum. Next, we show the typical forms of the expected response curves for the different excitations.

In Fig. 3.3, we show a typical frequency response of the structure in the case of primary-resonant excitations of the first mode. Here, the the detuning σ_1 is considered to be positive. Also, we assume that the excitation levels and damping values permit modulated motions to occur. The curves in the figure correspond to the fixed-point solutions of equations (3.58)-(3.61) obtained when σ_2 is used as a control parameter at a constant excitation level. In this and subsequent figures, the symbol a_j (which is a constant times $\sqrt{p_j^2 + q_j^2}$) represents the j th modal amplitude, the solid lines correspond to stable fixed points, the dashed and dotted lines correspond to unstable fixed points, and the label "Hopf bifurcation" marks the Hopf-bifurcation points. As σ_2 is increased gradually from a negative value, we have small-amplitude nonlinear periodic solutions (lower branches in the figure) until σ_2 reaches the value $\sigma_2^{(1)}$. At this control-parameter value, we have a saddle-node bifurcation. Subsequent to this bifurcation, the response follows the upper branches, which correspond to large-amplitude solutions. There is a jump from a small to a large amplitude motion at $\sigma_2^{(1)}$. It is noted that this point where an exchange of stability occurs from one branch to another is also a point of vertical tangency. When σ_2 is increased further, another critical value is reached

where the stable fixed point loses stability through a Hopf bifurcation. As discussed in Chapters 2 and 6, this bifurcation can be either a supercritical bifurcation or a subcritical bifurcation. In the region enclosed by the Hopf-bifurcation points, modulated motions are expected to occur. These modulated motions may be either quasiperiodic or chaotic. Another jump occurs at $\sigma_2 = \sigma_2^{(2)}$ during the forward sweep and this time one goes from a large-amplitude (upper branch) to a small-amplitude (lower branch) motion. During the reverse sweep of σ_2 saddle-node bifurcations (jumps) occur at $\sigma_2^{(3)}$ and $\sigma_2^{(4)}$ in that order. When there is perfect tuning (i.e., $\sigma_1 = 0$), the curves are symmetric about a vertical line at $\sigma_2 = 0$ and when $\sigma_1 < 0$, the skewing of the response curves is reversed.

Examining Fig. 3.3, we see that for $|\sigma_2| \gg 0$, $a_1 \gg a_2$ and this response is in close agreement with solutions of the linearized equations. Further, it is interesting to note that a local minimum amplitude occurs at $\sigma_2 = 0$, the point where linear theory predicts the local maximum in this approximation. Also, if the detuning σ_1 is increased, the curves resemble the solutions of the linearized equations. We also see from the figure that the response depends on the direction of sweep of the control parameter, that is, initial conditions. This feature is in contrast with linear theory which predicts that the long time response (time $t \rightarrow \infty$) is independent of the initial conditions.

We show typical frequency-response curves for a structure with positive internal detuning in the case of primary-resonant excitations of the second

mode in Fig. 3.4. Again here, we consider excitation levels and damping values that permit the occurrence of modulated motions. The curves plotted in the figure correspond to the fixed-point solutions of equations (3.70)-(3.73). We note that if $p_1, q_1, p_2,$ and q_2 is a solution then $-p_1, -q_1, p_2,$ and q_2 is also a solution. These solutions are denoted as Solution A and Solution B, respectively. As σ_2 is increased gradually from a negative value, we have linear periodic motions until the critical value $\sigma_2^{(1)}$ is reached. In this region, a_1 is zero and Solutions A and B are identical. A reverse pitchfork bifurcation occurs at $\sigma_2 = \sigma_2^{(1)}$, leading to nonlinear periodic motions. After this bifurcation, the reflection symmetry in the $p_1 - q_1$ plane is broken and Solutions A and B are no longer identical. We note that there is a change in the number of the fixed points after this bifurcation. As σ_2 is increased further, the nonlinear periodic response loses stability through a Hopf bifurcation. Both the Hopf-bifurcation points lie to the left of the vertical line at $\sigma_2 = 0$. In Chapter 6, we discuss how the nature of the Hopf bifurcation is determined. In the region enclosed by the Hopf bifurcation points modulated motions are expected to occur. These modulated motions may be quasiperiodic or chaotic. When σ_2 is increased further, a saddle-node bifurcation occurs at $\sigma_2 = \sigma_2^{(2)}$, resulting in a jump from a large- to a small-amplitude solution, which is linear and periodic. During the reverse sweep of σ_2 , a reverse pitchfork bifurcation occurs at $\sigma_2^{(3)}$ and a saddle-node bifurcation occurs at $\sigma_2^{(4)}$, respectively. Again, for perfect internal tuning, the frequency-response curves are symmetric about the vertical axis at $\sigma_2 = 0$.

We see from the figure that the response is sensitive to the direction of sweep of the control parameter (i.e., initial conditions). Further, the response curves have a local minimum at $\sigma_2 \simeq 0$, where linear theory predicts that a resonance would occur.

We show typical force-response curves for a structure with positive internal detuning in the case of primary-resonant excitations of the second mode in Fig. 3.5. The excitation frequency is held constant and is such that $\sigma_2(\sigma_1 + \sigma_2) < 0$, while the excitation amplitude F is varied. The curves in Fig. 3.5 correspond to fixed-point solutions of equations (3.70)-(3.73). As F is increased from zero, linear periodic responses occur until a critical value F_1 is reached. In this region, a_2 increases with F linearly, while a_1 remains trivial. At $F = F_1$, a supercritical pitchfork bifurcation occurs leading to nonlinear periodic responses. Again, as in the earlier figure, the solutions corresponding to $-a_1$ are not shown in the figure. The critical value F_1 depends on the damping and the detuning parameters and $F_1 \rightarrow 0$ as $\mu_i \rightarrow 0$ and $\sigma_i \rightarrow 0$. So, linear responses may not be possible in some cases. As F is increased further, a_2 remains constant while a_1 increases in a nonlinear manner until another critical value is reached. At $F = F_2$, a Hopf bifurcation causes the stable fixed point to lose stability. Consequently, the modal saturation breaks down and instead of periodic motions, we now have modulated motions. Also, during reverse sweeps of F the respective bifurcations occur at the same locations as they did during the forward sweeps. When the internal detuning is negative, for the same excitation

frequency, the transition from a linear periodic solution to a nonlinear periodic solution would occur via a reverse (subcritical) pitchfork bifurcation during forward sweeps of F .

In Fig. 3.6, we show typical force-response curves of a structure with positive internal detuning in the case of subharmonic excitations of order one-half of the higher mode. Again here, we consider an external detuning for which modulated motions occur. The excitation amplitude is used as a control parameter. The curves in the figure correspond to the fixed points of equations (3.87)-(3.90). Also, from the form of the equations we note that if p_1, q_1, p_2 , and q_2 is a solution then $-p_1, -q_1, p_2$, and q_2 is also a solution. This is one form of reflection symmetry exhibited by equations (3.87)-(3.90). They also exhibit other forms of reflection symmetries (Nayfeh and Nayfeh, 1990). As F is increased from zero, we only have stable trivial solutions (i.e., $a_1 = a_2 = 0$). At $F = F_1$, a supercritical pitchfork bifurcation occurs in the response of the first mode, leading to a nonlinear periodic response. Again here, the pitchfork bifurcation breaks the reflection symmetry in the $p_1 - q_1$ plane. It should be noted that the present analysis doesn't allow for stable single-mode solutions where $p_1 = q_1 = 0$ while $p_2 \neq 0$ and $q_2 \neq 0$. If one also considers cubic nonlinearities in the derivation, then these types of solution might be possible. As F is increased further, a_2 remains constant (modal saturation) while a_1 increases in a nonlinear manner until another critical value F_2 is reached. At this value, the fixed point loses stability through a Hopf bifurcation and consequently modulated motions follow. During the reverse

sweep of F the respective bifurcations occur at the same locations as in the forward sweep. If σ_1 and $\sigma_2(\sigma_1 + \frac{\sigma_2}{2})$ were both positive, the transition from trivial to nonlinear solutions would occur via a reverse pitchfork bifurcation during the forward sweeps of F .

A typical variation of the response amplitude with the excitation amplitude for the case of an external combination resonance of the additive type is shown in Fig. 3.7. We consider a positive internal detuning and a negative external detuning. The curves in the figure correspond to the fixed points of equations (3.96)-(3.99). There are three fixed point solutions for each value of F . One of them is a trivial fixed point while the other two are nontrivial fixed points. Further, one of the nontrivial fixed points is always unstable. As in the previous case, a certain critical threshold ($F = F_1$) has to be crossed before the trivial solution loses its stability through a reverse pitchfork bifurcation. This bifurcation leads to a change in the amplitude of the response. The solutions which correspond to $-a_1$ and $-a_2$ are not shown in the figure. When F is increased further, the amplitudes a_1 and a_2 increase nonlinearly. Further along in the forward sweep of F , at large excitation levels, the nonlinear periodic response may lose stability through a Hopf bifurcation, leading to modulated motions. During the reverse sweep of F , the transition from nonlinear to trivial solutions occurs at $F = F_2$ through a saddle-node bifurcation. It is also interesting to note that the first instability is due to a pitchfork bifurcation in the cases of primary resonance of the second mode, subharmonic resonance of the second mode, and external combination

resonance. However, in the case of primary resonance of the first mode, the first instability is due to a saddle-node bifurcation.

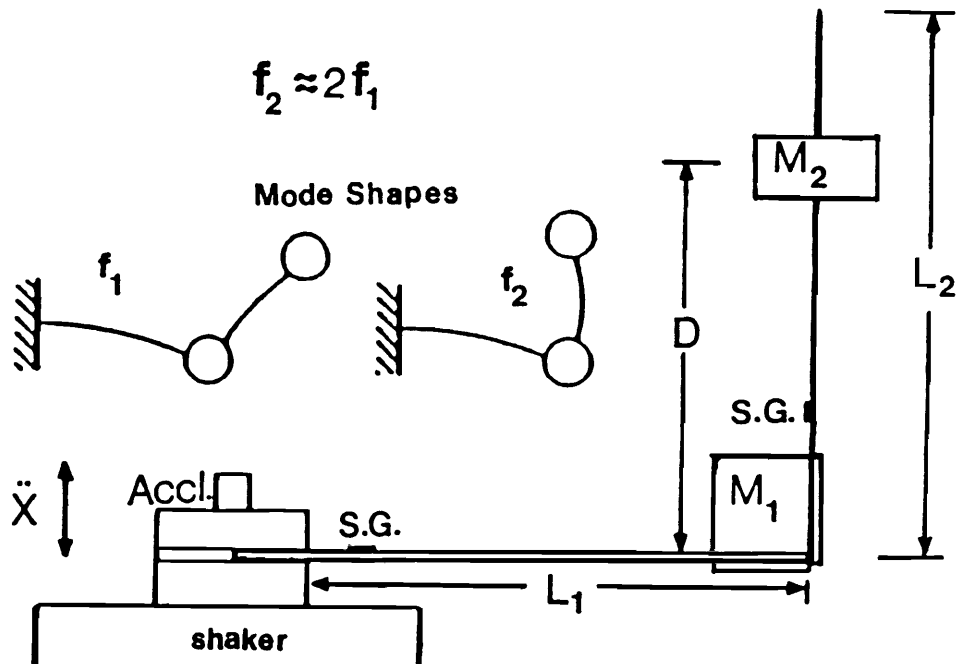


Figure 3.1. Beam-Mass structure and accompanying mode shapes of the first two modes of vibration. The second linear natural frequency f_2 is approximately twice the first linear natural frequency f_1 .

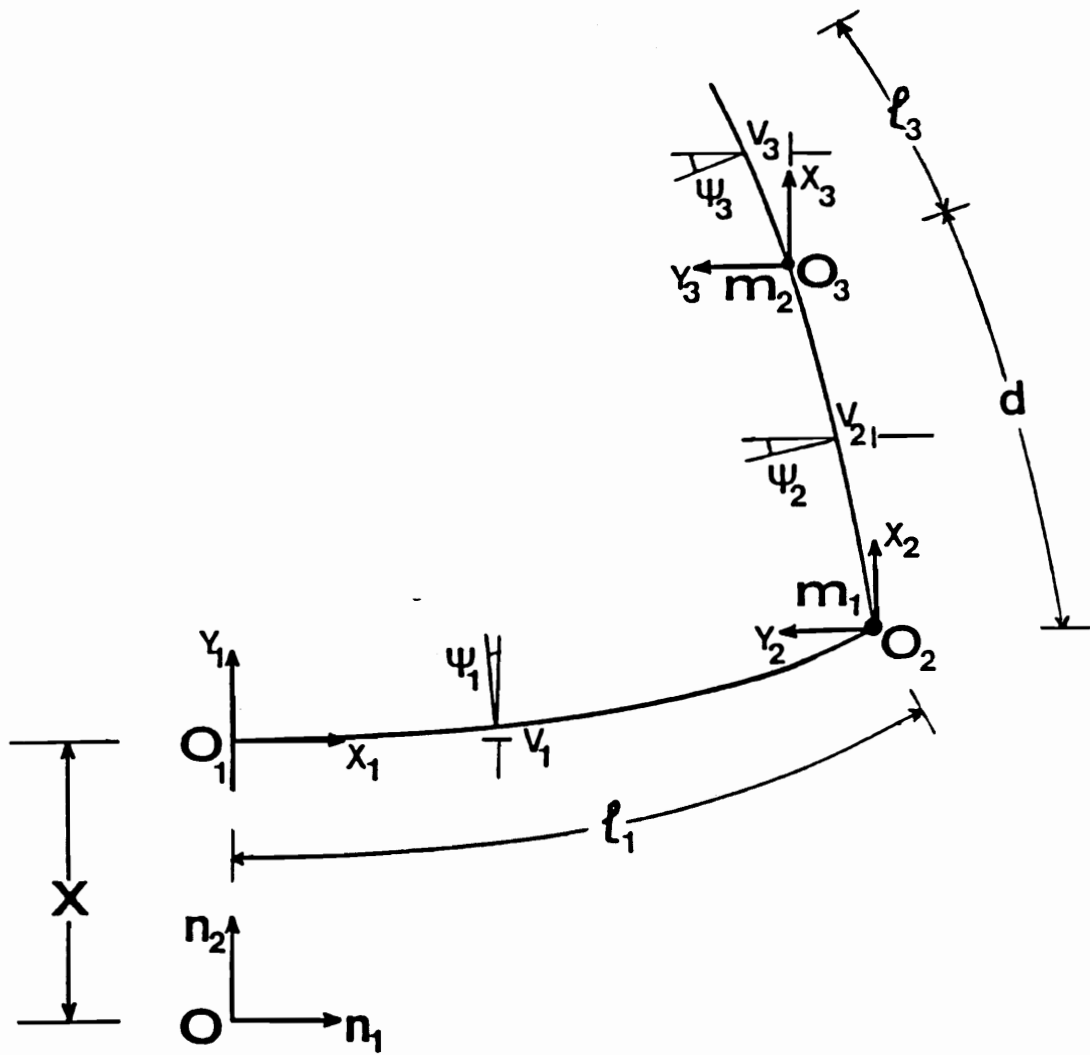


Figure 3.2. Coordinate systems and displacements used in the analysis : O n_1n_2 , Inertial frame of reference; $O_1X_1Y_1$, Origin fixed at the clamp at the left end of the horizontal beam; $O_2X_2Y_2$, Origin fixed at the junction of the horizontal and vertical beam and shares the translational motion of the horizontal beam; $O_3X_3Y_3$, Origin fixed at the center of the mass m_2 and shares the translational motion of this mass.

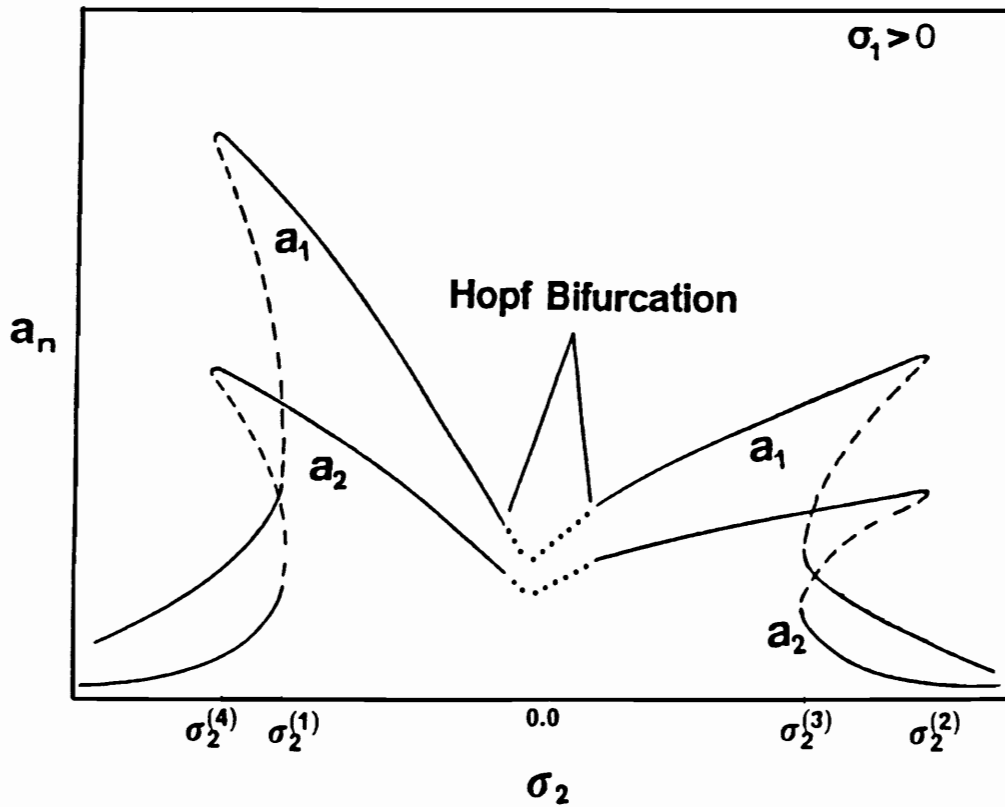


Figure 3.3. Typical frequency-response curves in the case of primary resonance of the first mode when $\sigma_1 > 0$: solid lines correspond to stable fixed points while dashed and dotted lines correspond to unstable fixed points.

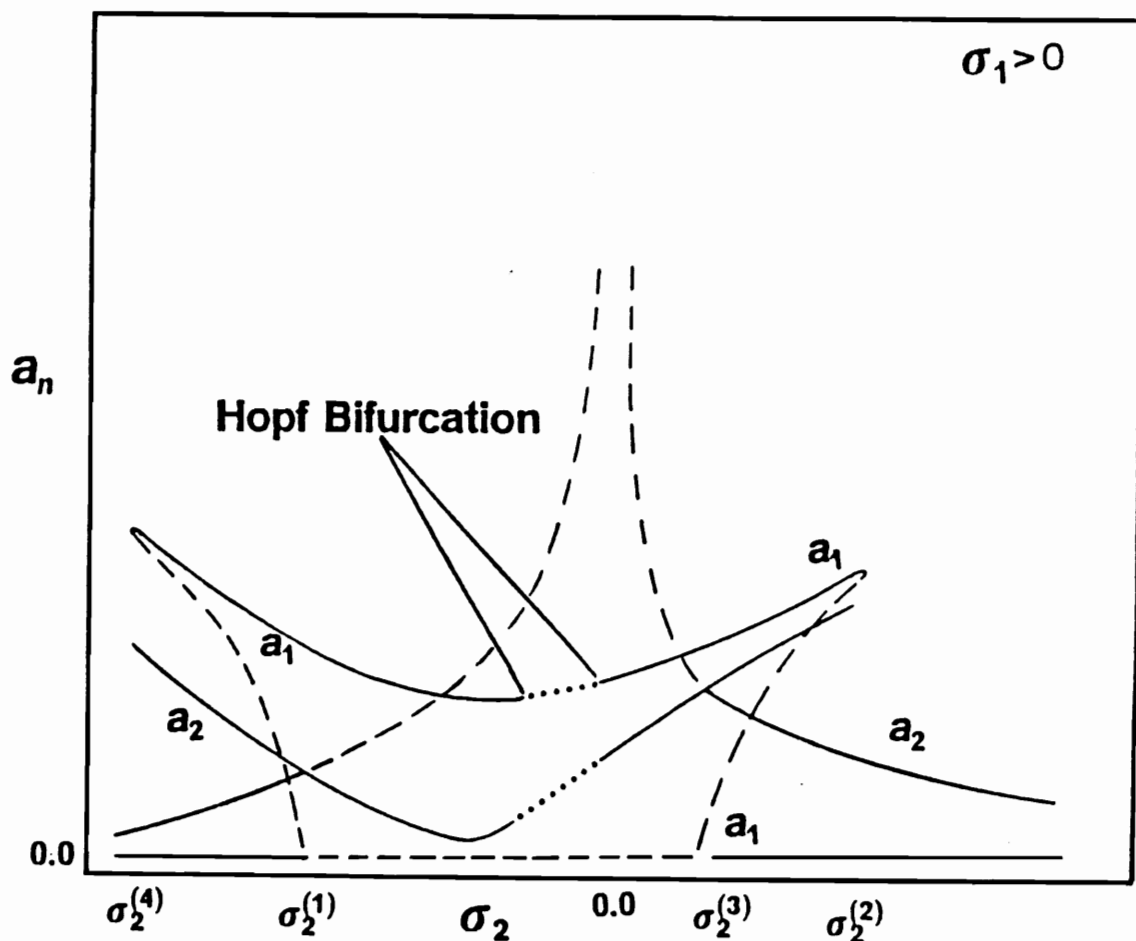


Figure 3.4. Typical frequency-response curves in the case of primary resonance of the second mode when $\sigma_1 > 0$: solid lines correspond to stable fixed points while dashed and dotted lines correspond to unstable fixed points.

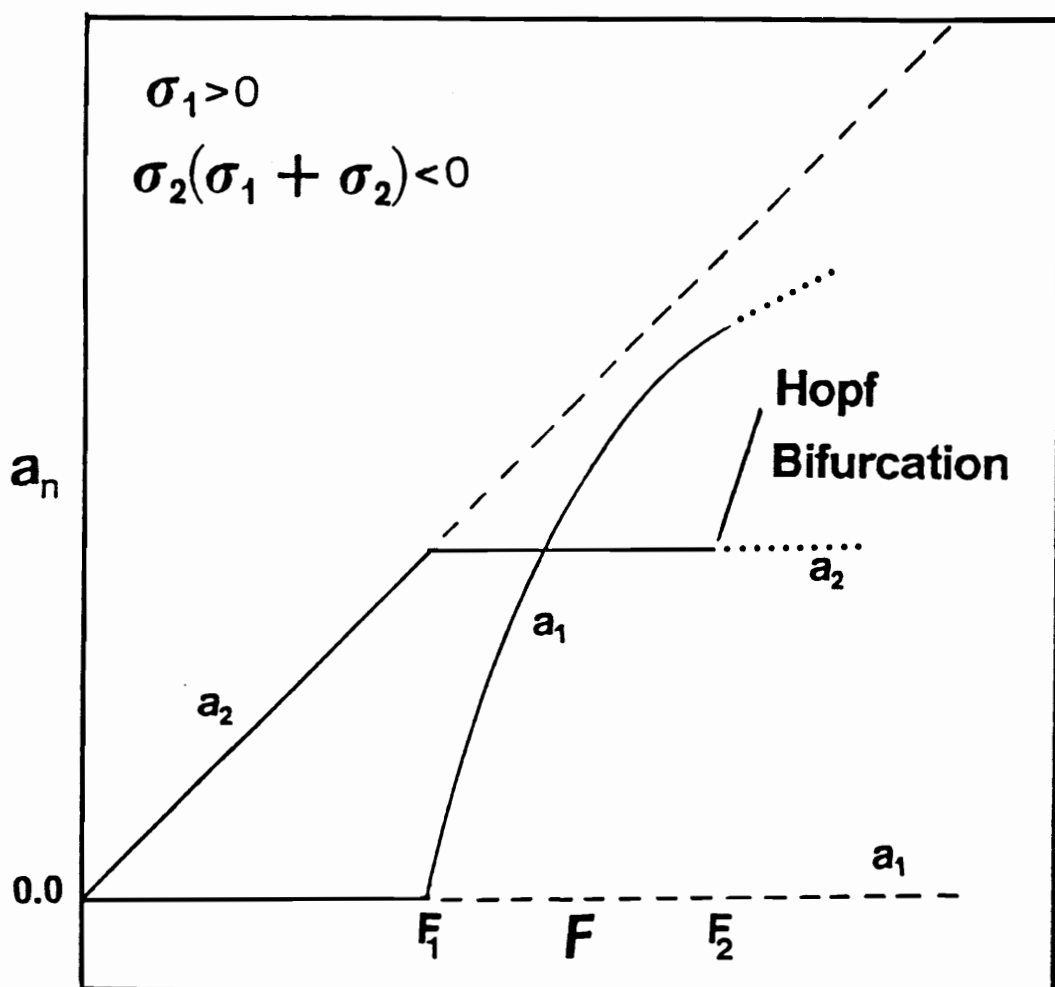


Figure 3.5. Typical force-response curves in the case of primary resonance of the second mode when $\sigma_1 > 0$ and $\sigma_2 < 0$: solid lines correspond to stable fixed points while dashed and dotted lines correspond to unstable fixed points.

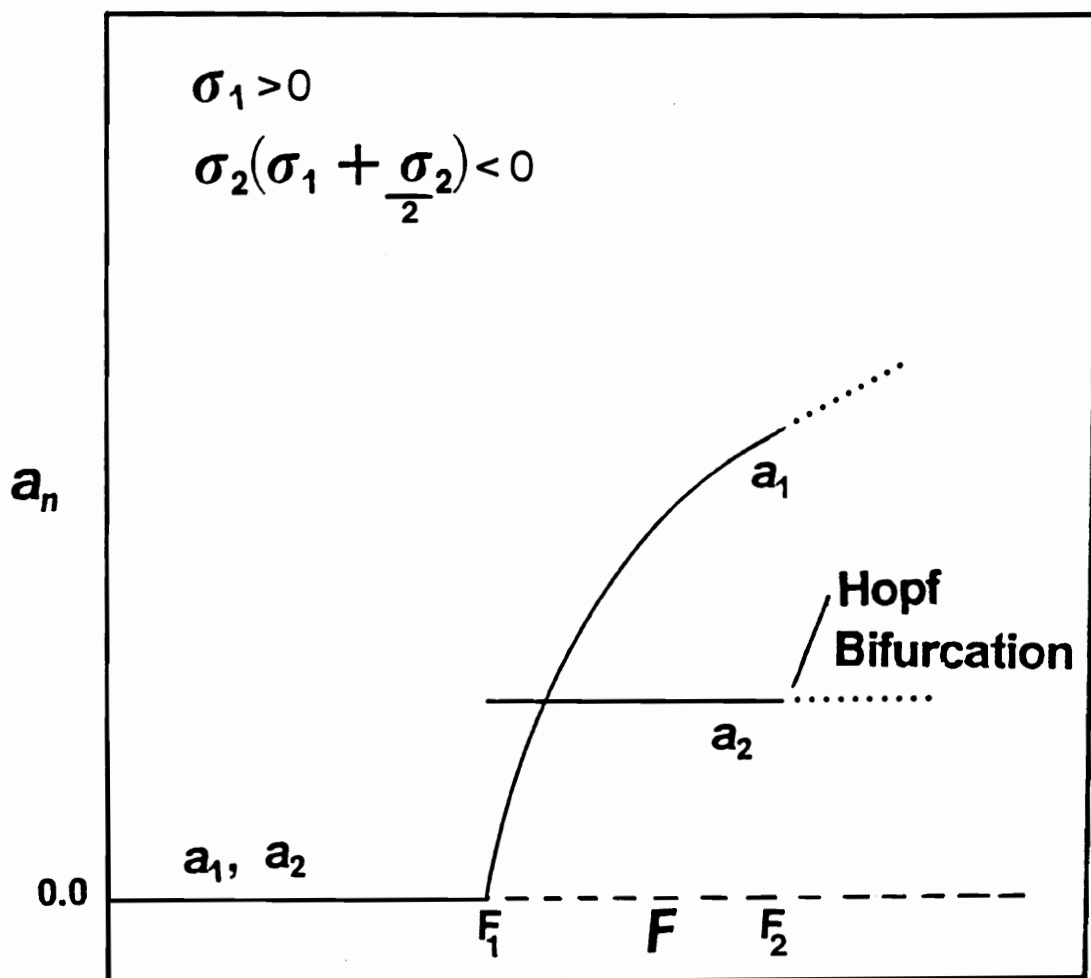


Figure 3.6. Typical force-response curves for the case of subharmonic resonance of order one-half of the second mode when $\sigma_1 > 0$: solid lines correspond to stable fixed points while dashed and dotted lines correspond to unstable fixed points.

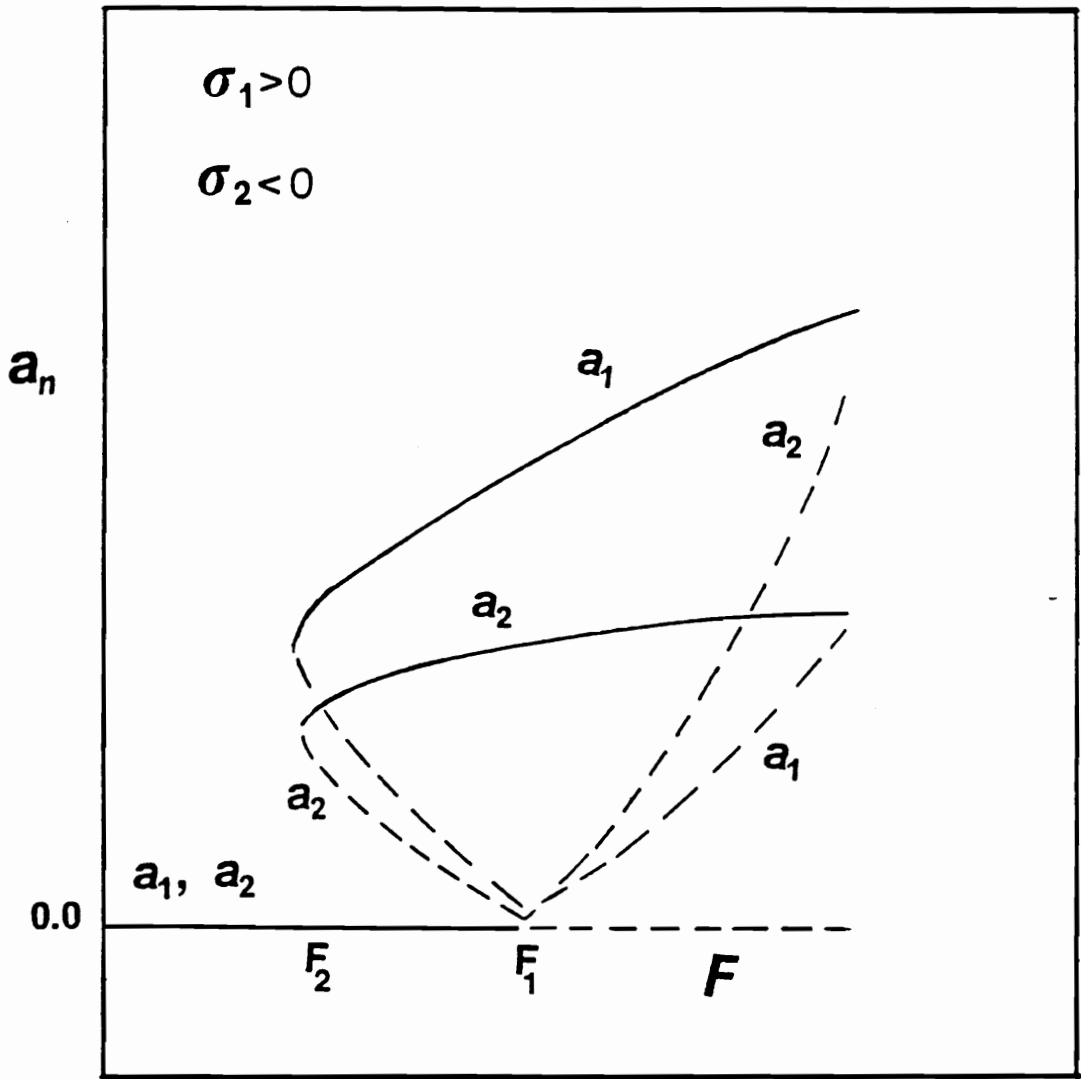


Figure 3.7. Typical force-response curves in the case of an external combination resonance of the additive type when $\sigma_1 > 0$: solid lines correspond to stable fixed points while dashed lines correspond to unstable fixed points.

4. Experiments with the Metallic Structures and Comparison with Theory

In this chapter, results of the experiments with the metallic structures are presented and compared with the analytical predictions. Two metallic structures were considered and the different settings of these structures resulted in Structures I, II, III, IV, V, and VI, whose properties and dimensions are provided in Appendix D. As mentioned in the preceding chapter, each metallic structure consisted of two flexible steel beams and two concentrated masses. Each of the Structures I and II weighed about 105.0 grams while each of the Structures III, IV, V, and VI weighed about 75.0 grams. So, these beam-mass structures are light-weight structures. Each of the Structures I, II, III, and VI was subjected to primary-resonant excitations while each of the Structures IV and V was subjected to secondary-resonant excitations. During primary-resonant excitations, a structure was excited at a frequency that was close to one of its natural frequencies, and during secondary-resonant

excitations the excitation frequency was far from a natural frequency of the structure. In the experiments, the responses were analyzed by using Fourier spectra, pseudo-phase planes, Poincaré sections, time-dependent modal decompositions, and dimension calculations. For all spectral analysis, a flat top window was used for windowing. The experimental results are compared with the analytical predictions discussed in Chapter 3.

The experimental set up is schematically represented in Fig. 4.1. It is, to a large extent, similar to that used by Zavodney (1987) and is discussed in detail in his work. Each L-shaped beam-mass structure was mounted on the table of a 250-pound modal shaker. A suspension system, not shown in the figure, bore the weight of the table and other accessories mounted on it. Besides serving as a platform, the table was also meant to act as a mechanical filter to reduce the feedback from the test structure to the actuator. The table consisted of four plates stacked in a sequence, each weighing about 16 pounds. A digital oscillator (Wavetek model 650) was used to drive the shaker and a quartz accelerometer (Sunstrand model QA-1400) mounted on the top plate of the table was used to measure the excitation amplitude F . During the experiments, a feedback loop was used to maintain the excitation amplitude at a desired level. In the loop, an IBM PC was interfaced through GPIB interfaces to the digital oscillator and a digital voltmeter (Brüel and Kjær model 2432), which read the root-mean-square value (rms value) of the shaker-table acceleration. Each voltmeter reading was a running average of five measurements. The PC monitored the voltmeter reading and made

necessary proportional changes in the signal sent from the oscillator to the shaker to maintain the excitation level. However, this feedback loop has some limitations. The loop was not effective in maintaining the excitation level during large modulated motions of a structure. Some of the limitations may be overcome if the frequency content of the excitation signal is also monitored in the feedback loop in addition to the rms amplitude of the excitation.

The beams of each structure were instrumented with dynamic strain gages to measure the displacements. Two 350-Ohm strain gages (Micro-Measurements WK-06-125AD-350) were mounted along the axes of the horizontal and vertical beams (referred to as Strain Gage H and Strain Gage V, respectively) to obtain a measure of the displacements due to flexural motions. In Fig. 3.1, the letters S.G. denote the strain gages. Strain Gage H was located at about an inch from the clamp while Strain Gage V was located at about an inch from the junction of the two beams. Each of these strain gages together with a 350-Ohm precision resistor (located on the shaker table) formed half a Wheatstone bridge (Horowitz and Hill, 1980), and was completed in a signal conditioner. For all cases, the bridge excitation level (after some experiments) was chosen as 2.7 volts. Later, the signals from the amplifiers were passed through Krohn-Hite elliptical low-pass filters (cut-off frequency = 50.0 Hz) and amplified again (post-amplifier gain either 10 dB or 20 dB) before their measurement. In one set of experiments with Structure I (primary resonance of the higher mode), the strain gage on the vertical beam was calibrated to measure transverse displacements of the tip of the vertical beam. For this

purpose, a micrometer head was used to provide known displacements. As the given displacement was gradually increased the beam bent and the micrometer head no longer made normal contact with the same point on the beam, and this restricted the range of the provided displacement to 6 millimeters for calibration. The sensitivity of the strain gage was found to be linear in this range and the static calibration factor for the strain-gage reading was 43.76 mm/volt. We assumed the strain-gage sensitivity to be linear for displacements higher than 6 millimeters as well. Strictly speaking, one should use dynamic calibration factors to calculate the displacements. During dynamic calibration, one excites a particular mode of the structure and measures the resulting displacement and the strain-gage reading, from which one obtains the corresponding calibration factor. However, due to the presence of the internal resonance in the structure and problems in accurately measuring the dynamic displacement, we opted to use the static calibration factor.

The strain-gage and accelerometer signals were analyzed by using a spectrum analyzer (GenRad system 2515). For periodic motions, a measure of the modal amplitudes was obtained from the frequency spectrum of the strain-gage signal. The displacement at a strain-gage location s on a structure and time t for primary-resonant excitations (see Chapter 3) can be written as

$$w(s, t) = \sum_i \dot{a}_i(t) \cos(\Omega_i t + \gamma_i) \quad (4.1)$$

where the $\dot{a}_i$ are the respective amplitudes of the peaks at the frequencies Ω_i , and the γ_i are the corresponding phase angles. The frequencies Ω_i are close to the structure's respective linear natural frequencies ω_i . During periodic motions, the amplitudes $\dot{a}_i$ are constants and are determined from the frequency spectrum. They are related to the modal amplitudes a_i and the mode shapes ϕ_i as

$$\dot{a}_i = C_a a_i \phi_i(s_d) \quad (4.2)$$

where C_a is a calibration factor (i.e., so many volts/mm) and s_d is a given location on the undeformed structure. During secondary-resonant excitations, the strain-gage signal consists of an additional peak at the forcing frequency in addition to the frequency components shown in equation (4.1). The procedures used to determine the linear resonant frequencies and modal-damping coefficients of the first two modes of the structure are discussed in Appendix E. The details of construction of the pseudo-phase planes and Poincaré sections, time-dependent modal decompositions, and pointwise dimension calculations are also discussed in the same appendix. All the observed forced responses of the metallic structures were planar.

4.1. Primary resonance of the second (higher) mode

Each of the Structures I, II, III, and VI, whose dimensions and other details are provided in Appendix D, was subjected to primary-resonant excitations of the second (higher) mode. The first and second damped-resonant frequencies were respectively 8.13 Hz and 16.44 Hz for Structure I, 8.16 Hz and 16.50 Hz for Structure II, 8.13 Hz and 16.45 Hz for Structure III, and 8.19 Hz and 16.72 Hz for Structure VI. For all these structures, the sign of the detuning of the internal resonance is positive. We also note that the internal detuning 0.34 Hz of Structure VI is large in comparison to the internal detunings of the other structures. Also in all four cases, the frequencies of the third and higher modes (third natural frequency is about 100 Hz) are far separated from the frequencies of the first and second modes of vibration.

In Fig. 4.2, we display the frequency-response curves of Structure I obtained by sweeping the frequency of excitation up and down at a constant excitation level of 16.67 mili g's rms, where mili stands for 10^{-3} and g stands for the acceleration due to gravity. The frequency of excitation f ranged from 15.5 Hz to 17.5 Hz. In this and subsequent figures, the C's (triangles) mark the observations made during forward (reverse) sweeps of the control parameter and the points where periodic motions ceased to exist are labeled "Hopf bifurcation". Also in the figures, the symbol a_i corresponds to the amplitude of the i th mode. In this and following chapters, curves are drawn through the

experimental observations of the force and frequency responses for the eye to follow the trend. The arrows in the figures are used to show the direction of sweep of a control parameter. As the driving frequency was increased from 15.5 Hz, linear responses were observed for values of f less than 16.20 Hz ; the amplitude of the first mode remained trivial. At $f = 16.20$ Hz, the linear response lost stability through a jump to a nonlinear periodic response. As f was increased further, this nonlinear response lost stability at 16.30 Hz; modulated motions ensued and were observed in the range $16.30 \text{ Hz} \leq f < 16.44 \text{ Hz}$. The observation of modulated motions in this region is consistent with the analysis (e.g., Nayfeh, 1989), which predicts that such motions are likely to occur when the condition $\sigma_2(\sigma_1 + \sigma_2) < 0$ is satisfied. As the excitation frequency was increased further, the structure displayed nonlinear periodic responses until the critical value 16.70 Hz was reached, beyond which a jump occurred. Thereafter, linear responses were observed. During the reverse sweep of f , jumps occurred at 16.47 Hz and 15.90 Hz. The form of the curves shown in Fig. 4.2 is similar to the analytically determined curves shown in Fig. 3.4.

The variation of the response amplitudes of Structure I with the excitation amplitude is shown in Fig. 4.3. During this experiment, the excitation frequency was held constant at 16.38 Hz. Here, the sign of the detuning of the external resonance is negative. As the excitation amplitude was increased from zero, a_2 increased linearly while a_1 remained trivial. This linear response lost stability when the excitation level reached a critical value. At an excitation

amplitude of 6.67 mili g's rms, the structure displayed a nonlinear periodic response. As the excitation amplitude was increased further, a_2 remained more or less constant while a_1 increased in a nonlinear manner. This modal saturation broke down at 12.00 mili g's rms. Beyond this excitation level periodic motions ceased to exist and modulated motions were observed. This experiment demonstrates that the saturation phenomenon can break down for some excitation-parameter values. Again here, the analytically obtained curves in Fig. 3.5 and experimentally obtained curves in Fig. 4.3 are similar in form.

Structure II was also subjected to primary-resonant excitations of the second mode. In experiments with this structure, the excitation amplitude was held constant at 33.33 mili g's rms, while the excitation frequency was varied. In Fig. 4.4, the observed response spectra, excitation spectra, and corresponding pseudo-phase planes (cross plots of strain-gage signals) are shown for typical linear periodic, nonlinear periodic, and modulated responses. When we examine the response spectrum in Fig. 4.4b, we see spectral lines at the excitation frequency f and its subharmonic $f/2$. Further, the corresponding cross plot (pseudo-phase plane) is an "eight-shaped" pattern. This pattern was stationary, when we watched it in real time. Further the "eight-shaped" pattern is also indicative of the presence of a one-half subharmonic in the response. The frequencies $f/2$ and f are close to the natural frequencies of the first and second modes of the structure, respectively. In Fig. 4.4c, discrete spectral lines appear at $f \pm n\delta f$ and $f/2 \pm n\delta f$, where $n = 0, 1, 2, 3, \dots$. The

sidebands uniformly spaced δf apart are due to the periodic variation of the amplitudes and phases. We also note sidebands around the excitation frequency in the corresponding table-accelerometer spectrum, indicating that there is feedback from the beam-mass structure to the top plate of the table. One should note that the plates on the shaker weigh about 64 pounds while the beam-mass structure only weighs about 105 grams. As mentioned before, the feedback loop may not be effective, wherever the structure exhibits large modulated responses. In Fig. 4.4c, we show the cross plot for a modulated motion. When we watched the motion in real time, we saw an evolving "eight-shaped" pattern. Amplitude- and phase-modulated motions were observed in the range $16.36 \text{ Hz} \leq f < 16.50 \text{ Hz}$. The nature of the modulated motions observed in this region is discussed in the next paragraph.

The response spectra were examined in a zoom span (1280 lines in a 2.5 Hz bandwidth) around the excitation frequency f to get a better idea of the modulated responses. In Figs. 4.5a-4.5c the response spectra are shown in a zoom span at the excitation frequencies 16.36 Hz, 16.37 Hz, and 16.41 Hz, respectively. They were obtained as f was gradually increased in the sweep. The uniformly spaced sidebands around the carrier frequency f correspond to periodically modulated motions. The sideband structure would be symmetric if we had a purely amplitude-modulated signal. Here, the asymmetric nature of the sideband structure indicates a periodic variation of the phases as well. As we proceed from Fig. 4.5a to Fig. 4.5b, we see distinct spectral lines appearing midway between the peak at the frequency f and the largest other

peaks. This is a consequence of the halving of the modulation frequency or the doubling of the modulation period. So, a period-doubling bifurcation took place when f was changed from 16.36 Hz to 16.37 Hz. In Fig. 4.5c, more subharmonics of the modulation frequency occur, and the spectral content indicates the occurrence of another period-doubling bifurcation. As f was increased further to 16.44 Hz, we observed what appears to be a chaotically modulated motion, which is suggested by the broadband structure around f in Fig. 4.6a. We verified that the motion was in fact chaotic with the help of dimension calculations. Examining the response spectrum in a baseband of 20.0 Hz bandwidth in Fig. 4.6b, we observe a broadband character around both the carrier frequencies, namely f and $f/2$, as well as low-frequency fluctuations.

The motion corresponding to each of the response spectra shown in Figs. 4.5a-4.5c is also known as two-period quasiperiodic motion because there are two incommensurate frequencies f and δf . The corresponding attractor is normally called a two torus (see Chapter 2). The spectrum of a modulated response could also show a broadband character around the carrier frequencies if one does not have enough frequency resolution. However, the 0.002 Hz resolution used here is sufficient for discerning the nature of the modulated motions.

The transition to the chaotically modulated motion at 16.44 Hz via quasiperiodic motions is shown in Fig. 4.7 by means of Poincaré sections, which are stroboscopic pictures of the motion obtained by using the excitation

frequency. The motion of the structure is composed of many basic frequencies, the excitation frequency being one of them. So, essentially the Poincaré sections present information related to the other basic frequencies of the motion. Here, in the first three sections, Figs. 4.7a-4.7c, the closed loops indicate that just one other basic frequency is present and that the motion of the structure is quasiperiodic. Also, this other basic frequency corresponds to a slow time scale (see the frequency spectrum). In the last section, Fig. 4.7d, the scatter in the points suggests a chaotic motion.

In Fig. 4.8a, the measured strain-gage signal $w_v(t)$ is shown at the excitation frequency 16.36 Hz. We used time-dependent modal decomposition to extract the modal amplitudes a_k^* from this signal. As described in Appendix E, two band-pass filters centered at 16.36 Hz and 8.18 Hz were used to digitally separate the first and second modes, respectively. The separated signals U_1 and U_2 , which correspond to the first and second modes, are shown in Figs. 4.8b and 4.8c, respectively. The modal amplitudes a_k^* , recovered from these signals, are shown in Figs. 4.8d and 4.8e. There appears to be a phase difference of about 90° between the two modal amplitudes. A slow periodic variation of the modal amplitudes is also evident in these figures. In Fig. 4.9 similar results are shown for the signal collected at the excitation frequency 16.37 Hz. These figures bring out the existence of two time scales in the motion: a fast time scale corresponding to the excitation frequency and multiples of it, and a slow time scale at which the amplitudes and phases of the interacting modes of motion evolve.

Projections of the motion on the $a_1 - a_2$ plane reveal information related to the slow time scale of the motion. When the structure's motion was periodic we observed a point in this plane. In Fig. 4.10, we show the projections on the $a_1 - a_2$ plane for the frequencies 16.36 Hz, 16.37 Hz, 16.46 Hz, and 16.47 Hz. They are limit cycles indicating that the corresponding motion of the beam-mass structure is two-period quasiperiodic. The numbers on the axes of the figures represent the measured values in volts. A period-doubling bifurcation occurs between Figs. 4.10a and 4.10b. A period-halving bifurcation occurs between Figs. 4.10c and 4.10d.

In Fig. 4.11, the chaotic motion at 16.44 Hz in the pseudo-phase plane is constructed by the method of delays (delay $T_d = 3.5$ seconds). This phase portrait is suggestive of a strange attractor having a dimension greater than 2. We determined the pointwise dimension for the motion observed at 16.44 Hz by the procedure discussed in Appendix E. In Fig. 4.12, we show the plot for the pointwise dimension for different values of the embedding dimension. Examining the different curves in this plot, we observe that the curves for embedding dimensions of three and above are parallel to each other in a certain region. This fact suggests that the pointwise dimension is close to three and that an embedding dimension of eight is large enough to capture the attractor. For an embedding dimension of eight, the pointwise dimension d is about 2.748. This fractal dimension indicates that the corresponding motion is on a strange attractor. The dimension d was found to vary within a range of ± 0.2 as T_d was varied from 1.6 to 4.2 seconds. The value of the dimension

also suggests that the structure's response can be modeled by a third- or higher-order autonomous system of equations. We recall that in Chapter 3, we had derived a fourth-order autonomous system to model the structure's behavior.

In Fig. 4.13, we compare the theoretical predictions and experimental observations for the frequency response of Structure I to a primary-resonant excitation of the second mode. During the experiment, the excitation amplitude F was held constant at 16.67 mili g's rms. The corresponding experimental observations are shown in Fig. 4.2. The unstable solutions predicted by the analysis are not shown in the figure. The solid lines in the figure correspond to the analytical predictions while the C's and triangles correspond to the experimental observations. The C's(triangles) in this figure and the following one mark the observations made during the forward(reverse) sweeps of the control parameter. The analytically predicted Hopf bifurcations are also marked in Fig. 4.13. In the experiments, jumps occurred at the excitation frequencies 16.20 Hz and 16.70 Hz during the forward sweep and at 16.47 Hz and 15.90 Hz during the reverse sweep. The analytical predictions of the jump locations (bifurcation points) are 16.160 Hz and 16.802 Hz during the forward sweep and 16.539 Hz and 15.880 Hz during the reverse sweep. Also, in the experiments modulated motions were observed in the frequency range $16.30 \text{ Hz} \leq f < 16.44 \text{ Hz}$, where again f is the excitation frequency. The analysis predicts Hopf bifurcations at 16.369 Hz and 16.424 Hz. From Fig. 4.13, we see that the analytical and experimental results are in good agreement. In

an experiment with Structure II, modulated motions were observed in the frequency range $16.36 \text{ Hz} \leq f < 16.50 \text{ Hz}$ at an excitation level of 33.33 mili g's rms. For this case, the analysis predicts Hopf bifurcations at 16.427 Hz and 16.485 Hz. Again in this case, the agreement between the analytical predictions and the experimental observations is good.

In Fig. 4.14, we show the experimental observations and analytical predictions for the force-response curves of Structure I. During the experiment, the excitation frequency was held constant at 16.38 Hz. The analysis predicts the occurrence of saturation at the excitation amplitude 4.83 mili g's rms and the breakdown of saturation due to a Hopf bifurcation at $F = 7.93$ mili g's rms. In the experiments, as F was increased the structure showed a saturated response at 6.67 mili g's rms. This saturated response lost stability at $F = 12.00$ mili g's rms, leading to modulated motions. Here, the analytical predictions show reasonable agreement with the experimental observations.

We also subjected Structure III to primary-resonant excitations of the second mode. At an excitation level of 33.33 mili g's rms, this structure displayed modulated motions in the range $16.28 \text{ Hz} \leq f < 16.38 \text{ Hz}$. In Fig. 4.15, the response spectra in a zoom span are shown around the excitation (carrier) frequency f at 16.30 Hz, 16.32 Hz, 16.34 Hz, 16.35 Hz, and 16.36 Hz. In each case, the response is periodically modulated as indicated by the discrete sideband structure around the carrier frequency. The period (frequency) of modulation undergoes period doubling (frequency halving) as f increases from

$f = 16.32$ Hz to $f = 16.34$ Hz. As f was increased further to 16.35 Hz and beyond, the modulations decreased in size and eventually died out giving way to periodic motions at 16.38 Hz.

At an excitation level of 166.67 mili g's rms and an excitation frequency of 16.36 Hz, we observed what appears to be a case of frequency locking. In this case, the modulation frequency was approximately 0.39 Hz. The corresponding response spectrum and Poincaré section are shown in Fig. 4.16. The clusters of points in the Poincaré section suggest that there is frequency locking (i.e., $\delta f/f$ is a rational number) and that the motion of the structure has just one basic frequency. However, this observation was a very rare one, and we could not obtain or repeat this experimental result during other experiments with this structure. Because there is an infinite number of irrational numbers between any two rational numbers, the experimental conditions have to be just right for the phenomenon of frequency locking to occur.

Structure VI was also subjected to primary-resonant excitations of the second mode. When the excitation level was 30.00 mili g's rms, the structure displayed modulated motions in the range $16.515 \text{ Hz} \leq f < 16.65 \text{ Hz}$. The excitation frequency was varied in this range, and the resulting responses were studied. In Figs. 4.17-4.22, the response spectra and corresponding Poincaré sections are displayed for the observed modulated motions. In Fig. 4.17, the response spectra in a zoom span around f is shown for the excitation

frequencies 16.515 Hz, 16.525 Hz, 16.54 Hz, and 16.56 Hz. In each of these cases, the uniformly spaced sidebands around f indicate that the corresponding motion is periodically modulated. The spectrum shown in Fig. 4.18a corresponds to the periodically modulated motion observed at the excitation frequency 16.57 Hz. The modulation frequency δf , determined from the sideband structure, is about 0.166 Hz. Here, the motion at $f = 16.57$ Hz is referred to as a period-one modulation. As f was increased to 16.576 Hz, a doubling of the modulation period occurred; this bifurcation is characterized by the appearance of peaks between the carrier frequency and the largest other peaks in Fig. 4.18b. The corresponding motion is called a period-two modulation.

When f was in the range from 16.573 Hz to 16.575 Hz, we observed noise at the subharmonics of the modulation frequency during the transient phase of the motions. We observed this feature in some of the other experiments too. In the present context, the appearance of noise can be considered as indicative of an impending period-multiplying bifurcation.

As we varied f to 16.582 Hz, another period-doubling bifurcation took place, and consequently, the resulting period of modulation is about four times that observed at 16.570 Hz. At $f = 16.582$ Hz, the motion takes a long time to settle down and drifts between period-two and period-four modulations during the transient phase. Another period-doubling bifurcation occurred at $f = 16.584$ Hz. This period-doubling bifurcation led to a period-eight modulation. As f

was increased further, higher-period modulations occurred (Fig. 4.19a) and the sequence culminated in chaotically modulated motions at $f = 16.61$ Hz. The corresponding spectrum in Fig. 4.19b has a broadband character. Similar motions were observed at 16.62 Hz (Fig. 4.19c) and 16.63 Hz. We observed periodically modulated motions again at $f = 16.64$ Hz. The spectral lines are sharper in Figs. 4.17-4.19 and 4.15 in comparison to those seen in Fig. 4.5. So, the appearance of the subharmonics of the modulation frequency is clearer in Figs. 4.17-4.19 and 4.15 in comparison to Fig. 4.5. A possible reason for the difference in clarity may be leakage (Harris, 1978).

In Fig. 4.20, the Poincaré sections are shown for driving frequencies of 16.515 Hz, 16.525 Hz, 16.54 Hz, 16.56 Hz, 16.57 Hz, and 16.576 Hz in that order. In all cases, the points form a closed curve indicating a two-period quasiperiodic motion. The period-doubling bifurcation which occurs at 16.576 Hz is not discernible in the Poincaré section. In Fig. 4.21, the sections are shown for driving frequencies of 16.581 Hz, 16.582 Hz, 16.584 Hz, and 16.586 Hz, respectively. At $f = 16.581$ Hz, there is a period-two modulation, while at $f = 16.582$ Hz we observed a period-four modulation. The Poincaré section at $f = 16.582$ Hz, has two loops (closed curves of points), each of which is identical to the single loop observed at $f = 16.581$ Hz. It appears that two Poincaré sections (each of which is identical to the one obtained $f = 16.581$ Hz and is a reflection of the other one) merge together at $f = 16.582$ Hz. In other words, one could construct the section in Fig. 4.21b by considering two objects of the form shown in Fig. 4.21a. This interesting structural feature is suggestive of a

period-doubling bifurcation. Figure 4.21c corresponds to a motion whose modulation period is twice that observed for the motion corresponding to Fig. 4.21b. This is discernible by comparing the structures of the Poincaré sections in the two cases. Figure 4.21d corresponds to possibly a higher period of motion. In Fig. 4.22, the Poincaré sections are shown for the motions observed at forcing frequencies of 16.59 Hz, 16.61 Hz, 16.62 Hz, and 16.64 Hz in that order. The first section corresponds to a higher period of motion, while the second and third sections correspond to chaotic motions. It takes about 61.6 seconds to collect 1024 points for each of the last two Poincaré sections. Strictly speaking, this time length is not long enough to decide if the motion is chaotic or not. However, there are two reasons for believing it is chaotic at these frequencies. The first reason is the increasing scatter in the location of the points in the sections in comparison to the earlier cases. The second reason is associated with an experimental observation; we observed that the structure in the Poincaré section did not repeat itself suggesting the absence of periodicity. The Poincaré section at $f = 16.64$ Hz is shown in Fig. 4.22d. It indicates that the corresponding motion is two-period quasiperiodic.

In general, during periodic responses, the feedback from the beam-mass structure to the table, as determined from the table acceleration spectrum (the ratio of the amplitude of the largest other peak to the amplitude of the peak at the excitation frequency) was less than -30.00 dB. This is considered reasonable because the excitation levels are small. During large modulated motions of the structure, the feedback from the structure resulted in

fluctuations in the excitation amplitude. These fluctuations, as determined from the voltmeter reading, were within a range of about 3.00 mili g's rms. As the feedback loop was not effective in controlling the excitation amplitude it was not used when modulated motions were observed. The experimentally observed frequency spectra of the linear periodic, nonlinear periodic, periodically modulated, and chaotically modulated responses are in agreement with the corresponding analytical predictions discussed in Chapter 3. The form of the frequency-response and force-response curves are in qualitative agreement with the analytical predictions given in Chapter 3. Whenever the structure displayed modulated motions, it was very sensitive to the experimental conditions. Runs at the same apparent excitation frequency and amplitude on two different days sometimes yielded two different responses. This aspect was not well documented for the present case, but care was taken to conduct the experiments in "nearly uniform" conditions.

4.2. Primary resonance of the first (lower) mode

Structure II, whose dimensions and details are provided in Appendix D, was subjected to primary-resonant excitations of the first mode. Three sets of experiments (referred to as A, B, and C in this section) were conducted with this structure and the observations made during these experiments are discussed next.

In Fig. 4.23, the frequency-response curves obtained by varying the excitation frequency at a constant excitation level of 150.0 mili g's rms are shown. The control parameter f was varied in the range running from 7.5 Hz to 8.9 Hz. As before, the C's (triangles) in the figure mark the observations made during the forward (reverse) sweeps of the control parameter and the points where periodic motions ceased to exist are labeled "Hopf bifurcation" in the figure. During the forward sweep of f , jumps occurred at 7.90 Hz and 8.70 Hz, and during the reverse sweep of f , jumps occurred at 8.35 Hz and 7.80 Hz. These jumps produce a large change in the magnitude of the structure's response. Amplitude- and phase-modulated motions were observed when the driving frequency was in the range $8.12 \text{ Hz} \leq f \leq 8.30 \text{ Hz}$. When we compare the forms of the analytically obtained curves shown in Fig. 3.3 and the experimentally obtained curves presented in Fig. 4.23, we observe that they are identical. In Fig. 4.24, experimental observations in the form of cross plots for a periodic response and a modulated response are given. The variation of the amplitudes and phases of the interacting modes causes an evolving "eight-shaped" pattern in Fig. 4.24b.

In Figs. 4.25-4.27, the observations made during Experiment A are presented. In Fig. 4.25, the response spectra are examined in a zoom span (1280 lines in 2.5 Hz) about excitation frequencies of 8.16 Hz, 8.18 Hz, and 8.20 Hz. In each of these cases, we observe discrete peaks at $f \pm n\delta f$ where $n = 0, 1, 2, 3 \dots$. The uniformly spaced sidebands are a result of the periodic variation (period = $2\pi/\delta f$) of the amplitudes and phases. In Fig. 4.26, the response spectrum is

given for the motion at $f = 8.22$ Hz in a zoom span and 20.0 Hz baseband. The broadband character around the carrier frequencies, namely f and $2f$, suggests that the motion is possibly chaotic at this frequency. The corresponding dimension calculations verify that the motion is chaotic at this frequency. We also note a few low-frequency peaks due to the modulation. During periodic motions, we observed a spectral line at zero frequency (d.c. component) in addition to the spectral lines at f and $2f$.

The projection of the attractor onto the $a_1 - a_2$ plane was obtained after extracting the a_i from the modulated signals by using the time-dependent modal decomposition procedure. When the observed response was periodic, the projection was a point in the $a_1 - a_2$ plane. In Fig. 4.27, the projections of the attractors are plotted at the excitation frequencies 8.16 Hz, 8.18 Hz, 8.20 Hz, 8.23 Hz, 8.26 Hz, and 8.28 Hz. The projections are limit cycles, indicating that the modulated motion corresponding to Figs. 4.25 and 4.27 is two-period quasiperiodic, with the incommensurable frequencies f and δf .

As stated earlier, to verify that the motion was chaotic at $f = 8.22$ Hz, we computed the pointwise dimension d . A delay of 1.00 second was chosen to construct the delayed coordinates of the embedding space. In Fig. 4.28, the logarithm of the number of points is plotted against the logarithm of the radius. From the scaling region, d was found to be about 2.737 for an embedding dimension of eight, verifying that the motion is chaotically modulated at 8.22

Hz. The dimension was found to vary within a range of ± 0.2 as the delay time was varied from 0.5 to 3.5 seconds.

In Figs. 4.29 and 4.30, the observations made during Experiment B are presented in the form of frequency spectra and Poincaré sections for the excitation frequencies 8.18 Hz, 8.20 Hz, and 8.22 Hz. The frequency spectra observed in Experiment B differ from those observed in Experiment A. The points in the Poincaré section for $f = 8.18$ Hz and $f = 8.20$ Hz are dispersed about a curve. Bergé et al. (1984) observed similar Poincaré sections in their studies of Rayleigh-Bernard convection with silicone oil. The Poincaré sections indicate that the modulated motions observed during experiment B are not two-period quasiperiodic motions and, hence, that the motion of the structure may be made up of more than two basic frequencies. However, we could only ascertain the presence of two basic frequencies from the frequency spectrum. Similar Poincaré sections were also seen at other values of f at which modulated motions were observed. The scatter of the points in Fig. 4.30c indicates that the motion is possibly chaotic at $f = 8.22$ Hz. In order to verify that the motion was chaotic, we determined the pointwise dimension d . In Fig. 4.31, d is plotted for different values of the embedding dimension. A delay of 0.83 seconds was used for the embedding. In the figure, the curves for embedding dimensions of four and above appear to run parallel to each other in a certain region. This suggests that an embedding dimension of eight should be adequate for the attractor to be embedded in it. The attractor dimension was found to be about 3.315, for an embedding dimension of eight.

In this case, as in Experiment A, the motion is seen to be chaotic at 8.22 Hz. However, the value of d is higher than that found in Experiment A. The value of the dimension suggests that the fourth-order autonomous system derived in Chapter 3 should be adequate to model the structure's behavior.

Experiment C was conducted in an effort to reproduce the observations made during Experiment A or B. In Fig. 4.32, the response spectra are shown for the excitation frequencies 8.12 Hz, 8.14 Hz, and 8.16 Hz. The modulated motions commence at $f = 8.12$ Hz and grow as f is increased further. The uniform sideband structure in Figs. 4.32b and 4.32c correspond to periodically modulated motions. In Fig. 4.33, we show the Poincaré sections obtained at the excitation frequencies 8.14 Hz, 8.16 Hz, 8.18 Hz, 8.20 Hz, and 8.22 Hz. The sections indicate that the motion is two-period quasiperiodic at the first four frequencies and possibly chaotic at 8.22 Hz. Unfortunately, we could not complete this set of experiments as the structure's linear frequencies changed. However the observations made during Experiment C agree qualitatively better with the observations made during Experiment A. All three sets of experiments were conducted with the same model. The observations made during Experiment B are not in qualitative agreement with the observations made during Experiment A or C. The differences in the observations from one set of experiments to another could be due to the sensitivity of the model to the experimental conditions (temperature, humidity).

Structure III was also subjected to primary-resonant excitations of the first mode. This structure exhibited modulated motions in the frequency range $8.12 \text{ Hz} \leq f \leq 8.18 \text{ Hz}$ for excitation levels starting from 66.67 mili g's rms. All the observed modulated motions were periodically modulated except for one case, which is shown in Fig. 4.34. For this case, the excitation frequency was 8.15 Hz while the excitation amplitude was 133.33 mili g's rms, and the modulation frequency was approximately 0.185 Hz. The clusters of points in the Poincaré section indicate that the frequency of modulation and the carrier frequency are locked; that is, the ratio of the two is a rational number. We could not repeat or obtain similar results during any other experiments with this structure.

Here, during periodic responses, the feedback from the beam-mass structure to the table, as determined from the table-acceleration spectrum (the ratio of the amplitude of the largest other peak to the amplitude of the peak at the excitation frequency), was again less than -30.00 dB. However, for some excitation-parameter values at which the structure displayed large-amplitude responses, the feedback was about -25.00 dB. During large modulated motions of the structure, the fluctuations in the excitation amplitude were within a range of 2.00 mili g's rms.

The experimentally observed frequency spectra of the nonlinear periodic, periodically modulated, and chaotically modulated responses are qualitatively in agreement with the corresponding analytical predictions discussed in

Chapter 3. The form of the frequency-response curves is also in qualitative agreement with the analytical predictions given in Chapter 3. The excitation levels required to produce chaotically modulated motions during primary-resonant excitations of the first mode are seen to be higher than those required during primary-resonant excitations of the second mode. Also, in the transition to chaotically modulated motions during excitations of the first flexural mode, we did not observe any period-doubling bifurcations.

Next, we compare the experimental observations (referring to Fig. 4.23) and analytical predictions for the frequency responses of Structure II to primary-resonant excitations of the first mode. In the experiment, jumps were observed at 7.90 Hz and 8.70 Hz during the forward sweep and at 8.35 Hz and 7.80 Hz during the reverse sweep. The corresponding analytical predictions for the locations of the jumps (saddle-node bifurcation points) are 7.928 Hz and 8.930 Hz during the forward sweep and 8.438 Hz and 7.381 Hz during the reverse sweep. Comparing the analytical and experimental results, we find good agreement for jumps from small-response amplitudes to large-response amplitudes. On the other hand, the agreement for jumps from large-response amplitudes to small-response amplitudes is not as good. In the experiment, we observed modulated motions in the frequency range $8.12 \text{ Hz} \leq f \leq 8.30 \text{ Hz}$. The analysis predicts Hopf bifurcations at 8.172 Hz and 8.310 Hz, showing good agreement with the experimental results.

4.3. Subharmonic resonance of order one-half of the second mode

During secondary-resonant excitations, the present case and the following one, two plates were taken off the shaker table, so that the shaker could provide high excitation levels. Structure V, whose dimensions and properties are provided in Appendix D, was subjected to a subharmonic-resonant excitation of order one-half of the second mode. The first two linear frequencies of this structure are 8.11 Hz and 16.36 Hz and the third and higher frequencies are far separated from these two frequencies. As in the earlier cases, the sign of the detuning of the internal resonance is positive here.

In Fig. 4.35, the experimentally obtained force-response curves at a constant excitation frequency $f = 32.65$ Hz are shown. For this value of f the sign of the detuning of the external resonance is negative. For this case, the analysis predicts that the structure's forced response should not exhibit any jumps and show a gradual transition from a linear to a nonlinear response (see Fig. 3.6). The symbols a_1 and a_2 are measures of the amplitudes of the first and second flexural modes of the structure, respectively. The amplitude of the component of the response at the excitation frequency is not shown in the figure. Circles (triangles) are used in Fig. 4.35 to mark the observations made during forward (reverse) sweeps. As we increased the excitation amplitude, the response remained linear (i.e., $a_1 = a_2 = 0$) until the excitation amplitude reached the

value 1.700 g's rms. At this excitation level, the linear response became unstable and a nonlinear periodic response, marked by the presence of spectral lines at $f/4$ and $f/2$, was observed. When we examine the response of the structure (see Fig. 4.35) at this excitation level, we see a large change in the magnitude of a_i . This indicates that the structure has experienced a jump. For excitation levels running from 1.70 g's rms to 1.775 g's rms, the value of a_2 remained more or less constant (i.e., it saturated) while the value of a_i increased in a nonlinear manner. As the excitation level was increased further to 1.8 g's rms, the nonlinear periodic response lost stability and periodically modulated motions were observed. Using the Fourier spectrum and Poincaré section, we ascertained that the periodically modulated motions were two-period quasiperiodic motions. These types of motion were observed all the way up to excitation levels of 2.0 g's rms. During reverse sweeps of the excitation amplitude, nonlinear periodic motions marked by spectral lines at $f/4$, $f/2$, and f (a fractional-harmonic pair; Nayfeh and Mook, 1979) were observed up to excitation levels of 1.65 g's rms. These nonlinear responses can be referred to as two-mode responses because two modes (the first and second modes) participate in the response. For the excitation levels ranging from 1.65 g's rms to 1.8 g's rms, a_2 remained more or less at a constant value while a_i varied monotonically in a nonlinear manner. So, in this region we observed a modal saturation. As the excitation amplitude was decreased further to 1.63 g's rms, a new form of response was observed. This response was nonlinear periodic and was characterized by spectral lines at $f/2$ and f . Because a single mode (the second mode) participates in the response, this

response can be called a single-mode response. It was observed down to excitation levels of 1.4 g's rms. In the range of excitation amplitudes running from 1.63 g's rms to 1.4 g's rms, a_2 decreased monotonically while a_1 assumed a trivial value. For excitation levels below 1.4 g's rms, we only observed linear responses. These responses were characterized by a spectral line at f .

The experimental observations of nonlinear responses beyond a certain threshold value of excitation amplitude, modal saturation, and breakdown of modal saturation leading to periodically modulated motions are in qualitative agreement with the analytical predictions. However, the jump in the response during the forward sweep from a linear to a nonlinear response and observations of nonlinear responses characterized by spectral lines at $f/2$ and f are not in accordance with the present analytical predictions. The experiments indicate that there are three types of responses: linear response ($a_1 = a_2 = 0$), single-mode response ($a_1 = 0$ and $a_2 \neq 0$), and two-mode response (a_1 and $a_2 \neq 0$). The current analysis can only predict stable linear and stable two-mode responses. Hence, further analyses and experiments are necessary to settle these differences between theory and experiment.

In Fig. 4.36, the response spectra, the excitation spectra, and the corresponding x-y plots of the strain-gage signals are presented for the two types of nonlinear response observed in the experiments. The results in Fig. 4.36a correspond to a nonlinear periodic response characterized by spectral lines at f and $f/2$, and those in Fig. 4.36b correspond to a nonlinear periodic

response characterized by spectral lines at $f/4$, $f/2$, and f . The response shown in Fig. 4.36a is not predicted by the analysis, but the response shown in Fig. 4.36b is. In Fig. 4.37, we show the response spectrum and corresponding Poincaré section for the periodically modulated response observed in the experiment at the excitation level of 1.95 g's rms. The observations shown in Fig. 4.37 are also typical of other observed periodically modulated responses. The response spectrum is shown in a zoom span (resolution 0.002 Hz) about the spectral line at $f/2$. From the uniform sideband structure in the response spectrum around $f/2$ and the loops of points in the Poincaré section, we conclude that the motion is a two-period quasiperiodic motion. The two loops of points in the Poincaré section indicate that the selected section intersects the two torus (attractor for a two-period quasiperiodic motion) at two locations.

4.4. Combination resonance of the additive type

Structure IV, whose dimensions and properties are provided in Appendix D, was subjected to combination-resonant excitations of the additive type. The first two linear frequencies of this structure are 8.11 Hz and 16.41 Hz and the third and higher frequencies are far separated from these two frequencies. As in the earlier cases, the sign of the detuning of the internal resonance is positive.

In Fig. 4.38, the experimentally obtained force-response curves at a constant excitation frequency of 24.2 Hz are shown. The symbols a_1^* and a_2^* correspond to the amplitudes of the first and second flexural modes, respectively. The amplitude of the component of the response at the excitation frequency is not shown in the figure. Circles (triangles) are used in Fig. 4.38 to mark the observations made during forward (reverse) sweeps. As we increased the excitation amplitude, the response remained linear (i.e., $a_1^* = a_2^* = 0$) until the excitation amplitude exceeded 2.35 g's rms where the linear response became unstable and nonlinear motions ensued. These nonlinear motions were found to be two-period quasiperiodic motions from the Poincaré sections. As the excitation amplitude was increased further, the modulations remained periodic and grew in size. However, we could not continue the experiment beyond the excitation level of 2.45 g's rms as it was not possible to maintain a constant excitation level.

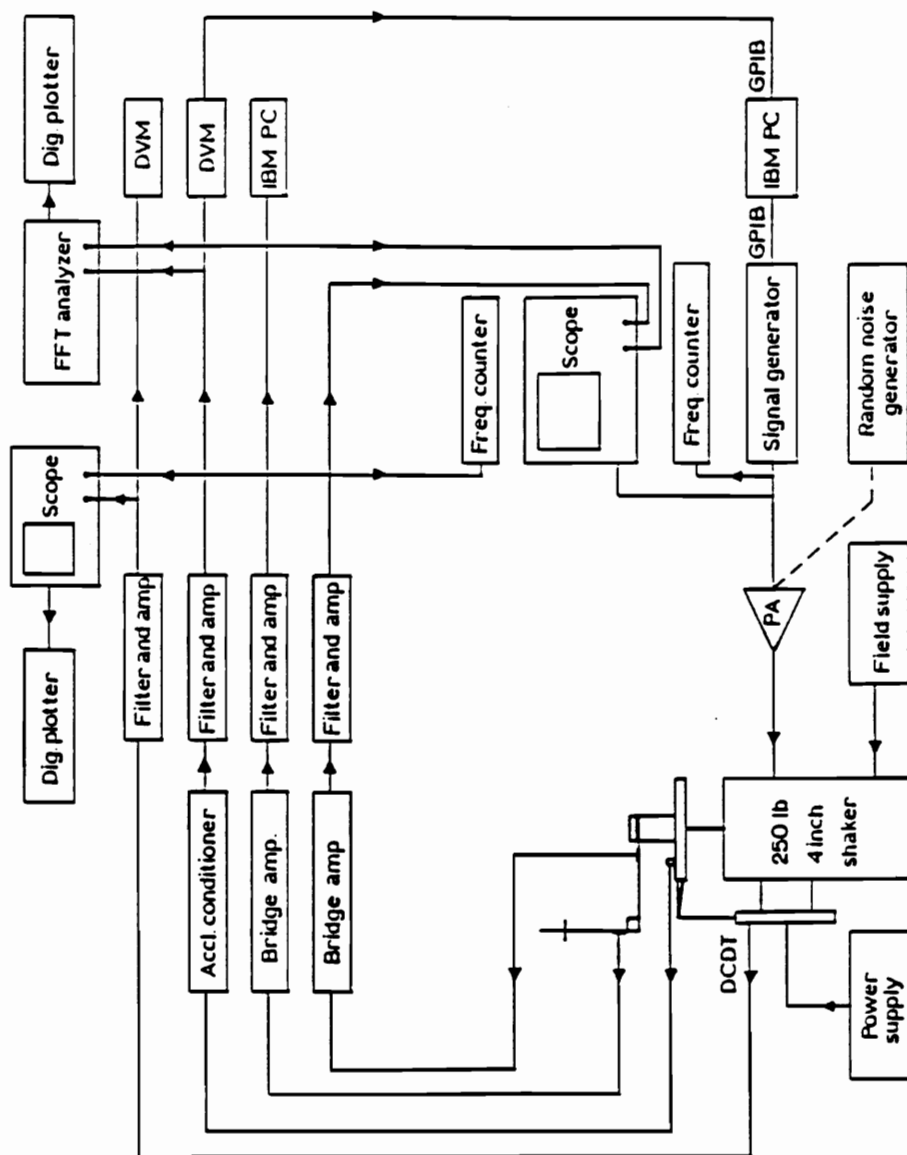
At this stage, we introduced a disturbance that resulted in the nonlinear periodic response of the structure and then we gradually decreased the excitation amplitude. During these nonlinear responses, the Poincaré sections consisted of three points and the response spectrum had peaks at $f/3$, $2f/3$, and f (a fractional-harmonic pair; Nayfeh and Mook, 1979). Initially, a_2^* decreased steadily, but below an excitation amplitude of 1.5 g's rms it saturated and remained more or less constant until the excitation amplitude decreased below 0.85 g's rms where the saturated response lost stability through a jump to the linear response (i.e., $a_1^* = a_2^* = 0$). In Fig. 4.39, the response spectra, the

excitation spectra, and the corresponding x-y plots of the strain-gage signals are shown for a linear response and a periodic nonlinear response at 24.2 Hz and 1.3 g's rms.

In another experiment, we examined the frequency response of the structure by varying the excitation frequency in the range $24.0 \text{ Hz} \leq f \leq 25.0 \text{ Hz}$ at the constant excitation amplitude 2.0 g's rms. As the frequency was increased from 24.00 Hz, the linear response lost stability at 24.26 Hz; two-period quasiperiodic motions developed and existed until 24.455 Hz. Chaotically modulated motions were observed in the range $24.458 \text{ Hz} \leq f \leq 24.468 \text{ Hz}$. Then, the structure exhibited periodically modulated motions at 24.47 Hz, which continued until 24.52 Hz. Linear responses were observed again at 24.53 Hz. In the case of linear responses, only a spectral line was present at the excitation frequency in the response spectrum. In Fig. 4.40 the response spectra in a baseband of 10.0 Hz and 40.0 Hz are shown for excitation frequencies of 24.45 Hz and 24.458 Hz; 1280 lines of resolution were used in each baseband. At 24.45 Hz, the uniformly spaced sidebands around the carrier frequencies $f/3$ and $2f/3$ indicate a periodically modulated response while at 24.458 Hz the broadband character around the carrier frequencies is indicative of a chaotically modulated response. The transition to the chaotic motion at $f = 24.458 \text{ Hz}$ via the quasiperiodic motion is shown in Fig. 4.41 by means of the Poincaré sections. The first three sections in the figure indicate that the motion is periodically modulated at excitation frequencies of 24.4 Hz, 24.45 Hz, and 24.455 Hz. In Fig. 4.42, the Poincaré sections obtained at

excitation frequencies of 24.465 Hz, 24.47 Hz, 24.48 Hz, and 24.51 Hz are shown. All these sections were obtained during the forward sweep of the excitation frequency. The section at $f = 24.465$ Hz suggests that the corresponding motion may be chaotically modulated. The sections at the other excitation frequencies in Fig. 4.42 indicate that Structure IV's motion is periodically modulated at the respective excitation frequencies.

All the experimental observations made during combination-resonant excitations are in qualitative agreement with the analytical predictions. During both subharmonic and combination excitations, the excitation was clean and there was hardly any feedback from the structure to the table. Typically, in all cases, the motion takes a long time (sometimes as much as 45 minutes or more) to settle down at places where a bifurcation (qualitative change) occurs in the response. In particular, during modulated motions the response takes a long time to settle down and the response drifts from an old form of response to a new form of response during the transient phase. The Poincaré sections were used to determine if a modulated response had reached a steady state or not.



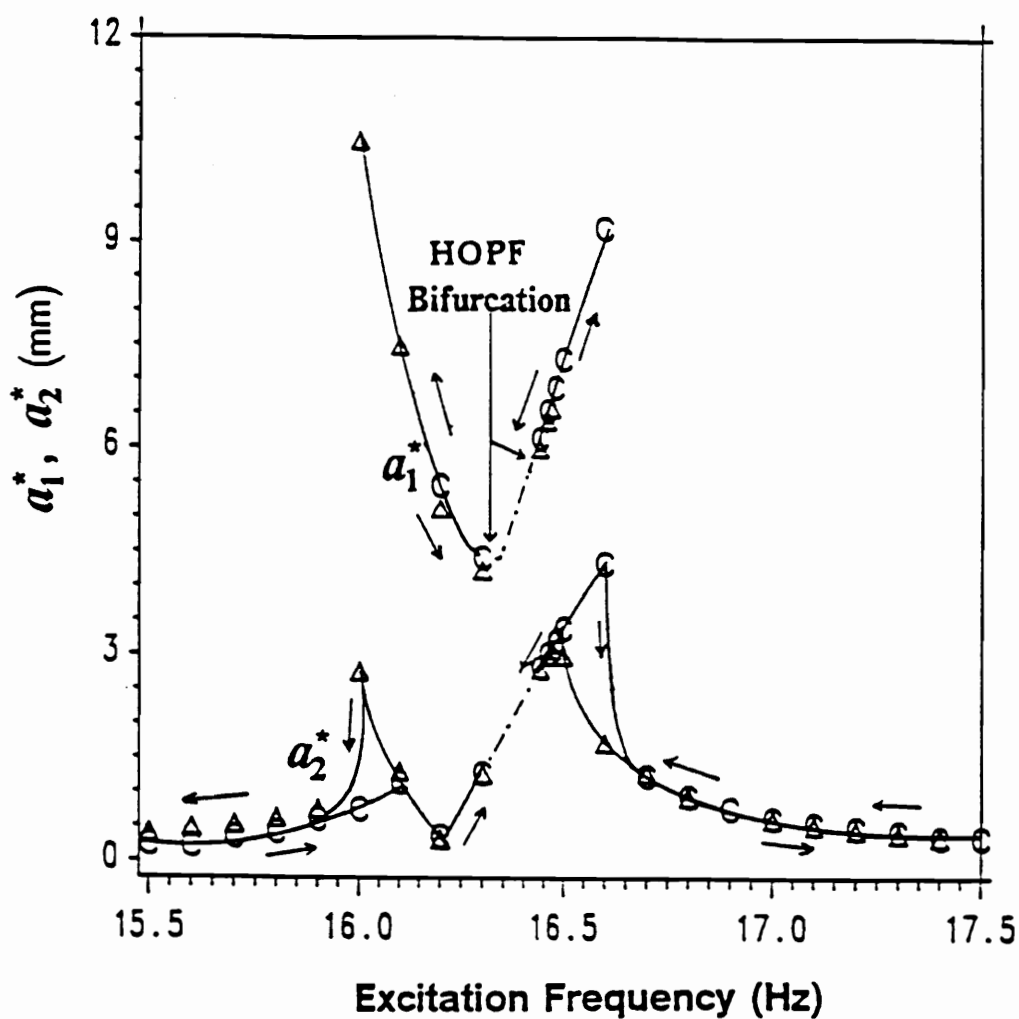


Figure 4.2. Experimentally obtained frequency-response curves of Structure I during primary resonance of the second mode: C's (triangles) correspond to experimental observations made during forward (reverse) sweep of the control parameter; solid (dotted) lines represent periodic (modulated) motions.

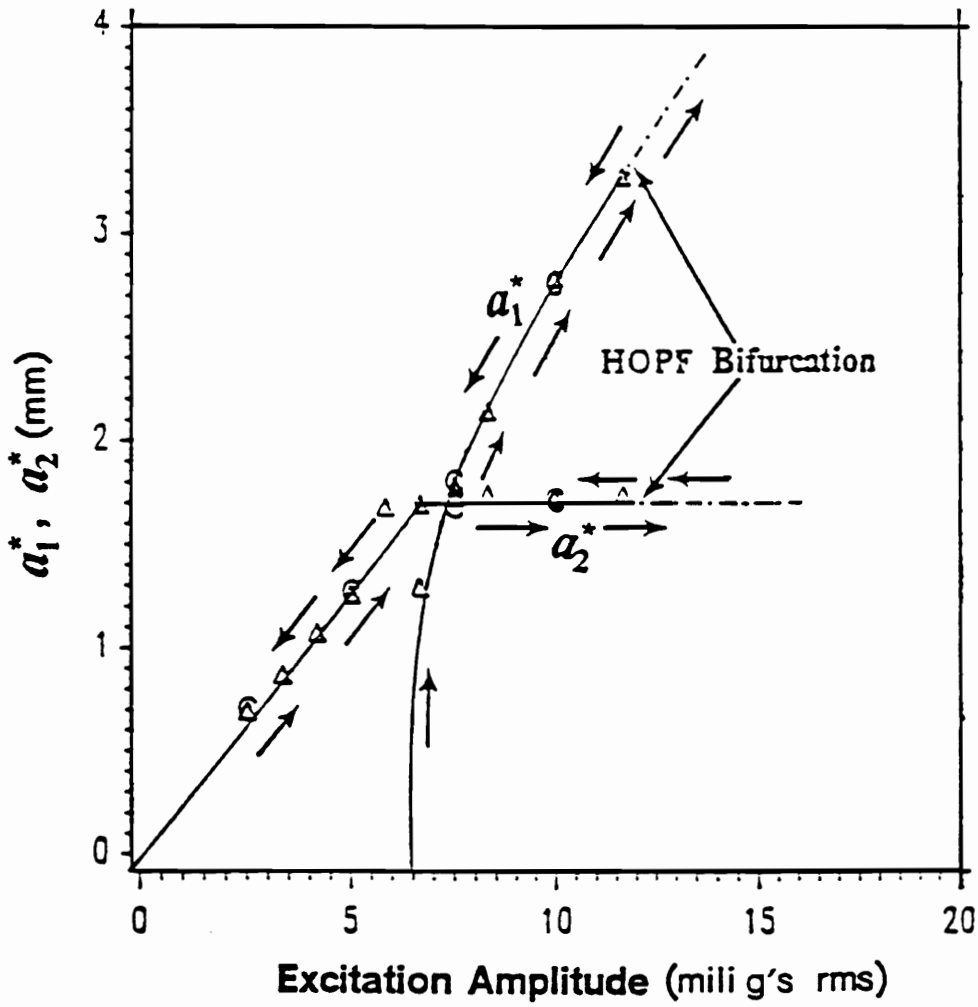


Figure 4.3. Experimentally obtained force-response curves of Structure I during primary resonance of the second mode: C's (triangles) correspond to experimental observations made during forward (reverse) sweep of the control parameter; solid (dotted) lines represent periodic (modulated) motions.

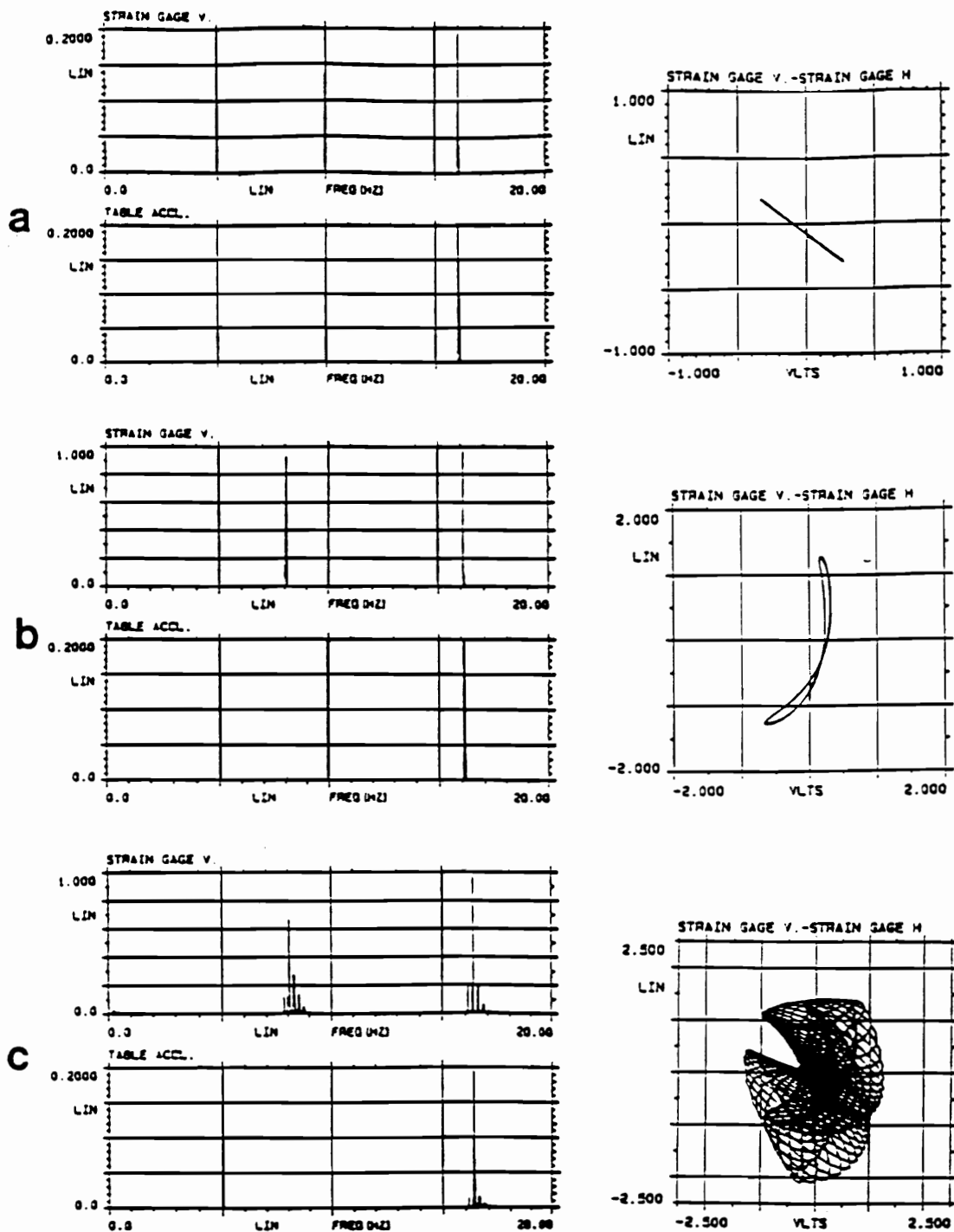


Figure 4.4. Spectra of the response and excitation and corresponding cross plots of strain-gage signals: a) $f = 16.00$ Hz, linear periodic response, b) $f = 16.10$ Hz, nonlinear periodic response, and c) $f = 16.36$ Hz, modulated response.

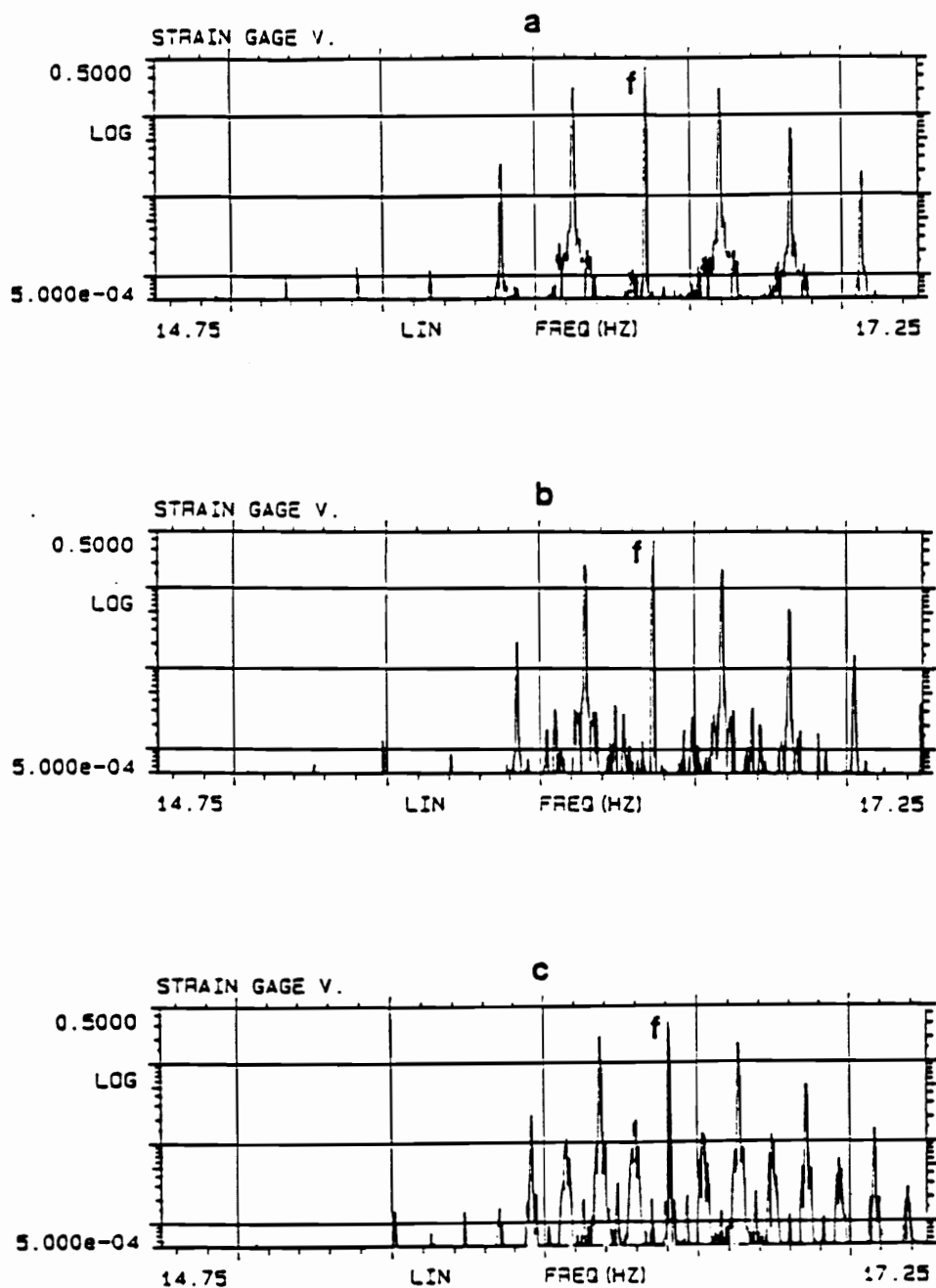


Figure 4.5. Response spectrum of Structure II on a log scale in a zoom span: a) $f = 16.36$ Hz, b) $f = 16.37$ Hz, and c) $f = 16.41$ Hz. We note the appearance of the subharmonics of the modulation frequency as we proceed from a) to b) to c).

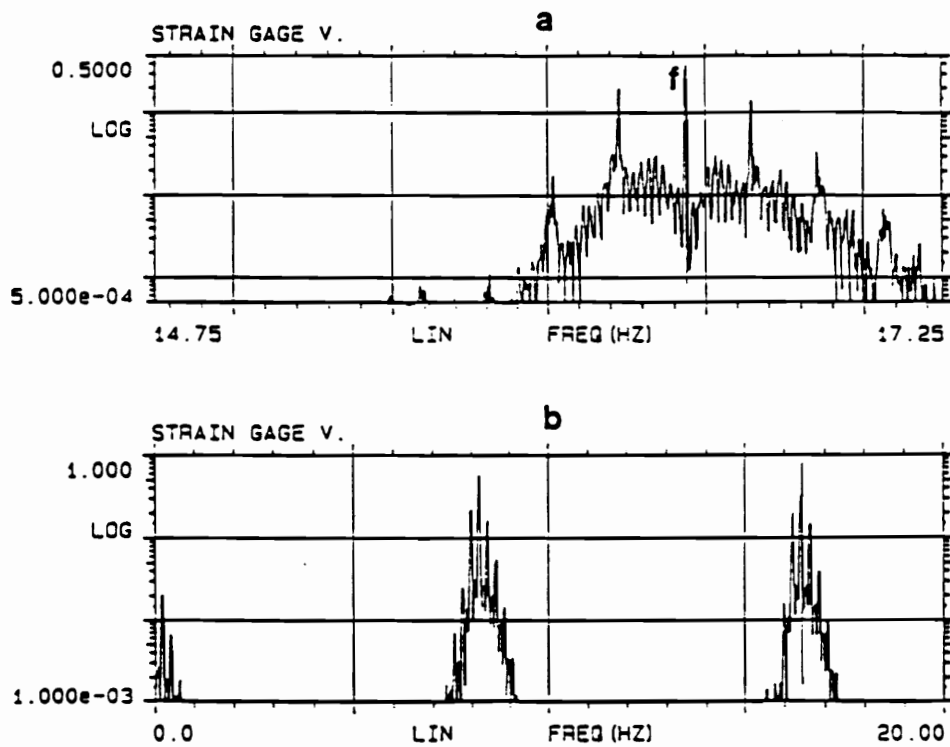


Figure 4.6. Response spectrum of Structure II at $f = 16.44$ Hz: a) zoom span of 2.5 Hz and b) baseband of 20.0 Hz. We note a broadband character around the carrier frequencies in a) and b).

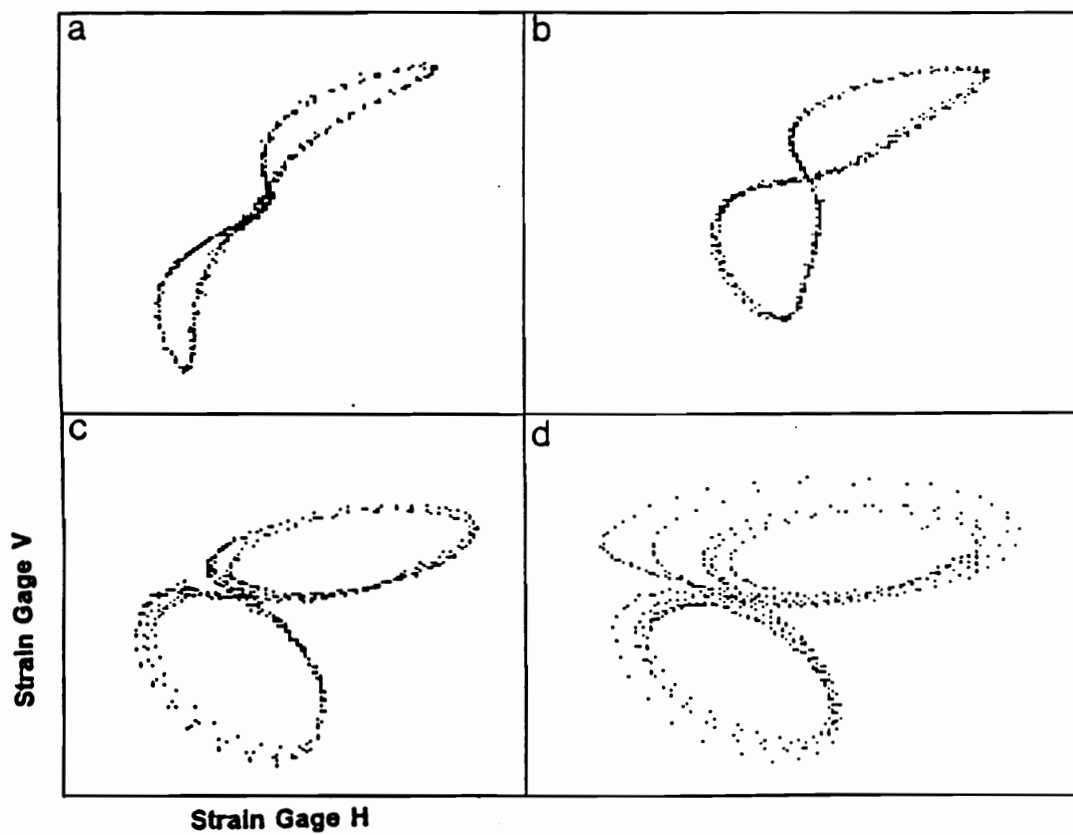


Figure 4.7. Poincaré sections of the modulated responses of Structure II: a) $f = 16.36$ Hz, b) $f = 16.40$ Hz, c) $f = 16.43$ Hz, and d) $f = 16.44$ Hz.

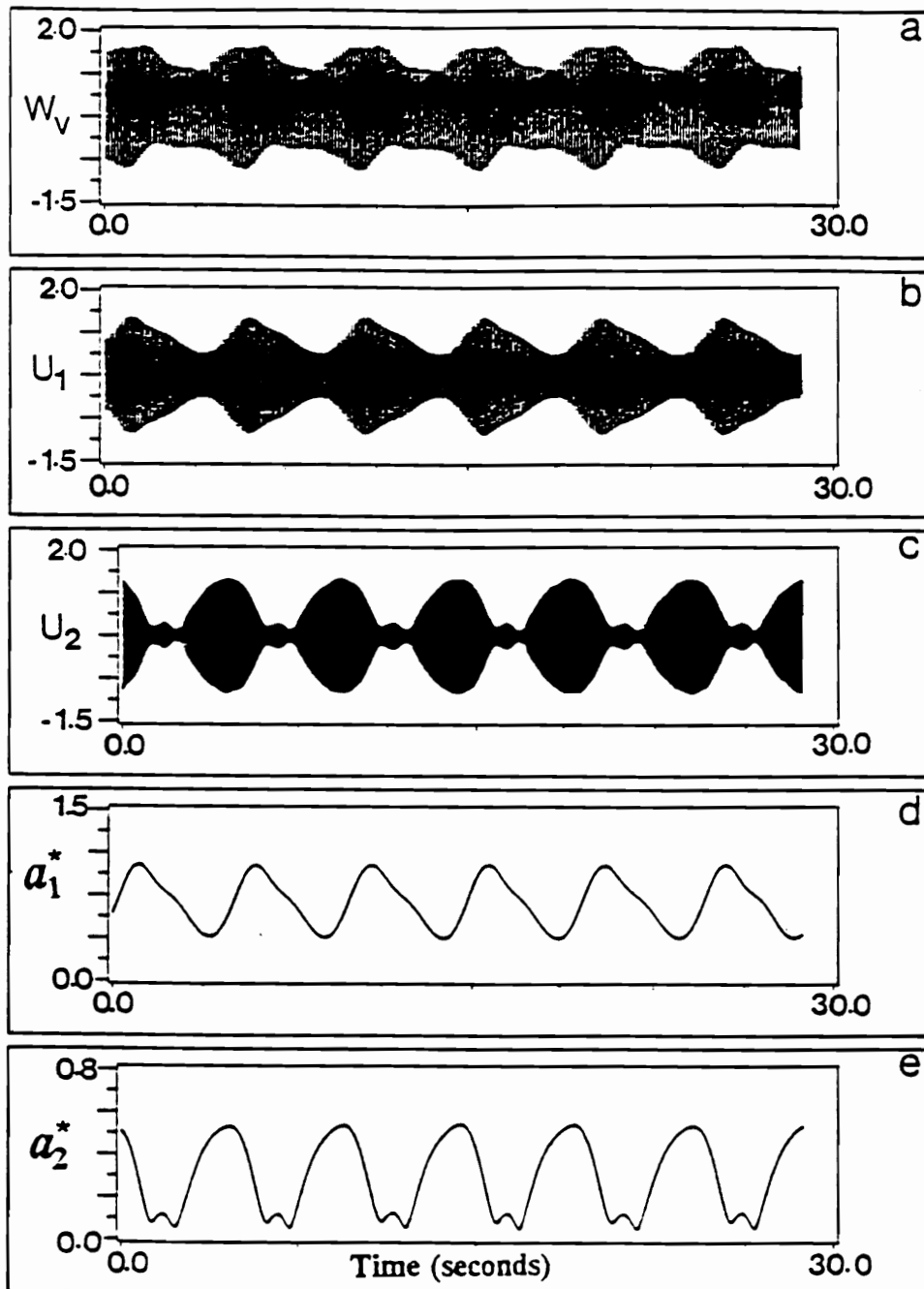


Figure 4.8. Time traces for motion at $f = 16.36$ Hz: a) signal collected from Strain Gage V, b) first mode, c) second mode, d) modal amplitude a_1^* , and e) modal amplitude a_2^* .

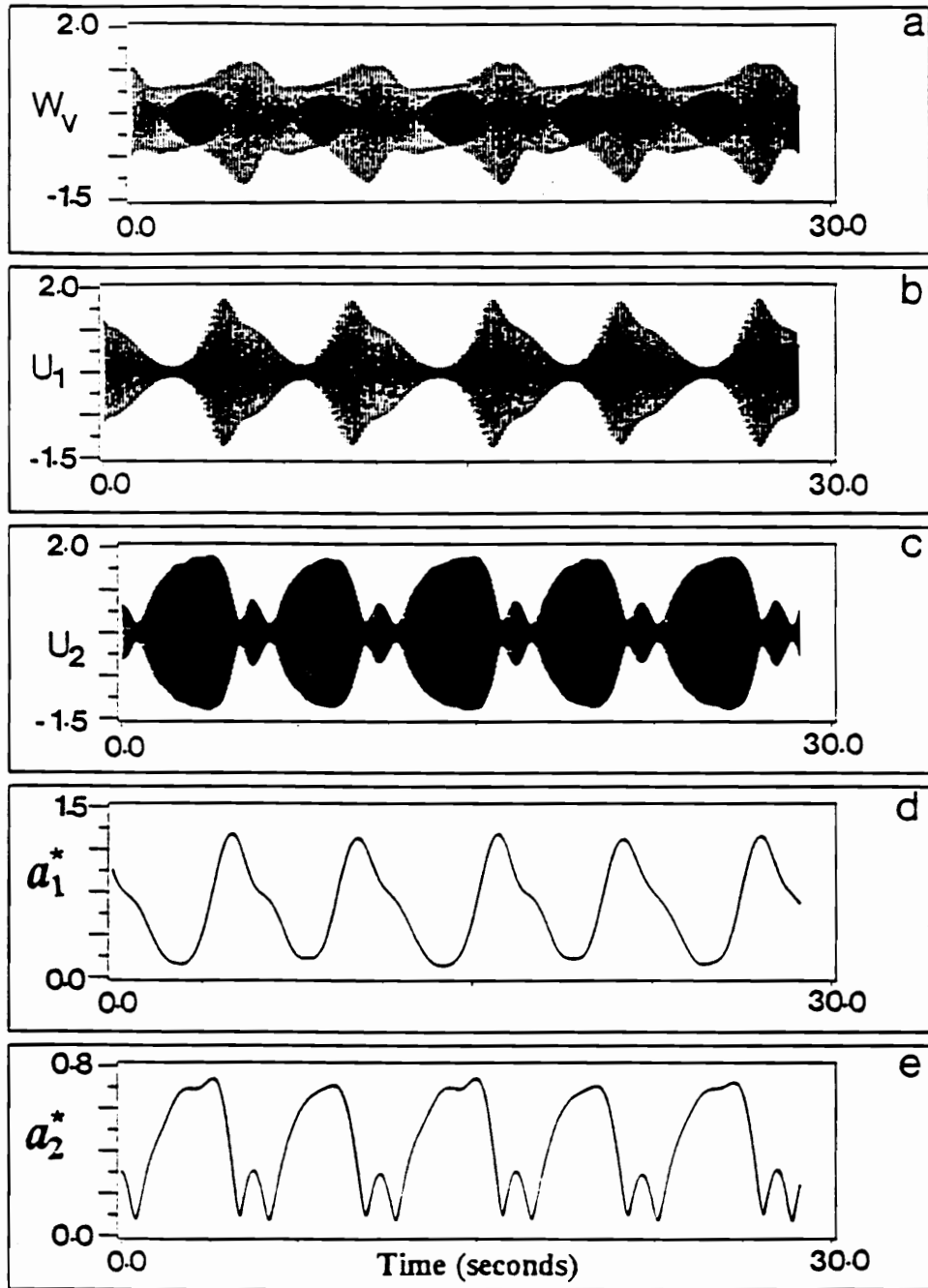


Figure 4.9. Time traces for motion at $f = 16.37$ Hz: a) signal collected from Strain Gage V, b) first mode, c) second mode, d) modal amplitude a_1^* , and e) modal amplitude a_2^* .

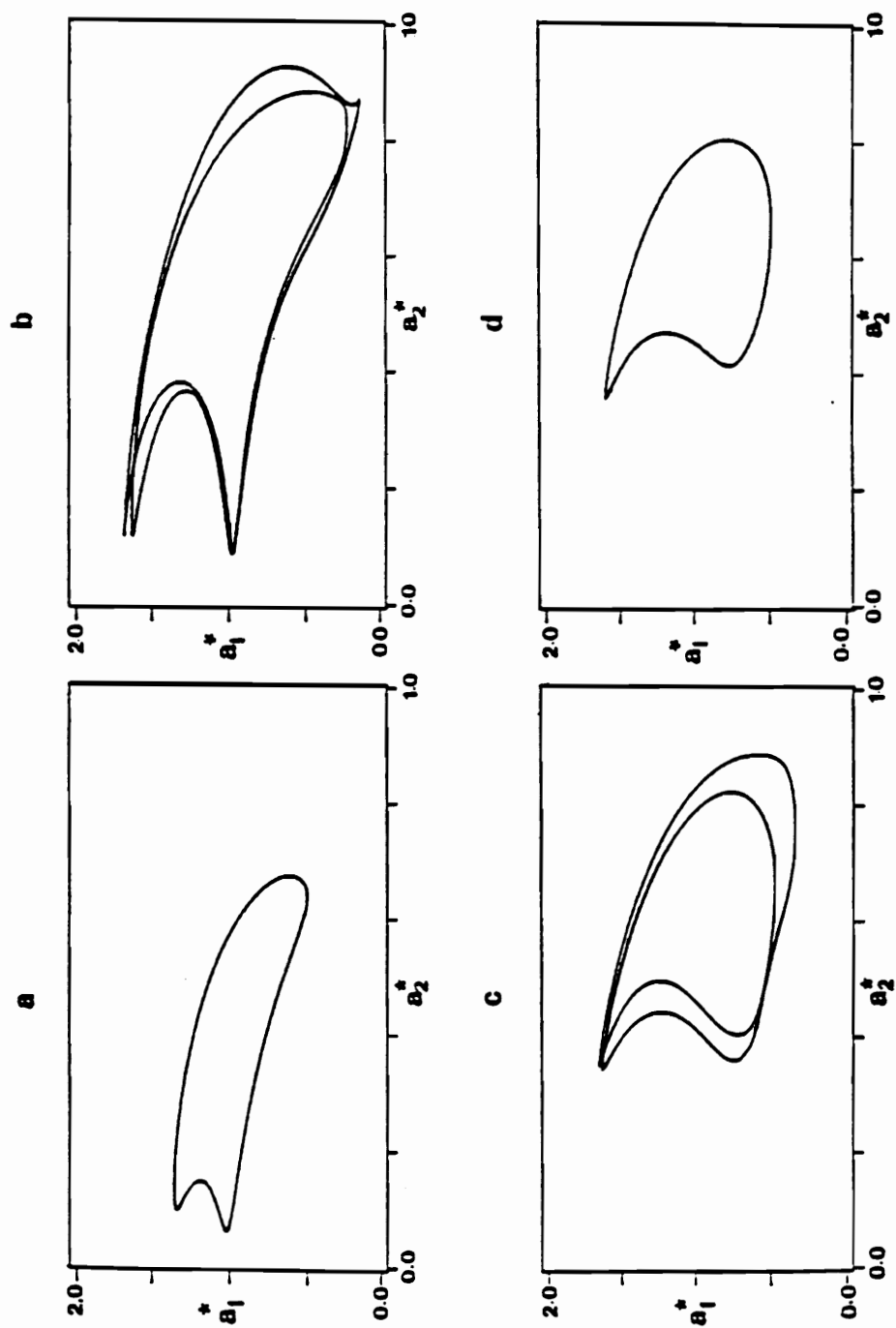


Figure 4.10. Projection of the asymptotic state of the motion in the a_1^* - a_2^* plane:
a) $f = 16.36$ Hz, b) $f = 16.46$ Hz, c) $f = 16.37$ Hz, and d) $f = 16.47$ Hz.

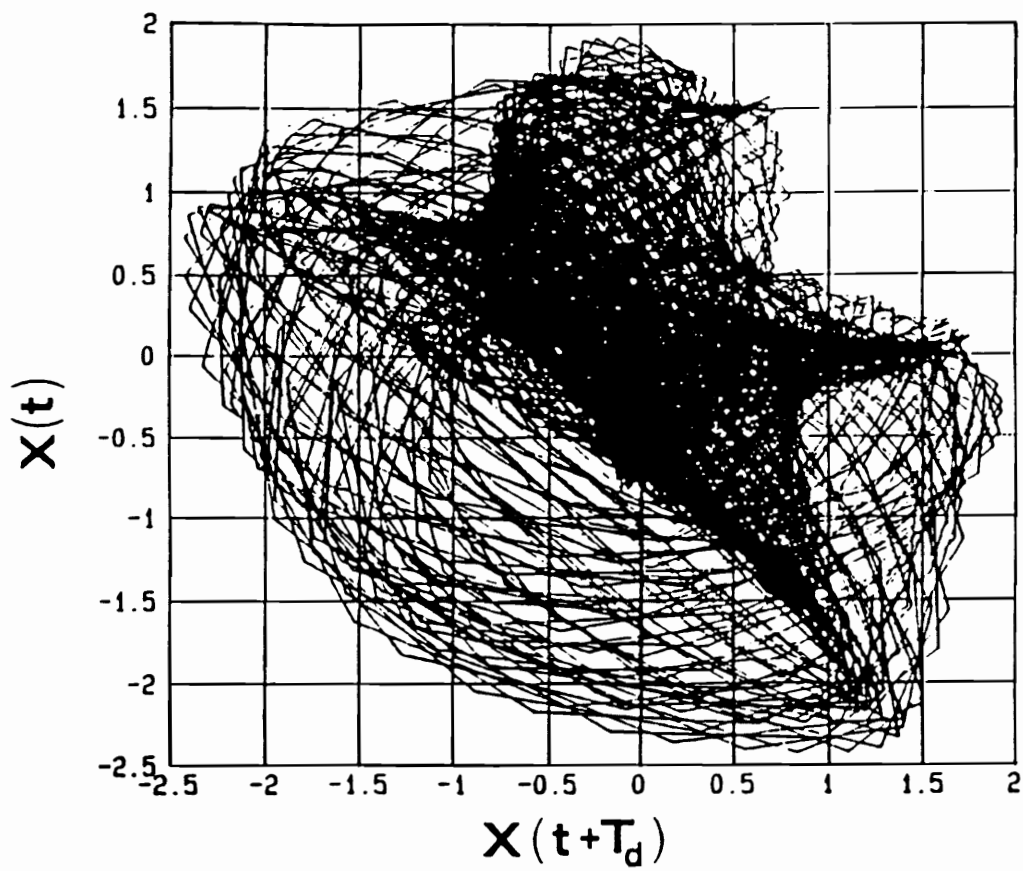


Figure 4.11. Projection of the chaotic motion at $f = 16.44$ Hz in the pseudo-phase plane.

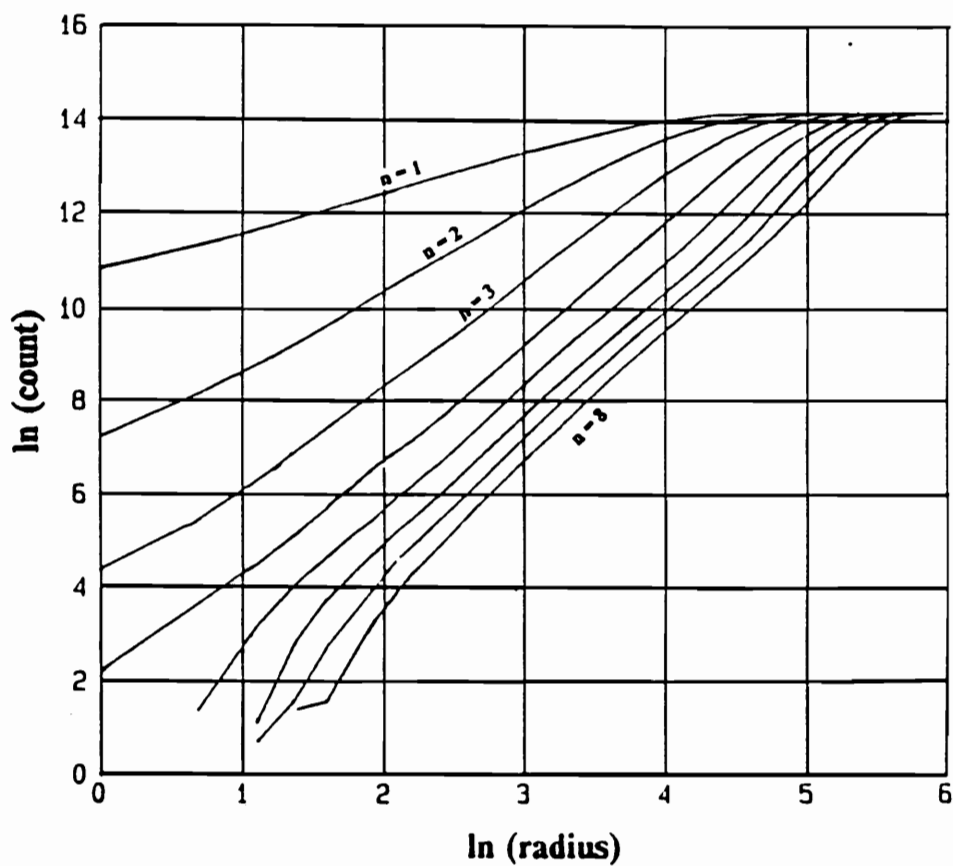


Figure 4.12. Plot for pointwise dimension for different values of the embedding dimension for the motion of Structure II at $f = 16.44$ Hz.

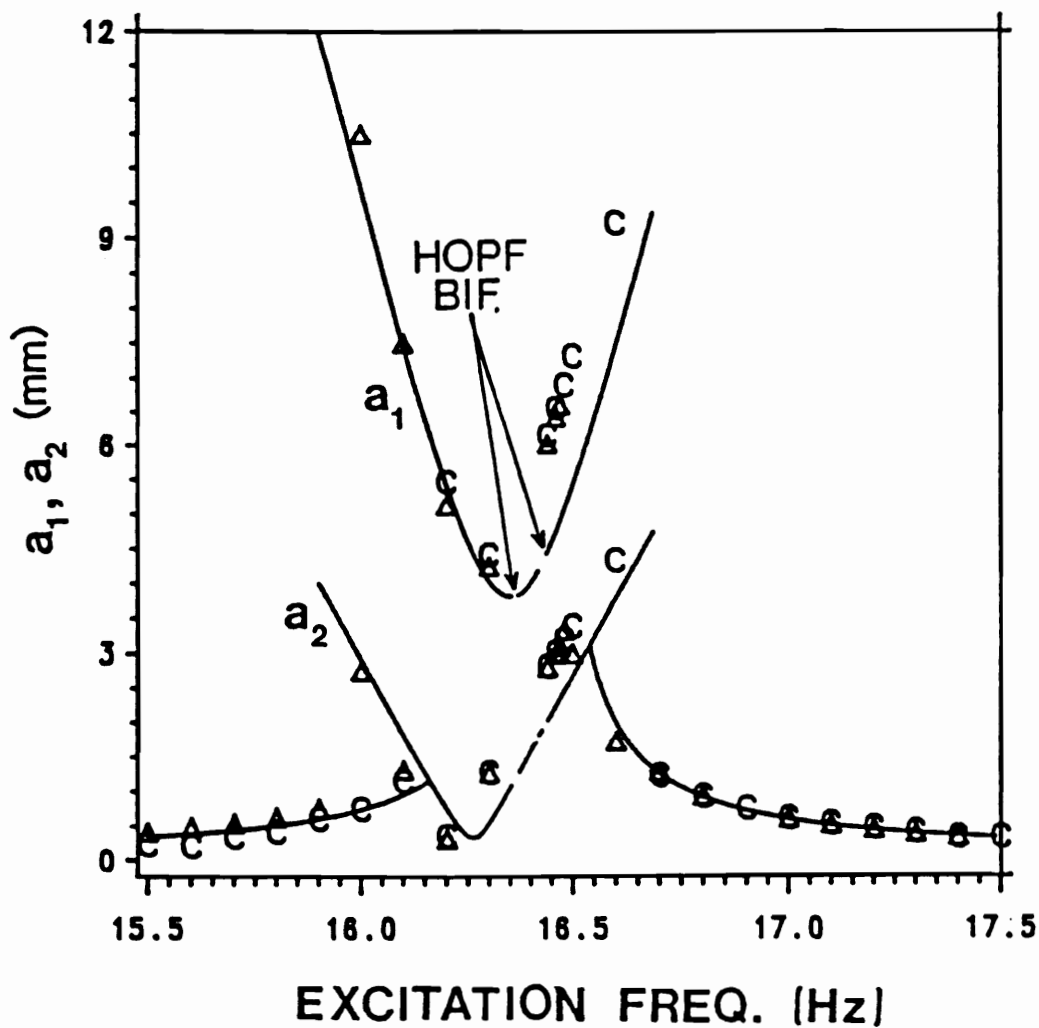


Figure 4.13. Comparison between theory and experiment for the frequency response of Structure I to primary-resonant excitations of the second mode: for the analysis $C_1 = 4.85$ and $C_2 = 1.04$; C 's (triangles) correspond to experimental observations made during forward (reverse) sweep of the control parameter.

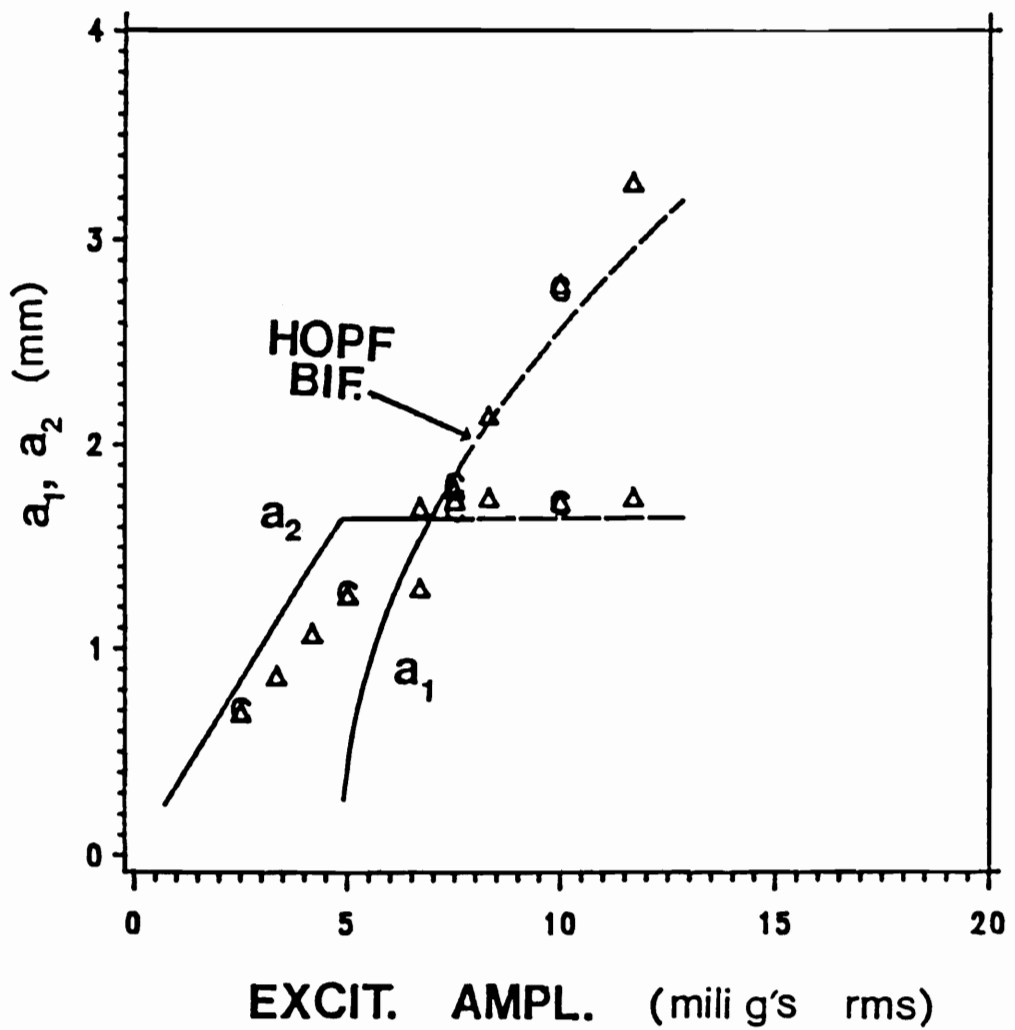


Figure 4.14. Comparison between theory and experiment for the force response of Structure I to primary-resonant excitations of the second mode: for the analysis $C_1 = 4.85$ and $C_2 = 1.25$; C's (triangles) correspond to experimental observations made during forward (reverse) sweep of the control parameter.

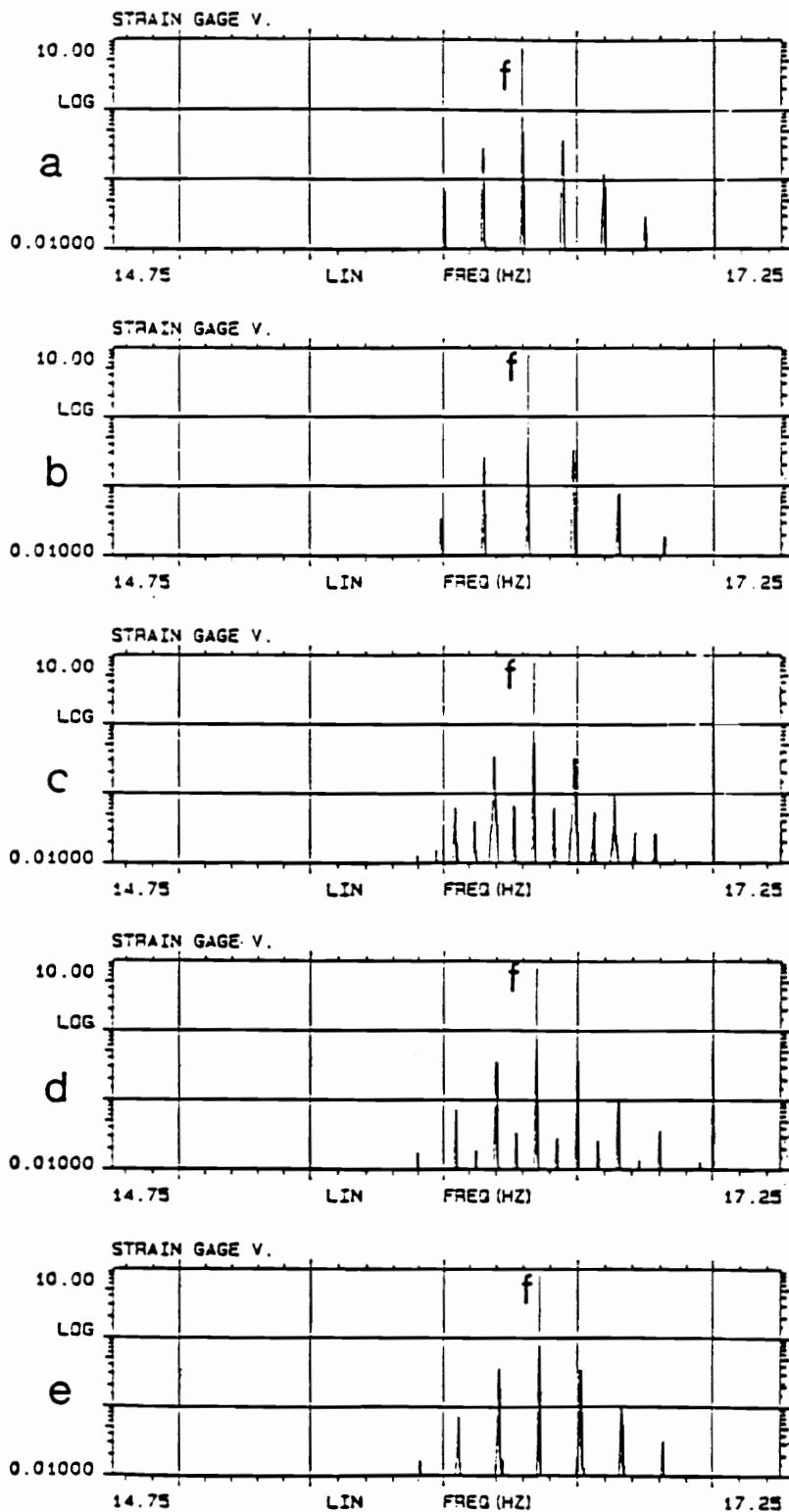


Figure 4.15. Response spectrum of Structure III on a log scale in a zoom span: a) $f = 16.30$ Hz, b) $f = 16.32$ Hz, c) $f = 16.34$ Hz, d) $f = 16.35$ Hz, and e) $f = 16.36$ Hz.

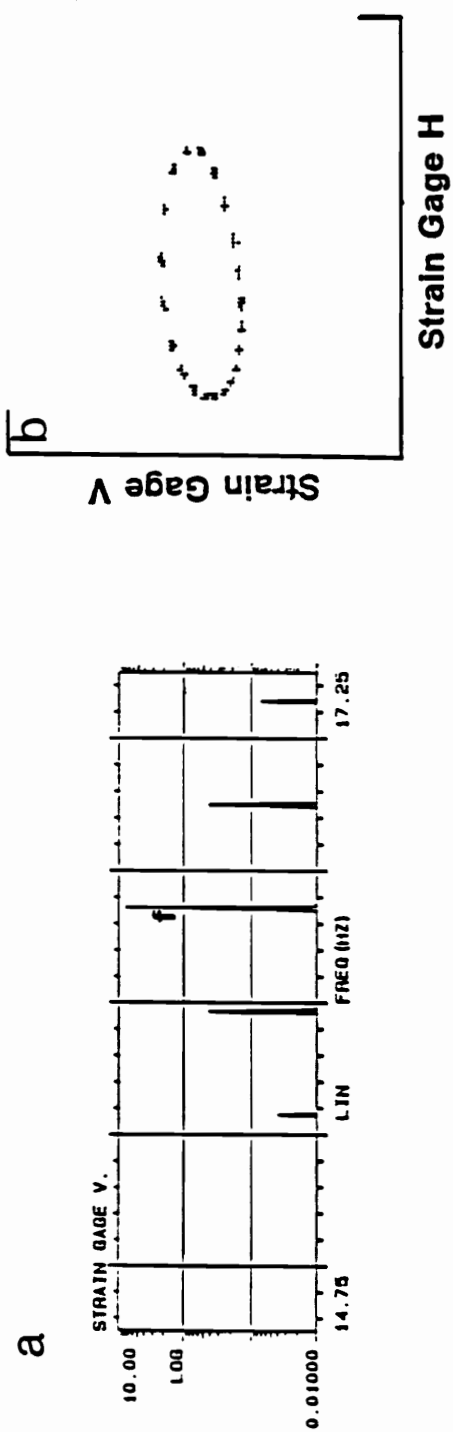


Figure 4.16. Response spectrum in a zoom span and Poincaré section for motion at $f = 16.36$ Hz and $F = 166.67$ mill g's rms.

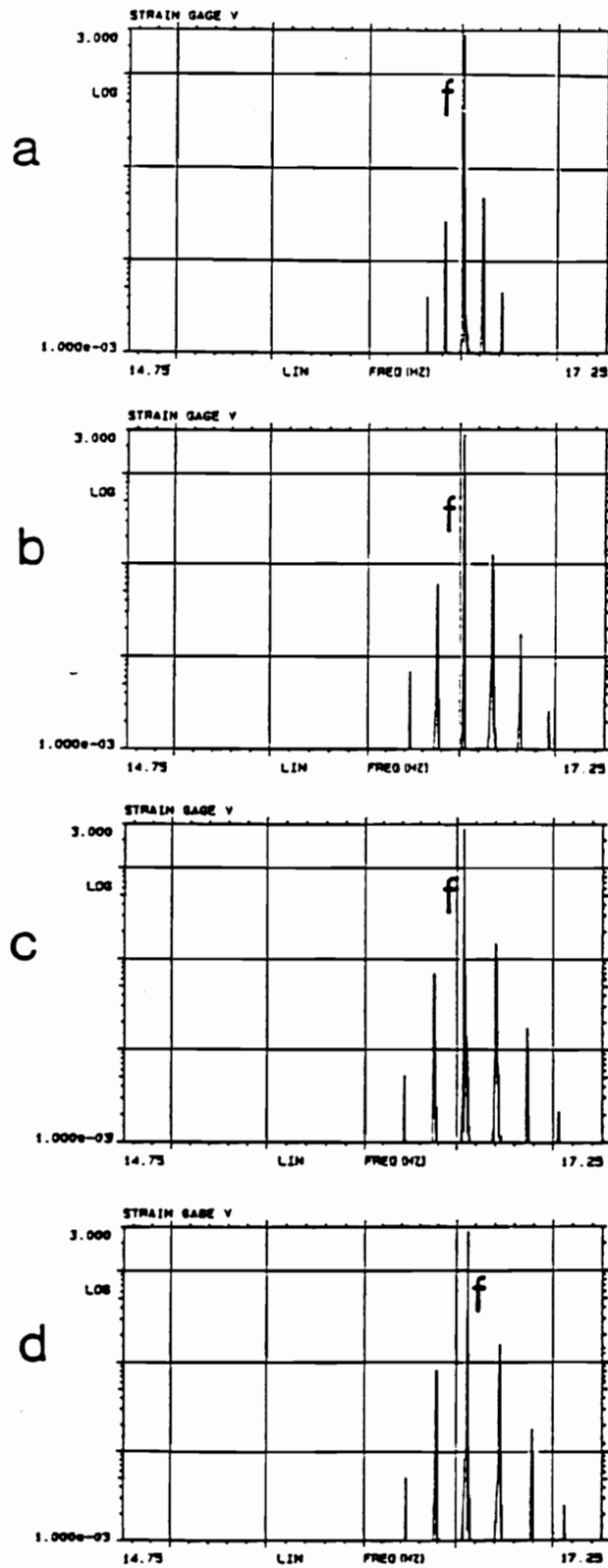


Figure 4.17. Response spectrum of Structure VI on a log scale in a zoom span:
a) $f = 16.515$ Hz, b) $f = 16.525$ Hz, c) $f = 16.54$ Hz, and d) $f = 16.56$ Hz.

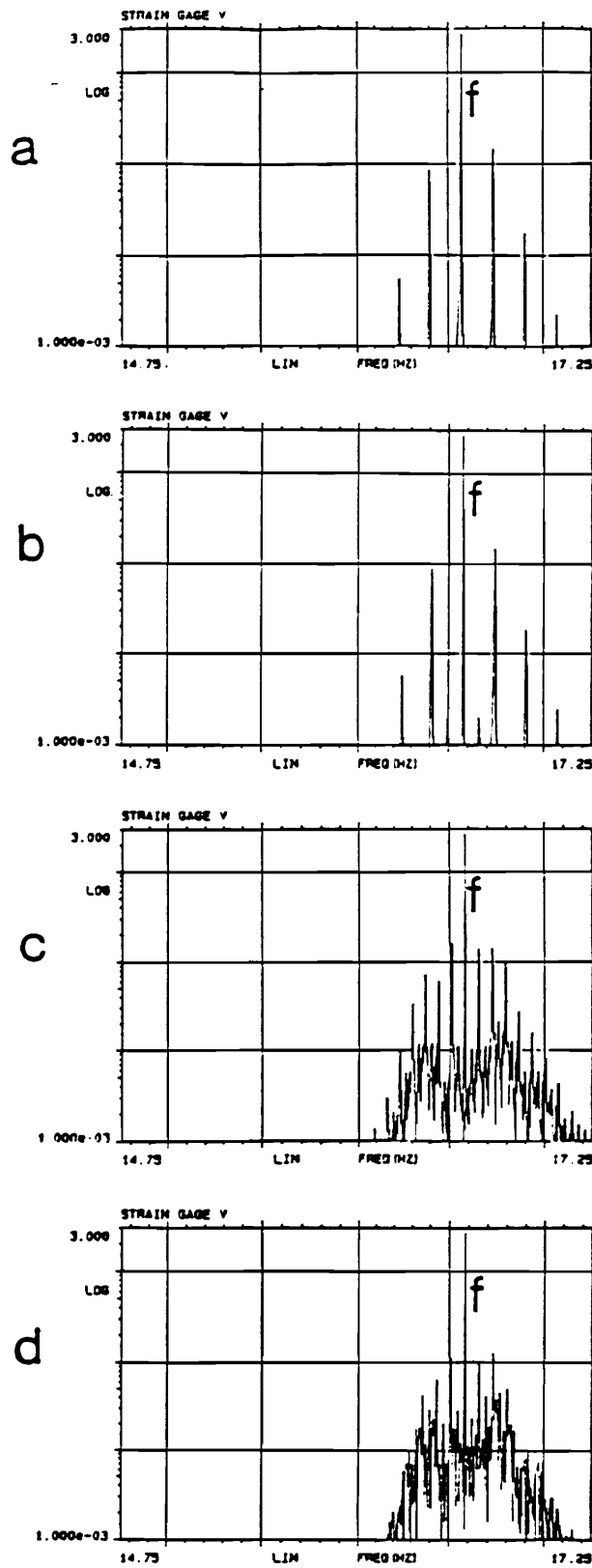


Figure 4.18. Response spectrum of Structure VI on a log scale in a zoom span:
a) $f = 16.57$ Hz, b) $f = 16.576$ Hz, c) $f = 16.582$ Hz, and d) $f = 16.584$ Hz.

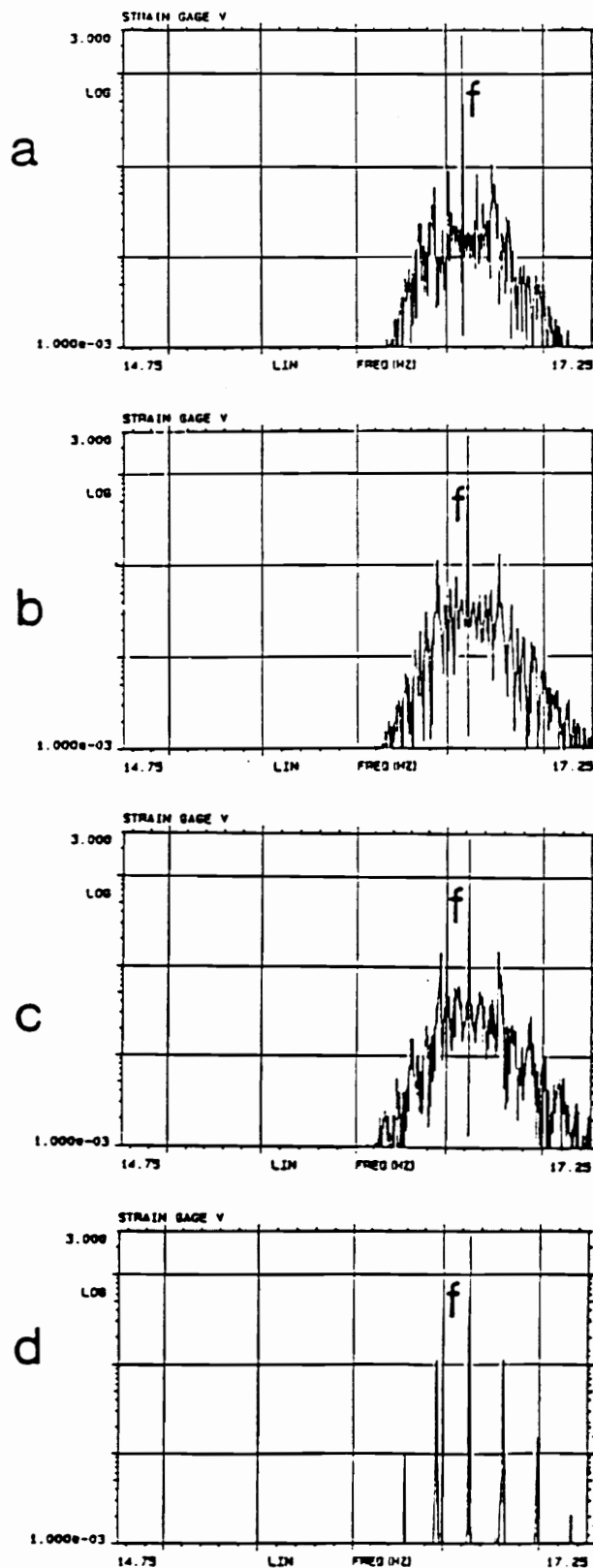


Figure 4.19. Response spectrum of Structure VI on a log scale in a zoom span:
a) $f = 16.586$ Hz, b) $f = 16.61$ Hz, c) $f = 16.62$ Hz, and d) $f = 16.64$ Hz.

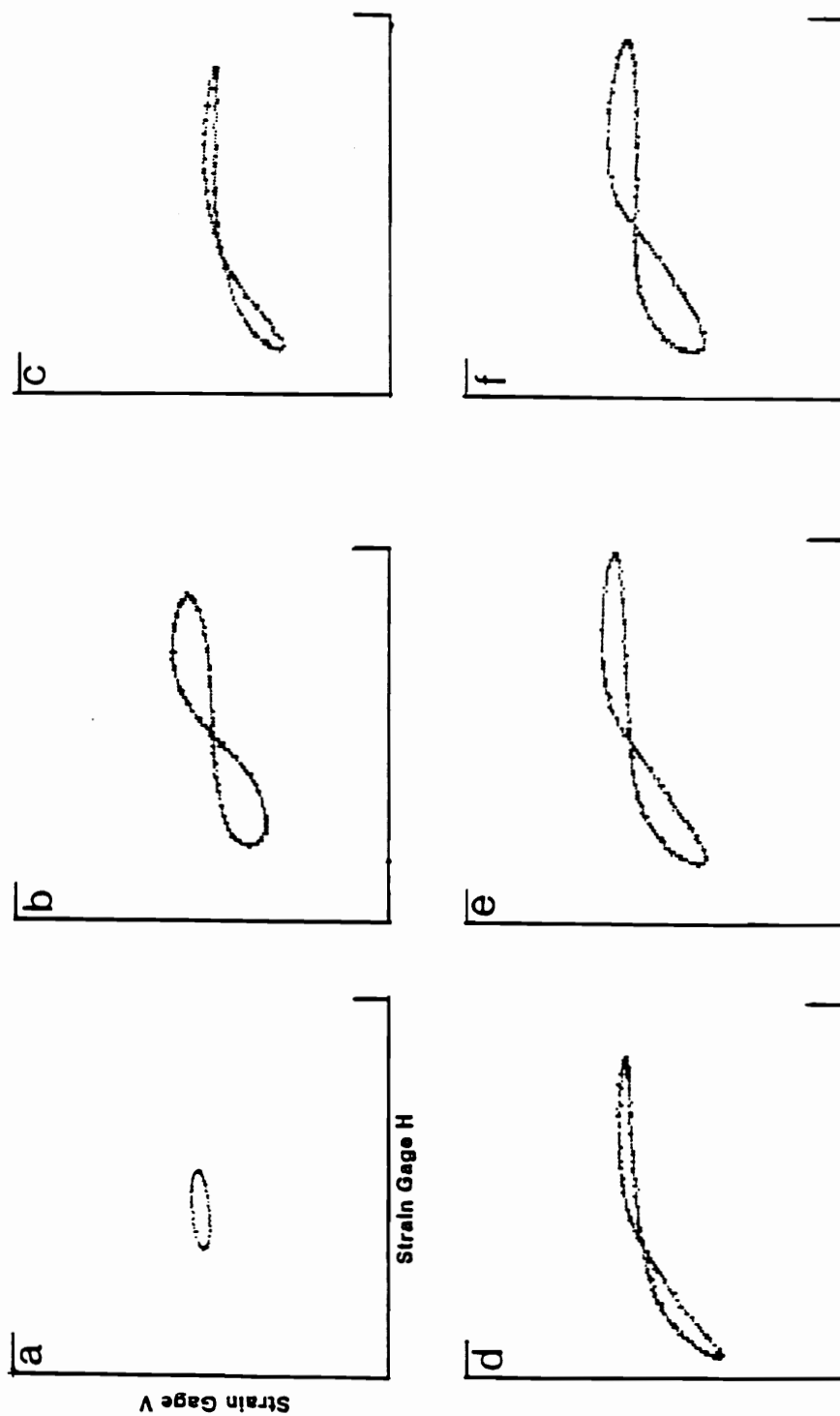


Figure 4.20. Poincaré sections of the modulated responses of Structure VI: a) $f = 16.515$ Hz, b) $f = 16.525$ Hz, c) $f = 16.54$ Hz, d) $f = 16.56$ Hz, e) $f = 16.57$ Hz, and f) $f = 16.576$ Hz.

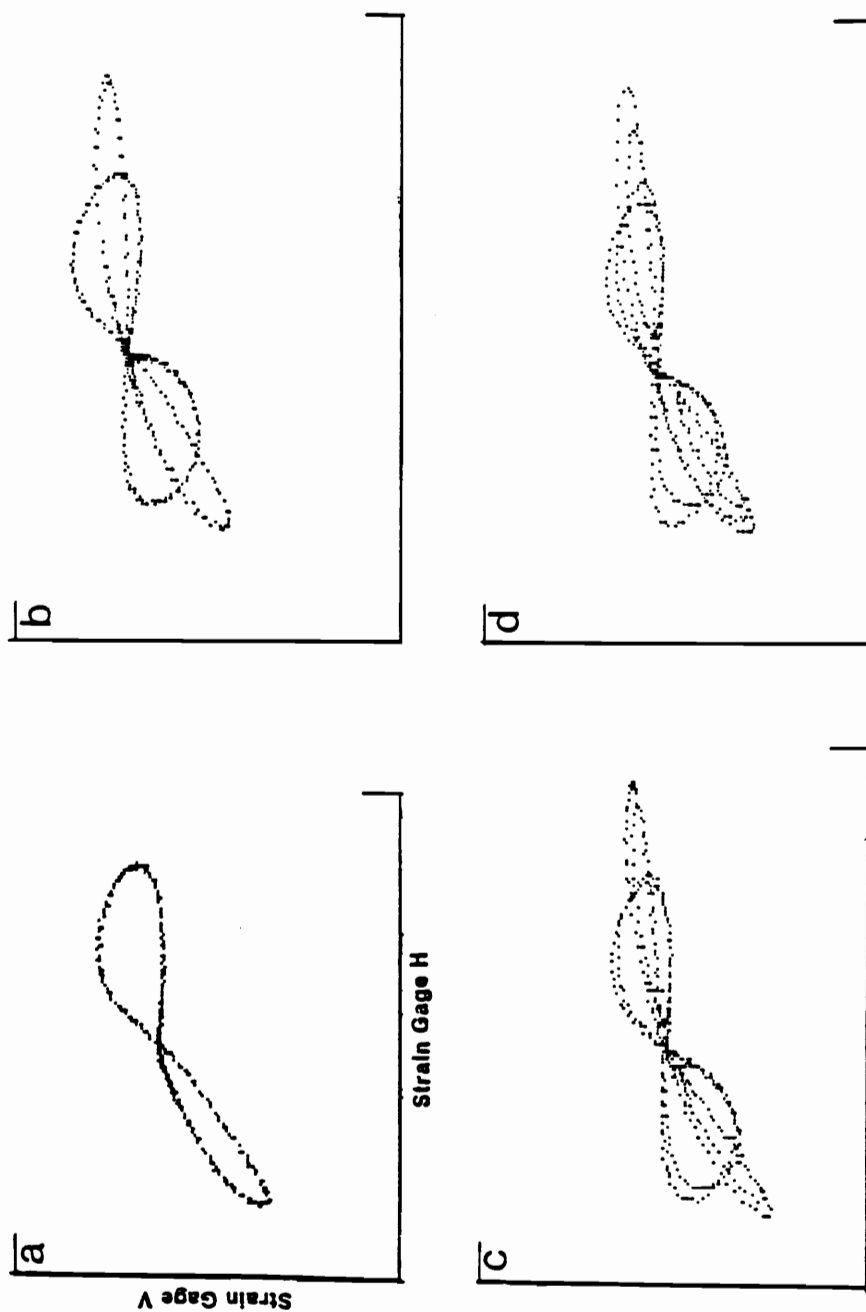


Figure 4.21. Poincaré sections of the modulated responses of Structure VI: a) $f = 16.581$ Hz, b) $f = 16.582$ Hz, c) $f = 16.584$ Hz, and d) $f = 16.586$ Hz.

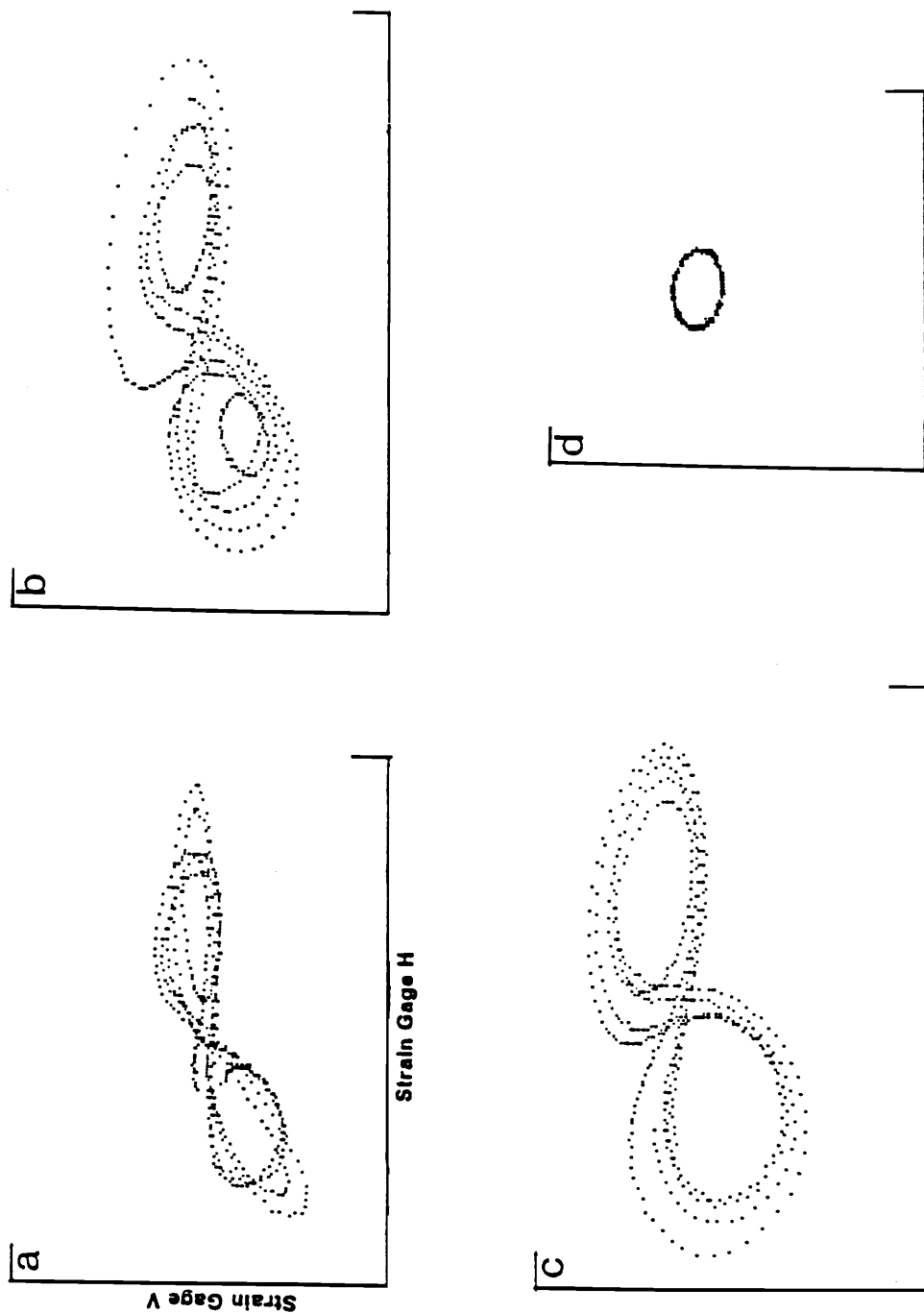


Figure 4.22. Poincaré sections of the modulated responses of Structure VI: a) $f = 16.59$ Hz, b) $f = 16.61$ Hz, c) $f = 16.62$ Hz, and d) $f = 16.64$ Hz.

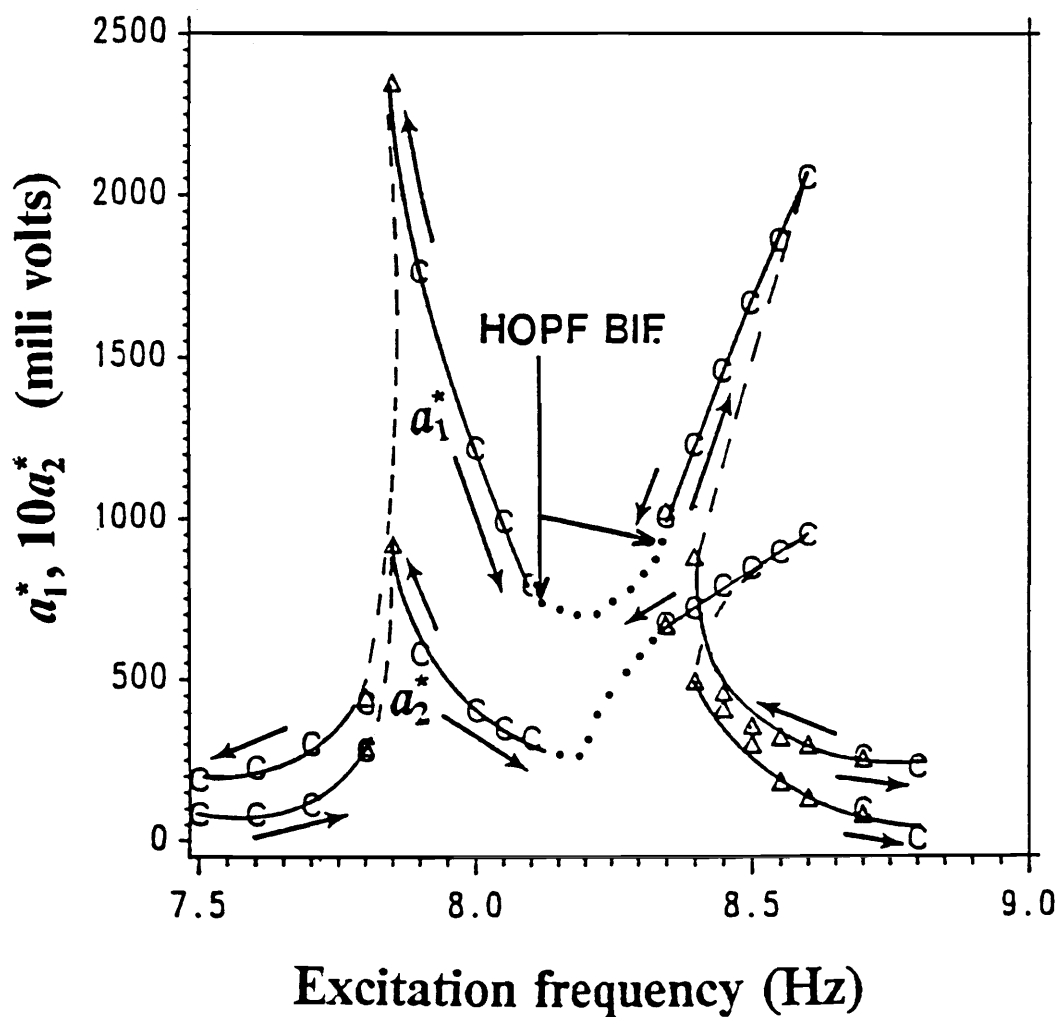


Figure 4.23. Experimentally obtained frequency-response curves of Structure II during primary-resonant excitations of the first mode: C's (triangles) represent experimental observations made during forward (reverse) sweep of the control parameter; solid lines denote periodic motions, and dotted lines denote modulated motions.

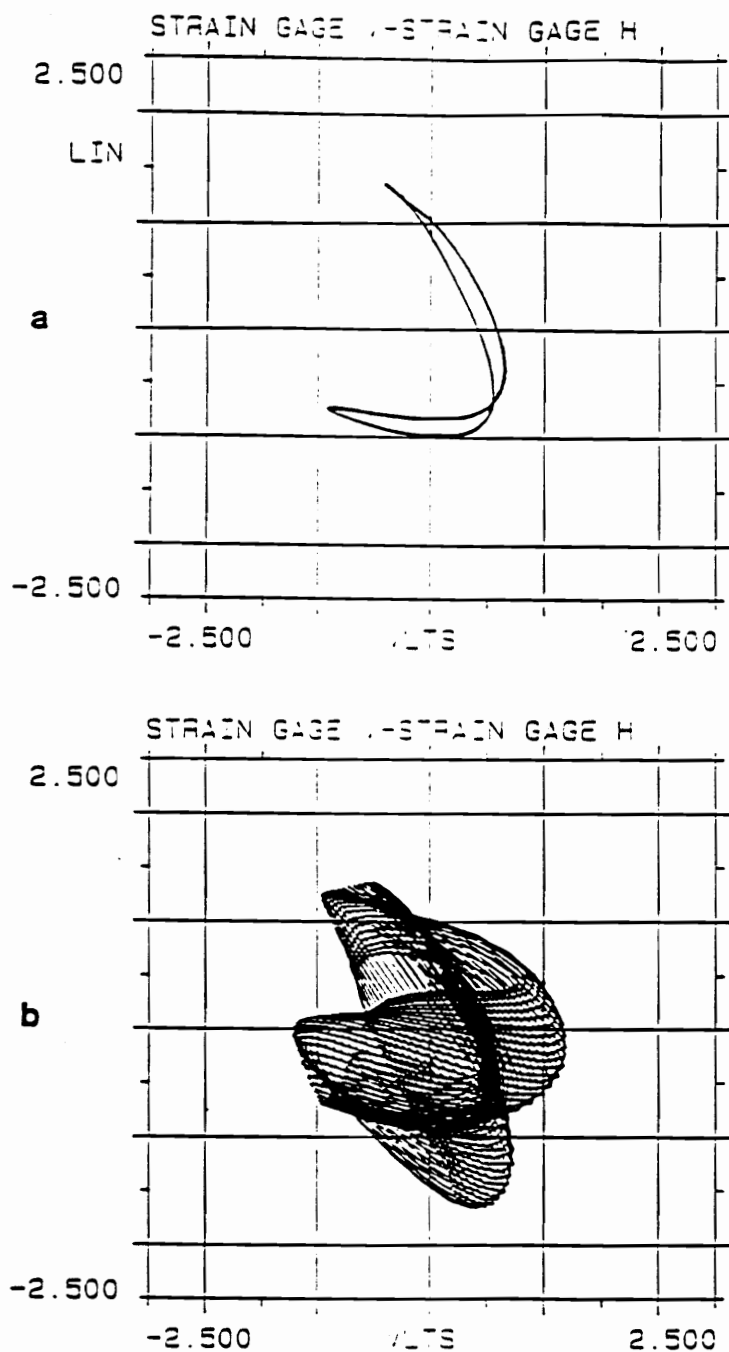


Figure 4.24. Cross plots of strain-gage signals: a) a typical periodic response and b) a typical modulated response.

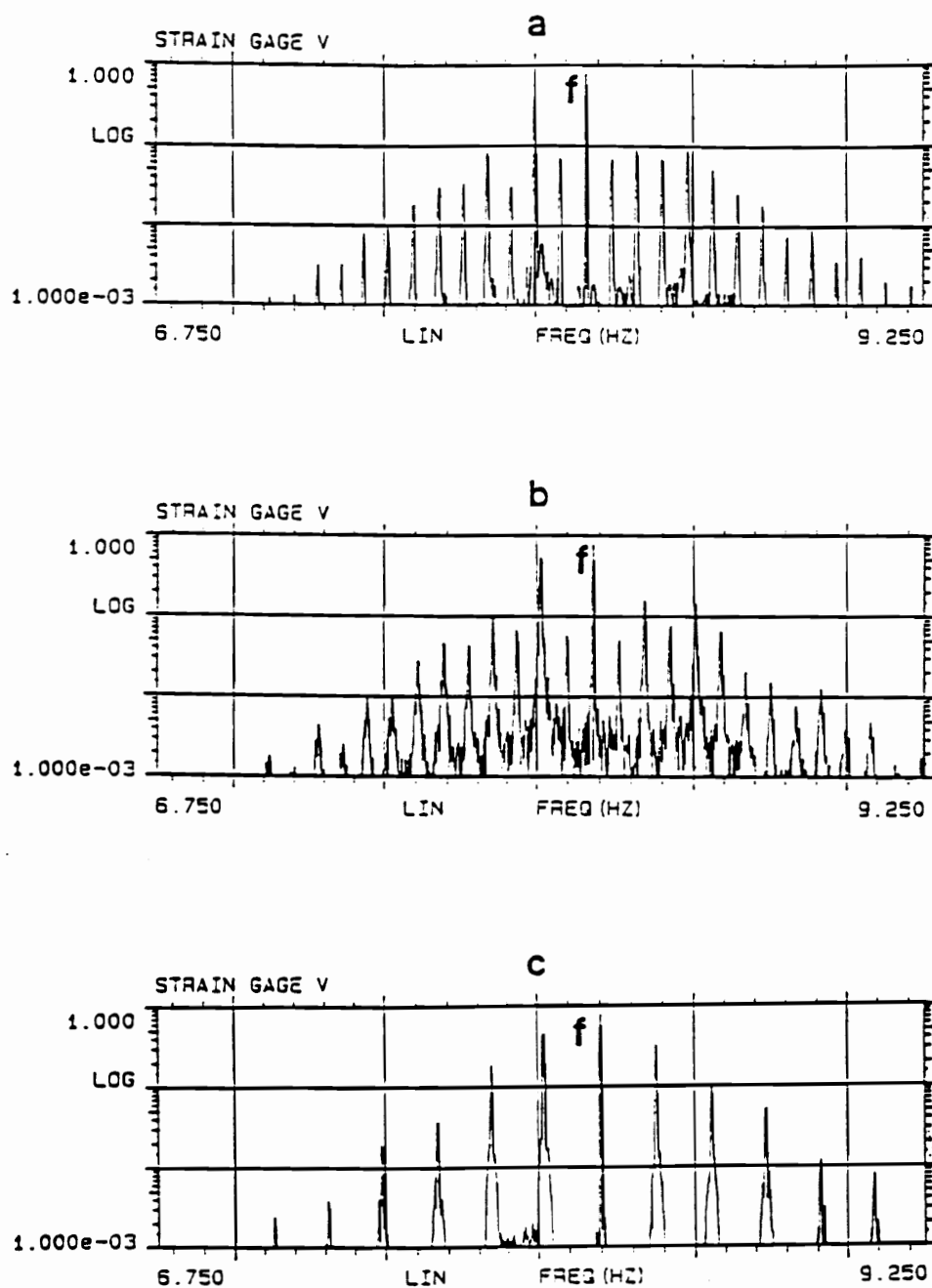


Figure 4.25. Experiment A, Structure II's response spectra on a log scale at the excitation frequencies: a) 8.16 Hz, b) 8.18 Hz, and c) 8.20 Hz.

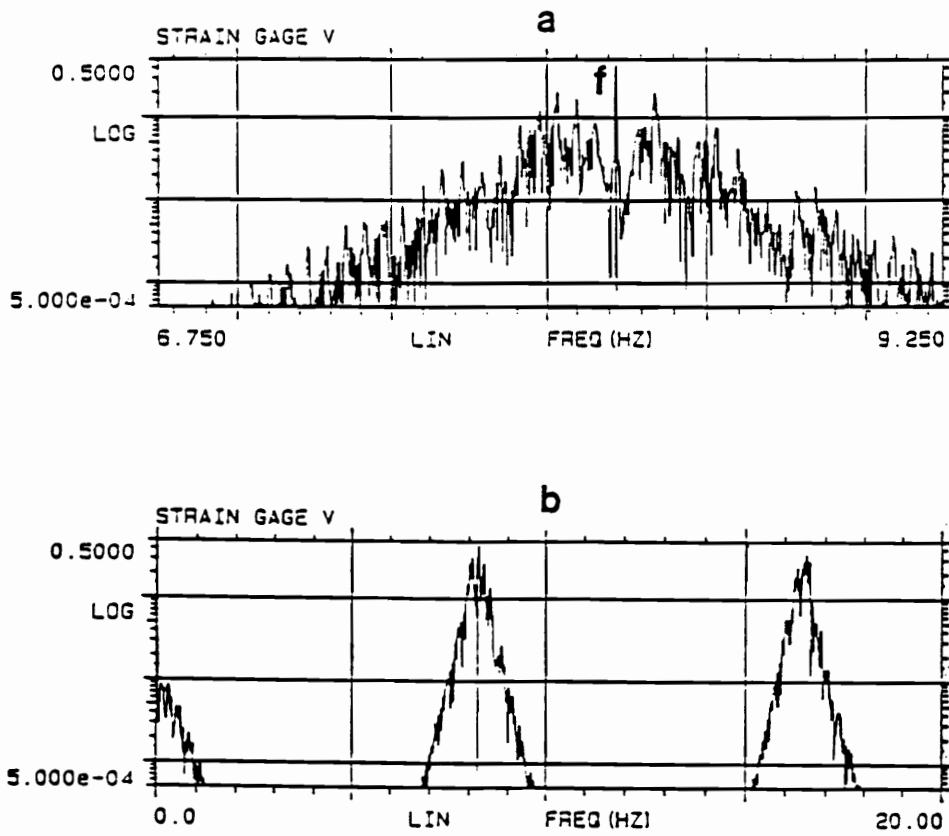


Figure 4.26. Experiment A, Structure II's response spectrum on a log scale for motion at $f = 8.22$ Hz: a) zoom span and b) baseband

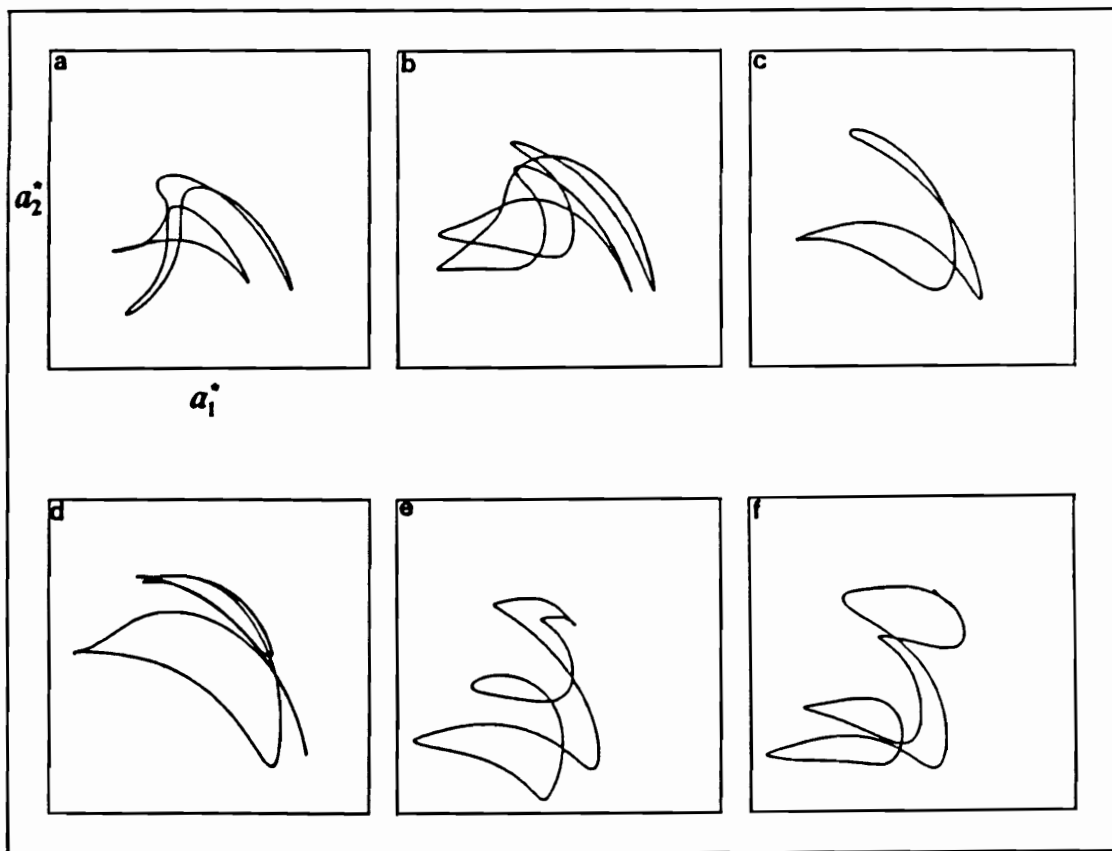


Figure 4.27. Experiment A, projection of the asymptotic state of motion in the $a_1^* - a_2^*$ plane: a) $f = 8.16$ Hz, b) $f = 8.18$ Hz, c) $f = 8.20$ Hz, d) $f = 8.23$ Hz, e) $f = 8.26$ Hz, and f) $f = 8.28$ Hz. In all cases, the scale of the x-axis runs from 0.0 to 2.5 volts, while that of the y-axis runs from 0.0 to 1.5 volts.

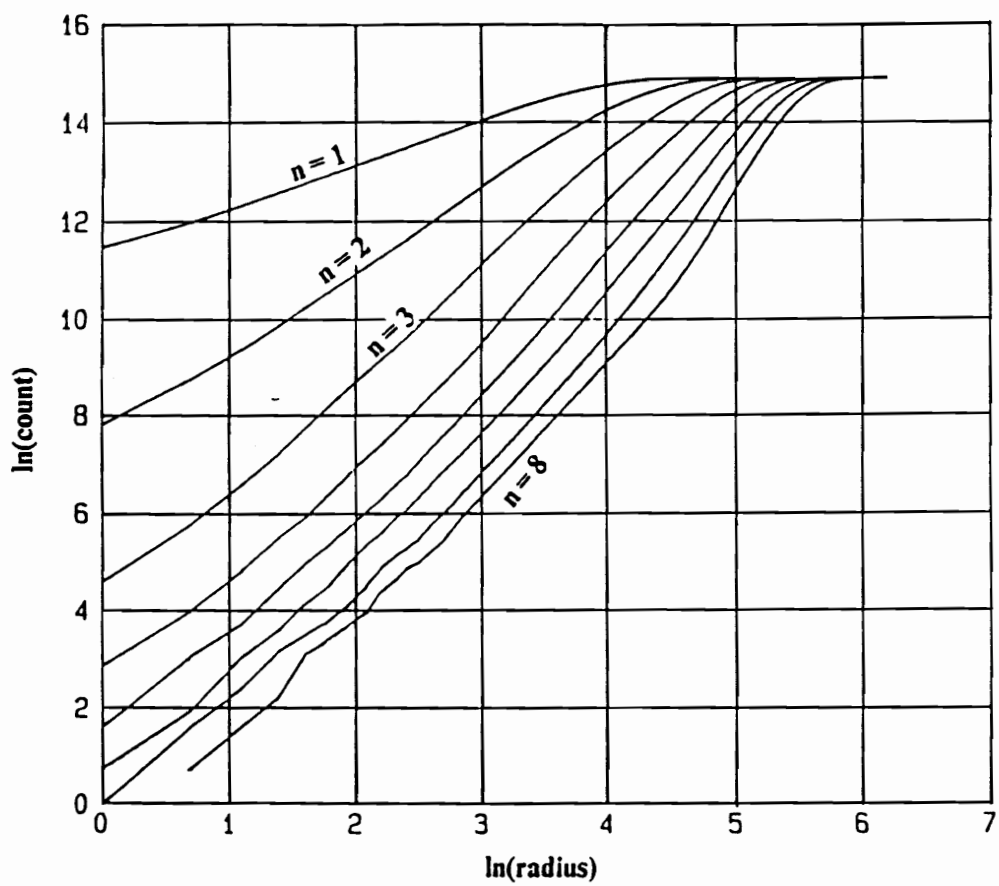


Figure 4.28. Experiment A, plot for pointwise dimension for different values of the embedding dimension for the motion at $f = 8.22$ Hz.

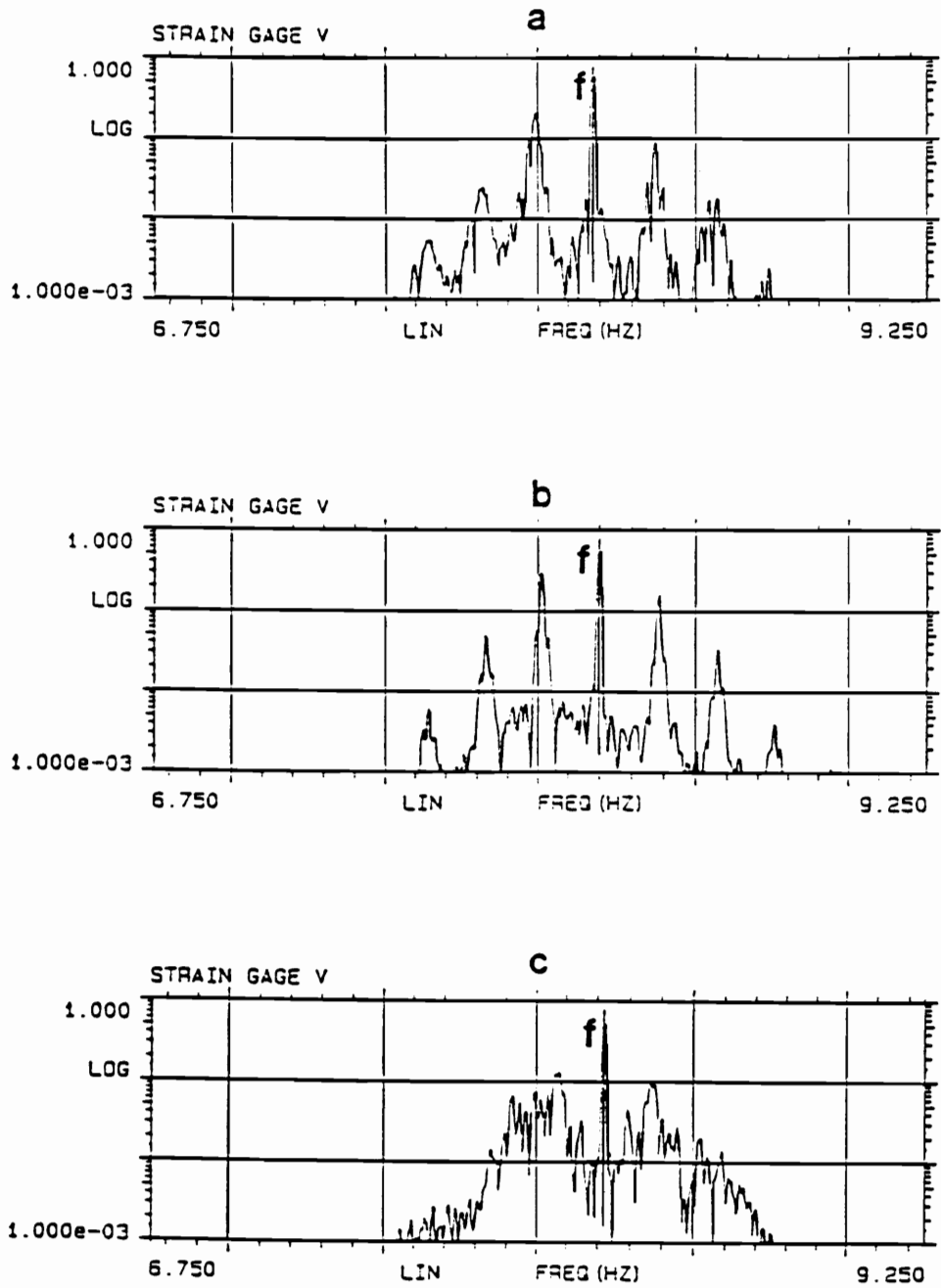


Figure 4.29. Experiment B, Structure II's response spectra on a log scale at the excitation frequencies: a) 8.18 Hz, b) 8.20 Hz, and c) 8.22 Hz.

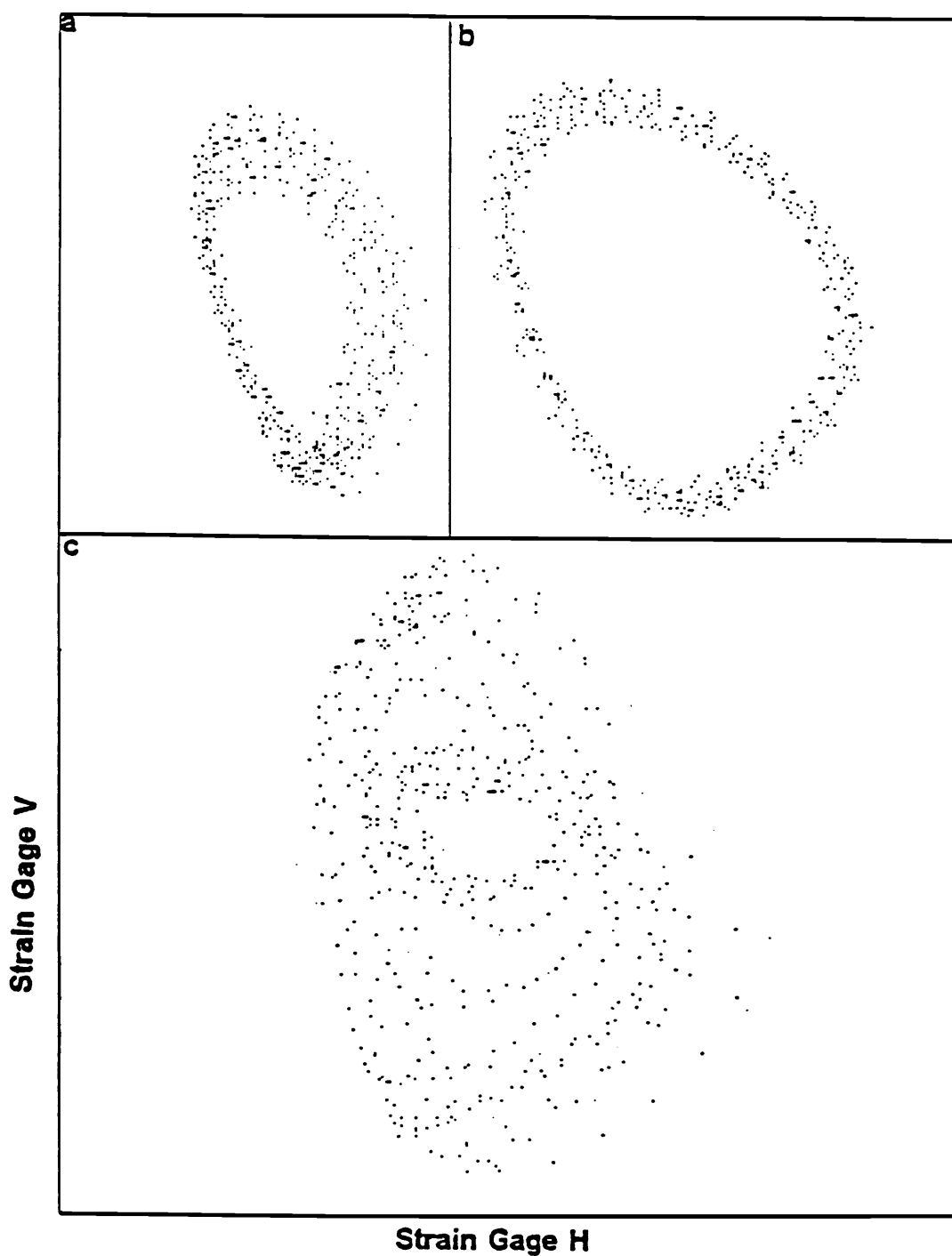


Figure 4.30. Experiment B, Poincaré sections of modulated responses of Structure II: a) $f = 8.18$ Hz, b) $f = 8.20$ Hz, and c) $f = 8.22$ Hz.

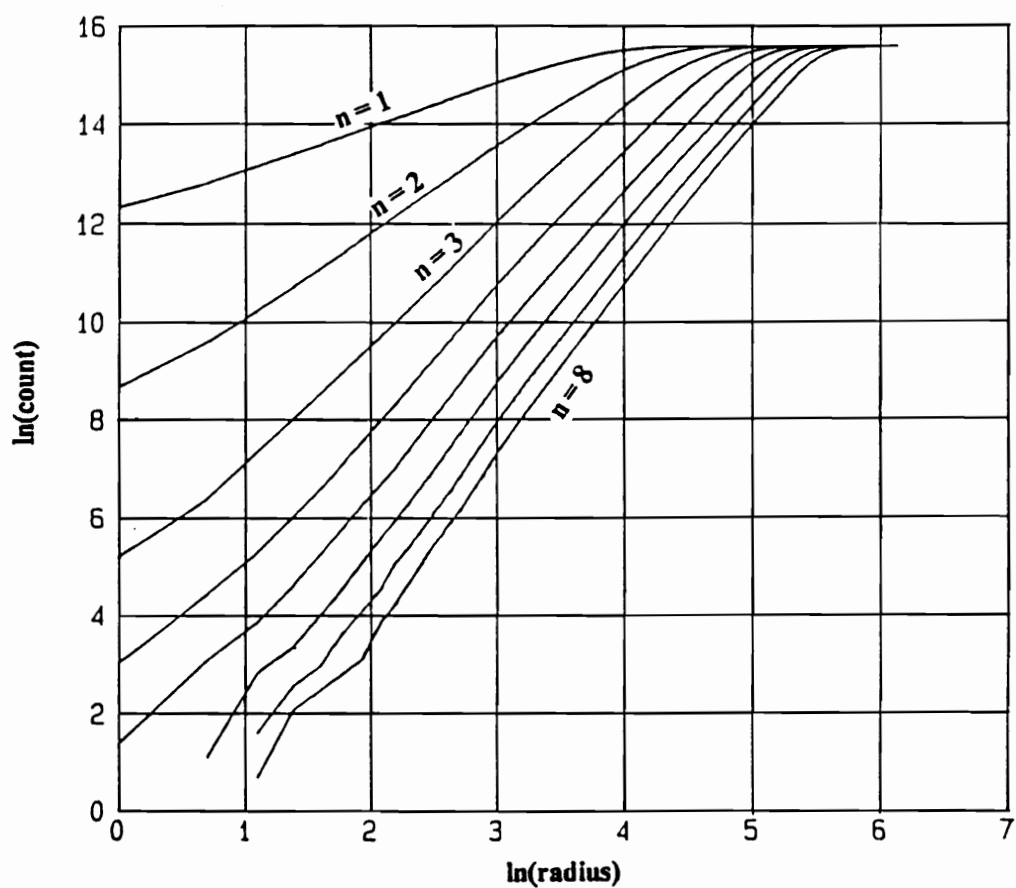


Figure 4.31. Experiment B, plot for pointwise dimension for different values of the embedding dimension for Structure II's motion at $f = 8.22$ Hz.

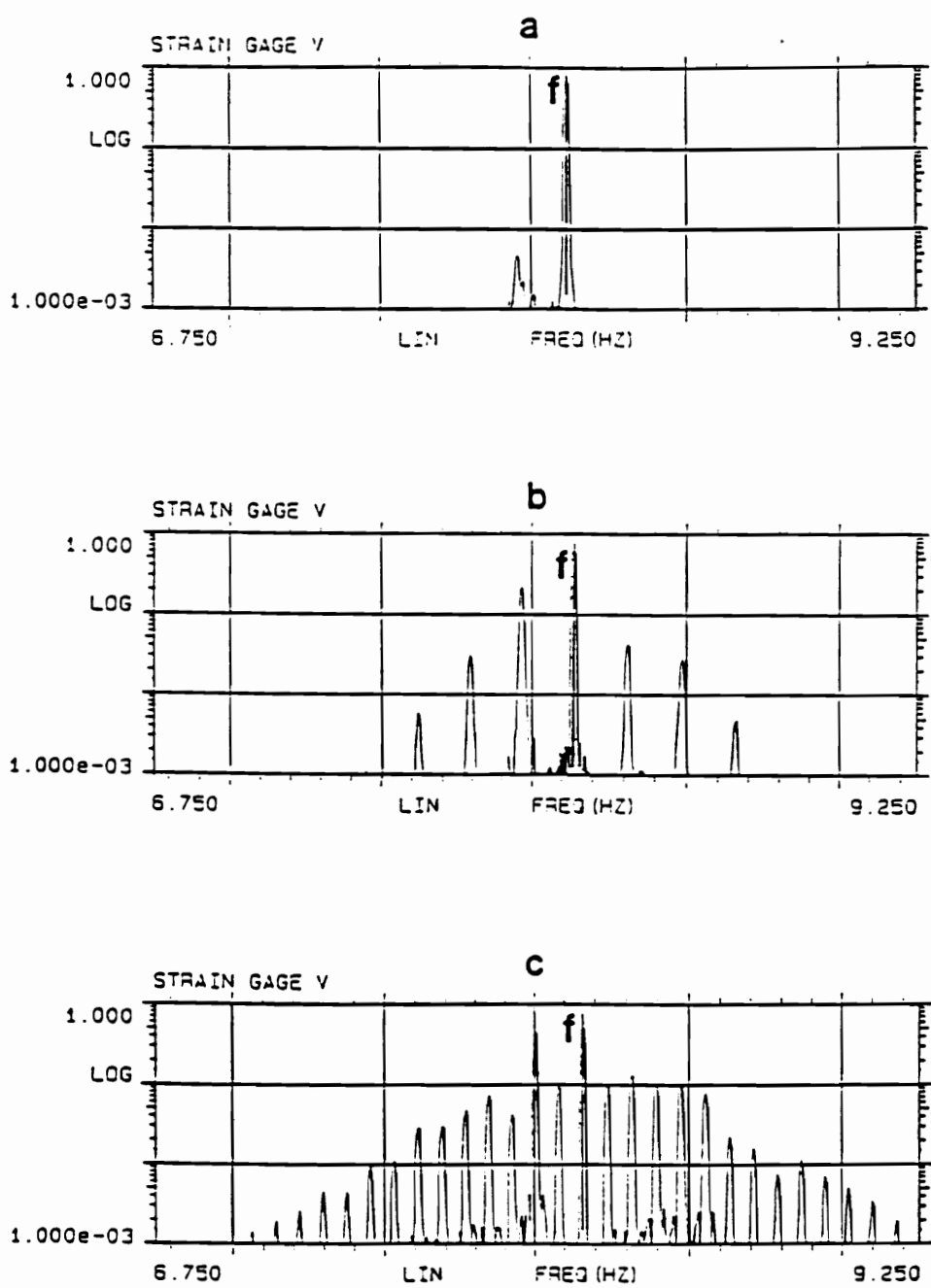


Figure 4.32. Experiment C, Structure II's response spectra on a log scale at the excitation frequencies: a) $f = 8.12$ Hz, b) $f = 8.14$ Hz, and c) $f = 8.16$ Hz.

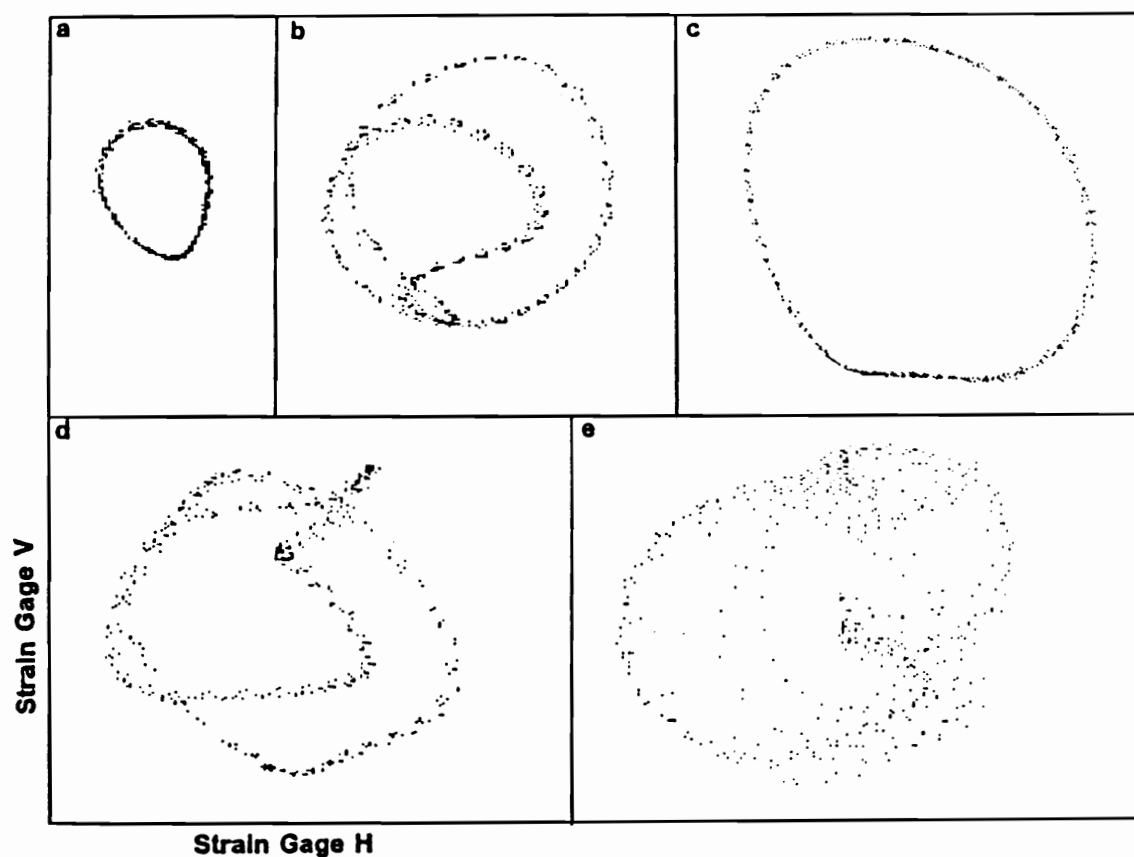


Figure 4.33. Experiment C, Poincaré sections of modulated responses of Structure II: a) $f = 8.14$ Hz, b) $f = 8.16$ Hz, c) $f = 8.18$ Hz, d) $f = 8.20$ Hz, and e) $f = 8.22$ Hz.

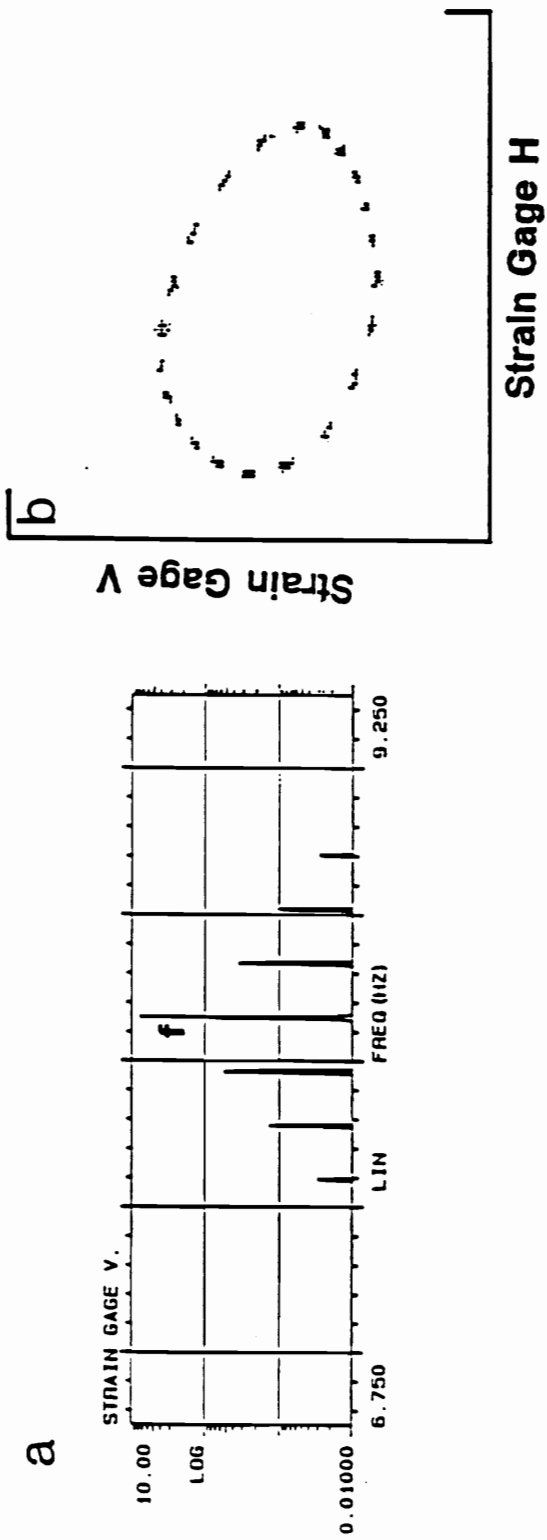


Figure 4.34. Response spectrum in a zoom span and Poincaré section for motion at $f = 8.15$ Hz and $F = 133.33$ mill g's rms.

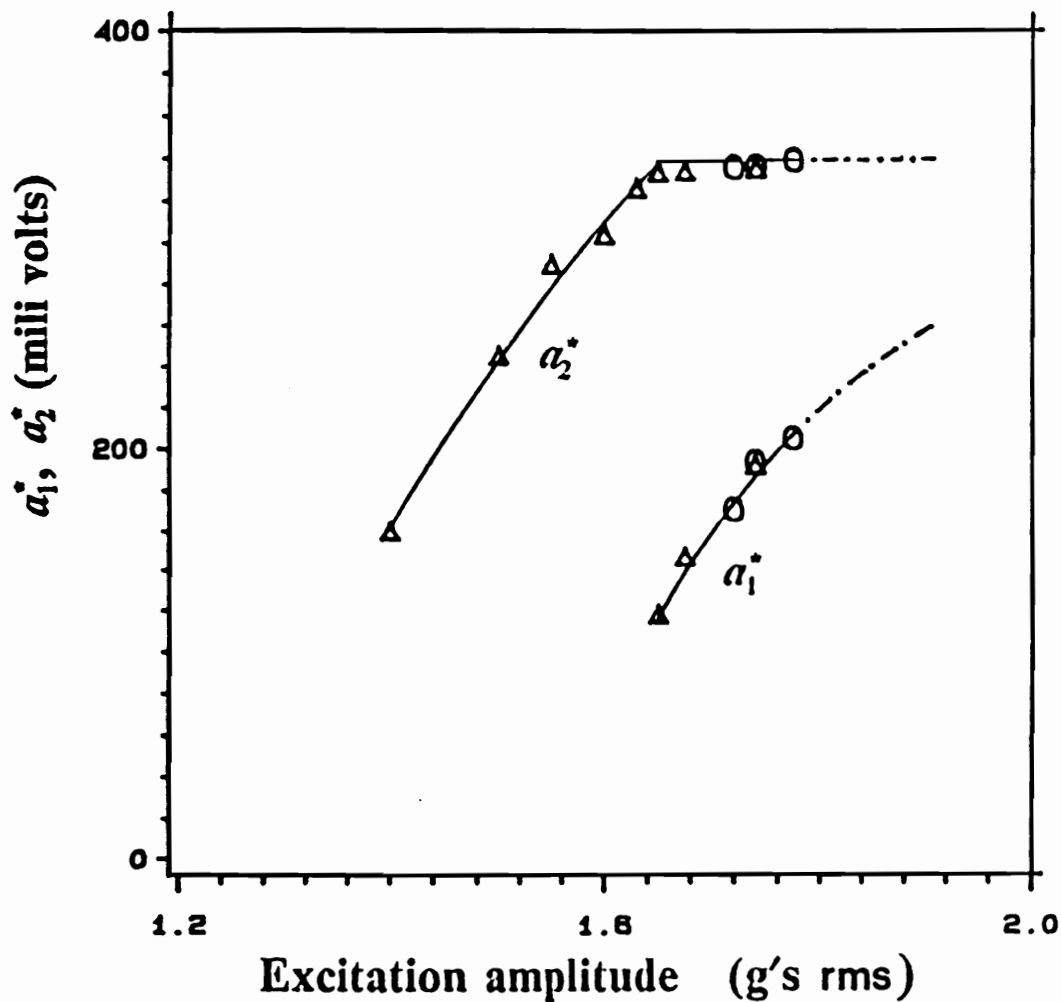


Figure 4.35. Experimentally obtained force-response curves of Structure V at $f = 32.65$ Hz for the case of subharmonic resonance of order one-half: circles (triangles) represent observations made during the forward (reverse) sweep of the control parameter; solid (dotted) lines denote periodic (modulated) motions.

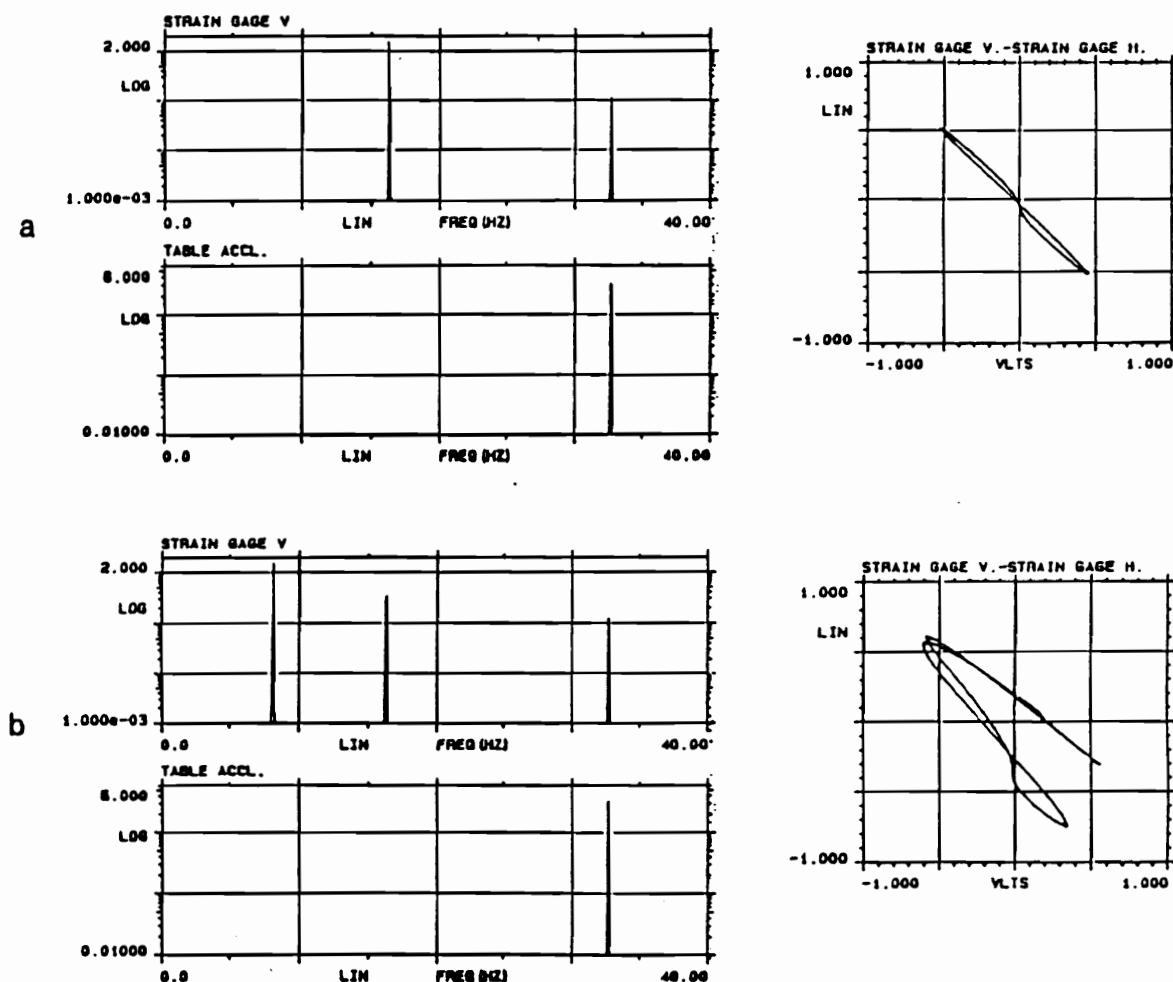


Figure 4.36. Spectra of the response and the excitation and corresponding cross plots of the strain-gage signals for two types of nonlinear periodic response: a) $f = 32.65$ Hz and an excitation level = 1.58 g's rms and b) $f = 32.65$ Hz and an excitation level = 1.72 g's rms.

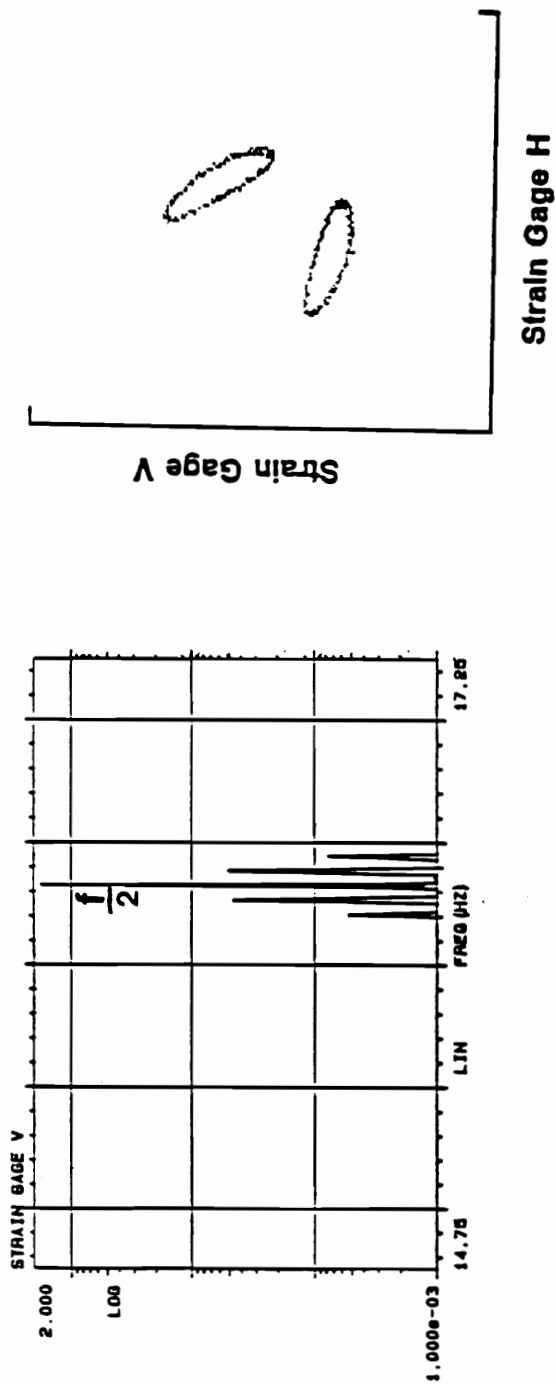


Figure 4.37. Response spectrum in a zoom span and Poincaré section for motion at $f = 32.65$ Hz and an excitation level of 1.95 g's rms.

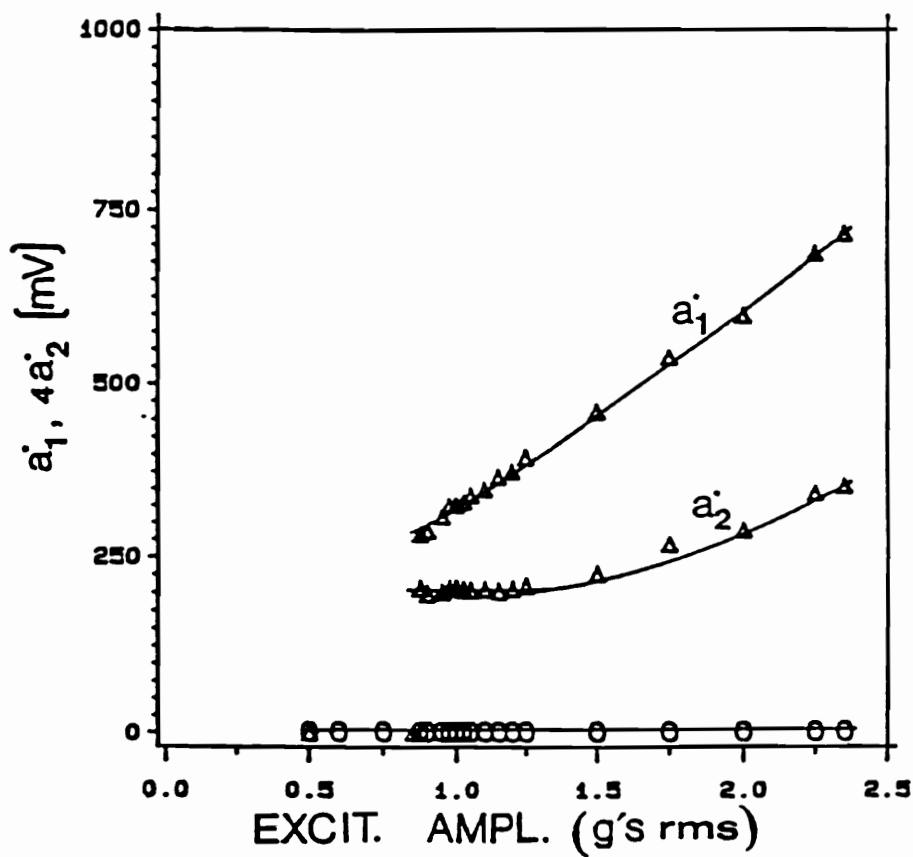


Figure 4.38. Experimentally obtained force-response curves of Structure IV at $f = 24.20$ Hz for the case of combination resonance of the additive type: circles (triangles) represent observations made during forward (reverse) sweep of the control parameter.

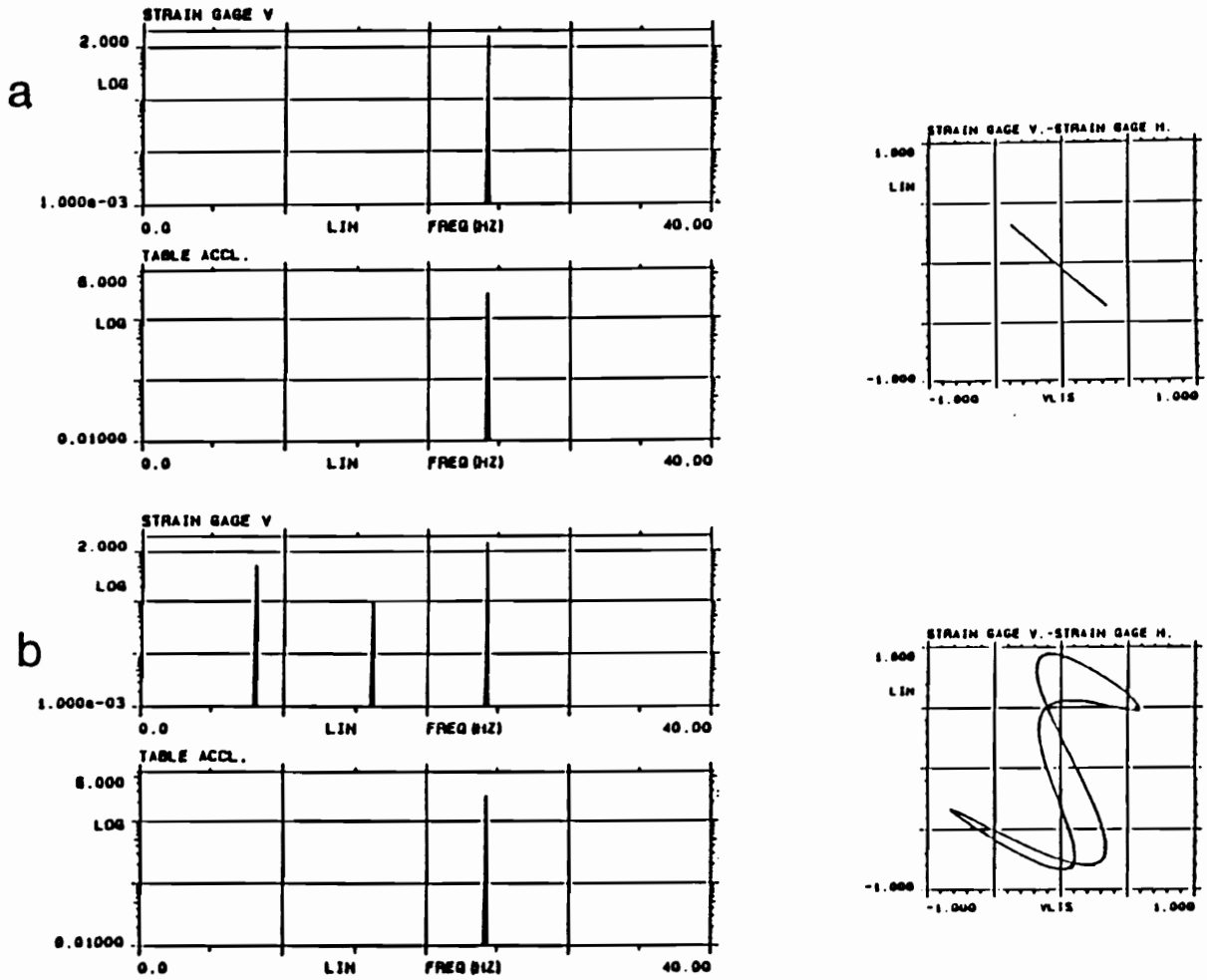


Figure 4.39. Spectra of the response and the excitation and corresponding cross plots of the strain-gage signals at $f = 24.2$ Hz and an excitation level of 1.3 g's rms: a) linear periodic response and b) nonlinear periodic response.

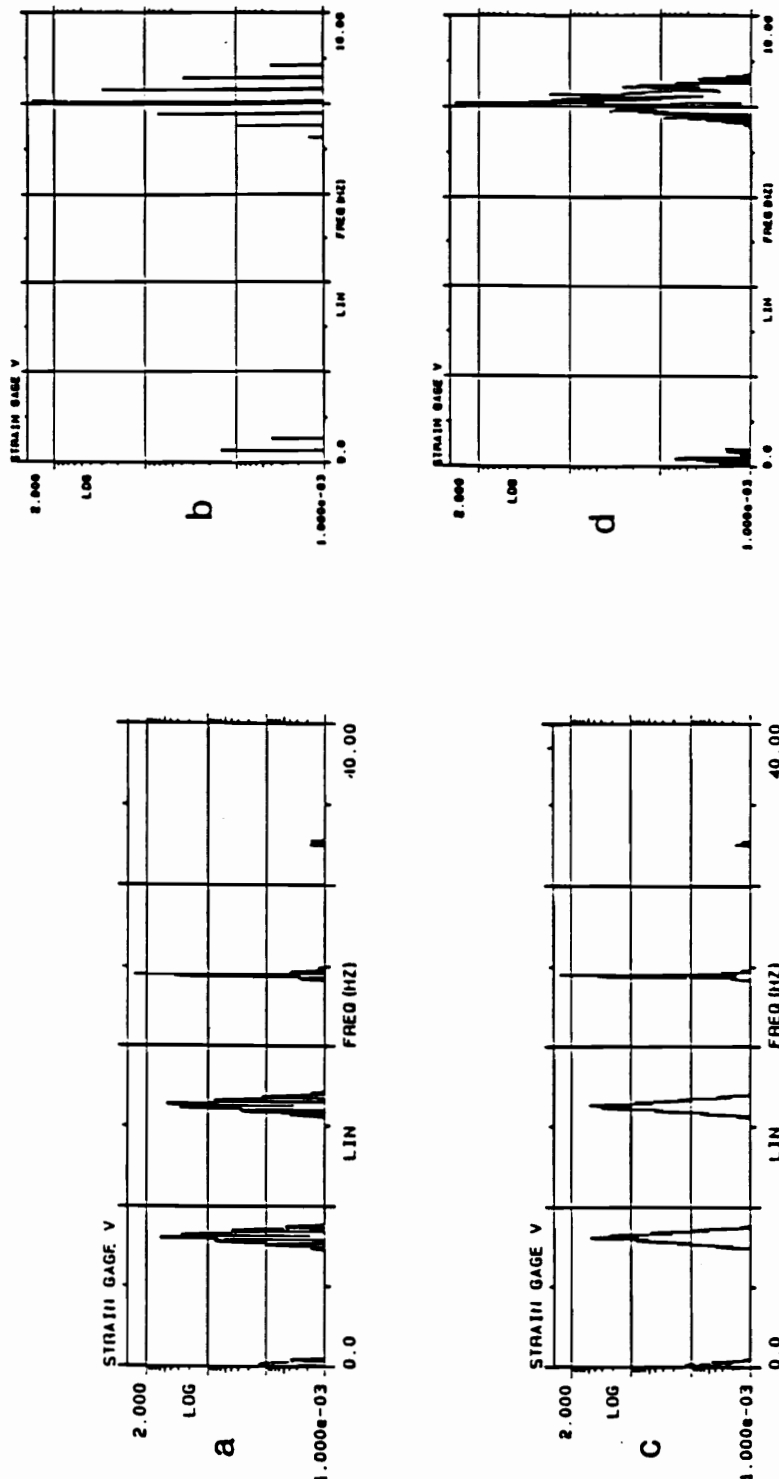


Figure 4.40. Spectra of the modulated responses of Structure IV on a log scale in a baseband for the case of combination resonance of the additive type: a) $f = 24.45$ Hz, b) $f = 24.45$ Hz, c) $f = 24.458$ Hz, and d) $f = 24.458$ Hz. We note a uniform sideband structure around the carrier frequencies in a) and b) and a broadband structure around the carrier frequencies in c) and d).

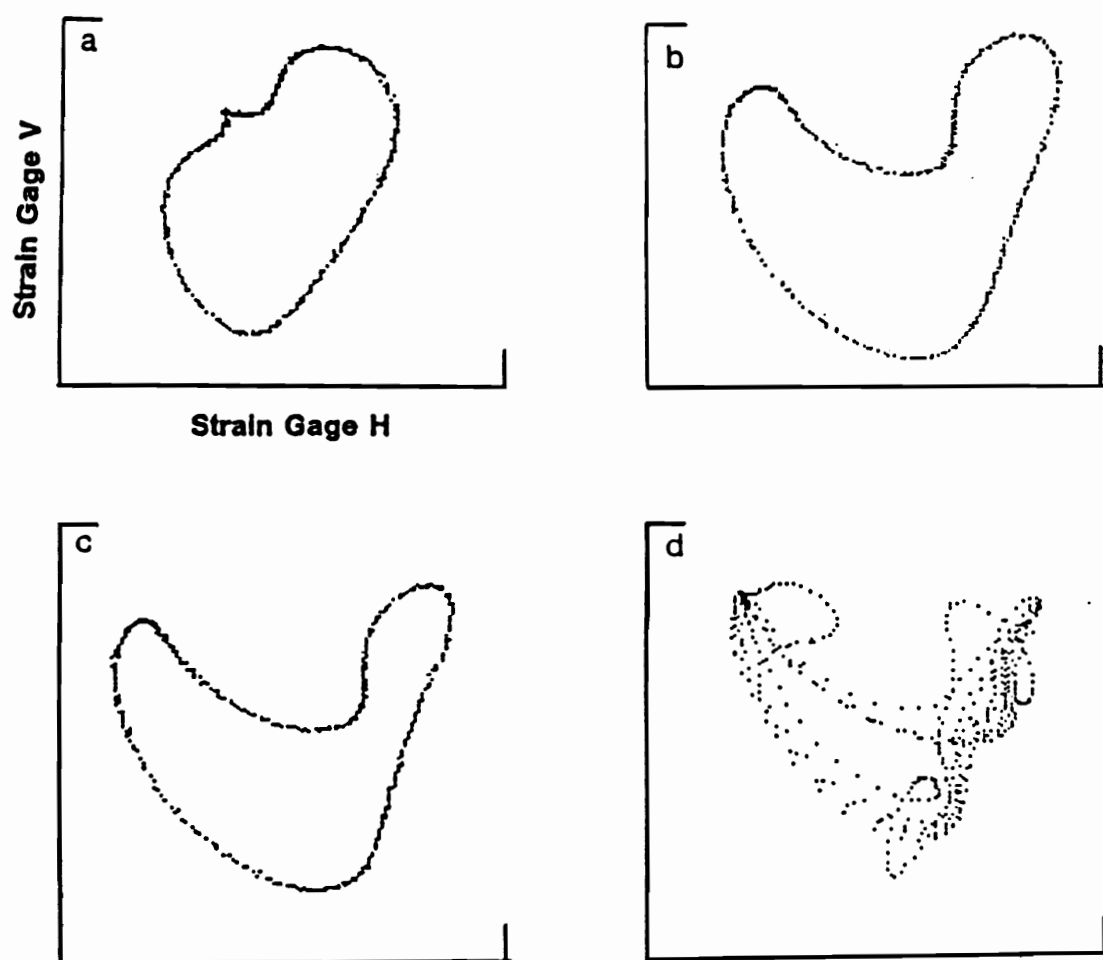


Figure 4.41. Poincaré sections of modulated responses in the case of combination resonance of the additive type: a) $f = 24.4$ Hz, b) $f = 24.45$ Hz, c) $f = 24.455$ Hz, and d) $f = 24.458$ Hz; sections a), b), and c) correspond to periodically modulated motions, and section d) may correspond to a chaotically modulated motion.

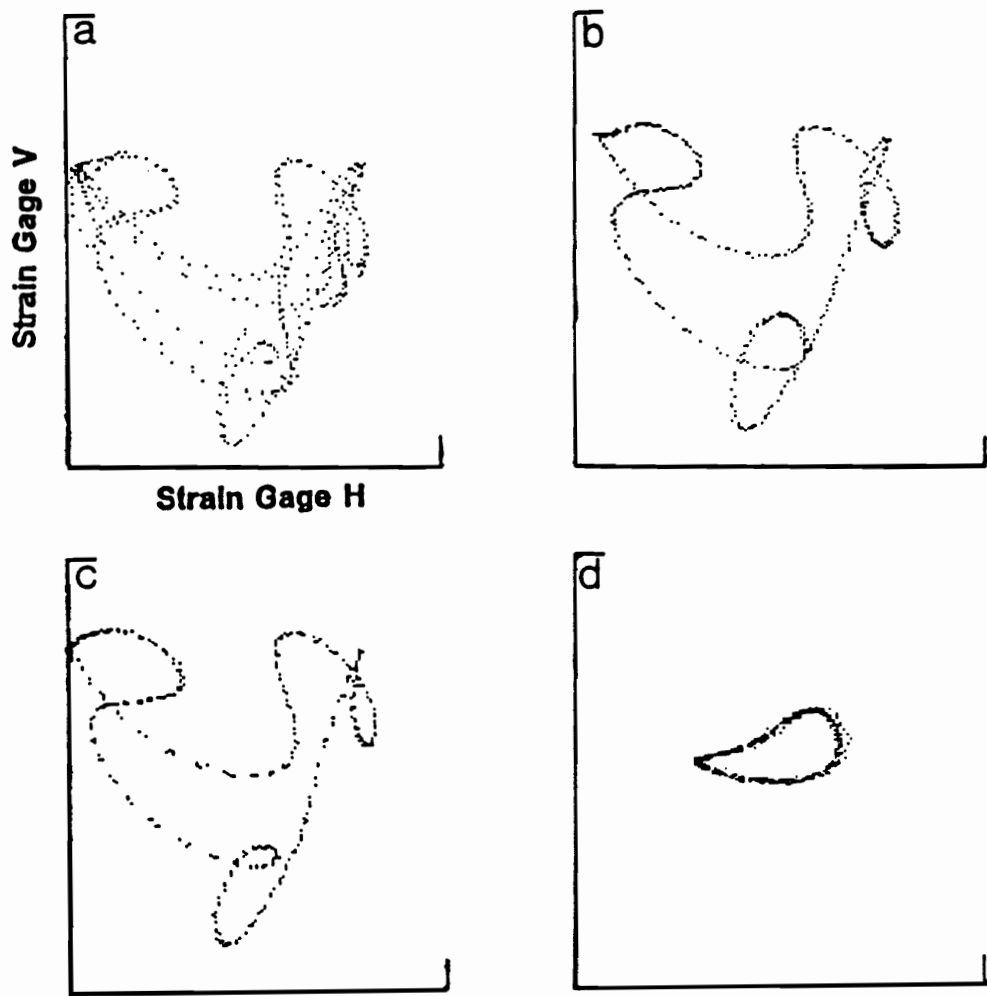


Figure 4.42. Poincaré sections of modulated responses in the case of combination resonance of the additive type: a) $f = 24.465$ Hz, b) $f = 24.47$ Hz, c) $f = 24.48$ Hz, and d) $f = 24.51$ Hz; sections b), c), and d) correspond to periodically modulated motions, and section a) may correspond to a chaotically modulated motion.

5. Experiments with the Composite Structures

In this chapter, first we present a short analysis that was conducted to analytically predict the dynamic response of the composite beam-mass structures. Subsequently, the experiments that were carried out and the experimental results obtained from them are discussed. These experimental results are also compared with those obtained in earlier experiments with metallic structures of similar form.

A typical composite beam-mass structure is shown in Fig. 5.1. It is made of two flexible composite beams and two concentrated aluminum masses. The mass M_1 at the junction of the horizontal and vertical beams is made of three pieces and acts as a C-clamp. Here, we preferred this construction to avoid drilling holes through the composite beams. Initially a different construction was used at the junction. The composite beams were bonded to perpendicular faces of a block of steel mass with an epoxy adhesive. The bonding provided by the adhesive did not hold when the structure went into nonplanar motions:

therefore, we resorted to the C-clamp construction. We conducted experiments with two structures: in the first case the composite beams were made from 7781/5245C glass-epoxy, $0^\circ/90^\circ$ woven fabric material with the lay-up $[0^\circ/90^\circ/45^\circ/-45^\circ/45^\circ/90^\circ/0^\circ]_s$, and in the second case the composite beams were made from 2TL/F584 graphite-epoxy nonwoven laminates with the lay-up $[0^\circ/45^\circ/-45^\circ/90^\circ]_s$. After the lay-up, each laminate was cured in an autoclave at the standard cure schedule for the resin system. Two different settings of the glass-epoxy composite-beam structure led to Structures A and C, and Structure B was made from graphite-epoxy composite beams. The dimensions and other details of these structures are provided in Appendix D. Each of the composite beam-mass structures weighed about 50.00 grams. Hence, these structures can be considered as light-weight structures. Structures A and B were subjected to primary-resonant excitations, and Structure C was subjected to secondary-resonant excitations. In the experiments, the responses were analyzed by Fourier spectra, pseudo-phase planes, Poincaré sections, and dimension calculations.

The experimental set up was the same as that discussed in the the previous chapter. During the experiments, the temperature and humidity were monitored at a location about two and a half feet away from the test structure. These measurements were used to ensure there were uniform conditions throughout a run. Again here, 350-Ohm strain gages (Micro-Measurements WK-06-125AD-350) were mounted along the axes of the horizontal and vertical beams (referred to as Strain Gage H and Strain Gage V, respectively) to obtain

a measure of the displacements due to flexural motions. The letters S.G. in Fig. 5.1 stand for strain gages. Strain Gage H was located about an inch from the clamp while Strain Gage V was located about an inch from the junction of the two beams. Each of these strain gages together with a 350-Ohm precision resistor (located on the shaker table) formed half a Wheatstone bridge, and was completed in a signal conditioner. An additional pair of strain gages was mounted on the top and bottom faces of the horizontal beam at 45° to its axis (referred to as Strain Gage II) about 3.0 inches from the clamp to obtain a measure of the displacements due to torsional motions. For all cases, the bridge excitation level (after some experiments) was fixed at 2.7 volts. Later, the signals from the amplifiers were passed through Krohn-Hite elliptical low-pass filters (cut-off frequency = 50.0 Hz) and amplified again (post-amplifier gain either 10 dB or 20 dB) before their measurement. The strain-gage and accelerometer signals were analyzed by a spectrum analyzer (GenRad system 2515). For periodic motions, a measure of the modal amplitudes was obtained from the frequency spectrum of the strain-gage signal. The damped resonant frequencies of each composite structure were determined as discussed in Appendix E.

5.1. *Brief Analysis*

As mentioned earlier, the composite structure consisted of two light-weight composite beams and two concentrated aluminum masses. The frequencies of the first three modes of Structures A, B, and C were well separated from the frequencies of the higher modes of vibration, and at low excitation frequencies these structures essentially behaved as three-degree-of-freedom systems. In contrast, the steel-beam structures were essentially two-degree-of-freedom systems. In each case, the first three natural frequencies of the composite structure correspond to the first flexural mode, the first torsional mode, and the second flexural mode, respectively. In the present study, the reason for the low torsional frequency of the composite structure is the relative weakness in shear of composite laminates, which is a consequence of their low transverse shear moduli. The composite structures have lower torsional stiffness than their metallic counterparts, and consequently their torsional modes have lower natural frequencies than the corresponding modes in metallic structures.

The observed mode shapes for the first two flexural modes are also shown in Fig. 5.1. When the torsional mode was present, the concentrated mass M_2 underwent motions in a plane normal to the plane of the flexural motions (out-of-plane motions). Denoting the undamped mode shapes of the first flexural mode, second flexural mode, and first torsional mode as $\phi_1(s)$, $\phi_2(s)$,

and $\phi_i(s)$, respectively, we can express the displacement $w(s,t)$ at any location s on the structure at time t by the following three-mode approximation:

$$w = u_1(t) \phi_1(s) + u_2(t) \phi_2(s) + v(t) \phi_l(s) \quad (5.1)$$

where u_1 and u_2 are the generalized coordinates corresponding to the displacements due to flexural motions and v is the generalized coordinate corresponding to an out-of-plane displacement due to torsional motion. As in Chapter 3, here we only include the modes of vibration that are directly excited or likely to be indirectly excited through the internal resonances. Here, we did not derive the governing partial-differential equations of motion. One can start from the mechanics and obtain equations such as those shown for the metallic structures in Appendix A. However, substituting equation (5.1) into the governing partial-differential equations and boundary conditions and using the Galerkin procedure, we expect to obtain a special form of the equations given below

$$\begin{aligned} \ddot{u}_1 + \omega_1^2 u_1 + 2\mu_1 \dot{u}_1 + \delta_4 \dot{u}_1^2 + \delta_5 \dot{u}_1 \dot{u}_2 + \delta_6 \dot{u}_2^2 + \delta_7 u_1 \ddot{u}_1 \\ + \delta_8 u_2 \ddot{u}_1 + \delta_9 u_1 \ddot{u}_2 + \delta_{10} u_2 \ddot{u}_2 + \delta_{11} u_1^3 + \delta_{12} u_1^2 u_2 \\ + \delta_{13} u_1 u_2^2 + \delta_{14} u_1 \dot{u}_1^2 + \delta_{15} u_1^2 \ddot{u}_1 + f_1(u_1, u_2, v) \\ + (h_{11} u_1 + h_{12} u_2) \cos(\Omega t) = F \cos(\Omega t) \end{aligned} \quad (5.2)$$

$$\begin{aligned} \ddot{u}_2 + \omega_2^2 u_2 + 2\mu_2 \dot{u}_2 + \alpha_4 \dot{u}_1^2 + \alpha_5 \dot{u}_1 \dot{u}_2 + \alpha_6 \dot{u}_2^2 + \alpha_7 u_1 \ddot{u}_1 \\ + \alpha_8 u_2 \ddot{u}_1 + \alpha_9 u_1 \ddot{u}_2 + \alpha_{10} u_2 \ddot{u}_2 + \alpha_{11} u_2^3 + \alpha_{12} u_2^2 u_1 \\ + \alpha_{13} u_2 u_1^2 + \alpha_{14} u_2 \dot{u}_2^2 + \alpha_{15} u_2^2 \ddot{u}_2 + f_2(u_1, u_2, v) \\ + (h_{21} u_1 + h_{22} u_2) \cos(\Omega t) = G \cos(\Omega t) \end{aligned} \quad (5.3)$$

$$\ddot{v} + \omega_t^2 v + 2\mu_t \dot{v} + f_3(u_1, u_2, v) = 0 \quad (5.4)$$

where the highest order of terms retained is cubic, the μ_n are the modal-damping coefficients, the ω_n are the linear natural frequencies, and F, G, h_{mn} , and Ω are constants, which depend on the excitation. The functions f_i consist of nonlinear terms which couple in-plane and out-of-plane displacements. The coefficients δ_i and α_i depend on the values of the concentrated masses, the mode shapes, and the dimensions and properties of the beams. A weakly nonlinear analysis of equations (5.2)-(5.4) would lead to a sixth-order autonomous system of equations governing the amplitudes and phases of the interacting modes of vibration.

For Structures A and C, the frequencies of the structure were such that $\omega_2 = 3\omega_1 + \sigma_1$ and $\omega_2 = 2\omega_1 + \sigma_2$, where the σ_n are small detuning parameters. So, in the case of the glass-epoxy composite structure there are both three-to-one and two-to-one internal resonances. In the case of Structure B, the frequencies were such that $\omega_1 = \omega_t + \sigma_1$ and $\omega_2 = 3\omega_1 + \sigma_2$, where again, the σ_n are small detuning parameters. Hence, in the case of the graphite-epoxy composite structure there are both three-to-one and one-to-one internal resonances. In the experiments, the detuning of the internal resonances was varied by varying the position of the concentrated mass on the vertical beam. During primary-resonant excitations, the excitation frequency was such that $\Omega = \omega_i + \sigma_3$ while during combination-resonant excitations, we had $\Omega = \omega_i + \omega_j + \sigma_3$. The parameter σ_3 is known as external detuning. Equations

(5.2) and (5.3) contain inertial quadratic nonlinearities because of the shape of the structure and cubic geometric and inertial nonlinearities. Besides these nonlinearities, there could also be material nonlinearities. We note that the periodic excitation terms are present only in equations (5.2) and (5.3). The excitation provided to the structure (see Fig. 5.1) can only excite the flexural modes. Hence, out-of-plane displacements (torsional motions) can only be excited indirectly through the autoparametric resonance.

Since the governing equations of the structures possess quadratic and cubic terms, we expect the internal resonances to be active. Based on previous theoretical and experimental findings for systems with a two-to-one internal resonance and quadratic nonlinearities (e.g., Nayfeh, 1989), during primary-resonant excitations of the second flexural mode, the glass-epoxy composite-beam structure is expected to exhibit the saturation phenomenon and, for some damping and excitation-parameter values, modulated motions. The structure is also expected to display nonlinear responses during the combination-resonant excitations. Also, the graphite-epoxy composite-beam structure is expected to exhibit nonlinear responses due to modal interactions.

5.2. Experiments

The linear resonant frequencies of the structure determined by a combination of free oscillations, random excitations, and tuned frequency sweeps were 5.84

Hz, 8.67 Hz, and 17.64 Hz for Structure A; 5.70 Hz, 8.58 Hz, and 17.22 Hz for Structure C; and 4.58 Hz, 4.77 Hz, and 13.41 Hz for Structure B. In each case, the three frequencies correspond to the first flexural mode, first torsional mode, and second flexural mode, respectively. We note that in Structures A and C, the second flexural frequency is approximately twice the lowest torsional frequency and three times the first flexural frequency, while in the case of Structure B, the first flexural frequency is approximately equal to the first torsional frequency and the second flexural frequency is approximately three times the first flexural frequency. Thus, each structure has two internal or autoparametric resonances.

In Fig. 5.2, we show the frequency-response curves of Structure A obtained by varying the excitation frequency f in a range around the first flexural frequency at a constant excitation level of 40.00 mili g's rms. In the figure, circles represent the experimental observations. The symbols a_1^* , a_2^* , and a_t^* correspond to the amplitudes of the first flexural mode, the second flexural mode, and the first torsional mode, respectively. The amplitudes a_1^* and a_2^* were obtained from the Strain Gage V signal while the amplitude a_t^* was obtained from the Strain Gage II signal. As f was varied from 5.2 Hz to 5.785 Hz the response was planar and nonlinear periodic, consisting of the two flexural modes. The amplitudes of the flexural modes increased with f in this range. The responses in the range $5.2 \text{ Hz} < f < 5.785 \text{ Hz}$ can be called a two-mode response because two modes of vibration participate in the forced response. In the range $5.79 \text{ Hz} \leq f \leq 5.83 \text{ Hz}$, the response was nonplanar and

nonlinear periodic, consisting of the two flexural and single torsional modes. Here, initially the amplitude a_2' of the second flexural mode decreased while the amplitude a_1' of the torsional mode increased; later a_2' increased while a_1' decreased. This is suggestive of an energy exchange between the second flexural and first torsional modes of the structure. The nonplanar response can be called a three-mode response as three modes of vibration take part in the forced response. We did not observe a three-mode response during other resonant excitations. In an another experiment f was held constant at 5.88 Hz while the excitation amplitude was varied. Nonplanar motions were observed when the rms value of the table acceleration exceeded a critical value of 98.50 mili g's rms.

In Fig. 5.3, the response spectra, the excitation spectra, and the corresponding x-y plots (pseudo-phase planes) of the strain-gage signals for planar and nonplanar nonlinear periodic responses are shown. The response at an excitation frequency of 5.88 Hz (5.795 Hz) and an excitation level of 14.00 mili g's rms (40.00 mili g's rms) is shown in Fig. 5.3a (Fig. 5.3b). The response spectrum in Fig. 5.3a has discrete spectral lines at f and $3f$, indicating that the response is nonlinear and periodic. The peak at f corresponds to the directly excited first flexural mode and the other strong peak at $3f$ corresponds to the second flexural mode, which is indirectly excited through the three-to-one internal resonance. For large planar responses, in addition to these peaks, a peak at the frequency $2f$ was also observed. The peak at $2f$ may be due to the presence of quadratic nonlinearities. In Fig. 5.3b, we see spectral lines at f ,

$3f/2$, $2f$, and $3f$ in the the response spectrum. The peak at $3f/2$ is close to the second damped-resonant frequency of the structure, and indicates the presence of the torsional mode in the nonplanar periodic response. This mode is indirectly excited through the two-to-one internal resonance. The cross plot in Fig. 5.3b differs from that observed in Fig. 5.3a as the period of the response increases during the nonplanar motions. When the response is planar and periodic, the period of motion is $\frac{1}{f}$ and when the response is nonplanar and periodic, the period of motion is $\frac{2}{f}$.

In Fig. 5.4, we show the experimentally obtained force-response curves of Structure A with the excitation amplitude at a constant excitation frequency of 17.53 Hz. For this excitation frequency, the detuning of the external resonance is negative. Again, the symbols a_2^* and a_1^* correspond to the amplitudes of the second flexural mode and the first torsional mode, respectively. They were obtained from the Strain Gage II signal. In this and subsequent figures, the observations made during forward and reverse sweeps are marked by circles and triangles, respectively. Also, the points where periodic motions ceased to exist are labeled "Hopf bifurcation".

As the excitation amplitude was increased from zero, a_2^* increased in a more or less linear manner until an excitation level of about 22.00 mili g's rms was reached. At this excitation level, a_1^* which had remained trivial till then assumed a nontrivial value and the structure displayed nonplanar motions. As the excitation level was increased further, a_2^* remained more or less

constant, while a_i^* increased in a nonlinear manner. So, we observed the saturation phenomenon (Nayfeh and Mook, 1979), which is known to occur in systems with a two-to-one internal resonance and quadratic nonlinearities. However, this modal saturation broke down when another critical excitation amplitude was reached. At a level of about 34.00 mili g's rms, the periodic motions ceased to exist and thereafter modulated motions were observed. The composite structure like its metallic counterpart exhibits modal saturation and its breakdown, but unlike the metallic structure its forced responses are nonplanar.

In Fig. 5.5, the frequency-response curves of Structure A are shown for primary-resonant excitations of the second flexural mode. During this experiment, the excitation frequency was varied in the range $17.1 \text{ Hz} \leq f \leq 17.9 \text{ Hz}$ at a constant excitation amplitude of 40.0 mili g's rms. Linear responses were observed up to 17.29 Hz ; the amplitudes of the other modes remained trivial. As the excitation frequency was increased to 17.295 Hz, the linear response lost stability through a jump to a nonlinear nonplanar periodic response. The motion of the structure consisted of the second flexural and first torsional modes. This nonlinear response lost stability at 17.518 Hz; modulated motions followed and they existed in the range $17.518 \text{ Hz} \leq f \leq 17.59 \text{ Hz}$. As f was increased further, nonlinear periodic responses were observed. At about 17.70 Hz, another jump occurred and thereafter, linear responses were observed. During the reverse sweep of the excitation frequency, jumps occurred at about 17.65 Hz and 17.15 Hz. The dashed lines

in the figure are drawn to indicate the jumps observed in the experiments. Also, the region of modulated motions remained the same during the reverse sweep. The nature of the observed modulated motions is discussed in the next paragraph. The form of the frequency-response curves seen here is similar to those observed in the response of the metallic structure (Fig. 4.2). However, in this case the observed responses are qualitatively different because they are nonplanar.

To determine the nature of the modulated motions displayed by Structure A, we examined the response spectrum in a zoom span (resolution 0.002 Hz) around the excitation (carrier) frequency f . In Figs. 5.6a to 5.6c, we show the response spectra at the excitation frequencies 17.539 Hz, 17.540 Hz, and 17.547 Hz. All three cases correspond to periodically modulated motions. In each case, the response spectrum has peaks at $f \pm n \delta f$, where $n=0,1,2,3 \dots$ and δf is the modulation frequency. The sharp spectral line at the middle of the structure corresponds to the excitation frequency and the uniformly spaced sidebands around it are due to a periodic variation of the amplitudes and phases of the interacting modes. As we proceed from Fig. 5.6a to Fig. 5.6b, we see a distinct peak appearing midway between the carrier frequency and the largest other peak, indicating that the modulation frequency has undergone a frequency halving and the period of the modulation has undergone a period doubling. So, a period-doubling bifurcation occurs as we increase f from 17.539 Hz to 17.540 Hz. In Fig. 5.6c, more subharmonics of the modulation frequency can be seen and the spectral content suggests that the

corresponding modulation period is about four times that observed in Fig. 5.6b. As f was increased further to 17.551 Hz, chaotically modulated motions occurred. This is indicated by the broadband structure around the carrier frequencies (f and $f/2$) in Figs. 5.7a and 5.7b.

Chaotically modulated motions were observed in the range $17.551 \text{ Hz} \leq f \leq 17.575 \text{ Hz}$. We verified that the motions were chaotic in this region using dimension calculations. When we increased f to 17.58 Hz, two-period quasiperiodic or periodically modulated motions were observed. In Fig. 5.7c the spectrum of the modulated response observed at 17.582 Hz is shown. This sequence to chaotically modulated motions via a period-doubling bifurcation is similar to that observed during primary-resonant excitations of the second flexural mode of the metallic structure. This sequence is analytically known to occur in systems with a two-to-one internal resonances and quadratic nonlinearities during such excitations (e.g., Nayfeh and Raouf, 1987b). The transition to a chaotically modulated motion via a quasiperiodic motion is shown in Fig. 5.8 by means of Poincaré sections. The first three sections and the last one correspond to periodically modulated motions. The sections shown in Figs. 5.8d and 5.8e correspond to chaotically modulated motions. In each of these cases, the scatter in the points that form the section is a characteristic of chaotic motions.

In Fig. 5.9, we show the plot for the pointwise dimension d for the motion of Structure A observed at 17.551 Hz. The plot has the logarithm of the number

of points $N(r)$ in an n -dimensional ball of radius r on the y -axis and the logarithm of the radius of the ball on the x -axis. Curves for different embedding dimensions n are shown in the plot. A delay of 0.5 seconds was used for the embedding. As stated earlier, the delay was varied to give as large a scaling region for $N(r)$ as possible. For embedding dimensions of four and above the curves run parallel in a certain region, indicating that an embedding dimension of eight should be adequate for the calculations. For an embedding dimension of eight, d was found to be 3.482 confirming the chaotic nature of the motion. Similar dimension calculations were made for the motions observed at 17.56 Hz and 17.57 Hz. Delays of 0.79 and 0.925 seconds were used for embedding at 17.56 Hz and 17.57 Hz, respectively. For an embedding dimension of eight, d was found to be about 4.154 at 17.56 Hz and 3.868 at 17.57 Hz. The corresponding plots for pointwise dimension are shown in Figs. 5.10 and 5.11, respectively. It is also interesting to note the change in the dimension value due to a change of 0.01 Hz in the excitation frequency. We note that chaotically modulated motions were observed at 40.00 mili g's rms, a small excitation level. The dimension values determined for the composite structure are in general higher than those determined for the metallic structure. In the present case, from the value of the dimension for the different cases of chaotic motions, we infer that we need at least a fifth-order autonomous system of equations to model the composite structure. This also suggests that equations (5.2)-(5.4) should be adequate to make qualitative predictions for the composite structure's behavior.

Again, as in the case of the steel beam-mass structures, we found the dynamic response of the glass-epoxy composite structure to be sensitive to the experimental conditions (temperature, humidity) in the region of modulated motions. In Fig. 5.12, the observed responses are shown at an excitation frequency of 17.543 Hz and an excitation amplitude of 40.00 mili g's rms under two different experimental conditions. The temperature and relative humidity measured at a certain location in the room are provided in the figure caption. In both cases, the motion is periodically modulated. However, these modulations are different as can be seen from the frequency spectra and the Poincaré sections.

Structure B was also subjected to primary-resonant excitations. During primary-resonant excitations of the first flexural mode, linear and nonlinear responses were observed at an excitation level of 40.0 mili g's rms. The motion was nonplanar during the nonlinear responses due to the presence of the torsional mode. Here, we could not use the response spectra to determine the modal amplitudes because the first flexural and torsional modes responded at the same frequency. This aspect points to a limitation of using strain gages for measuring displacements in such cases. Visual observations were used to determine if the responses were planar or nonplanar. In another experiment, during primary-resonant excitations of the second flexural mode, we could not excite any nonlinear responses for excitations levels up to 100.00 mili g's rms.

We conducted experiments with Structure C to study its response to combination-resonant excitations. In this case, the first three linear frequencies were 5.70 Hz, 8.58 Hz, and 17.22 Hz. During the combination-resonant excitations, we could not excite nonlinear responses of the composite structure up to excitation levels of 2.45 g's rms. We could not go to higher excitation levels because we could not maintain a constant level of excitation. A certain threshold has to be crossed to excite the nonlinear responses and for the composite structure this threshold is higher than 2.45 g's rms. In the case of the metallic structure (Structure IV), the threshold was about 0.85 g's rms. The higher threshold in the case of the composite structure may be due to its higher modal damping.

As in the case of the metallic structures, the response takes a long time to settle down at places where a qualitative change occurs (it takes about 45 minutes or more for the motion to settle down at transitions from periodic to modulated motions). The Poincaré sections proved to be useful in determining whether a modulated response had reached a steady state or not.

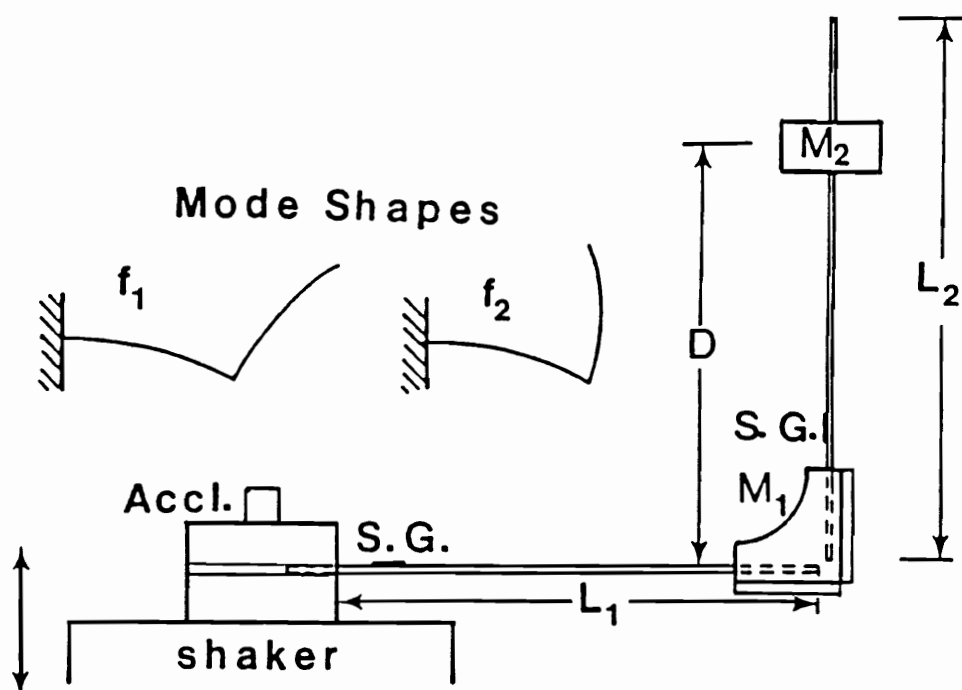


Figure 5.1. Composite beam-mass structure and accompanying mode shapes of first two flexural modes of vibration.

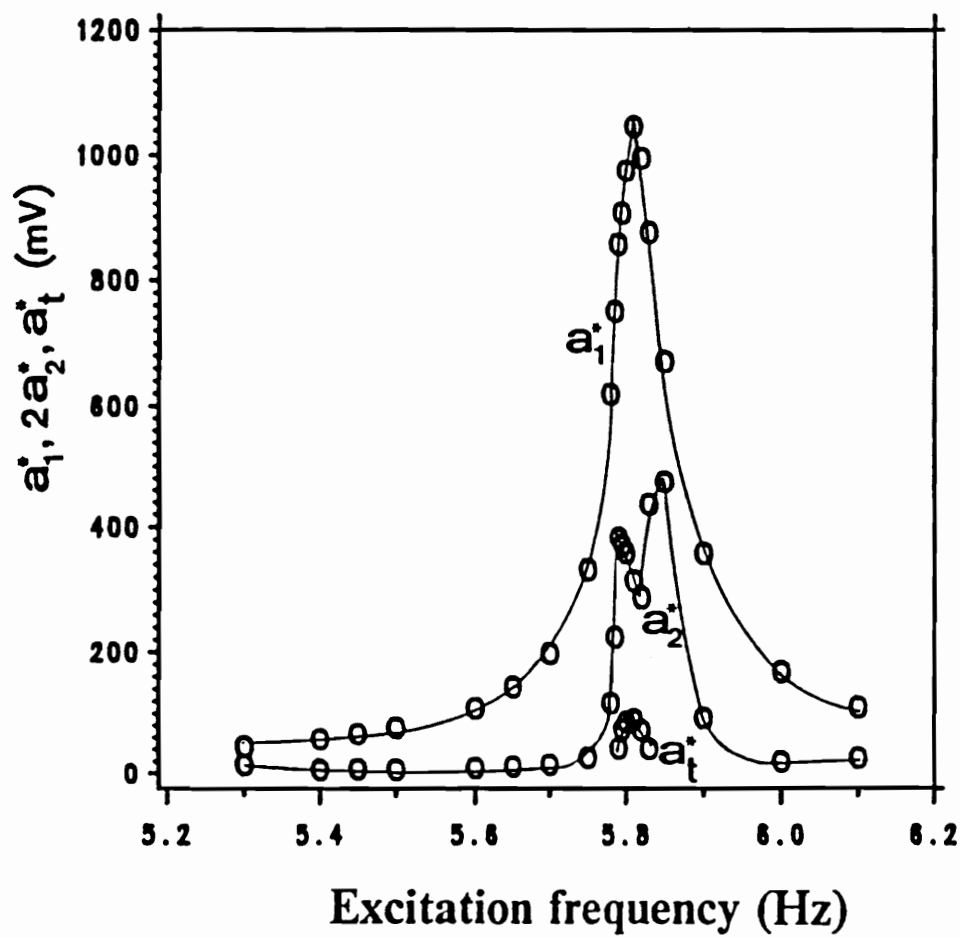


Figure 5.2. Experimentally obtained frequency-responses curves of Structure A, when the excitation frequency was close to the first flexural frequency.

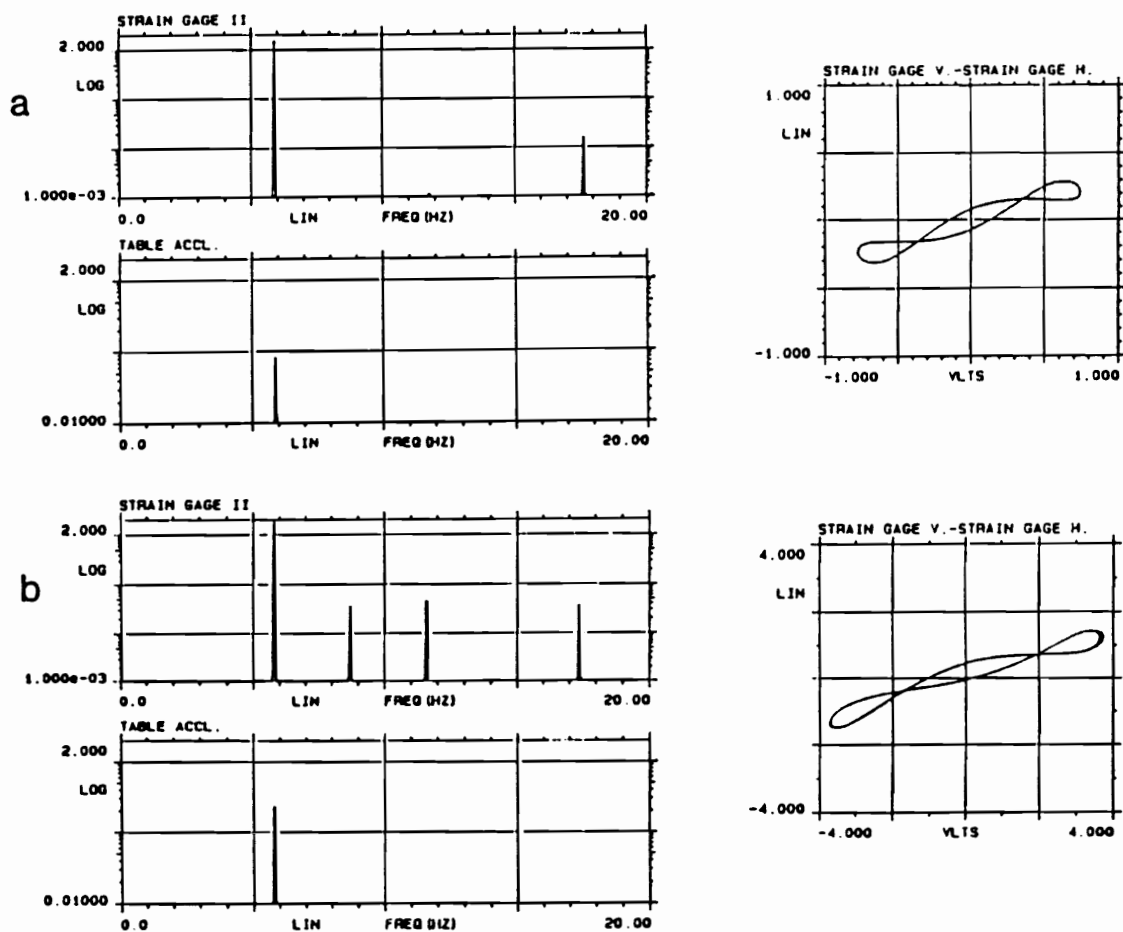


Figure 5.3. Spectra of the response and the excitation and corresponding cross plots of strain-gage signals: a) $f = 5.88$ Hz, planar and nonlinear periodic response and b) $f = 5.795$ Hz, nonplanar and nonlinear periodic response.

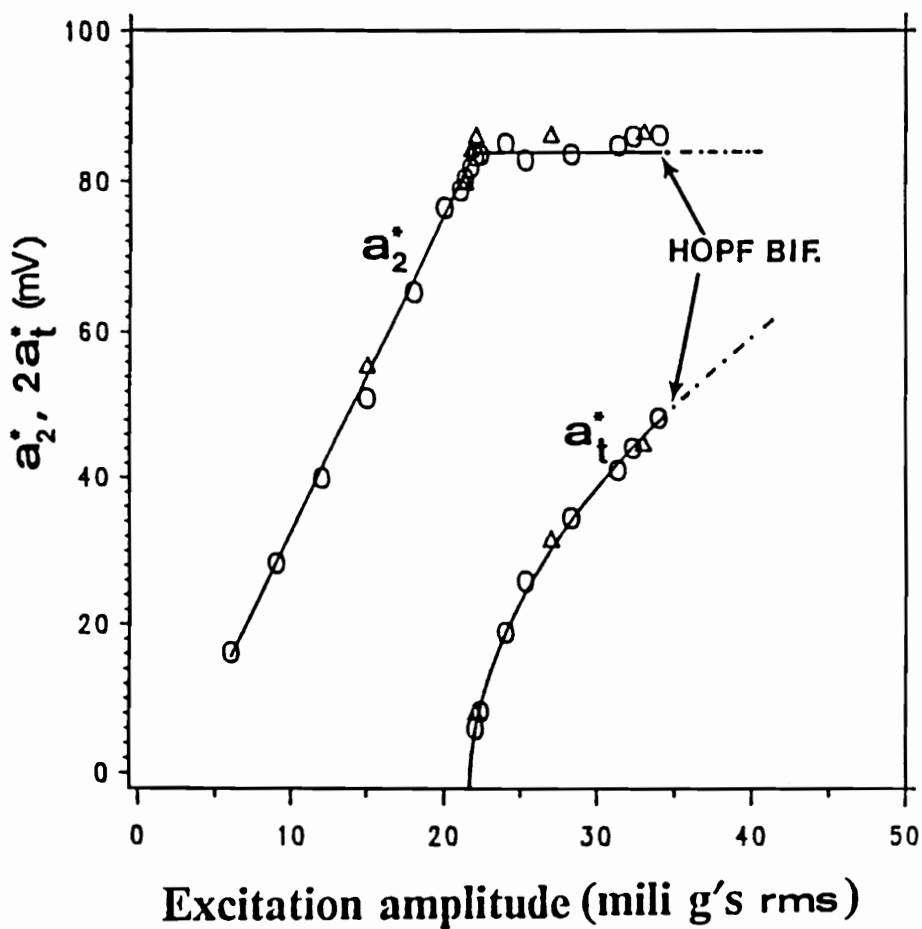


Figure 5.4. Experimentally obtained force-response curves of Structure A at $f = 17.53$ Hz: circles (triangles) represent observations made during forward (reverse) sweep of the control parameter; solid (dotted) lines represent periodic (modulated) motions.

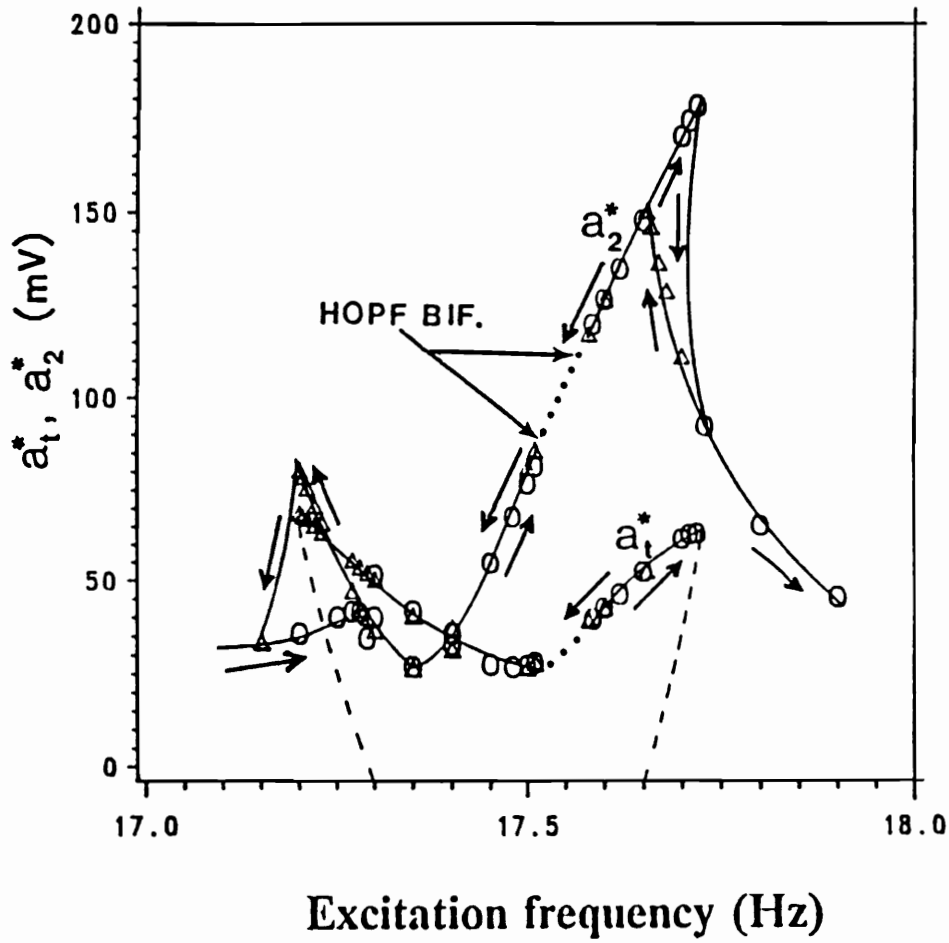


Figure 5.5. Experimentally obtained frequency-response curves of Structure A when the forcing frequency was close to the second flexural frequency: circles (triangles) represent observations made during forward (reverse) sweep of the control parameter; solid (dotted) lines represent periodic (modulated) motions.

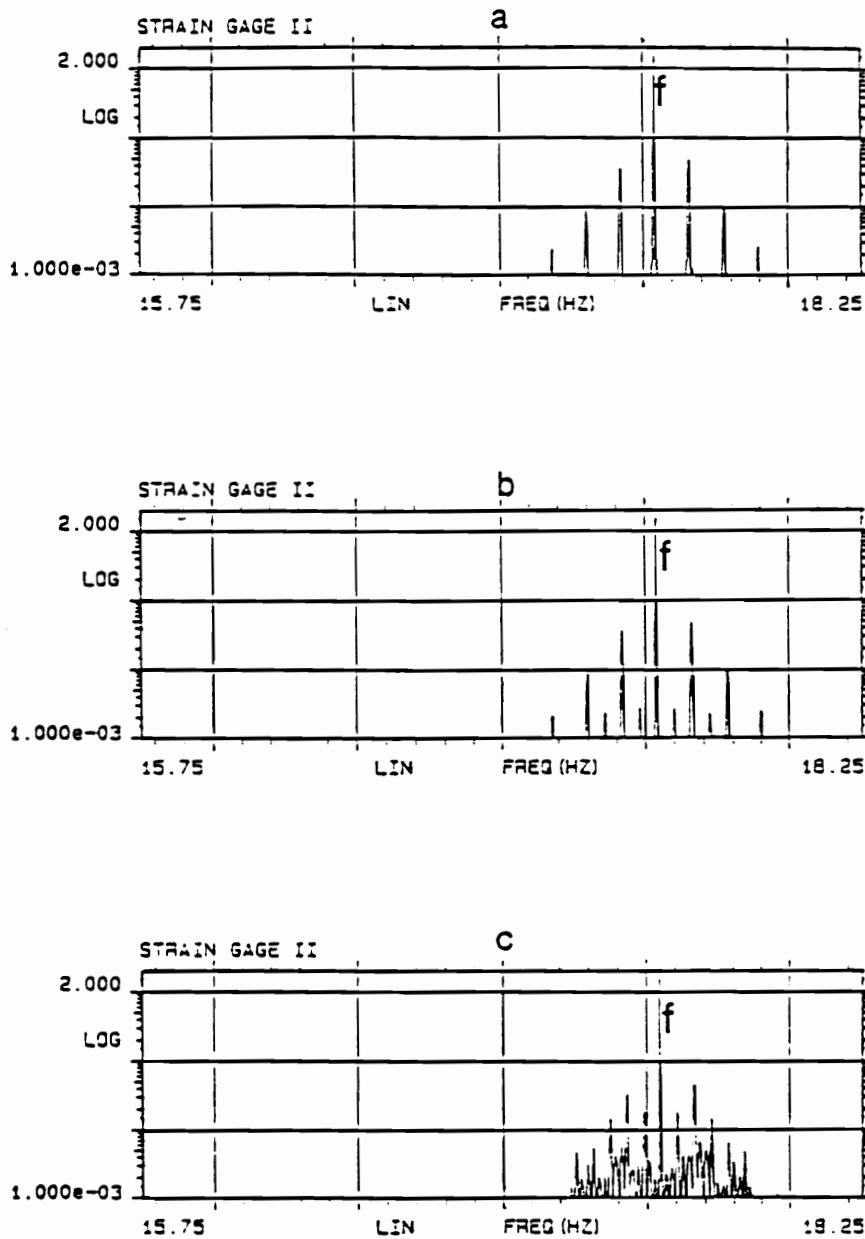


Figure 5.6. Spectra of the responses on a log scale in a zoom span: a) $f = 17.539$ Hz, b) $f = 17.540$ Hz, and c) $f = 17.547$ Hz. We note uniformly spaced sidebands around the carrier frequency and the appearance of the subharmonics of the modulation frequency as we proceed from a) to b) to c).

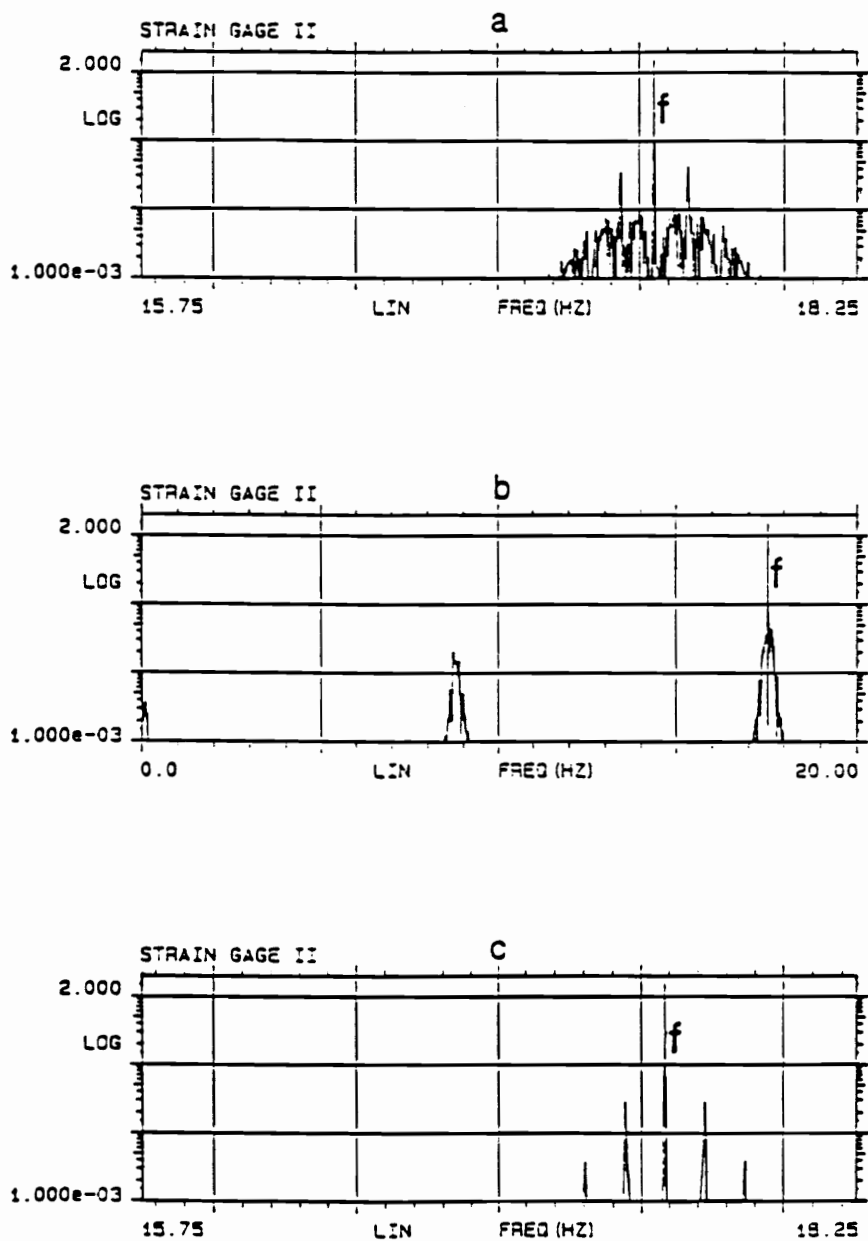


Figure 5.7. Spectra of the responses on a log scale: a) $f = 17.551$ Hz, b) $f = 17.551$ Hz, and c) $f = 17.582$ Hz. We note a broadband character around the carrier frequencies in a) and b).

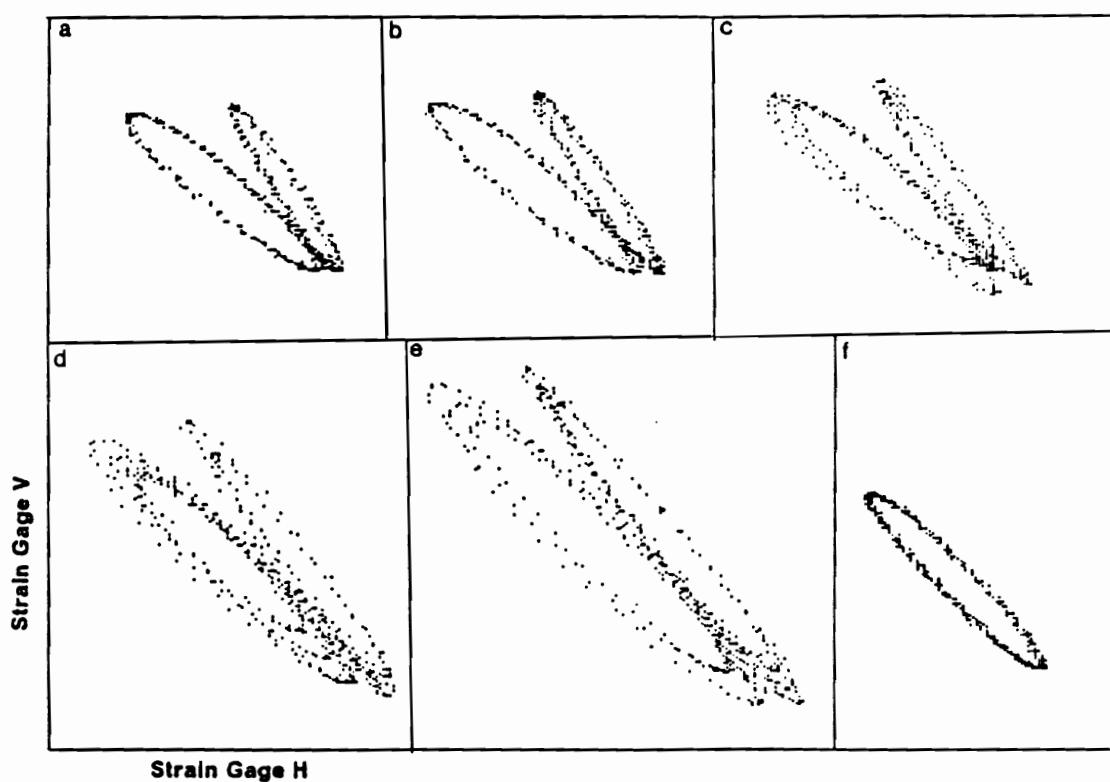


Figure 5.8. Poincaré sections of modulated motions: a) $f = 17.539$ Hz, b) $f = 17.54$ Hz, c) $f = 17.547$ Hz, d) $f = 17.551$ Hz, e) $f = 17.567$ Hz, and f) $f = 17.582$ Hz.

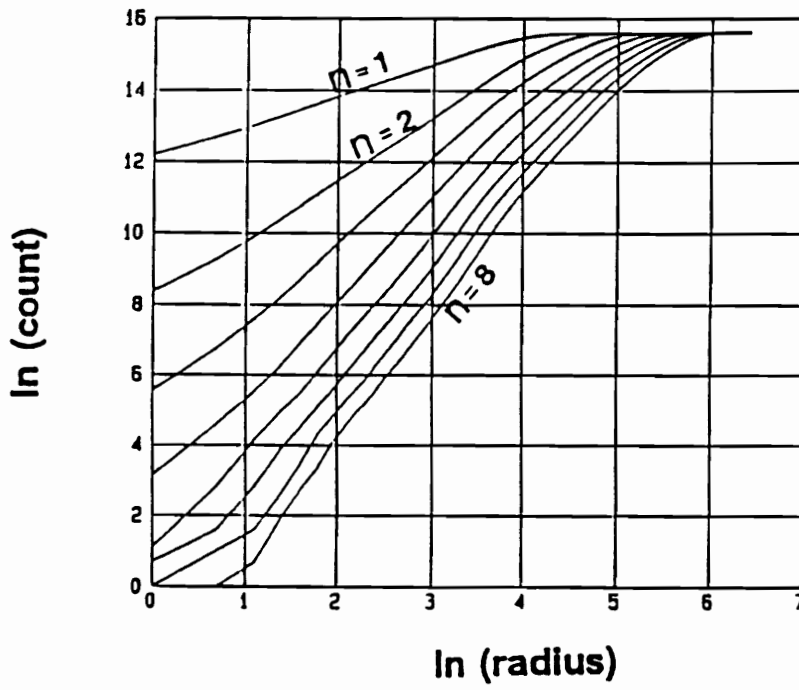


Figure 5.9. Plot for pointwise dimension for different values of the embedding dimension for the motion at $f = 17.551$ Hz: $d = 3.482$ for $n = 8$.

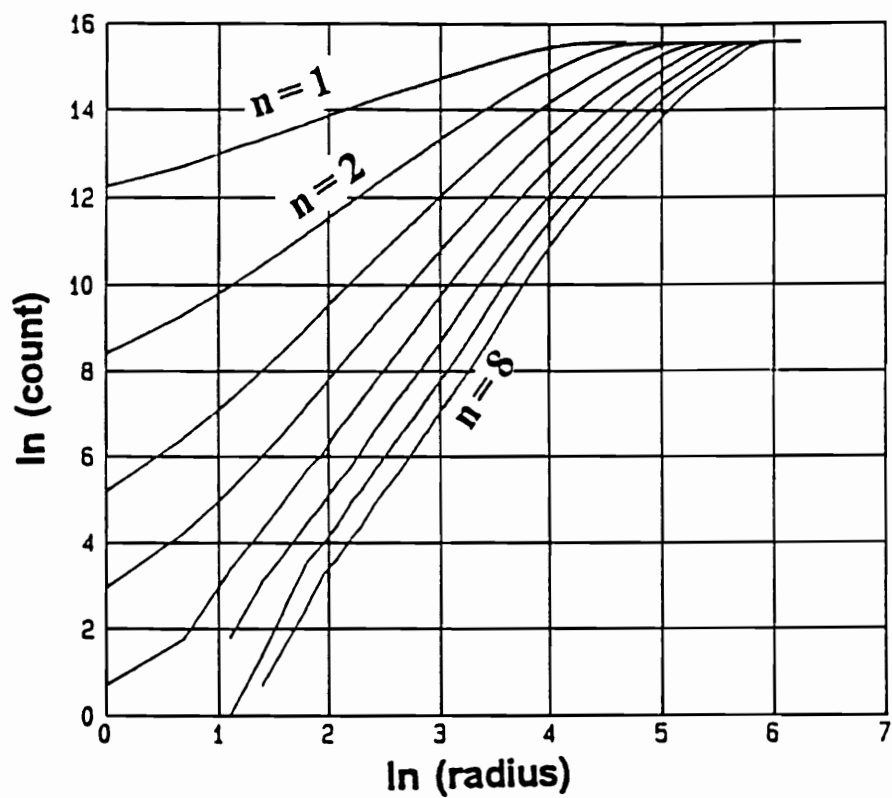


Figure 5.10. Plot for pointwise dimension for different values of the embedding dimension for the motion at $f = 17.56$ Hz: $d = 4.154$ for $n = 8$.

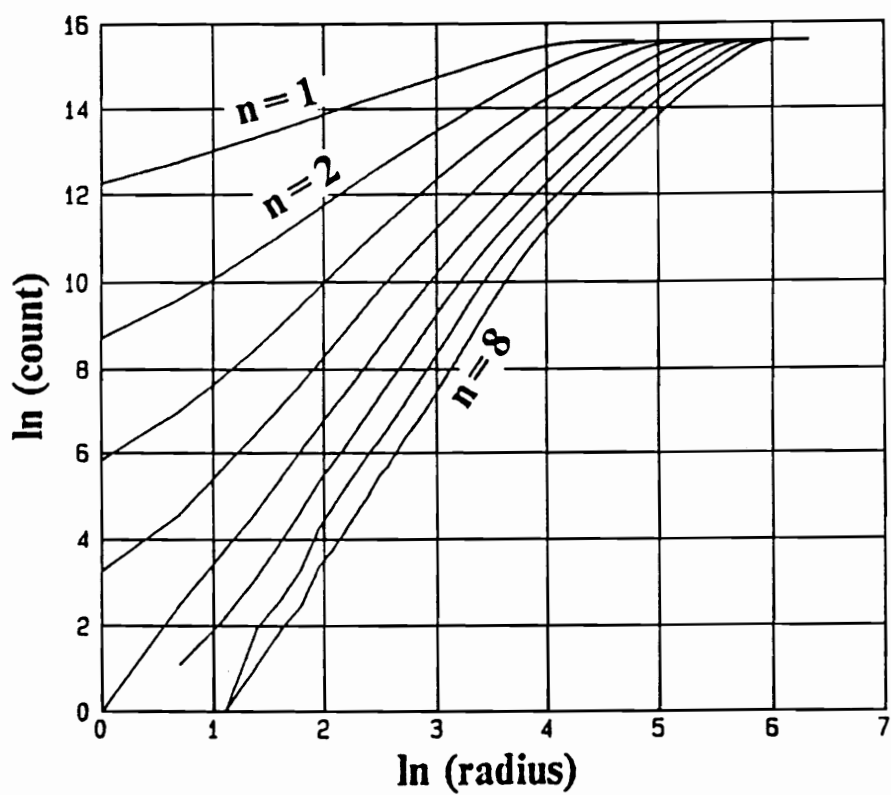


Figure 5.11. Plot for pointwise dimension for different values of the embedding dimension for the motion at $f = 17.57$ Hz: $d = 3.868$ for $n = 8$.

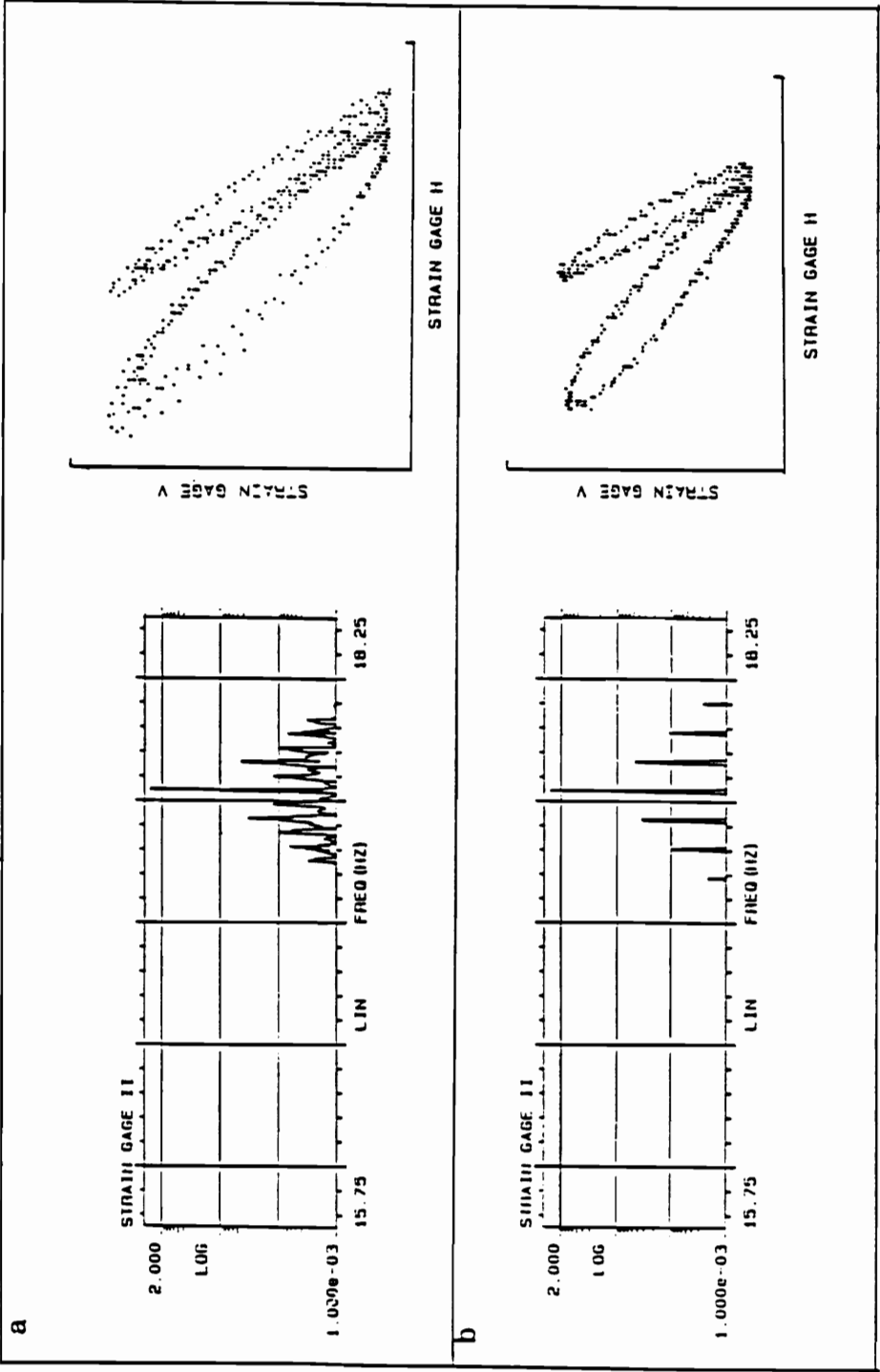


Figure 5.12. Response spectra and Poincaré sections at same excitation conditions but different experimental conditions: a) temperature = 76.6°F and relative humidity = 48.2 % and b) temperature = 74.2°F and relative humidity = 47.6 %.

6. Motion near a Hopf Bifurcation of Three- and Four-Dimensional Systems

In this chapter, three-dimensional and four-dimensional autonomous systems with quadratic nonlinearities are studied. The parameters in each system are such that they permit a Hopf bifurcation (known as flutter instability in aeroelasticity) to occur. We also assume that this bifurcation is a codimension-one bifurcation of a point attractor. As mentioned in Chapter 2, a Hopf bifurcation of a point attractor leads to a limit-cycle. This limit cycle is stable or unstable depending on the nature of the bifurcation (i.e., supercritical or subcritical). In Fig. 6.1, we show bifurcation diagrams for the supercritical and subcritical bifurcations. The amplitude of the limit-cycle motion is plotted against the control parameter μ in the bifurcation diagram. A Hopf bifurcation occurs at $\mu = 0$. In this figure, the symbol s (u) stands for stable (unstable) fixed points, and solid (dashed) lines represent stable (unstable) limit cycles. We draw closed curves in the figure to indicate limit-cycle motions. When the

Hopf bifurcation is supercritical, locally at the bifurcation point, we have stable fixed points on one side and stable limit cycles on the other side of the bifurcation point. As the transition from the stable fixed point to the stable limit cycle is smooth in this case, the supercritical Hopf bifurcation is also known as soft or benign flutter. In Fig. 6.1b, we show the scenario for a subcritical Hopf bifurcation. In this figure, we have stable fixed points, unstable limit cycles, and stable limit cycles to the left side of the bifurcation point at $\mu = 0$. Further, we have stable limit cycles and unstable fixed points to the right side of the bifurcation point. In this case, the transition from a stable fixed point to a stable limit cycle is accompanied by a large change in the amplitude of the motion and is not gradual as in the case of soft flutter. Hence, this bifurcation is also known as hard or explosive flutter.

Here, analytical approximations are obtained for the amplitude and frequency of the limit cycle near a Hopf-bifurcation point, from which the nature of the Hopf bifurcation (i.e., subcritical or supercritical) is ascertained. Information on the transient behavior leading to the limit-cycle motions can also be obtained from this analytical approximation. There are essentially two approaches to derive the analytical approximations. In the first approach, one uses a center manifold reduction (Carr, 1981; Rand and Armbruster, 1987) to reduce the order of the original autonomous system. In this process, one approximates the center manifold (a curved space which is tangent to the eigenvectors of the linearized system at the bifurcation point) by a power series and then projects the n -dimensional flow (in the present chapter $n = 3$

or 4) onto the center manifold. Here, the reduction results in a two-dimensional system. The reduced system can be studied by using the method of normal forms, the method of averaging, or the method of multiple scales (Nayfeh, 1973; Rand and Armbruster, 1987). In the second approach, one deals with the original autonomous system and studies it by using the method of normal forms, the method of strained parameters, the method of averaging (e.g., Sethna and Schapiro, 1977; Sethna and Bajaj, 1978), or the method of multiple scales (e.g., Smith and Morino, 1976). When the method of strained parameters is used, as in the study of Sethna and Schapiro (1977), one does not obtain any information on the transient motions leading to the final orbital motions. However, the other methods can provide this information.

Here, we follow the second approach and use the method of multiple scales to obtain an asymptotic expansion for the limit-cycle motion near the Hopf-bifurcation point. The three-dimensional system treated by Rand (1989) and a generalized version of two of the four four-dimensional autonomous systems derived in Chapter 3 are considered in this chapter. We also determine if the results of the analysis for the four-dimensional system can be verified experimentally.

6.1. Three-dimensional system

As mentioned earlier, we obtain an asymptotic expansion for the amplitude and frequency of the limit cycle near the Hopf-bifurcation point of the system. The resulting analytical approximation is valid only in a small range near the Hopf-bifurcation point. This range is ascertained by using numerical simulations and stability analysis. Floquet theory is used to determine the stability of the periodic solutions. Also, Rand's analysis is discussed in light of the present study. The perturbation analysis presented here can be easily applied to higher-dimensional systems. The discussion on the range of validity of the analytical approximation and the stability analysis are pertinent to higher-dimensional systems as well.

The problem studied by Rand (1989) is

$$\dot{x} = \mu x - y - xz \quad (6.1)$$

$$\dot{y} = \mu y + x \quad (6.2)$$

$$\dot{z} = -z + x^2 z + y^2 \quad (6.3)$$

where x , y , and z are three scalar variables, μ is a scalar control parameter, and the overdot indicates the derivative with respect to time t . The system may be thought of as a feedback control system made up of a damped linear oscillator in the x and y variables and a control variable z . The fixed point of

the system $x = y = z = 0$ is stable (unstable) when $\mu < 0$ ($\mu > 0$). A Hopf bifurcation occurs at $\mu = 0$. It follows from the form of the equations that if x , y , and z is a solution, then $-x$, $-y$, and z is also a solution. Using a center manifold reduction, Rand (1989) reduced equations (6.1)-(6.3) to a two-dimensional system. He first approximated the center manifold by a power series and then projected the three-dimensional flow onto this center manifold. The result is a two-dimensional system of equations. Later, he introduced a near-identity transformation (method of normal forms) to determine an approximation to the limit cycles near the Hopf-bifurcation point. Here, we use the method of multiple scales to determine a first-order approximation to the limit cycles near the Hopf-bifurcation point and determine its range of validity with the help of numerical simulations and stability analysis. Further, we show that as μ increases from zero, the limit cycle displays an inversion symmetry in the $x - y$ plane, increases in size, and deforms smoothly until μ reaches $\mu^{(1)} \approx 0.3002$. As μ increases beyond $\mu^{(1)}$, the limit cycle loses its inversion symmetry. Then, as μ increases further, the two resulting unsymmetric limit cycles deform smoothly until μ reaches the second critical value $\mu^{(2)} \approx 0.4405$. As μ exceeds $\mu^{(2)}$, the period of the unsymmetric limit cycles doubles. This sequence of bifurcations is expected in symmetric systems (Swift and Wiesenfeld, 1984). The results show that a general disturbance provided along all directions is necessary for the detection of the instabilities.

6.1.1. Perturbation Analysis

Using the method of multiple scales, we seek expansions for the three scalar variables in the form

$$x = \sum_{n=0}^3 \varepsilon^n x_n(T_0, T_1, T_2, \dots) + \dots \quad (6.4)$$

$$y = \sum_{n=0}^3 \varepsilon^n y_n(T_0, T_1, T_2, \dots) + \dots \quad (6.5)$$

$$z = \sum_{n=0}^3 \varepsilon^n z_n(T_0, T_1, T_2, \dots) + \dots \quad (6.6)$$

where $T_n = \varepsilon^n t$ and ε is a small dimensionless parameter that is used as a bookkeeping device and will be set equal to unity in the final analysis. In terms of the T_n , the time derivative becomes

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots \quad (6.7)$$

where $D_n = \partial/\partial t_n$. Also, the control parameter is expanded as

$$\mu = \varepsilon^2 \mu_2 + O(\varepsilon^3) \quad (6.8)$$

where μ is ordered at $O(\varepsilon^2)$ so that the influence of the nonlinear terms and the control parameter μ is realized at the same order.

Substituting equations (6.4)-(6.8) into equations (6.1)-(6.3) and equating coefficients of like powers of ε , we obtain the following equations:

$O(\varepsilon)$:

$$D_0 x_1 + y_1 = 0 \quad (6.9)$$

$$D_0 y_1 - x_1 = 0 \quad (6.10)$$

$$D_0 z_1 + z_1 = 0 \quad (6.11)$$

$O(\varepsilon^2)$:

$$D_0 x_2 + y_2 = -D_1 x_1 - x_1 z_1 \quad (6.12)$$

$$D_0 y_2 - x_2 = -D_1 y_1 \quad (6.13)$$

$$D_0 z_2 + z_2 = -D_1 z_1 + y_1^2 \quad (6.14)$$

$O(\varepsilon^3)$:

$$D_0 x_3 + y_3 = -D_1 x_2 - D_2 x_1 + \mu_2 x_1 - x_1 z_2 - x_2 z_1 \quad (6.15)$$

$$D_0 y_3 - x_3 = -D_1 y_2 - D_2 y_1 + \mu_2 y_1 \quad (6.16)$$

$$D_0 z_3 + z_3 = -D_1 z_2 - D_2 z_1 + x_1^2 z_1 + 2y_1 y_2 \quad (6.17)$$

The nondecaying solution of equations (6.9)-(6.11) is

$$x_1 = iA(T_1, T_2)e^{iT_0} + cc \quad (6.18)$$

$$y_1 = A(T_1, T_2)e^{iT_0} + cc \quad (6.19)$$

$$z_1 = 0 \quad (6.20)$$

where cc stands for the complex conjugate of the preceding terms and A is determined by imposing solvability conditions at the next level of approximation.

Substituting equations (6.18)-(6.20) into equations (6.12)-(6.14) and eliminating the terms that produce secular terms, we find that $D_1 A = 0$ and hence that $A = A(T_2)$. Then, the solution of equations (6.12)-(6.14) is

$$x_2 = y_2 = 0 \quad (6.21)$$

$$z_2 = A\bar{A} + \frac{A^2 e^{2iT_0}}{1 + 2i} + cc \quad (6.22)$$

where $\bar{A}$ is the complex conjugate of A . Also, in equations (6.21) and (6.22) because A is assumed to implicitly depend on the parameter ε , only the particular solution of equations (6.12)-(6.14) is considered.

Substituting equations (6.18)-(6.22) into equations (6.15) and (6.16) and eliminating the terms that produce secular terms, we obtain

$$2A' - 2\mu_2 A + \left(\frac{9}{5} + \frac{2}{5}i \right) A^2 \bar{A} = 0 \quad (6.23)$$

where the prime denotes the derivative with respect to T_2 . Substituting $A = \frac{1}{2} a e^{i\beta}$, where a and β are real quantities into equation (6.23), separating real and imaginary parts, and setting $\varepsilon = 1$, we obtain

$$\dot{a} = \mu_2 a - \frac{9}{40} a^3 \quad (6.24)$$

$$\dot{\beta} = -\frac{1}{20} a^2 \quad (6.25)$$

Equations (6.24) and (6.25) are equivalent to those obtained by Rand (1989) using normal forms. We note that the method of multiple scales involves less algebra than the method of normal forms.

It follows from equation (6.24) that $a \rightarrow 0$ if $\mu_2 < 0$. When $\mu_2 > 0$ there is a stable periodic orbit whose radius is a and frequency is $(1 + \beta)/2\pi$. The radius a of the limit cycle is $\frac{2}{3} \sqrt{10\mu_2}$. For $\mu > 0$, locally there is an exchange of stability between the limit-cycle branch and the fixed-point branch. Thus, $\mu = 0$ is a supercritical Hopf-bifurcation point. It follows from equations (6.4)-(6.6), (6.18)-(6.22), (6.24), and (6.25) that to second order the limit-cycle solution of equations (6.1)-(6.3) is given by

$$x = -a \sin \theta t + O(a^3) \quad (6.26)$$

$$y = a \cos \theta t + O(a^3) \quad (6.27)$$

$$z = \frac{1}{2} a^2 \left[1 + \frac{1}{5} \cos 2\theta t + \frac{2}{5} \sin 2\theta t \right] + O(a^3) \quad (6.28)$$

where

$$a = \frac{2}{3} \sqrt{10\mu} \quad \text{and} \quad \theta = \left(1 - \frac{1}{20} a^2 \right) + O(a^3) \quad (6.29)$$

It follows from equation (6.29) that the amplitude of the limit cycle is proportional to the square root of the control parameter as in other 'generic' systems discussed by Bergé et al. (1984) and Seydel (1988).

6.1.2. Comparison

The expansions (6.26)-(6.28) are determined for small values of μ and hence their accuracy is expected to deteriorate as μ becomes large. In Fig. 6.2, the analytical approximation (dashed lines) and the numerical solutions (solid lines) of equations (6.1)-(6.3) are given for comparison. We consider the projection of the motion on the $x - y$ plane for different values of μ . The numerical solutions are obtained by integrating equations (6.1)-(6.3) by using the Runge-Kutta-Verner fifth- and sixth-order method. The approximation deteriorates as μ increases to 0.2, and at $\mu = 0.3$ it is not even close to the

numerical solution in either form or magnitude. The analytical approximation is not shown for values of $\mu > 0.3$. Next, we present the stability analysis.

6.1.3. Stability Analysis

To describe the dynamics of the system (6.1)-(6.3), we need to determine the stable and unstable limit cycles. These limit cycles are located by using an algorithm originally proposed by Aprille and Trick (1972). It starts with a guess for the period and initial conditions of the limit cycle and then uses a combination of a numerical integration scheme and a Newton-Raphson iteration procedure. This algorithm is sensitive to the initial guesses and the step size of the numerical integration when multiple solutions coexist.

After a periodic solution $x(t)$, $y(t)$, and $z(t)$ with period T has been determined analytically or calculated numerically its stability is ascertained by Floquet theory. To this end, we superimpose on $x(t)$, $y(t)$, and $z(t)$ a general disturbance having the components $\xi_1(t)$, $\xi_2(t)$, and $\xi_3(t)$, respectively, substitute the total solution into equations (6.1)-(6.3), linearize the resultant equations, and obtain

$$\dot{\xi} = \begin{bmatrix} \mu - z & -1 & -x \\ 1 & \mu & 0 \\ 2xz & 2y & x^2 - 1 \end{bmatrix} \xi \quad (6.30)$$

where ξ is a column vector whose components are ξ_1 , ξ_2 , and ξ_3 and the 3x3 (Jacobian) matrix on the right-hand side of (6.30) is periodic with period T . A solution $x(t)$, $y(t)$, and $z(t)$ of equations (6.1)-(6.3) is asymptotically stable if and only if $\xi(t)$ decays with time. To determine the behavior of $\xi(t)$ as $t \rightarrow \infty$, we calculate three linearly independent solutions of (6.30) from $t = 0$ to $t = T$ by using the initial conditions

$$\xi_1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \xi_2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \xi_3(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (6.31)$$

Then, we form the monodromy matrix

$$M = [\xi_1(T) \quad \xi_2(T) \quad \xi_3(T)] \quad (6.32)$$

The eigenvalues of the 3x3 matrix M are called Floquet multipliers and $\xi(t) \rightarrow 0$ as $t \rightarrow \infty$ if and only if all Floquet multipliers are within the unit circle in the complex plane. For the numerical solutions, the monodromy matrix is obtained as a by-product of the algorithm used to determine the periodic solutions. Moreover, one of the Floquet multipliers is unity for all values of μ .

The analytical approximation, equations (6.26) and (6.27), corresponds to a circle in the x - y plane. It displays inversion-symmetry in the x - y plane; that is, $x(t) = -x(t + \frac{1}{2}T)$ and $y(t) = -y(t + \frac{1}{2}T)$ where T is the period of the solution. Rand (1989) studied the stability of the analytical approximation to a disturbance along the z -direction only; that is, $\xi_1 = \xi_2 = 0$ in equation (6.30)

and found that the solution loses stability at $\mu^{(3)} = 0.45$. However, if one considers a disturbance along all directions, a Floquet multiplier leaves the unit circle at $\mu \approx \mu^{(4)} = 0.194$, implying that the analytical solution is unstable beyond $\mu^{(4)}$.

Considering the stability of the numerical solution to a general disturbance, we find that the solution loses its inversion-symmetry in the $x - y$ plane at $\mu = \mu^{(1)} \approx 0.3002$ due to a symmetry-breaking bifurcation. Following this bifurcation, the symmetric solutions of equations (6.1)-(6.3) become unstable. Also, the frequency spectra of the x and y variables contain even harmonics. The symmetry-breaking bifurcation is detected by determining the Floquet multipliers of the matrix

$$M_2 = \left[\xi_1\left(\frac{1}{2}T\right) \quad \xi_2\left(\frac{1}{2}T\right) \quad \xi_3\left(\frac{1}{2}T\right) \right] \quad (6.33)$$

which is obtained by integrating over one-half the period of the limit cycle; that is, $\frac{1}{2}T$. At $\mu = \mu^{(1)}$ a Floquet multiplier corresponding to M_2 leaves the unit circle through 1, indicating the occurrence of the symmetry-breaking bifurcation. For the symmetric solutions, a multiplier always stays at -1. If, we had considered the stability of the numerical solution to a disturbance along the z -direction only, then we would not have detected this bifurcation.

As μ increases beyond $\mu^{(1)}$, the two clone unsymmetric limit cycles deform smoothly (one of these clones is shown in Figures 6.2d, 6.2e, and 6.2f; the other can be obtained by replacing x and y with $-x$ and $-y$, respectively) until

μ reaches the second critical value $\mu^{(2)} \approx 0.4405$. At this value, a Floquet multiplier corresponding to the matrix M leaves the unit circle through -1 , indicating the occurrence of the first period-doubling bifurcation.

Rand (1989) used the analytical approximation, discussed earlier, to predict the first period-doubling bifurcation. The form of the analytical approximation is incorrect for the unsymmetric solutions following the symmetry-breaking bifurcation. Moreover, in the present case, this bifurcation occurs at a value of μ which is beyond the range of validity of the analytical approximation. So, the analytical approximation is not good enough to predict the symmetry-breaking bifurcation and the subsequent bifurcations. In a recent paper, Rand (1990) treated the same system and carried out a revised version of his earlier analysis. The difference between the two versions is in the stability analysis. In the first paper, Rand used a variable which represented the distance perpendicular to the tangent space (here, the x - y plane) to the center manifold (a curved space) at the origin ($x = y = z = 0$). In the recent paper, Rand used a variable which represented the distance normal to the center manifold itself. This revision led to an improvement in his stability predictions and the revised version predicts the first instability following the Hopf bifurcation to occur at $\mu_{cr} = 0.25$, which is closer to the exact value of 0.3002 obtained numerically in the present work.

We have illustrated the use of the method of multiple scales to obtain asymptotic expansions for motion near a Hopf bifurcation of a

three-dimensional system. This perturbation analysis can be easily extended to higher-dimensional systems. It is extended to a four-dimensional system in the following section. The motion of the reduced system following the center manifold reduction usually displays an inversion-symmetry. In those cases, it is expected that the first period-doubling instability will be preceded by a loss of symmetry through a symmetry-breaking bifurcation. If the analytical approximation is symmetric, it can at most be used to detect the symmetry-breaking bifurcation preceding the first period-doubling bifurcation. In the stability analysis, a disturbance along all directions is necessary to detect the symmetry-breaking bifurcation.

6.2. *Four-dimensional system*

In this section, we consider motion near a Hopf-bifurcation point of a four-dimensional autonomous system. In Chapter 3, we derived equations (3.58)-(3.61) for the case of primary resonance of the first mode and equations (3.70)-(3.73) for the case of primary resonance of the second mode. Both systems of equations can be obtained as special cases of the four-dimensional system treated here. The primary interest is in using the analytical approximations to determine the nature of the Hopf bifurcation (i.e., subcritical or supercritical) and in validating the results experimentally. The region of validity of the analytical approximation is ascertained by carrying out an

analysis similar to that detailed for the three-dimensional system. The approach used here is also applicable to the other fourth-order autonomous systems derived in Chapter 3. The equations to be treated are of the form

$$p'_1 + \mu_1 p_1 + \nu_3 q_1 + \Lambda_1(p_2 q_1 - p_1 q_2) = 0 \quad (6.34)$$

$$q'_1 + \mu_1 q_1 - \nu_3 p_1 + \Lambda_1(p_1 p_2 + q_1 q_2) = f_1 \quad (6.35)$$

$$p'_2 + \mu_2 p_2 + \nu_4 q_2 - 2\Lambda_2 p_1 q_1 = 0 \quad (6.36)$$

$$q'_2 + \mu_2 q_2 - \nu_4 p_2 + \Lambda_2(p_1^2 - q_1^2) = f_2 \quad (6.37)$$

where the prime denotes a derivative with respect to time t . In equations (6.34)-(6.37), if f_2 is set equal to zero and f_1 , ν_3 , and ν_4 are replaced respectively by $\Lambda_3 F$, σ_2 , and $2\sigma_2 - \sigma_1$, we obtain equations (3.58)-(3.61) of Chapter 3. Similarly, setting $f_1 = 0$, replacing f_2 by $\Lambda_6 F$, putting $\nu_3 = \frac{1}{2}(\sigma_1 + \sigma_2)$, and setting $\nu_4 = \sigma_2$ in equations (6.34)-(6.37), we obtain equations (3.70)-(3.73) of Chapter 3. In the rest of the section, whenever we refer to primary resonance of the first or second mode it is to be understood that we make the respective changes in equations (6.34)-(6.37) to obtain the corresponding equations.

For the case of primary resonance of the second mode, examining equations (6.34)-(6.37), we see that, if p_1, q_1, p_2 , and q_2 is a solution, then $-p_1, -q_1, p_2$, and q_2 is also a solution. This symmetry property is similar to that observed in the previously treated three-dimensional system of equations. Equations (3.87)-(3.90) of Chapter 3 also display this reflection-symmetry

property. The method of multiple scales is used to obtain approximate expressions for the amplitude and frequency of the limit cycle near a Hopf bifurcation point of the system.

Using the external detuning σ_2 as the control parameter, we let $\sigma_2 = \sigma_{2c}$ at the Hopf-bifurcation point of the system and let p_{10}, q_{10}, p_{20} , and q_{20} be the corresponding fixed point of equations (6.34)-(6.37). Also, we let the critical values of v_i at the Hopf bifurcation point be v_{ic} . Next, we carry out the perturbation analysis.

6.2.1. Perturbation Analysis

We seek a third-order expansion around the Hopf-bifurcation point in the form

$$p_1 = p_{10} + \sum_{n=1}^{n=3} \varepsilon^n p_{1n}(T_0, T_1, T_2, \dots) + \dots \quad (6.38)$$

$$q_1 = q_{10} + \sum_{n=1}^{n=3} \varepsilon^n q_{1n}(T_0, T_1, T_2, \dots) + \dots \quad (6.39)$$

$$p_2 = p_{20} + \sum_{n=1}^{n=3} \varepsilon^n p_{2n}(T_0, T_1, T_2, \dots) + \dots \quad (6.40)$$

$$q_2 = q_{20} + \sum_{n=1}^{n=3} \varepsilon^n q_{2n}(T_0, T_1, T_2, \dots) + \dots \quad (6.41)$$

where $T_n = \varepsilon^n t$ are the different time scales and ε is a small dimensionless parameter that serves to establish the different orders of magnitude. The results obtained are independent of its choice and it is ultimately absorbed into the solution. This is equivalent to setting it equal to one in the final analysis. The transformation of the time derivative is given by equation (6.7), while the control parameter is expanded as

$$\sigma_2 = \sigma_{2c} + \varepsilon^2 \sigma + O(\varepsilon^3) \quad (6.42)$$

where the change σ about the critical value σ_{2c} is ordered at $O(\varepsilon^2)$ so that the influence of the nonlinear terms and the control parameter σ_2 is realized at the same order in the analysis. As a consequence of equation (6.42), we have

$$v_i = v_{ic} + \varepsilon^2 v_{ii} + O(\varepsilon^3) \quad \text{for } i = 3 \text{ and } 4 \quad (6.43)$$

where the v_{ii} are related to σ through equation (3.56) in the case of primary resonance of the first mode and through equation (3.69) in the case of primary resonance of the second mode.

Next, we substitute equations (6.38)-(6.43) into equations (6.34)-(6.37), expand the results, equate coefficients of like powers of ε , and obtain

$O(\varepsilon)$:

$$\begin{Bmatrix} D_0 p_{11} \\ D_0 q_{11} \\ D_0 p_{21} \\ D_0 q_{21} \end{Bmatrix} = [A] \begin{Bmatrix} p_{11} \\ q_{11} \\ p_{21} \\ q_{21} \end{Bmatrix} \quad (6.44)$$

$O(\varepsilon^2)$:

$$\begin{Bmatrix} D_0 p_{12} \\ D_0 q_{12} \\ D_0 p_{22} \\ D_0 q_{22} \end{Bmatrix} = [A] \begin{Bmatrix} p_{12} \\ q_{12} \\ p_{22} \\ q_{22} \end{Bmatrix} + \begin{Bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{Bmatrix} \quad (6.45)$$

$O(\varepsilon^3)$:

$$\begin{Bmatrix} D_0 p_{13} \\ D_0 q_{13} \\ D_0 p_{23} \\ D_0 q_{23} \end{Bmatrix} = [A] \begin{Bmatrix} p_{13} \\ q_{13} \\ p_{23} \\ q_{23} \end{Bmatrix} + \begin{Bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{Bmatrix} \quad (6.46)$$

where the elements a_{mn} of the 4×4 matrix $[A]$ are

$$a_{11} = -\mu_1 + \Lambda_1 q_{20}, \quad a_{12} = -v_{3c} - \Lambda_1 p_{20}, \quad a_{13} = -\Lambda_1 q_{10}, \quad a_{14} = \Lambda_1 p_{10} \quad (6.47)$$

$$a_{21} = v_{3c} - \Lambda_1 p_{20}, \quad a_{22} = -\mu_1 - \Lambda_1 q_{20}, \quad a_{23} = -\Lambda_1 p_{10}, \quad a_{24} = -\Lambda_1 q_{10} \quad (6.48)$$

$$a_{31} = 2\Lambda_2 q_{10}, \quad a_{32} = 2\Lambda_2 p_{10}, \quad a_{33} = -\mu_2, \quad a_{34} = -v_{4c} \quad (6.49)$$

$$a_{41} = -2\Lambda_2 p_{10}, \quad a_{42} = 2\Lambda_2 q_{10}, \quad a_{43} = v_{4c}, \quad a_{44} = -\mu_2 \quad (6.50)$$

The elements g_m are given by

$$g_1 = -D_1 p_{11} - \Lambda_1 (p_{21} q_{11} - p_{11} q_{21}) - v_{33} q_{10} \quad (6.51)$$

$$g_2 = -D_1 q_{11} - \Lambda_1 (p_{11} p_{21} + q_{11} q_{21}) + v_{33} p_{10} \quad (6.52)$$

$$g_3 = -D_1 p_{21} + 2\Lambda_2 p_{11} q_{11} - v_{44} q_{20} \quad (6.53)$$

$$g_4 = -D_1 q_{21} - \Lambda_2 (p_{11}^2 - q_{11}^2) + v_{44} p_{20} \quad (6.54)$$

and the elements h_m are given by

$$h_1 = -D_1 p_{12} - D_2 p_{11} - \Lambda_1 (p_{21} q_{12} + q_{11} p_{22} - p_{11} q_{22} - q_{21} p_{12}) - v_{33} q_{11} \quad (6.55)$$

$$h_2 = -D_1 q_{12} - D_2 q_{11} - \Lambda_1 (p_{11} p_{22} + p_{12} p_{21} + q_{11} q_{22} + q_{12} q_{21}) + v_{33} p_{11} \quad (6.56)$$

$$h_3 = -D_1 p_{22} - D_2 p_{21} + 2\Lambda_2 (p_{11} q_{12} + p_{12} q_{11}) - v_{44} q_{21} \quad (6.57)$$

$$h_4 = -D_1 q_{22} - D_2 q_{21} - 2\Lambda_2 (p_{11} p_{12} - q_{11} q_{12}) + v_{44} p_{21} \quad (6.58)$$

As a Hopf bifurcation of the system occurs at $\sigma_2 = \sigma_{2c}$, two of the eigenvalues of the matrix $[A]$ are purely imaginary and complex conjugates of each other, while the other two eigenvalues have negative real parts. The nondecaying solution of equations (6.44) can be expressed in the form

$$\begin{Bmatrix} p_{11} \\ q_{11} \\ p_{21} \\ q_{21} \end{Bmatrix} = A(T_1, T_2) e^{i\omega_m T_0} \begin{Bmatrix} K_{11} \\ K_{12} \\ K_{13} \\ K_{14} \end{Bmatrix} + cc \quad (6.59)$$

where the column K_{1j} is the eigenvector corresponding to the eigenvalue $i\omega_m$, cc stands for the complex conjugate of the preceding terms, and A is determined by imposing the solvability condition at the next level of approximation.

Substituting equations (6.59) into equations (6.45) and eliminating the source of secular terms, we arrive at $D_1 A = 0$ or $A = A(T_2)$. Then, the solution of equations (6.45) is determined to be

$$\begin{Bmatrix} p_{12} \\ q_{12} \\ p_{22} \\ q_{22} \end{Bmatrix} = A^2 e^{i2\omega_m T_0} \begin{Bmatrix} K_{21} \\ K_{22} \\ K_{23} \\ K_{24} \end{Bmatrix} + A\bar{A} \begin{Bmatrix} K_{31} \\ K_{32} \\ K_{33} \\ K_{34} \end{Bmatrix} + \begin{Bmatrix} K_{41} \\ K_{42} \\ K_{43} \\ K_{44} \end{Bmatrix} + cc \quad (6.60)$$

where again $\bar{A}$ is the complex conjugate of A . The elements K_{ij} are complex quantities, defined in Appendix F, and determined numerically.

Next, equations (6.59) and (6.60) are substituted into the right-side of equations (6.46) and the terms which produce secular terms are eliminated. Eliminating the secular terms is equivalent to determining the solvability conditions for the following system of equations:

$$[B] \underline{v} = - \underline{b} \quad (6.61)$$

where an underbar indicates an array and

$$[B] = [A] - i\omega_m [I] \quad (6.62)$$

in which $[I]$ is a 4 x 4 identity matrix, $\underline{v}$ is a 4 x 1 array, and $\underline{b}$ is made up of the elements b_i defined in Appendix F.

The required solvability condition is

$$- \underline{b}^T \underline{u} = 0 \quad (6.63)$$

where the superscript T indicates a transpose. The 4 x 1 array $\underline{u}$ is a solution of

$$[\overline{B}]^T \underline{u} = 0 \quad (6.64)$$

Equation (6.63) leads to an equation of the form

$$C_1 A' + C_2 A + C_3 A^2 \overline{A} = 0 \quad (6.65)$$

where the prime denotes a derivative with respect to the time scale T_2 and the C_j are complex quantities defined in Appendix F. We note that equation (6.65) is similar in form to equation (6.23) obtained for the three-dimensional system. Let R_j and I_j be respectively the real and imaginary parts of a complex quantity C_j . Substituting $A = \frac{1}{2} a e^{i\beta}$, where a and β are real quantities, into equation (6.65), separating real and imaginary parts, setting $\varepsilon = 1$, and assuming that R_1 is not equal to zero, we obtain

$$(R_1 + \frac{I_1^2}{R_1})\dot{a} + (R_2 + \frac{I_1 I_2}{R_1})a + \frac{1}{4}(R_3 + \frac{I_1 I_3}{R_1})a^3 = 0 \quad (6.66)$$

and

$$(R_1 + \frac{I_1^2}{R_1})a\dot{\beta} + (I_2 - \frac{I_1 R_2}{R_1})a + \frac{1}{4}(I_3 - \frac{I_1 R_3}{R_1})a^3 = 0 \quad (6.67)$$

Equations (6.66) and (6.67) are identical in form to equations (6.24) and (6.25) obtained for the three-dimensional system. They can be rewritten as

$$\dot{a} = \alpha_1 a + \alpha_2 a^3 \quad (6.68)$$

and

$$a \dot{\beta} = \alpha_3 a + \alpha_4 a^3 \quad (6.69)$$

where

$$\alpha_1 = - (R_2 + \frac{I_1 I_2}{R_1}) / (R_1 + \frac{I_1^2}{R_1}) \quad (6.70)$$

$$\alpha_2 = - (R_3 + \frac{I_1 I_3}{R_1}) / 4(R_1 + \frac{I_1^2}{R_1}) \quad (6.71)$$

$$\alpha_3 = - (I_2 - \frac{I_1 R_2}{R_1}) / (R_1 + \frac{I_1^2}{R_1}) \quad (6.72)$$

$$\alpha_4 = - (I_3 - \frac{I_1 R_3}{R_1}) / 4(R_1 + \frac{I_1^2}{R_1}) \quad (6.73)$$

It is assumed that the coefficient of a in equation (6.66) is not zero.

It is interesting to note that equation (6.68) is similar to the normal form for a pitchfork bifurcation (see Chapter 2). Nontrivial fixed-point solutions of equation (6.68) can occur if and only if α_1 and α_2 have opposite signs. When α_1 is positive and α_2 is negative, there is a stable limit cycle, whose radius is a and frequency is $(\omega_m + \beta)/2\pi$, and the Hopf bifurcation is supercritical. When α_2 is positive and α_1 is negative, the limit cycle is unstable and the Hopf bifurcation is subcritical. Equations (6.68) and (6.69) are similar in form to the equations which describe the amplitude and phase of limit cycles in other systems, for example, the van der Pol oscillator (Nayfeh and Mook, 1979; Seydel, 1988). Here, the coefficients in equations (6.68)-(6.73) are determined numerically.

The analytical approximation for a limit cycle near a Hopf bifurcation point is

$$\begin{aligned}
\begin{Bmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{Bmatrix} &= \begin{Bmatrix} p_{10} \\ q_{10} \\ p_{20} \\ q_{20} \end{Bmatrix} + a \begin{Bmatrix} \operatorname{Re}(K_{11}) \cos \phi - \operatorname{Im}(K_{11}) \sin \phi \\ \operatorname{Re}(K_{12}) \cos \phi - \operatorname{Im}(K_{12}) \sin \phi \\ \operatorname{Re}(K_{13}) \cos \phi - \operatorname{Im}(K_{13}) \sin \phi \\ \operatorname{Re}(K_{14}) \cos \phi - \operatorname{Im}(K_{14}) \sin \phi \end{Bmatrix} + \begin{Bmatrix} 2K_{41} \\ 2K_{42} \\ 2K_{43} \\ 2K_{44} \end{Bmatrix} \\
&+ \frac{1}{2} a^2 \begin{Bmatrix} \operatorname{Re}(K_{31}) + \operatorname{Re}(K_{21}) \cos 2\phi - \operatorname{Im}(K_{21}) \sin 2\phi \\ \operatorname{Re}(K_{32}) + \operatorname{Re}(K_{22}) \cos 2\phi - \operatorname{Im}(K_{22}) \sin 2\phi \\ \operatorname{Re}(K_{33}) + \operatorname{Re}(K_{23}) \cos 2\phi - \operatorname{Im}(K_{23}) \sin 2\phi \\ \operatorname{Re}(K_{34}) + \operatorname{Re}(K_{24}) \cos 2\phi - \operatorname{Im}(K_{24}) \sin 2\phi \end{Bmatrix} + \dots
\end{aligned} \tag{6.74}$$

where $\phi = \omega_m t + \beta$, $\beta = (\alpha_3 + \alpha_4 a^2)t + \beta_0$, $\operatorname{Re}(K_{ij})$ stands for the real part of K_{ij} , and $\operatorname{Im}(K_{ij})$ stands for the imaginary part of K_{ij} . The constant β_0 depends on the initial conditions and can be set equal to zero without any loss of generality. The transient motions can be obtained from equation (6.68).

Next, let us consider a case of primary resonance of the second mode, where we are interested in using f_2 instead of σ_2 as a control parameter. Then, in the perturbation analysis, equation (6.42) would be replaced by

$$f_2 = f_{2c} + \varepsilon^2 f_{22} + O(\varepsilon^3) \tag{6.75}$$

where f_{2c} is the critical value of f_2 at which a Hopf bifurcation of the system occurs. Also, the perturbation f_{22} is ordered at $O(\varepsilon^2)$ so that the influence of the nonlinear terms in equations (6.34)-(6.37) and the control parameter f_2 is realized at the same order in the analysis. Equations (6.44)-(6.46) will remain the same, while equations (6.51)-(6.54) would be replaced by

$$g_1 = -D_1 p_{11} - \Lambda_1(p_{21}q_{11} - p_{11}q_{21}) \quad (6.76)$$

$$g_2 = -D_1 q_{11} - \Lambda_1(p_{11}p_{21} + q_{11}q_{21}) \quad (6.77)$$

$$g_3 = -D_1 p_{21} + 2\Lambda_2 p_{11}q_{11} \quad (6.78)$$

$$g_4 = -D_1 q_{21} - \Lambda_2(p_{11}^2 - q_{11}^2) + f_{22} \quad (6.79)$$

Also, equations (F12) – (F15) in Appendix F would be replaced by the following equation:

$$g_{31} = g_{32} = g_{33} = 0; g_{34} = \frac{1}{2} f_{22} \quad (6.80)$$

Also, equations (6.55)-(6.58) would be replaced by

$$h_1 = -D_1 p_{12} - D_2 p_{11} - \Lambda_1(p_{21}q_{12} + q_{11}p_{22} - p_{11}q_{22} - p_{12}q_{21}) \quad (6.81)$$

$$h_2 = -D_1 q_{12} - D_2 q_{11} - \Lambda_1(p_{11}p_{22} + p_{12}p_{21} + q_{11}q_{22} + q_{12}q_{21}) \quad (6.82)$$

$$h_3 = -D_1 p_{22} - D_2 p_{21} + 2\Lambda_2(p_{11}q_{12} + p_{12}q_{11}) \quad (6.83)$$

$$h_4 = -D_1 q_{22} - D_2 q_{21} - 2\Lambda_2(p_{11}p_{12} - q_{11}q_{12}) \quad (6.84)$$

Equations (6.59)-(6.73) would remain as they were before. However, the column K_4 , would be different in this case.

We could also conduct a similar perturbation analysis for a case of primary resonance of the first mode, where we use f_1 instead of σ_2 as a control parameter. Further, one may also use a modal damping coefficient instead of the excitation frequency or amplitude as a control parameter. However, the approach would remain the same.

6.2.2. Numerical Results

In Cases I-III, we consider primary resonance of the second mode, while in Case IV we consider a primary resonance of the first mode.

Case I

Here, the Hopf bifurcations that arise during the frequency response (the external detuning σ_2 is used as a control parameter) of Structure I are considered. The values of the different parameters are $\sigma_1 = 1.131$, $\Lambda_1 = 293.613$, $\Lambda_2 = 2208.297$, $\Lambda_6 = 0.00854$, $\mu_1 = 0.09$, and $\mu_2 = 0.22$. The rms value of the excitation amplitude F is 16.667 mili g's and the corresponding value of f_2 is $0.231\Lambda_6$. For these values, Hopf bifurcations occur at $\sigma_2 \simeq \sigma_{2l} = -0.440965$ and $\sigma_2 \simeq \sigma_{2r} = -0.095820$. For motions around $\sigma_2 \simeq \sigma_{2l}$, $\alpha_1 = 2.61371 \times 10^{-5}$ and $\alpha_2 = -1.83195 \times 10^3$ when $\sigma = 5.0 \times 10^{-5}$. Therefore, at $\sigma_2 \simeq \sigma_{2l}$ there is a supercritical Hopf-bifurcation point. For motions around $\sigma_2 \simeq \sigma_{2r}$, $\alpha_1 = -2.84528 \times 10^{-5}$ and $\alpha_2 = 2.38070 \times 10^3$ when $\sigma = 5.0 \times 10^{-5}$. Hence, at $\sigma_2 \simeq \sigma_{2r}$ there is a subcritical Hopf-bifurcation point.

In Fig. 6.3, the analytical approximation (dashed lines) and the numerically obtained solutions (solid lines) of equations (6.34)-(6.37) are given for motion close to the supercritical Hopf-bifurcation point. We consider the projection of the motion on the $p_2 - q_2$ plane for four different values of the perturbation σ . In Figs. 6.3a and 6.3b, it is hard to distinguish the analytical approximation from the numerically obtained solutions; however, in Figs. 6.3c and 6.3d, the difference is discernible. As σ becomes larger, the difference between them increases. This is expected as the analytical approximation is an asymptotic expansion, which is valid only in a small neighborhood of the Hopf-bifurcation point. For values of σ of 10^{-6} or smaller, the numerical value of the analytical approximation is very sensitive to errors in the determination of σ_{2c} . The overhang near the subcritical Hopf-bifurcation point exists over the range $-0.09582 < \sigma_2 < 0.000$, which is fairly small to be discernible in a physical experiment. Hence, in the present case, it is very difficult to determine if the transition from periodic to quasiperiodic motions is due to a subcritical or supercritical Hopf bifurcation in a physical experiment. Also, as the real parts of some of the eigenvalues (those of matrix $[A]$) are close to zero (about 10^{-3} or smaller) near the bifurcation points, the motion takes a very long time to settle down.

Case II

Here, the Hopf bifurcations that arise during the frequency response (the external detuning σ_2 is used as a control parameter) of Structure II are

considered. The values of the different parameters are $\sigma_1 = 1.131$, $\Lambda_1 = 294.732$, $\Lambda_2 = 2213.729$, $\Lambda_6 = 0.00852$, $\mu_1 = 0.09$, and $\mu_2 = 0.22$. The rms value of the excitation amplitude F is 33.333 mili g's. The corresponding value of f_2 equals $0.462\Lambda_6$. Hopf bifurcations occur at $\sigma_2 \simeq \sigma_{2l} = -0.45865$ and $\sigma_2 \simeq \sigma_{2r} = -0.09437$. For motions around $\sigma_2 \simeq \sigma_{2l}$, $\alpha_1 = 2.84387 \times 10^{-5}$ and $\alpha_2 = -1.30135 \times 10^3$ when $\sigma = 5.0 \times 10^{-5}$. So, at $\sigma_2 \simeq \sigma_{2l}$ there is a supercritical Hopf-bifurcation point. For motions around $\sigma_2 \simeq \sigma_{2r}$, $\alpha_1 = -2.96005 \times 10^{-5}$ and $\alpha_2 = 1.15995 \times 10^3$ when $\sigma = 5.0 \times 10^{-5}$. Hence, at $\sigma_2 \simeq \sigma_{2r}$ there is a subcritical Hopf-bifurcation point.

Case III

Here, the Hopf bifurcations that arise during the forced response (the excitation amplitude F or $\Lambda_6 f_2$ is used as a control parameter) of Structure I are considered. The values of the different parameters are $\sigma_1 = 1.131$, $\Lambda_1 = 293.613$, $\Lambda_2 = 2208.297$, $\Lambda_6 = 0.00854$, $\mu_1 = 0.09$, and $\mu_2 = 0.22$. In this case, the excitation detuning $\sigma_2 = -0.37699$. A Hopf bifurcation occurs at $f_2 \simeq f_{2c} = 0.110\Lambda_6$. When $f_{22} = 5.0 \times 10^{-5}$, $\alpha_1 = 3.83016 \times 10^{-5}$ and $\alpha_2 = -8.19250 \times 10^3$; therefore, this bifurcation is supercritical.

Case IV

Here, the Hopf bifurcations that arise during the frequency response (the external detuning σ_2 is used as a control parameter) of Structure II are

considered. The values of the different parameters are $\sigma_1 = 1.131$, $\Lambda_1 = 294.732$, $\Lambda_2 = 2213.729$, $\Lambda_3 = 0.00082$, $\mu_1 = 0.09$, and $\mu_2 = 0.22$. The rms value of the excitation amplitude F is 150.00 mili g's. The corresponding value of f_2 is $2.081\Lambda_3$. In this case, the Hopf bifurcations occur at $\sigma_2 \simeq \sigma_{2l} = 0.076328$ and $\sigma_2 \simeq \sigma_{2r} = 0.944309$. For motions around $\sigma_2 \simeq \sigma_{2l}$, $\alpha_1 = 1.32294 \times 10^{-4}$ and $\alpha_2 = -1.50388 \times 10^4$ when $\sigma = 5.0 \times 10^{-5}$. Hence, at $\sigma_2 \simeq \sigma_{2l}$ there is a supercritical Hopf-bifurcation point. For motions around $\sigma_2 \simeq \sigma_{2r}$, $\alpha_1 = 1.52727 \times 10^{-4}$ and $\alpha_2 = -1.44835 \times 10^4$ when $\sigma = -5.0 \times 10^{-5}$. So, at $\sigma_2 \simeq \sigma_{2r}$ there is also a supercritical Hopf-bifurcation point.

In Fig. 6.4, the analytical approximation (dashed lines) and the numerically obtained solutions (solid lines) of equations (6.34)-(6.37) are shown for comparison for motions close to the supercritical Hopf-bifurcation point at σ_{2l} . We consider the projection of the motion on the $p_2 - q_2$ plane for four different values of the perturbation σ . In Figs. 6.4a and 6.4b, it is hard to distinguish the analytical approximation from the numerically obtained solutions; however, in Figs. 6.4c and 6.4d, the difference is discernible. In Fig. 6.4d, the numerical solution indicates the occurrence of a period-doubling bifurcation, while the analytical solution does not. As σ becomes larger, the difference increases. This is expected because the analytical approximation is an asymptotic expansion whose region of validity is only a small neighborhood of the Hopf-bifurcation point.

Using numerical integrations, for the same value of σ , we found two possible solutions in some regions. One of these numerical solutions is shown in Fig. 6.4. The other solution has a much larger amplitude and is distinct in character. In Fig. 6.5, we show the projection of this solution on the $p_2 - q_2$ plane for four different values of σ . For a given σ , the response is determined by the initial conditions. We conjecture that the large-amplitude solutions arise through a global bifurcation (see Chapter 2). The local analysis used in this chapter cannot predict these solutions. However, the coexistence of two solutions should be discernible in a physical experiment. When the experiments for the case of primary resonance of the first mode (see Chapter 4) were conducted, we were not aware of the existence of the two branches of modulated motions. Also from this case, one might see why it may be difficult to ascertain the nature of a Hopf bifurcation just by using numerical solutions. If one does not locate the small-amplitude branch, it is possible for one to conclude that the bifurcation is subcritical. So, analytical approximations are necessary to ascertain the nature of a Hopf bifurcation. In summary, the analytical approximation and numerical solutions are in good agreement near a Hopf-bifurcation point of the three- and four-dimensional systems discussed in this chapter. It should be possible to use the same perturbation analysis for higher-dimensional systems as well.

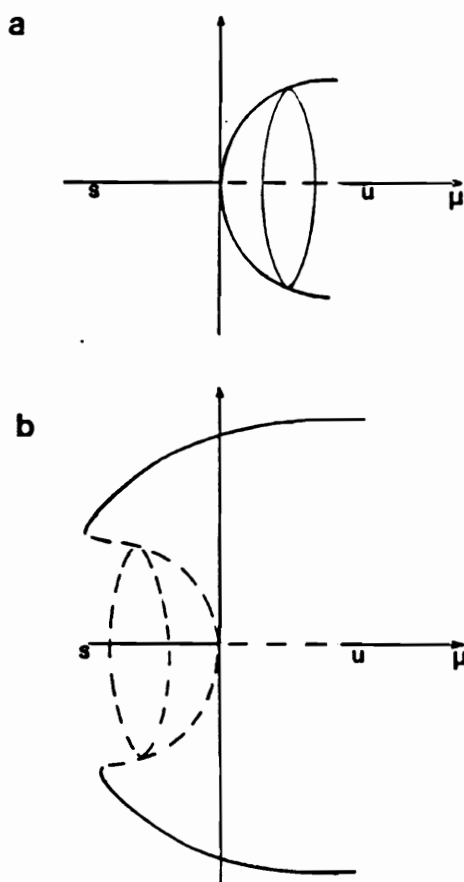


Figure 6.1. Amplitude of the limit cycle versus the control parameter: a) supercritical bifurcation and b) subcritical bifurcation; solid (dashed) lines correspond to stable (unstable) limit cycles, and the symbol s (u) represents stable (unstable) fixed points.

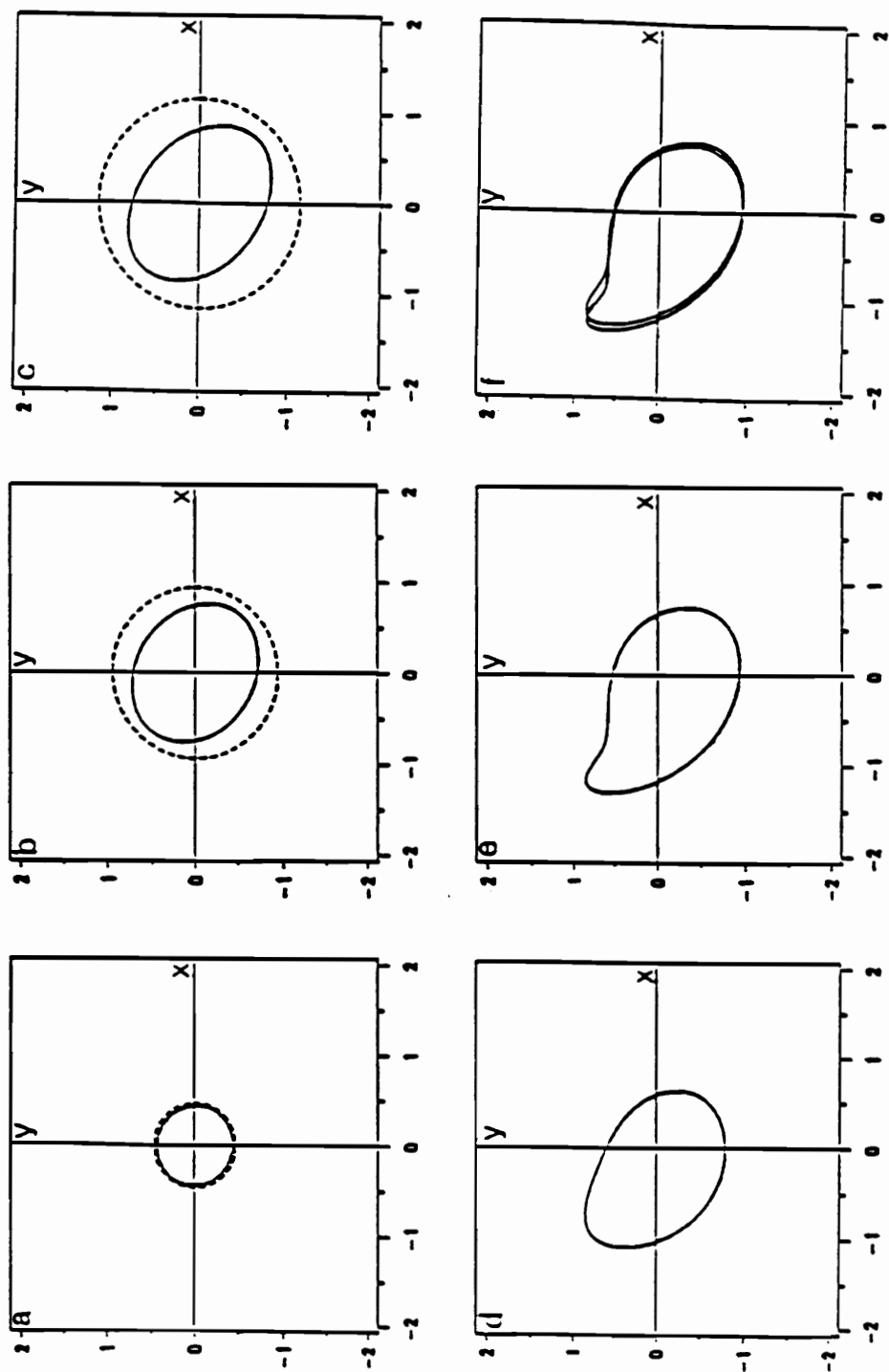


Figure 6.2. Three-dimensional system: a) $\mu = 0.05$, b) $\mu = 0.2$, c) $\mu = 0.3$, d) $\mu = 0.35$, e) $\mu = 0.44$, and f) $\mu = 0.445$; solid (dashed) lines correspond to numerical integration (analytical approximation).

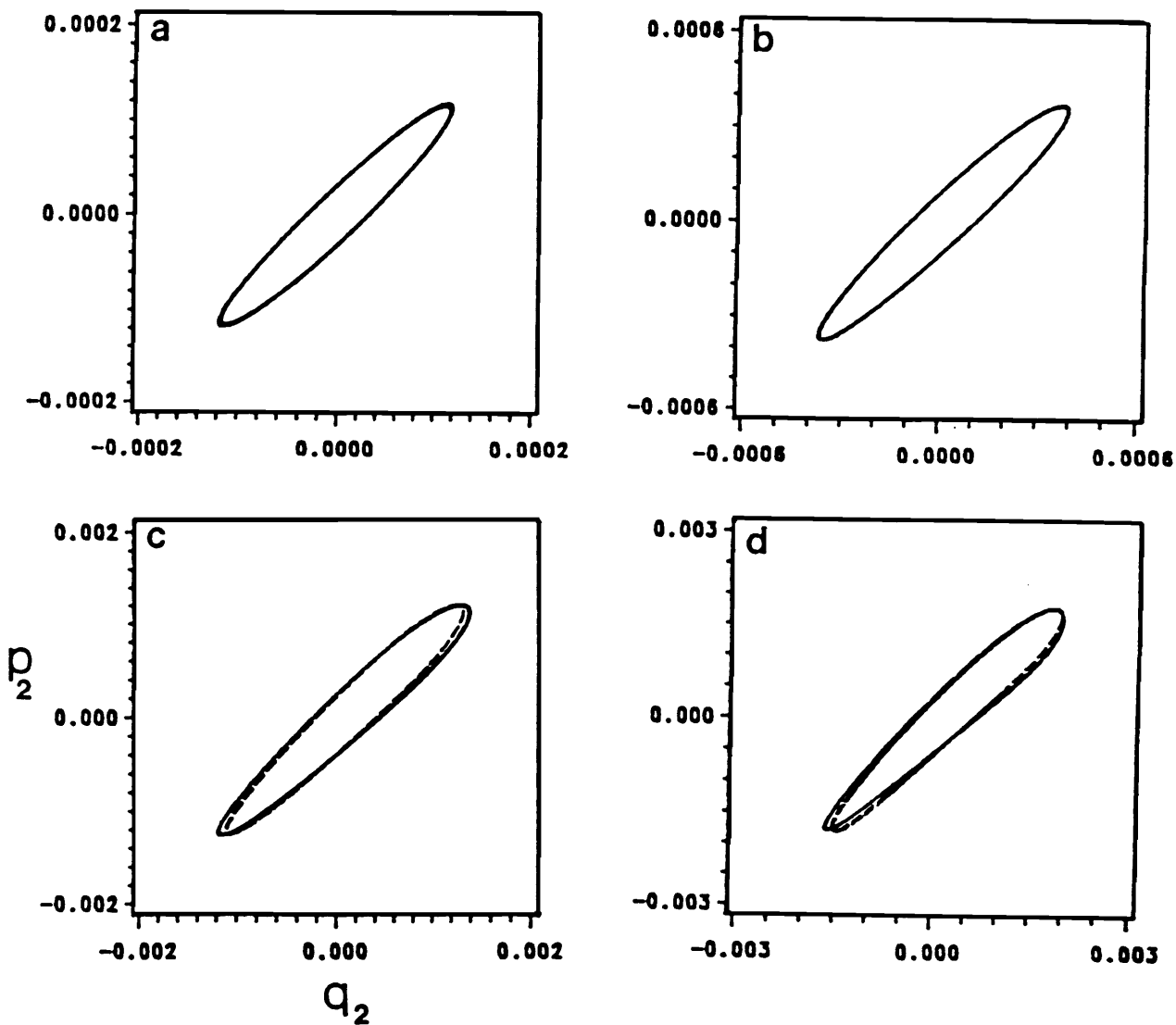


Figure 6.3. Four-dimensional system Case I: a) $\sigma = 5.0 \times 10^{-5}$, b) $\sigma = 5.0 \times 10^{-4}$, c) $\sigma = 5.0 \times 10^{-3}$, and d) $\sigma = 1.0 \times 10^{-2}$; solid (dashed) lines correspond to numerical integration (analytical approximation). In each case, the origin of the x-y plot is located at (p_{20}, q_{20}) .

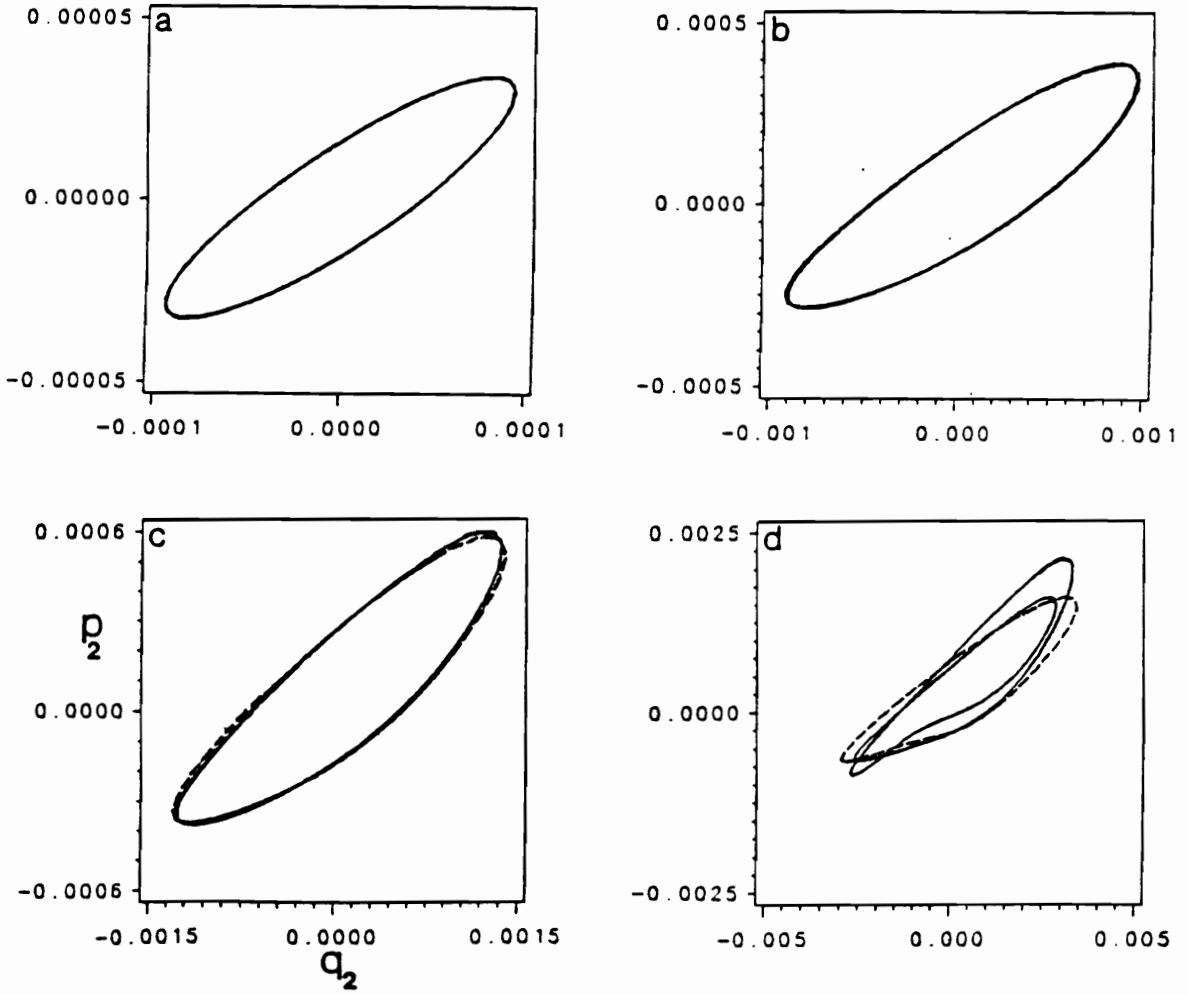


Figure 6.4. Four-dimensional system Case IV: a) $\sigma = 5.0 \times 10^{-5}$, b) $\sigma = 5.0 \times 10^{-3}$, c) $\sigma = 1.0 \times 10^{-2}$, and d) $\sigma = 5.0 \times 10^{-2}$; solid lines correspond to the small-amplitude numerical solution, while dashed lines correspond to the analytical approximation. In each case, the origin of the x-y plot is located at (p_{20}, q_{20}) .

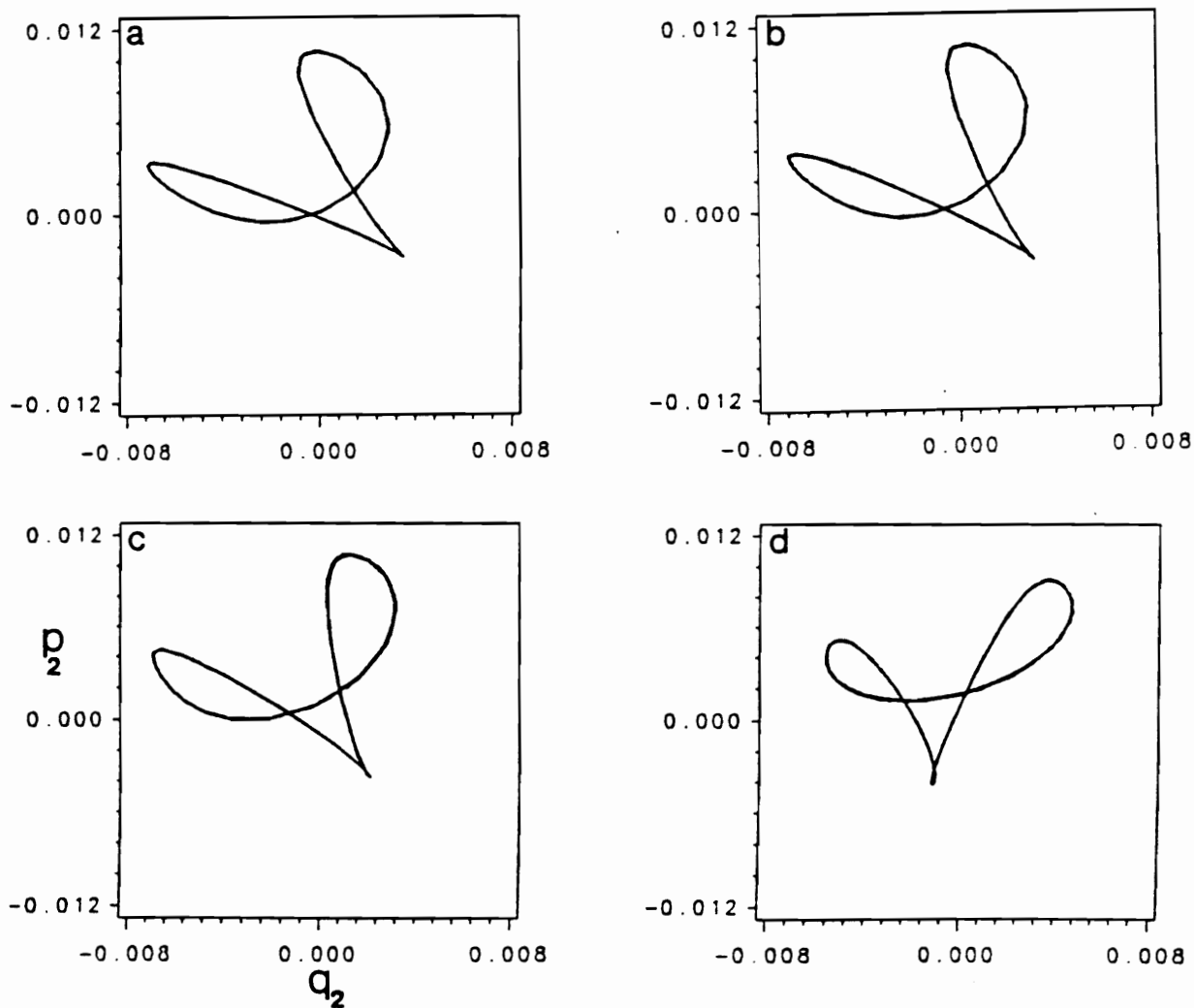


Figure 6.5. Four-dimensional system Case IV: a) $\sigma = 5.0 \times 10^{-5}$, b) $\sigma = 5.0 \times 10^{-3}$, c) $\sigma = 1.0 \times 10^{-2}$, and d) $\sigma = 5.0 \times 10^{-2}$; solid lines correspond to the large-amplitude numerical solution. In each case, the origin of the x-y plot is located at (p_{20}, q_{20}) .

7. Conclusions and Recommendations

In the present chapter, some concluding remarks and recommendations for future work are provided. The current work has dealt with the influence of modal interactions on the response of flexible metallic and composite structures to harmonic excitations. Both, analysis and experiments have been used in this study.

7.1. *Conclusions*

The good agreement between experiment and theory for the metallic structures indicates that the response of other structural systems to small primary-resonant excitations can also be modeled as shown in this work. Further, using the weakly nonlinear analysis (method of time-averaged Lagrangian), one can predict the Hopf bifurcations, thereby identifying the

control-parameter values for which periodically and chaotically modulated motions are likely to occur. Other structures like the large space structures envisaged in Wada and Fanson (1989) and McGowan, Edighoffer, and Wallace (1989) may also exhibit similar dynamic behavior when subjected to low excitation levels. Following the approach illustrated in the present study, one could formulate an analytical model to predict the dynamic behavior of these structures. Typically, in these large structural systems about ten to twenty modes of vibration occur in a range of 0-50 Hz. Some of the modal frequencies are commensurable and this may lead to internal resonances and consequent modal interactions. The good qualitative agreement between experiment and theory for the metallic structures subjected to secondary-resonant excitations also suggests that a weakly nonlinear analysis can be used to predict the behavior of a structural system to such excitations.

The experiments conducted with both the metallic and composite structures illustrate that the modal interactions can lead to nonlinear periodic, quasi-periodic, and chaotically modulated motions under a wide range of resonant excitations. Further, the occurrence of chaotically modulated motions at small excitation levels in these structures deserves attention as systems which do not possess any internal resonances have to be excited at relatively high excitation levels to produce such motions. As the structures treated here can be considered as mechanical analogues of other nonlinear coupled oscillators and internally-resonant physical systems, the qualitative nature of the experimental results also apply to these other systems. The

display of nonplanar motions by each composite structure under planar excitations is also worth noting. This fact may have important implications for other composite structural systems and has to be factored into their design.

Further, we have also illustrated the use of the method of multiple scales to obtain asymptotic expansions for motions near a Hopf bifurcation of three- and four-dimensional autonomous systems. The perturbation analysis can be easily extended to higher-dimensional systems. Further, the validity of the analytical approximations has been ascertained by comparing them with numerical simulations.

7.2. *Recommendations*

Some fundamental issues such as the determination of the linear frequencies and the modal-damping coefficients of an internally-resonant structural system need to be addressed. The material damping associated with composite beams is believed to be a nonlinear quantity and needs to be studied.

In the present study, either the excitation frequency or excitation amplitude is used as a control parameter. It would be interesting to study the qualitative changes in the structure's response as an another parameter such as damping is varied. Also, further studies are needed to ascertain the range of the detuning of the internal resonances for which the modal interactions are likely

to occur. In some of the experiments, the response is found to be sensitive to the experimental conditions, especially in the region of modulated motions. The reason for this sensitivity requires investigation.

In some of the experiments, we observed a feedback from a light-weight structure to the shaker table during the structure's nonlinear motions. This fact might have important implications for other situations (control-structure interactions), where much larger structures than those used in the present experiments are controlled by actuators, which are much smaller than the shaker used in the present work.

The influence of modal interactions in composite structures in the case of primary- and secondary-resonant excitations also requires further investigations. There are some studies which show that delamination in composite beams causes the linear frequencies to change and it would be interesting to study how these changes affect modal interactions.

References

Anand, G. V. (1966), Nonlinear resonance in stretched strings with viscous damping, *Journal of Acoustical Society of America* 40, 1517-1528.

Aprille, T. J. and Trick, T. N. (1972), A computer algorithm to determine the steady-state response of nonlinear oscillators, *IEEE Transactions on Circuit Theory* CT-19, 354-360.

✓ Bergé, P., Pomeau, Y., and Vidal, C. (1984), **Order within Chaos: Towards a deterministic approach to turbulence**, John Wiley and Sons, New York.

Bert, C. W. (1973), Material damping : An introductory review of mathematical models, measures, and experimental techniques, *Journal of Sound and Vibration* 29(2), 129-153.

Carr, J. (1981), **Applications of Centre Manifold Theory**, Springer-Verlag, New York.

Chen, J. C. and Babcock, C. D. (1975), Nonlinear vibration of cylindrical shells, *AIAA Journal* 13, 868-876.

Chia, C. J. (1980), **Nonlinear Analysis of Plates**, McGraw-Hill, New York.

Ciliberto, S. and Gollub, J. P. (1985), Chaotic mode competition in parametrically forced surface waves, *Journal of Fluid Mechanics* 158, 381-398.

Crespo da Silva, M. R. M. and Glynn, C. C. (1978), Nonlinear flexural-flexural-torsional dynamics of inextensional beams, II. Forced motions, *Journal of Structural Mechanics* 6, 449-461.

Crespo da Silva, M. R. M. (1980), On the whirling of a base-excited cantilever beam, *Journal of Acoustical Society of America* 67, 704-707.

Crespo da Silva, M. R. M. and Zaretzky, C. L. (1988), Nonlinear modal coupling in the response of inextensional beams. In *Proceedings of the Second*

Conference on Nonlinear Vibrations, Stability, and Dynamics of Structures and Mechanisms, Virginia Polytechnic Institute and State University, Blacksburg, VA.

- ✓ Eckmann, J. P. (1981), Roads to turbulence in dissipative dynamical systems, *Reviews of Modern Physics* 53(4), 643-654.
- Eckmann, J. P. and Ruelle, D. (1985), Ergodic theory of chaos and strange attractors, *Reviews of Modern Physics* 57(3), 617-656.
- Evensen, D. A. (1966), Nonlinear flexural vibrations of thin circular rings, *Journal of Applied Mechanics* 33, 553-560.
- ✓ Farmer, J. D., Ott, E., and Yorke, J. A. (1983), The dimension of chaotic attractors, *Physica* 7D, 153-180.
- Fraser, A. M. and Swinney, H. L. (1986), Independent coordinates for strange attractors from mutual information, *Physics Reviews A* 33, 1134-1140.
- Froehling, H., Crutchfield, J. P., Farmer, D., Packard, N. H., and Shaw, R. (1981), On determining the dimension of chaotic flows, *Physica* 3D, 605-617.
- Froude, W. (1863), Remarks on Mr. Scott Russell's paper on rolling, *Transactions of Institute of Naval Architects* 4, 232-275.
- Goodier, J. N. and McIvor, I. K. (1964), The elastic cylindrical shell under nearly uniform radial impulse, *Journal of Applied Mechanics* 31, 259-266.
- Gorelik, G. and Witt, A. (1933), Swing of an elastic pendulum as an example of two parametrically bound linear vibration systems, *Journal of Technical Physics (USSR)* 3, 244-307.
- Grassberger, P. and Procaccia, I. (1983), Measuring the strangeness of strange attractors, *Physica* 9D, 189-208.
- Grebogi, C., Ott, E., Pelikan, S., and Yorke, J. A. (1984), Strange attractors that are not chaotic, *Physica* 13D, 261-268.
- ✓ Grebogi, C., Ott, E. and Yorke, J. A. (1985), Attractors on a n-torus: Quasiperiodicity versus chaos, *Physica* 15D, 354-373.
- Gu, X. M. and Sethna, P. R. (1987), Resonant surface waves and chaotic phenomena, *Journal of Fluid Mechanics* 183, 543-565.
- Guckenheimer, J. and Holmes, P. (1983), **Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields**, Springer-Verlag, New York.

- Haddow, A. G. (1983), Theoretical and experimental study of modal interaction in a two degree of freedom nonlinear structure, Ph. D. Dissertation, The University of Dundee, Scotland.
- Haddow, A. G., Barr, A. D. S., and Mook, D. T. (1984), Theoretical and experimental study of modal interaction in a two-degree-of-freedom structure, *Journal of Sound and Vibration* 97, 451-473.
- Haight, E. C. and King, W. W. (1972), Stability of non-linear oscillations of an elastic rod, *Journal of Acoustical Society of America* 52, 899-911.
- ✓ Haken, H. (1983) **Advanced Synergetics**. Springer-Verlag, Berlin.
- Hale, J. K. (1963), **Oscillations in Nonlinear Systems**, McGraw-Hill, New York.
- Harris, F. J. (1978), On the use of windows for harmonic analysis with the discrete fourier transform, In *Proceedings of the IEEE* 66(1), 51-83.
- Hatwal, H., Mallik, A. K., and Ghosh, A. (1983a), Forced nonlinear oscillations of an autoparametric system - Part 1: Periodic responses, *Journal of Applied Mechanics* 50, 657-662.
- Hatwal, H., Mallik, A. K., and Ghosh, A. (1983b), Forced nonlinear oscillations of an autoparametric system - Part 2: Chaotic responses, *Journal of Applied Mechanics* 50, 663-668.
- Ho, C. H., Scott, R. A., and Easley, J. G. (1975), Non-planar, non-linear oscillations of a beam - I. Forced motions, *International Journal of Non-Linear Mechanics* 10, 113-127.
- Holmes, P. J. (1986), Chaotic motions in a weakly nonlinear model for surface waves, *Journal of Fluid Mechanics* 162, 365-388.
- ✓ Holmes, P. (1990), Nonlinear dynamics, chaos, and mechanics, *Applied Mechanics Reviews* 43(5), 23-39.
- Horowitz, P. and Hill, W. (1980), **The Art of Electronics**, Cambridge University Press, Cambridge.
- Ibrahim, R. A. and Barr, A. D. S. (1975a), Autoparametric resonance in a structure containing a liquid. Part I: Two mode interaction, *Journal of Sound and Vibration* 42, 159-179.
- Ibrahim, R. A. and Barr, A. D. S. (1975b), Autoparametric resonance in a structure containing a liquid. Part II: Three mode interaction, *Journal of Sound and Vibration* 42, 181-200.

- Jackson, E. A. (1989), **Perspectives of Nonlinear Dynamics, Volume 1**, Cambridge University Press, Cambridge.
- Johnson, J. M. and Bajaj, A. K. (1989), Amplitude modulated and chaotic dynamics in resonant motion of strings, *Journal of Sound and Vibration* 128, 87-107.
- Kambe, T. and Umeki, M. (1990), Nonlinear dynamics of two-mode interactions in parametric excitation of surface waves, *Journal of Fluid Mechanics* 212, 373-393.
- Kim, Y. C., Khadra, L., and Powers, E. J. (1980), Wave modulation in a nonlinear dispersive medium, *Physics of Fluids* 23(11), 2250-2257.
- Liebert, W. and Schuster, H. G. (1989), Proper choice of the time delay for the analysis of chaotic time series, *Physics Letters A* 142, 107-111.
- Lobitz, D. W., Nayfeh, A. H., and Mook, D. T. (1977), Nonlinear analysis of vibrations of irregular plates, *Journal of Sound and Vibration* 49, 203-217.
- Maewal, A. (1986a) Miles' evolution equations for axisymmetric shells: simple strange attractors in structural dynamics. *International Journal of Non-Linear Mechanics* 21, 433-438.
- Maewal, A. (1986b), Chaos in a harmonically excited elastic beam, *Journal of Applied Mechanics* 53, 625-632.
- McGowan, P. E., Edighoffer, H. H., and Wallace, J. W. (1989), Development of an experimental space station model for structural dynamics research, In *Proceedings of the 60th Shock and Vibration Symposium*, Virginia Beach, Virginia, Vol. II, 13-29.
- McIvor, I. K. and Lovell, E. G. (1968), Dynamic response of finite-length cylindrical shells to nearly uniform radial impulse, *AIAA Journal* 6, 2346-2351.
- McIvor, I. K. and Sonstegard, D. A. (1966), Axisymmetric response of a closed spherical shell to a nearly uniform radial impulse, *Journal of Acoustical Society of America* 40, 1540-1547.
- Mees, A. and Sparrow, C. (1987), Some tools for analyzing chaos, In *Proceedings of the IEEE* 75(8), 1058-1070.
- Meirovitch, L. (1980), **Computational Methods in Structural Dynamics**, Sijthoff & Noordhoff International Publishers. The Netherlands.

- Miles, J. W. (1962), Stability of forced oscillations of a spherical pendulum, *Quarterly of Applied Mathematics* 20, 21-32.
- Miles, J. W. (1965), Stability of forced oscillations of a vibrating string, *Journal of Acoustical Society of America* 38, 855-861.
- Miles, J. W. (1976), Nonlinear surface waves in closed basins, *Journal of Fluid Mechanics* 75, 419-448.
- Miles, J. W. (1984a), Resonant motion of a spherical pendulum, *Physica* 11D, 309-323.
- Miles, J. W. (1984b), Resonantly forced surface waves in a circular cylinder, *Journal of Fluid Mechanics* 149, 15-31.
- Miles, J. W. (1984c), Resonant, nonplanar motion of a stretched string, *Journal of Acoustical Society of America* 75, 1505-1510.
- Miles, J. W. (1984d), Resonantly forced motion of two quadratically coupled oscillators, *Physica* 13D, 247-260.
- Miles, J. W. (1984e), Nonlinear Faraday resonance, *Journal of Fluid Mechanics* 146, 285-302.
- Miles, J. W. (1985), Parametric excitation of an internally resonant double pendulum, *Journal of Applied Mathematics and Physics (ZAMP)* 36, 337-345.
- Miles, J. and Henderson, D. (1990), Parametrically forced surface waves, *Annual Review of Fluid Mechanics* 22, 143-165.
- Mook, D. T., Plaut, R. H., and HaQuang, N. (1985), The influence of an internal resonance on non-linear structural vibrations under subharmonic resonance conditions, *Journal of Sound and Vibration* 102, 473-492.
- Mook, D. T., HaQuang, N., and Plaut, R. H. (1986), The influence of an internal resonance on non-linear structural vibrations under combination resonance conditions, *Journal of Sound and Vibration* 104, 229-241.
- ✓ Moon, F. C. (1987), **Chaotic Vibrations: An introduction for applied scientists and engineers**, Wiley-Interscience, New York.
- Narasimha, R. (1968), Nonlinear vibration of an elastic string, *Journal of Sound and Vibration* 8, 134-136.
- Nayfeh, A. H. (1971), Two-to-one resonances near the equilateral libration points, *AIAA Journal* 9(1), 23-27.

- Nayfeh, A. H. (1973), **Perturbation Methods** , Wiley-Interscience, New York.
- Nayfeh, A. H. (1981), **Introduction to Perturbation Techniques** , Wiley-Interscience, New York.
- Nayfeh, A. H. (1983a), The response of a two-degree-of-freedom systems with quadratic nonlinearities to a parametric excitation, *Journal of Sound and Vibration* 88, 547-557.
- Nayfeh, A. H. (1983b), The response of multi-degree-of-freedom systems with quadratic nonlinearities to a harmonic parametric resonance, *Journal of Sound and Vibration* 90, 237-244.
- Nayfeh, A. H. (1987a), Parametric excitation of two internally resonant oscillators, *Journal of Sound and Vibration* 119, 95-109.
- Nayfeh, A. H. (1987b), Surface waves in closed basins under parametric and internal resonances, *Physics of Fluids* 30, 2976-2982.
- Nayfeh, A. H. (1988), On the undesirable roll characteristics of ships in regular seas, *Journal of Ship Research* 32, 92-100.
- Nayfeh, A. H. (1989), Application of the method of multiple scales to nonlinearly coupled oscillators, In *Lasers, Molecules, and Methods*, edited by J. O. Hirschfelder, R. E. Wyatt, and R. D. Coalson, Wiley, New York, 137-196.
- Nayfeh, A. H. and Mook, D. T. (1978), A saturation phenomenon in the forced response of systems with quadratic nonlinearities, In *Proceedings of the VIIth International Conference on Nonlinear Oscillations*, Prague, 511-516.
- Nayfeh, A. H. and Mook, D. T. (1979), **Nonlinear Oscillations** , Wiley-Interscience, New York.
- Nayfeh, A. H., Mook, D. T., and Lobitz, D. W. (1974a), Numerical-perturbation method for the nonlinear analysis of structural vibrations, *AIAA Journal* 12, 1222-1228.
- Nayfeh, A. H., Mook, D. T., and Marshall, L. R. (1973), Nonlinear coupling of pitch and roll modes in ship motions, *Journal of Hydronautics* 7, 145-152.
- Nayfeh, A. H., Mook, D. T., and Nayfeh, J. F. (1987), Some aspects of modal interactions in the response of beams, *AIAA Paper No. 87-0777*, AIAA 28th Structures, Structural Dynamics and Materials Conference, Monterey, CA.

Nayfeh, A. H., Mook, D. T., and Sridhar, S. (1974b), Nonlinear analysis of the forced response of structural elements, *Journal of Acoustical Society of America* 55, 281-291.

Nayfeh, A. H. and Nayfeh, J. F. (1990), Surface waves in closed basins under principal and autoparametric resonances, accepted for publication, *Physics of Fluids*.

Nayfeh, A. H. and Pai, P. F. (1989), Non-linear non-planar parametric responses of an inextensional beam, *International Journal of Non-Linear Mechanics* 24, 139-158.

Nayfeh, A. H. and Raouf, R. A. (1987a), Non-linear oscillation of circular cylindrical shells, *International Journal of Solids and Structures* 23, 1625-1638.

Nayfeh, A. H. and Raouf, R. A. (1987b), Nonlinear forced response of infinitely long circular cylindrical shells, *Journal of Applied Mechanics* 54, 571-577.

Nayfeh, A. H., Raouf, R. A., and Nayfeh, J. F. (1990), Nonlinear response of infinitely long circular cylindrical shells to subharmonic radial loads, accepted for publication, *Journal of Applied Mechanics*.

Nayfeh, A. H. and Zavodney, L. D. (1986), The response of two-degree-of-freedom systems with quadratic non-linearities to a combination parametric resonance, *Journal of Sound and Vibration* 107, 329-350.

Nayfeh, A. H. and Zavodney, L. D. (1988), Experimental observation of amplitude- and phase-modulated responses of two internally coupled oscillators to a harmonic excitation, *Journal of Applied Mechanics* 55, 706-711.

Nayfeh, J. F., Nayfeh, A. H., and Mook, D. T. (1988), Modal interactions in the response of beams to a harmonic excitation, In *Proceedings of the Second Conference on Nonlinear Vibrations, Stability, and Dynamics of Structures and Mechanisms*, Virginia Polytechnic Institute and State University, Blacksburg, VA.

Nayfeh, T. A., Nayfeh, A. H., and Mook, D. T. (1990), A theoretical-experimental investigation of a three-degree-of-freedom structure, *AIAA Paper No. 90-1081*, AIAA 31st Structure, Structural Dynamics and Materials Conference, Long Beach, CA.

✓ Newhouse, S. E., Ruelle, D., and Takens, F. (1978), Occurrence of strange axiom A attractors near quasiperiodic flows on T^m , $m \geq 3$, *Communications in Mathematical Physics* 64, 35-40.

- Pai, P. F. and Nayfeh, A. H. (1990), Non-linear non-planar oscillations of a cantilever beam under lateral base excitations, accepted for publication, International Journal of Non-Linear Mechanics.
- Parker, T. S. and Chua, L. O. (1987), Chaos : A tutorial for engineers, In Proceedings of the IEEE 75(8), 982-1008.
- Rand, R. H. (1989), Analytical approximation for period-doubling following a Hopf bifurcation, Mechanics Research Communications 16(2), 117-123.
- Rand, R. H. (1990), Center manifold approach to post-Hopf behavior, preprint.
- Rand, R. H. and Armbruster, D. (1987), **Perturbation Methods, Bifurcation Theory, and Computer Algebra**, Springer-Verlag, New York.
- Raouf, R. A. and Nayfeh, A. H. (1990a), One-to-one autoparametric resonances in infinitely long cylindrical shells, submitted for publication, Computers and Structures.
- Raouf, R. A. and Nayfeh, A. H. (1990b), Nonlinear axisymmetric response of spherical shells to a radial harmonic excitation, submitted for publication International Journal of Non-Linear Mechanics.
- Riesenthel, P. (1987,1988), Fluid Dynamics Research Center, IIT, Chicago, private communication.
- Roux, J. C., Simoyi, R. H., and Swinney, H. L. (1983), Observation of a strange attractor, Physica 8D, 257-266.
- ✓ Ruelle, D. (1989), **Chaotic Evolution and Strange Attractors**, Cambridge University Press, Cambridge.
- Schmidt, G. and Tondl, A. (1986), **Non-Linear Vibrations**, Cambridge University Press, Cambridge.
- Sethna, P. R. (1963), Coupling in certain classes of weakly nonlinear vibrating systems, In **Nonlinear Differential Equations and Nonlinear Mechanics** , edited by J. P. Lasalle and S. Lefschetz, Academic Press, New York, 58-70.
- Sethna, P. R. (1965), Vibrations of dynamical systems with quadratic nonlinearities, Journal of Applied Mechanics 32, 576-582.
- Sethna, P. R. and Bajaj, A. K. (1978), Bifurcations in dynamical systems with internal resonance, Journal of Applied Mechanics 45, 895-902.

- Sethna, P. R. and Schapiro, S. M. (1977), Nonlinear behavior of flutter unstable dynamical systems with gyroscopic and circulatory forces, *Journal of Applied Mechanics* 44, 755-762.
- Seydel, R. (1988), **From Equilibrium to Chaos: Practical bifurcation and stability analysis**, Elsevier, New York.
- Smith, L. L. and Morino, L. (1976), Stability analysis of nonlinear differential autonomous systems with applications to flutter, *AIAA Journal* 14(3), 333-341.
- Sridhar, S., Mook, D. T., and Nayfeh, A. H. (1975), Non-Linear resonances in forced responses of plates, Part I: Symmetric responses of circular plates, *Journal of Sound and Vibration* 41, 359-373.
- Sridhar, S., Mook, D. T., and Nayfeh, A. H. (1978), Non-Linear resonances in forced responses of plates, Part II: Asymmetric responses of circular plates, *Journal of Sound and Vibration* 59, 159-170.
- Sridhar, S., Nayfeh, A. H., and Mook, D. T. (1975), Nonlinear resonances in a class of multi-degree-of-freedom systems, *Journal of Acoustical Society of America* 58, 113-123.
- Streit, D. A., Bajaj, A. K., and Krousgrill, C. M. (1988), Combination parametric resonance leading to periodic and chaotic response in two-degree-of-freedom systems with quadratic non-linearities, *Journal of Sound and Vibration* 124, 297-314.
- ✓ Swinney, H. L. (1983), Observations of order and chaos in nonlinear systems, *Physica* 7D, 3-15.
- Swift, J. W. and Wiesenfeld, K. (1984), Suppression of period doubling in symmetric systems, *Physics Review Letters* 52(9), 705-708.
- Takens, F. (1981), Detecting strange attractors in turbulence, In *Dynamical Systems and Turbulence*, edited by D. A. Rand and L. S. Young, Lecture notes in Mathematics 898, 366-381, Springer-Verlag, New York.
- Tezak, E. G., Mook, D. T., and Nayfeh, A. H. (1978), Nonlinear analysis of the lateral response of columns to periodic loads, *Journal of Mechanical Design* 100, 651-659.
- ✓ Thompson, J. M. T. and Stewart, H. B. (1986), **Nonlinear Dynamics and Chaos**, John Wiley and Sons, Chichester.
- Tondl, A. (1965), **Some Problems of Rotor Dynamics**, Chapman and Hall, London.

- Tousi, S. and Bajaj, A. K. (1985), Period-doubling bifurcations and modulated motions in forced mechanical systems, *Journal of Applied Mechanics* 52, 446-452.
- Tso, W. K. and Asmis, K. G. (1974), Multiple parametric resonance in a non-linear two-degree-of-freedom system, *International Journal of Non-Linear Mechanics* 9, 269-277.
- Wada, B. K. and Fanson, J. L. (1989), Adaptive structures to enable future missions by relaxing ground test requirements, In *Proceedings of the 60th Shock and Vibration Symposium*, Virginia Beach, Virginia, Vol. I, 67-85.
- Yamamoto, T. and Yasuda, K. (1977), On the internal resonance in a nonlinear two-degree-of-freedom system (forced vibrations near the lower resonance point when the natural frequencies are in the ratio 1:2), *Bulletin of Japanese Society of Mechanical Engineering* 20(140), 168-175.
- Yamamoto, T., Yasuda, K., and Nagasaka, I. (1977), On the internal resonance in a nonlinear two-degree-of-freedom system (forced vibrations near the higher resonance point when the natural frequencies are in the ratio 1:2), *Bulletin of the Japanese Society of Mechanical Engineering* 20(147), 1093-1101.
- Yasuda, K. and Kushida, G. (1984), Nonlinear forced oscillations of a shallow spherical shell, *Bulletin of the Japanese Society of Mechanical Engineering* 27, 2233-2240.
- Yasuda, K. and Uno, H. (1983), Multimode response of a circular membrane, *Bulletin of the Japanese Society of Mechanical Engineering* 26(216), 1050-1058.
- Zavodney, L. D. (1987), A theoretical and experimental investigation of parametrically excited nonlinear mechanical systems, Ph. D. Dissertation, Virginia Polytechnic University and State University, Blacksburg, Virginia.

Table 1. Interaction Coefficients

	Λ_1	Λ_2	Λ_3	Λ_6
Structure I	293.613	2208.297	0.00082	0.00854
Structure II	294.732	2213.729	0.00082	0.00852

Appendix A. Metallic structures: Equations of motion

The steps leading to the linear and nonlinear equations of motion of the metallic structure are outlined in this appendix.

Application of Hamilton's principle results in

$$\int_{t_1}^{t_2} \delta L_{aug} dt = 0 \quad (A1)$$

where the times t_1 and t_2 are arbitrary times. Equation (A1) is rewritten as

$$\int_{t_1}^{t_2} \left\{ \int_0^{\ell_1} \delta \hat{L}_1 ds_1 + \int_0^d \delta \hat{L}_2 ds_2 + \int_0^{\ell_3} \delta \hat{L}_3 ds_3 + \delta \hat{L}_0 \right\} dt = 0 \quad (A2)$$

where the $\hat{L}_i$ are the Lagrangian densities. For primary-resonant excitations, X is $O(\varepsilon^2)$, and for secondary-resonant excitations X is $O(\varepsilon)$. Assuming X is $O(\varepsilon)$, expanding, and retaining terms up to $O(\varepsilon^3)$ in equation (3.5), we obtain

$$\begin{aligned} \hat{L}_1 = & \frac{\rho_1}{2} (\dot{v}_1 + \dot{X})^2 - \frac{(EI)_1}{2} \left(\frac{\partial \psi_1}{\partial s_1} \right)^2 - \frac{\lambda_1}{2} \left[2 \frac{\partial u_1}{\partial s_1} + \left(\frac{\partial v_1}{\partial s_1} \right)^2 \right] \\ & - \rho_1 g(X + v_1) \end{aligned} \quad (A3)$$

$$\begin{aligned} \hat{L}_2 = & \frac{\rho_2}{2} \dot{v}_2^2 - \frac{(EI)_2}{2} \left(\frac{\partial \psi_2}{\partial s_2} \right)^2 - \rho_2 g u_2 - \frac{\lambda_2}{2} \left[2 \frac{\partial u_2}{\partial s_2} + \left(\frac{\partial v_2}{\partial s_2} \right)^2 \right] \\ & + \frac{\rho_2}{2} \left[2 \dot{u}_2 \left(\dot{X} + \dot{v}_1 \Big|_{s_1 = \ell_1} \right) - 2 \dot{v}_2 \dot{u}_1 \Big|_{s_1 = \ell_1} \right] \end{aligned} \quad (A4)$$

$$\begin{aligned} \hat{L}_3 = & \frac{\rho_2}{2} \left[\dot{v}_3^2 + 2 \dot{v}_3 \dot{v}_2 \Big|_{s_2 = d} \right] - \frac{(EI)_2}{2} \left(\frac{\partial \psi_3}{\partial s_3} \right)^2 - \rho_2 g u_3 \\ & - \frac{\lambda_3}{2} \left[2 \frac{\partial u_3}{\partial s_3} + \left(\frac{\partial v_3}{\partial s_3} \right)^2 \right] \\ & + \frac{\rho_2}{2} \left[2 \dot{u}_3 \left(\dot{X} + \dot{v}_1 \Big|_{s_1 = \ell_1} \right) - 2 \dot{v}_3 \dot{u}_1 \Big|_{s_1 = \ell_1} \right] \end{aligned} \quad (A5)$$

$$\begin{aligned}
\hat{L}_0 = & \frac{1}{2} [m_1 + m_2 + \rho_2(d + \ell_3)] \left(\dot{v}_1 \Big|_{s_1 = \ell_1} + \dot{X} \right)^2 + \frac{1}{2} J_1 (\dot{\psi}_1)^2 \Big|_{s_1 = \ell_1} \\
& + \frac{1}{2} [m_2 + \rho_2 \ell_3] \dot{v}_2^2 \Big|_{s_2 = d} + \frac{1}{2} J_2 (\dot{\psi}_2)^2 \Big|_{s_2 = d} \\
& - [m_1 + m_2 + \rho_2(d + \ell_3)] g (X + v_1 \Big|_{s_1 = \ell_1}) - (m_2 + \rho_2 \ell_3) g u_2 \Big|_{s_2 = d} \\
& + \frac{1}{2} (m_2 + \rho_2 \ell_3) \left[2 \dot{u}_2 \Big|_{s_2 = d} \left(\dot{X} + \dot{v}_1 \Big|_{s_1 = \ell_1} \right) - 2 \dot{u}_1 \Big|_{s_1 = \ell_1} \dot{v}_2 \Big|_{s_2 = d} \right]
\end{aligned} \tag{A6}$$

After expanding the variations $\delta\psi$, in terms of $\delta u_i'$ and $\delta v_i'$, the variations of the Lagrangian densities $\delta\hat{L}_k$ take the form

$$\begin{aligned}
\delta\hat{L}_1 = & \frac{\partial\hat{L}_1}{\partial v_1} \delta v_1 + \frac{\partial\hat{L}_1}{\partial \dot{v}_1} \delta \dot{v}_1 + \frac{\partial\hat{L}_1}{\partial v_1'} \delta v_1' + \frac{\partial\hat{L}_1}{\partial v_1''} \delta v_1'' + \frac{\partial\hat{L}_1}{\partial u_1} \delta u_1 + \frac{\partial\hat{L}_1}{\partial u_1'} \delta u_1' \\
& + \frac{\partial\hat{L}_1}{\partial \lambda_1} \delta \lambda_1
\end{aligned} \tag{A7}$$

$$\begin{aligned}
\delta\hat{L}_2 = & \frac{\partial\hat{L}_2}{\partial u_1} \delta u_2 + \frac{\partial\hat{L}_2}{\partial \dot{u}_2} \delta \dot{u}_2 + \frac{\partial\hat{L}_2}{\partial u_2'} \delta u_2' + \frac{\partial\hat{L}_2}{\partial v_2} \delta v_2 + \frac{\partial\hat{L}_2}{\partial \dot{v}_2} \delta \dot{v}_2 + \frac{\partial\hat{L}_2}{\partial v_2'} \delta v_2' \\
& + \frac{\partial\hat{L}_2}{\partial v_2''} \delta v_2'' + \frac{\partial\hat{L}_2}{\partial \lambda_2} \delta \lambda_2 + \frac{\partial\hat{L}_2}{\partial u_1} \delta u_1 + \frac{\partial\hat{L}_2}{\partial \dot{u}_1} \delta \dot{u}_1 + \frac{\partial\hat{L}_2}{\partial v_1} \delta v_1 + \frac{\partial\hat{L}_2}{\partial \dot{v}_1} \delta \dot{v}_1
\end{aligned} \tag{A8}$$

$$\begin{aligned}
\delta\hat{L}_3 = & \frac{\partial\hat{L}_3}{\partial u_3} \delta u_3 + \frac{\partial\hat{L}_3}{\partial u_3'} \delta u_3' + \frac{\partial\hat{L}_3}{\partial \dot{u}_3} \delta \dot{u}_3 + \frac{\partial\hat{L}_3}{\partial v_2} \delta v_2 + \frac{\partial\hat{L}_3}{\partial \dot{v}_2} \delta \dot{v}_2 + \frac{\partial\hat{L}_3}{\partial v_3} \delta v_3 \\
& + \frac{\partial\hat{L}_3}{\partial \dot{v}_3} \delta \dot{v}_3 + \frac{\partial\hat{L}_3}{\partial v_3'} \delta v_3' + \frac{\partial\hat{L}_3}{\partial v_3''} \delta v_3'' + \frac{\partial\hat{L}_3}{\partial \lambda_3} \delta \lambda_3 + \frac{\partial\hat{L}_3}{\partial u_1} \delta u_1 \\
& + \frac{\partial\hat{L}_3}{\partial \dot{u}_1} \delta \dot{u}_1 + \frac{\partial\hat{L}_3}{\partial v_1} \delta v_1 + \frac{\partial\hat{L}_3}{\partial \dot{v}_1} \delta \dot{v}_1
\end{aligned} \tag{A9}$$

$$\begin{aligned}\delta\hat{L}_0 = & \frac{\partial\hat{L}_0}{\partial v_1} \delta v_1 + \frac{\partial\hat{L}_0}{\partial \dot{v}_1} \delta \dot{v}_1 + \frac{\partial\hat{L}_0}{\partial \dot{v}_1'} \delta \dot{v}_1' + \frac{\partial\hat{L}_0}{\partial v_2} \delta v_2 + \frac{\partial\hat{L}_0}{\partial \dot{v}_2} \delta \dot{v}_2 + \frac{\partial\hat{L}_0}{\partial \dot{v}_2'} \delta \dot{v}_2' \\ & + \frac{\partial\hat{L}_0}{\partial u_1} \delta u_1 + \frac{\partial\hat{L}_0}{\partial \dot{u}_1} \delta \dot{u}_1 + \frac{\partial\hat{L}_0}{\partial u_2} \delta u_2 + \frac{\partial\hat{L}_0}{\partial \dot{u}_2} \delta \dot{u}_2\end{aligned}\quad (A10)$$

where the overdot indicates the derivative with respect to time and the prime indicates the derivative with respect to the spatial variable defined over a particular beam section. Substituting equations (A3)-(A10) into equation (A2), carrying out integration by parts with respect to space and time, and equating the coefficients of δu_i , δv_i , and $\delta \lambda_i$ to zero leads to the nonlinear equations. We drop the forcing and nonlinear terms from these equations to obtain the equations for linear free oscillations. This leads to

$$\frac{\partial u_k}{\partial s_k} = -\frac{1}{2} \left(\frac{\partial v_k}{\partial s_k} \right)^2 \quad \text{for } k = 1, 2, 3 \quad (A11)$$

$$\lambda_1 = 0 \quad (A12)$$

$$\lambda_2 = -[m_2 + \rho_2(d + \ell_3 - s_2)]g \quad (A13)$$

$$\lambda_3 = -\rho_2 g(\ell_3 - s_3) \quad (A14)$$

and consequently, to equations (3.6)-(3.20).

Appendix B. Perturbation analysis to determine frequency correction

In this appendix, the perturbation analysis used to determine the correction to the natural frequencies due to the underlined terms in equations (3.7) and (3.8) is outlined.

First, the underlined terms are assumed to be of higher order than the rest of the terms. In order to make this explicit, a dimensionless parameter ε is considered and these terms are ordered at $O(\varepsilon)$. Considering the method of strained parameters (Nayfeh 1973,1981), we expand the displacements v_i in equations (3.6)-(3.20) and the natural frequency ω as

$$\omega = \omega_0 + \varepsilon \omega_1 + \dots \quad (B1)$$

$$v_j = v_{j0} + \varepsilon v_{j1} + \dots \quad (B2)$$

where ω_1 is the correction we seek. Considering displacements about the static equilibrium position, substituting equations (B1) and (B2) into equations (3.6)-(3.20), and making the time dependence of v_j explicit, we rewrite equations (3.6)-(3.8) as

$$-\rho_1(\omega_0 + \varepsilon\omega_1 + \dots)^2(V_{10} + \varepsilon V_{11} + \dots) + (EI)_1 \frac{d^4}{ds_1^4} (V_{10} + \varepsilon V_{11} + \dots) = 0 \quad (B3)$$

$$\begin{aligned} & -\rho_2(\omega_0 + \varepsilon\omega_1 + \dots)^2(V_{20} + \varepsilon V_{21} + \dots) + (EI)_2 \frac{d^4}{ds_2^4} (V_{20} + \varepsilon V_{21} + \dots) \\ & + m_2 g \frac{d^2}{ds_2^2} (V_{20} + \varepsilon V_{21} + \dots) + \varepsilon \rho_2 g(d + \ell_3 - s_2) \frac{d^2}{ds_2^2} (V_{20} + \varepsilon V_{21} + \dots) \\ & - \varepsilon \rho_2 g \frac{d}{ds_2} (V_{20} + \varepsilon V_{21} + \dots) = 0 \end{aligned} \quad (B4)$$

$$\begin{aligned} & -\rho_2(\omega_0 + \varepsilon\omega_1 + \dots)^2(V_{40} + \varepsilon V_{41} + \dots) + (EI)_2 \frac{d^4}{ds_3^4} (V_{40} + \varepsilon V_{41} + \dots) \\ & + \varepsilon \rho_2 g(\ell_3 - s_3) \frac{d^2}{ds_3^2} (V_{40} + \varepsilon V_{41} + \dots) - \varepsilon \rho_2 g \frac{d}{ds_3} (V_{40} + \varepsilon V_{41} + \dots) = 0 \end{aligned} \quad (B5)$$

where $V_1 = V_1(s_1)$, $V_2 = V_2(s_2)$, and $V_4 = V_3(s_3) + V_2(s_2 = d)$. The time dependence of v_j is also made explicit in the boundary conditions (3.9)-(3.20). Equating coefficients of like powers of ε , we obtain equations to solve for the zeroth- and first-order terms. The zeroth-order problem consists of equations (3.6)-(3.20) without the underlined terms in equations (3.7) and (3.8). This resulting

system of equations is a linear eigenvalue problem. At $O(\varepsilon)$, we have the following equations and boundary conditions:

Equations :

$$-\rho_1 \omega_0^2 V_{11} + (EI)_1 \frac{d^4 V_{11}}{ds_1^4} = 2\rho_1 \omega_0 \omega_1 V_{10} \quad (B6)$$

in the region $0 < s_1 < \ell_1$,

$$\begin{aligned} -\rho_2 \omega_0^2 V_{21} + (EI)_2 \frac{d^4 V_{21}}{ds_2^4} + m_2 g \frac{d^2 V_{21}}{ds_2^2} = & -\rho_2 (d + \ell_3 - s_2) g \frac{d^2 V_{20}}{ds_2^2} \\ & + \rho_2 g \frac{dV_{20}}{ds_2} + 2\rho_2 \omega_0 \omega_1 V_{20} \end{aligned} \quad (B7)$$

in the region $0 < s_2 < d$, and

$$\begin{aligned} -\rho_2 \omega_0^2 V_{41} + (EI)_2 \frac{d^4 V_{41}}{ds_3^4} = & -\rho_2 (\ell_3 - s_3) g \frac{d^2 V_{20}}{ds_3^2} + \rho_2 g \frac{dV_{40}}{ds_3} \\ & + 2\rho_2 \omega_0 \omega_1 V_{40} \end{aligned} \quad (B8)$$

in the region $0 < s_3 < \ell_3$.

Boundary conditions:

$$V_{11}(s_1 = 0) = 0 \quad (B9)$$

$$\frac{dV_{11}}{ds_1}(s_1 = 0) = 0 \quad (B10)$$

$$V_{21}(s_2 = 0) = 0 \quad (B11)$$

$$\frac{dV_{11}}{ds_1}(s_1 = \ell_1) = \frac{dV_{21}}{ds_2}(s_2 = 0) \quad (B12)$$

$$\begin{aligned} (EI)_1 \frac{d^3 V_{11}}{ds_1^3}(s_1 = \ell_1) + [m_1 + m_2 + \rho_2(d + \ell_3)]\omega_0^2 V_{11}(s_1 = \ell_1) \\ = -2\omega_0\omega_1[m_1 + m_2 + \rho_2(d + \ell_3)]V_{10}(s_1 = \ell_1) \end{aligned} \quad (B13)$$

$$\begin{aligned} (EI)_1 \frac{d^2 V_{11}}{ds_1^2}(s_1 = \ell_1) - J_1\omega_0^2 \frac{dV_{11}}{ds_1}(s_1 = \ell_1) - (EI)_2 \frac{d^2 V_{21}}{ds_2^2}(s_2 = 0) \\ = 2\omega_0\omega_1 J_1 \frac{dV_{10}}{ds_1}(s_1 = \ell_1) \end{aligned} \quad (B14)$$

$$V_{41}(s_3 = 0) = V_{21}(s_2 = d) \quad (B15)$$

$$\frac{dV_{21}}{ds_2}(s_2 = d) = \frac{dV_{41}}{ds_3}(s_3 = 0) \quad (B16)$$

$$\begin{aligned} m_2 g \frac{dV_{21}}{ds_2}(s_2 = d) + (EI)_2 \frac{d^3 V_{21}}{ds_2^3}(s_2 = d) - (EI)_2 \frac{d^3 V_{41}}{ds_3^3}(s_3 = 0) \\ + m_2 \omega_0^2 V_{21}(s_2 = d) = -2\omega_0\omega_1 m_2 V_{20}(s_2 = d) \end{aligned} \quad (B17)$$

$$\begin{aligned} (EI)_2 \frac{d^2 V_{21}}{ds_2^2}(s_2 = d) - J_2\omega_0^2 \frac{dV_{21}}{ds_2}(s_2 = d) - (EI)_2 \frac{d^2 V_{41}}{ds_3^2}(s_3 = 0) \\ = 2\omega_0\omega_1 J_2 \frac{dV_{20}}{ds_2}(s_2 = d) \end{aligned} \quad (B18)$$

$$(EI)_2 \frac{d^3 V_{41}}{ds_3^3} (s_3 = \ell_3) = 0 \quad (B19)$$

$$(EI)_2 \frac{d^2 V_{41}}{ds_3^2} (s_3 = \ell_3) = 0 \quad (B20)$$

Because the homogeneous equations (B6)-(B20) have the same form as the zeroth-order problem, the inhomogeneous first-order problem has a solution only if a solvability condition (Nayfeh, 1981) is satisfied. The solvability condition is determined by defining an adjoint problem. We consider three adjoint functions Z_1 , Z_2 , and Z_3 defined over the regions $0 \leq s_1 \leq \ell_1$, $0 \leq s_2 \leq d$, and $0 \leq s_3 \leq \ell_3$, respectively. Carrying out an analysis similar to that shown in (Nayfeh, 1981), we find that $Z_1 = V_{10}$, $Z_2 = V_{20}$, $Z_3 = V_{40}$, and

$$\omega_1 = \frac{NUMI}{DENI} \quad (B21)$$

where

$$\begin{aligned} NUMI = & \int_0^d \rho_2 (d + \ell_3 - s_2) g V_{20} \frac{d^2 V_{20}}{ds_2^2} ds_2 + \int_0^{\ell_3} \rho_2 (\ell_3 - s_3) g V_{40} \frac{d^2 V_{40}}{ds_3^2} ds_3 \\ & - \int_0^d \rho_2 g V_{20} \frac{dV_{20}}{ds_2} ds_2 - \int_0^{\ell_3} \rho_2 g V_{40} \frac{dV_{40}}{ds_3} ds_3 \end{aligned} \quad (B22)$$

and

$$\begin{aligned}
 DENI = 2\omega_0 & \left\{ \int_0^{\ell_1} \rho_1 V_{10}^2 ds_1 + \int_0^d \rho_2 V_{20}^2 ds_2 + \int_0^{\ell_3} \rho_2 V_{40}^2 ds_3 \right. \\
 & + [m_1 + m_2 + \rho_2(d + \ell_3)] V_{10}^2(s_1 = \ell_1) \\
 & \left. + J_1 \left(\frac{dV_{10}}{ds_1} \right)^2 (s_1 = \ell_1) + m_2 V_{20}^2(s_2 = d) + J_2 \left(\frac{dV_{20}}{ds_2} \right)^2 (s_2 = d) \right\}
 \end{aligned} \tag{B23}$$

Appendix C. Metallic structures analysis: Integrals

The Γ_i are defined in this appendix. They depend on the mode shapes of the structure. Simpson's rule is used for the numerical integrations.

$$\Gamma_1 = -g \left\{ \int_0^{\ell_1} \rho_1 \phi_{11} ds_1 + [m_1 + m_2 + \rho_2(d + \ell_3)] \phi_{11} \Big|_{s_1 = \ell_1} \right\} \quad (C1)$$

$$\Gamma_2 = -g \left\{ \int_0^{\ell_1} \rho_1 \phi_{12} ds_1 + [m_1 + m_2 + \rho_2(d + \ell_3)] \phi_{12} \Big|_{s_1 = \ell_1} \right\} \quad (C2)$$

$$\begin{aligned}
\Gamma_3 = & -\frac{1}{2} \left\{ \int_0^{\ell_1} \left[(EI)_1 \left(\frac{d^2 \phi_{11}}{ds_1^2} \right)^2 + \lambda_1 \left(2 \frac{d\eta_{11}}{ds_1} + \left(\frac{d\phi_{11}}{ds_1} \right)^2 \right) \right] ds_1 \right. \\
& + \int_0^d \left[(EI)_2 \left(\frac{d^2 \phi_{21}}{ds_2^2} \right)^2 + \lambda_2 \left(2 \frac{d\eta_{21}}{ds_2} + \left(\frac{d\phi_{21}}{ds_2} \right)^2 \right) + 2\rho_2 g \eta_{21} \right] ds_2 \\
& + \int_0^{\ell_3} \left[(EI)_2 \left(\frac{d^2 \phi_{31}}{ds_3^2} \right)^2 + \lambda_3 \left(2 \frac{d\eta_{31}}{ds_3} + \left(\frac{d\phi_{31}}{ds_3} \right)^2 \right) + 2\rho_2 g \eta_{31} \right] ds_3 \\
& \left. + 2(m_2 + \rho_2 \ell_3) g \eta_{21} \right|_{s_2=d} \Big\} \quad (C3)
\end{aligned}$$

$$\begin{aligned}
\Gamma_4 = & -\frac{1}{2} \left\{ \int_0^{\ell_1} \left[(EI)_1 \left(\frac{d^2 \phi_{12}}{ds_1^2} \right)^2 + \lambda_1 \left(2 \frac{d\eta_{12}}{ds_1} + \left(\frac{d\phi_{12}}{ds_1} \right)^2 \right) \right] ds_1 \right. \\
& + \int_0^d \left[(EI)_2 \left(\frac{d^2 \phi_{22}}{ds_2^2} \right)^2 + \lambda_2 \left(2 \frac{d\eta_{22}}{ds_2} + \left(\frac{d\phi_{22}}{ds_2} \right)^2 \right) + 2\rho_2 g \eta_{22} \right] ds_2 \\
& + \int_0^{\ell_3} \left[(EI)_2 \left(\frac{d^2 \phi_{32}}{ds_3^2} \right)^2 + \lambda_3 \left(2 \frac{d\eta_{32}}{ds_3} + \left(\frac{d\phi_{32}}{ds_3} \right)^2 \right) + 2\rho_2 g \eta_{32} \right] ds_3 \\
& \left. + 2(m_2 + \rho_2 \ell_3) g \eta_{22} \right|_{s_2=d} \Big\} \quad (C4)
\end{aligned}$$

$$\begin{aligned}
\Gamma_5 = & \int_0^{\ell_1} \left[- (EI)_1 \frac{d^2 \phi_{11}}{ds_1^2} \frac{d^2 \phi_{12}}{ds_1^2} - \lambda_1 \left(\frac{2d\eta_{13}}{ds_1} + \frac{d\phi_{11}}{ds_1} \frac{d\phi_{12}}{ds_1} \right) \right] ds_1 \\
& + \int_0^d \left[- (EI)_2 \frac{d^2 \phi_{21}}{ds_2^2} \frac{d^2 \phi_{22}}{ds_2^2} - \lambda_2 \left(\frac{2d\eta_{23}}{ds_2} + \frac{d\phi_{21}}{ds_2} \frac{d\phi_{22}}{ds_2} \right) - 2\rho_2 g \eta_{23} \right] ds_2 \\
& + \int_0^{\ell_3} \left[- (EI)_2 \frac{d^2 \phi_{31}}{ds_3^2} \frac{d^2 \phi_{32}}{ds_3^2} - \lambda_3 \left(\frac{2d\eta_{33}}{ds_3} + \frac{d\phi_{31}}{ds_3} \frac{d\phi_{32}}{ds_3} \right) - 2\rho_2 g \eta_{33} \right] ds_3 \\
& - 2(m_2 + \rho_2 \ell_3) g \eta_{23} \Big|_{s_2=d}
\end{aligned} \tag{C5}$$

$$\begin{aligned}
\Gamma_6 = & \frac{1}{2} \left\{ \int_0^{\ell_1} \rho_1 \phi_{11}^2 ds_1 + \int_0^d \rho_2 \phi_{21}^2 ds_2 + \int_0^{\ell_3} \rho_2 \phi_{31}^2 ds_3 + 2 \int_0^{\ell_3} \rho_2 \phi_{31} \phi_{21} \Big|_{s_2=d} ds_3 \right. \\
& + [m_1 + m_2 + \rho_2(d + \ell_3)] \phi_{11}^2 \Big|_{s_1=\ell_1} + J_1 \left(\frac{d\phi_{11}}{ds_1} \right)^2 \Big|_{s_1=\ell_1} \\
& \left. + J_2 \left(\frac{d\phi_{21}}{ds_2} \right)^2 \Big|_{s_2=d} + (m_2 + \rho_2 \ell_3) \phi_{21}^2 \Big|_{s_2=d} \right\}
\end{aligned} \tag{C6}$$

$$\begin{aligned}
\Gamma_7 = & \frac{1}{2} \left\{ \int_0^{\ell_1} \rho_1 \phi_{12}^2 ds_1 + \int_0^d \rho_2 \phi_{22}^2 ds_2 + \int_0^{\ell_3} \rho_2 \phi_{32}^2 ds_3 + 2 \int_0^{\ell_3} \rho_2 \phi_{32} \phi_{22} \Big|_{s_2=d} ds_3 \right. \\
& + [m_1 + m_2 + \rho_2(d + \ell_3)] \phi_{12}^2 \Big|_{s_1=\ell_1} + J_1 \left(\frac{d\phi_{12}}{ds_1} \right)^2 \Big|_{s_1=\ell_1} \\
& \left. + J_2 \left(\frac{d\phi_{22}}{ds_2} \right)^2 \Big|_{s_2=d} + (m_2 + \rho_2 \ell_3) \phi_{22}^2 \Big|_{s_2=d} \right\}
\end{aligned} \tag{C7}$$

$$\begin{aligned}
\Gamma_8 = & \int_0^{\ell_1} \rho_1 \phi_{11} \phi_{12} ds_1 + \int_0^d \rho_2 \phi_{21} \phi_{22} ds_2 + \int_0^{\ell_3} \rho_2 \left[\phi_{31} \phi_{32} + \phi_{31} \phi_{22} \Big|_{s_2=d} + \phi_{32} \phi_{21} \Big|_{s_2=d} \right] ds_3 \\
& + [m_1 + m_2 + \rho_2(d + \ell_3)](\phi_{11} \phi_{12}) \Big|_{s_1=\ell_1} + J_1 \left(\frac{d\phi_{11}}{ds_1} \frac{d\phi_{12}}{ds_1} \right) \Big|_{s_1=\ell_1} \quad (C8) \\
& + (m_2 + \rho_2 \ell_3)(\phi_{21} \phi_{22}) \Big|_{s_2=d} + J_2 \left(\frac{d\phi_{21}}{ds_2} \frac{d\phi_{22}}{ds_2} \right) \Big|_{s_2=d}
\end{aligned}$$

$$\Gamma_9 = \int_0^{\ell_1} \rho_1 \phi_{11} ds_1 + [m_1 + m_2 + \rho_2(d + \ell_3)] \phi_{11} \Big|_{s_1=\ell_1} \quad (C9)$$

$$\Gamma_{10} = \int_0^{\ell_1} \rho_1 \phi_{12} ds_1 + [m_1 + m_2 + \rho_2(d + \ell_3)] \phi_{12} \Big|_{s_1=\ell_1} \quad (C10)$$

$$\begin{aligned}
\Gamma_{11} = & \int_0^d 2\rho_2 \left[\eta_{21} \phi_{11} \Big|_{s_1=\ell_1} - \eta_{11} \Big|_{s_1=\ell_1} \phi_{21} \right] ds_2 \\
& + \int_0^{\ell_3} 2\rho_2 \left[\eta_{31} \phi_{11} \Big|_{s_1=\ell_1} - \eta_{11} \Big|_{s_1=\ell_1} \phi_{31} \right] ds_3 \quad (C11) \\
& + 2(m_2 + \rho_2 \ell_3) \left[\eta_{21} \Big|_{s_2=d} \phi_{11} \Big|_{s_1=\ell_1} - \eta_{11} \Big|_{s_1=\ell_1} \phi_{21} \Big|_{s_2=d} \right]
\end{aligned}$$

$$\begin{aligned}
\Gamma_{12} = & \int_0^d 2\rho_2 \left[\eta_{21}\phi_{12} \Big|_{s_1=\ell_1} + \eta_{23}\phi_{11} \Big|_{s_1=\ell_1} - \phi_{22}\eta_{11} \Big|_{s_1=\ell_1} - \phi_{21}\eta_{13} \Big|_{s_1=\ell_1} \right] ds_2 \\
& + \int_0^{\ell_3} 2\rho_2 \left[\eta_{31}\phi_{12} \Big|_{s_1=\ell_1} + \eta_{33}\phi_{11} \Big|_{s_1=\ell_1} - \phi_{32}\eta_{11} \Big|_{s_1=\ell_1} - \phi_{31}\eta_{13} \Big|_{s_1=\ell_1} \right] ds_3 \\
& + 2(m_2 + \rho_2\ell_3) \left[\eta_{21} \Big|_{s_2=d} \phi_{12} \Big|_{s_1=\ell_1} + \eta_{23} \Big|_{s_2=d} \phi_{11} \Big|_{s_1=\ell_1} - \eta_{11} \Big|_{s_1=\ell_1} \phi_{22} \Big|_{s_2=d} \right. \\
& \left. - \eta_{13} \Big|_{s_1=\ell_1} \phi_{21} \Big|_{s_2=d} \right] \quad (C12)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{13} = & \int_0^d 2\rho_2 \left[\eta_{22}\phi_{12} \Big|_{s_1=\ell_1} - \eta_{12} \Big|_{s_1=\ell_1} \phi_{22} \right] ds_2 \\
& + \int_0^{\ell_3} 2\rho_2 \left[\eta_{32}\phi_{12} \Big|_{s_1=\ell_1} - \eta_{12} \Big|_{s_1=\ell_1} \phi_{32} \right] ds_3 \\
& + 2(m_2 + \rho_2\ell_3) \left[\eta_{22} \Big|_{s_2=d} \phi_{12} \Big|_{s_1=\ell_1} - \eta_{12} \Big|_{s_1=\ell_1} \phi_{22} \Big|_{s_2=d} \right] \quad (C13)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{14} = & \int_0^d 2\rho_2 \left[\eta_{22}\phi_{11} \Big|_{s_1=\ell_1} + \eta_{23}\phi_{12} \Big|_{s_1=\ell_1} - \phi_{21}\eta_{12} \Big|_{s_1=\ell_1} - \phi_{22}\eta_{13} \Big|_{s_1=\ell_1} \right] ds_2 \\
& + \int_0^{\ell_3} 2\rho_2 \left[\eta_{32}\phi_{11} \Big|_{s_1=\ell_1} + \eta_{33}\phi_{12} \Big|_{s_1=\ell_1} - \phi_{31}\eta_{12} \Big|_{s_1=\ell_1} - \phi_{32}\eta_{13} \Big|_{s_1=\ell_1} \right] ds_3 \\
& + 2(m_2 + \rho_2\ell_3) \left[\eta_{22} \Big|_{s_2=d} \phi_{11} \Big|_{s_1=\ell_1} + \eta_{23} \Big|_{s_2=d} \phi_{12} \Big|_{s_1=\ell_1} \right. \\
& \left. - \phi_{21} \Big|_{s_2=d} \eta_{12} \Big|_{s_1=\ell_1} - \phi_{22} \Big|_{s_2=d} \eta_{13} \Big|_{s_1=\ell_1} \right] \quad (C14)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{15} = & \int_0^d 2\rho_2 \left[\eta_{23} \phi_{11} \Big|_{s_1=\ell_1} - \phi_{21} \eta_{13} \Big|_{s_1=\ell_1} \right] ds_2 \\
& + \int_0^{\ell_3} 2\rho_2 \left[\eta_{33} \phi_{11} \Big|_{s_1=\ell_1} - \phi_{31} \eta_{13} \Big|_{s_1=\ell_1} \right] ds_3 \\
& + 2(m_2 + \rho_2 \ell_3) \left[\phi_{11} \Big|_{s_1=\ell_1} \eta_{23} \Big|_{s_2=d} - \phi_{21} \Big|_{s_2=d} \eta_{13} \Big|_{s_1=\ell_1} \right]
\end{aligned} \tag{C15}$$

$$\begin{aligned}
\Gamma_{16} = & \int_0^d 2\rho_2 \left[\eta_{23} \phi_{12} \Big|_{s_1=\ell_1} - \phi_{22} \eta_{13} \Big|_{s_1=\ell_1} \right] ds_2 \\
& + \int_0^{\ell_3} 2\rho_2 \left[\eta_{33} \phi_{12} \Big|_{s_1=\ell_1} - \phi_{32} \eta_{13} \Big|_{s_1=\ell_1} \right] ds_3 \\
& + 2(m_2 + \rho_2 \ell_3) \left[\eta_{23} \Big|_{s_2=d} \phi_{12} \Big|_{s_1=\ell_1} - \phi_{22} \Big|_{s_2=d} \eta_{13} \Big|_{s_1=\ell_1} \right]
\end{aligned} \tag{C16}$$

$$\Gamma_{17} = -g \{ \rho_1 \ell_1 + [m_1 + m_2 + \rho_2(d + \ell_3)] \} \tag{C17}$$

$$\Gamma_{18} = \frac{1}{2} \{ \rho_1 \ell_1 + [m_1 + m_2 + \rho_2(d + \ell_3)] \} \tag{C18}$$

$$\Gamma_{19} = -2 \left\{ \int_0^d \rho_2 \eta_{21} ds_2 + \int_0^{\ell_3} \rho_2 \eta_{31} ds_3 + (m_2 + \rho_2 \ell_3) \eta_{21} \Big|_{s_2=d} \right\} \tag{C19}$$

$$\Gamma_{20} = -2 \left\{ \int_0^d \rho_2 \eta_{22} ds_2 + \int_0^{\ell_3} \rho_2 \eta_{32} ds_3 + (m_2 + \rho_2 \ell_3) \eta_{22} \Big|_{s_2=d} \right\} \tag{C20}$$

$$\Gamma_{21} = -2 \left\{ \int_0^d \rho_2 \eta_{23} ds_2 + \int_0^{\ell_3} \rho_2 \eta_{33} ds_3 + (m_2 + \rho_2 \ell_3) \eta_{23} \Big|_{s_2=d} \right\} \quad (\text{C21})$$

Appendix D. Dimensions and properties of structures

In this appendix, the dimensions and properties of the steel and composite beam-mass structures are provided. The dimensions of the metallic structures are given with respect to Figs. 3.1 and 3.2 and the dimensions of the composite structures are given with respect to Fig. 5.1.

D.1. Dimensions and properties of metallic structures

The value of the Young's modulus of elasticity E for the steel beams is $2 \times 10^{11} \text{ N/m}^2$. Essentially, two metallic beam-mass structures were constructed and the different settings of these two structures resulted in Structures I, II, III, IV, V, and VI. Structures I and II have the following common properties:

Horizontal Beam : width x thickness = 12.83 mm x 1.68 mm ;

$$\rho_1 = 161.9 \text{ grams/m}$$

Vertical Beam : width x thickness = 12.80 mm x 0.56 mm ;

$$\rho_2 = 49.9 \text{ grams/m}$$

$$L_1 = 154.51 \text{ mm} ; L_2 = 152.40 \text{ mm}$$

$$M_1 = 31.1 \text{ grams} ; M_2 = 40.0 \text{ grams}$$

$$m_1 = 33.8 \text{ grams} ; m_2 = 40.6 \text{ grams}$$

$$J_1 = 2.161 \times 10^{-6} \text{ Kg m}^2$$

$$J_2 = 1.593 \times 10^{-6} \text{ Kg m}^2$$

The mass M_1 is made of three steel pieces. Two of these pieces are identical: each weighs 8.4 grams and has the dimensions 0.500"x0.500"x0.250". The third one has the dimensions 0.500"x0.500"x0.500" and weighs 14.3 grams. The masses of the screws used to put these masses together are also included in the numbers given above.

The mass M_2 is made of two steel pieces. One piece has the dimensions 0.375"x0.500"x1.005" and weighs 24.6 grams, while the other piece has the dimensions 0.250"x0.500"x1.005" and weighs 15.4 grams.

Structures III, IV, V, and VI have the following common properties:

Horizontal Beam : width x thickness = 12.73 mm x 1.35 mm ;

$$\rho_1 = 128.5 \text{ grams/m}$$

Vertical Beam : width x thickness = 13.11 mm x 0.56 mm ;

$$\rho_2 = 55.8 \text{ grams/m}$$

$$M_1 = 31.1 \text{ grams} ; M_2 = 15.5 \text{ grams}$$

$$m_1 = 32.7 \text{ grams} ; m_2 = 16.0 \text{ grams}$$

$$J_1 = 2.102 \times 10^{-6} \text{ Kg m}^2$$

$$J_2 = 0.580 \times 10^{-6} \text{ Kg m}^2$$

The mass M_1 is identical to that used in Structures I and II, while the mass M_2 is made of two identical aluminum pieces each having the dimensions 0.380"x0.350"x1.253" and weighing 15.5 grams, screws included.

The following properties and dimensions distinguish Structures I, II, III, IV, V, and VI from each other.

Structure I

$$D = 90.53 \text{ mm} ; \ell_1 = 150.80 \text{ mm} ; \ell_3 = 61.90 \text{ mm} ; d = 87.40 \text{ mm}$$

$$\text{Damped resonant frequencies : } f_1 = 8.13 \text{ Hz} ; f_2 = 16.44 \text{ Hz.}$$

Structure II

$D = 90.45 \text{ mm} ; \ell_1 = 150.80 \text{ mm} ; \ell_3 = 61.98 \text{ mm} ; d = 87.32 \text{ mm}$

Damped resonant frequencies : $f_1 = 8.16 \text{ Hz} ; f_2 = 16.50 \text{ Hz}$.

Structure III

$L_1 = 147.37 \text{ mm} ; L_2 = 152.40 \text{ mm} ; D = 118.72 \text{ mm}$

Damped resonant frequencies : $f_1 = 8.13 \text{ Hz} ; f_2 = 16.45 \text{ Hz}$.

Structure IV

$L_1 = 145.95 \text{ mm} ; L_2 = 152.40 \text{ mm} ; D = 124.21 \text{ mm}$

Damped resonant frequencies : $f_1 = 8.11 \text{ Hz} ; f_2 = 16.41 \text{ Hz}$.

Structure V

Structure V has the same dimensions as Structure IV. Initially, what was Structure IV was subjected to high levels (one to two g's) of combination-resonant excitations. After a week of experiments, the damped resonant frequencies were determined to be $f_1 = 8.11 \text{ Hz}$ and $f_2 = 16.36 \text{ Hz}$. The structure with these frequencies became Structure V.

Structure VI

$$L_1 = 144.81 \text{ mm} ; L_2 = 152.40 \text{ mm} ; D = 124.02 \text{ mm}$$

Damped resonant frequencies : $f_1 = 8.19 \text{ Hz}$; $f_2 = 16.72 \text{ Hz}$.

D.2. Dimensions of composite structures

Essentially, two composite beam-mass structures were used in the experiments. One structure was made of glass-epoxy composite beams, while the other structure was made of graphite-epoxy composite beams. Two different settings of the glass-epoxy composite structure resulted in Structures A and C while a setting of the graphite-epoxy composite structure resulted in Structure B.

Structure A

Horizontal Beam : width x thickness = 12.95 mm x 2.16 mm ;

$$\rho_1 = 58.6 \text{ grams/m}$$

Vertical Beam : width x thickness = 12.95 mm x 2.16 mm ;

$$\rho_2 = 53.1 \text{ grams/m}$$

$$L_1 = 184.75 \text{ mm} ; L_2 = 203.20 \text{ mm} ; D = 152.80 \text{ mm}$$

$$M_1 = 10.9 \text{ grams} ; M_2 = 14.9 \text{ grams}$$

Structure B

$$\text{Horizontal Beam : width x thickness} = 12.95 \text{ mm} \times 1.19 \text{ mm} ;$$

$$\rho_1 = 24.1 \text{ grams/m}$$

$$\text{Vertical Beam : width x thickness} = 13.08 \text{ mm} \times 1.19 \text{ mm} ;$$

$$\rho_2 = 24.1 \text{ grams/m}$$

$$L_1 = 176.80 \text{ mm} ; L_2 = 203.20 \text{ mm} ; D = 171.45 \text{ mm}$$

$$M_1 = 10.9 \text{ grams} ; M_2 = 14.9 \text{ grams}$$

Structure C

$$\text{Horizontal Beam : width x thickness} = 12.95 \text{ mm} \times 2.16 \text{ mm} ;$$

$$\rho_1 = 58.6 \text{ grams/m}$$

$$\text{Vertical Beam : width x thickness} = 12.95 \text{ mm} \times 2.16 \text{ mm} ;$$

$$\rho_2 = 53.1 \text{ grams/m}$$

$$L_1 = 181.18 \text{ mm} ; L_2 = 203.20 \text{ mm} ; D = 155.78 \text{ mm}$$

$$M_1 = 10.9 \text{ grams} ; M_2 = 14.9 \text{ grams}$$

Both masses M_1 and M_2 are made of aluminum. The mass M_2 is made of two identical pieces each having the dimensions 0.380"x0.350"x1.253". The mass M_1 is made of three pieces and is shaped like a C-clamp. Two of these pieces are identical and have the dimensions 0.864"x0.525"x 0.125" each.

Appendix E. Experimental procedures

Here, the procedures used to determine the damped resonant frequencies and damping, construct pseudo-phase planes and Poincaré sections, perform time-dependent modal decompositions, and calculate pointwise dimensions are discussed.

E.1. Determination of linear frequencies

For the experiments, the linear frequencies had to be determined as accurately as possible and we sought an accuracy of at least two decimal places. Each structure's response was examined in a 20.0 Hz base band with 1280 lines of resolution. First, a structure was subjected to a low level random excitation. The peaks in the response spectrum correspond to the different modes of the structure. In Fig. E.1, we show a typical response spectrum of a

metallic structure to a random excitation. Symbols f_i are used to mark the respective modal frequencies in this figure. When the modes of vibration are coupled through an internal resonance these peaks are not sharp and distinct (e.g., peak f_2 in Fig. E.1) and hence, the procedure is not accurate enough to determine the damped resonant frequencies. However, they help in establishing the location of the modal frequencies. Subsequent to the random excitation, we subjected the structure to a nonstationary excitation, where the frequency was linearly swept in a short span of frequency (about 0.5 Hz to 1.0 Hz) around a desired frequency, identified from the random excitation, in a short time (typically 1.0 to 2.0 seconds). The dominant peak in the decaying free response, following the tuned frequency sweep, corresponds to the damped resonant frequency of a particular mode. This method is useful in isolating the mode of interest when the modes are coupled through an internal resonance. During the experiments, we ensured that the excitation level was low enough so that the other coupled modes were not excited during the sweep. However, the modes of a structure have to be "sufficiently" separated in frequency for one to use this procedure. Further, one should note that the type of excitation used here can only excite the flexural modes of the metallic/composite structures.

We also excited each of the metallic (composite) structures manually and obtained their first two (three) linear frequencies for each of them by carefully examining the frequency spectrum of their free response. The resulting frequencies agreed with those found using the former procedure. For each

composite structure, the frequency of the first torsional mode was determined from the free response, after providing the structure an out-of-plane disturbance. The spectral lines in the free response shifted as the amplitude of the motion decayed, displaying frequency dependence on amplitude, a common characteristic of nonlinear systems. The damped resonant frequencies for the metallic and composite structures are provided in Appendix D.

E.2. Determination of modal-damping coefficients

The damping coefficients were determined only for Structures I and II. This paragraph is written in their context. As the damping factors for the beam-mass structure were small they were determined using the log decrement method (Bert, 1973). We captured the structure's free response (time length of 5.0 seconds) on a digital storage oscilloscope. When the structure was excited manually its free response contained both the first and second modes (see Fig. E.2). The coupling between the modes made the free response of the internally-resonant structure unsuitable for determining the damping coefficients of the respective modes by the log decrement method. Although the last few cycles of the decaying response is dominated by the mode with the lowest natural frequency, the time length and magnitude are not large enough to allow for a reasonable estimate of the damping coefficient of

the respective mode. So, the modal-damping coefficients were determined indirectly by determining the modal-damping coefficients of a detuned structure. For the detuned structure, the first two frequencies were no longer commensurable or nearly commensurable. The structure was detuned by moving the rigid mass M_2 on the vertical beam, see Fig. 3.1. Several cases of the detuned structure were considered. The mass M_2 was moved on the vertical beam, such that its mass center was above or below the corresponding location in the internally-resonant case. For each of these cases, we excited each mode and determined the corresponding modal-damping coefficient from the free response using the log decrement method. Based on the values of these modal-damping coefficients the corresponding quantities for the internally-resonant case were found by interpolation to be $\mu_1 = 0.090$ and $\mu_2 = 0.220$. These damping coefficients were used for both Structures I and II in the analyses. The difference between the natural and resonant frequencies due to damping was less than our measurement resolution and hence it was neglected.

E.3. Pseudo-phase plane and Poincaré section

In general, a phase plane is constructed with the coordinates $x(t)$ and its time derivative $\dot{x}(t)$, where $x(t)$ is a measured displacement of a system while $\dot{x}(t)$ is obtained by using a differentiator (Horowitz and Hill, 1980) or a numerical

scheme. On the other hand, the coordinates of a pseudo-phase plane are a variable, say $x(t)$ and its time-delayed version $x(t + T_d)$, where T_d is a "suitably chosen" time delay (see Moon, 1987). Here, the signals w_H and w_V from Strain Gage H and Strain Gage V, respectively, were used as coordinates to construct the so-called "pseudo-phase plane". Roughly speaking, each of the two strain-gage signals, $w_V(t)$ and $w_H(t)$ can be construed as a time-delayed version of the other signal, i.e., $w_H(t) = w_V(t \pm T_d)$ where T_d is the time delay. This time delay is a result of phase differences between respective spectral components of the two signals which in turn is a consequence of the different strain-gage locations (see equations (4.1) and (4.2)). In this study, each structure's response evolves in a phase space whose dimension is greater than two. So, one should bear in mind that the x-y plots or cross plots in the pseudo-phase plane are projections of the response on a two-dimensional plane. These cross plots can also be referred to as Lissajous figures. Modulated responses were distinguished from periodic responses by examining the cross plots or x-y plots in the pseudo-phase plane. When the motion of the structure was periodic the cross plot remained stationary while when it was modulated an evolving pattern was observed in this plane. We used the Poincaré section to ascertain if a response had reached a steady state and to examine the nature of a modulated motion, that is, two-period quasi-periodic or otherwise (Bergé, Pomeau, and Vidal, 1984 and Moon, 1987). The excitation frequency f was used as a clock frequency to construct a Poincaré section. Like a cross plot, this section is also a projection on a two-dimensional plane. Each section is made up of points $\{w_H, w_V, t | t \bmod 1/f = t_0\}$, where t_0 is the initial time of

collection. The Poincaré sections were made on a digital storage oscilloscope (Tektronix model 2230) and 1024/4096 points were collected for each section. However, each displayed or plotted Poincaré section had only 1024 points in it. When we collected 4096 points, we chose a window of 1024 points from it for display.

E.4. Time-dependent modal decompositions

The objective of this procedure is to obtain the time-dependent modal amplitudes (here, a_k) of the respective modes. The projection of the asymptotic state (as time $t \rightarrow \infty$) of motion in the plane of a_k was used in studying modulated motions. The projections of periodic and two-period quasi-periodic motions are respectively, fixed points and limit cycles in this phase plane. This procedure was used only in experiments where the metallic structure was subjected to primary-resonant excitations.

If the motion was just amplitude modulated, one could obtain the modal amplitudes at different times from the power spectrum as in the study of Ciliberto and Gollub (1985). However, here, the observed motions were amplitude- and phase-modulated. The time-dependence of the phases prevented us from directly obtaining the time-dependent modal amplitudes from the frequency spectrum or the corresponding time series. An amplitude- and phase-modulated signal can be demodulated using a phase detector

(Horowitz and Hill, 1980), if the signal has just one carrier frequency. Essentially, the phase detector averages over the fast time scale or carrier frequency and results in a signal composed of just the slowly varying quantities. As the signals considered here had more than one carrier frequency, we had to resort to the procedure discussed in this section. The hardware and software to implement this procedure was developed by Mike Colbert and Mahir Nayfeh, at the Vibration Laboratory in the department. This procedure is very similar to the digital complex demodulation technique used by Kim, Khadra, and Powers (1980) to study wave modulations in a weakly ionized plasma.

The measured signal (from Strain Gage H or Strain Gage V) during primary-resonant excitations of the metallic structure is of the form

$$w(t) = a_1(t) \cos[\Omega_1 t + \gamma_1(t)] + a_2(t) \cos[\Omega_2 t + \gamma_2(t)] \quad (E1)$$

where Ω_i and γ_i are the respective radian modal (carrier) frequencies and phases associated with the modes. The amplitudes a_k and phases γ_k vary at a low frequency, which is the frequency of modulation. In all the experiments, the frequencies Ω_i were less than 40.0 Hz. The signal was low-pass filtered (cut-off frequency = 40.0 Hz) to eliminate line noise, amplified, and sent to an IBM PC which acquired data through an 8-bit analog-to-digital converter. At the time, the hardware was made, this was the only available converter in the laboratory. Despite its limitations, we found it useful in our study. A quartz clock was used to set the desired sampling frequency. We used a sampling

frequency of 244.285 Hz and collected 8192 points per frame. A fast Fourier transform (FFT) algorithm was used to transform the data from the time domain to the frequency domain. From the frequency spectrum, we determined the spectral width Δf of the sideband structure. Two band-pass filters centered at $\Omega_1/2\pi$ and $\Omega_2/2\pi$ and each having the width Δf were used for the digital filtering, which separated the two modes. Here, this procedure was used because the modes were "well" separated. An inverse FFT (IFFT) algorithm was used to transform the separated frequency components back into the time domain.

Subsequently, each separated signal $a_k(t) \cos[\Omega_k t + \gamma_k(t)]$ was modulated by $\sin(\Omega_k t)$ and $\cos(\Omega_k t)$ functions. The resulting signals, which consist of a high-frequency and low-frequency components, have the following form:

$$a_k(t) \cos[\Omega_k t + \gamma_k(t)] \cos(\Omega_k t) = \frac{1}{2} a_k(t) \{ \cos[2\Omega_k t + \gamma_k(t)] + \cos[\gamma_k(t)] \} \quad (E2)$$

$$a_k(t) \cos[\Omega_k t + \gamma_k(t)] \sin(\Omega_k t) = \frac{1}{2} a_k(t) \{ \sin[2\Omega_k t + \gamma_k(t)] - \sin[\gamma_k(t)] \} \quad (E3)$$

We used a low-pass filter to eliminate the $2\Omega_k$ components in the signals shown above and obtained

$$m(t) = \frac{1}{2} a_k(t) \cos[\gamma_k(t)] \quad (E4)$$

and

$$n(t) = -\frac{1}{2} \dot{a}_k(t) \sin[\gamma_k(t)] \quad (E4)$$

from which we determined the time-dependent amplitudes and phases according to

$$\dot{a}_k(t) = 2\sqrt{m^2 + n^2} \quad , \quad \gamma_k(t) = \tan^{-1}(-n/m) \quad (E5)$$

E.5. Pointwise dimension calculations

One of the geometrical properties that characterizes an attractor (some subset of phase space to which the initial conditions are attracted as $t \rightarrow \infty$) is its dimension (Farmer, Ott, and Yorke, 1983). It is a measure of the minimum number of essential variables necessary to model the dynamics of a system. Attractors that have a noninteger dimension are called strange attractors while attractors with sensitive dependence on initial conditions are called chaotic (Grebogi, Ott, Pelikan, and Yorke, 1984). Normally, one evaluates the Lyapunov exponents (Farmer, Ott, and Yorke, 1983) to determine the sensitive dependence on initial conditions. In general, the adjective strange is associated with the geometry of the attractor while the adjective chaotic is used in context of the dynamics on the attractor. However, most strange attractors are chaotic, and hence these words are used interchangeably in the literature. There are examples of strange nonchaotic attractors. An example

which arises in the context of discrete dynamical systems, the logistic maps (one-dimensional noninvertible maps), is the Feigenbaum attractor (Ruelle, 1989). Another, in the context of continuous-time dynamical systems (i.e. flows), occurs in a nonlinear oscillator driven at two incommensurate frequencies (Grebogi, Ott, Pelikan, and Yorke, 1984). In their study, Grebogi et al. conjecture that, in general systems, strange nonchaotic attractors are less likely to occur. We extend their conjecture to the present study and expect attractors with a noninteger or fractal dimension to be chaotic as well. Here, pointwise dimension (Farmer et al. 1983, Moon, 1987) calculations were used in conjunction with the frequency spectrum and the Poincaré section to ascertain if some of the modulated motions were chaotic.

For all dimension calculations, 60,000 points of either Strain Gage H or Strain Gage V signal were collected at a sampling frequency of 120.0 Hz. The data acquisition and the calculations were done on a Masscomp laboratory computer (model MC5500). From the collected signal $x(t)$, an N -dimensional phase space with coordinates $x(t), x(t + T_d), \dots, x(t + (N - 1)T_d)$ was constructed according to the method of delays, Ruelle, (1989). The time delay T_d is normally chosen in an arbitrary manner while N is the dimension of the embedding space in which the attractor is embedded. As discussed in Takens (1981), loosely speaking an m -dimensional surface can be embedded in a space whose dimension is greater than $2m + 1$. In our context, we had to take a sufficiently large N to ensure that the embedding was possible, and it was chosen to be eight for the dimension calculations. Here, T_d was chosen

based on the considerations discussed in Roux, Simoyi, and Swinney (1983) and Froehling, Crutchfield, Farmer, Packard, and Shaw (1981). The delay was also varied to obtain a large scaling region in the plot for pointwise dimension. The criteria for choosing a proper T_d are also provided in the papers of Fraser and Swinney (1986) and Liebert and Schuster (1989). One hundred reference points on the constructed N -dimensional phase space were chosen in a random manner, and for each one the number of points (of the 60,000 data points) that were located within a ball of radius r centered at it were counted. The counts determined for each of the reference points were averaged over the total number of such points, in this case 100. Then the logarithm of this count was plotted against the logarithm of the radius of the ball for the different embedding dimensions. According to the definition for the dimension, the count should scale as r^d , where d is the pointwise dimension as $r \rightarrow 0$. However, because the data collected from the experiment are not free from noise, it is not possible to determine the value of d from the log-log plot for small values of r . Hence, we located the scaling region "above" the noise floor. For a particular embedding dimension, the slope of the curve in this region gives the pointwise dimension. The dimension calculation can also be used to determine if the experimental data are just made up of random noise or otherwise. If the data collected from an experiment are made up of just random noise, then the count would scale as r^N , where N is the embedding dimension.

For the dimension calculations, a 176 line program written in C, a modified version of the program provided by Riesenethel (1987, 1988) was run on the Masscomp MC5500. On the average, for 60,000 data points and an embedding dimension of eight, for each reference point the program took about a minute for the computations. The program was initially tested by considering a quasi-periodic signal with two incommensurable frequencies 16.40 Hz and 0.33 Hz. Thirty thousand points of the quasi-periodic signal were collected at a sampling frequency of 120.0 Hz and a delay of 0.83 seconds was used for the embedding. For an embedding dimension of 6 and 100 reference points, the pointwise dimension was found to be about 2.014. This is in good agreement with the actual dimension of a two-period quasi-periodic attractor, which is 2.000. For other two-period quasiperiodic signals, the program yielded a dimension of about two. Strictly speaking, for the dimension calculations (as per the definition) we should consider a large number of points (i.e., the number of reference points = number of data points and they should $\rightarrow \infty$). However here, the use of the dimension to verify whether the attractor was chaotic or not was considered more important than determining the precise value of the dimension.

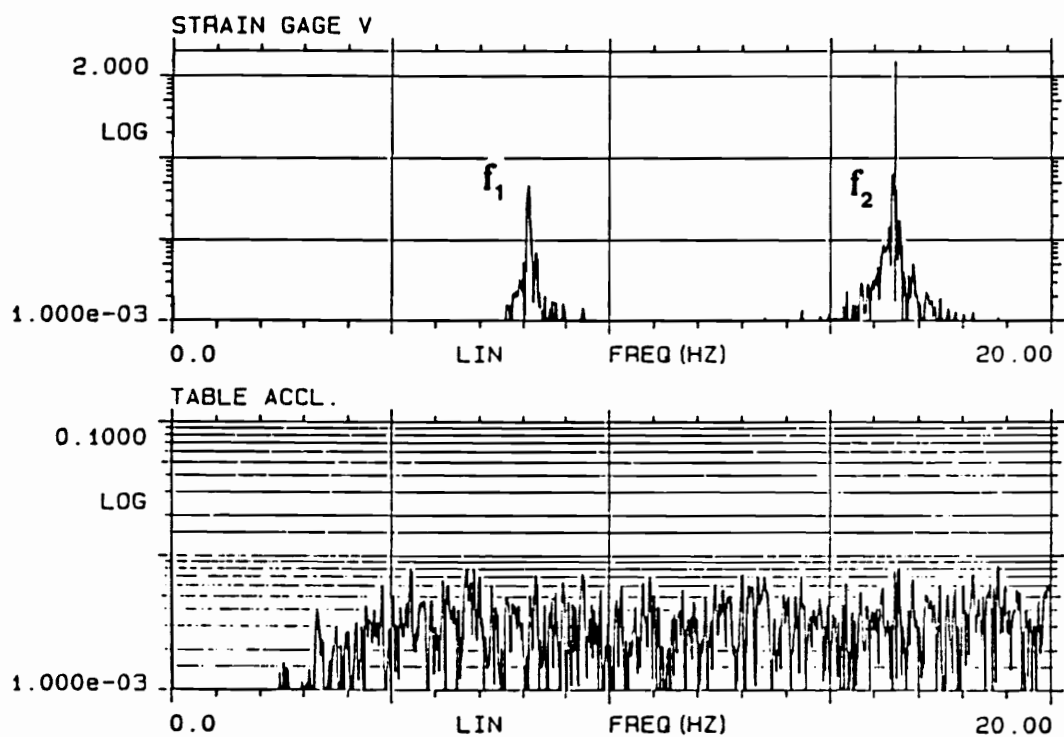


Figure E.1. Upper figure shows the response spectrum of a metallic structure to a random excitation, whose spectrum is shown in the lower figure.

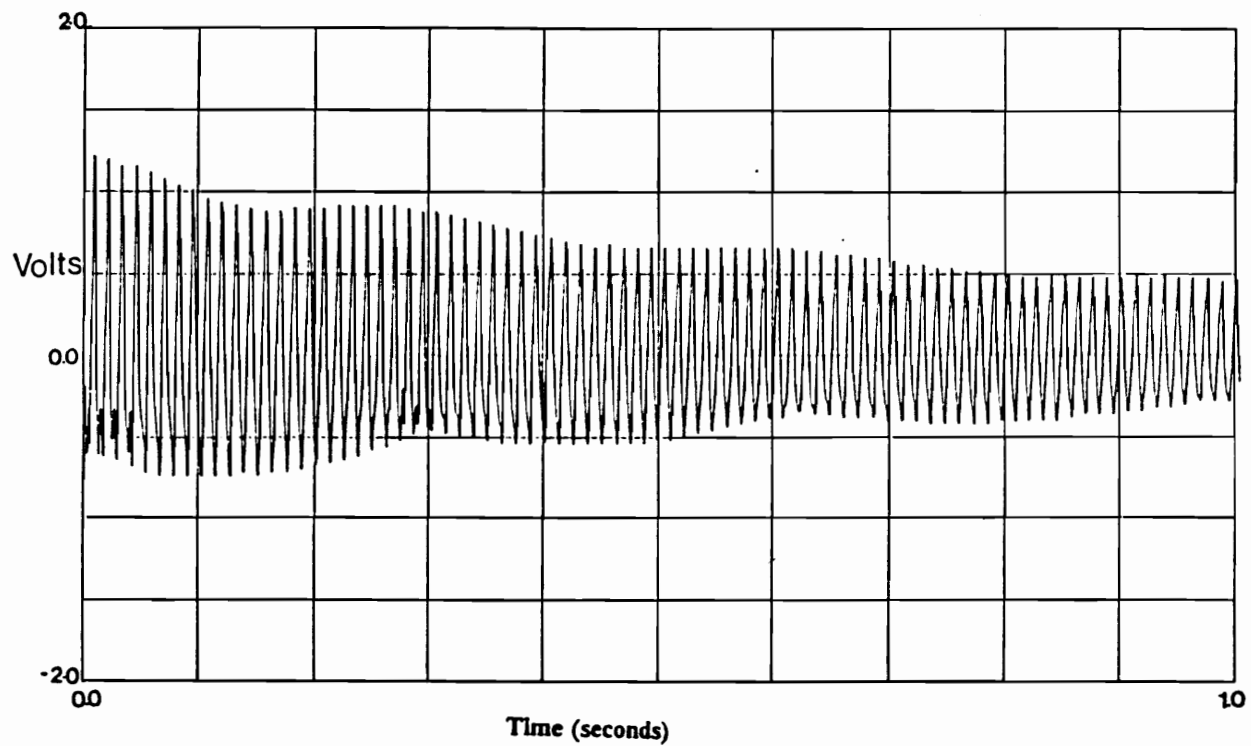


Figure E.2. Free response of a metallic structure.

Appendix F. Coefficients of Section 6.2.

Here, the quantities K_{ij} , b_i , and C_j used in the analysis in Section 6.2 are provided. The elements K_{2j} are determined from

$$[AB] \begin{Bmatrix} K_{21} \\ K_{22} \\ K_{23} \\ K_{24} \end{Bmatrix} = \begin{Bmatrix} g_{11} \\ g_{12} \\ g_{13} \\ g_{14} \end{Bmatrix} \quad (F1)$$

where the 4×4 matrix $[AB]$ is $-[\Lambda] + 2i\omega_m[I]$, the matrix $[\Lambda]$ is defined in Chapter 6, and the matrix $[I]$ is the identity matrix. The elements g_{ij} are given by

$$g_{11} = -\Lambda_1(K_{12}K_{13} - K_{11}K_{14}) \quad (F2)$$

$$g_{12} = -\Lambda_1(K_{11}K_{13} + K_{12}K_{14}) \quad (F3)$$

$$g_{13} = 2\Lambda_2 K_{11}K_{12} \quad (F4)$$

$$g_{14} = -\Lambda_2(K_{11}^2 - K_{12}^2) \quad (F5)$$

The elements K_{3j} are the solution of

$$[AC] \begin{Bmatrix} K_{31} \\ K_{32} \\ K_{33} \\ K_{34} \end{Bmatrix} = \begin{Bmatrix} g_{21} \\ g_{22} \\ g_{23} \\ g_{24} \end{Bmatrix} \quad (F6)$$

where the 4 x 4 matrix $[AC]$ is $-[\Lambda]$ and the elements g_{2j} are given by

$$g_{21} = -\Lambda_1(K_{12}\overline{K_{13}} + \overline{K_{12}}K_{13} - K_{11}\overline{K_{14}} - \overline{K_{11}}K_{14})/2 \quad (F7)$$

$$g_{22} = -\Lambda_1(K_{11}\overline{K_{13}} + \overline{K_{11}}K_{13} + K_{12}\overline{K_{14}} + \overline{K_{12}}K_{14})/2 \quad (F8)$$

$$g_{23} = \Lambda_2(K_{11}\overline{K_{12}} + \overline{K_{11}}K_{12}) \quad (F9)$$

$$g_{24} = -\Lambda_2(K_{11}\overline{K_{11}} - K_{12}\overline{K_{12}}) \quad (F10)$$

In the above and following equations $\overline{K_{ij}}$ is the complex conjugate of K_{ij} .

The elements K_{4j} are obtained from

$$[AC] \begin{Bmatrix} K_{41} \\ K_{42} \\ K_{43} \\ K_{44} \end{Bmatrix} = \begin{Bmatrix} g_{31} \\ g_{32} \\ g_{33} \\ g_{34} \end{Bmatrix} \quad (F11)$$

where the elements g_{3j} are given by

$$g_{31} = -v_{33}q_{10}/2 \quad (F12)$$

$$g_{32} = v_{33}p_{10}/2 \quad (F13)$$

$$g_{33} = -v_{44}q_{20}/2 \quad (F14)$$

$$g_{34} = v_{44}p_{20}/2 \quad (F15)$$

The complex quantities b_i are given by

$$\begin{aligned} b_1 = & -K_{11}D_2A - \left\{ \Lambda_1[K_{13}(K_{42} + \overline{K_{42}}) + K_{12}(K_{43} + \overline{K_{43}}) - K_{11}(K_{44} + \overline{K_{44}}) \right. \\ & \left. - K_{14}(K_{41} + \overline{K_{41}})] + v_{33}K_{12} \right\} A - \left\{ \Lambda_1[\overline{K_{13}}K_{22} + K_{13}(K_{32} + \overline{K_{32}}) \right. \\ & + \overline{K_{12}}K_{23} + K_{12}(K_{33} + \overline{K_{33}}) - \overline{K_{11}}K_{24} - K_{11}(K_{34} + \overline{K_{34}}) - \overline{K_{14}}K_{21} \\ & \left. - K_{14}(K_{31} + \overline{K_{31}})] \right\} A^2\overline{A} \end{aligned} \quad (F16)$$

$$\begin{aligned} b_2 = & -K_{12}D_2A - \left\{ \Lambda_1[K_{11}(K_{43} + \overline{K_{43}}) + K_{13}(K_{41} + \overline{K_{41}}) + K_{12}(K_{44} + \overline{K_{44}}) \right. \\ & \left. + K_{14}(K_{42} + \overline{K_{42}})] - v_{33}K_{11} \right\} A - \left\{ \Lambda_1[\overline{K_{11}}K_{23} + K_{11}(K_{33} + \overline{K_{33}}) \right. \\ & + \overline{K_{13}}K_{21} + K_{13}(K_{31} + \overline{K_{31}}) + \overline{K_{12}}K_{24} + K_{12}(K_{34} + \overline{K_{34}}) + \overline{K_{14}}K_{22} \\ & \left. + K_{14}(K_{32} + \overline{K_{32}})] \right\} A^2\overline{A} \end{aligned} \quad (F17)$$

$$\begin{aligned} b_3 = & -K_{13}D_2A + \left\{ 2\Lambda_2[K_{11}(K_{42} + \overline{K_{42}}) + K_{12}(K_{41} + \overline{K_{41}})] - v_{44}K_{14} \right\} A \\ & + \left\{ 2\Lambda_2[\overline{K_{11}}K_{22} + K_{11}(K_{32} + \overline{K_{32}}) + \overline{K_{12}}K_{21} + K_{12}(K_{31} + \overline{K_{31}})] \right\} A^2\overline{A} \end{aligned} \quad (F18)$$

$$\begin{aligned} b_4 = & -K_{14}D_2A - \left\{ 2\Lambda_2[K_{11}(K_{41} + \overline{K_{41}}) - K_{12}(K_{42} + \overline{K_{42}})] - v_{44}K_{13} \right\} A \\ & - \left\{ 2\Lambda_2[\overline{K_{11}}K_{21} + K_{11}(K_{31} + \overline{K_{31}}) - \overline{K_{12}}K_{22} - K_{12}(K_{32} + \overline{K_{32}})] \right\} A^2\overline{A} \end{aligned} \quad (F19)$$

The quantities C_1 , C_2 , and C_3 are, respectively, the coefficients of D_2A , A , and $A^2\bar{A}$ in the following equation:

$$\bar{u}_1 b_1 + \bar{u}_2 b_2 + \bar{u}_3 b_3 + \bar{u}_4 b_4 = 0 \quad (F20)$$

where $\bar{u}_i$ is an element of the vector $\bar{u}$ which is defined in Chapter 6.

Vita

The author was born in Nellore, Andhra Pradesh, India on the fourth of February, 1963. He received a Bachelor of Technology degree in Naval Architecture from the Indian Institute of Technology, Madras, India in July, 1985. He had worked as a Naval Architect at Chowgule and Co., Goa, India from January 1985 to July 1985. In September 1985, he joined the master's program in the Aerospace and Ocean Engineering Department at Virginia Polytechnic Institute and State University (VPI&SU), Blacksburg, VA, U.S.A. and received a Master of Science degree in Aerospace Engineering in December, 1986. Later, he started on his Doctoral Studies in Engineering Mechanics in the Engineering Science and Mechanics Department at VPI&SU in January, 1987.

B. Balakumar
11/26/90