

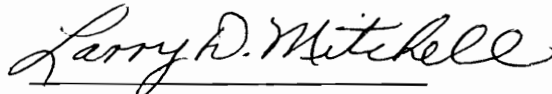
A Method of Determining Modal Residues Using an Improved Residual Model and Least Squares

by

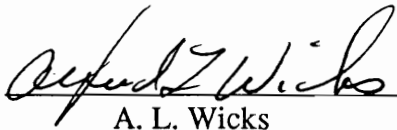
Kevin B. Kochersberger

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in partial fulfillment of the requirements for the degree of
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in
Mechanical Engineering

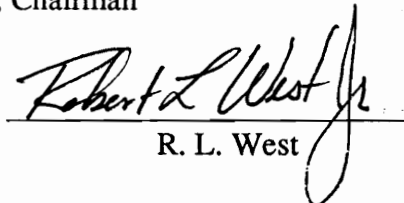
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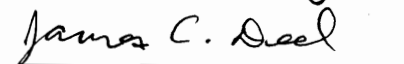
A. L. Wicks



R. L. West



H. H. Robertshaw



J. C. Deel

March 17, 1994
Blacksburg, Virginia

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by

Kevin Kochersberger

Committee Chairman: Larry D. Mitchell
Mechanical Engineering

(ABSTRACT)

A new approach to determining mode vectors is presented which uses pre-determined global parameters and an improved residual model to iteratively determine modal residues. The motivation for such a technique is to determine modal parameters rapidly so that, as data acquisition techniques become faster, more structural degrees of freedom can be measured without significantly slowing down the parameter estimation process.

The technique requires an accurate determination of the global parameters of natural frequency and damping by means of an FRF curve fit. More than one structural point is recommended to determine the global parameters since they will be used in determining the mode vectors. A structurally damped curve fitter which uses one or two FRF's is described and can be used for determining the global parameters. Examples of curve fitting simulated and measured data are presented and a comparison is made to a commercially available curve-fitter.

Once a frequency range-of-interest is selected, frequencies will be chosen at which the mobility is measured using sine excitation. The in-range modal response is represented by a matrix-vector product where the vector contains the residues for the modes of

interest. The out-of-range modal content is also represented by a matrix-vector product and forms the improved residual model. The residual content is removed from the measured mobility by an iterative technique which allows for an accurate determination of the residues of interest.

An evaluation of the technique is carried out by simulating a dynamic system including the shaker and power supply. The simulated system is closely modeled after a real system used to evaluate the technique on experimental data. Convergence rates are shown for cases of close modes, low amplitude modes and errors in the global parameters. The results of using the technique on experimental data shows that convergence typically occurs in under 15 iterations. Regenerating the FRF from the modal parameters shows close agreement to the original FRF and better agreement than the regeneration from modal parameters derived from a commercially available curve fitter.

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Nomenclature

Roman

A	Modal residue
a	Polynomial coefficient
B	Magnetic field
b	Polynomial coefficient
C	Viscous damping matrix
c	Shaker constant
D	Admittance matrix
E	Error
e	Residual coefficient
F	Force vector
f_p	Force delivered to structure
f_s	Force at shaker armature
$f_{1,2,3}$	Matrices containing weighting and polynomial functions
$g_{1,2}$	Intermediate matrix functions
H	Frequency response function
h_m	Frequency response function
h_n	High frequency residual numerator coefficient
h_d	High frequency residual denominator coefficient
i	Current
K	Impedance matrix
k	Stiffness
k_t	Stinger stiffness
k_s	Shaker stiffness
l_n	Low frequency residual numerator coefficient
l_d	Low frequency residual denominator coefficient
l	Wire length
M	Mass matrix
m	Mass
n	Noise strength
n_{dr}	Number of averages of random data

n_{ds}	Number of averages of sine data
P	Dynamic function
P_n	Numerator polynomial
P_d	Denominator polynomial
Q	Generalized force
R_s	Shaker electrical resistance
R_g	Voltage generator resistance
R^l	Low frequency residual
R^h	High frequency residual
R^{l*}	Low frequency residual matrix
R^{h*}	High frequency residual matrix
R_{dl}	Low frequency residual denominator
R_{dh}	High frequency residual denominator
S	Auto and Cross Spectra
s	Signal strength
s^2	Sample Variance
T	Chebyshev Polynomial
t	Time
u	Unscaled mode shape
v	Modal participation vector
v_g	Voltage generator signal
v_s	Shaker voltage
W_a	Weighting function
W_b	Weighting function
X	Displacement vector
x	Displacement
Y	Mobility

Greek

α_i	Trial force vector
ε	Error vector

ε'	Modified error vector
ϕ	Mass normalized mode vector
φ	Mode vector
γ_{xy}^2	Coherence function
η	Structural damping coefficient
λ	Eigenvalue
σ	Singular value
Σ	Singular value matrix
ω	Radian frequency
ω_n	Natural frequency
ξ	Viscous damping coefficient
χ^2	Probability function

Chapter 1

Introduction

1.1 Overview

Modern modal analysis has evolved from sine sweep/sine dwell techniques in the 1950's [1] to a mathematically rigorous technique of analyzing frequency response functions (FRF). The evolution has been hastened by the rapid development of fast digital signal processors and the theory of the fast Fourier transform (FFT). While frequency response function techniques work well for smaller structures, the energy required to excite large structures over a broad frequency range is excessive. This has resulted in the continued use of sine-sweep/sine-dwell methods with improved analysis techniques for large structures, such as aircraft, automobiles, and spaceborne equipment.

The experimentalist is interested in acquiring the modal parameters of natural frequency, damping, and mode shape from a structure. Obtaining this information over an entire structure can be very time consuming, particularly if the structure is large. The modal test is usually compromised because of the time required to map large numbers of spatial points, and the result is a coarse modal model with the possibility of spatial aliasing caused by the sparse spatial sampling. As techniques of finite element model correlation improve, there is a continuing need for higher spatial density of modal data. In addition, the successful measurement of higher order modes can only be realized if the

spatial density of data is great. Clearly, one of the important driving forces in modal analysis evolution has been the need for fast data acquisition so that higher density modal models can be realized.

The introduction of the scanning laser vibrometer (SLV) [2] has given a new perspective on data acquisition since a historical bottleneck in modal analysis is eliminated: the physical attachment of transducers to the structure. With the SLV, very high spatial density data can be obtained at high speeds with the added benefit of not mass loading the structure with conventional response transducers. The large amount of rapidly acquired data using the SLV has, thus, created a need for the quick determination of the modal parameters.

Modal analysis can be described as a three-step process. First, it is necessary to measure the time-history response of the structural point of interest when the structure is excited. Next, the time-domain data must be converted to the frequency domain. Finally, the frequency-domain data is analyzed to determine the modal parameters of natural frequency, damping, and residue. There are many ways to accomplish these tasks. A few will be discussed here.

Excitation of a structure can be accomplished with a shaker to apply either a random or sinusoidal force. If random excitation is used, the input force is a random signal with a frequency range determined by the dynamic characteristics of the structure. If sinusoidal excitation is used, the input force is a harmonic, single-frequency signal. Random has the advantage of covering the range of frequencies of interest with each sample, however the data is non-deterministic and processing techniques must be statistically based. Sinusoidal excitation is deterministic so the processing is easy, but with only one frequency input at a time, the analysis of a range of frequencies can be excessive.

Random excitation is typically used for structures which are small, or have low damping. These structures take less energy to excite, and so it is possible to use random excitation to provide excitation energy over the range of frequencies of interest. The input signal consists of a randomly varying amplitude and phase in the time domain, which ideally in the long-run average has a uniform distribution of energy over the input frequency range. After every sample of data, the cross spectra and autospectra of the input and output signals are typically determined by using the Fast Fourier Transform (FFT) technique. Most FFT techniques default to using 1024 time-domain data points at the structural point of interest. An estimator is then used which will determine the frequency content of the ratio of output to input, which is typically the quantity of interest. For 1024 time-domain data points, 400 non-aliased complex frequency-domain data points are obtained. Determining the modal parameters is then accomplished by curve fitting the frequency response function data, or FRF.

Sinusoidal excitation is preferred for structures which are large or highly damped where it is difficult to impart enough energy using random. Typically, sinusoidal excitation takes the form of resonance sine-dwell, where the frequency of excitation is the natural frequency of a single mode. Many techniques are available to find the natural frequency, which will be discussed in the next section. The forcing vector is usually determined on the assumption that there is no complexity in the mode shape. Complexity is defined as a response in modal coordinates which is not completely in-phase or out-of-phase at resonance. This assumption, although not valid for many structures, has not hurt the aircraft industry which uses the technique because an extensive amount of flight testing is performed to correct for deficiencies in the modal data [3].

1.2 Review of Dynamic Modeling

All physical structures with mass and stiffness will respond to a time-varying input forcing function in a dynamic manner. It is the challenge of the analyst to model the structural properties correctly so that when the response is measured, the modal parameters can be correctly extracted.

The basis for multi-degree-of-freedom behavior is found in the analysis of a single-degree-of-freedom system. The dynamic equation for the system shown in Figure 1.1 is as follows:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) \quad (1.1)$$

where m represents the mass, c represents the viscous damping factor, and k represents the stiffness. The displacement is indicated by x and the forcing function is f . Given that the input forcing function is harmonic and the response is assumed to be harmonic, then:

$$(-m\omega^2 + ic\omega + k)X = F \quad (1.2)$$

The terms X and F are measurable zero-peak quantities in a modal test and represent the amplitudes of oscillation and force, respectively. Typically, the analytical parameters are determined from a measurement of the ratio of $\frac{X}{F}$, $\frac{V}{F}$ or $\frac{A}{F}$, where V is the structure

velocity and A is the acceleration. Since a harmonic response has been assumed, displacement is obtained from velocity and acceleration by dividing by $i\omega$ and $(i\omega)^2$, respectively. Rewriting equation (1.2) so that the unknown terms are on the right-hand side of the equation:

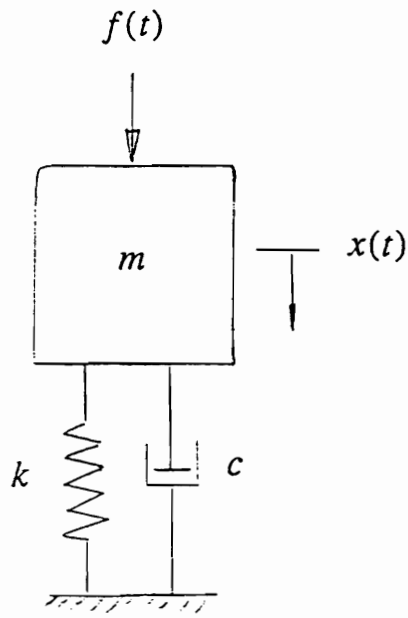


Figure 1.1 Forced Single-Degree-of-Freedom System

$$\frac{X}{F} = \frac{1}{m(-\omega^2 + \omega_n^2 + i2\zeta\omega\omega_n)} \quad (1.3)$$

where:

$$\omega_n = \sqrt{\frac{k}{m}}; \text{ natural frequency} \quad (1.4)$$

$$\zeta = \frac{c}{2\sqrt{mk}}; \text{ viscous damping ratio}$$

Equation (1.3) represents the dynamic compliance of the system. If the velocity were measured instead of displacement, the ratio of velocity to force would be defined by Y , or mobility:

$$Y = \frac{V}{F} = \frac{i\omega}{m(-\omega^2 + \omega_n^2 + i2\zeta\omega\omega_n)} \quad (1.5)$$

The above equations have assumed viscous damping represents the damping behavior of the structure. Another dynamic model considers complex stiffness instead of viscous damping, which results in hysteretic, or structural damping:

$$Y = \frac{i\omega}{m(-\omega^2 + \omega_n^2(1 + i\eta))} \quad (1.6)$$

The term η is the structural damping coefficient, or modal loss factor. It is noted that the imaginary term associated with damping in this case is frequency independent. Materials such as elastomers and polymers behave in this manner [4], and other materials will behave similarly over a limited frequency range.

The model used for modal correlation will depend on the composition of the structure. It may be desired to compare the two models to see which one more closely

resembles the measured behavior. In reality, damping and stiffness behavior may be a combination of the two models, or may follow the behavior of a third model. This is especially true if bolted joints are involved, where cubic stiffness is typical [5].

In multi-degree-of-freedom systems, the dynamic behavior in modal coordinates closely resembles the above single-degree-of-freedom analysis. Assuming for now that the system we are measuring has a finite number of degrees of freedom l , a matrix differential equation can be written. Defining a displacement response vector as $\{x\}$, and the force vector as $\{f\}$, the differential equation of motion becomes:

$$[M]\{\ddot{x}(t)\} + [C]\{\dot{x}(t)\} + [K]\{x(t)\} = \{f(t)\} \quad (1.7)$$

The matrices $[M]$, $[C]$ and $[K]$ represent the mass, damping and stiffness of the structure. These are discrete values which are intended to approximate the true values within the structure. If the damping matrix is non-zero and proportional to a linear combination of the stiffness and mass matrices, then the solution to this equation is:

$$\{x\} = \{X\}e^{i\omega t} \quad (1.8)$$

where:

$$X_i = \sum_{r=1}^l \frac{{}_r\varphi_i \quad {}_r\varphi_j F_j}{\hat{m}_r \left((i\omega)^2 + i2\zeta_r \omega \omega_r + \omega_r^2 \right)} \quad (1.9)$$

The eigenvector terms ${}_r\varphi_i$ and ${}_r\varphi_j$ refer to the mode amplitudes for mode r at structural points i and j . The modal mass, $\hat{m}_r$, represents the mode-shape weighted mass

of the structure. This solution applies to viscous damping, since damping is a function of frequency in the original differential equation. In the case where the damping is assumed to be structural, $[C]$ is zero and $[K]$ is complex. This leads to a different solution:

$$X_i = \sum_{r=1}^l \frac{{}_r\Phi_i {}_r\Phi_j F_j}{\hat{m}_r \left((i\omega)^2 + \omega_r^2 (1 + i\eta_r) \right)} \quad (1.10)$$

The dynamic compliance is then expressed as:

$$\frac{X_i}{F_j} = \sum_{r=1}^l \frac{{}_r\Phi_i {}_r\Phi_j}{\hat{m}_r \left((i\omega)^2 + \omega_r^2 (1 + i\eta_r) \right)} \quad (1.11)$$

The above equations describe the motion of a structure in terms of displacement. It is again possible to write the differential equation in terms of velocity:

$$\frac{V_i}{F_j} = Y_{ij} = i\omega \sum_{r=1}^l \frac{{}_r\Phi_i {}_r\Phi_j}{(i\omega)^2 + \omega_r^2 (1 + i\eta_r)} \quad (1.12)$$

In this case, the modal mass has been incorporated into the mode vectors so that new mode vector terms are created, ${}_r\phi_i$ and ${}_r\phi_j$.

Further reduction of the above equation is accomplished by substituting a single variable for the two mode vector terms:

$$Y_{ij} = i\omega \sum_{r=1}^l \frac{{}_rA_{ij}}{(i\omega)^2 + \omega_r^2 (1 + i\eta_r)} \quad (1.13)$$

The new term $, A_{ij}$ is called the residue, a name which is derived from the special case of a Laurent Series where the single-valued function $f(z)$ has only one term:

$$f(z) = \frac{a_{-1}}{(z - a)} \quad (1.14)$$

Equation (1.13) has the same form as equation (1.14), hence the name residue. The residue represents the mode vector for the structure at point i scaled by a complex factor due to the mode vector at j and the modal mass. Note that l modes have been included in the analytical developments above. In reality, a structure has many more modes than can be accounted for in the model, so a means of modeling the out-of-range modes must be used.

Ewins [6] presents a technique of modeling the out-of-range response using the behavior of a single-degree-of-freedom system. The residual model is created by assuming single-degree-of-freedom behavior for the modes above and below the range of interest. For the high frequency modes, the mobility is simplified to:

$$Y = -\frac{i}{m\omega} \quad (1.15)$$

Likewise, the modes below the range of interest are modeled by:

$$Y = \frac{i\omega}{k(1 + i\eta)} \quad (1.16)$$

Using these two models, the high and low out-of-range modes that are not modeled can be curve fit. Equation (1.15) is used to model modes below the analysis range and

equation (1.16) models the modes above the analysis range. Since there are more modes than just two which will be out-of-range, this technique is only approximate.

This "conventional" residual model is the most commonly used in parameter estimation; other models exist, however. Leuridan [7] presents a residual model which determines "pseudo poles" above and below the range of interest which more accurately describe the behavior of the out-of-range modes. These functions represent non-physical modes that serve only to fit the data within the measurement range. Because they are non-physical, care must be taken in their interpretation.

Chapter 2 presents yet another residual model developed by the author which is shown to model the out-of-range content better than the conventional model.

1.3 Modal Testing Using Sine-Dwell Excitation

The first step in a sine-dwell test is to locate the system natural frequencies. Crude techniques were used in the 1950's and continue to be popular in certain aircraft industries which ignore modal complexity and assumed single-degree-of-freedom behavior. More rigorous techniques are now available which use multiple shakers to vary the force vector and increase the available excitation energy to the structure. For structures such as aircraft symmetric and antisymmetric natural frequencies, the damping and mode shapes are obtained using multiple shakers. The ability to detect the presence of modes has also evolved from the use of orthogonal forcing vectors to excite a structure.

Historically, attempts at determining the mode shapes have been made by using force appropriation [8,9]. Force appropriation is the determination of a force vector (magnitude and phase) which has been found to isolate a single mode from all other modes. The skill required to tune modes has been supplanted since the 1950's with analytical techniques such as the Mode Indicator Function which give the exact force ratios needed for optimal single mode excitation [10 - 14]. However, the operator must still use several shakers to isolate modes which have the potential for becoming coupled and difficult to separate. This is a time consuming process.

Lewis and Wrisley [8] were the first to propose force appropriation as a method of exciting structures which would find the natural modes of vibration. The theory basically stated that if excitation forces exactly canceled the modal damping forces, then the natural mode shape would be described by the resulting oscillating displacement. The forcing vector was determined by equating the excitation power to the power absorbed by an assumed distribution of damping forces on the structure. The structure used to prove the concept was a system of nine lumped masses interconnected by stiffness elements and electrodynamic dampers located at the masses.

The result of this work was significant in that if a force vector could be determined that would excite a single mode, the forced oscillation would be the mode shape and the free decay of the system would determine the damping. The system was known in this work, so determining the force vector was easy. No attempt was made to determine the force vector from an unknown system, so the results presented were still incomplete concerning the analysis of an unknown system.

Asher [9] in 1958 approached the mode isolation problem from the standpoint of minimizing the out-of-phase response. Asher also defined force vectors that would excite only the normal modes of the structure, thereby allowing the measurement of the

normal modes. In addition, a technique for determining the force vector was presented which was experimentally based. He formulated the equations of motion in matrix notation, and defined the admittance matrix as the matrix inverse of the mass, damping, and stiffness influence coefficient matrix, or

$$[D] = \left([-\omega^2[M] + [K]] + i[C] \right)^{-1} \quad (1.17)$$

Next, the equations of motion can be rewritten in terms of admittance:

$$\left[[D_{real}] + i[D_{imag}] \right] \{F\} = \{X\} \quad (1.18)$$

where $\{F\}$ is the force vector applied to the structure and $\{X\}$ is the measured response vector. The response locations are the same as the force locations, so the $[D]$ matrix is square. Exciting the structure at a natural frequency will result in a response which is 90 degrees out of phase, or a null real response, or:

$$[D_{real}]\{F\} = \{0\} \quad (1.19)$$

In reality, this is only approximate because of unmodeled damping behavior, and the influence of other modes not accounted for. Asher solved equation (1.17) for the admittance functions by exciting the structure with several trial force vectors and measuring the response. The natural frequencies were determined by plotting the determinant of the real part of the admittance matrix and finding the minima. These would correspond to the frequencies where the excitation vectors show maximum

orthogonality. The force vectors at the natural frequencies were determined by solving equation (1.18) and the natural modes of vibration were subsequently measured.

Iterative improvement of the force vector was proposed by Morosow and Ayre [15] in 1978, and J. E. Anderson [16] continued work in this area. The first step in an iterative analysis is to experimentally determine the admittance matrix at resonance frequencies (assumed to be the natural frequencies) with unique force vectors. The imaginary part of the admittance matrix multiplied by an isolating force vector will result in the corresponding i^{th} mode shape, or:

$$[D_{imag}(\omega_{ni})]\{\tilde{\alpha}_i\} \approx \{\tilde{\phi}_i\} \quad (1.20)$$

Equation (1.19) is approximate because the forcing vector $\{\tilde{\alpha}_i\}$ has dimension equal to (number of shakers $\times$ 1) which is (of course) much less than the total number of degrees-of-freedom of the structure. Solving this for the approximate mode shape $\{\tilde{\phi}_i\}$ will be one step in the iteration process to obtain the best estimate for an isolating force vector.

The first-iteration force vectors are selected to be consistent with the modes being excited. The estimated mode shape matrix $[\tilde{\phi}]$ is determined using the trial force vectors at the resonance frequencies corresponding to the modes of interest. A refined force vector is then found by expressing the generalized force $\{\tilde{Q}_i\}$ as a matrix-vector product:

$$\{\tilde{Q}_i\} = [\tilde{\phi}]^T \{\alpha_i\} \quad (1.21)$$

An updated forcing vector for the i^{th} mode is obtained in equation (1.21) where $\{Q_i\}$ is a unit vector with one non-zero value at the i^{th} position:

$$\{\tilde{\alpha}_i\} = [\tilde{\phi}]^{-T} \{Q_i\} \quad (1.22)$$

This process is repeated until $\{\tilde{Q}_i\}$ computed in equation (1.20) becomes close by some criteria to a unit vector.

Kabe [17, 18] expanded on a mode isolation logic called selective orthogonal excitation (SOREX) which was introduced by Anderson [16]. This technique uses equation (1.21) to find updated excitation vectors similar to what Morosow and Ayre proposed. In this case, the initial force vector is obtained by exciting the structure with each shaker individually at the resonance peaks. The amplitudes at the driving points are used as the initial force vector. The frequency of excitation is adjusted to maximize the response and the mode shape is again measured to determine an updated force vector. The unique part of this test philosophy is that a random signal can be used to accomplish what had only been done with sine in the past, because the frequency response $\frac{A}{F}$ is measured .

Although some of the earliest work in force apportioning to excite normal modes was done in 1958 [8], it is this basis which has led to some of the most important developments in modal analysis. Ibanez [19] in 1976 extended Asher's work to account for more measurement locations than shakers by multiplying the admittance matrix by its transpose, forming a square matrix. The determinant of the square matrix is plotted as a function of frequency, indicating the existence of modes by a significant dip in the plot. Modes will occur where the singular values are minimum. As before, the admittance matrix can be experimentally determined by exciting the structure with varying forcing functions over the range of frequencies. Ibanez had introduced the concept which would become known as the Mode Indicator Function.

In 1984 [20] and 1985 [10], Vold showed that if an estimated mass matrix was introduced into Ibanez's eigenvalue formulation, then a Raleigh's Quotient could be formed. The mass-weighted 2-norm of the real response vector is expressed as:

$$\|X_r\|_2 = X_r^T M X_r \quad (1.23)$$

Minimizing this vector in relation to the total response vector results in a real normal mode of the structure. If the ratio of the real response to the total response is formed, then a Rayleigh's Quotient results:

$$\min \frac{\|X_r\|_2}{\|X_r + iX_i\|_2} = \lambda \quad (1.24)$$

which is equivalent to:

$$\min \frac{\|F^T (H_r^T M H_r) F\|_2}{\|F^T (H_r^T M H_r + H_i^T M H_i) F\|_2} = \lambda \quad (1.25)$$

The smallest eigenvalue λ and the corresponding eigenvector F are the solutions to equation (1.24). Equation (1.25) shows the form of the eigenfunction.

$$(H_r^T M H_r) F = (H_r^T M H_r + H_i^T M H_i) F \lambda \quad (1.26)$$

Plotting values of λ as a function of frequency will reveal the natural frequencies wherever the values are minimum. Repeated roots are detected if the next largest eigenvalue shows a minimum at the same frequency as the smallest eigenvalue. This plot

is referred to as the Mode Indicator Function, or MIF. The eigenvectors F associated with the eigenvalues represent the orthogonal forcing vectors needed to excite the modes.

Another technique developed by Shih et al. [11, 21, 22] allows complexity in the modes and similarly indicates the presence of modes with a plot of singular values as a function of frequency. The Complex Mode Indicator Function (CMIF) is a plot of the singular values of $[H(i\omega)]^H [H(i\omega)]$ for each spectral line. Maximum values in the plot indicate the existence of modes and the located frequencies give the corresponding damped natural frequencies. The decomposition of the normal matrix is expressed as:

$$[H(i\omega)]^H [H(i\omega)] = [V(i\omega)] [\Sigma^2(i\omega)] [V(i\omega)]^H \quad (1.27)$$

The eigenvectors $[V(i\omega)]^H$ correspond to the modal participation vectors and the unscaled mode shapes can be found from:

$$\{u(i\omega)\}_k = [H(i\omega)] \{v(i\omega)\}_k \sigma^2(i\omega)_k^{-1} \quad (1.28)$$

Here, $\sigma^2(i\omega)_k$ is the k^{th} singular value where the largest value corresponds to the dominant mode. If a repeated root exists, it will be indicated by the presence of the next largest singular value. An enhanced FRF is obtained by pre- and post-multiplying the FRF matrix $[H]$ by the vectors obtained from SVD at the peak CMIF values:

$$[H]_p = \{u(i\omega_p)\}^H [H] \{v(i\omega_p)\} \quad (1.29)$$

This isolates the p^{th} mode from the others and allows a simple SDOF curve-fitting technique to be used on the enhanced FRF. The use of CMIF was intended mainly to

reduce the parameter estimation workload for the Spatial Sine Testing (SST) [11 - 14] technique.

Spatial Sine Testing is a simplifying philosophy of obtaining modal parameters where large numbers of degrees of freedom are involved. Using sine excitation at arbitrary frequencies, the response data is obtained with a 4-point DFT and is used to build a FRF database. The CMIF technique is then used to find natural frequencies, damping and mode vectors without having to employ an estimator on the entire data set. Despite the analytical simplifications, the test setup could be extensive as several shakers (up to 32) and response transducers (up to 1024) were envisioned in the concept.

1.4 Modal Analysis Using Sine-Dwell Techniques and Pre-Determined Global Parameters

While resonance sine-dwell excitation has been an ongoing challenge for the aircraft industry as well as other industries that encounter large structures, smaller structures can be measured at every structural point of interest using random excitation and an estimation technique. Ewins [23] mentions several techniques including circle fitting and the complex exponential time-domain technique. Ibrahim [24] and Pappa [25] have also developed time-domain techniques, and Han [26] has shown that use of Forsythe Polynomials in the frequency domain will yield a good fit of the data.

In frequency-domain curve fitting, the FRF is curve-fit to a model which represents structural dynamic behavior. Typically, a ratio of polynomials is used as the model, and once a good curve fit is obtained the modal parameters are extracted by partial fraction expansion. At every structural point, modal parameters of natural

frequency, damping and residue are obtained. The natural frequency and damping for a structure are theoretically position independent, so the values obtained at the structural points should approach the same value. This being the case, it is possible to use values of natural frequency and damping from previously measured structural points to make the curve fitting for subsequent points easier [27]. It is still necessary to curve-fit all of the frequency-domain data points, which is typically 400 complex values.

Estimation techniques using sine-dwell are computationally more efficient than FRF curve-fitting because single-degree-of-freedom behavior is usually assumed. However, an assumption of no modal complexity is usually made in sine-dwell which is not made when the FRF is curve fit. Furthermore, the techniques of determining the natural frequencies using sine-dwell are flawed because of the "no complexity" assumption. If a curve fit technique could be employed to obtain the natural frequency and damping values for the structure and it was possible to measure complex mode shapes using sine-dwell, then the best of both techniques could be realized: The accuracy of FRF curve-fitting and the simplicity of sine-dwell.

Stahle [28] recognized a technique of utilizing the global properties of natural frequency and damping to find the residues at the structural points. He proposed first to find the natural frequencies by locating the quadrature peaks of response. Next, he estimated the damping, and then measured the total quadrature response at the quadrature peak frequencies. Since the response is the arithmetic sum of all the modal contributions, a matrix equation could be written to describe the total response. In the equations shown below, only the quadrature (imaginary) response is considered:

$$[R_Q] = \begin{bmatrix} 1 & C_{12} & \cdots & C_{1n} \\ C_{21} & 1 & & \vdots \\ \vdots & & 1 & \vdots \\ C_{m1} & & & 1 \end{bmatrix} [R_Q^*] \quad (1.30)$$

$$R_{Q_{ij}} = \frac{\ddot{X}_{ij}}{F} \quad (1.31)$$

$$C_{ij} = \frac{(i\omega_i)^2 (-i\eta_j \omega_j^2)}{\left((i\omega_i)^2 + \omega_j^2 (1 + i\eta_j) \right)^2} \quad (1.32)$$

$$R_{Q_{ji}}^* = \text{residue for mode } j, \text{ point } i \quad (1.33)$$

Solving for the residues, or complex scaled mode shapes, is a simple matter of inverting the C matrix and multiplying by the measured accelerance:

$$[R_Q^*] = [C]^{-1} [R_Q] \quad (1.34)$$

Since only the quadrature response was considered, the modes are not complex. Furthermore, the number of modes are restricted to equal the number of measurement locations because of the matrix inverse.

Alampalli [29] used a similar technique to find the modes of a concrete retaining wall. In this case, however, a curve-fitting algorithm found the natural frequencies and damping. The structure was excited at the resonance frequencies and the response was measured to calculate the individual modal amplitudes at several points.

Ewins [30] also proposed a method similar to Stahle, where the natural frequencies are assumed to be located at the quadrature peak responses. Ewins' model had no damping, making the formulation much easier. He did not restrict the frequency of dwell to be at the resonance and, in fact, specified that it would be off-resonance making an unbiased measurement of the response easier. This was the first attempt to obtain modal parameters from a forced response that was not at a resonance frequency.

Moser [31] presented a technique in 1993 that allowed complexity in the mode shape and, unlike Ewins, included damping of the model. Using a viscously damped model, the residues were obtained using a few measurements of acceleration along with the pre-determined natural frequencies and damping.

The optimal model for the data $\hat{h}_m$ is found by minimizing the error in the Chi-squared merit function for each residue A :

$$\chi^2(A_i) = \sum_{m=1}^M \left(\frac{\hat{h}_m - h_m}{\sigma_m} \right)^2 \quad (1.35)$$

Minimizing this function at each measurement frequency m corresponds to the maximum likelihood estimate, or the optimal solution of A [32]. The model for the data, h_m , is expressed as a matrix-vector product of the global parameters $[B]$ and the residues, $[A]$:

$$[B]\{A\} = \{\hat{H}\} \quad (1.36)$$

Weighting the row entries of $[B]$ and $\{\hat{H}\}$ by the standard deviations and solving for the residues is the maximum likelihood estimate. Although the theory was presented in this paper, no suggestion was made as to how the standard deviation of the measurements

would be obtained. Additionally, the technique has no means of accounting for the residuals, or modal content from the modes not modeled. Despite these shortcomings, good agreement between the measured mode shapes and the analytical mode shapes were obtained.

The recognition of natural frequency and damping as global modal parameters can save significant time in the analysis procedure. As presented, some work has been accomplished in this area. Since the global parameters are determined only once, a technique which will accurately determine these parameters is desired. Obtaining the residues at the structural points of interest is then accomplished by measuring the mobility at a few frequencies with sine excitation which provides a high signal to noise ratio. A significant time savings can be realized if the number of frequencies at which mobilities are obtained is small. This is the modal concept presented in Chapter 2.

Chapter 2

A Method for Parameter Estimation Using Pre-Determined Global Parameters

2.1 Overview

As was presented in Chapter 1, there have been papers published on the use of global modal parameters and discrete response measurements to facilitate the determination of mode vectors. Although some of the work presents good ideas, none of the literature fully addresses the complete process. It is the goal of this chapter to describe a process which can be used to obtain accurate residues for a set of modes which have been selected from the structure's modal content. This is usually the case in a modal test, where only the modes within a certain frequency range need to be considered.

As an additional benefit, the technique can amount to a significant time savings due to the reduced amount of frequency-domain data that needs to be acquired; a full 400 line FRF is no longer required at every point. Even though the test procedure uses sine excitation exclusively, none of the limitations of sine-dwell testing exist. The modes can be complex and use of a force apportioning technique is not necessary. A multi-sine signal can be used to further speed up the data acquisition process. If data is acquired rapidly, such as with a scanning laser vibrometer, then hardware and software can be developed to determine parameters while the laser is moving to a new structural point.

One of the main elements of the technique is the use of improved residual

models. High-frequency and low-frequency residual models account for modal behavior above and below the modes of interest and must be included when modeling the dynamic response of the structure. Although the theory of the new residual model is simple, the only other residual model in the literature is the one commonly used and presented by Ewins [33]. These two models will be compared in the next section.

2.2 An Improved Residual Model

The residuals are modeled by first assuming that the dynamic behavior occurring for the out-of-range modes also occurs for the in-range modes; equation (1.13) shows the assumed dynamic behavior. Given $(m-1)$ modes below the frequency range of interest, and from $n+1$ to p modes above, the measured mobility can be written as follows:

$$Y|_{\omega} = R^l|_{\omega} + i\omega \sum_{r=m}^n \frac{rA}{(i\omega)^2 + \omega_r^2(1 + i\eta_r)} + R^h|_{\omega} \quad (2.1)$$

where the low frequency residual is:

$$\begin{aligned} R^l|_{\omega} &= i\omega \sum_{r=1}^{m-1} \frac{rA}{(i\omega)^2 + \omega_r^2(1 + i\eta_r)} \\ &= \frac{(i\omega)(l_{n1} + l_{n2}(i\omega)^2 + \dots + l_{n(m-1)}(i\omega)^{2m-4})}{l_{d1} + l_{d2}(i\omega)^2 + \dots + l_{dm}(i\omega)^{2m-2}} \end{aligned} \quad (2.2)$$

and the high frequency residual is:

$$\begin{aligned}
 R^h \Big|_{\omega} &= i\omega \sum_{r=n+1}^p \frac{rA}{(i\omega)^2 + \omega_r^2(1+i\eta_r)} \\
 &= \frac{(i\omega)(h_{n1} + h_{n2}(i\omega)^2 + \dots + h_{n(p-n)}(i\omega)^{2(p-n)-2})}{h_{d1} + h_{d2}(i\omega)^2 + \dots + h_{d(p-n+1)}(i\omega)^{2(p-n)}}
 \end{aligned} \tag{2.3}$$

Writing the residuals as a ratio of polynomials allows for some simplification. As previously stated, the residuals account for the out-of-range modal behavior which appears in the range where the measurements occur. The poles of the residual terms lie outside of the range of interest, so simplification of the equations can occur without losing information at the measurements. Truncating the numerator and denominator polynomials of the low-frequency residual so that only the highest three terms remain will adequately represent the low-frequency data. Likewise, the lowest three terms of the high-frequency residual numerator and denominator are all that is needed to represent the high-frequency data. This is shown in equations (2.4) and (2.5).

$$R^l \Big|_{\omega} = \frac{l_{n1}(i\omega)^{-1} + l_{n2}(i\omega) + l_{n3}(i\omega)^3}{l_{d(m-2)} + l_{d(m-1)}(i\omega)^2 + l_{dm}(i\omega)^4} \tag{2.4}$$

$$R^h \Big|_{\omega} = \frac{h_{n1}(i\omega) + h_{n2}(i\omega)^3 + h_{n3}(i\omega)^5}{h_{d1} + h_{d2}(i\omega)^2 + h_{d3}(i\omega)^4} \tag{2.5}$$

Simplifying the residual equations by replacing the denominator function with one term leads to:

$$\begin{aligned}
R^l|_{\omega} &= \frac{l_{n1}(i\omega)^{-1} + l_{n2}(i\omega) + l_{n3}(i\omega)^3}{R_{dl}|_{\omega}} \\
R^h|_{\omega} &= \frac{h_{n1}(i\omega) + h_{n2}(i\omega)^3 + h_{n3}(i\omega)^5}{R_{dh}|_{\omega}}
\end{aligned} \tag{2.6}$$

In these equations, the denominator terms $R_{dl}|_{\omega}$ and $R_{dh}|_{\omega}$ are functions of the global parameters only and therefore will be invariant over the structure. They can be determined at one structural point if the measured FRF has modal content for all the dominant modes in the structure. This implies that the input and output locations are not at nodal positions, so pre-testing may be necessary to determine the optimal locations. The use of a spatially invariant denominator is part of the modal scheme that will be introduced in section 2.4.

A comparison of this residual model to the conventional model which is described in Chapter 1 is made by simulating five out-of-range modes below the range of interest and five modes above. There are no modes within the range to be curve fit, so the equation for the measured mobility is:

$$Y|_{\omega} = R^l|_{\omega} + R^h|_{\omega} \tag{2.7}$$

The conventional model reduces to:

$$Y|_{\omega} = \frac{ie_1}{\omega} + ie_2\omega \tag{2.8}$$

Table 2-1 shows the data used to generate the simulated out-of-range modal response and Figure 2.1 shows the simulated mobility FRF.

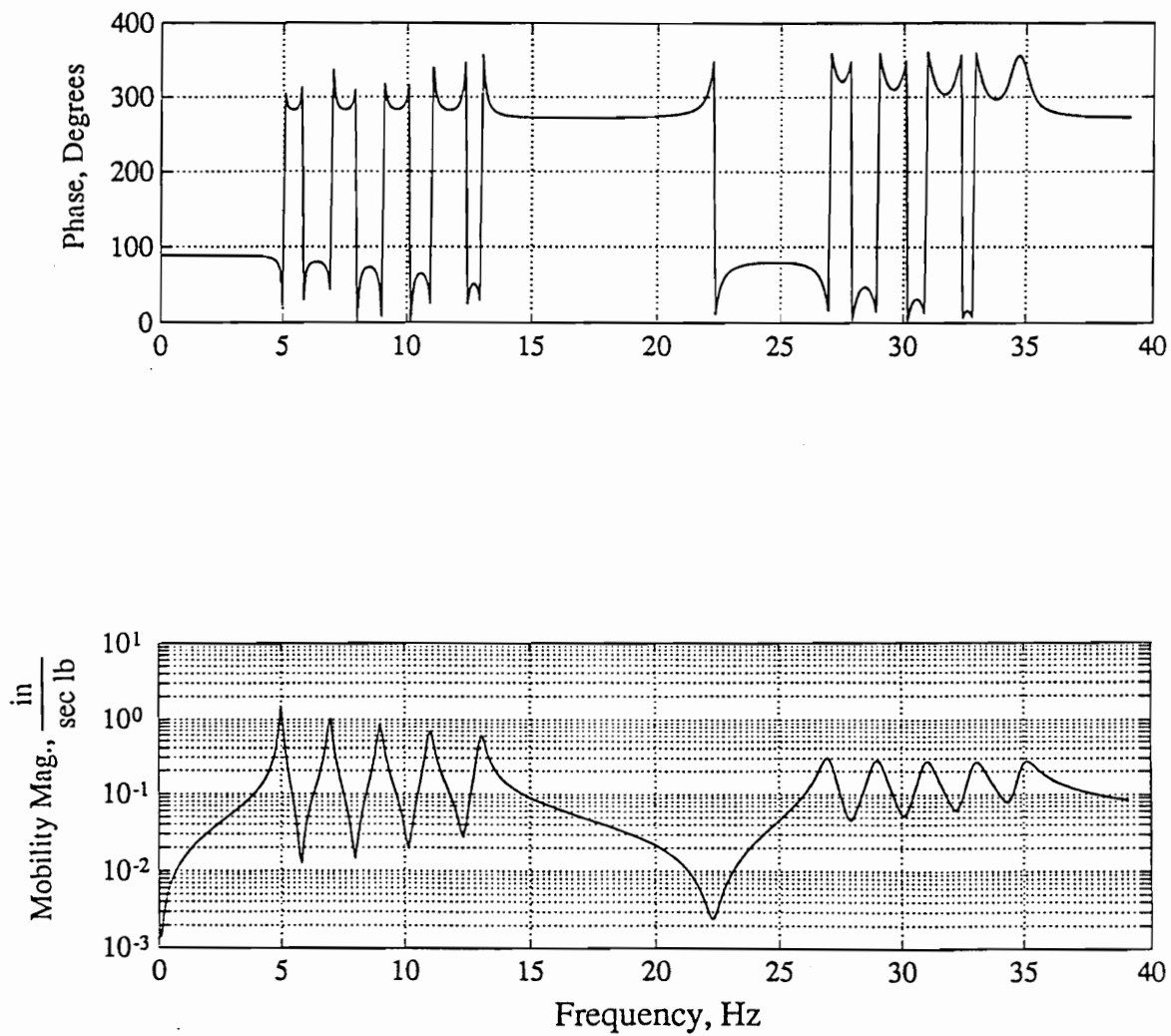


Figure 2.1 Simulated FRF for Residual Evaluation

Table 2-1 Data Used to Test the Residual Curve Fitter			
Natural Frequencies, Hz	Damping Coefficient, η	Residue	Measurement Frequencies, Hz
5	0.02	1.0	
7	0.02	1.0	
9	0.02	1.0	
11	0.02	1.0	
13	0.02	1.0	
			16.1
			17.6
			19.0
			20.5
			22.0
			23.4
			24.9
27	0.02	1.0	
29	0.02	1.0	
31	0.02	1.0	
33	0.02	1.0	
35	0.02	1.0	

In order to curve fit the FRF to the above models, seven mobilities were used in the range where the modes of interest would normally lie. This will over-determine the proposed residual model by one, since there are six unknowns. The conventional model will be over determined by a factor of five. The seven mobilities were chosen at equally spaced intervals, and are also shown in Table 2-1. For the proposed residual model, the matrix equation relating the measured mobilities to the unknown coefficients is:

$$\{Y\} = \begin{bmatrix} [R^{*}] & [R^{**}] \end{bmatrix} \begin{Bmatrix} \{l_n\} \\ \{h_n\} \end{Bmatrix} \quad (2.9)$$

where:

$$\begin{bmatrix} [R^{l*}] & [R^{h*}] \end{bmatrix} = \begin{bmatrix} \frac{(i\omega_1)^{-1}}{R_{dl}|_{\omega_1}} & \frac{(i\omega_1)}{R_{dl}|_{\omega_1}} & \frac{(i\omega_1)^3}{R_{dl}|_{\omega_1}} & \frac{(i\omega_1)^{-1}}{R_{dh}|_{\omega_1}} & \frac{(i\omega_1)}{R_{dh}|_{\omega_1}} & \frac{(i\omega_1)^3}{R_{dh}|_{\omega_1}} \\ \frac{(i\omega_2)^{-1}}{R_{dl}|_{\omega_2}} & \frac{(i\omega_2)}{R_{dl}|_{\omega_2}} & \frac{(i\omega_2)^3}{R_{dl}|_{\omega_2}} & \frac{(i\omega_2)^{-1}}{R_{dh}|_{\omega_2}} & \frac{(i\omega_2)}{R_{dh}|_{\omega_2}} & \frac{(i\omega_2)^3}{R_{dh}|_{\omega_2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(i\omega_p)^{-1}}{R_{dl}|_{\omega_p}} & \frac{(i\omega_p)}{R_{dl}|_{\omega_p}} & \frac{(i\omega_p)^3}{R_{dl}|_{\omega_p}} & \frac{(i\omega_p)^{-1}}{R_{dh}|_{\omega_p}} & \frac{(i\omega_p)}{R_{dh}|_{\omega_p}} & \frac{(i\omega_p)^3}{R_{dh}|_{\omega_p}} \end{bmatrix} \quad (2.10)$$

$$\begin{Bmatrix} \{l_n\} \\ \{h_n\} \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} l_{n1} \\ l_{n2} \\ l_{n3} \end{Bmatrix} \\ \begin{Bmatrix} h_{n1} \\ h_{n2} \\ h_{n3} \end{Bmatrix} \end{Bmatrix} \quad (2.11)$$

The number of measurements p is equal to seven in the above case. A similar matrix equation can be written using the conventional form of the residuals. To solve for the unknown coefficient vectors, a QR factorization technique called Modified Gram Schmidt is used [34].

The results of the curve fits are shown in Figures 2.2 and 2.3. The error in the new residual model is considerably less than in the traditional model, which is seen in Figure 2.4. The maximum error for the new model within the range of the measurements is 23% as opposed to 280% for the traditional model.

A more rigorous comparison of the models is obtained by using an F test to determine if the new residual model is better from the standpoint of error variance. The F test takes into consideration the degrees of freedom of the model which may show that

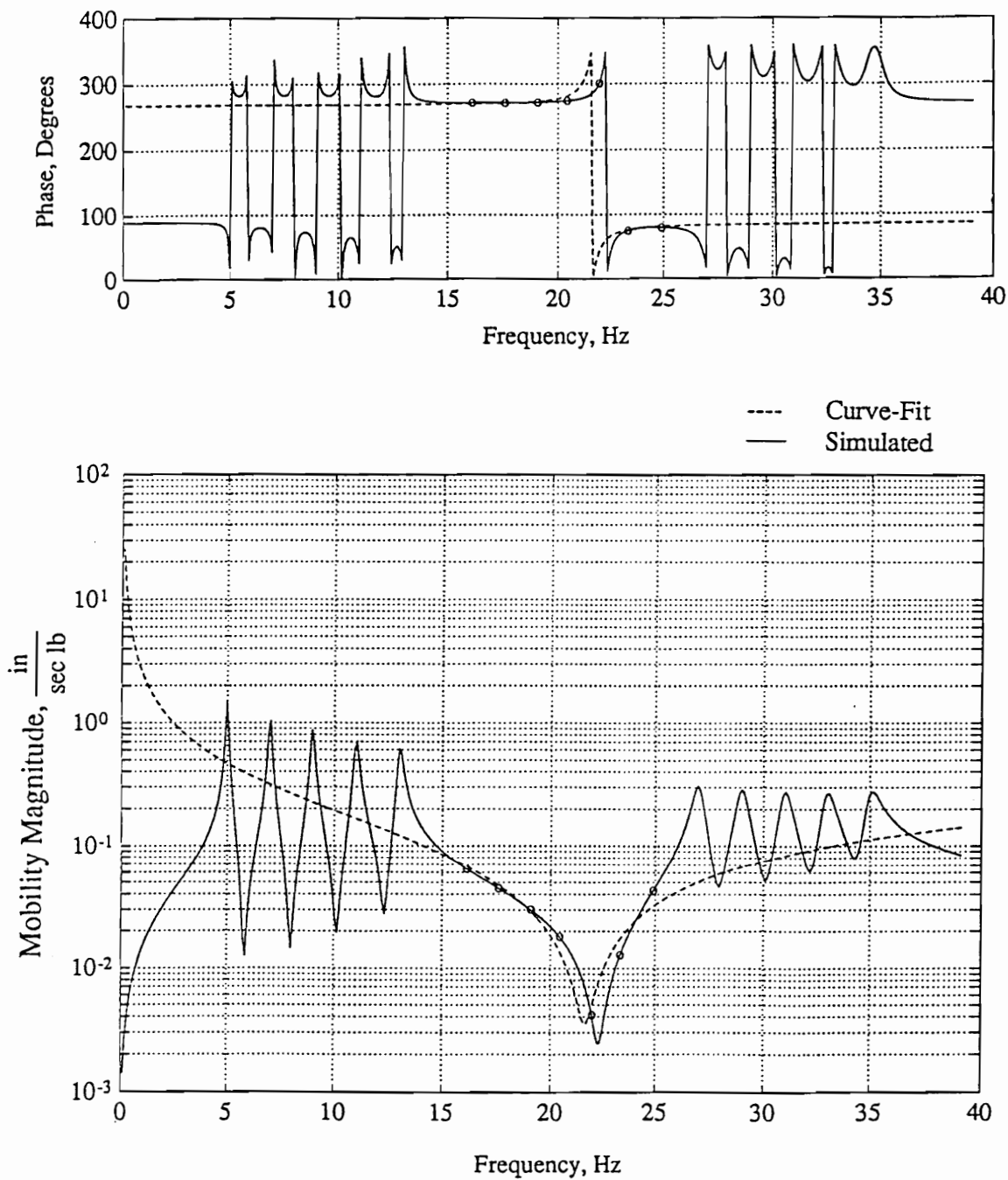


Figure 2.2 Simulated and Curve-Fit FRF's for Residual Evaluation - Ewin's Model

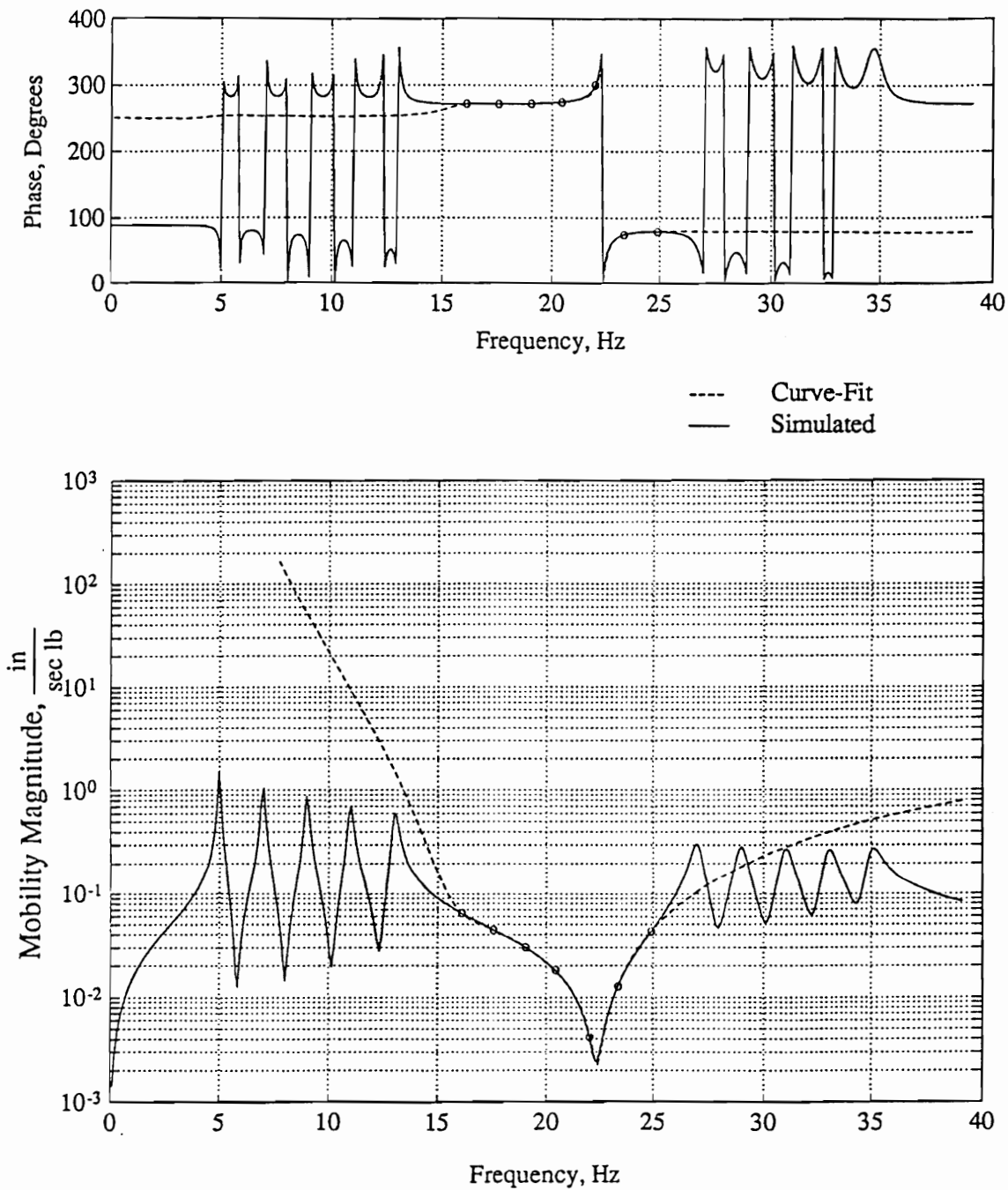


Figure 2.3 Simulated and Curve-Fit FRF's for Residual Evaluation - Improved Model

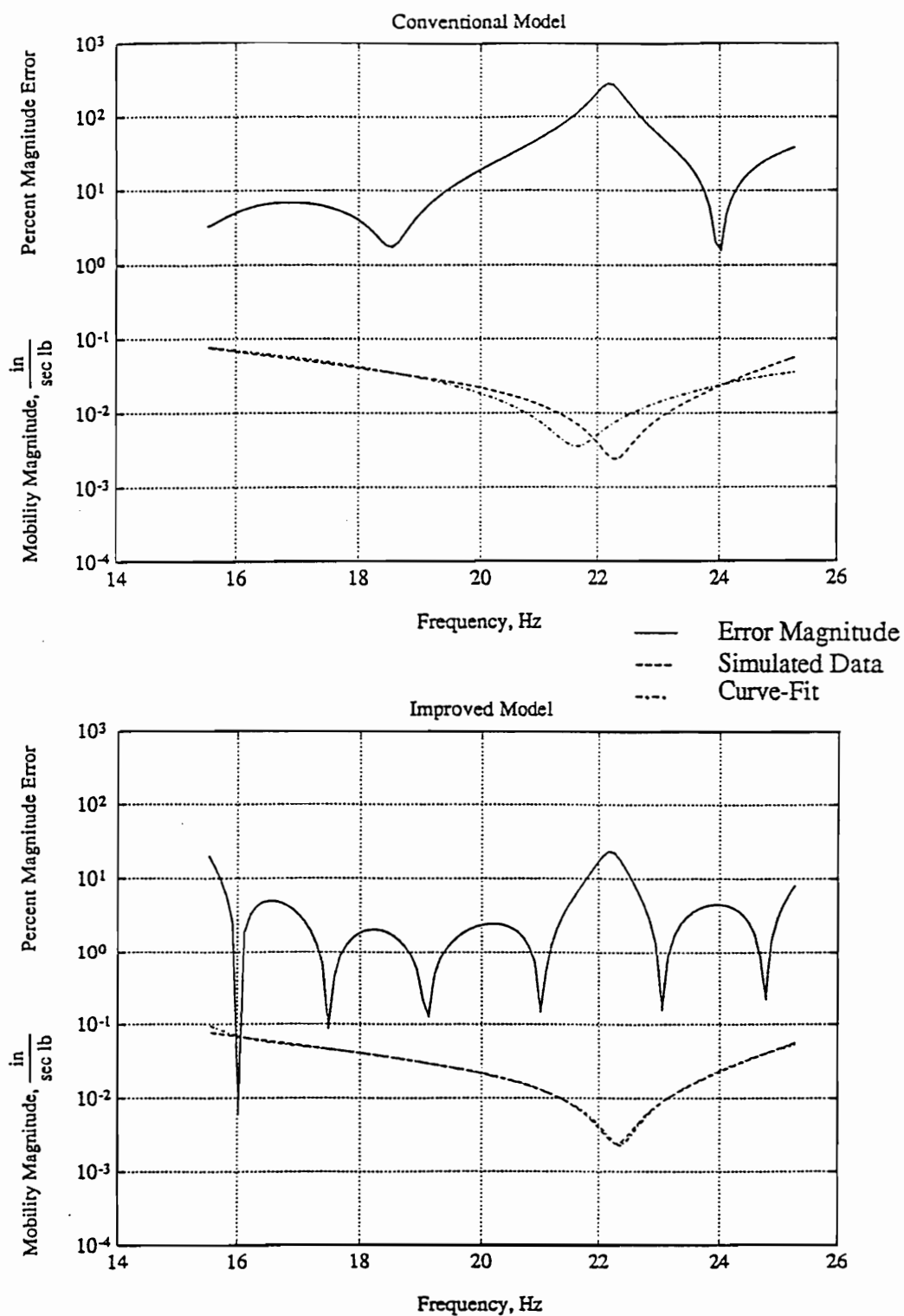


Figure 2.4 Error in Residual Models

added degrees of freedom do not result in a better model. Using eight equally spaced measurements between 15 Hz and 25 Hz, the sample variances for the conventional "c" and improved "i" models are:

$$\begin{aligned} s_c^2 &= 2.66e-5 \\ s_i^2 &= 2.52e-7 \end{aligned} \tag{2.12}$$

The F test requires the effective degrees of freedom for the two residual models. These are obtained by subtracting the degrees of freedom of the model from the number of measurements:

$$\begin{aligned} n_c &= 8 - 2 = 6 \\ n_i &= 8 - 6 = 2 \end{aligned} \tag{2.13}$$

The F statistic pertaining to the above degrees of freedom and a rejection criteria of 2.5% is 39.33. If the computed F statistic for the data is greater than 39.33, the new residual model is superior to the conventional model from the standpoint of error variance. The computational form for the F statistic is:

$$F = \frac{\frac{n_c s_c^2}{(n_c - 1)}}{\frac{n_i s_i^2}{(n_i - 1)}} \tag{2.14}$$

This results in a computed F value of 63.3. Since this is larger than 39.33, it is concluded that the new residual model is truly improved.

Further evaluation of the residual models is achieved by adding two modes to the range of interest for the system in Table 2-1; one at 19 Hz and one at 21 Hz. Nine measurements are used in both residual cases to determine the residues and then to regenerate the FRF. The original and regenerated FRF for the conventional model is shown in Figure 2.5; Figure 2.6 shows the percent magnitude error. The regenerated FRF for the improved residual model case is shown in Figure 2.7 and the error is shown in Figure 2.8. Similar to the cases with no modes in the measurement range, the regenerated FRF error for the improved model is less by about an order of magnitude compared to the conventional model. Chapter 3 will further evaluate the use of the improved residual model by examining the curve fit-derived residues.

2.3 A Maximum Likelihood Estimator of Residues

From Chapter 1, it was seen that the minimization of the Chi-squared function in equation (1.27) would give the maximum likelihood estimate of a set of parameters describing the measured behavior. If the global parameters that describe modal behavior are considered known, the residues and residuals are the only unknowns to be determined. The residues in equation (1.13) can be separated in a matrix-vector product to facilitate their solution:

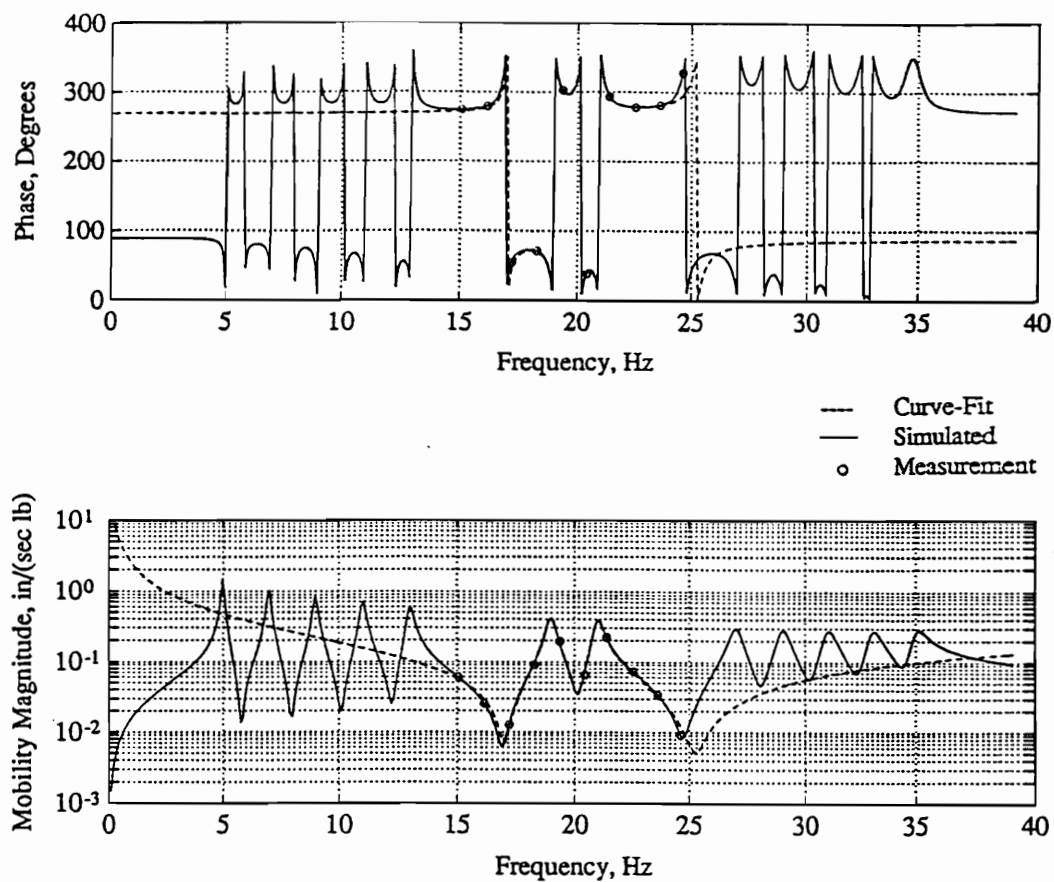


Figure 2.5 Simulated and Curve-Fit FRF's for Residual Evaluation - Conventional Model With Two In-Range Modes

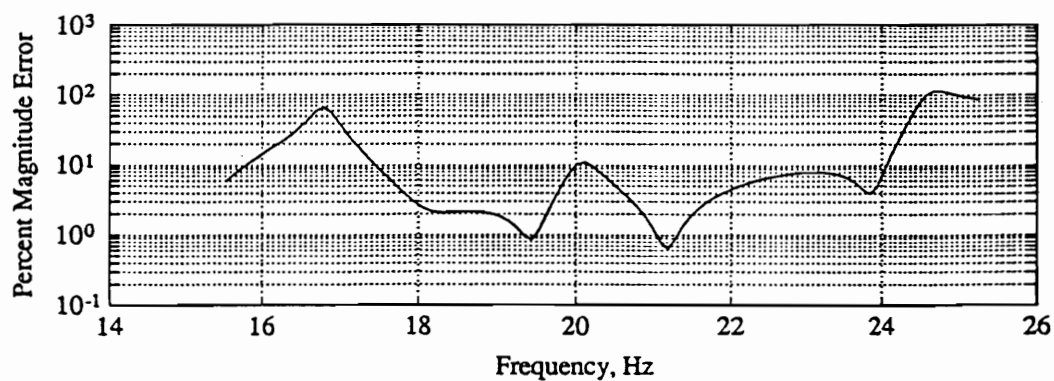


Figure 2.6 Magnitude Error for Conventional Model

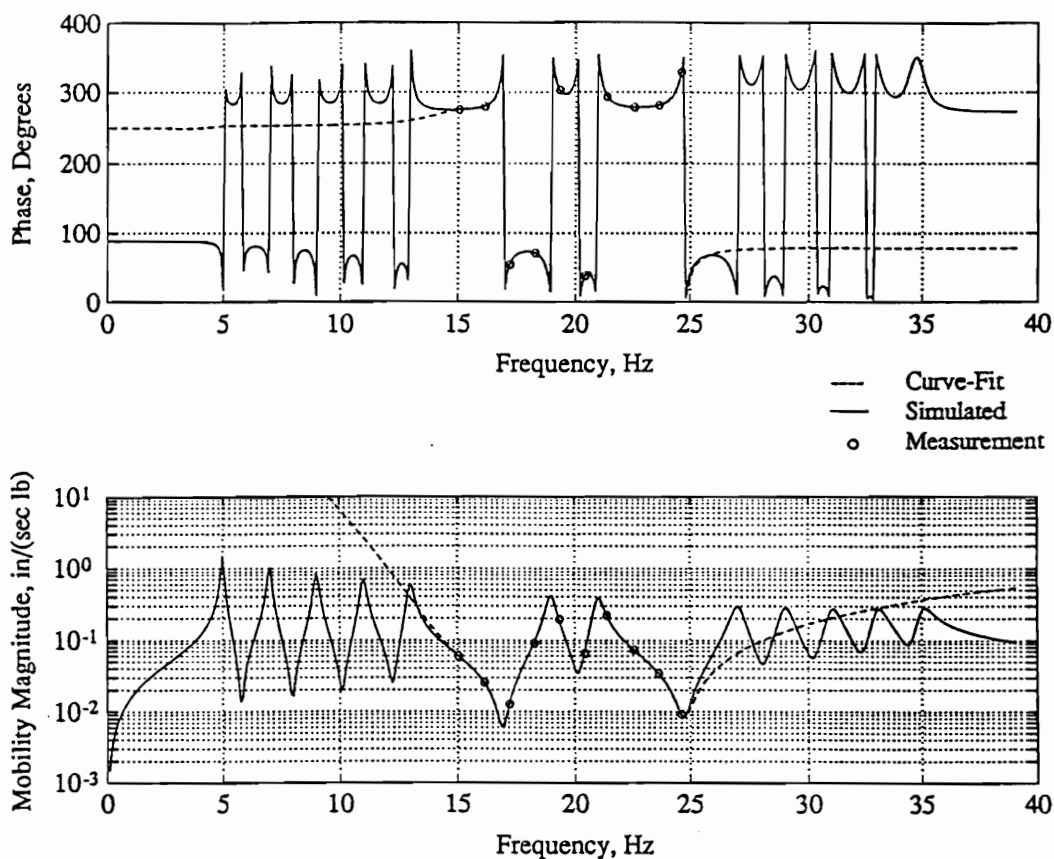


Figure 2.7 Simulated and Curve-Fit FRF's for Residual Evaluation - Improved Model With Two In-Range Modes

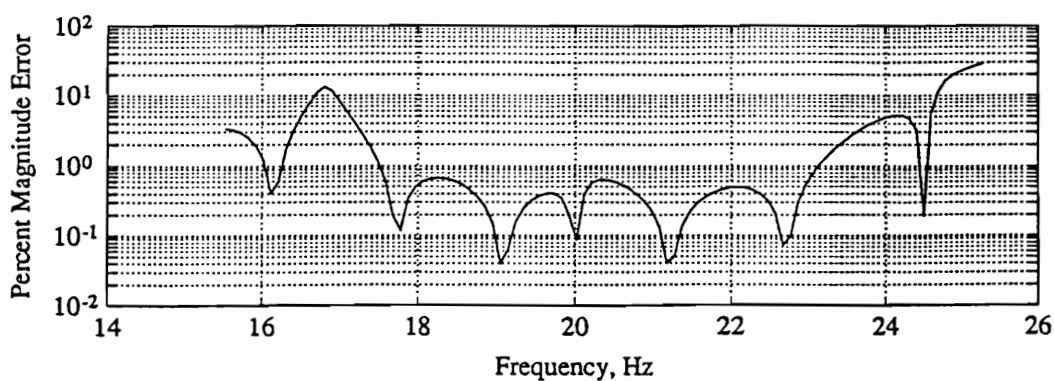


Figure 2.8 Magnitude Error for Improved Model

$$\begin{Bmatrix} Y_{\omega 1} \\ Y_{\omega 2} \\ \vdots \\ Y_{\omega p} \end{Bmatrix} = \begin{bmatrix} \frac{i\omega_1}{{}_1P_{\omega 1}} & \frac{i\omega_1}{{}_2P_{\omega 1}} & \dots & \frac{i\omega_1}{{}_lP_{\omega 1}} \\ \frac{i\omega_2}{{}_1P_{\omega 2}} & \frac{i\omega_2}{{}_2P_{\omega 2}} & \dots & \frac{i\omega_2}{{}_lP_{\omega 2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{i\omega_p}{{}_1P_{\omega p}} & \frac{i\omega_p}{{}_2P_{\omega p}} & \dots & \frac{i\omega_p}{{}_lP_{\omega p}} \end{bmatrix} \begin{Bmatrix} {}_1A \\ {}_2A \\ \vdots \\ {}_lA \end{Bmatrix} \quad (2.15)$$

where:

$${}_rP_{\omega_l} = (i\omega_l)^2 + \omega_r^2(1 + i\eta_r) \quad (2.16)$$

Once again, the maximum likelihood estimate of the residues is obtained by minimizing the Chi-squared equation below:

$$\chi^2 = \sum_{k=1}^p \left(\frac{Y_k - \begin{bmatrix} \frac{i\omega_k}{{}_1P_{\omega k}} & \dots & \frac{i\omega_k}{{}_lP_{\omega k}} \end{bmatrix} \begin{Bmatrix} {}_1A \\ \vdots \\ {}_lA \end{Bmatrix}}{\sigma_k} \right)^2 \quad (2.17)$$

Note that minimizing the Chi-squared function is the same as a least squares minimization problem with a σ weighting at each data point:

$$\min \left\| \frac{\{Y\} - \begin{bmatrix} i\omega \\ P \end{bmatrix} \{A\}}{\sigma} \right\|_2 \quad (2.18)$$

Given this, a maximum likelihood estimate of the residues can be obtained by solving the least squares problem in (2.18). A practical solution to this is not easy, however.

The only accurate determination of the variances at the measured k mobilities is obtained from the long run averaging of the data and the calculation of the sample variances. This would require extensive sampling and an unacceptable amount of time. It would be possible to obtain an analytical estimate of the variance from the system, however this would require the measurement of noise on the system, and monitoring of the input and output signal levels to determine signal/noise ratios. Most likely, the analytical technique would be inaccurate.

Because of problems in determining the sample variance it will be assumed that the variance is constant for all of the measurement points, which will eliminate the σ in equations (2.17) and (2.18). This returns the problem of determining the residues to a standard least squares problem shown in equation (2.19). The risk in doing this is minimized because, in determining the residues, sine excitation will be used which gives high signal/noise ratios of the mobility data.

$$\min \left\| \{Y\} - \begin{bmatrix} i\omega \\ P \end{bmatrix} \{A\} \right\|_2 \quad (2.19)$$

There are many techniques for solving this equation for the residues, A , such as SVD, normal equations and QR factorization. The technique chosen for this work is a QR factorization technique called Gram-Schmidt Orthogonalization [34]. This technique is about twice as efficient as Householder Orthogonalization, however it needs vectors which are fairly independent. It requires on the order of $2mn^2$ flop¹, where m and n are the rows and columns of the matrix in equation (2.19).

¹flop is the number of FLoating point OPerations to complete a calculation

2.4 Determination of Residues Using Direct and Iterative Techniques

The previous two sections have presented in-range and out-of-range modal models which separate the global and local parameters in a matrix-vector product. This form of the dynamic equations will be used to accurately determine in-range modal parameters rapidly. The in-range modal content is defined as the dynamic response due to modes within a frequency range where the modal parameters are desired. Modal parameters for the out-of-range modes are unimportant, however the behavior must still be modeled because it influences the measured dynamic response; the high- and low-frequency residual models account for this.

The logical way of solving for the residues of interest is to combine the in-range and out-of-range models and use least squares to determine the unknown parameters. Combining the previous equations results in the following overdetermined equation:

$$\{Y\} = \begin{bmatrix} [R^{*}] & [K] & [R^{**}] \end{bmatrix} \begin{Bmatrix} \{l_n\} \\ \{A\} \\ \{h_n\} \end{Bmatrix} \quad (2.20)$$

where:

$$[K] = \begin{bmatrix} \frac{i\omega_1}{1P_{\omega 1}} & \frac{i\omega_1}{2P_{\omega 1}} & \dots & \frac{i\omega_1}{lP_{\omega 1}} \\ \frac{i\omega_2}{1P_{\omega 2}} & \frac{i\omega_2}{2P_{\omega 2}} & \dots & \frac{i\omega_2}{lP_{\omega 2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{i\omega_p}{1P_{\omega p}} & \frac{i\omega_p}{2P_{\omega p}} & \dots & \frac{i\omega_p}{lP_{\omega p}} \end{bmatrix} \quad (2.21)$$

$$\{Y\} = \begin{Bmatrix} Y_{\omega 1} \\ Y_{\omega 2} \\ \vdots \\ Y_{\omega p} \end{Bmatrix} \quad (2.22)$$

This will yield an unbiased estimate of the residues A provided the following is true:

- a). The condition number of the $\begin{bmatrix} [R'^*] & [K] & [R^{**}] \end{bmatrix}$ matrix is low enough and the processor precision is high enough that errors propagated in the data will not significantly affect the residues.
- b). The data has a high enough signal/noise ratio so that the residual model does not inadvertently model the in-range content.

Item a). concerns the capability of the processor and the method of solution. The processor has a finite precision value μ which depends on the length of the binary representation of numeric data used in computing. Less expensive hardware will have less precision, and this must be considered when selecting a solution method. Normal equations is computationally efficient, but errors in the residues are on the order of $\mu \kappa_2 \left(\begin{bmatrix} [R'^*] & [K] & [R^{**}] \end{bmatrix} \right)^2$, where $\kappa_2(A)$ is the 2-norm condition number of a matrix A . The condition number squared can be very large which suggests that a processor with a small μ will be desirable if normal equations is used. If it is not possible to increase the processor precision (smaller μ), then using a QR factorization is recommended, although QR is computationally less efficient. Factorization in this manner results in errors on the order of $\mu \kappa_2 \left(\begin{bmatrix} [R'^*] & [K] & [R^{**}] \end{bmatrix} \right)$.

Item b) requires that the signal/noise ratio of the data be large enough to reduce residue error caused by the residual terms erroneously modeling the in-range modes. In

certain cases, noisy data will lead to a best fit which includes residual contributions that do not exist.

In order to reduce the likelihood of errors caused by large condition numbers and noisy data, an iterative scheme can be employed. In this case, equation (2.20) is rewritten so that the in-range modal content is considered separately from the out-of-range content:

$$\{Y\} = [K]\{A\} + \begin{bmatrix} [R'^*] & [R^{h*}] \end{bmatrix} \begin{Bmatrix} \{l_n\} \\ \{h_n\} \end{Bmatrix} \quad (2.23)$$

In this expression, the condition numbers of the two parts can be orders of magnitude less than when the terms are combined. This will reduce residue error caused by processor precision. Additionally, iterative estimates of the residues have less uncertainty than when the direct approach is used on mobility data with signal/noise ratios of less than 100. It may be difficult to excite larger structures with enough energy to achieve high signal/noise ratios, making the iterative technique attractive.

The first step of the iterative scheme is to assume the residuals are negligible and solve for the in-range residues:

$$\{Y\} = [K]\{A\} \quad (2.24)$$

The difference between the data and the curve-fit from equation (2.24) now becomes the data for solving the residual coefficients. The least squares equation for finding the residual coefficients is:

$$\{Y\} - [K]\{A\} = \begin{bmatrix} [R'^*] & [R^{h*}] \end{bmatrix} \begin{Bmatrix} \{l_n\} \\ \{h_n\} \end{Bmatrix} \quad (2.25)$$

Now that the residual content has been fit, a better model of the modal response due to the in-range modes can be obtained. The residual content is subtracted from the measured response, and again the residues for the in-range modes are solved for:

$$\{Y\} - \begin{bmatrix} [R'^*] & [R^{h*}] \end{bmatrix} \begin{Bmatrix} \{l_n\} \\ \{h_n\} \end{Bmatrix} = [K]\{A\} \quad (2.26)$$

By obtaining successively better estimates of the residuals using the above technique, the residues in the range-of-interest will converge to a solution. If the number of discrete frequencies p used to excite the structure is increased, the convergence will be faster. This will result in longer test time, however, so it is desirable to keep the number of discrete frequencies to a minimum to minimize lab testing time. The minimum number of mobility measurements is equal to the number of unknown residual coefficients (six), plus the number of modes that are in the range, plus one. This over determines the system by one. It has been found that one or two additional measurements beyond the minimum are usually adequate when using either the direct technique or the iterative technique.

Both the direct and iterative approaches require the system global parameters to solve for the unknown coefficients: the residues and the residual numerator coefficients. Therefore, a good estimate of the natural frequencies and damping coefficients is necessary. Determining these is typically accomplished with a curve-fitting algorithm, and previously mentioned algorithms [23 - 26] will accomplish this. To improve the global parameter estimate, more than one response point FRF can be measured to lessen

the possibility of missing a mode due to the measurement of a nodal point. In addition, more than one force location can be used for the same reason, however care must be taken not to alter the system by moving force transducers on the structure.

Chapter 4 will present a curve fitter based on a structurally damped model that has the capability to simultaneously fit two FRF's. Use of this fitter has shown to accurately determine global parameters where structural damping is a good assumption.

The FRF's which will be curve fit must be accurate, and so step-sine excitation is recommended over the frequency range encompassing the modes to be mapped. Since a residual model is being used, the FRF should extend beyond the range of mapped modes to determine the residual denominator terms. A minimum of three poles above and below the range of interest are needed to determine the residuals based on the derivation in section 2.2. If there are fewer than three low-frequency poles, the residual equation must be further truncated to account for this. When necessary, a low-frequency rigid-body mode and a high-frequency pseudo-mode or residual should be included. A pseudo-mode is a mode determined by a curve fitter which lies outside of the frequency range of the measured data; it can be interpreted as another form of a residual.

Step-sine excitation is superior to random or swept-sine excitation, but requires more time than the other techniques so it usually is not employed. Since the whole FRF is measured only once, the extra time is not considered a major factor against the improved quality. Using step-sine excitation will give the best quality data over the frequency range and thus allow the most accurate determination of global parameters.

The frequencies at which the mobilities are measured are selected based on the behavior of the FRF; this will be discussed in the next chapter. In any case, the measurement range should exceed the range of mapped modes by $\pm 10 - 20\%$, depending on the damping. Lightly damped structures ($<1.0\%$) require less overlap of the

measurement range than what is required for moderately damped structures. Higher damping requires a wider measurement range to capture the available dynamic range of the FRF.

Modes near zero-frequency require measurements that extend as low in frequency as possible. The low-frequency measurement limit is typically driven by the dynamics of the shaker and the power supply characteristics. Large shaker displacements are necessary at low frequencies to reach the force and acceleration levels necessary for measurement. This can cause harmonics of the fundamental excitation frequency which are larger than the fundamental. In addition to this, a voltage-controlled power supply will not be able to excite a structure at the low frequencies attainable with a current-controlled power supply because the shaker force is a function of current.

2.5 Mobility Measurement Techniques

Measuring the mobilities required for the iterative technique can be accomplished in a number of ways. Since sine excitation will be used, the data processing is much simpler than if random excitation was used. High signal-to-noise ratios of the data can eliminate the need for averaging, reducing the data acquisition time. Using a deterministic signal opens the possibility of inputting all the frequency components of excitation simultaneously and extracting the respective mobility components with a DFT, FFT, or least squares technique. In the case of the DFT or FFT technique, the frequencies can be cell-centered which eliminates the need for windowing; using least squares puts no restriction on the frequencies. The relatively small number of mobility

measurements will greatly speed up the overall parameter estimation process, making real-time mode vector determination a possibility.

2.5.1 Discrete Fourier Transform Techniques

The Discrete Fourier Transform, or DFT, is a discretized version of the Fourier Transform which is used to find harmonic components of time-domain data. The Fourier Transform is an integral equation in the time domain which brings the data into the frequency domain:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{i\omega t} dt \quad (2.27)$$

The discrete version is:

$$X(\omega) = \sum_{k=1}^p [x(t_k) \cos(\omega_o t_k) + x(t_k) \sin(\omega_o t_k)] \quad (2.28)$$

where:

$$\begin{aligned} X(\omega) &= \text{spectral component at } \omega \\ \omega_o &= \frac{2\pi}{T} \\ T &= \text{period of signal} \end{aligned}$$

If the sampling time t is equal to $\frac{T}{4}$, then the sine and cosine terms become either 1 or 0.

This simplified version of the DFT is referred to as a four-point DFT since the number of samples per period is equal to four. The spectral components become:

$$\begin{aligned} X(\omega)_{imag} &= x(t_2) - x(t_4) \\ X(\omega)_{real} &= x(t_1) - x(t_3) \end{aligned} \quad (2.29)$$

By increasing the sampling rate to a multiple of the fundamental, more spectral components can be determined in a similar fashion. A vector of frequency-domain terms can be created from a binary matrix multiplied by a vector of time-domain points if multi-sine excitation is used. The restriction of frequencies can be a problem, however, and it is this reason that makes a least squares fit attractive.

2.5.2 Least Squares Fitting of Time-Domain Data

Least squares solutions have been used for fitting data to a variety of models. Yet another least squares model is obtained by using sines and cosines as the functions to be fit. The real and imaginary components of mobility (or any other transfer function) can be found by relating the input and output harmonic coefficients obtained in the least squares formulation. Unlike the DFT technique, there is no restriction on periodicity of the data or sampling interval. More calculations are necessary, but they are only done once based on the excitation frequencies.

A matrix equation can be written describing measured data in terms of a single harmonic content:

$$\begin{Bmatrix} V_1 \\ V_2 \\ \vdots \\ V_p \end{Bmatrix} = \begin{bmatrix} \cos \omega t_1 & \sin \omega t_1 & 1 \\ \cos \omega t_2 & \sin \omega t_2 & 1 \\ \vdots & \vdots & \vdots \\ \cos \omega t_p & \sin \omega t_p & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad (2.30)$$

where V_i is the velocity at time increment i . The measured velocity at p time-domain points will be used to determine the a coefficients. A normal equations solution can be used to accomplish this. If the input forcing function is harmonic at frequency ω , then the sine, cosine, and D.C. terms above will fully describe the data in the frequency domain. Using multi-sine results in a "wider" matrix above with additional sine and cosine terms at other frequencies. Like all least squares solutions, the minimum amount of data needed is at least greater than the number of coefficients to find. In this case, two times the number of frequencies that are input plus two (one for D.C. and one for over determination) will be the minimum amount of data required.

The same form of equation is used on the input force so that two sets of harmonic coefficients are obtained. Similar to what would be accomplished in the DFT case, the frequency-domain terms on the input and output are used to find the mobility magnitude and phase:

$$0 - \text{peak mag. velocity} = \sqrt{S_w} = \sqrt{a_1^2 + a_2^2} \quad (2.31)$$

$$\text{mobility magnitude} = \frac{\sqrt{S_v}}{\sqrt{S_f}}$$

$$\text{mobility phase} = \tan^{-1} \left(\frac{\text{imag}(S_v)}{\text{real}(S_v)} \right) \quad (2.32)$$

Since one average of data is taken, the calculation of the mobility magnitude and phase will not be different when using different estimators. If it is necessary to average the data, then an appropriate estimator should be selected to get an unbiased estimate of the magnitude and phase.

Reiterating, periodicity of data is not required and equal sampling interval is not critical using the least squares technique. If bad data is sampled and there is a means of detecting it, rejection can occur and the remaining data can be used for the fit without difficulty; DFT analysis cannot do this. Kochersberger [35] describes a method of detecting outliers in the data and using an efficient scheme for determining frequency-response data.

2.5.3 Fast Fourier Transform Techniques

Use of the FFT is almost universal in random data processing. It accomplishes what the DFT accomplishes with many fewer flop; the number of calculations is reduced from N^2 to $N \log_2 N$, where N is the number of time-domain data points. For determining harmonic components over a broad frequency range it is clearly the preferred choice. The restriction on the FFT technique lies in the number of time-domain data points; they must be a multiple of 2^n where n is an integer. The other consideration is that the sampling rate required for the FFT must be at least twice the analysis frequency range. There will undoubtedly be harmonic components provided by the FFT which are not needed in least squares because only a few harmonics are needed. This will likely make the FFT technique the most costly in time among the three options presented here.

Chapter 3

Practical Considerations for Residue Determination

3.1 Determining Residues for Simulated Cases

In order to evaluate direct and iterative least squares for finding residues, typical situations encountered in modal analysis are simulated and the results presented. Although it is impossible to analyze every situation that will be encountered, general trends can be estimated by studying a few cases. A simulation based on the modal characteristics of a wind tunnel model is used to show a case where the iterative technique is preferred over the direct technique. Several other cases study convergence rates and final errors encountered when using the iterative technique. The iterative study examines the following:

1. The effects of variations in the global parameters.
2. The effects of measurement location and the number of measurements on convergence speed and residue error.
4. Considerations for modes which are closely spaced.
5. Considerations for modes which are much smaller in amplitude than adjacent modes.
6. The effects of system damping on residue error and convergence speed.

All of the simulations include a shaker dynamic model to represent the force drop-off typical in a real system.

3.2 A Realistic Shaker Model

The block diagram in Figure 3.1 shows the overall model for the electro-dynamic system. The free-body diagram in Figure 3.2 will be used to determine the dynamic interaction of the shaker with the structure. Rao [37], Olson [38] and Tomlinson [39] have investigated shaker force behavior for single- and two-degree-of-freedom systems, but have not extended this to multi-degree-of-freedom systems. As Figure 3.2 shows, there is no assumption made about the number of degrees of freedom. In order to determine the force applied to the shaker armature, a Kirchoff Loop Equation for the electrical circuit must be written. Referring to Figure 3.3, the current around the loop will be:

$$i = \frac{v_g - v_s}{R_s + R_g} \quad (3.1)$$

The voltage v_s is the shaker back-EMF caused by the shaker acting as a generator as it moves. This can be modeled as a function of the armature velocity, $\dot{x}_s$, the magnetic field strength, B , and the length of the wire in the coil, l . The force applied to the armature, f_s , will be the EMF generated by the resulting current in the magnetic field:

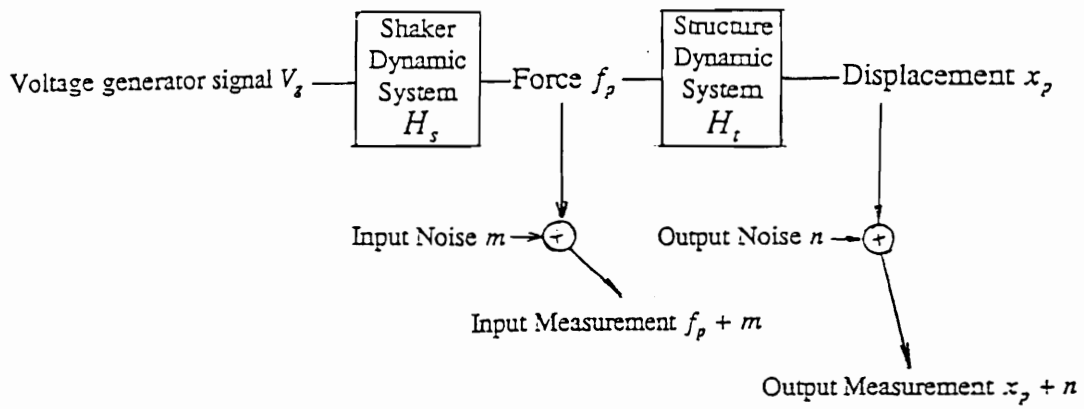


Figure 3.1 Block Diagram of Electro-Dynamic System

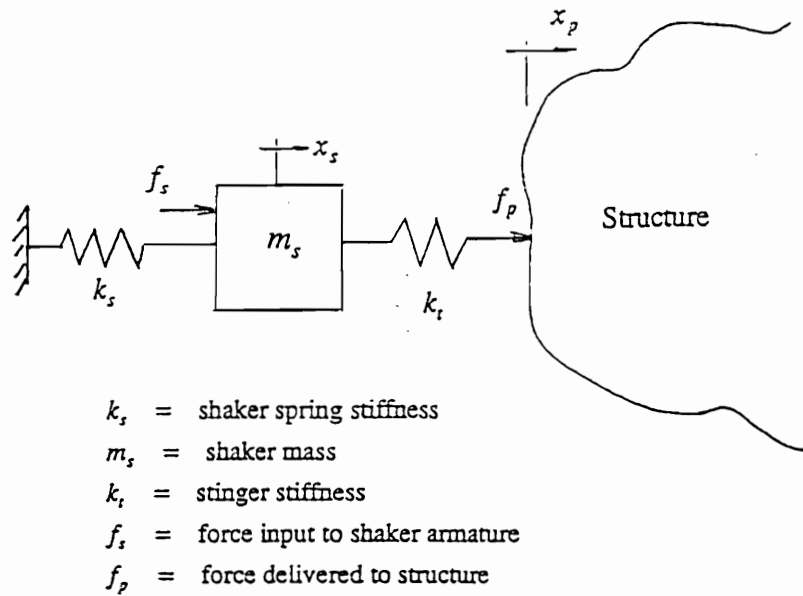
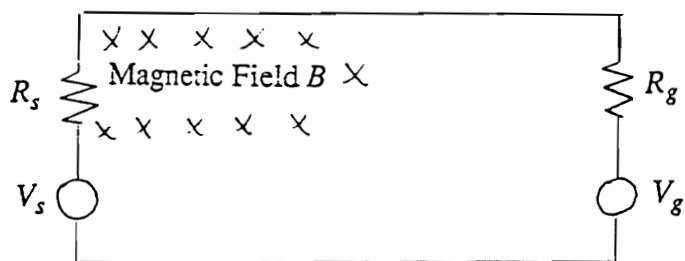


Figure 3.2 Free-Body Diagram of Shaker-Structure System



- V_s = Shaker EMF
- R_s = Shaker Resistance
- V_g = Signal Generator Resistance
- V_s = Shaker Resistance

Figure 3.3 Shaker Electrical Schematic

$$f_s = ilB = lB \left[\frac{v_g - Bl\dot{x}_s}{R_s + R_g} \right] \quad (3.2)$$

As was described by Tomlinson [39], the field strength B will vary in a quadratic manner through the stroke of the shaker due to the non-linear nature of the magnetic field inside the armature. This results in the following equation for the armature force:

$$f_s = c(1 - \alpha x^2) \left[\frac{v_g - c(1 - \alpha x^2)\dot{x}_s}{R_s + R_g} \right] \quad (3.3)$$

The force delivered to the structure is simply:

$$f_p = k_t(x_s - x_p) \quad (3.4)$$

where x_p is the displacement of the structure. To determine the force applied to the structure as a function of the voltage generator signal, the shaker displacement must first be found. The force equations above and the following dynamic equation for the structure are used:

$$\frac{x_p}{f_p} = \sum_{r=1}^n \frac{A_r}{(i\omega)^2 + \omega_r^2(1 + i\eta_r)} \quad (3.5)$$

The resulting FRF for shaker displacement and input voltage is:

$$\frac{X_s}{V_g} = \frac{\frac{c(1 - \alpha X_s^2 e^{i2\omega t})}{R_s + R_g}}{-m_s \omega^2 + \frac{i\omega c^2 (1 - \alpha X_s^2 e^{i2\omega t})^2}{R_s + R_g} + k_t \left[1 - k_t \left(k_t + \left(\sum_{r=1}^n \frac{A_r}{(i\omega)^2 + \omega_r^2 (1 + i\eta_r)} \right)^{-1} \right)^{-1} \right] + k_s} \quad (3.6)$$

The capitalized variables X_s and V_g refer to the magnitudes where harmonic motion has been assumed:

$$\begin{aligned} x_s &= X_s e^{i\omega t} \\ v_g &= V_g e^{i\omega t} \end{aligned} \quad (3.7)$$

Unlike other familiar FRFs, equation (3.6) has harmonic components that are present in the magnitude ratio. This is the source of harmonic distortion in the input force from a shaker. Note that when the frequency is at a structural resonance, equation (3.6) shows an increased presence of the second and fourth harmonics in the amplitude, causing harmonic distortion. Typically these contributions are much smaller than the fundamental, and so it is acceptable to ignore them. Rewriting equation (3.6) where only the fundamental is considered:

$$\frac{X_s}{V_g} = \frac{\frac{c}{R_s + R_g}}{-m_s \omega^2 + \frac{i\omega c^2}{R_s + R_g} + k_t \left[1 - k_t \left(k_t + \left(\sum_{r=1}^n \frac{A_r}{(i\omega)^2 + \omega_r^2 (1 + i\eta_r)} \right)^{-1} \right)^{-1} \right] + k_s} \quad (3.8)$$

The structure force/input voltage FRF is then obtained by using equation (3.4):

$$H_s = \frac{F_p}{V_s} = \frac{k_t \left(\frac{c}{R_s + R_g} \right)}{\left(-m_s \omega^2 + \frac{i\omega c^2}{R_s + R_g} + k_s \right) \left(k_t \sum_{r=1}^n \frac{A_r}{(i\omega)^2 + \omega_r^2 (1 + i\eta_r)} + 1 \right) + k_t} \quad (3.9)$$

This equation results in the glitch that is typically observed in the force in a dynamic test. The term "force glitch" was first used by Rao [36] and describes the rapid rise and fall of the force level autospectra at a structural resonance. If it is desired to simulate the force levels occurring for a particular shaker-structure system, then the parameters in equation (3.9) must be found. The structure parameters can be found from an estimation technique, and most of the other parameters are measurable or known quantities. The shaker constant c can be experimentally determined by measuring the force input to the real system and correlating with the simulated force.

3.3 Direct vs. Iterative Determination of Residues

The direct least squares solution is the preferred method of finding residues from measured data, provided the signal/noise ratio is high enough. There are cases however when the iterative technique yields better results. Generally, structures with low natural frequencies and signal/noise ratios of less than 100 are candidates for the iterative technique. One structure that showed this quality was a low speed wind tunnel model of a Cessna Model 650 Corporate Jet. An FRF was simulated that had characteristics similar to that of the actual model to verify that an iterative technique was preferred to the direct technique.

The simulated FRF and the mobility measurement locations used in least squares are shown in Figure 3.4; Table 3-1 contains the data used in the simulation. The signal and noise autospectra for the simulated case are shown in Figure 3.5. Signal/noise ratios of approximately 50 and 100 were used for the force and velocity, respectively.

Table 3-1 Simulated Modal Parameters		
Natural frequency	Damping Coefficient*	Residue
0 Hz (rigid body)	0.04	-1.00
5.0	0.04	-0.10
6.5	0.04	-1.00 - 0.02i
7.0	0.04	0.80 + 0.04i
10.0	0.04	-0.25 - 0.02i
11.0	0.04	0.10 + 0.01i
12.0	0.04	-0.15 - 0.01i
13.0	0.04	-0.15 - 0.01i
15.0	0.04	-0.15 - 0.01i
17.0	0.04	-0.20 - 0.01i
19.0	0.04	0.20 + 0.01i
21.0	0.04	-0.10

Twenty-five samples of data were taken at each measurement frequency and the residues were determined for each sample using direct and iterative least squares. In the iterative case, 15 iterations were used which provided adequate convergence of the residues (less than 0.1% magnitude change). The average values and 95% confidence bounds were determined for each residue and are shown in Figures 3.6 - 3.9. Overall, the confidence bounds were smaller for the iterative case and the iterative-derived residues were closer overall to the actual residues than the direct-solved residues. This is typical

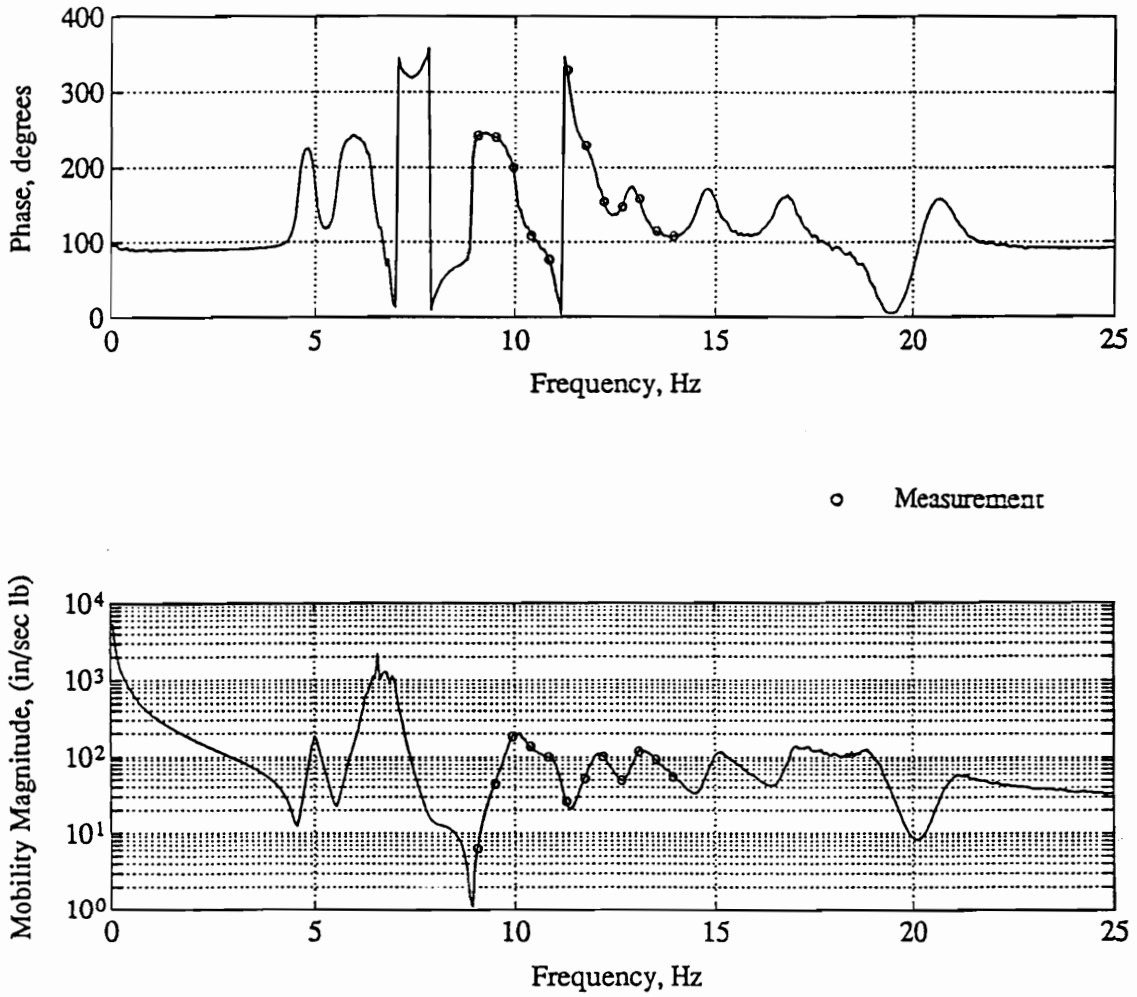


Figure 3.4 Simulated FRF and Mobility Measurement Locations For Iterative Least Squares Evaluation

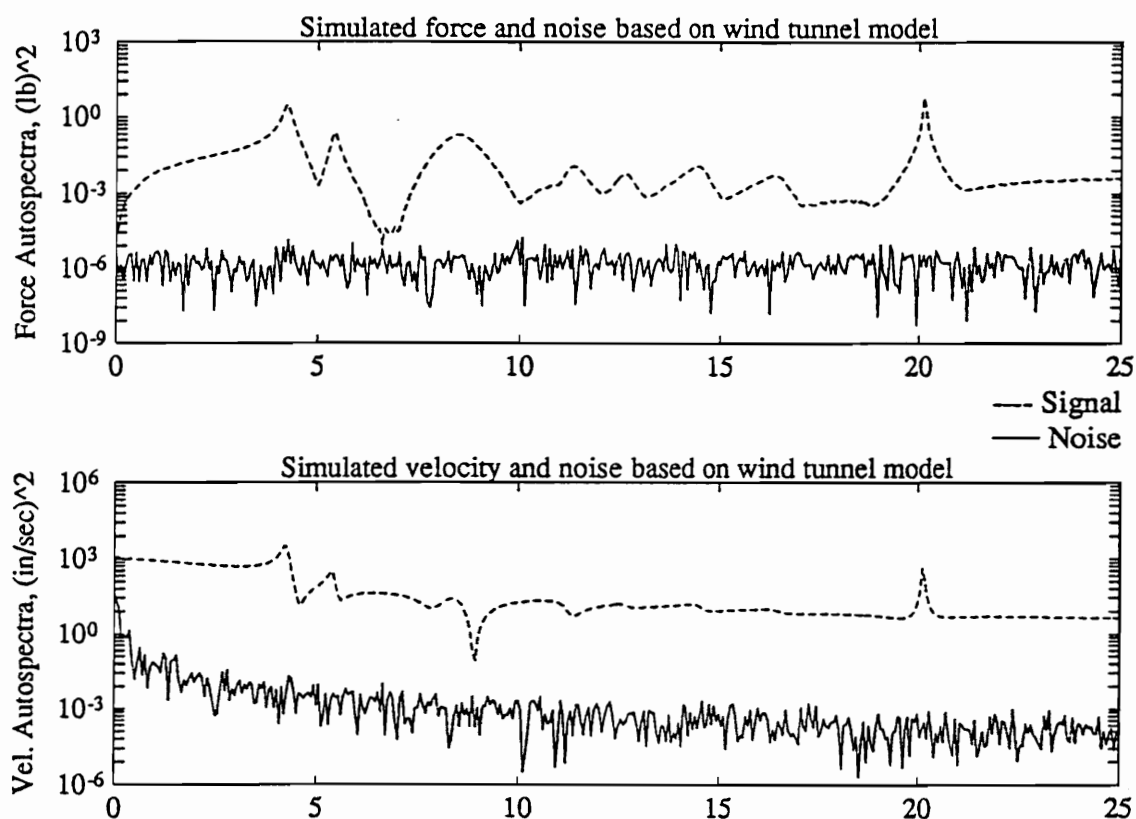


Figure 3.5 Simulated Force and Velocity Autospectra for Iterative Least Squares Evaluation; Noise Autospectra Defined As $(PSD)(\Delta f)(BW \text{ factor})(2)$

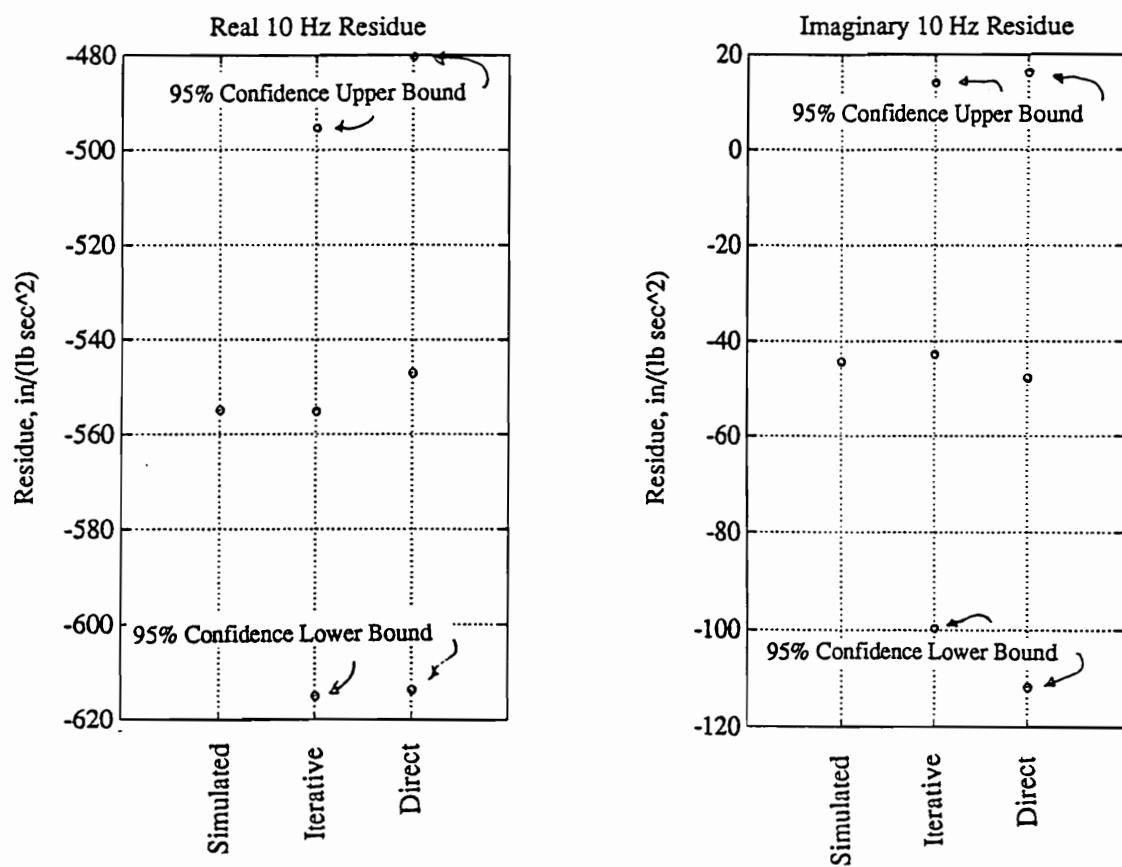


Figure 3.6 Comparison of Simulated Residues, Direct Least Squares Residues and Iterative-Derived Residues

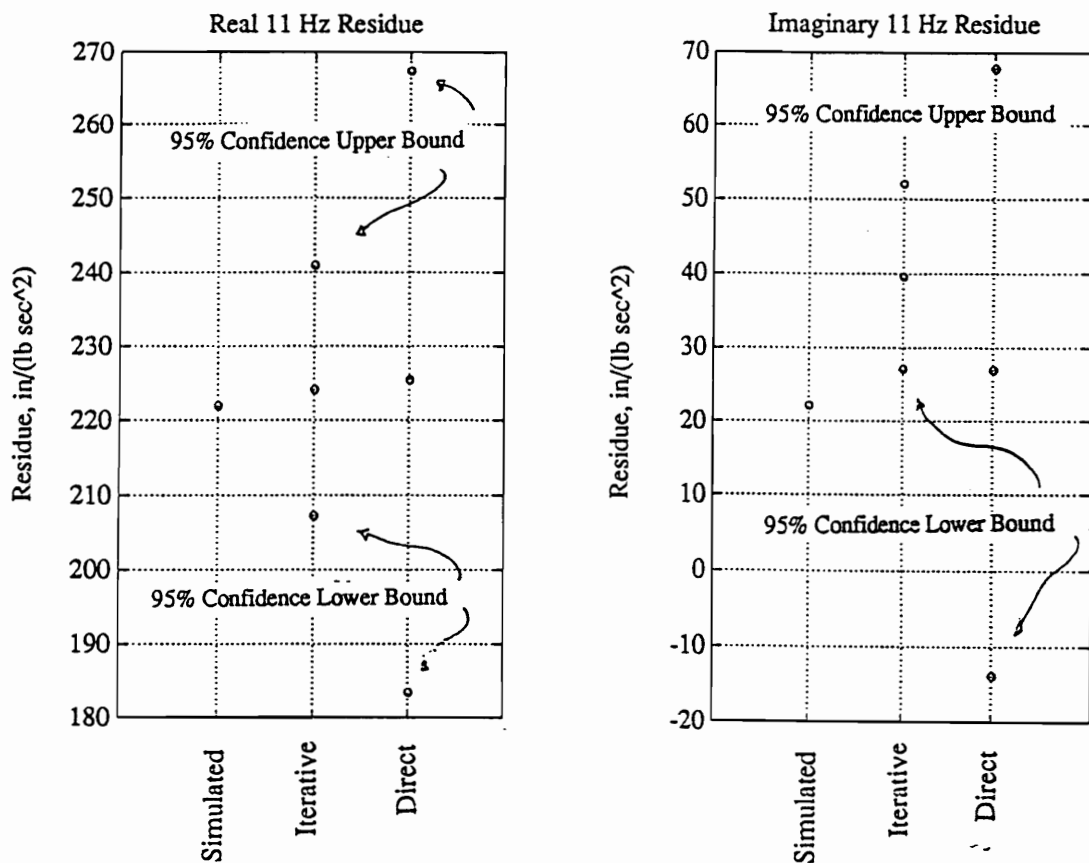


Figure 3.7 Comparison of Simulated Residues, Direct Least Squares Residues and Iterative-Derived Residues

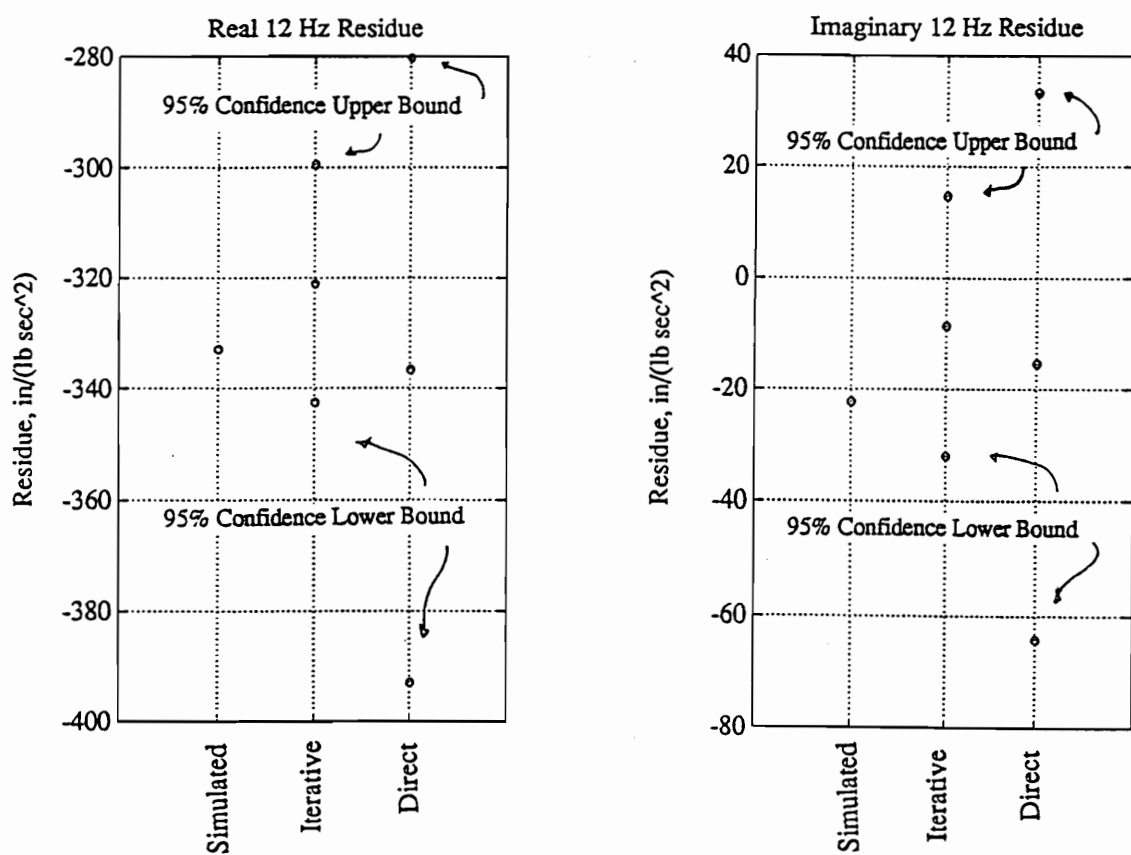


Figure 3.8 Comparison of Simulated Residues, Direct Least Squares Residues and Iterative-Derived Residues

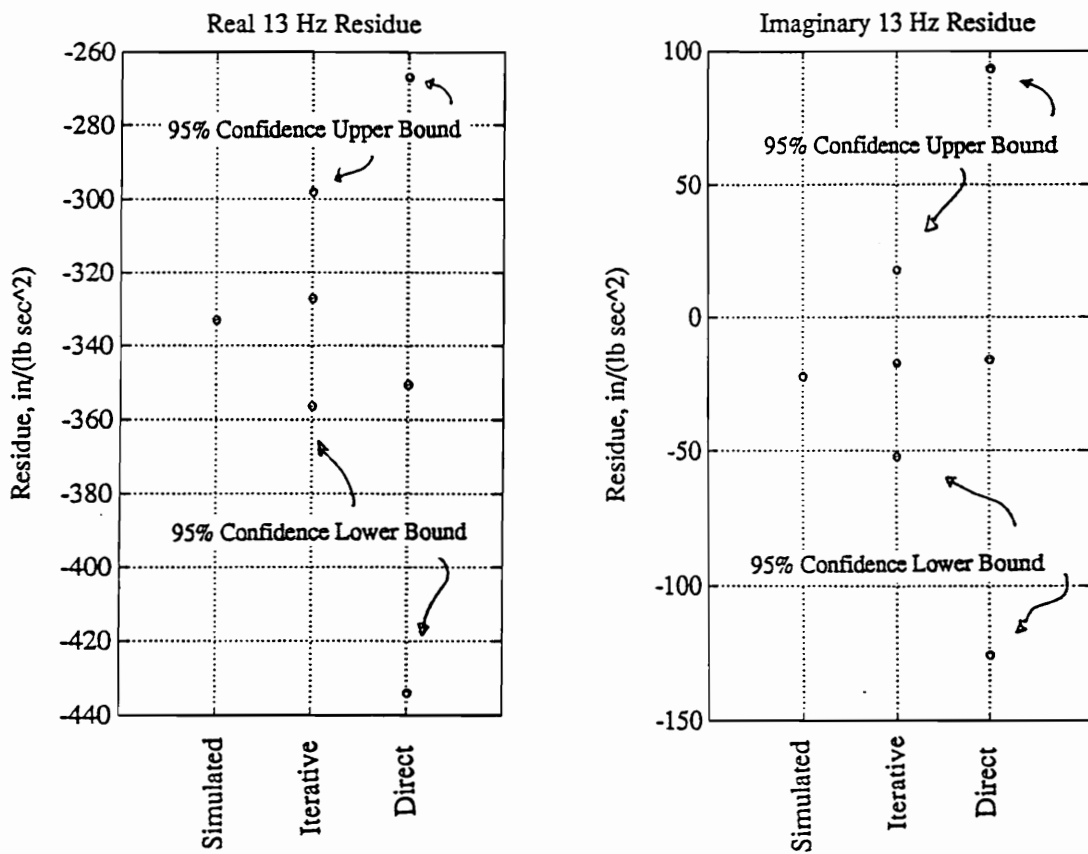


Figure 3.9 Comparison of Simulated Residues, Direct Least Squares Residues and Iterative-Derived Residues

in the presence of the noise levels used for the simulation. Higher signal/noise ratios show that the direct method is superior, however, the use of random excitation and the analysis of larger structures can result in measurement noise similar to this case, favoring the iterative technique.

3.4 The Effects of Incorrect Damping and Natural Frequency

If the curve fitting used to find the global parameters does not accurately identify the parameters, errors can result in residue determination. Since a system rarely conforms to the analytical model exactly, some error is to be expected. The amount of resulting error in the residues will be a function of the relative location of the measured mobilities and the system dynamic parameters. The residue sensitivity can be analytically derived starting with the functional relationship of the residue total derivative with respect to measurement frequency ω_j (the independent variable):

$$\frac{dA}{d\omega_j} = \frac{\partial A}{\partial \omega_j} + \frac{\partial A}{\partial \omega_r} \frac{\partial \omega_r}{\partial \omega_j} + \frac{\partial A}{\partial \eta_r} \frac{\partial \eta_r}{\partial \omega_j} + \frac{\partial A}{\partial Y} \frac{\partial Y}{\partial \omega_j} \quad (3.10)$$

where:

$$A = f(\omega_j, \omega_r(\omega_j), \eta_r(\omega_j), Y)$$

This will result in a vector of sensitivities of the residues to *one jth* measurement frequency. In the event that there is uncertainty in the excitation frequency, this information will be useful. The general form above could be applied to the extraction of the parameters from the full 400-line FRF. In extracting the global parameters from the

FRF, the terms $\frac{\partial \omega_r}{\partial \omega_j}, \frac{\partial \eta_r}{\partial \omega_j}$ will exist because there is a functional relationship between

the natural frequencies and dampings and the measurement frequency. There is no functional relationship between the measured mobilities Y and the measurement frequency because the mobilities represent raw data. Therefore, $\frac{\partial Y}{\partial \omega_j} = 0$, and the last

term in the total derivative is zero.

For the least squares determination of the residues, the global parameters have already been extracted so there is no functional relationship $\omega_r(\omega_j)$ and $\eta_r(\omega_j)$ at the time of residue extraction. This results in $\frac{\partial \omega_r}{\partial \omega_j}, \frac{\partial \eta_r}{\partial \omega_j} = 0$, and the total derivative

becomes:

$$\frac{dA}{d\omega_j} = \frac{\partial A}{\partial \omega_j} \quad (3.11)$$

Determining this function requires expressing A in the form of the normal equations and defining a vector notation as shown below:

$$\{A\} = ([K]^T [K])^{-1} [K] \{Y\} \quad (3.12)$$

or:

$$\bar{A} = (K^T K)^{-1} K^T \bar{Y} \quad (3.13)$$

Now:

$$\frac{d\bar{A}}{d\omega_j} = \frac{d}{d\omega_j} \left[(K^T K)^{-1} K^T \bar{Y} \right] \quad (3.14)$$

where:

$$[K] = \begin{bmatrix} \frac{i\omega_1}{(i\omega_1)^2 + \omega_1^2(1+i\eta_1)} & \dots & \frac{i\omega_1}{(i\omega_1)^2 + \omega_1^2(1+i\eta_1)} \\ \vdots & \frac{i\omega_j}{(i\omega_j)^2 + \omega_r^2(1+i\eta_r)} & \vdots \\ \frac{i\omega_p}{(i\omega_p)^2 + \omega_1^2(1+i\eta_1)} & \dots & \frac{i\omega_p}{(i\omega_p)^2 + \omega_1^2(1+i\eta_1)} \end{bmatrix} \quad (3.15)$$

The frequency of excitation ω_j is known to approximately 1 part in 10^7 when using a frequency synthesizer, making the frequency errors very small; this derivative will not be calculated as a result.

At the residue determination stage, the terms of the total derivative that are of interest are $\frac{\partial A}{\partial \omega_r}$ and $\frac{\partial A}{\partial \eta_r}$. Given uncertainty in the global parameters, the uncertainty in

the residues can be determined from these derivatives. These terms can be computed directly, again by starting with the normal equations:

Writing the derivative $\frac{\partial \bar{A}}{\partial \omega_r}$:

$$\frac{\partial \bar{A}}{\partial \omega_r} = -(K^T K)^{-1} \frac{\partial (K^T K)}{\partial \omega_r} (K^T K)^{-1} K^T \bar{Y} + (K^T K)^{-1} \left(\frac{\partial K}{\partial \omega_r} \right)^T \bar{Y} \quad (3.16)$$

assuming $\bar{Y} = K\bar{A}$ exactly and expanding the above derivative,

$$\frac{\partial \bar{A}}{\partial \omega_r} = (K^T K)^{-1} \left[- \left(\left(\frac{\partial K}{\partial \omega_r} \right) K + K^T \frac{\partial K}{\partial \omega_r} \right) (K^T K)^{-1} K^T + \left(\frac{\partial K}{\partial \omega_r} \right)^T \right] K\bar{A} \quad (3.17)$$

Simplifying:

$$\frac{\partial \vec{A}}{\partial \omega_r} = -\left(K^T K\right)^{-1} K^T \frac{\partial K}{\partial \omega_r} \vec{A} \quad (3.18)$$

This is the sensitivity of the residues to variations in natural frequency ω_r . Similarly, the sensitivity of the residues to variations in the damping η_r is:

$$\frac{\partial \vec{A}}{\partial \eta_r} = -\left(K^T K\right)^{-1} K^T \frac{\partial K}{\partial \eta_r} \vec{A} \quad (3.19)$$

These derivatives are unchanged in the event that the residual terms are included.

To verify these equations, a 2-DOF system is simulated and a least squares solution is used to determine the residues. Small perturbations of the global parameters are input and the resulting residues are determined. Once the equations are verified, two cases are examined: measurement frequencies close to the resonances and measurement frequencies away from the resonances.

The system parameters are shown in Table 3-2 along with the measurement frequencies for the two cases. The perturbed frequency and damping parameters used to test the analytical sensitivities are as follows:

$$\omega'_1 = 1.0001\omega_1$$

$$\eta'_1 = 1.01\eta_1$$

For equation verification, the exact mobilities Y (no noise) at the close measurement frequencies are used to determine the residues. The sensitivities and residues from the least squares fit are presented in Table 3-3. The sensitivities are

multiplied by the frequency and damping perturbations and the result is subtracted from the perturbed residues to obtain the sensitivity-derived residues. Table 3-3 shows the residues which should be close to the original residues used to generate the data.

Table 3-2 Simulated 2-DOF Parameters and Measurement Frequencies				
f_r	η_r	A_r	"Near" Measurement Frequencies	"Far" Measurement Frequencies
750 Hz	0.02	1	700 Hz	600 Hz
1250 Hz	0.02	1	800	900
			1200	1100
			1300	1400

Table 3-3 Comparison of Exact Residues with Sensitivity-Derived Residues - "Near" Measurements, Variation in ω_1, η_1			
Exact Residue, A	Frequency-perturbed residue, A'	Sensitivities, $\frac{dA}{d\omega_1}$	$A'' = A' - \frac{dA}{d\omega_1} \Delta\omega_1$
$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 0.9999426 - 6.61e^{-4}i \\ 1.000114 - 1.80e^{-5}i \end{Bmatrix}$	$\begin{Bmatrix} -1.12e^{-4} - 1.40e^{-3}i \\ 2.42e^{-4} - 3.84e^{-5}i \end{Bmatrix}$	$\begin{Bmatrix} 0.9999954 + 7.25e^{-8}i \\ 0.9999999 + 1.04e^{-7}i \end{Bmatrix}$
Exact Residue, A	Damping-perturbed residue, A'	Sensitivities, $\frac{dA}{d\eta_1}$	$A'' = A' - \frac{dA}{d\eta_1} \Delta\eta_1$
$\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 1.000660 - 6.63e^{-5}i \\ 1.000020 - 1.14e^{-4}i \end{Bmatrix}$	$\begin{Bmatrix} 3.307 - 0.264i \\ 0.0904 + 0.571i \end{Bmatrix}$	$\begin{Bmatrix} 1.000003 - 1.35e^{-5}i \\ 1.000002 - 5.16e^{-7}i \end{Bmatrix}$

The sensitivity-derived residues A'' are close to the original residues and therefore confirm that the analytical sensitivity model is correct. It is now desired to compare the sensitivities $\frac{dA}{d\omega_r}$ and $\frac{dA}{d\eta_r}$ for the cases of "near" and "far" measurements.

Furthermore, the effects of damping will be studied by examining the system described in Table 3-2 with damping coefficients of 0.005 (light) and 0.05 (moderate). This exercise will serve as a guide as to where the measurements should be taken for an MDOF system. Table 3-4 shows the sensitivities for the lightly damped system, and Table 3-5 shows the sensitivities for the moderately damped system.

Table 3-4 Sensitivities for Lightly Damped ($\eta_r = 0.005$) 2-DOF System				
	$\frac{dA}{d\omega_1}$	$\frac{dA}{d\omega_2}$	$\frac{dA}{d\eta_1}$	$\frac{dA}{d\eta_2}$
"Near" Measurements	$\begin{Bmatrix} -1.04e^{-4} - 3.54e^{-4}i \\ 2.44e^{-4} - 8.28e^{-6}i \end{Bmatrix}$	$\begin{Bmatrix} -4.04e^{-4} - 1.01e^{-5}i \\ -7.04e^{-5} - 5.91e^{-4}i \end{Bmatrix}$	$\begin{Bmatrix} 0.833 - 0.245i \\ 0.0195 + 0.575i \end{Bmatrix}$	$\begin{Bmatrix} 0.0396 - 1.59i \\ 2.32 - 0.276i \end{Bmatrix}$
"Far" Measurements	$\begin{Bmatrix} -8.89e^{-5} - 3.73e^{-5}i \\ 2.89e^{-4} - 7.56e^{-6}i \end{Bmatrix}$	$\begin{Bmatrix} -4.58e^{-4} - 7.12e^{-6}i \\ -1.14e^{-4} - 6.12e^{-5}i \end{Bmatrix}$	$\begin{Bmatrix} 0.0879 - 0.210i \\ 0.0178 + 0.680i \end{Bmatrix}$	$\begin{Bmatrix} 0.0280 - 1.80i \\ 0.240 - 0.448i \end{Bmatrix}$

Table 3-5 Sensitivities for Moderately Damped ($\eta_r = 0.05$) 2-DOF System				
	$\frac{dA}{d\omega_1}$	$\frac{dA}{d\omega_2}$	$\frac{dA}{d\eta_1}$	$\frac{dA}{d\eta_2}$
"Near" Measurements	$\begin{Bmatrix} -1.87e^{-4} - 3.41e^{-3}i \\ 1.43e^{-4} - 2.12e^{-4}i \end{Bmatrix}$	$\begin{Bmatrix} -5.04e^{-4} - 1.37e^{-4}i \\ 3.67e^{-4} - 5.94e^{-3}i \end{Bmatrix}$	$\begin{Bmatrix} 8.02 - 0.440i \\ 0.500 + 0.337i \end{Bmatrix}$	$\begin{Bmatrix} 0.536 - 1.98i \\ 23.3 - 1.44i \end{Bmatrix}$
"Far" Measurements	$\begin{Bmatrix} -1.01e^{-4} - 3.70e^{-4}i \\ 2.88e^{-4} - 8.27e^{-5}i \end{Bmatrix}$	$\begin{Bmatrix} -4.67e^{-4} - 7.10e^{-5}i \\ -1.25e^{-4} - 5.99e^{-4}i \end{Bmatrix}$	$\begin{Bmatrix} 0.873 - 0.238i \\ 0.195 + 0.678i \end{Bmatrix}$	$\begin{Bmatrix} 0.279 - 1.84i \\ 2.35 - 0.492i \end{Bmatrix}$

The tables show that sensitivity is generally less when the measurements are taken away from the resonances. The lightly damped case has overall less sensitivity to variations of the parameters than the moderately damped case. From these results, measurements are recommended to be away from the resonances to minimize errors caused by inexact global parameters.

It is desired to plot the variations in residues as a function of measurement location and global parameter error for a simulated single degree-of-freedom system. These results will confirm the derivative results above and give a graphical presentation of the effects of measurement location on residue error. The simulated SDOF system parameters are derived from the real system shown in Figure 4.1. Realistic noise levels and the shaker characteristics are included in the simulation. Considering just one mode, the measured mobility will be:

$$Y = \frac{(i\omega)A}{(i\omega)^2 + \omega_n^2(1 + i\eta)} \quad (3.20)$$

Writing an SDOF version of equation (2.12) results in:

$$\begin{Bmatrix} Y_{\omega_1} \\ Y_{\omega_2} \end{Bmatrix} = \begin{bmatrix} \frac{i\omega_1}{(i\omega_1)^2 + \omega_n^2(1 + i\eta)} \\ \frac{i\omega_2}{(i\omega_2)^2 + \omega_n^2(1 + i\eta)} \end{bmatrix} \{A\} \quad (3.21)$$

Since there are no residual terms in the simulation and only one mode of interest, two simulated mobility measurements are sufficient to obtain the residues. A normal equations solution is used to find the residue for each case analyzed. The parameters used in the model were:

$$\begin{aligned}
f_n &= 1500 \text{ Hz} \\
\eta &= 0.00276 \\
A &= 66.58 + 1.32i
\end{aligned}
\tag{3.22}$$

The simulated measurements were obtained by using the system model in Figure 3.1. Referring to Figure 3.1, the shaker frequency response function H_s is given in equation (3.9) and the structure frequency response function H_t is just the SDOF version of equation (3.5). The voltage signal V_s is adjusted to achieve force levels F close to observed force levels when using sine excitation. As will be discussed in Chapter 4, the step-sine signal amplitude was controlled to take advantage of the dynamic range of the transducers without over-driving the structure. This would keep the signal/noise ratio high.

The simulated noise levels were obtained from actual force and accel signal measurements of the real system with the shaker off. Although the actual distribution was not measured, the noise was assumed to be gaussian. Figures 3.10 and 3.11 compare 10 averages of the simulated noise autospectra to the measured noise autospectra, showing that the overall levels are close. Local deviations did occur in the measured noise, however these could not be predicted and were therefore not modeled.

Figure 3.12 shows the simulated mobility and Figures 3.13 and 3.14 show the respective force and velocity spectra along with the simulated noise for one sample. The large signal-to-noise ratio allows that just one simulated sample be used to get a good estimate of the mobility; this will hold true for the experimental results presented in Chapter 5. Four measurement cases were investigated for damping and natural frequency variations. Each case pertained to a different set of frequencies used to find the residues; Table 3-6 summarizes the cases analyzed. The mobilities were obtained by

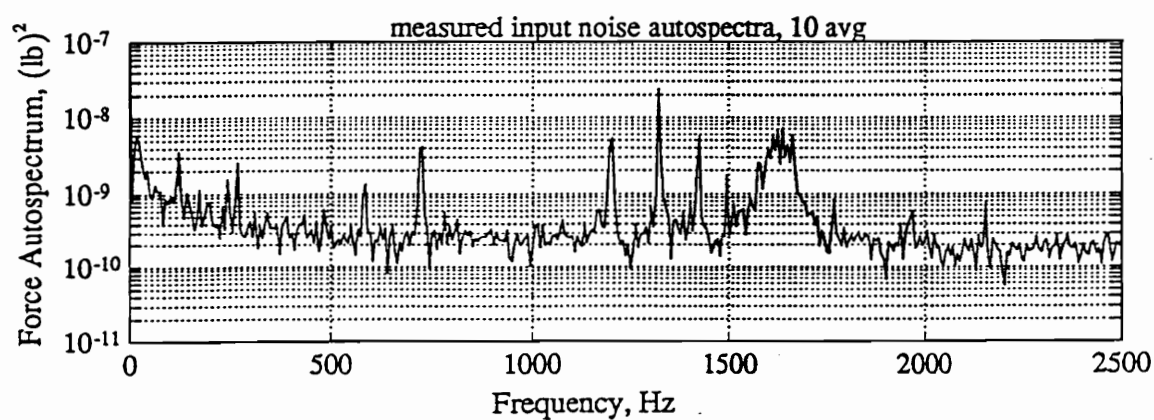
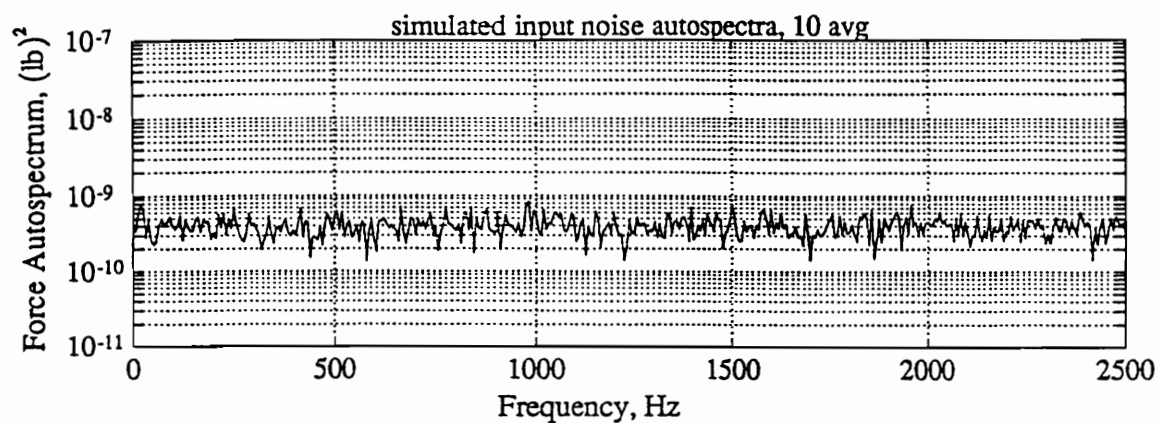


Figure 3.10 Simulated and Measured Input Noise Autospectra,
10 Averages - Noise is $(PSD)(\Delta f)(BW \text{ factor})(2)$

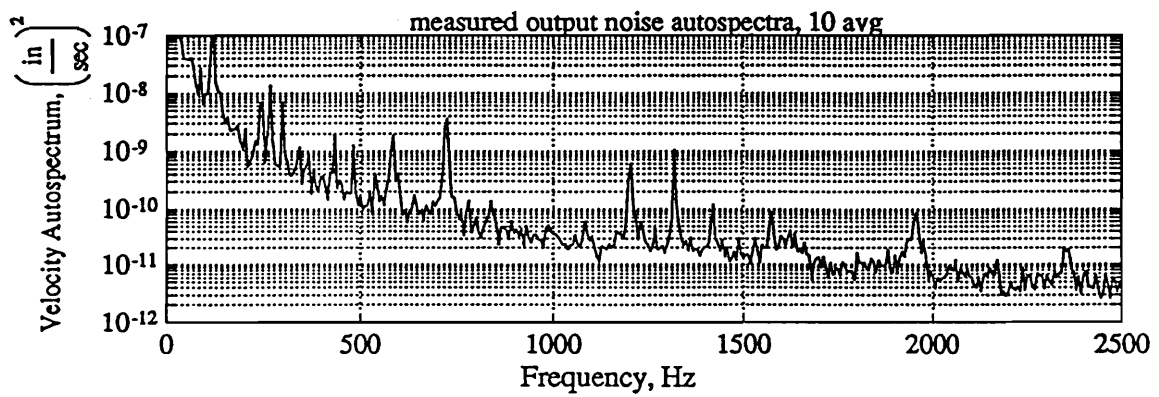
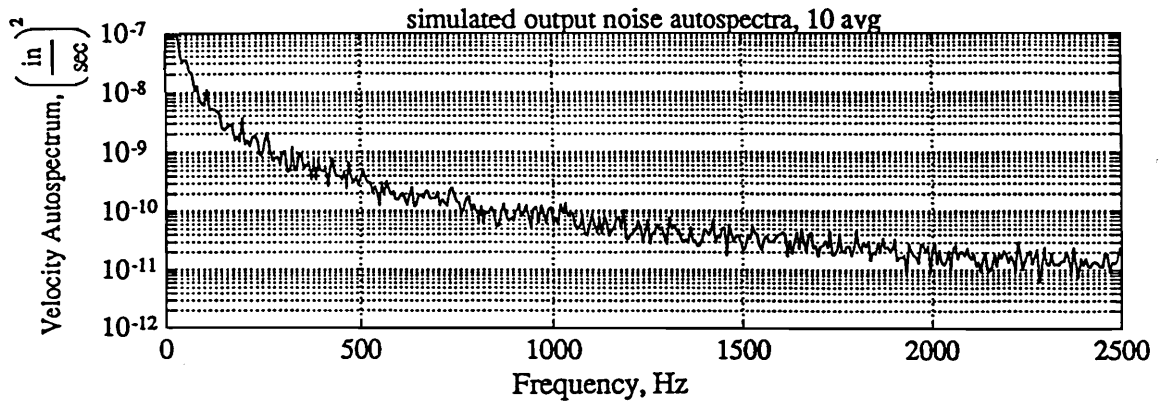


Figure 3.11 Simulated and Measured Output Noise Autospectra,
10 Averages - Noise is $(PSD)(\Delta f)(BW \text{ factor})(2)$

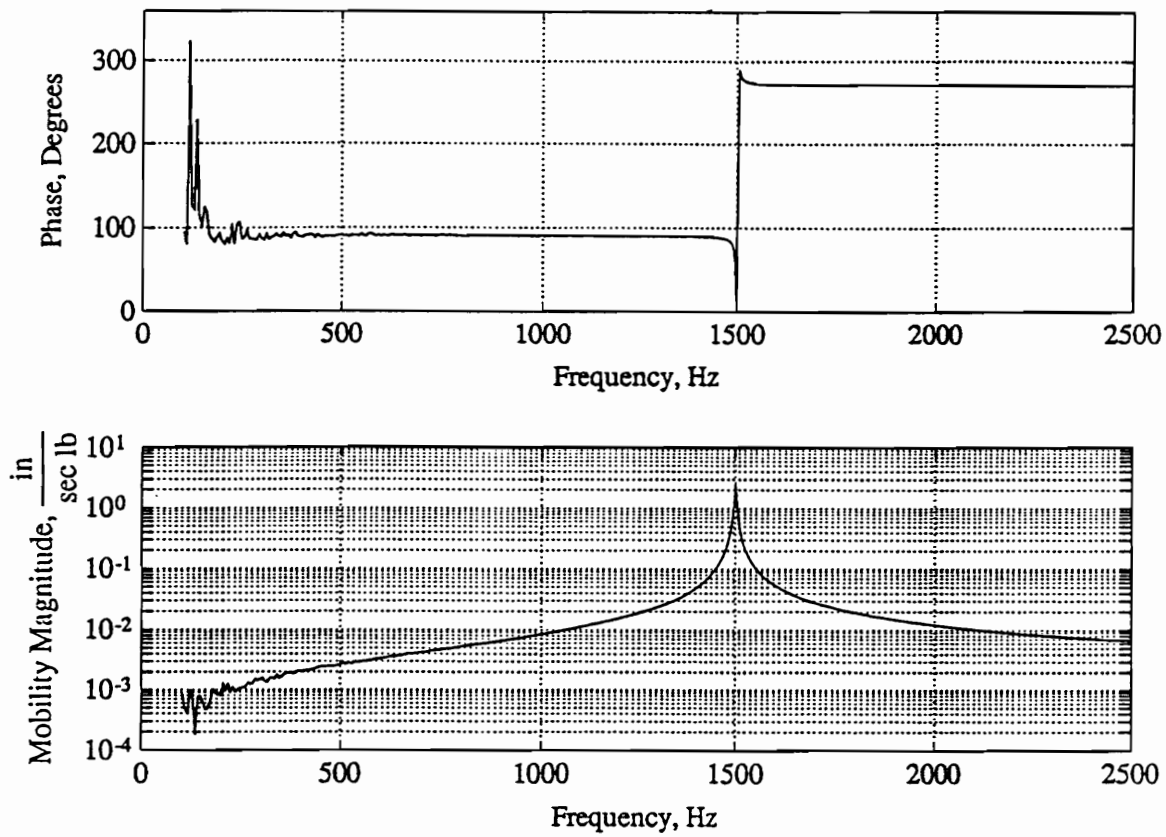


Figure 3.12 Simulated Single-Degree-of-Freedom FRF with Simulated Noise

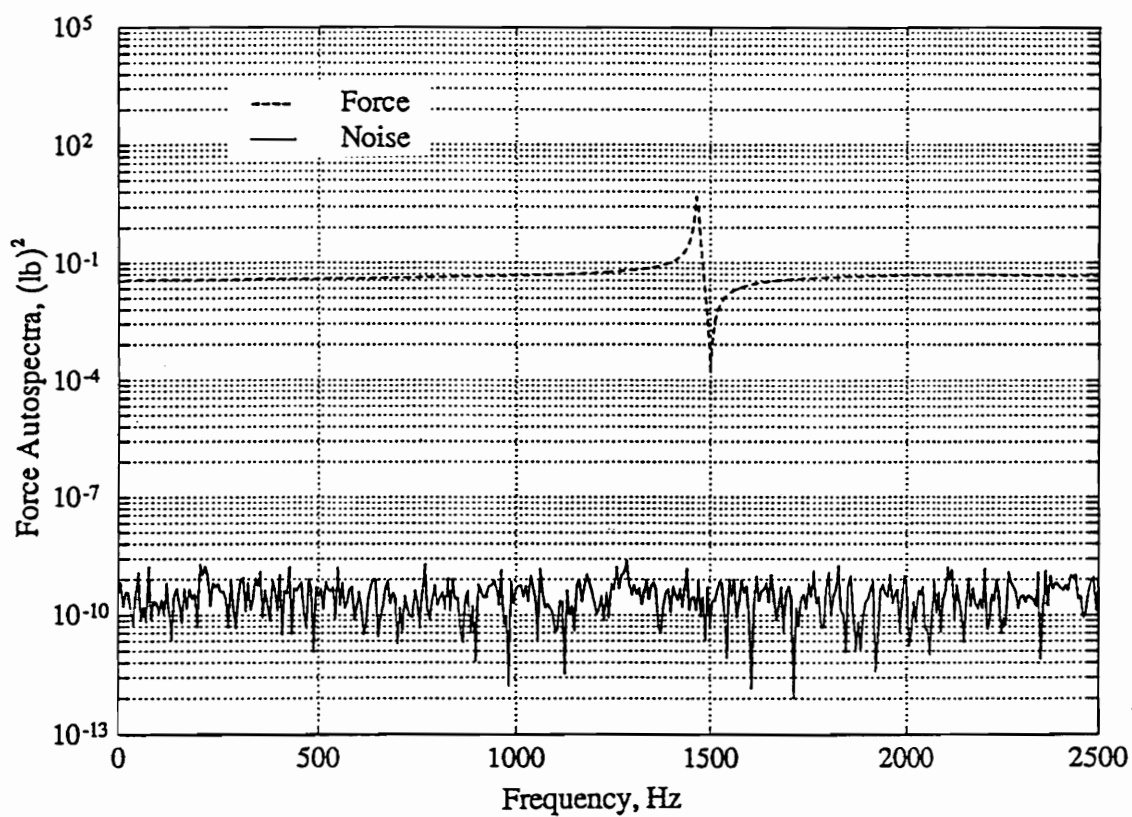


Figure 3.13 Simulated One-DOF Autospectra of Force and Realistic Input Noise - Force is $(0 - \text{peak})^2$ and Noise is $(PSD)(\Delta f)(BW \text{ factor})(2)$

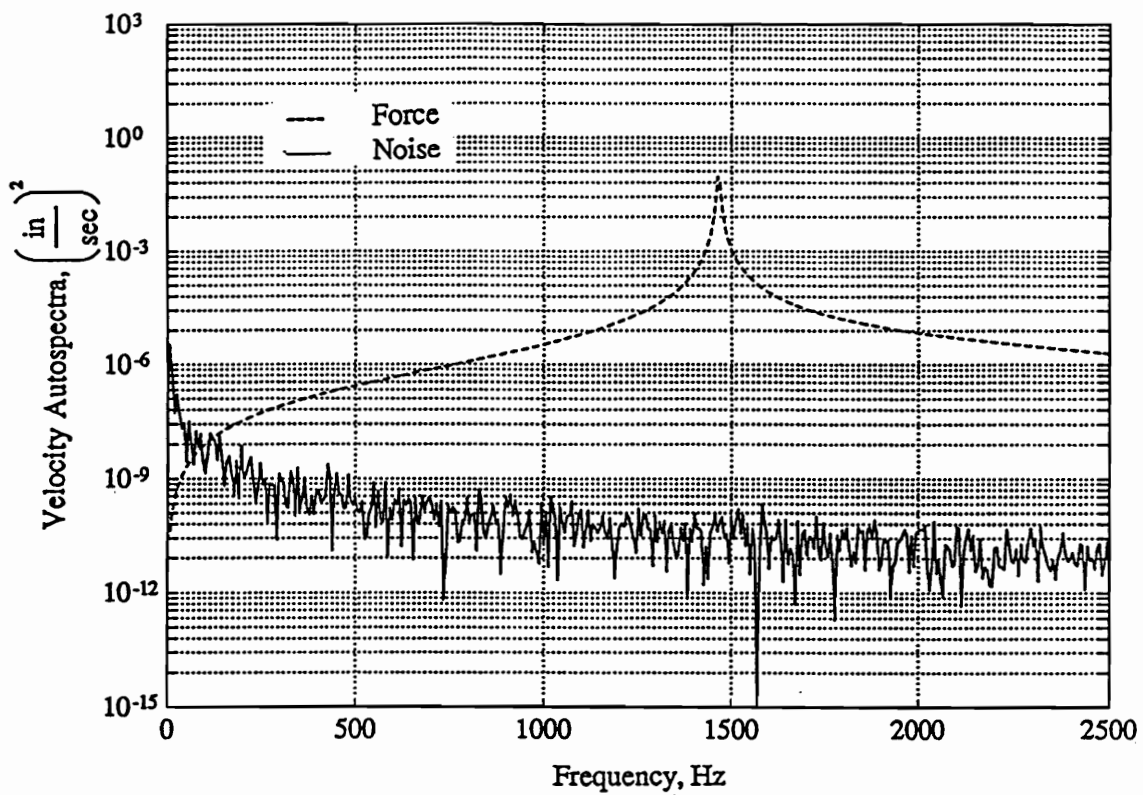


Figure 3.14 Simulated One-DOF Autospectra of Velocity and Realistic Output Noise - Velocity is $(0 - peak)^2$ and Noise is $(PSD)(\Delta f)(BW \text{ factor})(2)$

dividing one sample of the simulated frequency-domain velocities by one sample of the simulated forces. This is how the mobilities would be obtained in practice, where one sample of data would alleviate the need for determining cross and autospectra.

Table 3-6		
Measurement Frequencies for Sensitivity Evaluation		
	Measurement Frequencies	
Case	Low	High
1	1300 Hz	1700 Hz
2	1350	1650
3	1400	1600
4	1450	1550

Figure 3.15 shows the effects of variations of $\pm 50\%$ in the damping coefficient for the four cases and Figure 3.16 shows the effects of variations in natural frequency of $\pm 0.02\%$. As was indicated in the sensitivity analysis, measurements that are farther away from the resonance result in a residue less sensitive to variations in the global parameters.

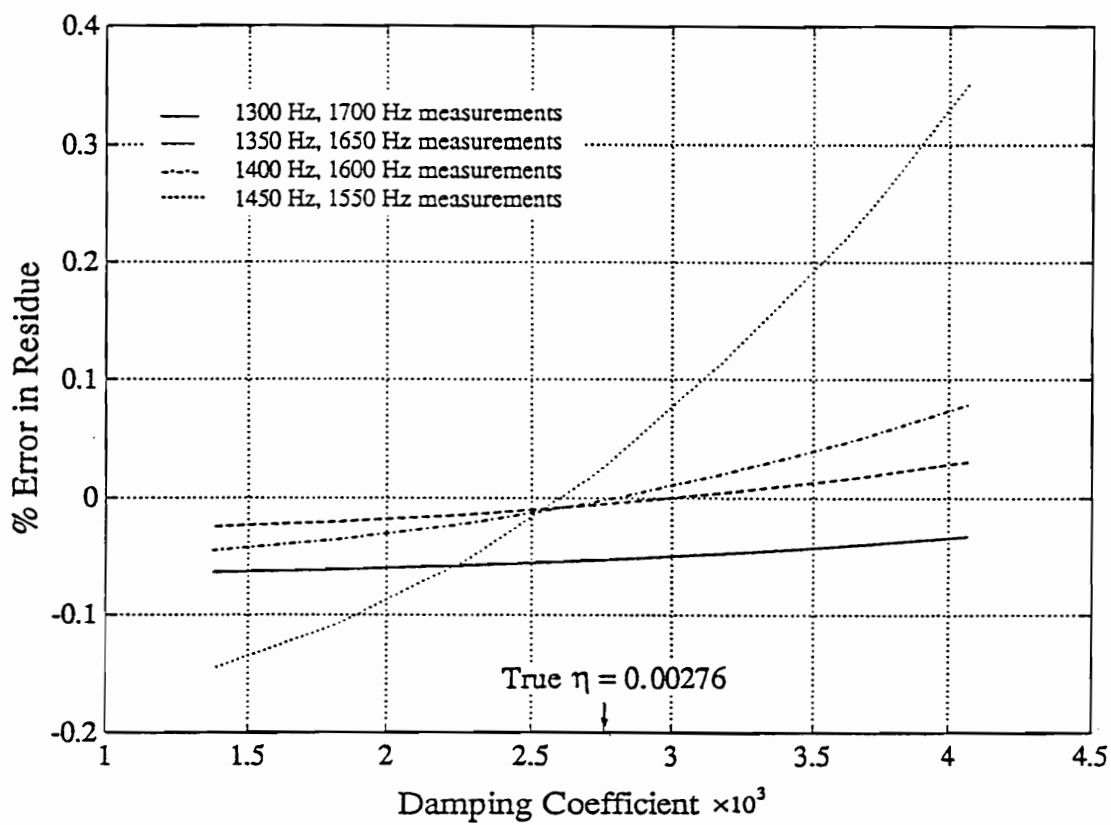


Figure 3.15 The Effects of Damping Variations on Residue Determination

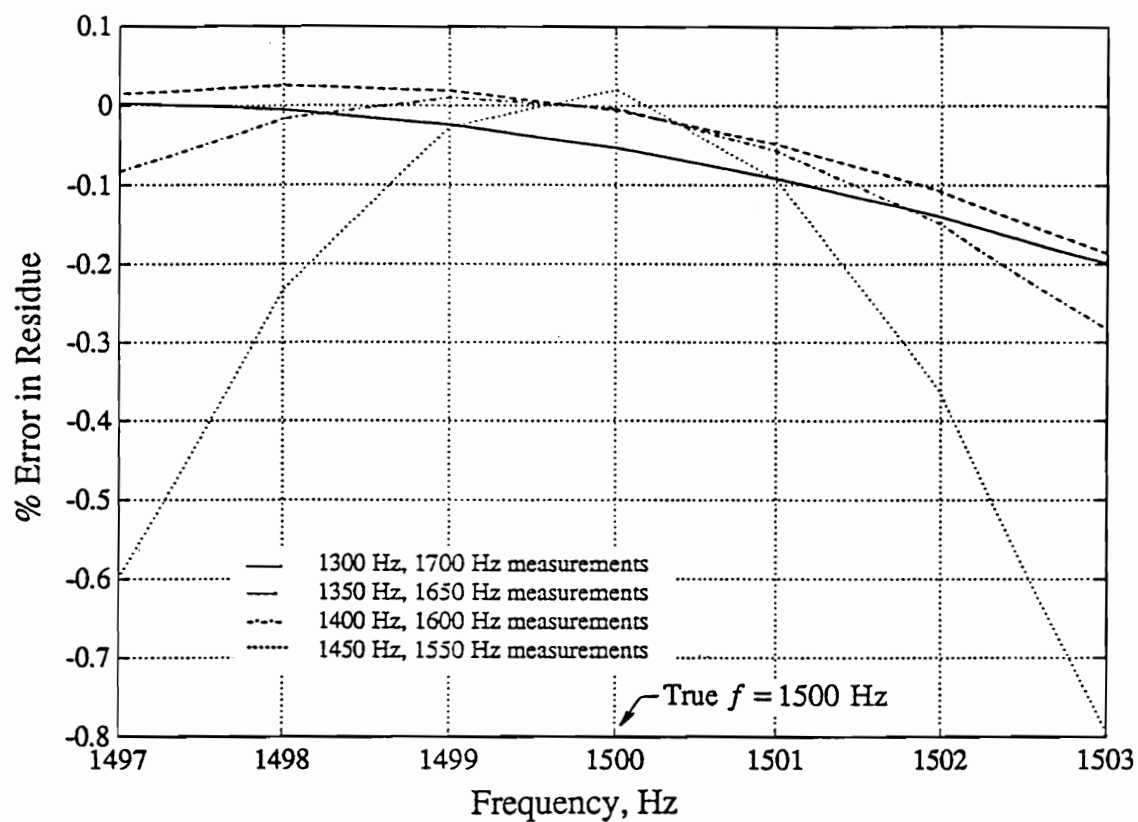


Figure 3.16 The Effects of Frequency Variations on Residue Determination

3.5 Measurement Frequencies For Multi-Degree-of-Freedom Systems

The choice of measurement frequencies for the mobilities affects the outcome of the residues due to the sensitivity discussed above and also due to the relative behavior of the modes. When using the iterative technique, convergence must be considered as well as final residue error. Representative cases of FRF's that have global parameter errors, close modes and low amplitude modes will be examined for convergence rate and final residue error. The number of measurements used to determine the residues will also be examined. Only the iterative technique will be used to find the residues since the final errors after convergence will be nearly identical to those from the direct method. This is due to the high signal/noise ratio assumed for the data. Therefore, the final residue errors presented apply to the iterative technique as well as the direct technique.

An eight degree-of-freedom system is simulated with the shaker model to obtain realistic FRF data. This system is again closely modeled after a real system which will be used in Chapters 4 and 5 to experimentally demonstrate the technique. Figure 3.17 shows the simulated FRF and Figures 3.18 and 3.19 show the autospectra of the input force and output velocity. Similar to the previous SDOF case, the input force autospectra was adjusted to a level that closely matched the real levels by adjusting the signal voltage variable in the algorithm. The noise autospectra is the same that was used in the SDOF case and represents the noise levels that occurred for the actual system. Figures 3.18 and 3.19 also show the noise autospectra. Table 3-7 shows the simulated parameters; a rigid-body coefficient is also present which was observed in the real system.

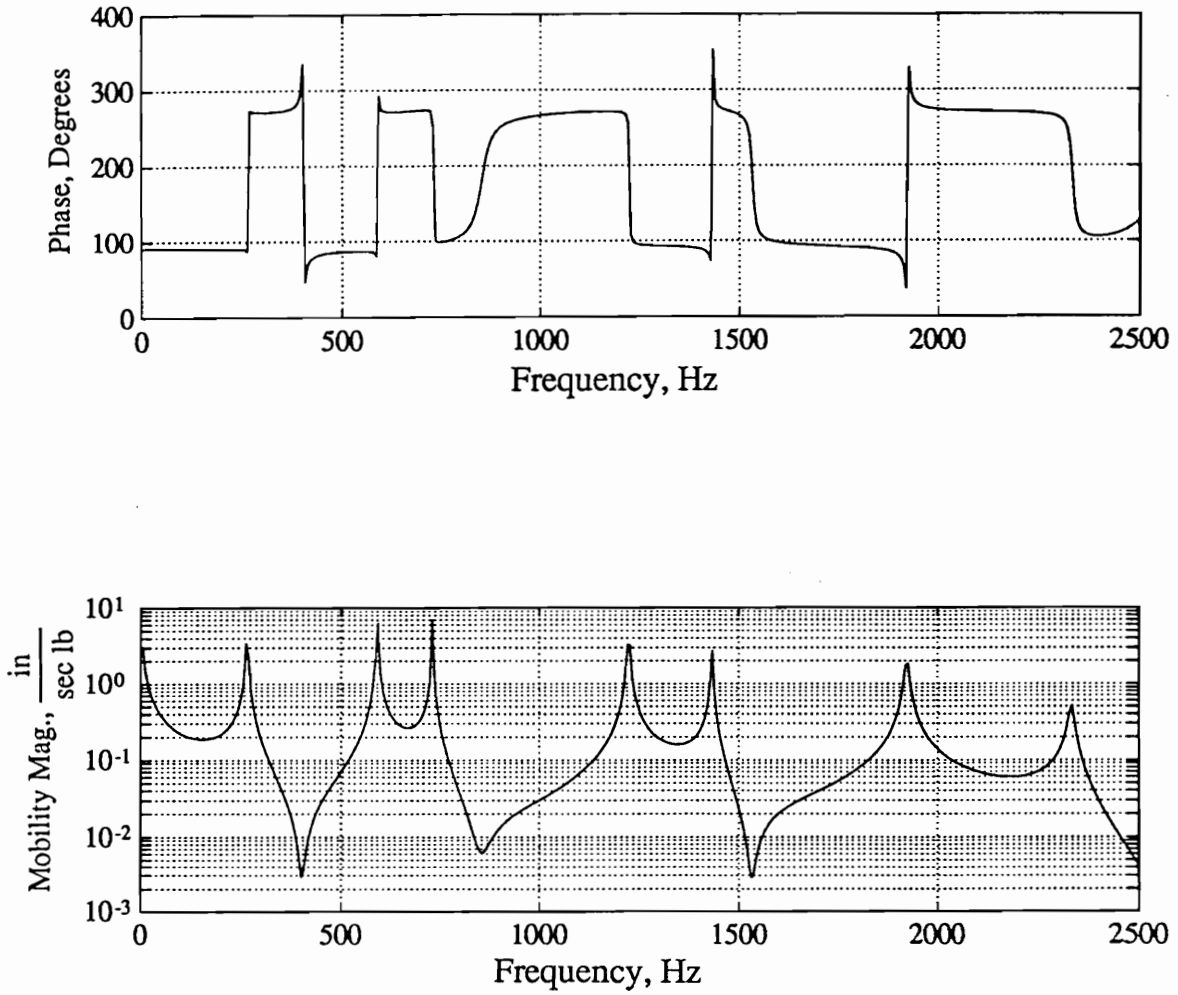


Figure 3.17 Simulated Eight-DOF System Used to Evaluate ILS²

² "Iterative least squares" is abbreviated ILS in the subsequent figure titles

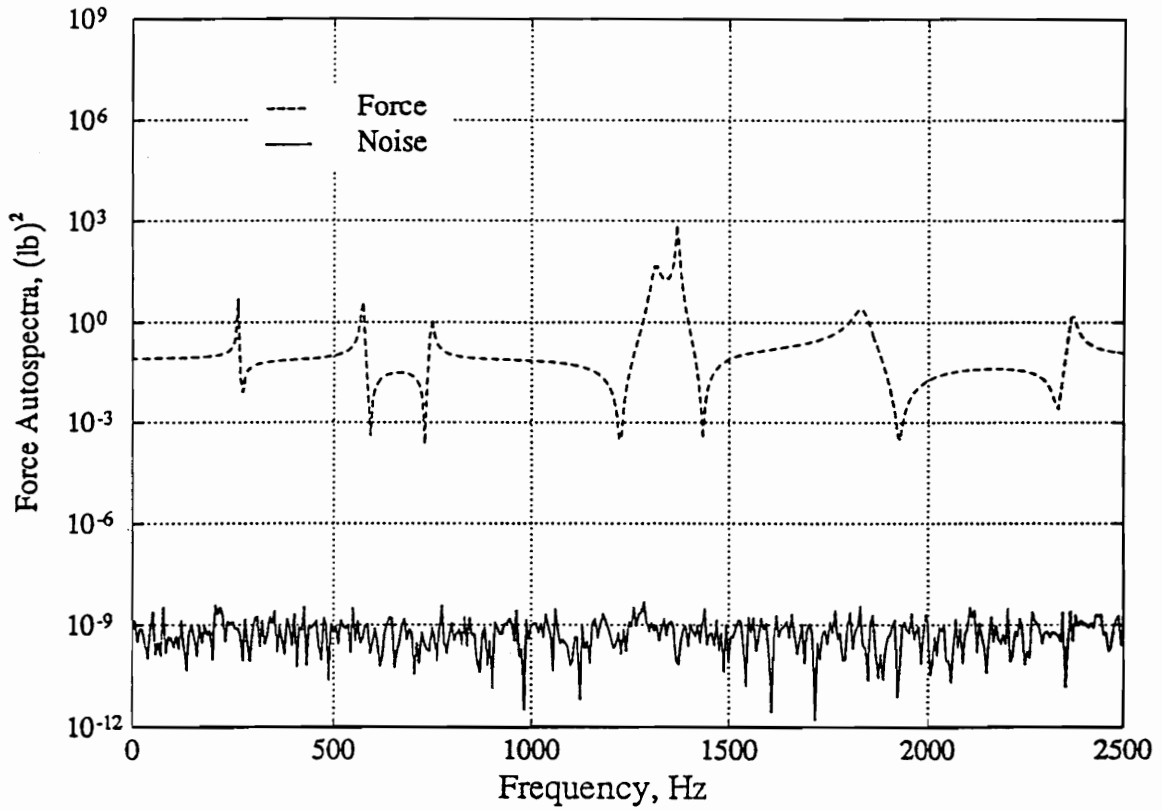


Figure 3.18 Simulated Eight-DOF Autospectra of Force and Realistic Input Noise - Force is $(0 - peak)^2$ and Noise is $(PSD)(\Delta f)(BW \text{ factor})(2)$

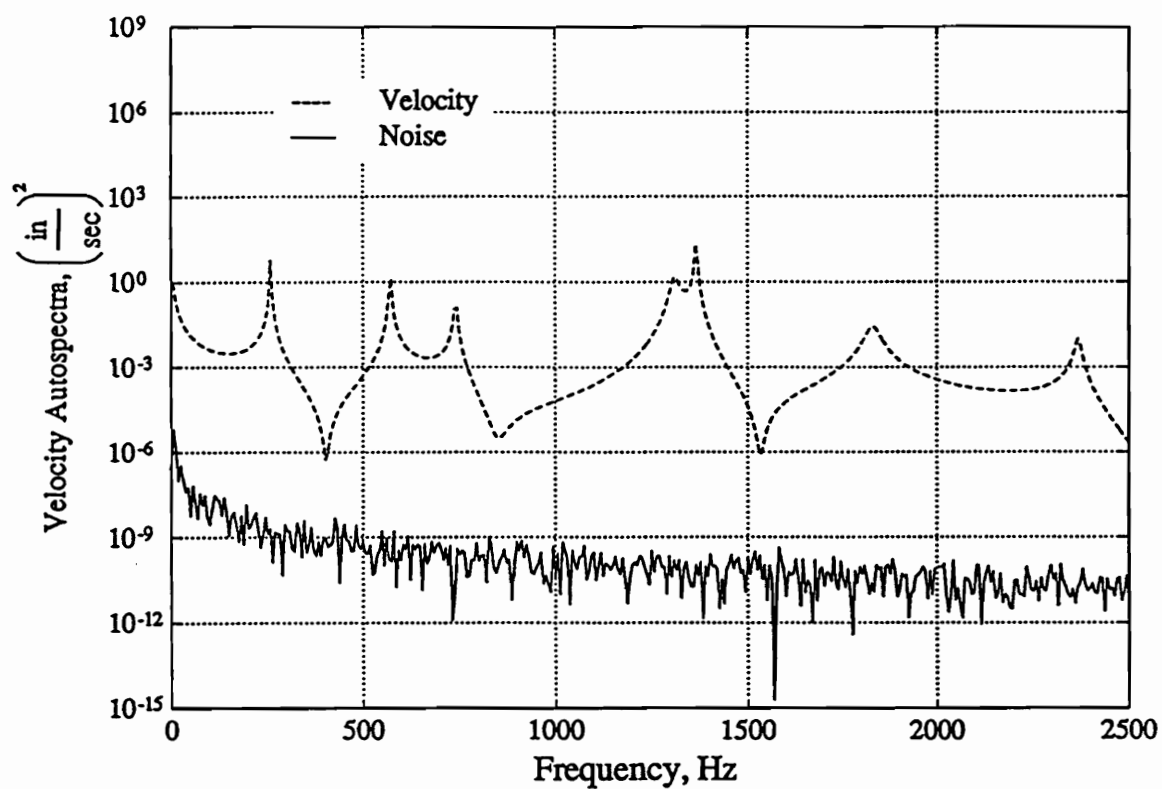


Figure 3.19 Simulated Eight-DOF Autospectra of Velocity and Realistic Output Noise - Velocity is $(0 - peak)^2$ and Noise is $(PSD)(\Delta f)(BW \text{ factor})(2)$

Table 3-7 Simulated Modal Parameters		
Natural frequency	Damping Coefficient*	Residue
0 Hz (rigid body)	0	-120.22 - 2.22i
265.08	0.00137	109.42 + 0.825i
592.16	0.00233	137.07 - 2.24i
732.16	0.00110	-87.38 - 8.65i
1221.8	0.00150	-134.30 - 6.42i
1431.0	0.00273	67.61 + 0.111i
1746.1	0.00301	-0.036 - 0.293 i
1922.1	0.00491	128.89 + 3.06i
2337.1	0.00554	-41.90 - 2.10i

* The lightly damped coefficients are shown. These are increased by a factor of 10 to obtain moderately damped coefficients used later

The four modes of interest which will be investigated are the 592, 732, 1221, and 1431 Hz modes. The other modes are considered out-of-range modes and will be treated as the residual contribution. With four modes to measure and five residual constants to determine, a minimum of 10 mobility measurements will be necessary; 12 equally spaced mobility measurements will be used as a baseline case. Exact values of the global parameters have been assumed which will not be the case in a real experiment.

The measurements will be made at equal intervals which will neither have resonance bias nor off-resonance bias. The measurement range is extended by $\pm 10\%$ of the frequency range containing the modes of interest which will allow the residual content to be adequately represented. Table 3-8 lists the frequencies used for the baseline case. As was done in the SDOF case, the simulated mobility measurements are obtained by dividing one sample of the simulated velocity by one sample of force.

Table 3-8 Simulated Frequencies for Residue Determination	
1	500.00 Hz
2	593.75
3	687.50
4	781.25
5	875.00
6	968.75
7	1062.5
8	1156.25
9	1250.0
10	1343.8
11	1437.5
12	1531.25

The iterative convergence to the residues is shown in Figure 3.20 in terms of percent error at each iteration. The maximum error magnitude after 15 iterations is 0.15% and occurs in the 1431 Hz residue. Regenerating the FRF from the in-range parameters and the residual content results in a fit to the original FRF which does not exceed 3% magnitude error within the range of interest. Figure 3.21 shows the original FRF and the regenerated FRF, and Figure 3.22 shows the resulting magnitude of error in mobility. Table 3-9 shows the iterative-derived residues along with the errors associated with them.

The curve fit is valid only within the measurement range, however the regenerated FRF also shows reasonably good data fitting for the low-frequency modes outside of the measurement range. This is attributed to the fact that the poles for these modes are included in the residual model. The high frequency portion of the FRF does

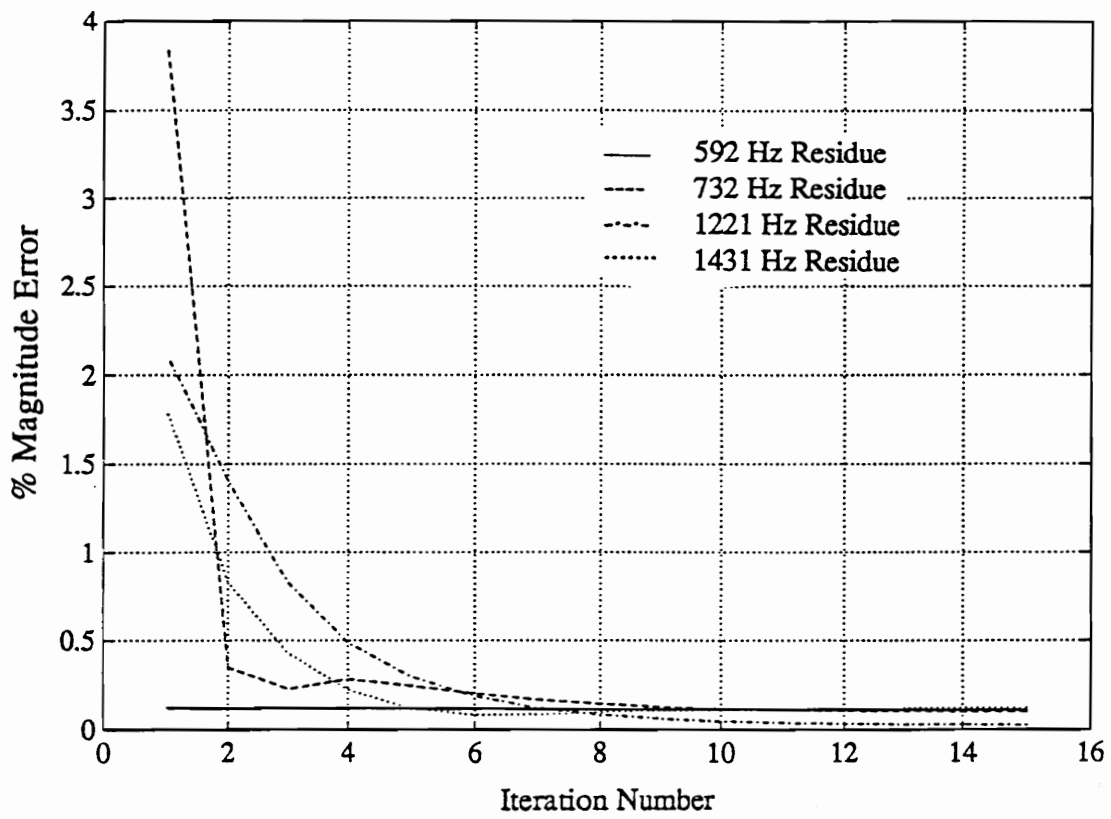


Figure 3.20 Convergence of Residues for Baseline Case

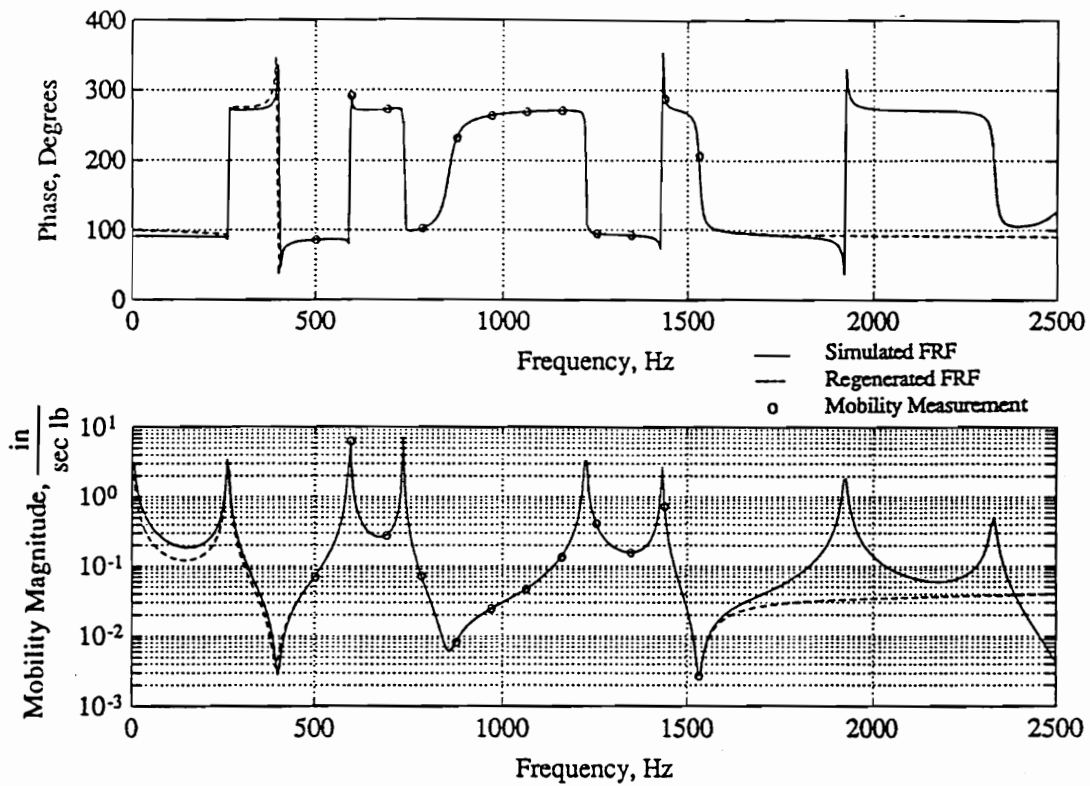


Figure 3.21 Simulated and Regenerated FRF's Using ILS, 12 Equally Spaced Measurements

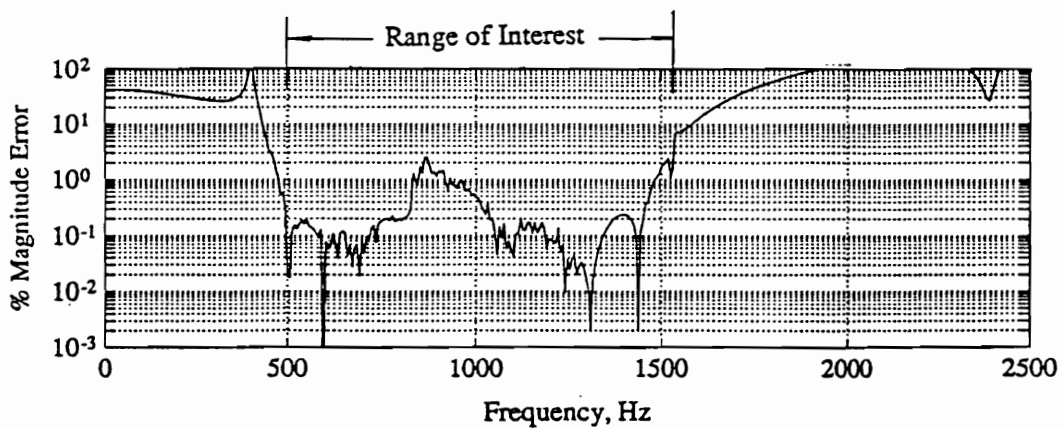


Figure 3.22 Magnitude Mobility Error, 12 Equally Spaced Measurements

not show as good a curve fit even though the high frequency poles are also in the residual model. Regardless of this, the only important part of the fit is in the range of interest and this appears to be very good.

Table 3-9 Error in Iteratively Derived Residues			
Mode	Real Residue	Imaginary Residue	Percent Magnitude Error
592 Hz	137.05	-2.14	0.14%
732	-87.25	-8.65	0.14
1221	-134.34	-6.40	0.03
1431	67.67	0.033	0.15

Since the residues were obtained in the presence of simulated random noise, the values will change whenever a new measurement is taken. For this reason, the residues need to be analyzed from a statistical standpoint. To accomplish this, 100 simulations of the FRF estimate were generated and the residues were extracted. Average residue values and their respective sample variances were calculated using the following equations:

$$\begin{aligned}\bar{X} &= \frac{\sum_{r=1}^{100} A}{100} \\ s^2 &= \frac{\sum_{r=1}^{100} (Y_r - \bar{Y})^2}{99}\end{aligned}\tag{3.23}$$

Table 3-10 shows the extracted residues when no noise is present and when 100 averages of noisy data is taken. The errors between the simulated residues and the calculated residues are also shown. In addition, the standard deviations are listed.

Table 3-10 Residues for Simulated Data					
Mode (Hz)	Noise - Free Residues	Noise - Free Residue Error	100 - Average Residues	100 - Average Residue Error	100 - Average Standard Dev.
592	137.07 - 2.24i	-6.90e-3+3.23e-3i	137.07 - 2.25i	-1.17e-2-1.21e-2i	1.56e-1 + 1.36e-1i
732	-87.29 - 8.65i	9.11e-2 - 1.67e-3i	-87.29 - 8.66i	9.33e-2 - 5.66e-3i	3.14e-2 + 3.11e-2i
1221	-134.34 - 6.42i	-3.79e-2+3.68e-3i	-134.34 - 6.42i	-3.76e-2+3.78e-3i	1.52e-2 + 1.59e-2i
1431	67.66 + 0.097i	4.60e-2 - 1.45e-2i	67.66 + 0.101i	4.21e-2 - 1.06e-2i	2.30e-2 + 2.40e-2i

From Table 3-10 it is evident that even with no random error on the measured data, errors do exist in the determination of the residues. These bias errors are largest for the imaginary terms corresponding to the modes at the extreme frequencies; most likely caused by the residual model. Although the residual model is better than currently used models, it is still only an approximation of the out-of-range modes. When the residues are averaged 100 times in the presence of random noise, the residue values appear to approach the noise-free residue values.

Figures 3.23 through 3.26 show the simulated (true) residues, the iteratively-derived residues with and without noise, and the 95% confidence bounds for 100 samples of data. Generally, the averaged residues from the 100 samples were very close to the noise-free residues. The 100 sample-derived imaginary residue for the 592 Hz mode had the greatest uncertainty, and this was due partly to the location of one of the mobility

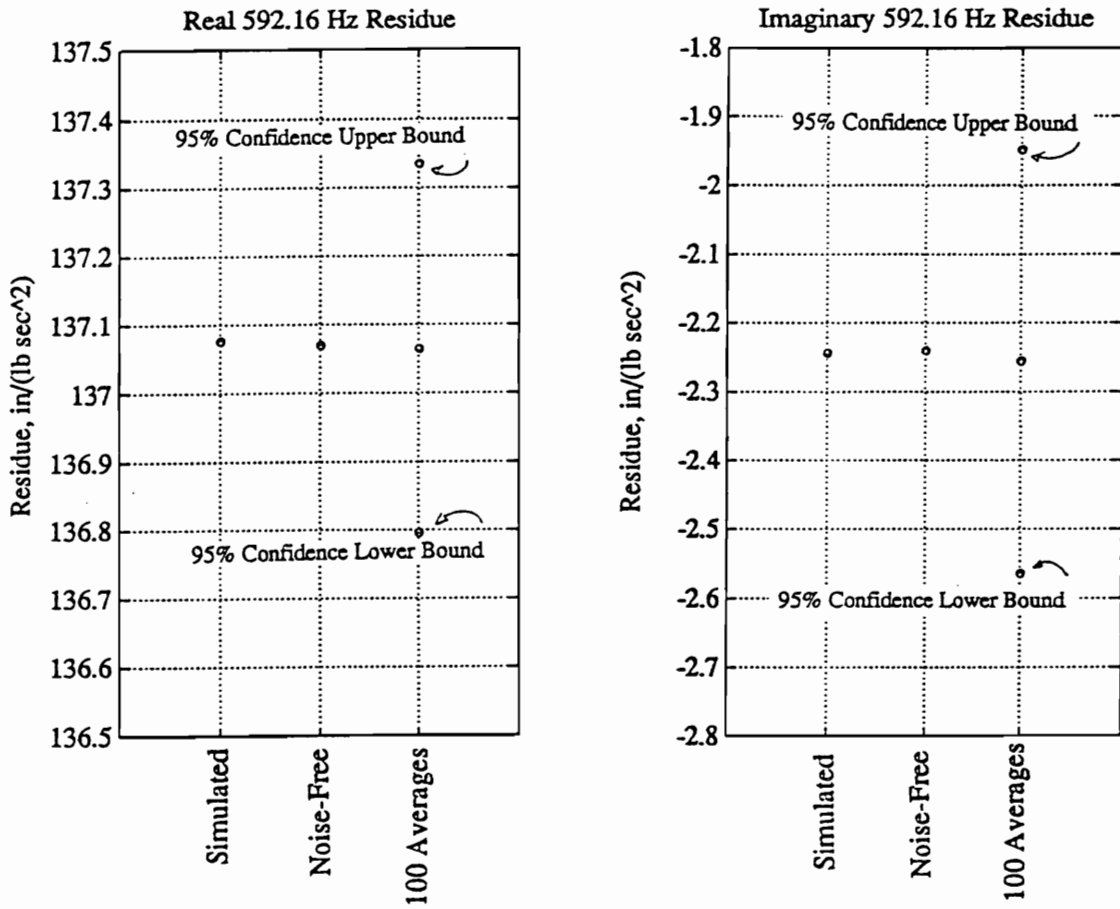


Figure 3.23 Comparison of Simulated Residues, Noise-Free ILS-Derived Residues, and Residues from Noisy Data, 100 Averages

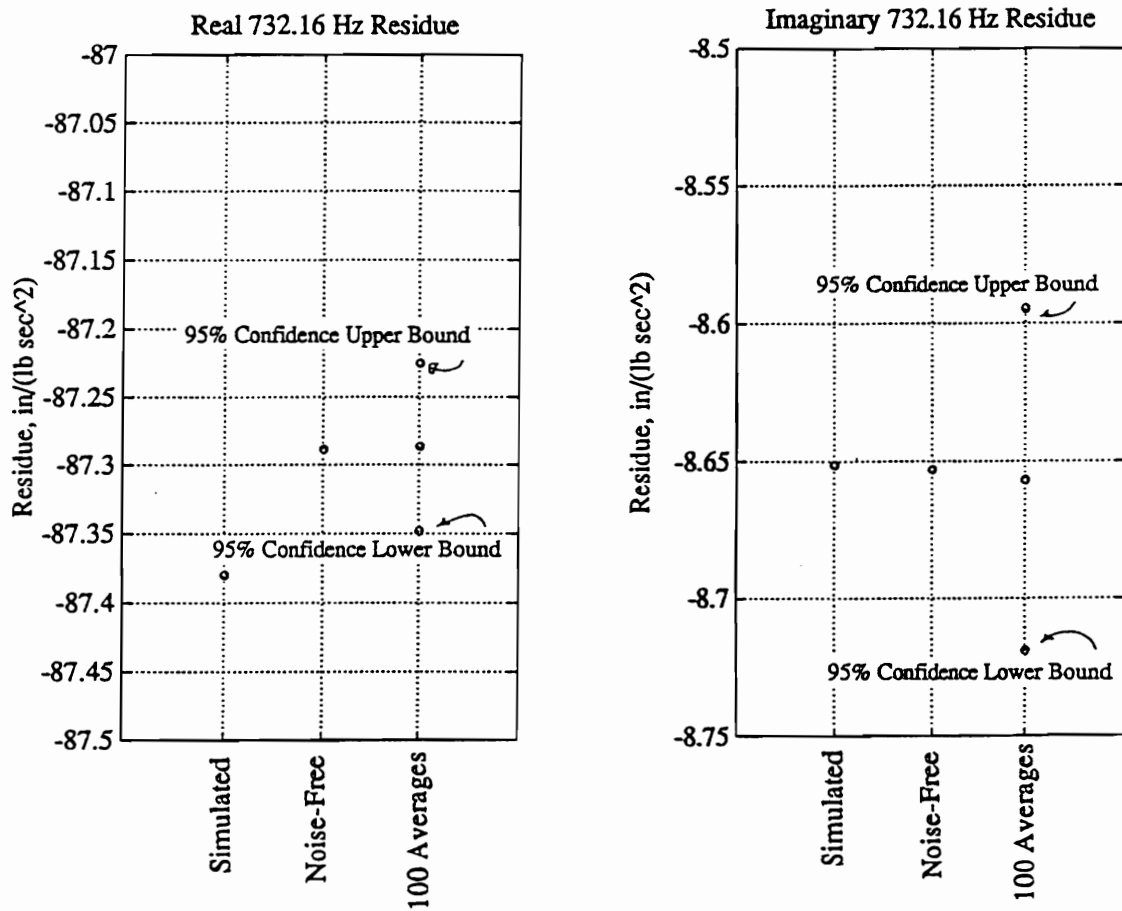


Figure 3.24 Comparison of Simulated Residues, Noise-Free ILS-Derived Residues, and Residues from Noisy Data, 100 Averages

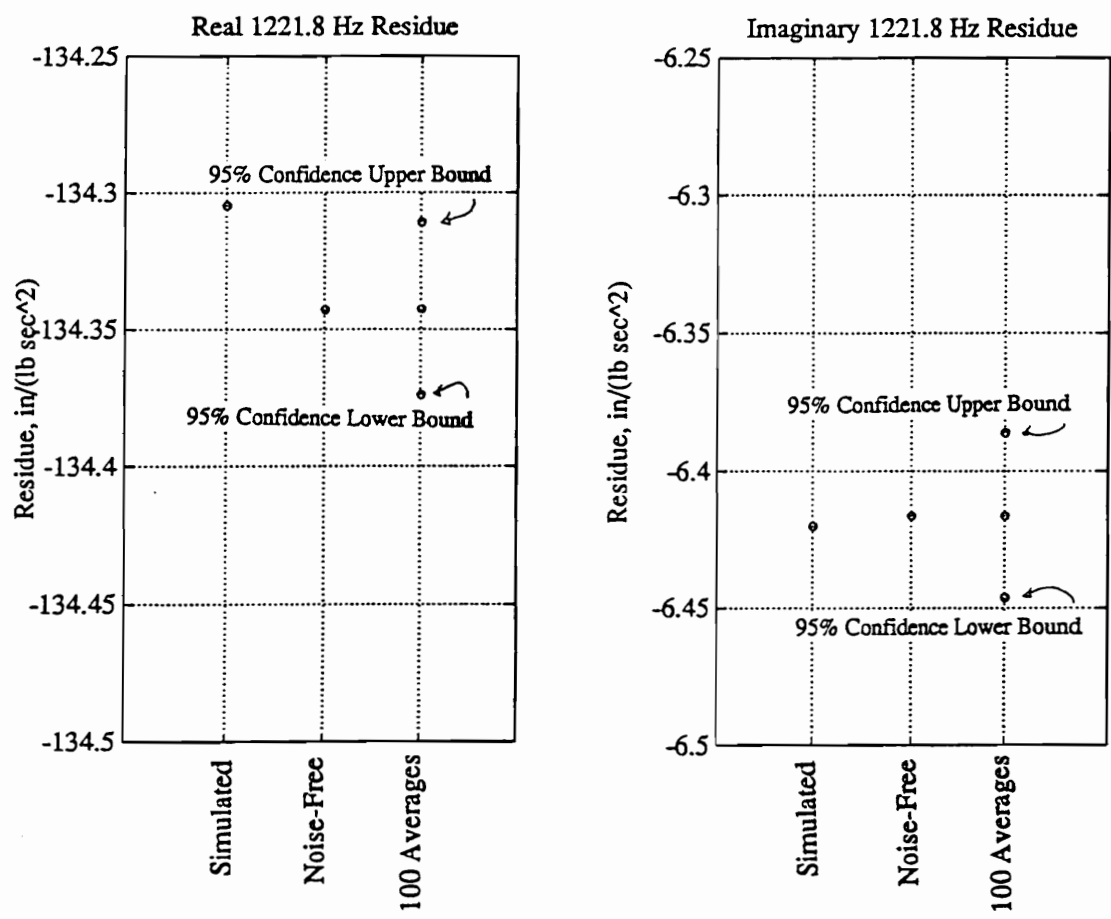


Figure 3.25 Comparison of Simulated Residues, Noise-Free ILS-Derived Residues, and Residues from Noisy Data, 100 Averages

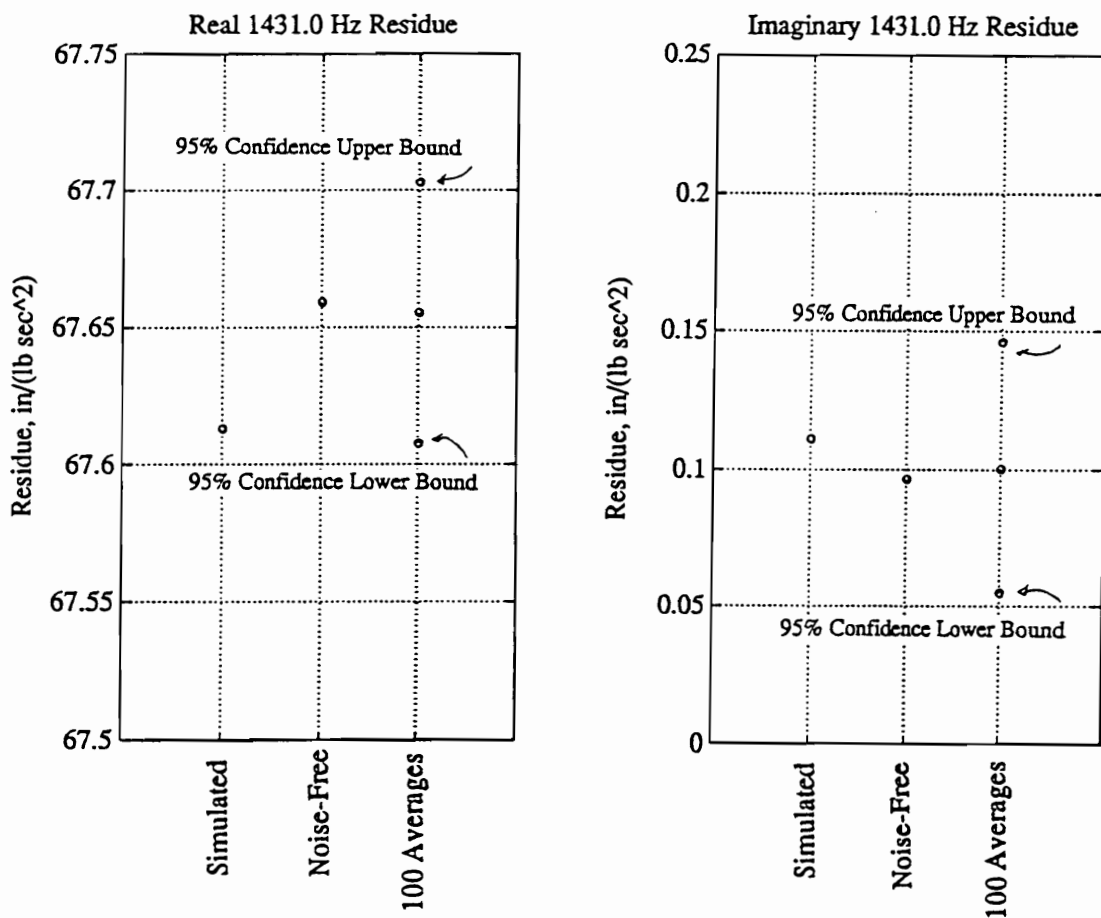


Figure 3.26 Comparison of Simulated Residues, Noise-Free ILS-Derived Residues, and Residues from Noisy Data, 100 Averages

measurements. As shown in Figure 3.21, one of the measurements was directly on the resonance of the mode, making it susceptible to more noise due to force drop-off at resonance. Since this point contributes heavily to the determination of the 592 Hz residue, the measurement error will carry through to the residue.

Given that the residues will have random and bias errors, the confidence bounds shown in Figs. 3.23 - 3.26 can be used to determine the amount of error that is due to random error, or noise. Table 3-11 shows the differences between the average values and the 95% confidence limit values for the real and imaginary residues (only one-half the interval is considered). Note that for each residue, the values are about the same for the real and imaginary parts which indicates that noise equally affects the real and imaginary residues. Taking the magnitude of these values, dividing by the nominal residue magnitude and multiplying by 100 gives a 95% confidence level of percent magnitude error due to noise. As shown in Table 3-11, the largest error due to noise occurs for the 592 Hz mode with 0.30% error.

From these results, it can be determined that residue errors larger than 0.30% are most likely bias errors caused by something other than noise. This will be a helpful result in the analyses to follow.

Table 3-11 Random Error Content in Extracted Residues				
Mode	95% Confidence Bound	100-Average Residue	Difference	95% Confidence Percent Magnitude Error
592 Hz	137.33 - 1.95i	137.07 - 2.26i	0.27 + 0.31i	0.30 %
732	-87.22 - 8.59i	-87.29 - 8.66i	0.062 + 0.062i	0.10
1221	-134.31 - 6.39i	-134.34 -6.42i	0.032 + 0.030i	0.032
1431	67.70 + 0.15i	67.66 + 0.10i	0.048 + 0.046i	0.097

3.5.1 The Effects of the Number of Mobility Measurements

Increasing the number of measurements to 30 results in a decrease in the maximum magnitude residue error, from 0.15% to 0.07%. The residue errors and the regenerated FRF are shown in Figures 3.27 and 3.28, respectively, and the mobility error is shown in Figure 3.29. Comparing Figures 3.20 and 3.27 show that the real benefit of using additional measurements is realized in the rate of convergence. Four iterations are necessary in the 30-measurement case to achieve a rate of change in the residues of less than 0.1%, whereas seven iterations are needed to achieve this rate in the 12-measurement case. The regenerated FRF's show nearly the same error over the range of interest, although since these errors are for one sample of data, a statistical analysis will be helpful in determining if there is any benefit in the 30-measurement curve fit.

Simulating 25 mobility sample sets for the 12- and 30-measurement cases and regenerating the FRF for each sample allows the error variances for the cases to be determined. Figure 3.30 shows the error variances for both cases as well as the 95% confidence intervals. Unlike the previous error plots, these show the variances determined from the difference of the regenerated FRF and the noise-free FRF. The confidence interval (based on a chi-square Function) has not significantly changed as a result of the increased number of measurements. This may be expected since the noise level is the same. More important is that the increased number of measurements has not increased or decreased the overall error level, but has redistributed the levels in the frequency domain as seen in Figure 3.31. From these results, it appears that 12 measurements are adequate in finding the residues.

It should be noted that using the minimum number of measurements (10) for over determination will cause a significant increase in the regenerated FRF error. Eleven measurements has about the same magnitude of error as the 12 measurement case, so it is

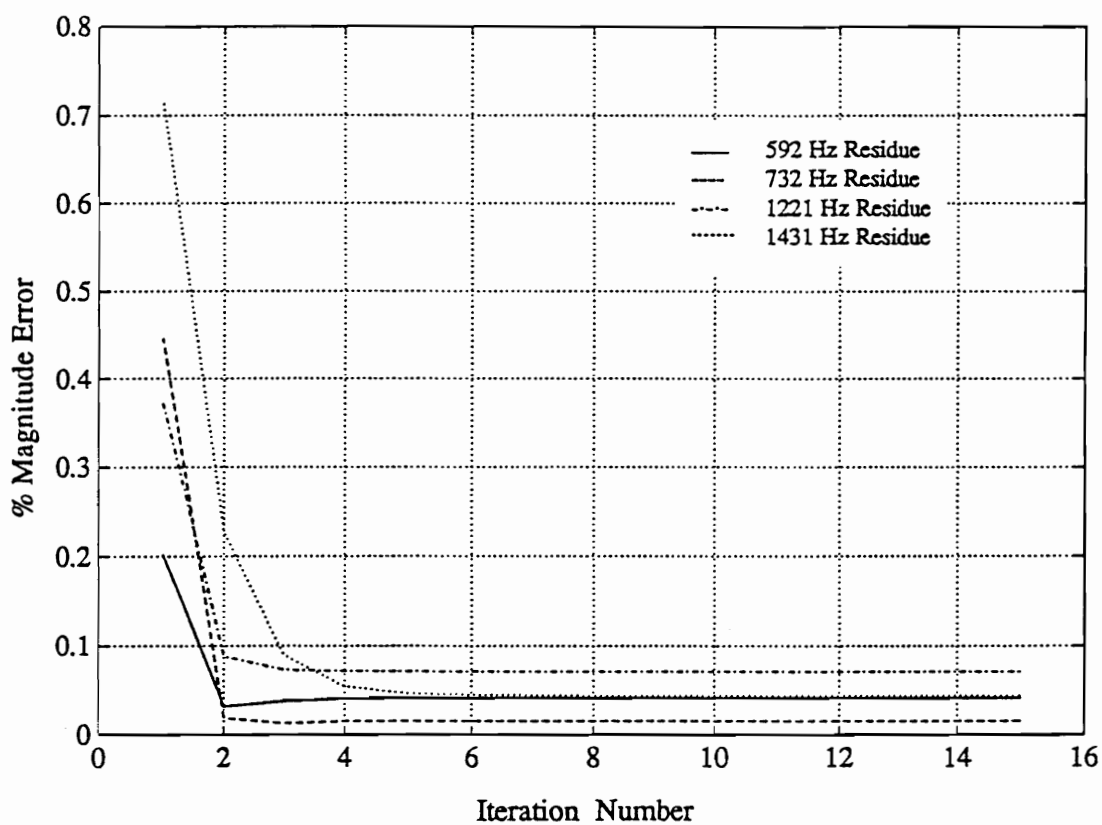


Figure 3.27 Convergence of Residues for 30 Measurements

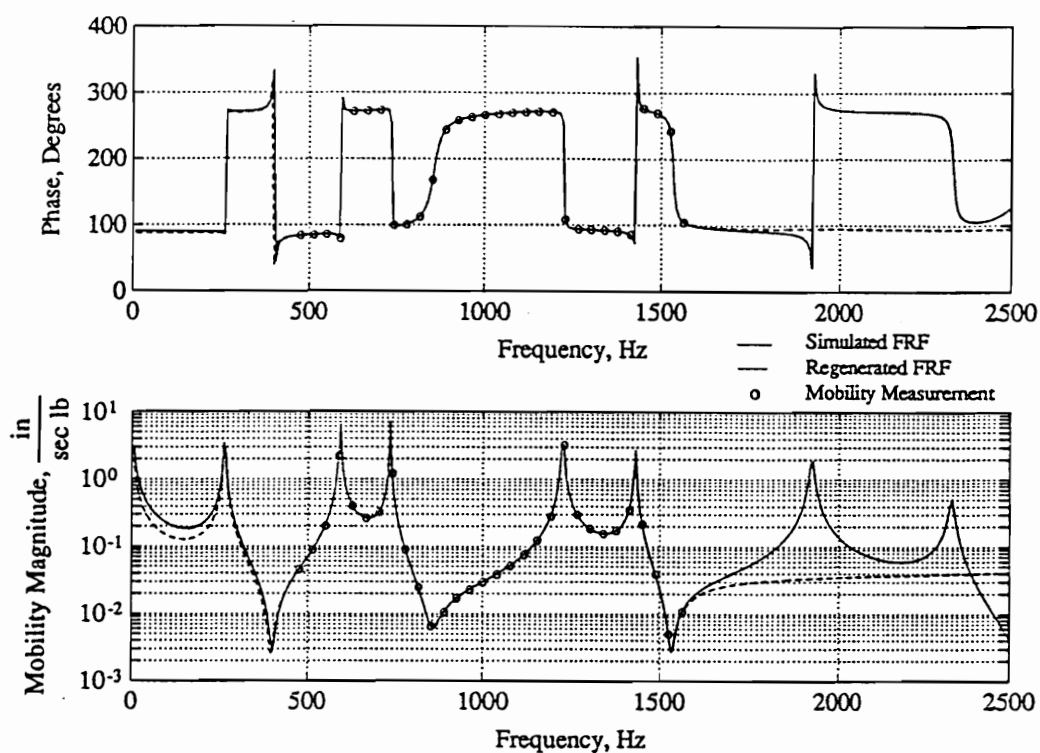


Figure 3.28 Simulated and Regenerated FRF's Using ILS, 30 Equally Spaced Measurements

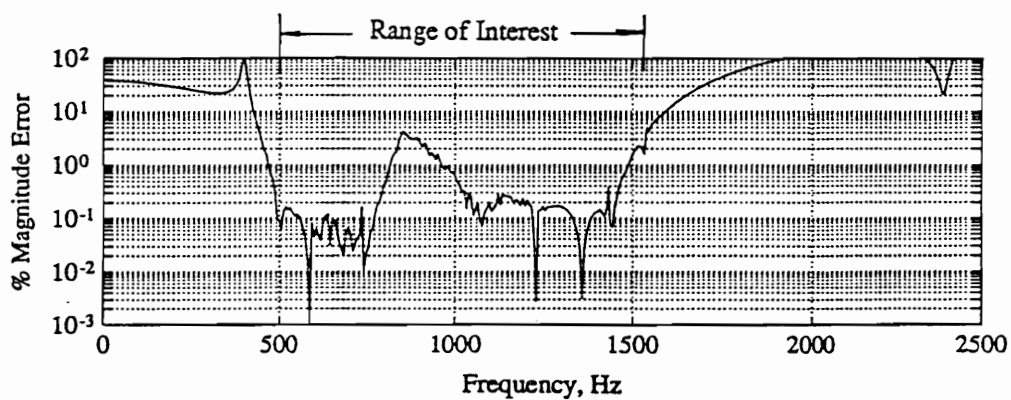


Figure 3.29 Magnitude Mobility Error, 30 Equally Spaced Measurements

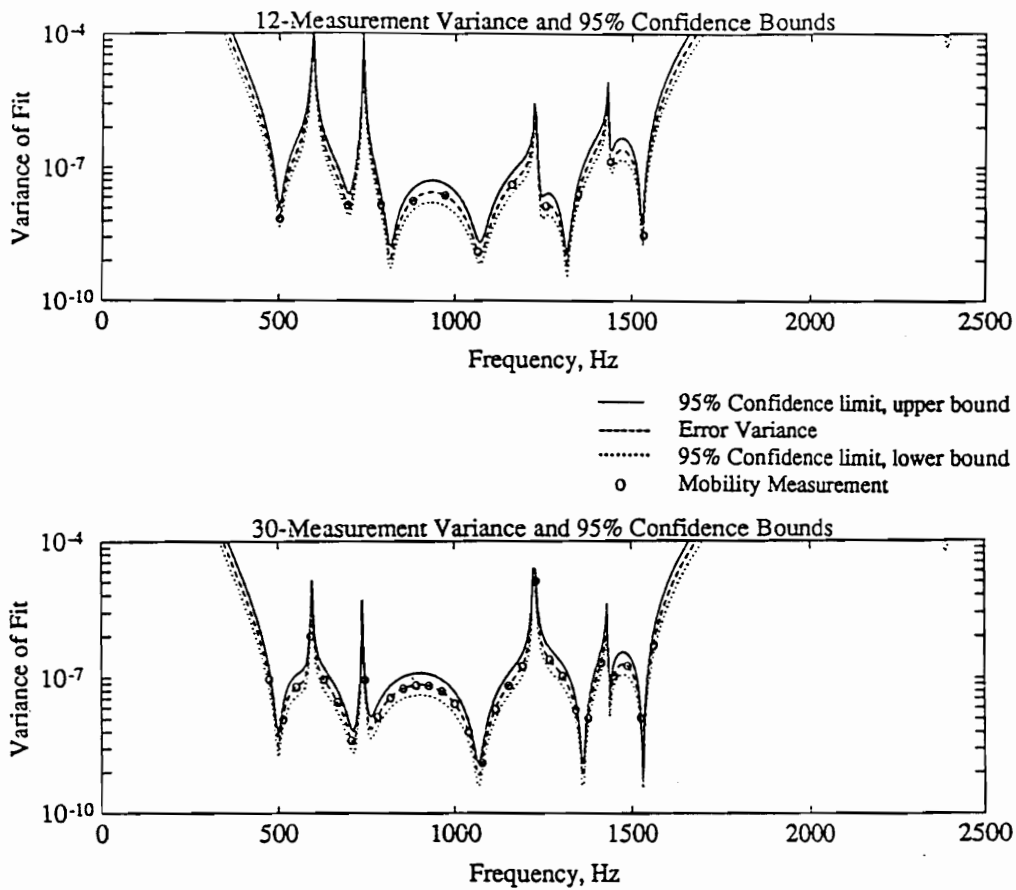


Figure 3.30 Error Variance and 95% Confidence Intervals for 12- and 30-Measurement Cases

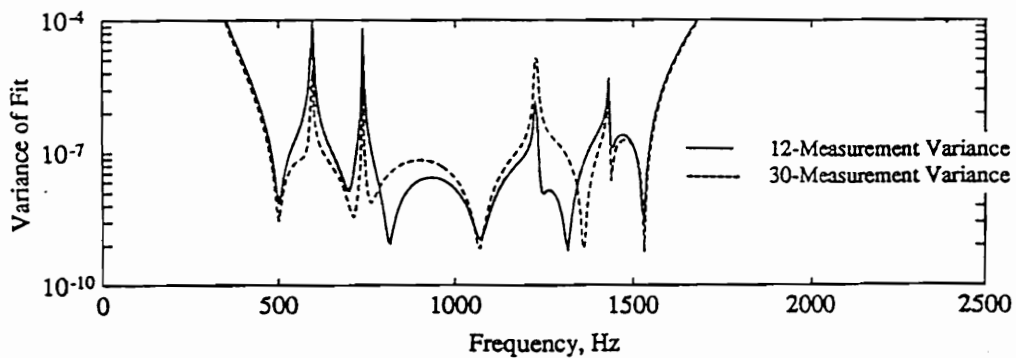


Figure 3.31 Twelve- and 30-Measurement Error Variances from Simulated and Regenerated Data

recommended to use at least one more measurement than the minimum for over determination.

3.5.2 The Effects of Measurement Frequency Location

Previously it was shown that measuring the mobility away from a resonance would reduce errors due to variations in the global parameters. For the iterative technique, convergence speed must also be considered when picking frequencies. To examine the effects of measurement location on convergence and accuracy, the mobility measurements are biased near and away from the resonances in the previously used multi-DOF simulation.

Biasing near resonance, 12 measurements are taken around the resonance peaks as shown in Figure 3.32. At five iterations the residues are changing less than 0.1%; Figure 3.33 shows the convergence and the maximum residue error of 0.21%. Measuring away from the resonances, Figure 3.34 shows 12 anti-resonance measurement locations and Figure 3.35 show the convergence. The final maximum residue error is 0.33%, slightly higher than the on-resonance case with nine iterations necessary for convergence. These results indicate that measurements near resonance will speed up the convergence. From the final residue errors, bias errors are not likely to exceed 0.33% in either case. As previously shown, if the global parameters are in error it is advantageous to measure the mobilities away from the resonances and sacrifice the faster convergence rates.

The presence of natural frequency and damping errors will be a factor in selecting the measurement frequencies, as was shown in Section 3.4. Three different measurement

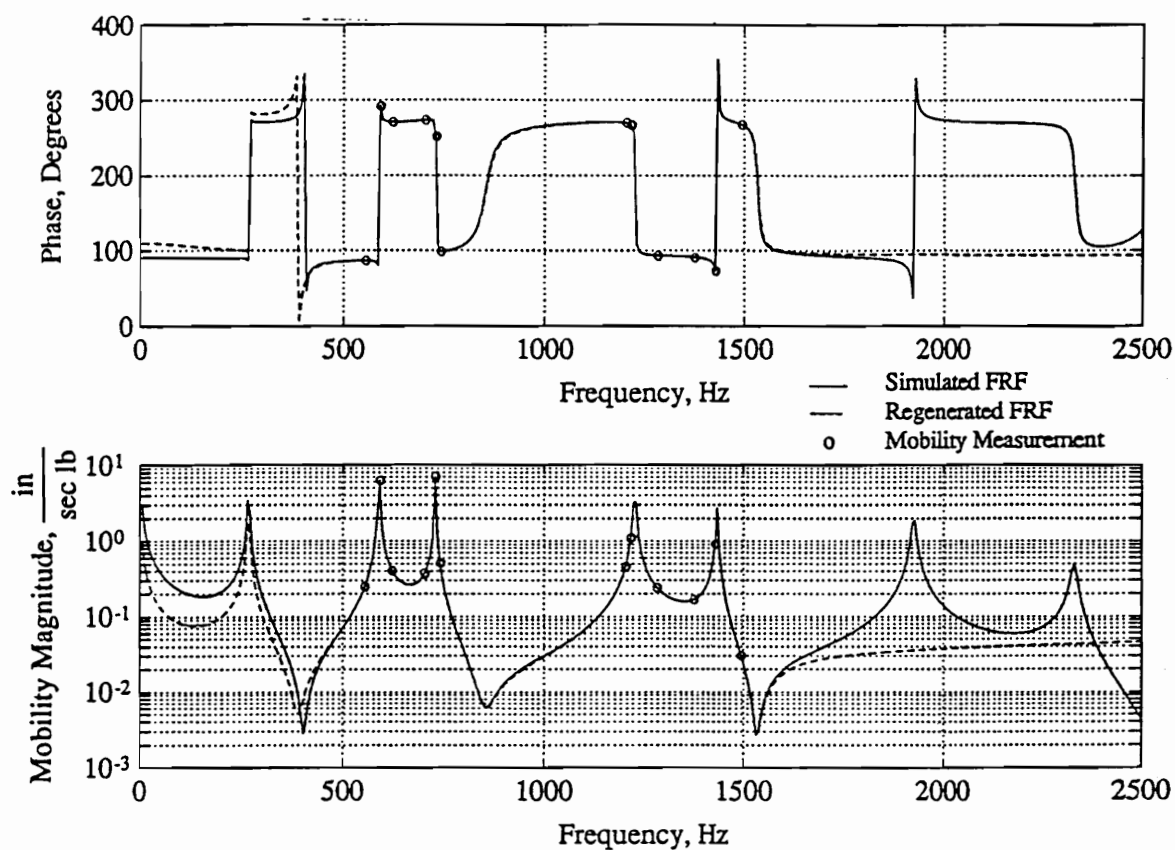


Figure 3.32 Simulated and Regenerated FRF's Using ILS, 12 Measurements Biased at Resonances

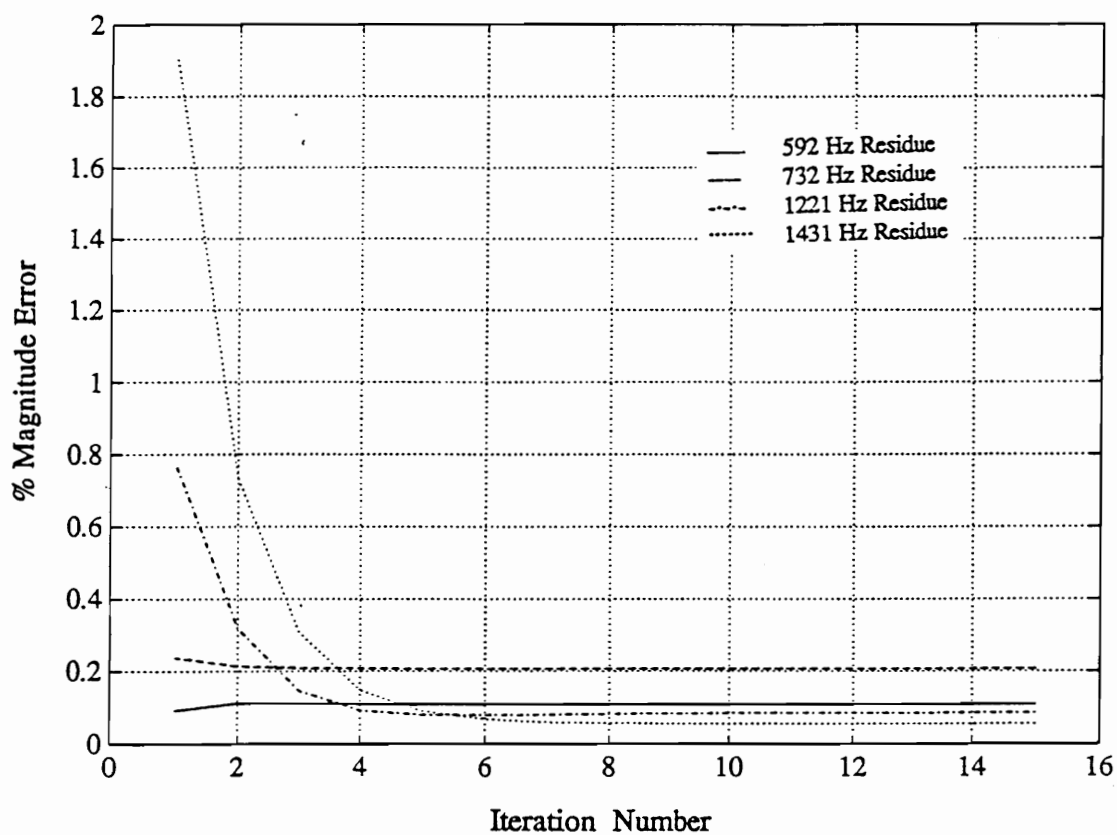


Figure 3.33 Convergence of Residues for 12 Measurements Biased Near Resonance

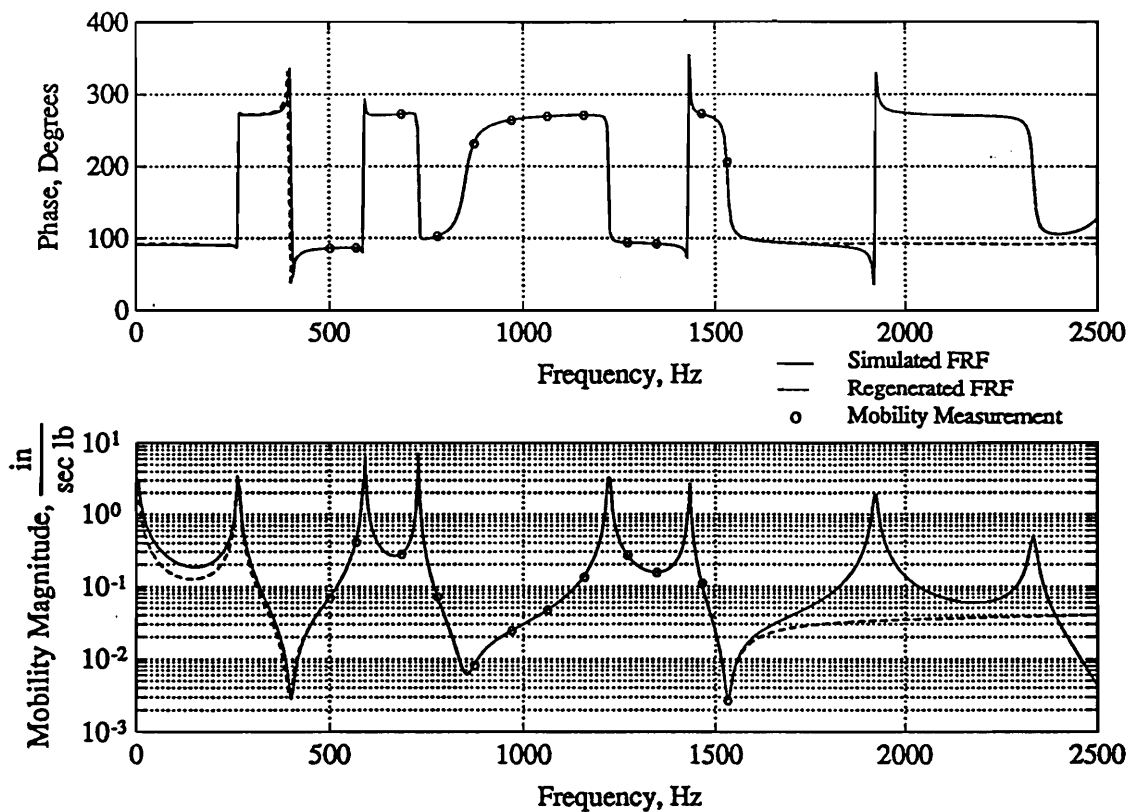


Figure 3.34 Simulated and Regenerated FRF's Using ILS, 12 Measurements Biased at Anti-Resonances

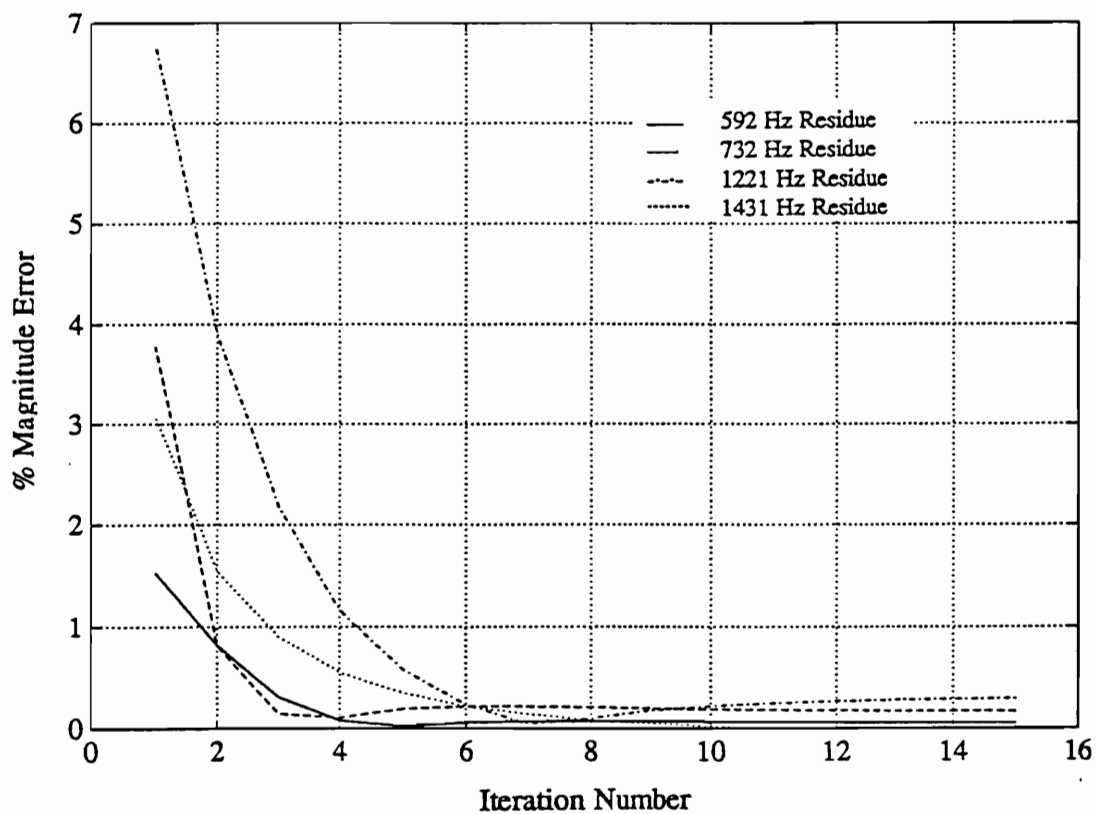


Figure 3.35 Convergence of Residues for 12 Measurements Biased Away from Resonances

sets will be simulated to investigate natural frequency and damping error effects: the baseline case with evenly spaced measurements, the anti-resonance bias case and resonance bias case. Each case is subjected to natural frequency variations of $\pm 0.01\%$ with the damping held constant. Additionally, with the natural frequencies exact, the damping coefficients are varied $\pm 50\%$. The 592 Hz and 1221 Hz modes will have positive errors and the 732 Hz and 1431 Hz modes will have negative errors. Twelve measurements are used on the previously presented MDOF system.

Although the variation in natural frequency is very low, it is the same order of variation that was observed in the real system. With low damping, a very exact determination of the natural frequencies is possible from the curve fitter.

For the case with no bias in the measurements, varying the natural frequencies led to the FRF fit shown in Figure 3.36. Figure 3.37 shows the residue errors converging after about seven iterations with the maximum magnitude error being 3.4%. Although the errors in the natural frequencies are small, the light damping of the structure effectively amplifies the resulting mobility errors, causing greater error in the residues. If the measurements are biased near the resonances, the residues are more sensitive to incorrect natural frequencies causing a regenerated FRF with greater error as shown in Figure 3.38. Figure 3.39 presents the residue error and shows changes of less than 0.1% after eight iterations and a final maximum residue error of 7.2%. Once again, errors greater than 0.3% are most likely bias errors and are predominant in this simulation.

Biasing the measurements away from the resonances will minimize the error due to natural frequency variations. Figure 3.40 shows the simulated and regenerated FRF's when the measurements are away from the resonances, and Figure 3.41 shows the residue errors with a maximum error of 0.70%. Note that the error in the 1431 Hz residue increases after iteration six, similar to what happened in Figure 3.35 where the

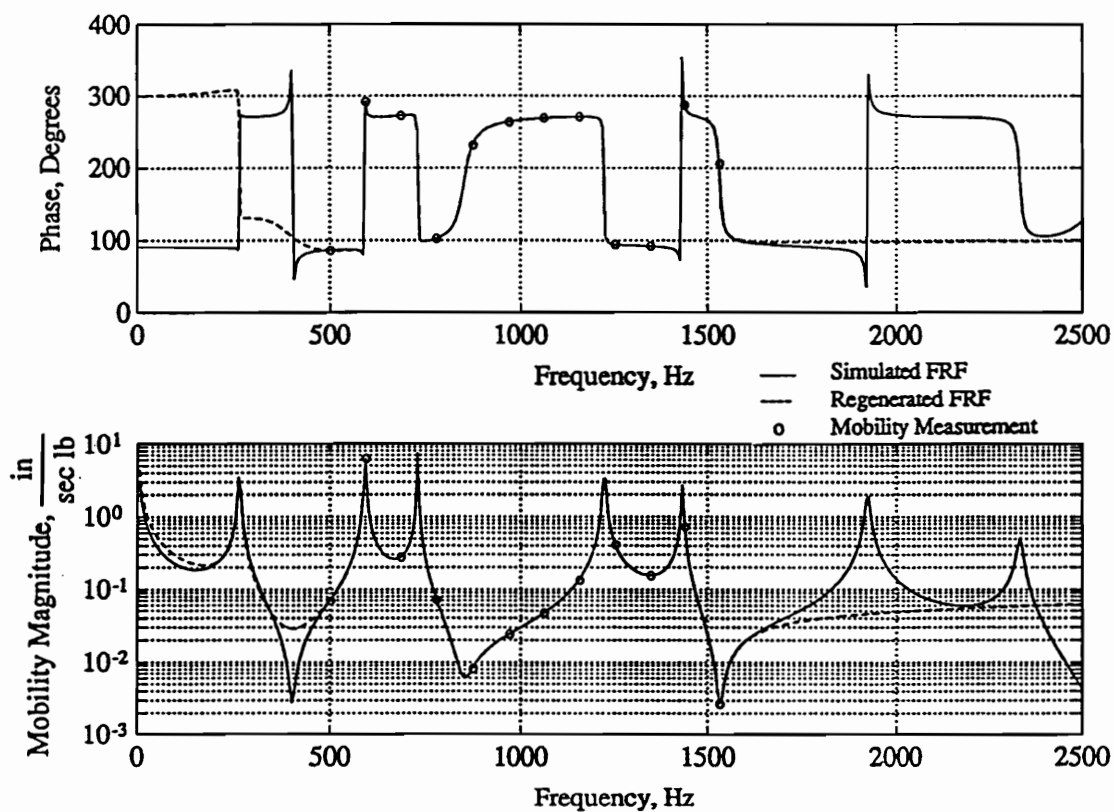


Figure 3.36 Simulated and Regenerated FRF's Using ILS, 12 Equally Spaced Measurements and Global Natural Frequencies Varied $\pm 0.01\%$

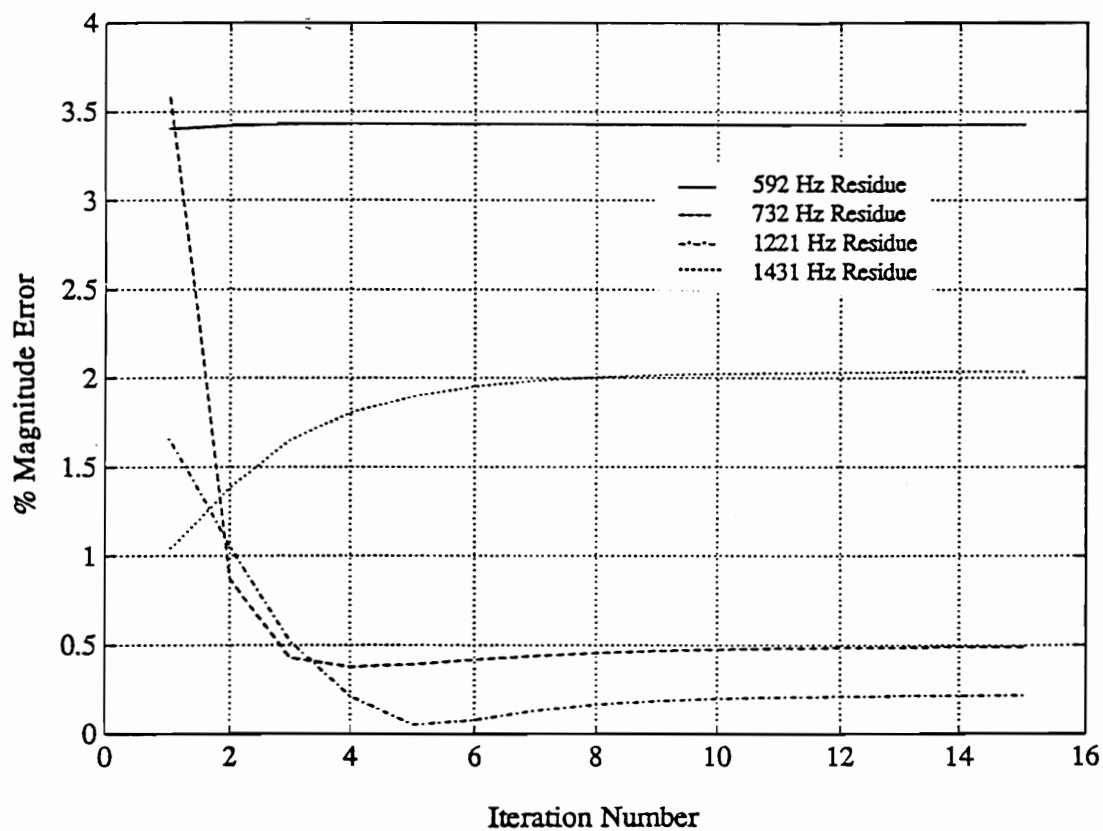


Figure 3.37 Convergence of Residues for 12 Equally Spaced Measurements and Global Natural Frequencies Varied $\pm 0.01\%$

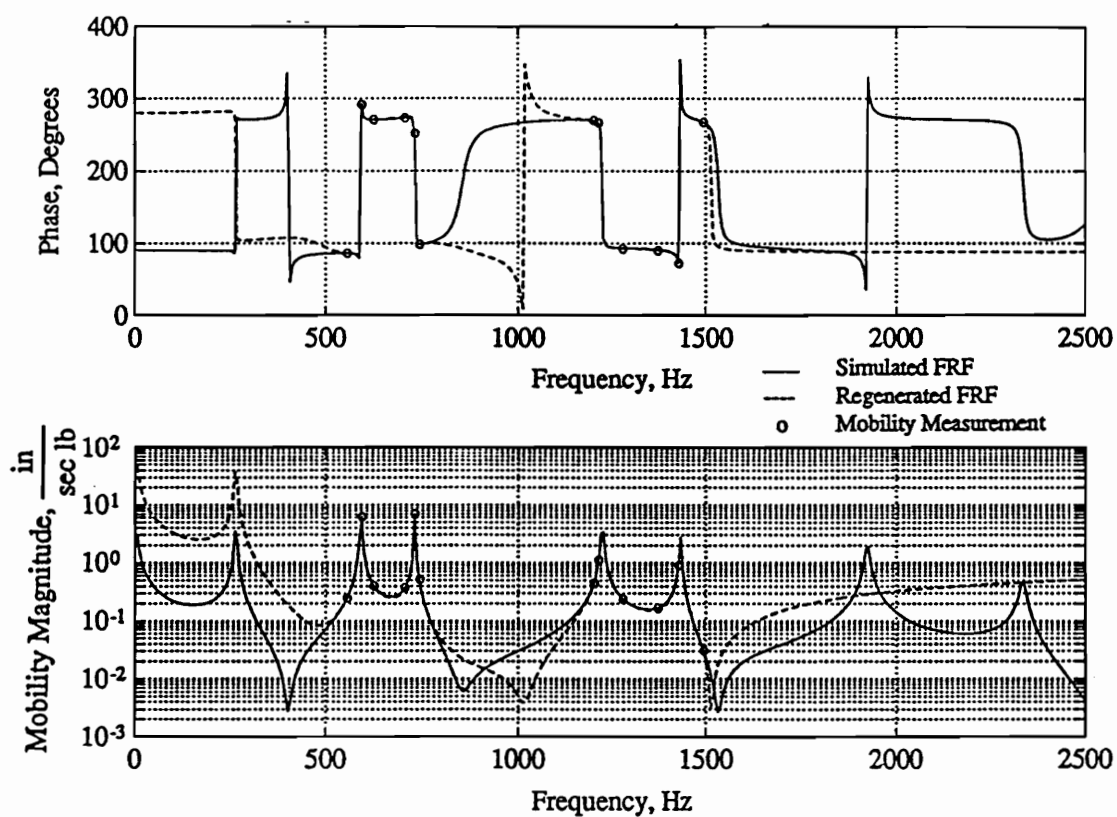


Figure 3.38 Simulated and Regenerated FRF's Using ILS, 12 Measurements Biased Near Resonances and Global Natural Frequencies Varied $\pm 0.01\%$

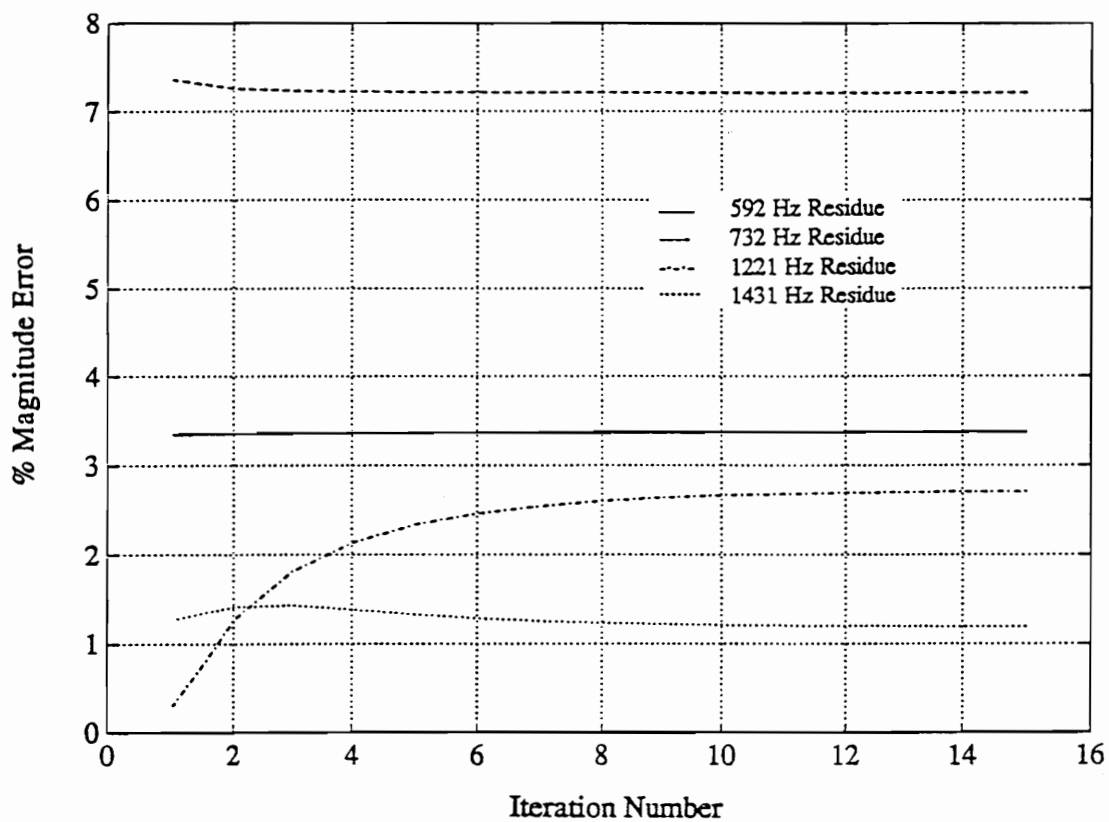


Figure 3.39 Convergence of Residues for 12 Measurements Biased Near Resonances and Global Natural Frequencies Varied $\pm 0.01\%$

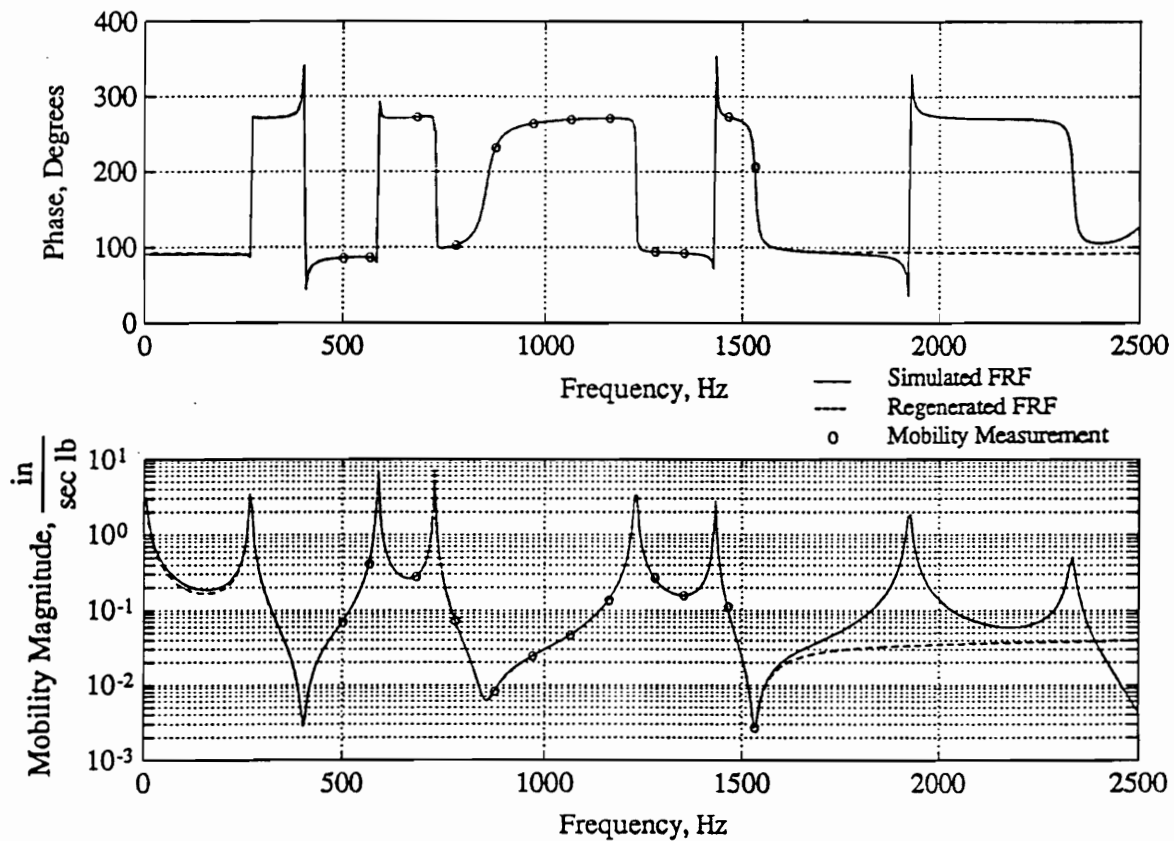


Figure 3.40 Simulated and Regenerated FRF's Using ILS, 12 Measurements Biased Away From Resonances and Global Natural Frequencies Varied $\pm 0.01\%$

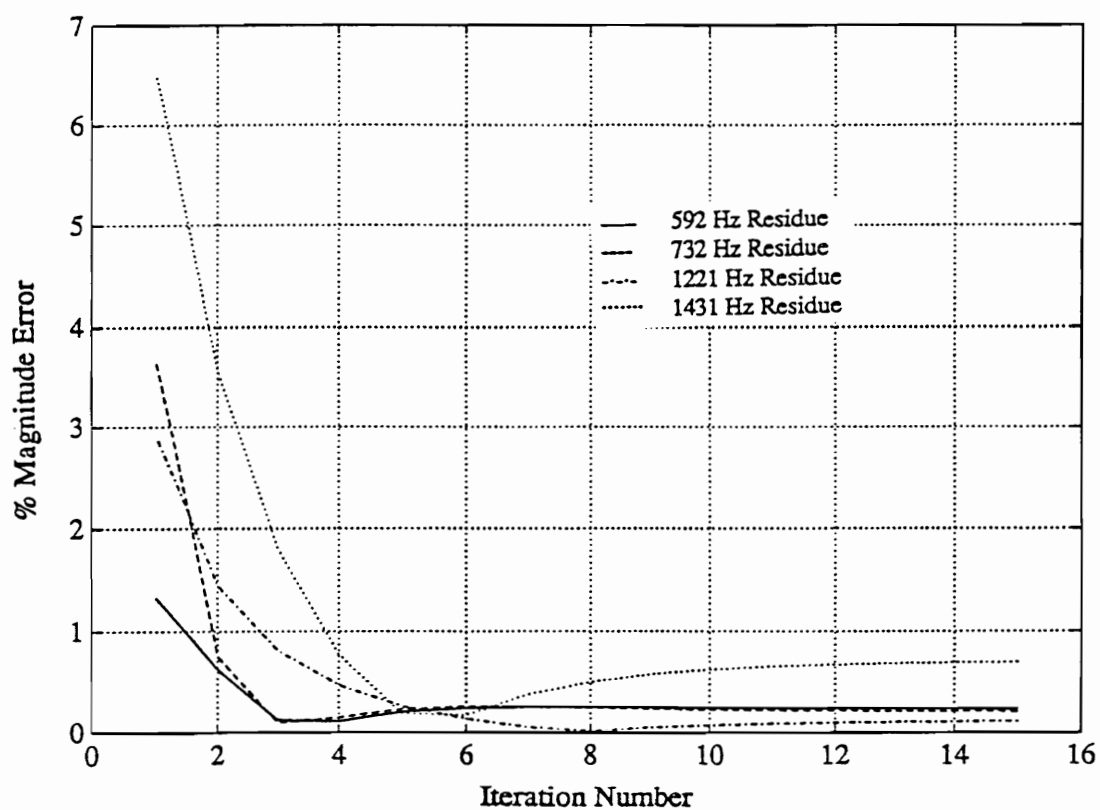


Figure 3.41 Convergence of Residues for 12 Measurements Biased Away From Resonances and Global Natural Frequencies Varied $\pm 0.01\%$

measurements were also away from the resonances. This is caused by the similarity in residual behavior with the in-range mode behavior at the saddles in the FRF. The distinction between the models is lost at the measurement points and a "drift" will appear in the convergence. Nevertheless, the error in the residues is significantly less than when the measurements were made near the resonances.

When damping is varied in the case of 12 evenly spaced measurements, the resulting maximum error in the residues is 20%. Concentrating the measurements near the resonances results in about the same error (19.7%) which is shown in Figure 3.42. In this case, the damping variations have a significant effect on the mobility amplitudes around the resonances due to the form of the mobility equation. The amplified mobility error propagates to a large error in the residues. Figure 3.43 shows the original and regenerated FRF's, with a particularly bad fit evident in the regenerated curve. Moving the measurements away from the resonances results in a maximum residue error of 2.6%, shown in Figure 3.44, and a much better fit of the regenerated FRF shown in Figure 3.45. Biasing the measurements away from the resonances results in smaller residue errors due to damping errors.

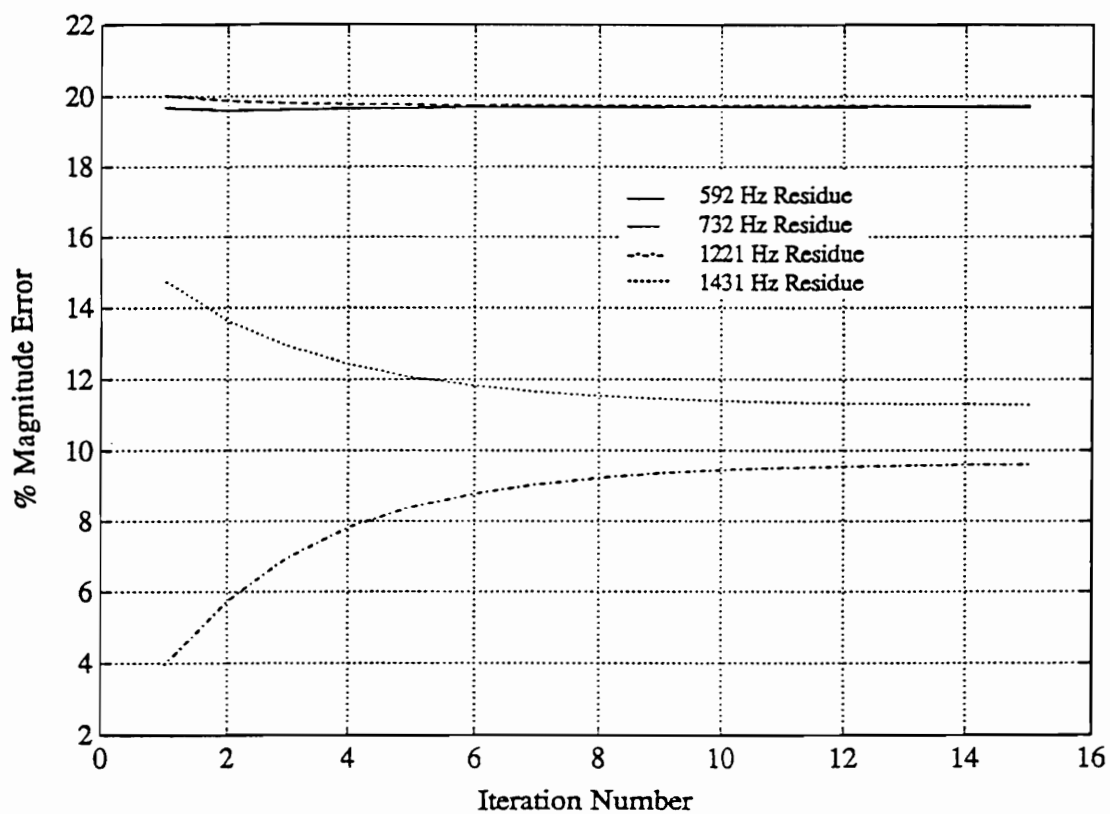


Figure 3.42 Convergence of Residues for 12 Measurements Biased Near Resonances and Damping Varied $\pm 50\%$

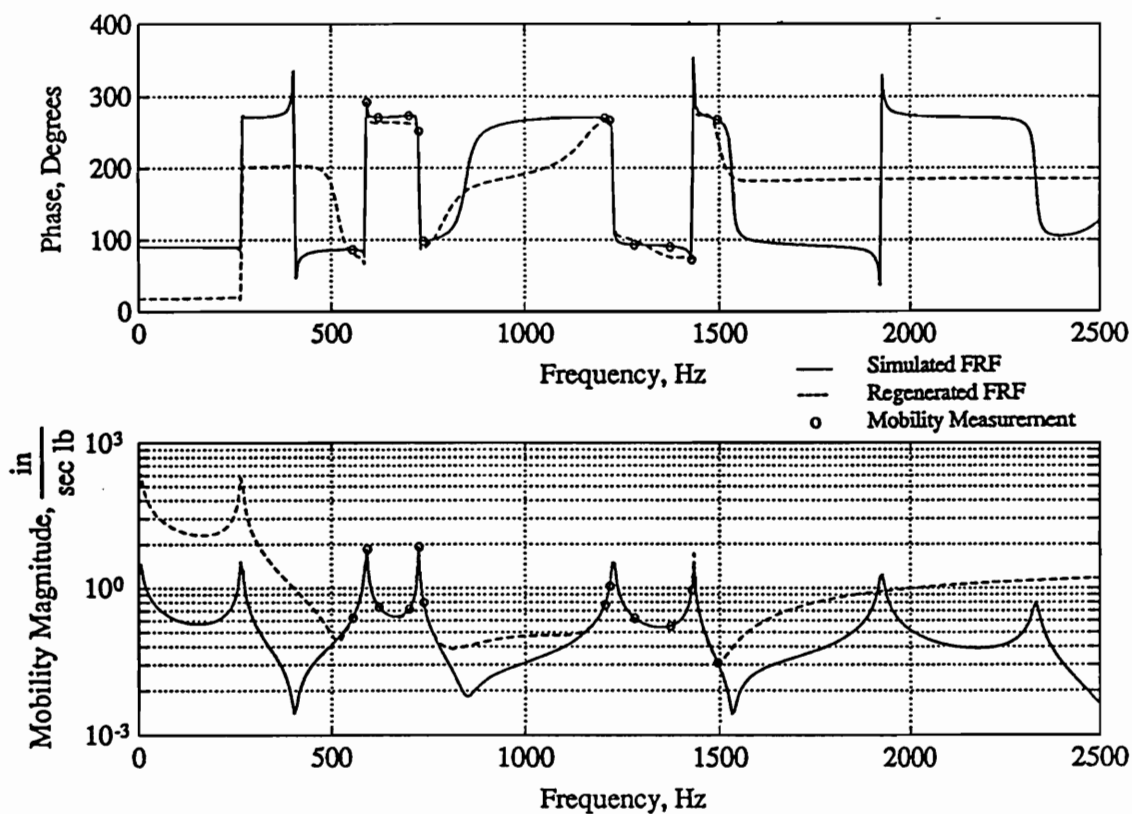


Figure 3.43 Simulated and Regenerated FRF's Using ILS, 12 Measurements Biased Near Resonances and Damping Varied $\pm 50\%$

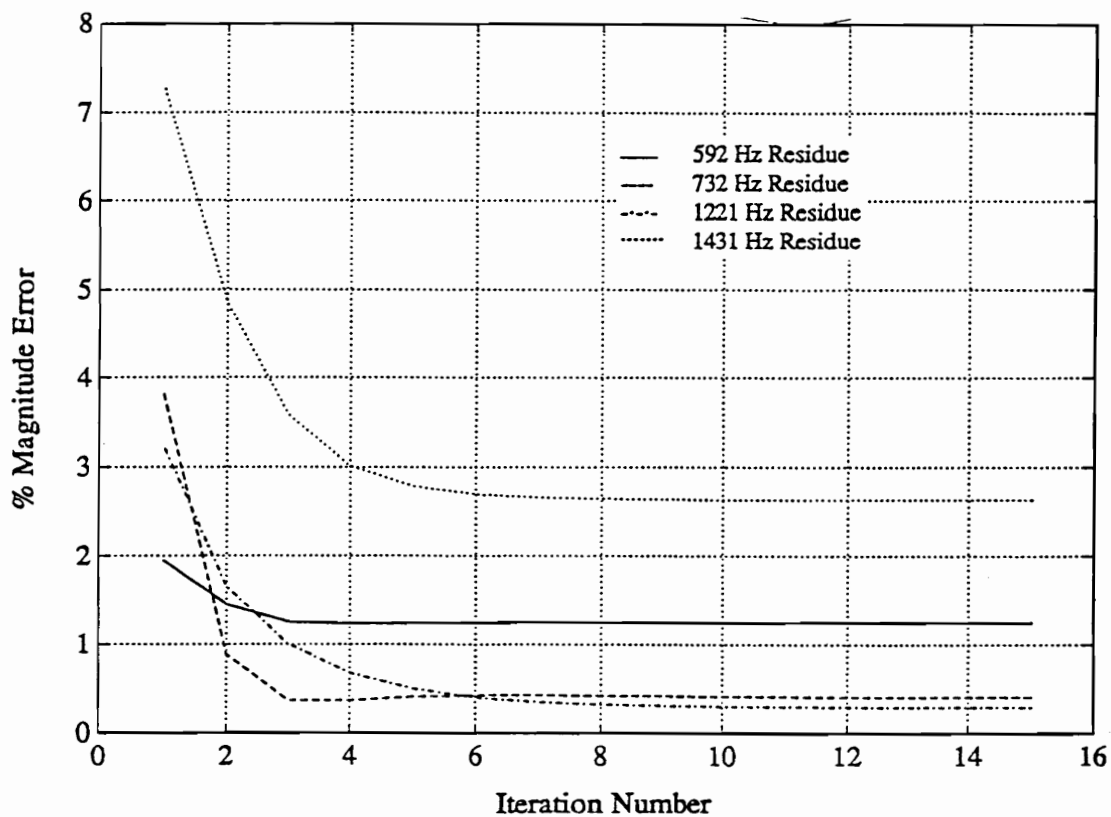


Figure 3.44 Convergence of Residues for 12 Measurements Biased Away From Resonances and Damping Varied $\pm 50\%$

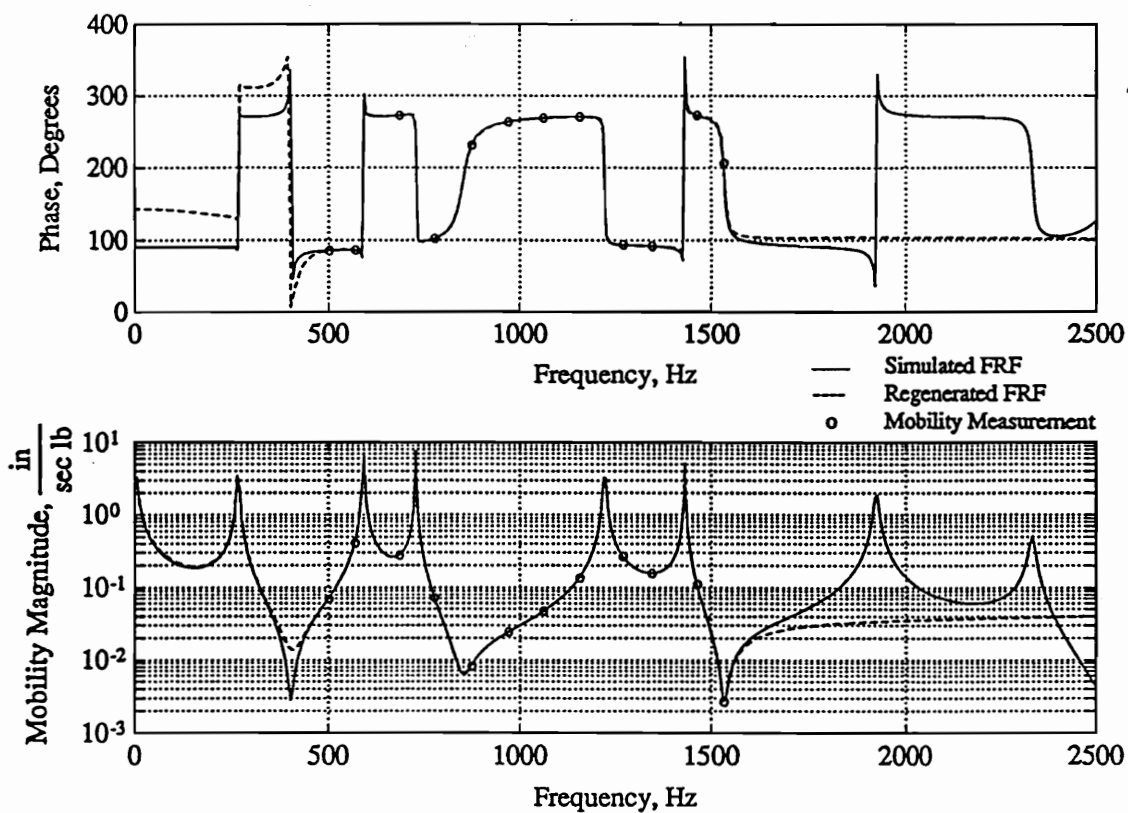


Figure 3.45 Simulated and Regenerated FRF's Using ILS, 12 Measurements Biased Away From Resonances and Damping Varied $\pm 50\%$

3.5.3 The Effect of Closely Spaced Modes

When modes are not well separated, biasing the measurements around the close modes will improve the curve fit. Figure 3.46 shows the previously used MDOF system with the third mode (1221 Hz) being moved to 1410 Hz, only 21 Hz away from the fourth mode. The 12 measurement frequencies shown in Table 3-3 and the regenerated FRF are also indicated in Figure 3.46, and the residue error as a function of iteration number is shown in Figure 3.47. Modes three and four (the close modes) have significantly more error than modes one and two, and the error is increasing slightly after 45 iterations. The convergence does not appear stable in this case. Biasing four measurements around the close modes results in faster convergence which is stable and has less error as shown in Figure 3.48. The residue error is changing less than 0.1% after five iterations and the final maximum error is 0.22%. Figure 3.49 shows the simulated FRF and the regenerated FRF.

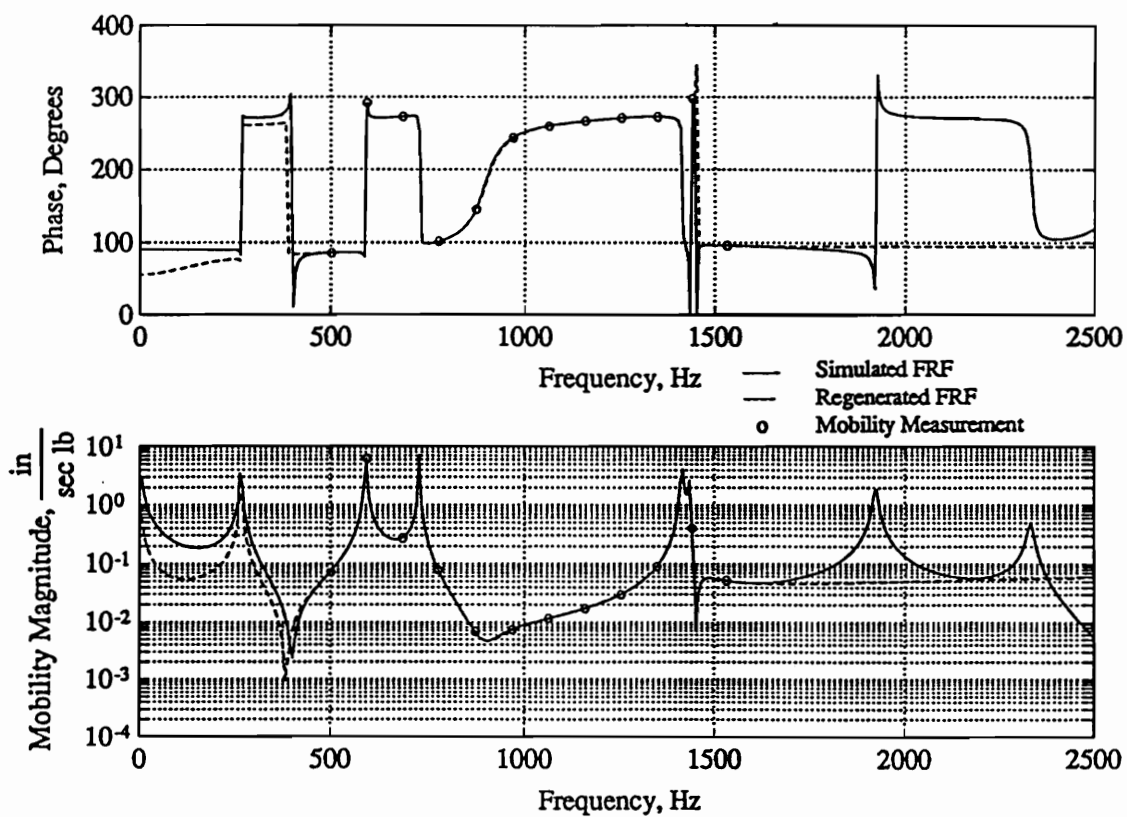


Figure 3.46 Simulated and Regenerated FRF's Using ILS, 12 Equally Spaced Measurements With Two Modes Close in Frequency

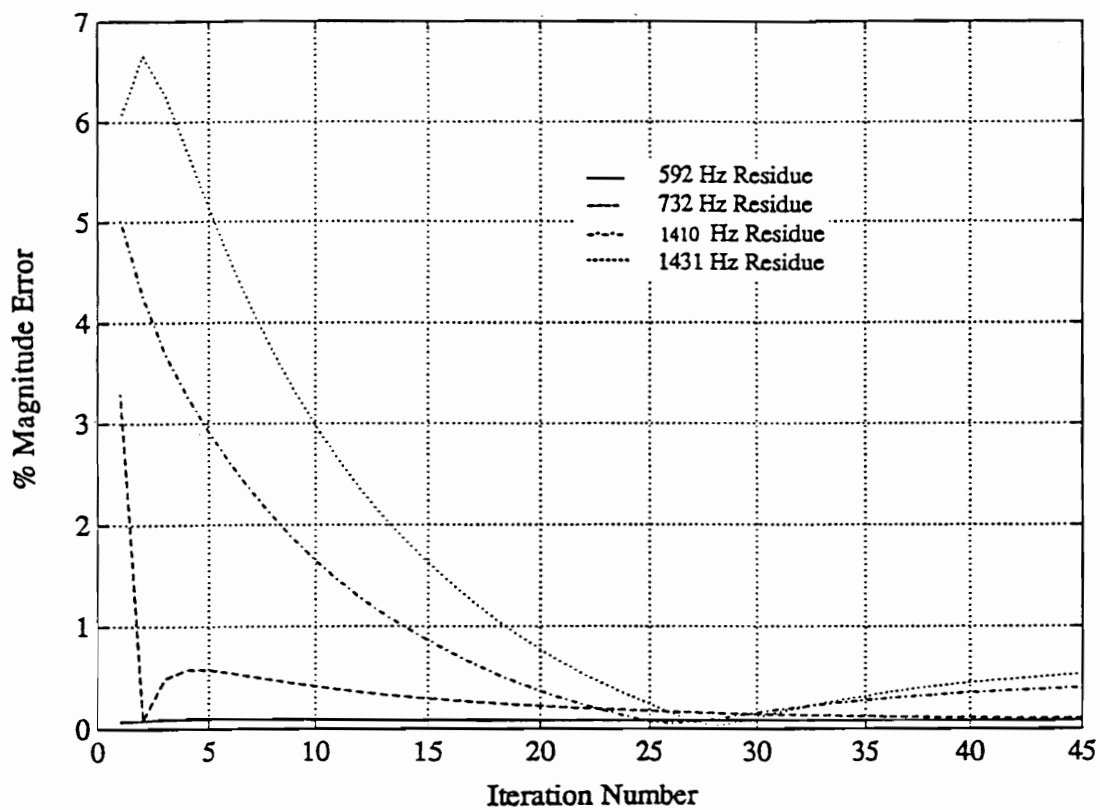


Figure 3.47 Convergence of Residues for 12 Equally Spaced Measurements and Two Modes Close in Frequency

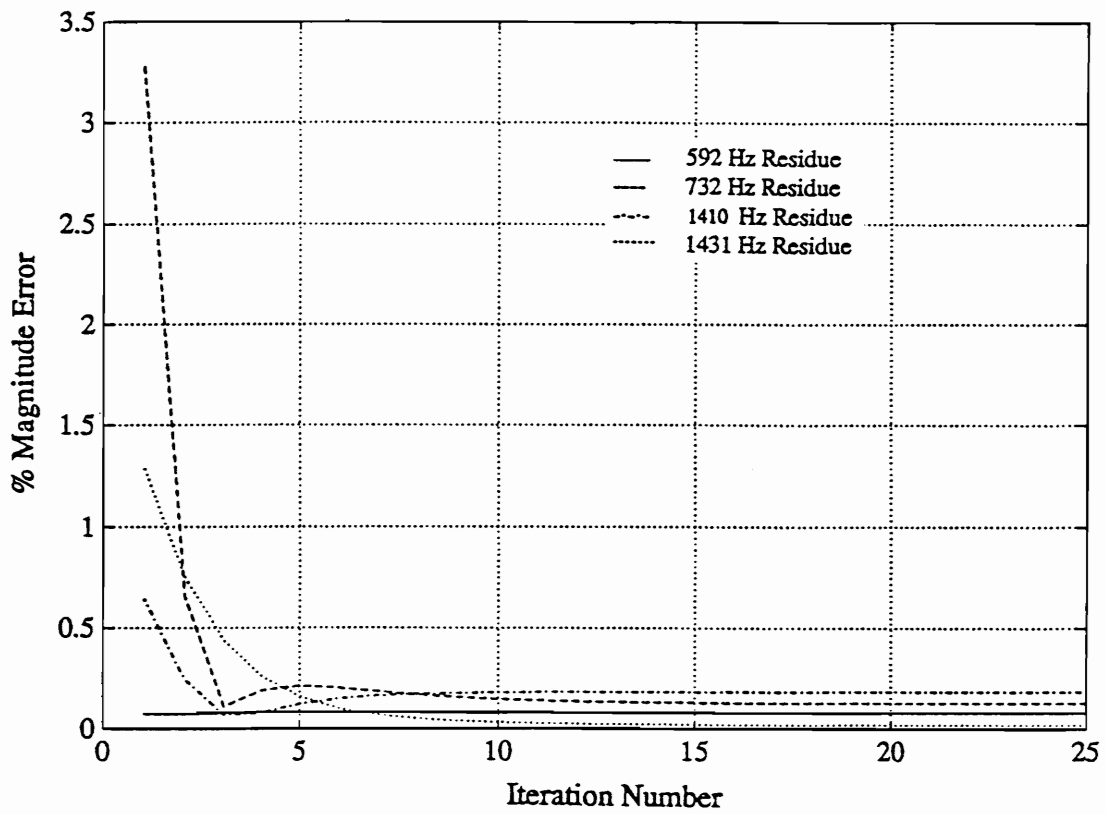


Figure 3.48 Convergence of Residues for 12 Measurements, Four Measurements Biased Near Modes Close in Frequency

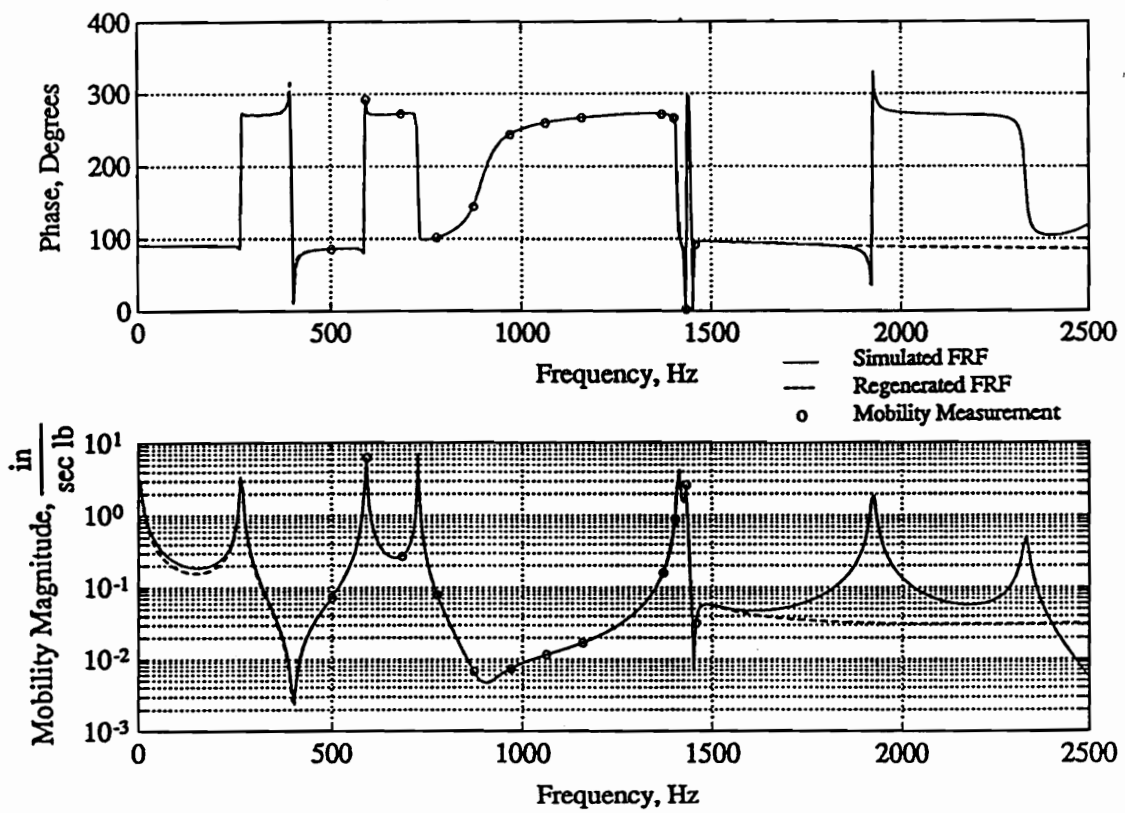


Figure 3.49 Simulated and Regenerated FRF's Using ILS, 12 Measurements with Four Measurements Biased Near Modes Close in Frequency

3.5.4 The Effects of Low Amplitude Modes

As in the close modes case, a low amplitude mode requires biasing of the mobility measurements to reduce the iterations and residue error. Figure 3.50 shows the MDOF system with mode 3 having an amplitude 10% of its original amplitude. Again, the same 12 equally spaced measurements are used and the resulting regenerated FRF is also shown in Figure 3.50. Residue error in Figure 3.51 shows significant error in mode three up to 10 iterations, but convergence eventually occurs. If three measurements are biased around the low amplitude mode as in Figure 3.52, there is much more rapid convergence of mode three. Figure 3.53 shows that convergence to less than 0.1% change occurs at six iterations and the final residue error is 0.29%.

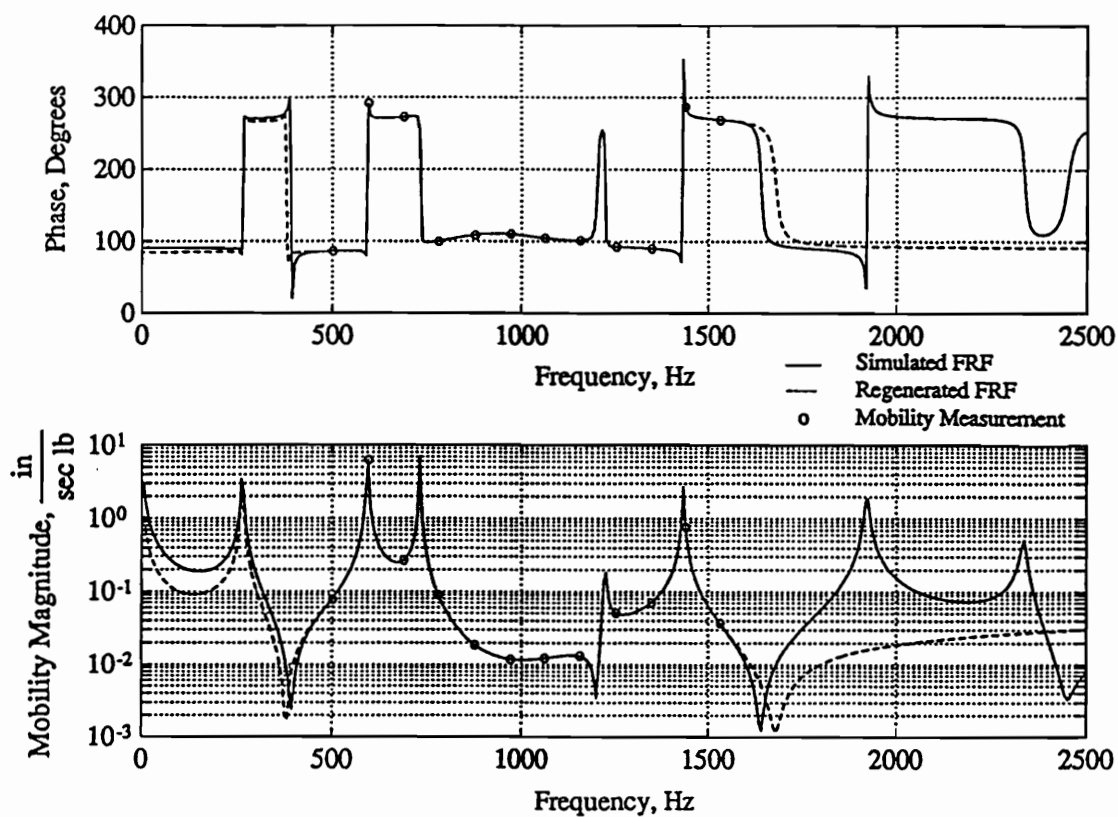


Figure 3.50 Simulated and Regenerated FRF's Using ILS, 12 Equally Spaced Measurements With One Low-Amplitude Mode

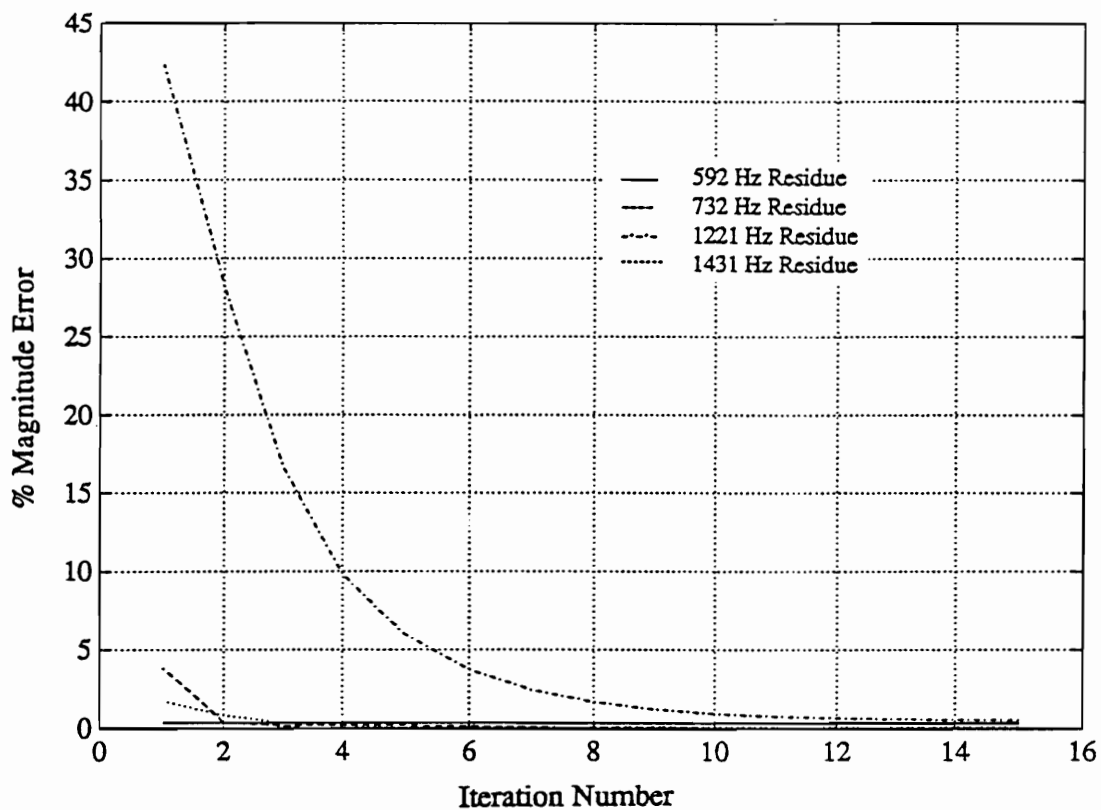


Figure 3.51 Convergence of Residues for 12 Equally Spaced Measurements, One Low-Amplitude Mode

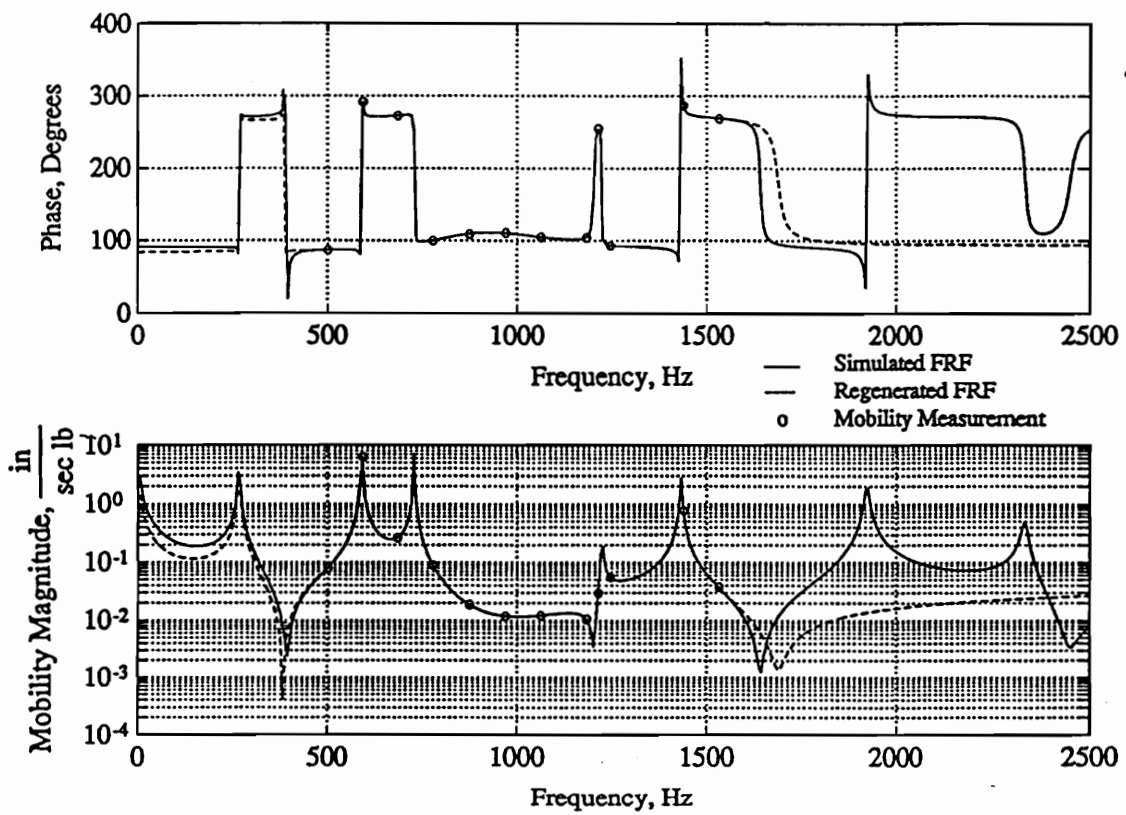


Figure 3.52 Simulated and Regenerated FRF's Using ILS, 12 Measurements With Three Measurements Biased Near Low-Amplitude Mode

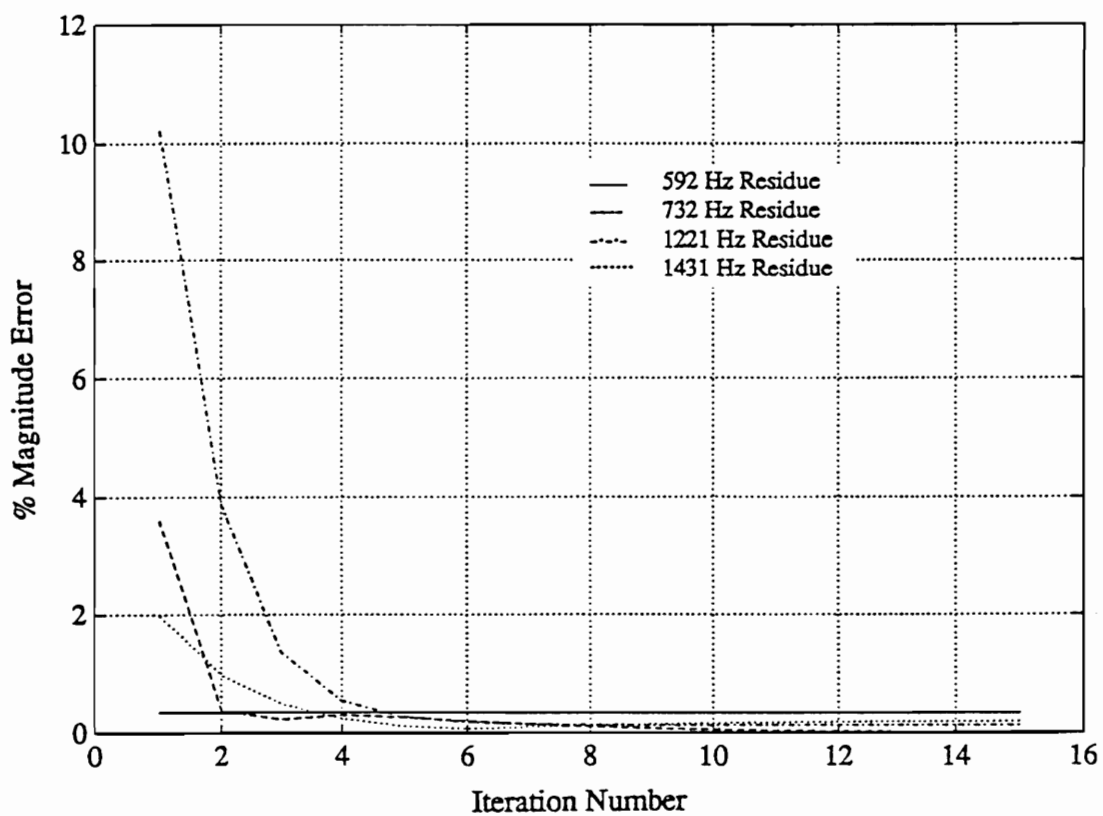


Figure 3.53 Convergence of Residues for 12 Measurements, Three Measurements Biased Near Low-Amplitude Mode

3.5.5 The Effects of System Damping on Convergence Speed

The last system to consider is one which has the same natural frequencies and residues as the previously analyzed system but damping which is 10x the previous level. Figure 3.54 shows the moderately damped system with the baseline measurement frequencies and the resulting regenerated FRF. Figure 3.55 shows residue convergence occurring in nine iterations, slightly more than for the lightly damped case in Figure 3.20. The final maximum magnitude error is slightly higher than the lightly damped case at 0.21%, but overall it appears that the increase in system damping has little effect on the iterative process.

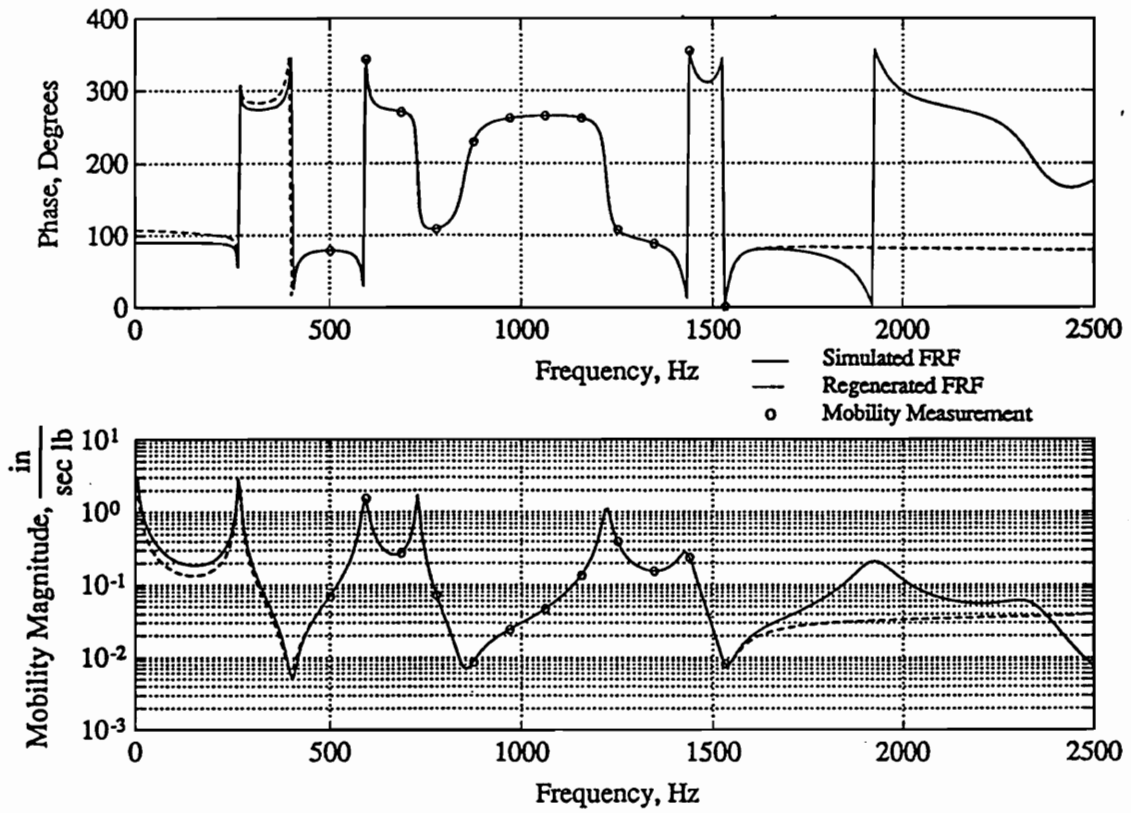


Figure 3.54 Simulated and Regenerated FRF's Using ILS, 12 Equally Spaced Measurements on a Moderately Damped System

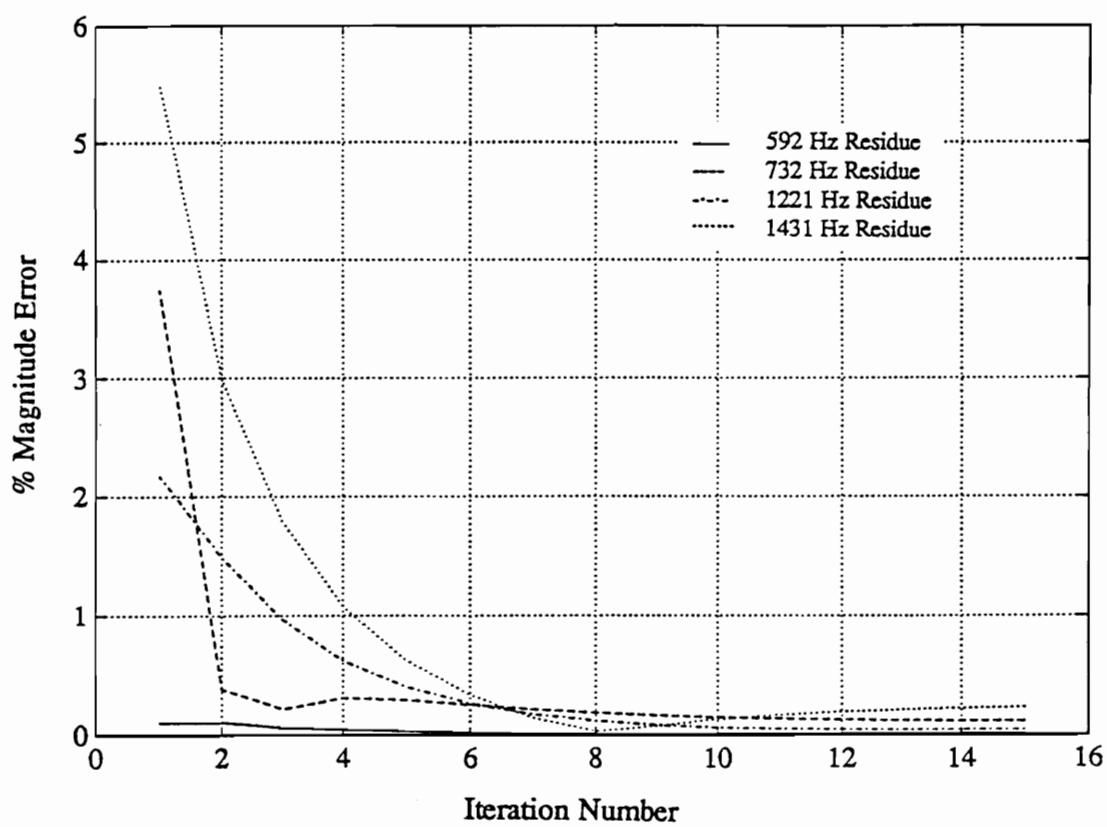


Figure 3.55 Convergence of Residues for 12 Equally Spaced Measurements, Moderately Damped System

3.6 Some Notes on Convergence Using the Iterative Technique

With the exception of the close modes case, the iterative technique has been shown to converge to a solution and convergence speed has been shown to increase as the number of mobility measurements increases. In the event that the system is not over determined, convergence will not occur. As was shown in the case of using equally spaced measurements for a system with closely spaced modes, convergence may not occur in a reasonable number of iterations. This is due to the similarity in frequency-domain behavior for close modes which can cause large residue errors for small mobility errors. If the iterative process is allowed to proceed for close modes, the final answer will be very close to the direct least squares answer, although it may take over 100 iterations to achieve convergence.

As the noise content on the data increases, the convergence rate decreases. Less residual content is removed with every iteration because the data does not fit the model as well, and so the convergence is slower. For typical levels of noise including the higher noise levels found in random excitation, convergence has always been found to occur.

3.7 Summary

The examples presented were intended to show general trends, such as where the measurement frequencies should be biased. The errors obtained were relative to the problem, and should not be taken as typical errors for all structures. Residue errors resulting from the direct least squares solution will be close to the iterative errors

presented because of the high signal/noise ratios. An assumption of Gaussian noise was made which is not the actual distribution of noise on the signals, but it provides an adequate model to estimate actual noise-induced residue variation. The cases examined showed the following:

For a lightly damped structure,

1. The number of measurements needed to adequately determine residues is at least one greater than the minimum required for over determination.
2. Measurements near resonance result in faster convergence but also cause larger errors in the residues if the global parameters are in error.
3. Measurements near closely spaced modes will decrease the final residue error and increase the rate of convergence for the close modes.
4. Measurements near low amplitude modes will decrease the final residue error and increase the rate of convergence for the low amplitude mode.

For a moderately damped structure,

1. The convergence rate and final errors are similar to the lightly damped case, although cases of global parameter error were not investigated.

Reducing error in the global parameters will allow more flexibility in the selection of the frequencies, so the choice of a good curve fitter is important. The next chapter presents a curve fitter which uses a structurally damped model instead of the more commonly used viscous model.

Chapter 4

Structurally Damped Curve Fitter

4.1 Theory

Most frequency-domain curve fitters assume dynamic behavior which is described by a viscous damping model [40, 41]. Although this is adequate for many structures, some structures can be better represented with a structural damping model. The aluminum plate test structure used to validate the least squares technique is an example of a structure with damping characteristics which are more structural than viscous. For this reason, a curve fitter that assumes structural damping has been developed.

Typical frequency response function curve fitters fit data by finding the coefficients to polynomials which are expressed in a ratio. The type of polynomial used and the method for determining the coefficients defines the curve fitter. Ordinary polynomials are not orthogonal and show a high degree of sensitivity to the coefficients, making them unsuitable to FRF fitting [40]. Chebechev polynomials are orthogonal and, therefore, can describe a vector space with minimum dimension. These polynomials maintain their orthogonality evenly over the domain, whereas the power polynomials are weighted heavier towards higher frequencies. Chebechev Polynomials are the result of minimizing the maximum value of the polynomial over the domain $\langle -1, 1 \rangle$. Mathematically, it can be proven that if Q_n is a polynomial of order n , and if

$\alpha_n = \max_{-1 \leq x \leq 1} |Q_n(x)|$, then the minimum value α_n for all n is obtained using Chebechev

Polynomials [42].

Another polynomial commonly used in curve fitting is the Forsythe Polynomial [43]. Like the Chebechev polynomial it is orthogonal over the domain $\langle -1, 1 \rangle$, but orthogonality is limited to the discrete points of measurement. This means that arbitrary measurements (non-equally spaced) may be used over the domain and orthogonality of the polynomials will still be maintained. This is beneficial if a small section of the FRF is curve fit with more than one polynomial and it is desired to maintain linear independence of the polynomials. Both Chebechev and Forsythe Polynomials exhibit symmetry about the origin, and as a result show orthogonality on either the intervals $\langle -1, 0 \rangle$ and $\langle 0, 1 \rangle$. This is well suited to FRF curve fitting where only positive frequencies exist in the domain.

Regardless of the polynomials used, they can always be related to the power polynomial form used to derive the modal parameters. The model for a multi-degree-of-freedom system where structurally damped behavior is assumed is:

$$Y_{ij} = \frac{i\omega(a_0 + a_1(i\omega)^2 + \dots + a_{n-1}(i\omega)^{2n-2})}{b_0 + b_1(i\omega)^2 + \dots + b_n(i\omega)^{2n}} \quad (4.1)$$

The number of modes is n and i and j refer to the response and reference points on the structure. This equation is equivalent to the traditional form of mobility:

$$Y_{ij} = \sum_{r=1}^n \frac{i\omega_r A_{ij}}{(i\omega)^2 + \omega_r^2(1 + i\eta_r)} \quad (4.2)$$

The unknowns in equation (4.1) are the polynomial coefficients a_i and b_i . The common method for finding the coefficients from a measured FRF is to solve a weighted least squares problem:

$$\{\epsilon\}_p = \{W\}_p * \left\{ Y - \frac{P_n}{P_d} \right\}_p \quad (4.3)$$

where W is a weighting function, P_n is the numerator polynomial and P_d is the denominator polynomial. The error vector $\{\epsilon\}_p$ is the error to be minimized over the frequency domain whose length is equal to the number of frequency measurements, p . The operator $*$ represents the row-by-row multiplication of the vector entries. Minimizing this error is a non-linear least squares problem which cannot be efficiently evaluated. Therefore, the definition of the error is changed by multiplying by P_d to obtain a weighted linear least squares problem:

$$\{\epsilon'\}_p = \{W\} * \{P_d\} * \{Y\} - \{W\} * \{P_n\}_p \quad (4.4)$$

Note that if $W = \frac{1}{P_d}$, the error becomes the original error to be minimized;

obtaining an estimate of W will be a part of the curve-fitting process.

The polynomial coefficients are included in equation (4.4) by making the following substitutions:

$$\begin{aligned} [\phi](a) &= \{P_n\} \\ [\psi](b) &= \{P_d\} \end{aligned} \quad (4.5)$$

The polynomial matrix $[\phi]$ has dimension $p \times (n-1)$ where $2n$ is the order of the denominator polynomial. Likewise, $[\psi]$ has dimension $p \times n$. From equation (4.1) it is seen that the denominator polynomial is an even-order polynomial and the numerator is an odd order. Expressing the error again:

$$\begin{aligned} \{\epsilon'\}_p &= \{W\}^* \{Y\}^* [\psi](b) - \{W\}^* [\phi](a)_p \\ \text{or} \\ \{\epsilon'\}_p &= \{W_b\}^* [\psi](b) - \{W_a\}^* [\phi](a)_p \end{aligned} \quad (4.6)$$

$$\begin{aligned} \text{where:} \quad \{W_b\} &= \{W\}^* \{Y\} \\ \{W_a\} &= \{W\} \end{aligned} \quad (4.7)$$

Expanding this equation in real and imaginary components and writing the squared error as $E = \epsilon' \epsilon'^*$ results in:

$$\begin{aligned} E|_{\omega} &= \left(W_{b\omega} [\psi_{\omega}] \{b_r\} - W_{b\omega i} [\psi_{\omega}] \{b_i\} - W_{a\omega} [\phi_{\omega}] \{a_r\} + W_{a\omega i} [\phi_{\omega}] \{a_i\} \right)^2 \\ &\quad + \left(W_{b\omega i} [\psi_{\omega}] \{b_r\} + W_{b\omega} [\psi_{\omega}] \{b_i\} - W_{a\omega i} [\phi_{\omega}] \{a_r\} - W_{a\omega} [\phi_{\omega}] \{a_i\} \right)^2 \end{aligned} \quad (4.8)$$

For clarity, this has been written to express the error at a single frequency in the FRF, ω . Allowing the coefficients to be complex gives complexity to the residues. Simplifying this equation so that the first and second terms squared are respectively g_1^2 and g_2^2 :

$$E|_{\omega} = g_1^2 + g_2^2 \quad (4.9)$$

If coefficients a and b are to be solved for,

$$\begin{aligned}
\left. \frac{\partial E}{\partial a_r} \right|_{\omega} &= -2W_{a\omega r}[\phi_{\omega}]^T g_1 - 2W_{a\omega i}[\phi]^T g_2 = 0 \\
\left. \frac{\partial E}{\partial a_i} \right|_{\omega} &= 2W_{a\omega i}[\phi_{\omega}]^T g_1 - 2W_{a\omega r}[\phi]^T g_2 = 0 \\
\left. \frac{\partial E}{\partial b_r} \right|_{\omega} &= 2W_{b\omega r}[\psi_{\omega}]^T g_1 + 2W_{b\omega i}[\psi]^T g_2 = 0 \\
\left. \frac{\partial E}{\partial b_i} \right|_{\omega} &= -2W_{b\omega i}[\psi_{\omega}]^T g_1 + 2W_{b\omega r}[\psi]^T g_2 = 0
\end{aligned} \tag{4.10}$$

This problem can be expressed in matrix form:

$$\begin{bmatrix} [f_1] & [f_2] \\ [f_2]^T & [f_3] \end{bmatrix} \begin{Bmatrix} \{b_r\} \\ \{b_i\} \\ \{a_r\} \\ \{a_i\} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix} \tag{4.11}$$

where:

$$[f_1] = \begin{bmatrix} 2(W_{br}^2 + W_{bi}^2)[\psi]^T[\psi] & 0 \\ 0 & 2(W_{br}^2 + W_{bi}^2)[\psi]^T[\psi] \end{bmatrix} \tag{4.12}$$

$$[f_2] = \begin{bmatrix} 2(W_{ar}W_{br} + W_{ai}W_{bi})[\psi]^T[\phi] & 2(W_{ai}W_{br} - W_{ar}W_{bi})[\psi]^T[\phi] \\ 2(W_{ar}W_{bi} - W_{ai}W_{br})[\psi]^T[\phi] & -2(W_{ai}W_{bi} + W_{ar}W_{br})[\psi]^T[\phi] \end{bmatrix} \quad (4.13)$$

$$[f_3] = \begin{bmatrix} 2(W_{ar}^2 + W_{ai}^2)[\phi]^T[\phi] & 0 \\ 0 & 2(W_{ai}^2 + W_{ar}^2)[\phi]^T[\phi] \end{bmatrix} \quad (4.14)$$

Setting the highest order a coefficient to $(1 + i)$ and returning the polynomial matrices to their original definition, the matrix equation becomes:

$$\begin{bmatrix} [f_1] & [f_2] \\ [f_2]^T & [f_3] \end{bmatrix} \begin{Bmatrix} \{b_r\} \\ \{b_i\} \\ \{a_r\} \\ \{a_i\} \end{Bmatrix} = \begin{bmatrix} 2(W_{ar}W_{br} + W_{ai}W_{bi})[\psi]^T\{\phi_{n-1}\} + 2(W_{ar}W_{bi} - W_{ai}W_{br})[\psi]^T\{\phi_{n-1}\} \\ 2(W_{ar}W_{br} - W_{ar}W_{bi})[\psi]^T\{\phi_{n-1}\} + 2(W_{ai}W_{bi} + W_{ar}W_{br})[\psi]^T\{\phi_{n-1}\} \\ -2(W_{ar}^2 + W_{ai}^2)[\phi]^T\{\phi_{n-1}\} \\ 2(W_{ai}^2 + W_{ar}^2)[\phi]^T\{\phi_{n-1}\} \end{bmatrix} \quad (4.15)$$

A modified Gram Schmidt Technique is then used to solve for the unknown coefficients.

Expressing the mobility in terms of Chebechev Polynomials, a modified form of equation (4.1) is obtained:

$$Y_{ij} = \frac{a_0 T_1 + a_1 T_3 + \cdots + a_{n-1} T_{2n-1}}{b_0 + b_1 T_2 + \cdots + b_n T_{2n}} \quad (4.16)$$

The i th order Chebechev Polynomial is T_i . The polynomials are generated by the following recursive algorithm:

$$\begin{aligned} T_0 &= 1 \\ T_1 &= \omega \\ T_2 &= 2\omega T_{n-1} - T_{n-2} \\ &\vdots \end{aligned} \quad (4.17)$$

The polynomials are in terms of ω , not $i\omega$, so a transformation is made after the polynomial coefficients are found. The ordinary power series equivalent coefficients must be found so that the roots and then the values of natural frequency and damping can be extracted. The Appendix contains the listing for an algorithm which accomplishes this task.

Returning to equation (4.6), $W_a = \frac{1}{P_d}$ and $W_b = \frac{Y}{P_d}$ is the weighting which returns the problem to the original minimization problem. Since the system poles P_d are not known a-priori, estimates for the poles will be obtained from the curve-fitting itself. Better estimates of the poles are obtained as the curve-fitting over a frequency interval is repeated. Using the curve-fit pole data, the weighting functions become:

$$W_{a_j} = \frac{1}{\sum_{r=1}^n (i\omega_j)^2 + \omega_r^2 (1 + i\eta_r)}$$

$$W_{b_j} = \frac{Y_j}{\sum_{r=1}^n (i\omega_j)^2 + \omega_r^2 (1 + i\eta_r)} \quad (4.18)$$

ω_j = circular frequency of analysis

The weights W_{a_j} and W_{b_j} are the weighting terms applied at frequency j . If it is desired to curve-fit only a portion of the FRF, then W_{a_j} and W_{b_j} are modified to eliminate the FRF where the data is not desired. Both terms are set to zero outside of the frequency range being curve fit. It is necessary to subtract the modal content due to modes outside of the range being curve fit, since the poles and residues obtained will pertain only to the portion of the FRF with a non-zero weight.

Subtracting out a portion of the FRF allows an easier identification of the modes that will be curve fit since they are not contaminated by the content of other modes. Frequently when the out-of-range content is subtracted, modal behavior not evident from the original FRF is seen, making it clear where the modes exist. Sub-range fitting in this manner is preferred since it is easier for the curve fitter to obtain accurate estimates of the parameters.

4.2 Curve Fitting Two FRF's Simultaneously

Selecting a single reference input and response output to determine a system's parameters is risky if the mode shapes of the structure are not already identified. This is

because a node of vibration may exist at either of these points, causing a false indication that the mode does not exist. To reduce the chances of this occurring, it is recommended that "pre-modal" testing be performed to determine where modes exist on the structure (frequency and spatial domains). If this cannot be done, a better estimate of the modal parameters can be obtained by measuring two responses instead of one. In this case, if one response is on the node of a mode the other response will still detect the mode.

Considering another response point in the curve-fitting algorithm amounts to minimizing the error for two FRF's simultaneously:

$$E = \epsilon_1' \epsilon_1'' + \epsilon_2' \epsilon_2'' \quad (4.19)$$

As a result, there will be two more partial derivative expressions added to equation (4.9). The sensitivity of the error to the denominator coefficients gains two terms from the additional FRF:

$$\begin{aligned} \left. \frac{\partial E}{\partial b_r} \right|_{\omega} &= 2W_{b_{1r}}[\psi]^T g_{11} + 2W_{b_{1i}}[\psi]^T g_{21} + 2W_{b_{2r}}[\psi]^T g_{12} + 2W_{b_{2i}}[\psi]^T g_{22} \\ \left. \frac{\partial E}{\partial b_i} \right|_{\omega} &= -2W_{b_{1i}}[\psi]^T g_{11} + 2W_{b_{1r}}[\psi]^T g_{21} - 2W_{b_{2i}}[\psi]^T g_{12} + 2W_{b_{2r}}[\psi]^T g_{22} \end{aligned} \quad (4.20)$$

The new subscripted variables g_{ij} refer to the i^{th} constant as previously defined and the j^{th} FRF to be minimized. Equation (4.15) will have one more set of numerator coefficients making a total of four numerator coefficients and two denominator coefficients. The solution of the coefficients remains the same.

4.3 An Example of Curve Fitting on Measured Data - Single FRF

To demonstrate the operation of the structurally damped curve fitter, three FRF's are measured from an aluminum plate 24.0 in long, 6.0 in wide and 0.5 in thick. The test setup is shown in Figure 4.1. The accelerometers are PCB™ type 307M02 with a nominal sensitivity of 43 mV / g. The force gage is a PCB™ type 208B, with a sensitivity of 502 mV / lb. One of the non-driving point FRF's at location 3 is used to demonstrate the single-point curve-fitting capabilities. Another non-driving point FRF at location 2 and the driving-point FRF at location 1 are used to demonstrate curve fitting using two FRF's.

The location of the shaker was chosen to ensure that all modes of interest would be excited; the free-free rectangular plate shows experimentally that all modes being measured have motion at the plate corners. Sine excitation is stepped in bin-centered increments to excite the structure with a high signal-to-noise ratio. This allows an accurate FRF to be obtained without windowing using an FFT. In the event that the noise is significant and averaging must occur, a commonly used estimator could be employed.

The step-sine test is executed with a Zonic™ System 7000 FFT using a batch program written within the Zeta™ Operating System. The batch program takes advantage of amplitude feedback to adjust the excitation voltage so that neither the force nor acceleration signals exceed limits which would damage equipment or cause erroneous measurements. The Appendix contains a listing for the batch program. A shaker voltage limit of 150.0 mV ensures that the power limit of the shaker is not exceeded although this may result in lower than desired acceleration signals at anti-resonances.

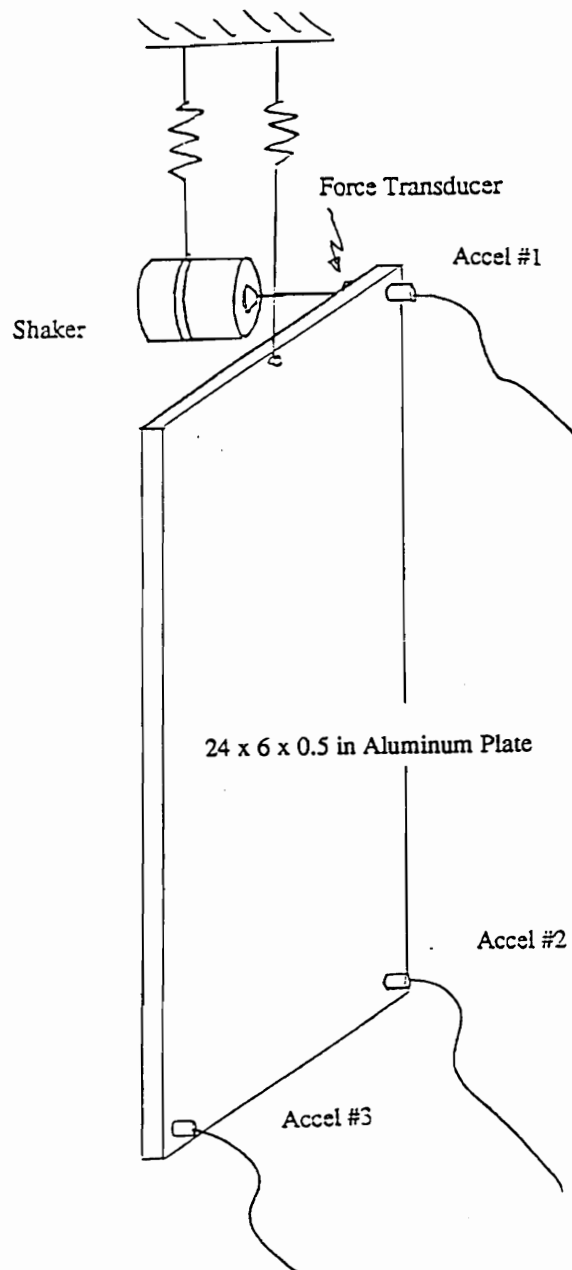


Figure 4.1 Experimental Test Set-Up

The measurements start at 100 Hz since the shaker amplitude can become excessive at lower frequencies. Although data for a single frequency is desired at each frequency step, the FFT technique yields data for all frequencies in the analysis range. Only the bins corresponding to the excitation frequency are stored and the rest of the frequency-domain data is discarded. Although this is wasteful, the ability to use existing hardware with FFT algorithm speed makes the option feasible.

To ensure the structure was linear, measurements of accelerance were taken at two frequencies of excitation as the signal voltage was varied. Figure 4.2 shows the linearity plots for the driving point (location 1) for a 725 Hz and an 1193.73 Hz excitation signal. These frequencies were selected because they both correspond to high amplitude force and acceleration signal where nonlinearities would most likely occur.

Figure 4.3 shows the force and acceleration autospectra at the driving point resulting from the step-sine test; the frequency locations used for the linearity check are also shown. The "flat spots" that exist in the acceleration autospectrum correspond to changes in the shaker voltage signal in an attempt to keep the acceleration above a minimum value. When the shaker voltage reaches 150 mV, the signal is no longer increased and the acceleration is allowed to decrease below what was considered minimum acceptable. Figure 4.4 shows the shaker signal voltage which can be correlated to the amplitudes seen in Figure 4.3.

The measured accelerance for point 3 is shown in Figure 4.5. This will be the FRF that is fit using the structurally damped curve fitter. Typical signal-to-noise ratios are shown in Figure 4.6. The ratios are obtained by measuring the autospectra of the force and acceleration at the excitation frequency and dividing this value by the same quantity five bins away from the excitation frequency. The square root of the ratio is the signal/noise ratio. Most of the analysis range shows signal/noise ratios greater than 1000, which demonstrates the advantage of step-sine excitation. These levels could not be

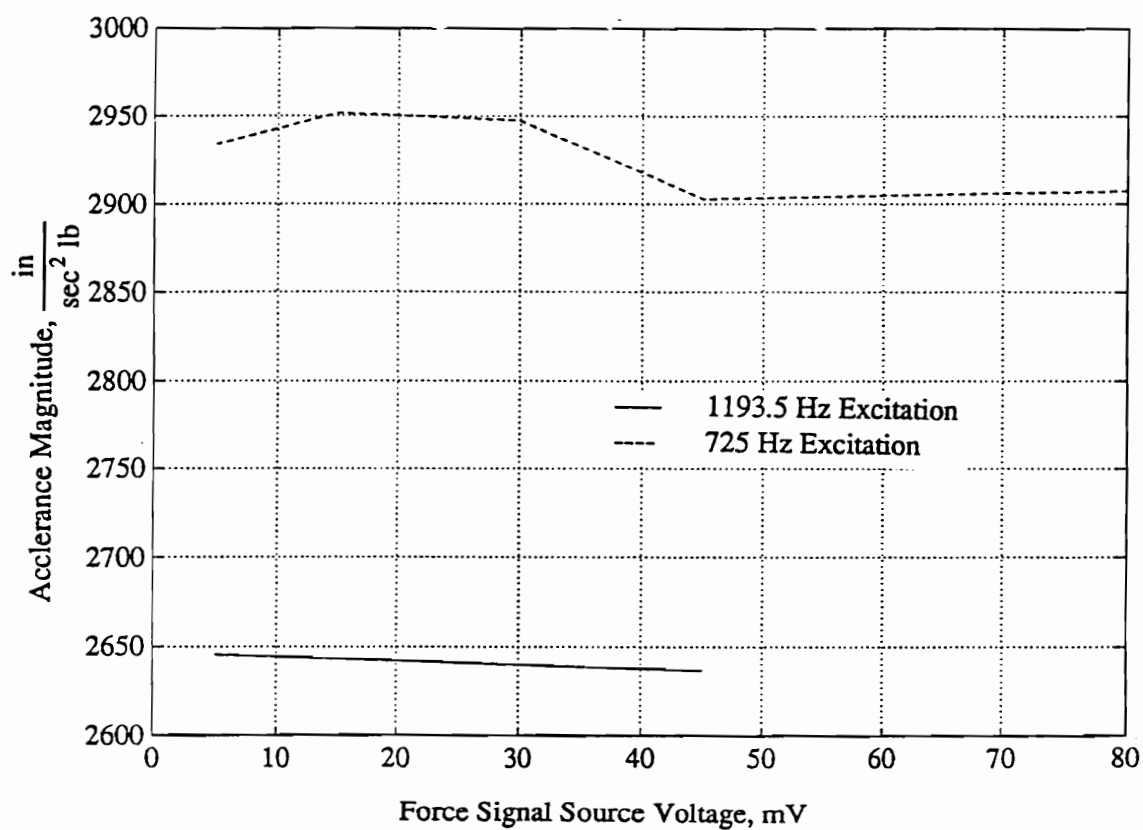


Figure 4.2 Linearity Check of Test Structure at High-Amplitude Response Frequencies

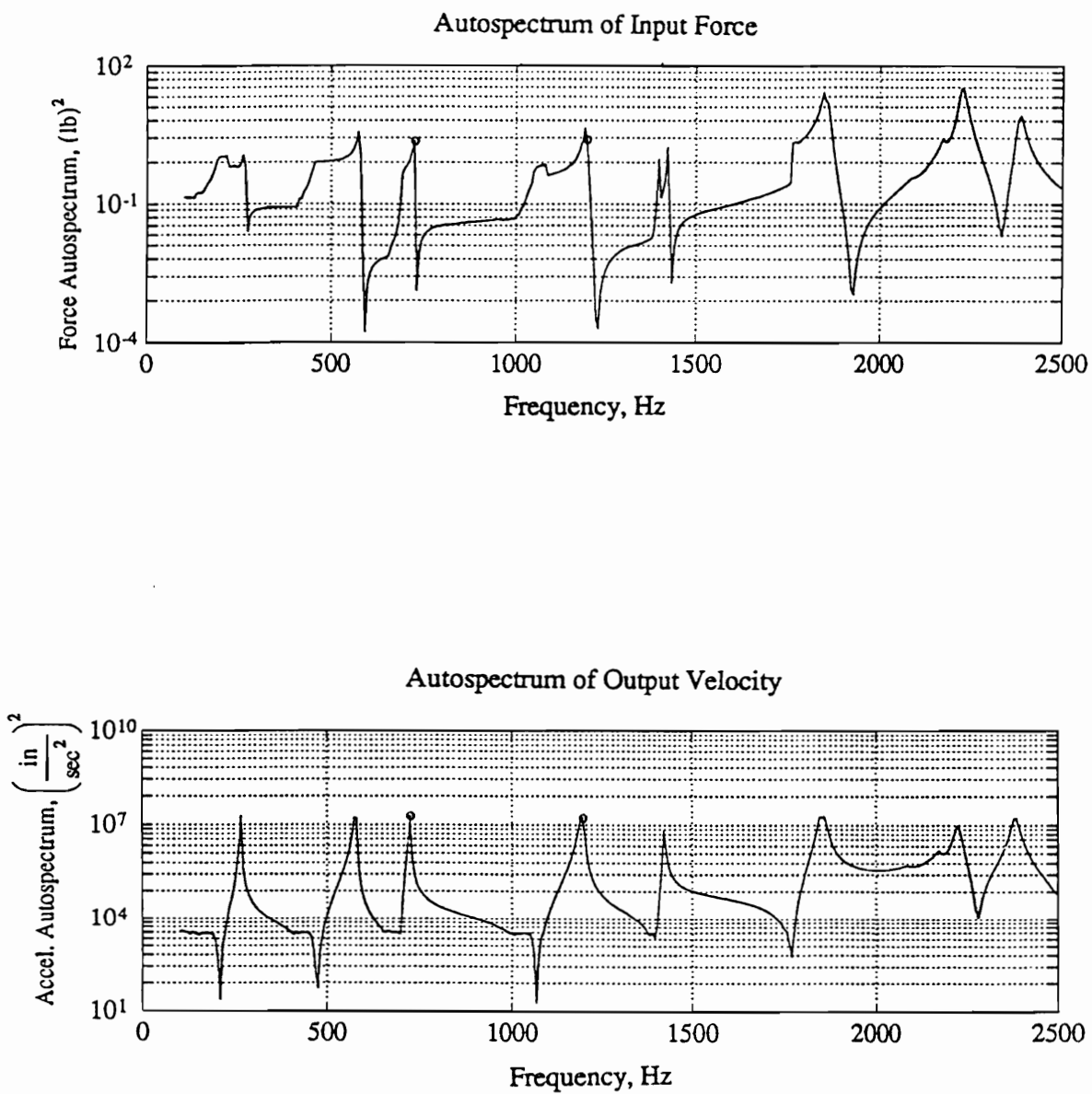


Figure 4.3 Force and Acceleration Autospectra of the Test Structure

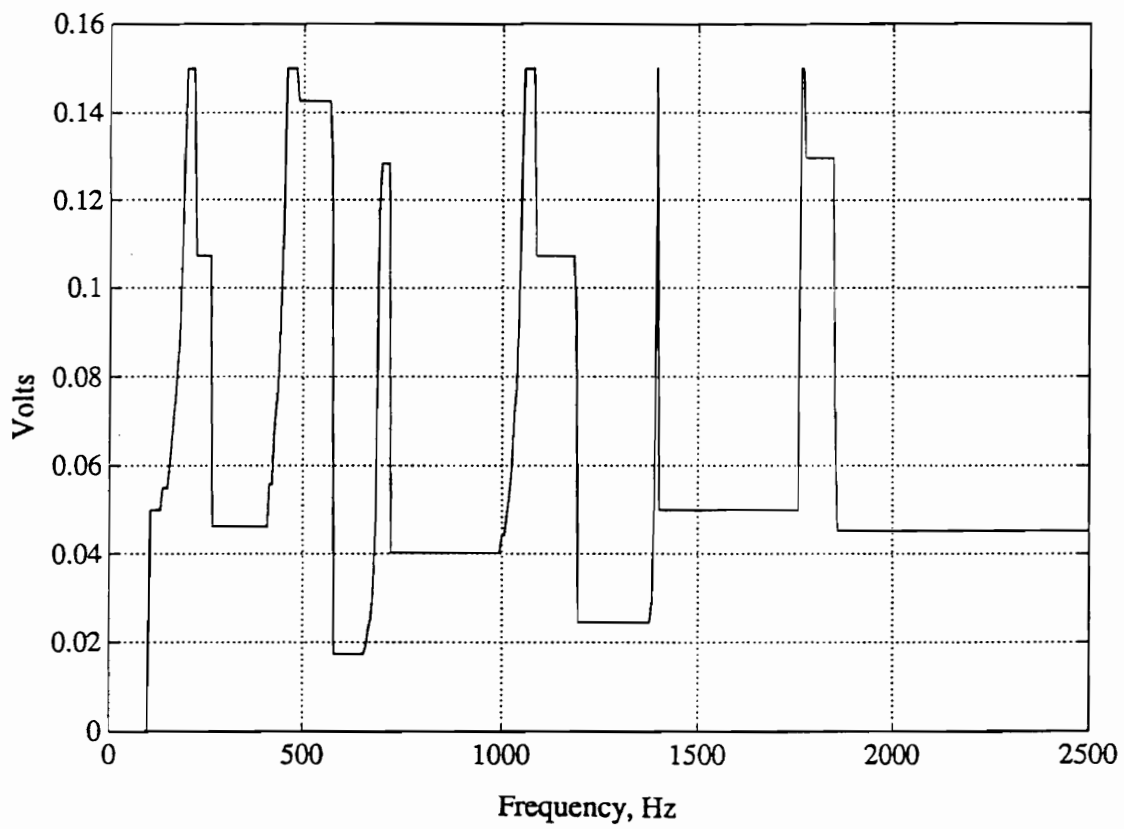


Figure 4.4 Excitation Voltage Signal for Experiment

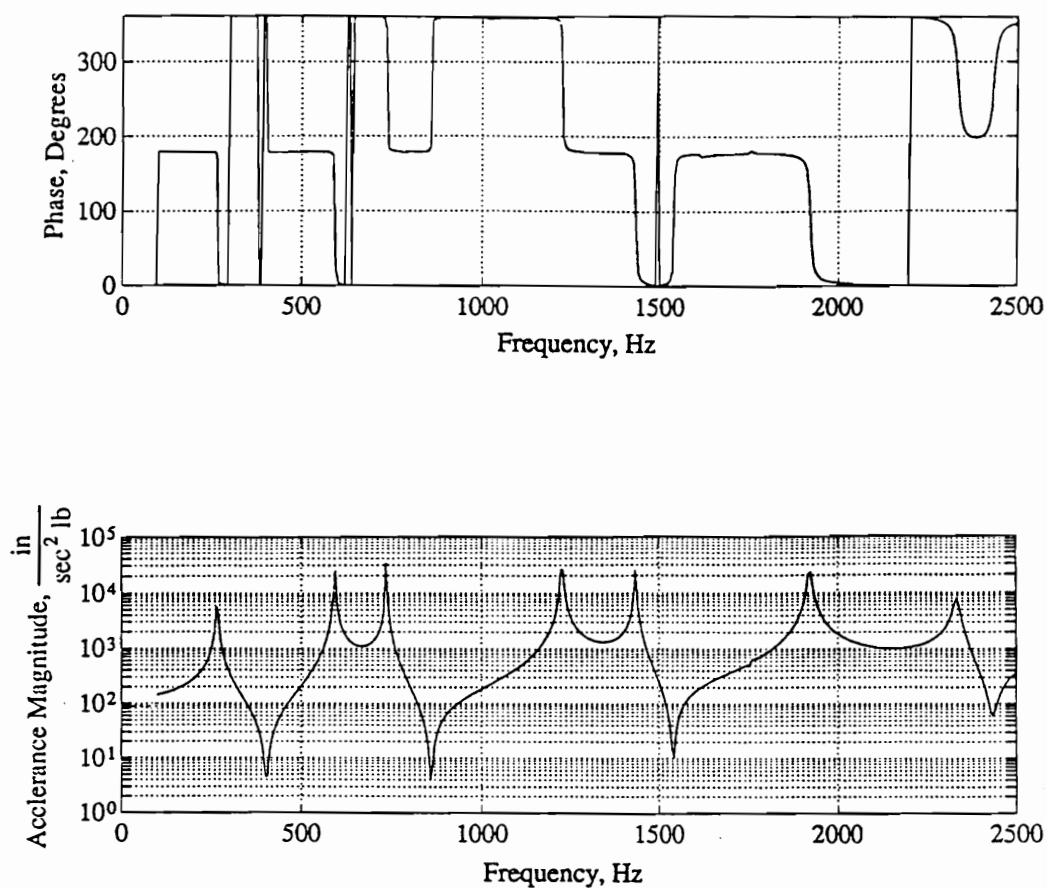


Figure 4.5 Measured FRF from Structure Using Step-Sine Excitation

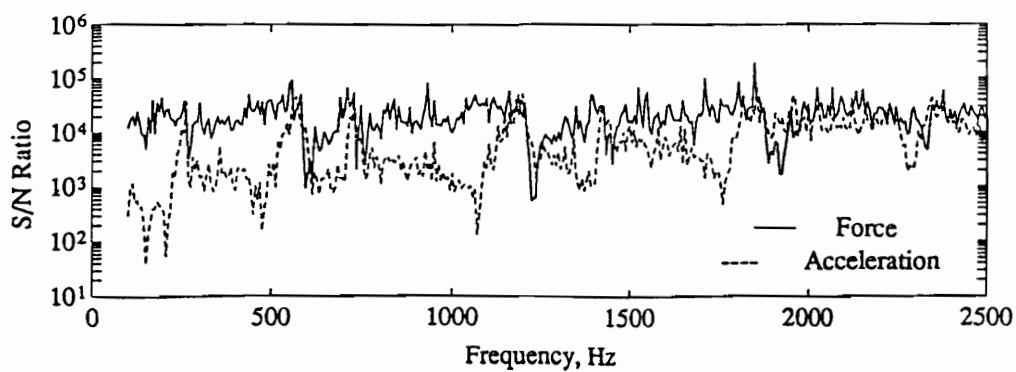


Figure 4.6 Signal/Noise Ratio of Measured Force and Acceleration Using Step-Sine Excitation

realistically achieved with random excitation since the energy required would be excessive.

Using sub-range fitting, the modal parameters are extracted from point 3 and are shown in Table 4-1. A separate program is executed to determine the rigid-body mode by subtracting the curve-fit modal content from the FRF and then fitting what is left to a rigid-body mode model. A "pseudo-mode" has also been used to model the dynamic response of the system above the measured frequency range. This is a mode above the measured frequency range which is found from the curve-fitting of the measured response data at the high end of the frequency range. Figure 4.7 shows the original mobility FRF and the regenerated FRF from the extracted parameters. Figure 4.8 shows the percent magnitude error in the regenerated FRF as compared to the measured FRF. Although most of the frequency range shows errors below 10%, there are some higher errors around the anti-resonances.

Table 4-1 Structurally Damped Curve Fitter Results for Location 3			
Mode	ω_r	η_r	A_r
rigid body	0 Hz	0	-127.8 - 0.803i
1	265.09	1.37e-3	109.45 + 0.812i
2	592.19	2.05e-3	133.10 + 1.65i
3	732.20	8.19e-4	-88.00 + 0.174i
4	1221.8	1.50e-3	-134.41 - 6.37i
5	1431.0	2.73e-3	67.45 + 0.357i
6	1759.3	1.98e-4	-0.230 + 1.06e-3i
7	1922.1	4.89e-3	128.36 + 4.31i
8	2337.9	6.56e-3	-51.01 - 0.265i
Pseudo-mode			
9	2720.2	3.81e-2	-91.62 - 13.87i

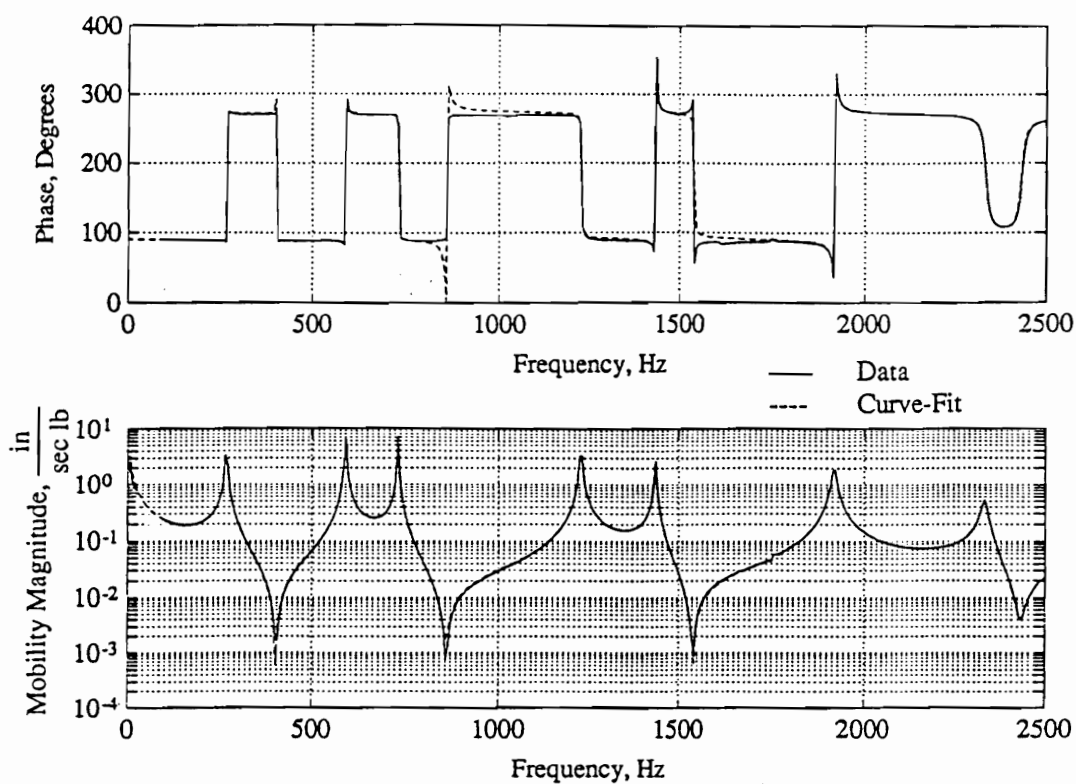


Figure 4.7 Measured and Regenerated FRF Using the Structurally Damped Curve Fitter

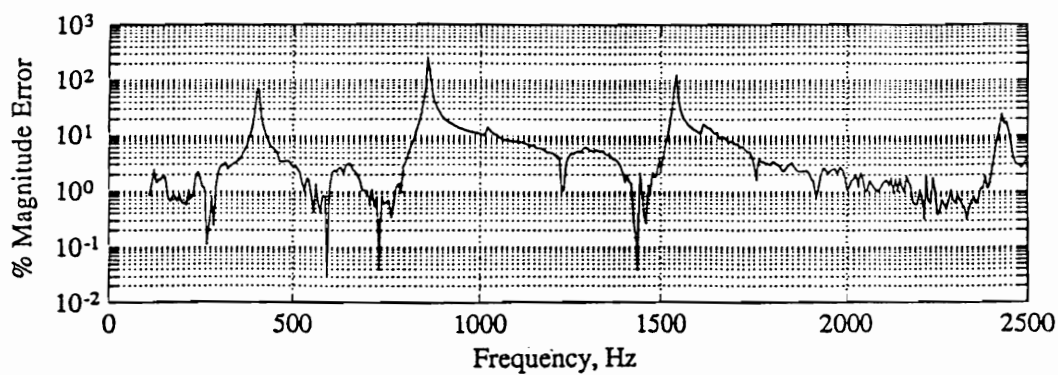


Figure 4.8 Magnitude Error Between Measured FRF and Regenerated FRF

These errors can be further examined to get a better idea of their cause and whether or not they should be considered acceptable. Figure 4.9 shows the remaining mobility content in the data after the curve-fit content is subtracted. The largest error occurs in the real mobility from 1190 Hz to 1260 Hz. To see where the error originates, Figure 4.10 shows the data and the curve fit for this range. It appears that the curve-fitter is correctly minimizing the error between the data and the curve fit since the error is not biased. The deviation seen from 1190 Hz to 1210 Hz is further examined in Figure 4.11.

The real mobility trend seen in the curve fit but not in the data in Figure 4.11 comes from the imaginary part of the residue. The imaginary residue can be considered a weighted contribution of imaginary mobility behavior to the real mobility curve fit. If the imaginary part is omitted, the fit from 1190 Hz to 1210 Hz is improved however significant errors occur near the peak of the real mobility. The curve fitter appears to be working correctly, so a likely sources of error could be the data itself.

If the simulated system used in Chapter 3 is curve fit, then parameters are obtained which are very close to the simulated parameters. Table 4-2 shows the simulated and curve-fit parameters along with the errors, and Figure 4.12 shows the simulated and curve-fit data. The fit is so close that it cannot be seen on the regenerated plot. The resulting magnitude error in the regenerated FRF is shown in Figure 4.13. As in Chapter 3, a level of noise was used in the simulation which was actually measured. Since the curve fitter was able to fit simulated data with the same noise levels as in the real test, the most likely cause of error in the curve fit of the measured data is the non-structurally damped behavior of the real structure.

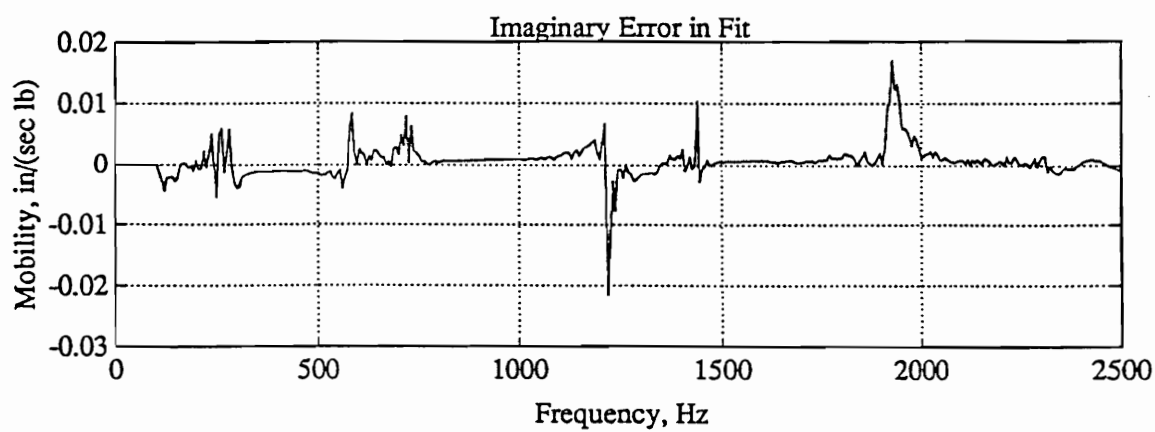
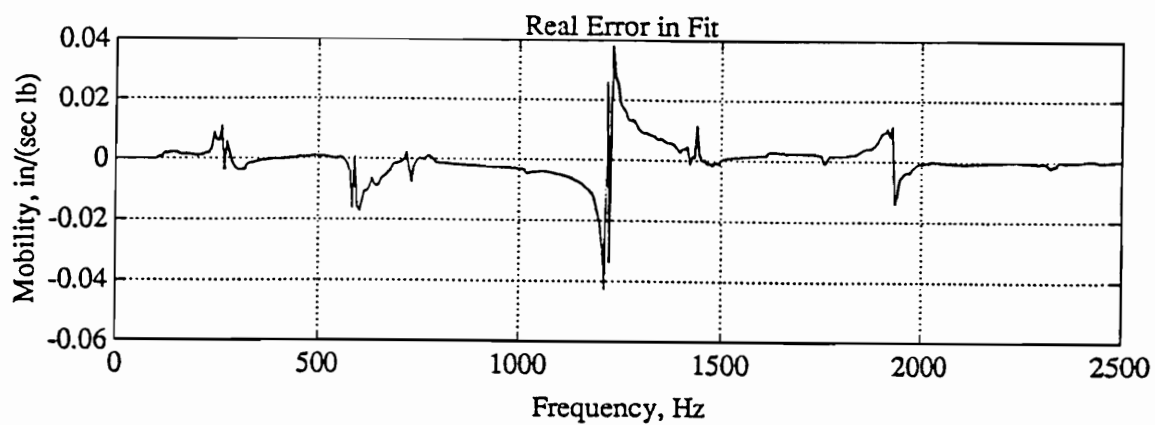


Figure 4.9 Remaining Mobility Content After Curve-Fit is Subtracted

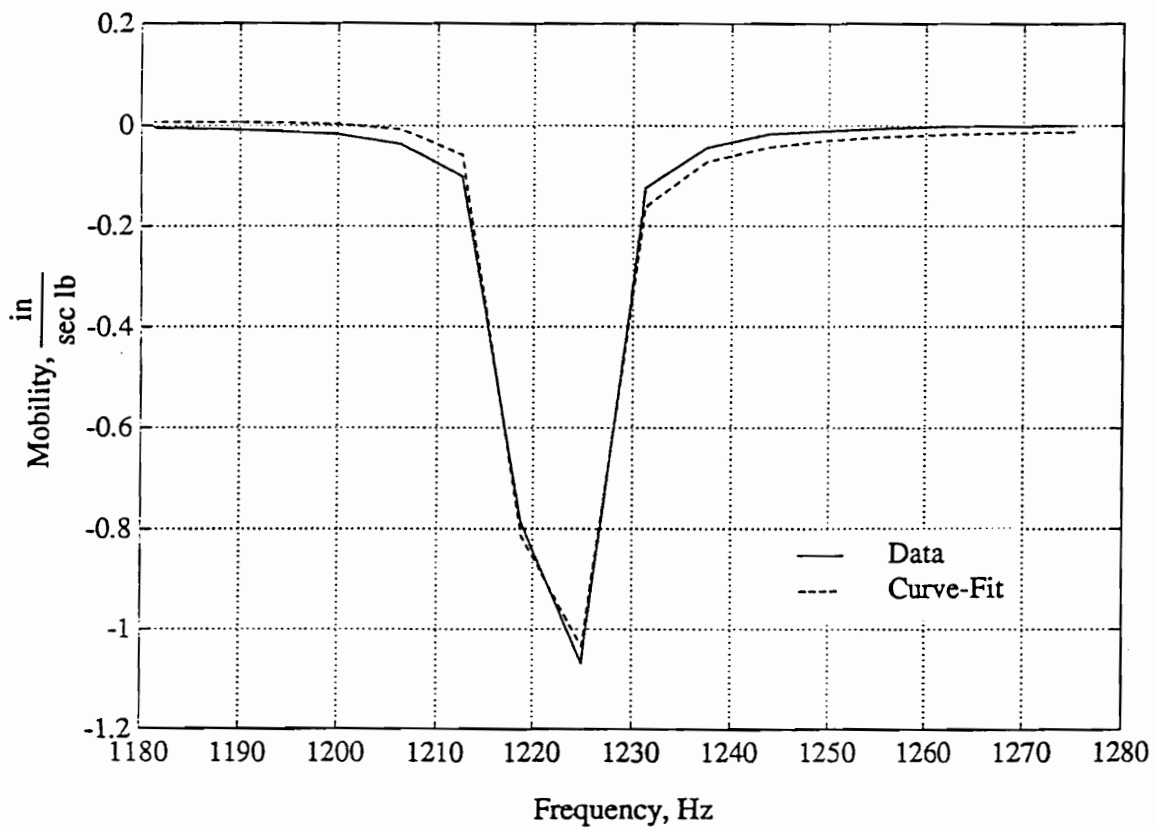


Figure 4.10 Measured Mobility and Curve-Fit Mobility, Real Part

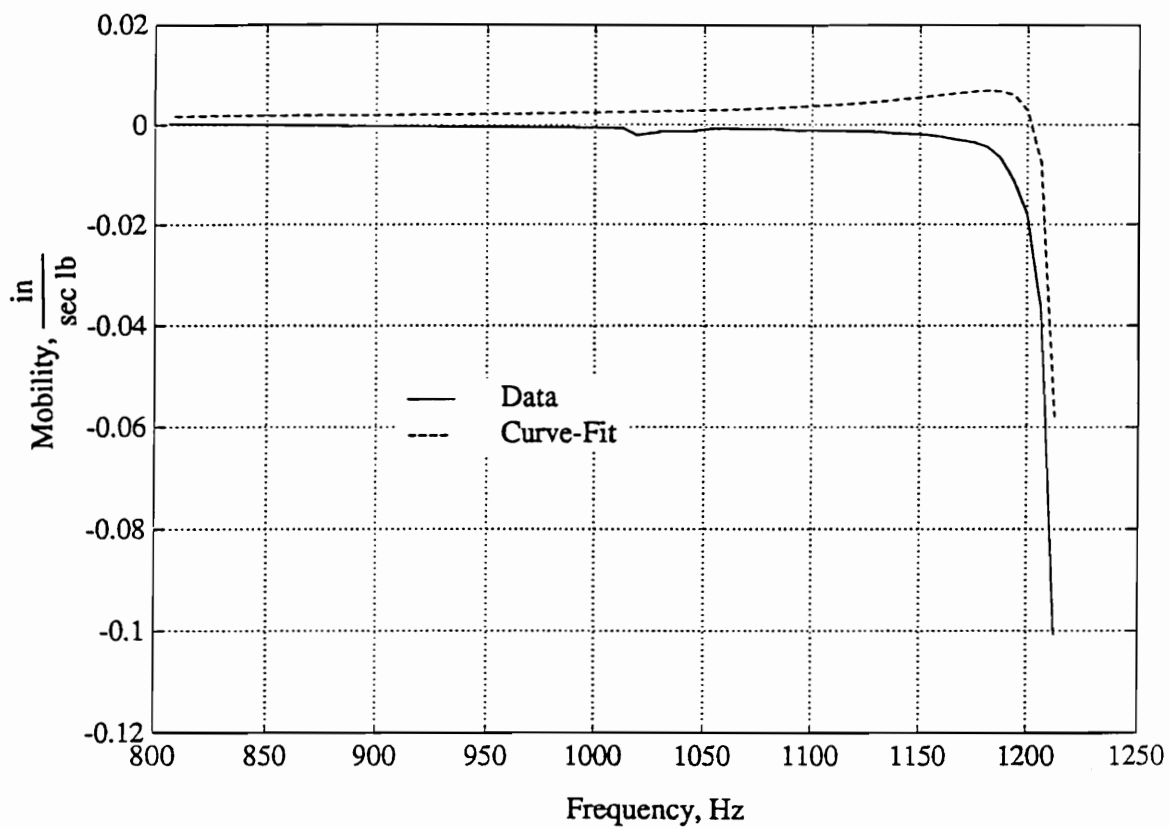


Figure 4.11 Measured Mobility and Curve-Fit Mobility, Real Part

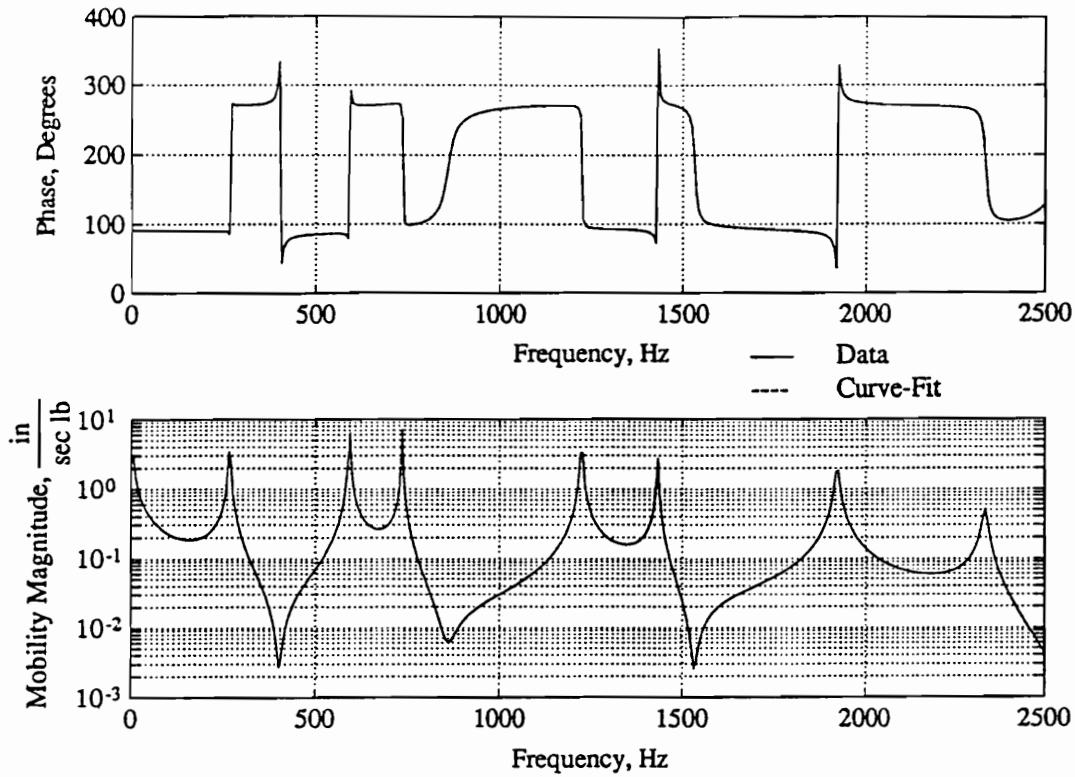


Figure 4.12 Simulated and Curve-Fit FRF's Using The Structurally Damped Curve Fitter

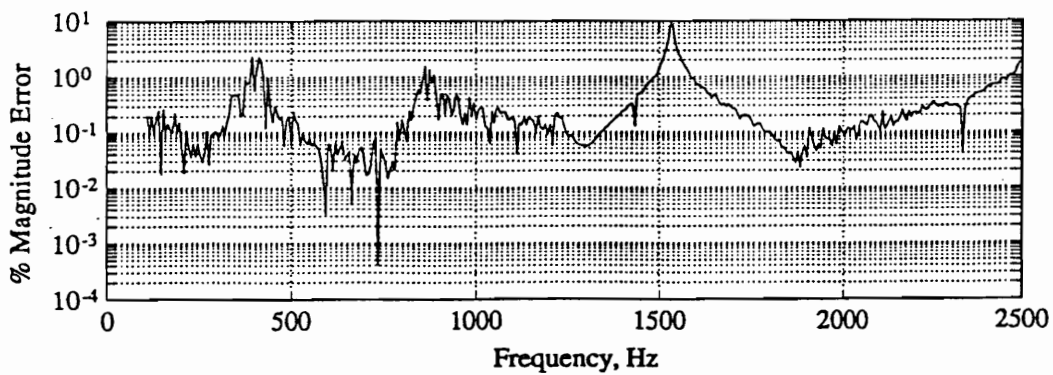


Figure 4.13 Magnitude Error Between Simulated Data and Curve-Fit Data

Table 4-2									
Simulated and Curve-Fit Parameters									
	Simulated Parameters			Curve-Fit Parameters			% Magnitude error		
Mode	f_r	η_r	A_r^{**}	f_r	η_r	A_r^{**}	f_r	η_r	A_r
R.B.	0 Hz	0	-.0271	0 Hz	0	-2.71e-2 - 4.83e-4i	N/A	N/A	<1e-1
1	265.08	1.37e-3	2.47e-2 + 1.86e-4i	265.08	1.38e-3	2.46e-2 + 1.88e-4i	<1e-3	0.73	0.40
2	592.16	2.33e-3	3.09e-2 - 5.05e-4i	592.16	2.34e-3	3.09e-2 - 5.07e-4i	<1e-3	0.43	<1e-1
3	732.16	1.10e-3	-1.97e-2 - 1.95e-3i	732.17	1.11e-3	-1.97e-2 - 1.94e-3i	1.37e-3	0.91	<1e-1
4	1221.8	1.50e-3	-3.03e-2 - 1.45e-3i	1221.8	1.51e-3	-3.03e-2 - 1.45e-3i	<1e-3	0.66	<1e-1
5	1431.0	2.74e-3	1.52e-2 + 2.50e-5i	1431.0	2.73e-3	1.52e-2 + 6.54e-5i	<1e-3	0.36	<1e-1
6	1746.1	3.01e-3	-8.19e-6 - 6.61e-5i	1750.0	1.89e-6	< 1e-10	*	*	*
7	1922.1	4.91e-3	2.90e-2 + 6.90e-4i	1922.1	4.91e-3	2.90e-2 + 7.02e-4i	<1e-3	<0.10	<1e-1
8	2337.1	5.54e-3	-9.44e-3 - 4.73e-4i	2337.1	5.55e-3	-9.45e-3 - 5.09e-4i	<1e-3	0.18	0.12

* Since the magnitude of the mode at resonance is less than the noise magnitude, these parameters amount to noise.

** These residues have been scaled down by a factor of 4438.6, the conversion factor from volts/volt to acceleration

Another measure of curve-fitter performance is obtained by comparing the structurally damped curve fit to a fit obtained with a viscous model. Using SDRC I-DEAS™ Software, the Orthogonal Polyreference Curve Fitter is used to fit the data. This curve fitter is a robust parameter estimator capable of using multiple references to obtain better estimates of the global parameters; in this application only one reference is used. Like the previous curve fitting example, sub-range fitting occurs to get better estimates of the parameters. There is also a low-frequency and high-frequency residual capability in the fitter which is used to obtain the fit shown in Figure 4.14. The viscous model also shows deficiencies in the fit, especially at the anti-resonances. Figure 4.15 compares the percent magnitude errors in the two curve fits and shows that the structurally damped

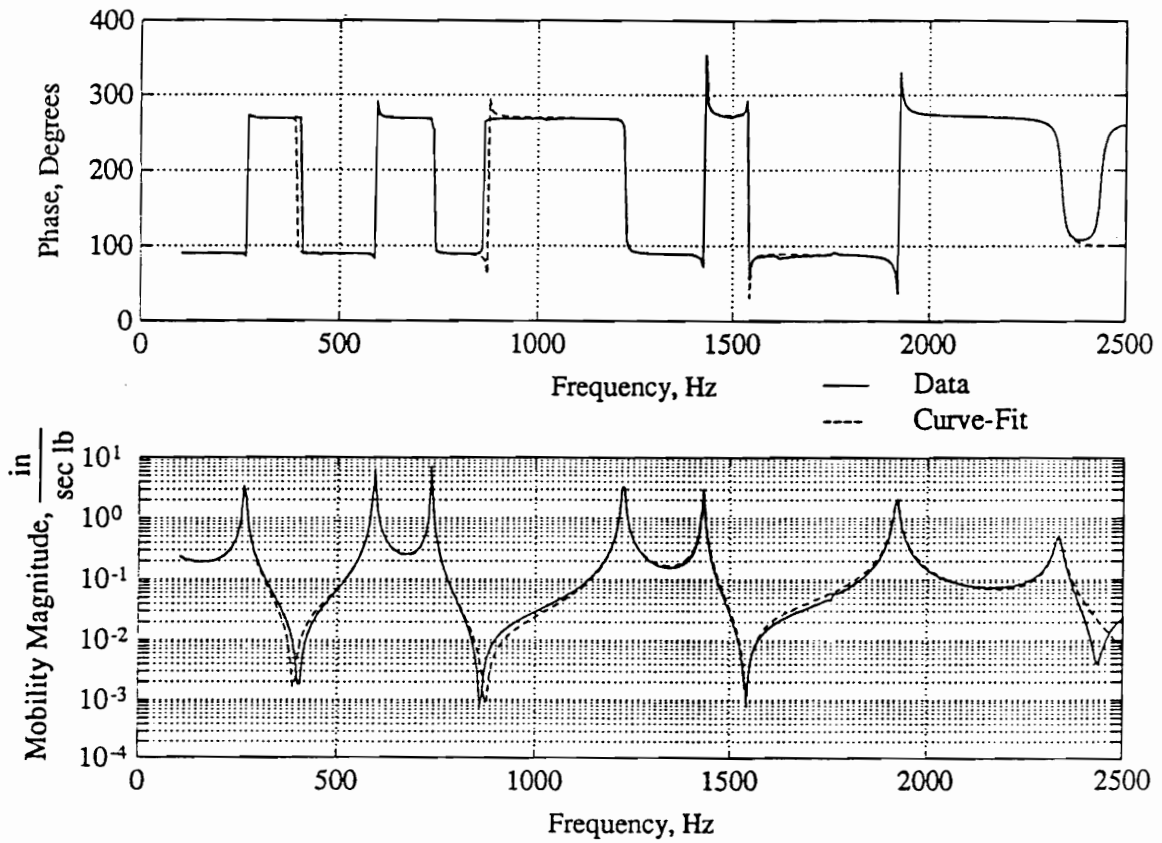


Figure 4.14 Measured FRF and SDRC Curve-Fit FRF

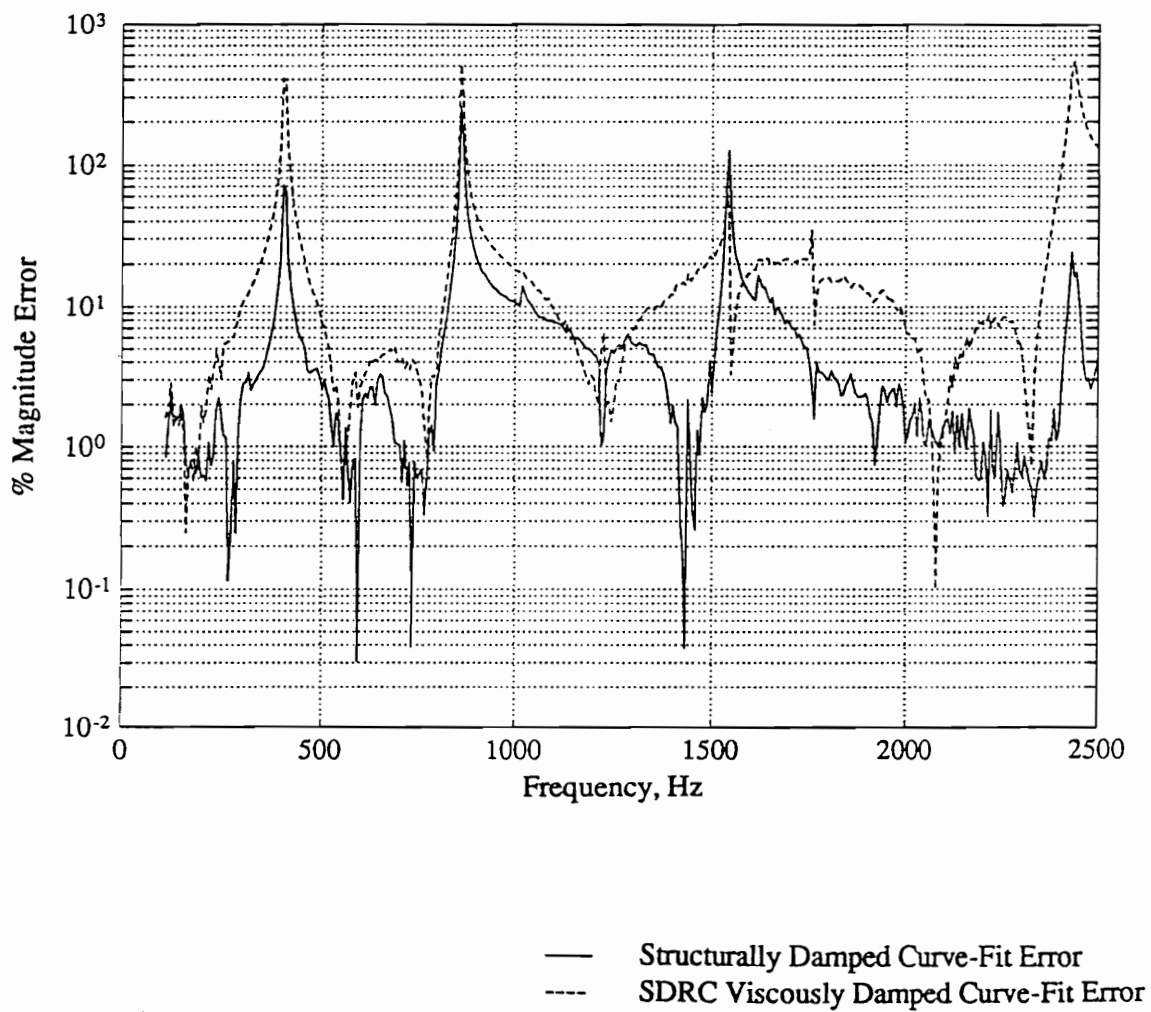


Figure 4.15 Comparison of Structurally Damped Curve-Fit Error and SDRC Curve-Fit Error

fitter has less error over the range of frequencies, indicating it is a better model of the physical structure.

The 1759 Hz mode was not fit using Orthogonal Polyreference because the fitter would not work on a mode of such small amplitude. The curve-fit amplitude would always be much greater than what the data showed. Table 4-3 shows the viscous model parameters. In general, it appears that the structurally damped fitter performs equally well or slightly better (certainly for low amplitude modes) than the commercially available viscous model fitter.

Table 4-3 I-DEAS Orthogonal Polynomial Curve Fitter Extracted Parameters for Location 3			
Mode	f_r	ζ_r	A_r
Low F. Res.	0 Hz	0	-2.93e-2 + 1.7e-5i
1	265.09	7.2e-4	-8.66e-5 - 3.10e-2i
2	592.22	1.00e-3	2.10e-5 - 1.74e-2i
3	732.21	4.0e-4	7.55e-5 + 9.21e-3i
4	1221.8	7.2e-4	-6.90e-6 + 8.67e-3i
5	1431.0	1.38e-3	-2.51e-5 - 4.33e-3i
6	1922.1	2.44e-3	7.24e-5 - 5.93e-3i
7	2338.0	3.28e-3	-3.91e-5 + 1.79e-3i
High F. Res.	-	-	-1.9e-10 + 1.8e-12i

4.4 An Example of Curve Fitting on Measured Data - Two FRF's

Obtaining good global parameter estimates is important in iterative least squares, so the use of two FRF's in the determination of the natural frequency and damping is highly beneficial. Using the FRF's from points 1 and 2 in Figure 4.1, the curve fits are obtained using the structurally damped fitter and are shown in Figures 4.16 and 4.17. Like the single FRF case, there is phase error in the fit which is likely attributed to the non-structurally damped behavior of the plate. Figure 4.16 shows magnitude error at the higher frequencies due to error in the pseudo-mode out of the range of the FRF. The resulting global parameters are shown in Table 4-4 and are used in a demonstration of iterative least squares in Chapter 5.

Table 4-4 Extracted Global Parameters Using The Structurally Damped Curve Fitter and Two FRF's		
Mode	f_r	η_r
Rigid Body	0 Hz	0
1	265.10	1.41e-3
2	592.19	2.05e-3
3	732.21	8.34e-4
4	1221.8	1.50e-3
5	1431.0	2.72e-3
6	1759.2	1.50e-3
7	1922.1	4.89e-3
8	2337.9	6.59e-3
Pseudo Mode		
9	2677.6	1.87e-2

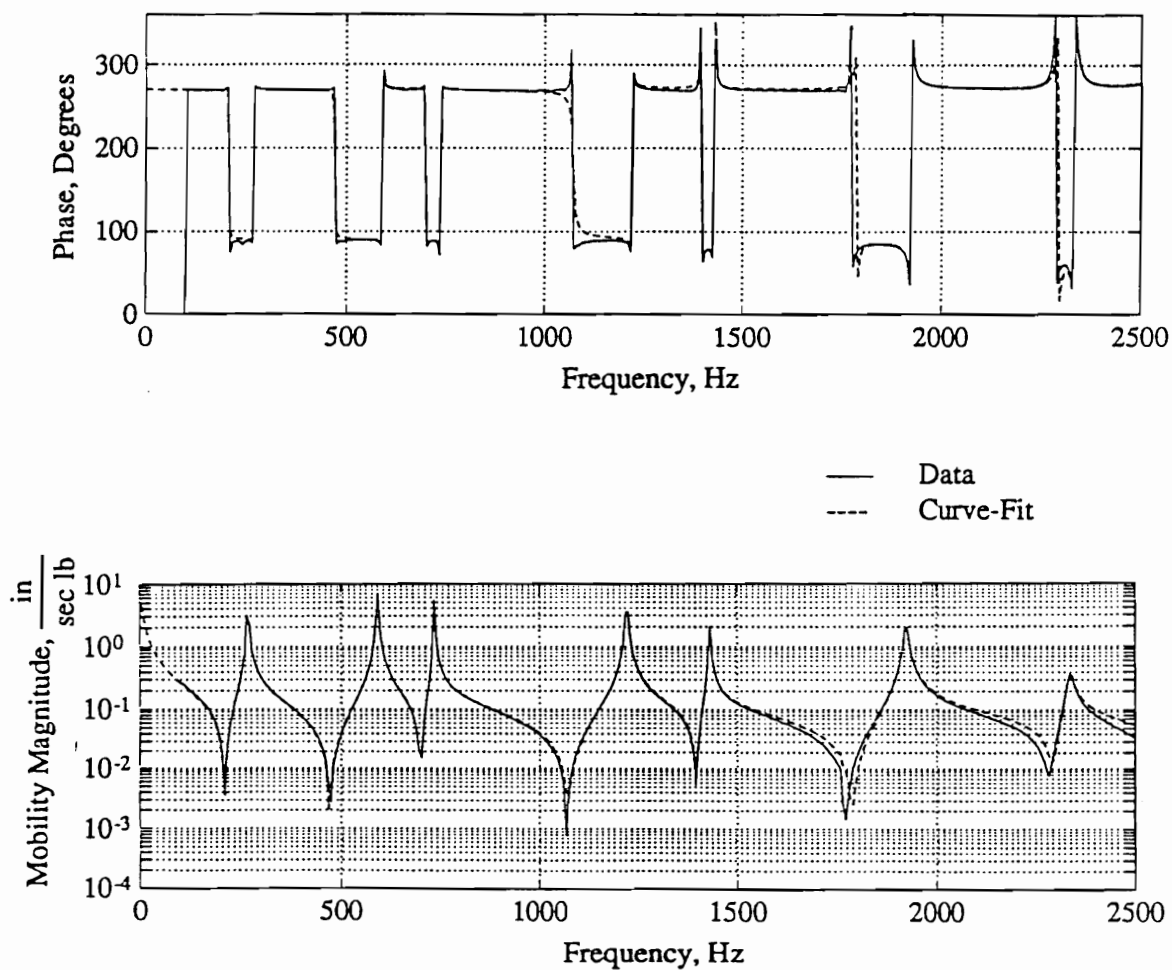


Figure 4.16 Curve-Fit of Driving Point FRF Using Structurally-Damped Curve Fitter

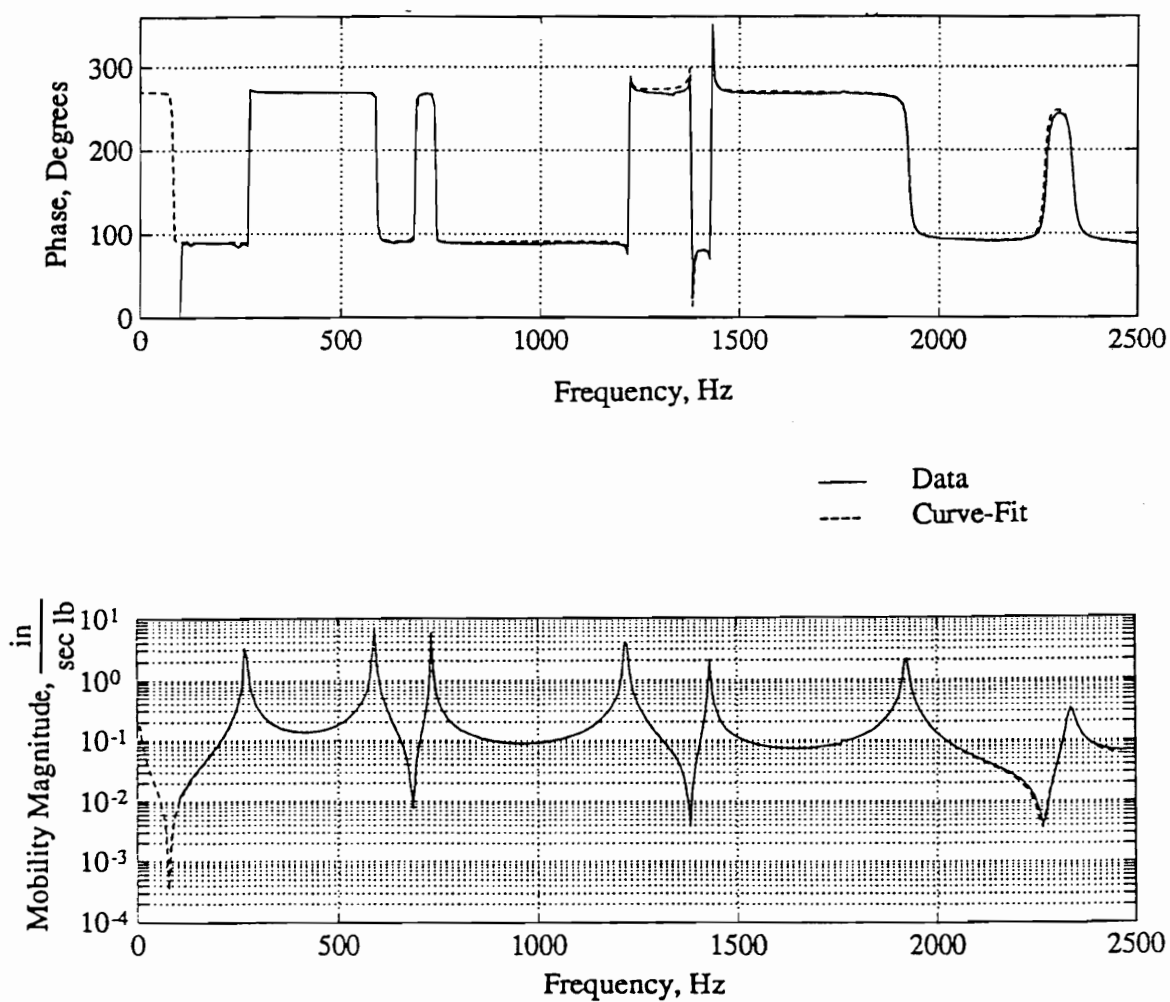


Figure 4.17 Curve-Fit of Point 2 FRF Using Structurally-Damped Curve Fitter

The natural frequencies in Table 4-4 compare favorably with those for the curve fit of point 3, shown in Table 4-1. The frequencies are within 0.006% of the frequencies at point 3. The damping coefficients for points 1 and 2 are within 2.8% of the damping coefficients for point 3.

4.5 Summary

The structurally damped fitter described in this chapter has shown good performance on an aluminum plate test structure. Use of two FRF's to determine the global parameters gives a reliable measure of the natural frequency and damping of the structure provided the forcing location and responses do not lie on modal nodes. The parameters obtained using this fitter will be used in the next chapter to demonstrate the iterative technique on real data.

Chapter 5

Experimental Results Using Iterative Least Squares

5.1 Demonstration on a Real Structure

To demonstrate the use of pre-determined global parameters on real data, the residues for the 592 Hz, 732 Hz, 1221 Hz and 1431 Hz modes of the structure presented in Chapter 4 will be extracted at location 3. The previously acquired global parameters in Table 4-4 will be used as well as 12 mobility measurements. The mobility measurements are taken using sine excitation at equally spaced frequencies which exceed the range of modes by $\pm 10\%$. The frequencies of excitation are shown in Table 5-1. From Chapter 3, equal frequency spacing is not optimal if variations in the global parameters are expected. However, the global parameters in Tables 4-1 and 4-4 compare very closely and the signal/noise ratio is mostly greater than 1,000 for acceleration and greater than 10,000 for the force as shown in Figure 4.6. These two facts give confidence that the global parameters are accurately known. Equal frequency spacing will be a compromise between convergence speed and insensitivity to global parameter variations.

Table 5-1	
Frequencies of Excitation for Least Squares	
1	500.00 Hz
2	593.75
3	687.50
4	781.25
5	875.00
6	968.75
7	1062.5
8	1156.25
9	1250.00
10	1343.80
11	1437.50
12	1531.25

The percent change in the residues is checked at each iteration, and once the magnitude change is less than 0.1%, the residues are considered to have converged. Five iterations were necessary for this example, as shown in Figure 5.1. Once again, a measure of adequacy of the parameter extraction is obtained by comparing the regenerated FRF with the measured FRF. Figure 5.2 shows the two curves and Figure 5.3 shows the magnitude error between the measured and curve-fit FRF's. The extracted residues are shown in Table 5-2 along with the residues obtained previously in Chapter 4 by using the structurally damped curve fitter.

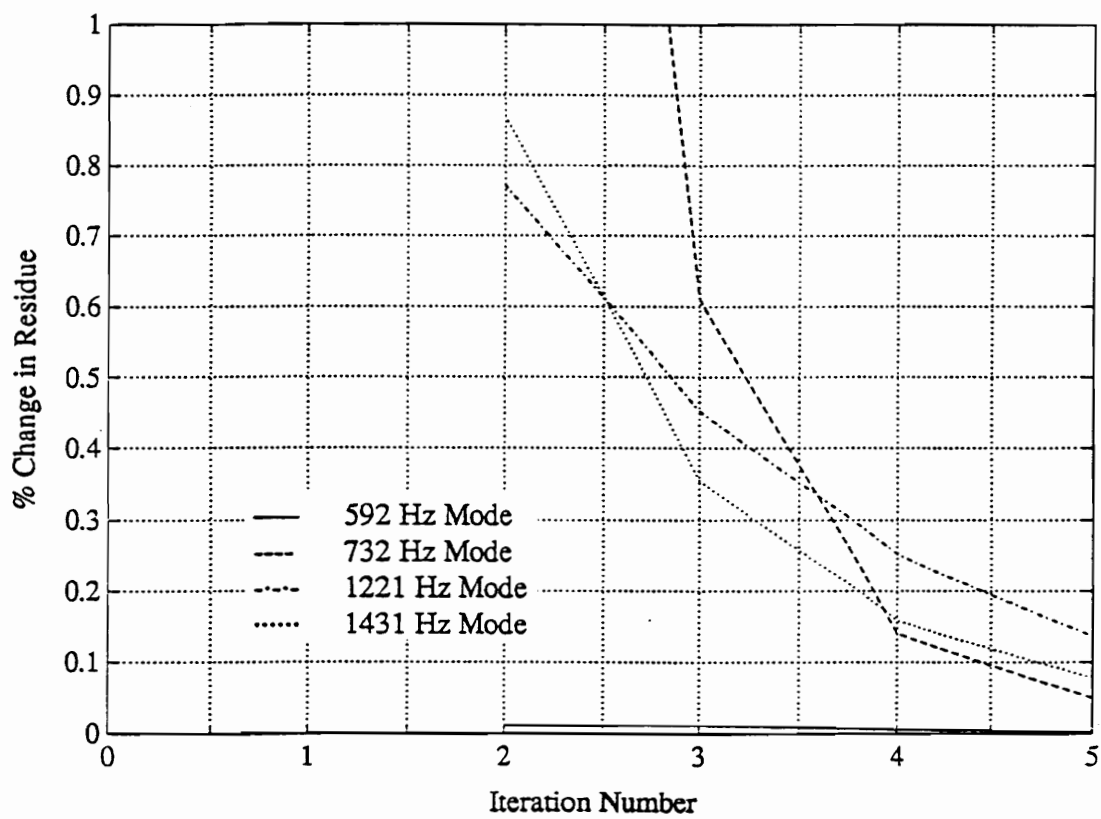


Figure 5.1 Convergence of Residues Using Iterative Least Squares on Measured Data

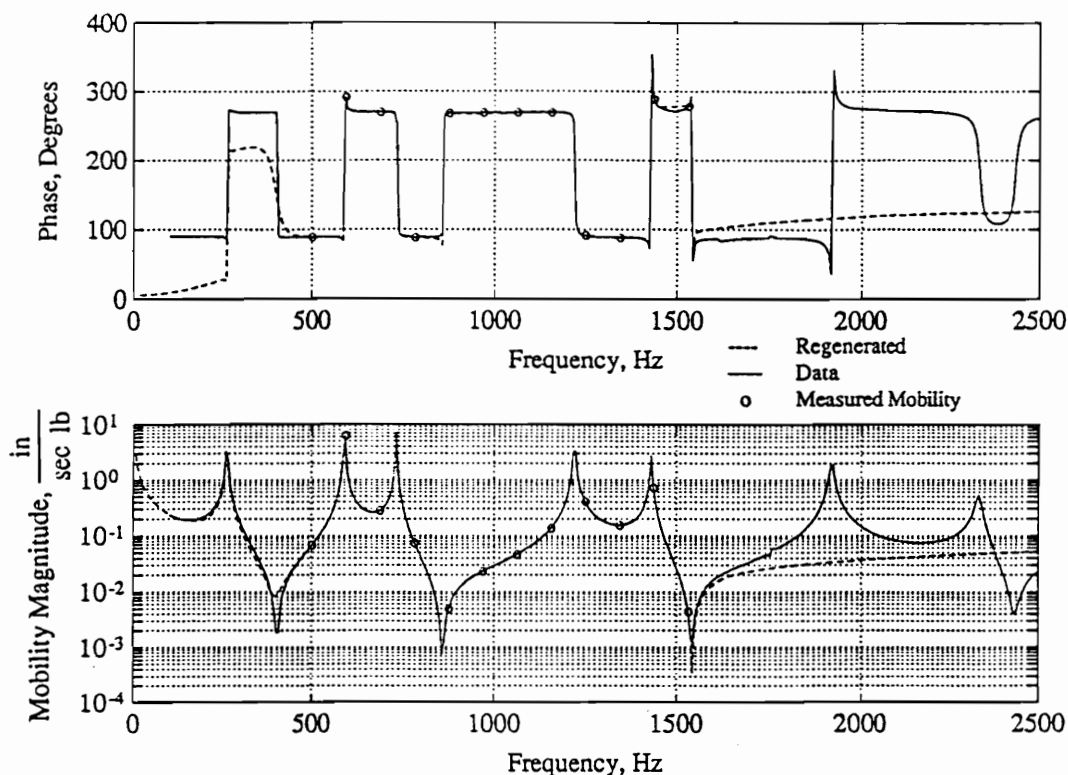


Figure 5.2 Measured FRF and Regenerated FRF Using Iterative Least Squares

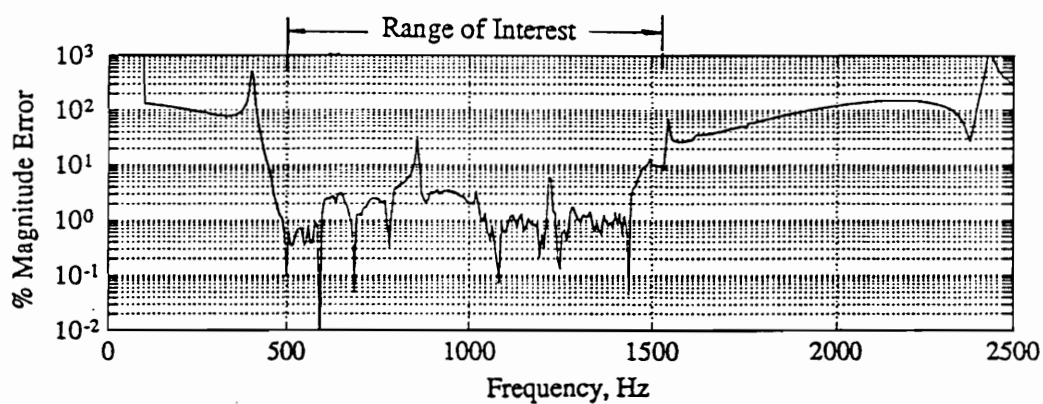


Figure 5.3 Magnitude Error Between Measured FRF and Regenerated FRF Using Iterative Least Squares

Table 5-2 Extracted Residues for Four Modes of Interest			
Mode	f_r	Iterative Residues	Struct. Damp. Fit Residues
1	592 Hz	133.2 + 1.75i	133.1 + 1.65i
2	732	-87.8 + 1.89i	-88.0 + .174i
3	1221	-133.7 - .143i	-134.4 - 6.37i
4	1431	67.0 + 1.32i	67.4 + .357i

The maximum magnitude error in the range of interest is about 30% and occurs at the lowest amplitude anti-resonance (approximately 830 Hz). Although the percent error is high at the anti-resonance, the absolute error at this point is about two orders of magnitude smaller than the error at the resonances. Most of the error over the range is less than 3%, which is significantly less than the typical errors from FRF curve fitting presented in Chapter 4.

The real residues in Table 5-2 show agree closely (< 0.6% difference), however the imaginary residues do not show close agreement. As observed in Chapter 3, inadequacies in the residual model can cause errors in the imaginary residues. More importantly, Chapter 4 showed that the structure does not totally conform to a structurally damped model, causing errors in the residues derived from the structurally damped curve fitter. Because of the non-structural behavior of the structure, the residues may be more correct than the full FRF curve-fit residues because of the ability of the residual model to minimize error without regard for a specific behavior. This leaves only four modes to be

fit to a structurally damped model instead of all the modes plus a rigid-body mode and a high-frequency pseudo mode as is the case with the full FRF curve fit.

A further comparison between the least squares-derived residues and the FRF curve-fit residues is obtained by substituting the least squares residues into the FRF curve-fit parameter set. Regenerating the FRF results in the fit shown in Figure 5.4, which has about the same error that was obtained when using the structurally damped fitter alone (Figure 4.8). The error is shown in Figure 5.5. This gives reassurance that the least squares residues are at least as good as the residues obtained using the full FRF curve fitter.

It has already been observed in Chapter 4 that the real system does not conform to a structural or viscous model, so the use of a good residual model may be the best way to handle modes that won't be mapped. The low error seen in Figure 5.3 is likely the result of modeling only the modes of interest and keeping flexibility in the curve fit for the other modes.

5.2 Comparison of Least Squares Residues to Curve Fit Residues Using a Commercially Available Technique

As discussed in Chapter 2, direct or iterative least squares is well suited to modal analysis of systems with large numbers of degrees of freedom because of the reduced mobility measurements required to solve for the residues. The most common existing technique to map large DOF systems is to use random excitation and statistical means of determining the FRF and then curve fit the entire FRF. Iterative least squares will be compared to this technique to further evaluate its capabilities.

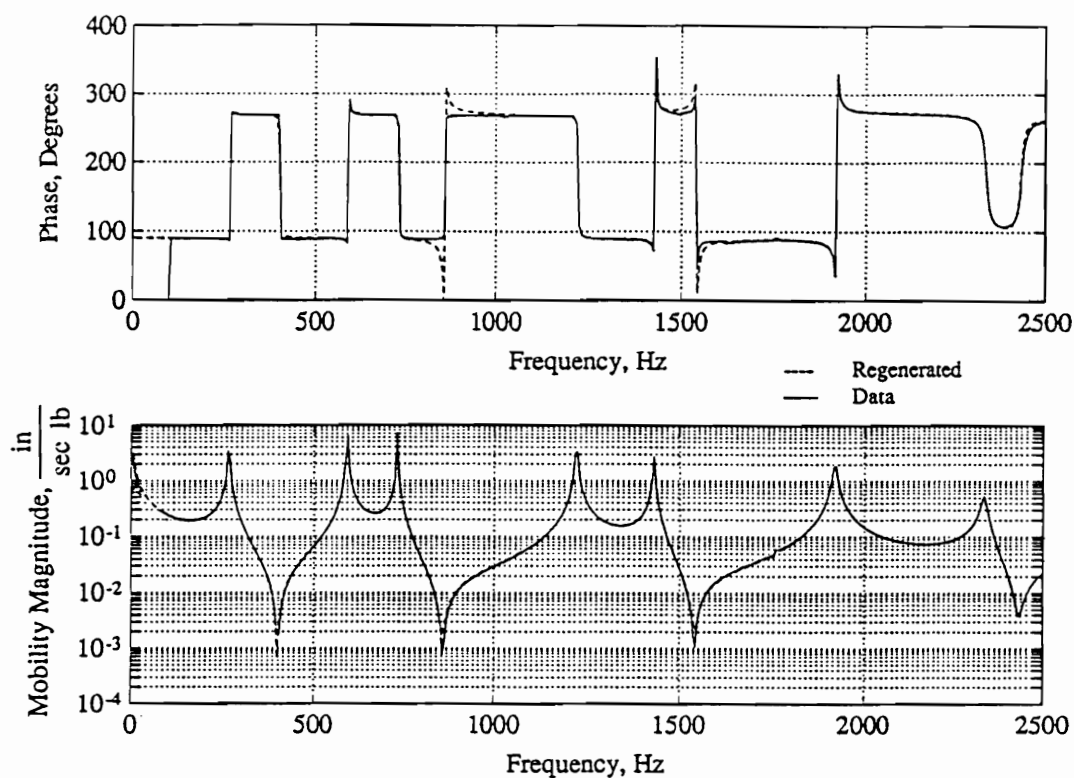


Figure 5.4 Measured FRF and Regenerated FRF From Curve-Fit Data with Iterative-Derived Residues Substituted For Curve-Fit Residues

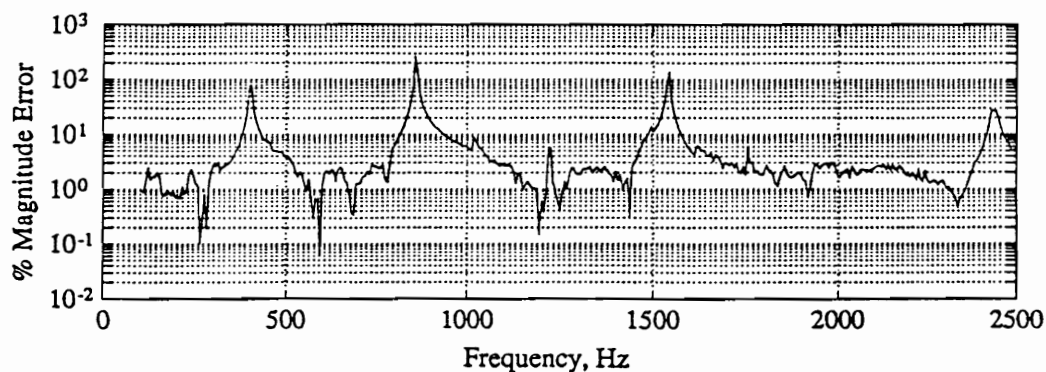


Figure 5.5 Magnitude Error Between Measured and Regenerated FRF's in Figure 5.4

Once again, location 3 on the structure in Figure 4.1 is used as the test point for evaluation. A ZonicTM System 7000 FFT driven by ZetaTM Software on an HPTM 375 Workstation is the platform used to conduct the test. Burst random excitation and exponential windows on the input and output are used to obtain the best possible FRF with random excitation. Figure 5.6 shows the FRF which exhibits more damping than the real system possesses due to the exponential window [44]. A correction is applied after the parameters are extracted based on the decay constant of the window to return to the actual parameters. For the System 7000, the exponential window was determined experimentally to be the following function:

$$W(t) = e^{-39.3t} \quad (5.1)$$

Returning the parameters to their correct values involves adding a term to the measured damping, ζ_r' :

$$\zeta_r = \zeta_r' + \frac{-39.3}{\omega_r} \quad (5.2)$$

Coherence was better than 0.99 over the range of frequencies except at the anti-resonances, where it dropped below 0.70. Ten averages of the data were taken.

After measuring the FRF, the parameters are extracted using the Orthogonal Polyreference Curve Fitter that was used in Chapter 4 on step-sine data. Table 5-3 shows the extracted parameters, which show some difference from the parameters in Table 4-3 for the step-sine case. As before, most of the variability is in the smaller of the two residue components; in the case of viscous damping it is the real part.

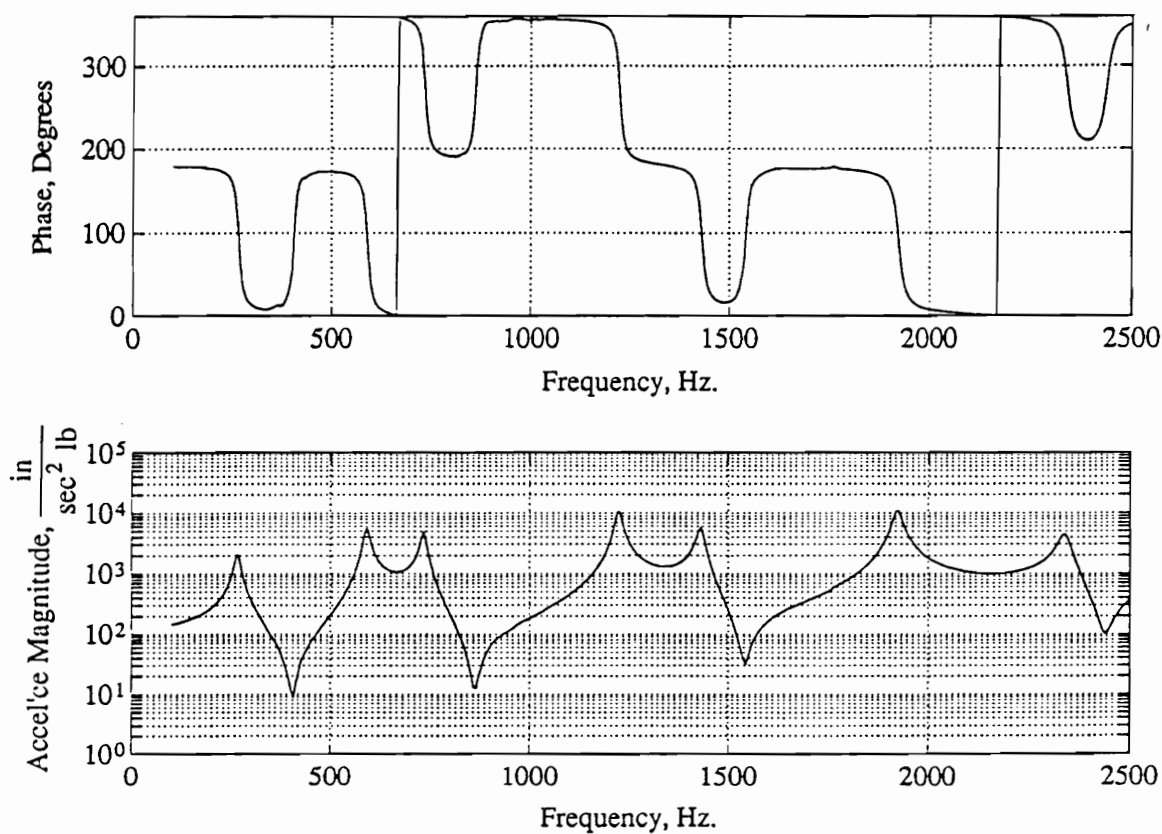


Figure 5.6 Accelerance of Point 3 Using Burst Random Excitation With Exponential Window

Table 5-3 Orthogonal Polyreference Curve Fit Using Burst Random				
Mode	f_r	ζ'_r	ζ_r	A_r
Low F. Res.	0 Hz	0	0	-2.93e-2 + 7.8e-4i
1	265.03	2.48e-2	1.18e-3	4.67e-4 - 3.07e-2i
2	592.18	1.16e-2	9.9e-4	-1.02e-4 - 1.76e-2i
3	732.25	9.04e-3	5.1e-4	-3.49e-5 + 9.20e-3i
4	1221.7	5.69e-3	5.8e-4	3.68e-5 + 8.75e-3i
5	1430.9	5.76e-3	1.39e-3	7.38e-5 - 4.29e-3i
6	1921.9	5.65e-3	2.40e-3	3.91e-5 - 5.43e-3i
7	2337.3	5.74e-3	3.07e-3	-8.71e-5 + 1.82e-3i
High F. Res.	-	-	-	3.17e-10 - 2.2e-11i

Figure 5.7 shows the regenerated FRF from the burst random-derived parameters and the original data. The magnitude error between the curves is shown in Figure 5.8 and closely resembles the error in Figure 4.15 which resulted from using the same fitter on step-sine data. However, the error at that resonances is greater and may be attributed to the use of the exponential window. From Table 5-3, it can be seen that most of damping in the parameter extraction is due to the window, so small errors in the window function can lead to significant errors in the damping. The potential for errors always exist when windows are used and they can be exaggerated for lightly damped structures; using sine excitation throughout eliminates this problem.

Comparing Figures 5.3 and 5.8 shows that the iterative regenerated curve has much less error than what is achieved with a robust commercial fitter. Furthermore, the computations required in the iterative technique after the global parameters have been identified is much less than for the Orthogonal Polynomial Fitter. In the latter case, 400

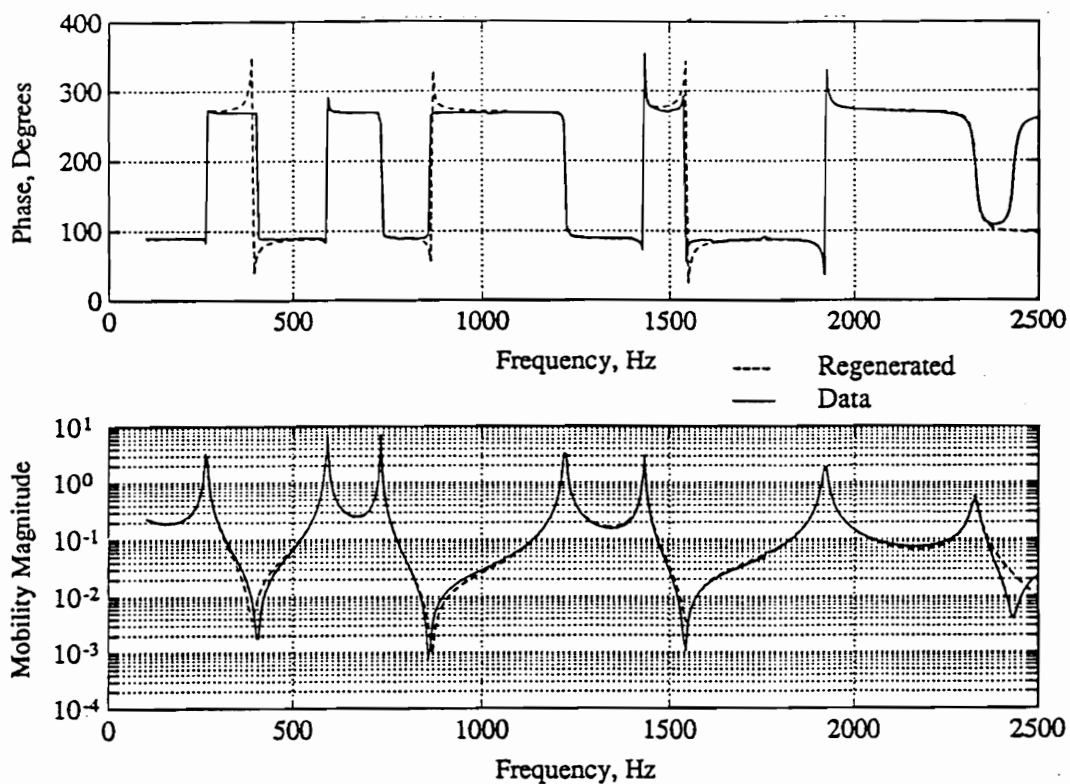


Figure 5.7 Measured and Regenerated FRF from SDRC Viscously Damped Curve Fit

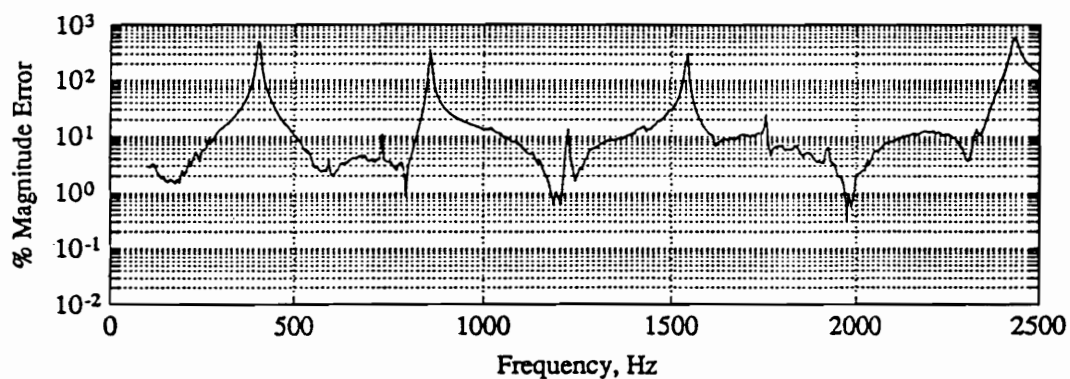


Figure 5.8 Magnitude Error Between Measured FRF and Regenerated FRF From SDRC Viscously Damped Curve-Fit

frequency-domain points must be fit at every structural point and this can require sub-range fitting if the modes are not well separated or they are low in amplitude. Iterative least squares requires many fewer measurements and the execution of just two Gram-Schmidt Factorizations and back-substitutions at every iteration to find the residues. These operations are simple vector operations which can be programmed at a low level to speed the execution.

5.3 Some Notes About Data Quality

The use of sine excitation in the determination of residues has a significant advantage over the use of random excitation. Sine excitation has been shown in the examples here to give average signal/noise ratios of 1000 and 10,000 in the acceleration and force signals, respectively. The burst random test presented in the previous section showed average signal/noise ratios of only 10 and 100 for the acceleration and force signals. These low values were a result of using low force levels to keep the lightly damped structure amplitudes reasonable. Based on these levels of signal strength, a measure of the reduction in frequency-domain measurements can be made for sine as opposed to random excitation.

From Bendat and Piersol [45], the variance of an FRF measurement is:

$$\text{Var}(|\hat{H}_{xy}|) = \frac{(1 - \gamma_{xy}^2) |H_{xy}|}{2\gamma_{xy}^2 n_d} \quad (5.1)$$

where x and y are the measured input and output signals. This can be calculated for the sine excitation case and the random excitation case. The number of samples of data at each frequency-domain measurement using sine is $n_{d_s} = 1$, since the data is of such high quality. It is desired to determine the number of samples of random-derived FRF data which would be necessary to achieve the same variance as in the sine case. Solving for the number of samples in the random case yields:

$$n_{d_r} = \frac{\gamma_{xy_s}^2 (1 - \gamma_{xy_r}^2)}{\gamma_{xy_r}^2 (1 - \gamma_{xy_s}^2)} \quad (5.2)$$

where the sub-subscripts r and s refer to random and sine respectively. The coherence functions γ_{xy_r} and γ_{xy_s} can be expressed in terms of signal/noise ratios:

$$\begin{aligned} \gamma_{xy_r}^2 &= \frac{1}{\left(1 + \left(\frac{n_{input_r}}{s_{input_r}}\right)^2\right) \left(1 + \left(\frac{n_{output_r}}{s_{output_r}}\right)^2\right)} \\ \gamma_{xy_s}^2 &= \frac{1}{\left(1 + \left(\frac{n_{input_s}}{s_{input_s}}\right)^2\right) \left(1 + \left(\frac{n_{output_s}}{s_{output_s}}\right)^2\right)} \end{aligned} \quad (5.3)$$

where n and s are the noise and signal magnitudes, respectively. Using the average measured signal/noise ratios results in the following coherence functions:

$$\gamma_{xy_s} = 0.999999$$

$$\gamma_{xy_r} = 0.990000$$

From equation 5.2, the number of averages in the random case needed to achieve the same variance as in the sine case is $n_{d_r} = 10100$. If there are 400 frequency-domain data points used in the curve-fit, then n_{d_r} can be divided by 400 to get the number of random averages of the FRF per each sine mobility measurement. In doing this, the total number of degrees of freedom does not change, only the distribution of the measurements in the frequency domain. The resulting number of averages using random is equal to $25 \times$ (the number of sine mobility measurements). Thus, the use of sine excitation in the least squares approach can significantly reduce the number of averages of data which will speed up the data acquisition process.

Chapter 6

Conclusions and Recommendations

6.1 Conclusions

Developments of equipment capable of rapid point-to-point velocity measurement for modal analysis have created a need for the rapid determination of modal parameters. As the speed of data acquisition increases, the numbers of structural points can increase and the calculations required will also increase. Existing estimation techniques which use random excitation require a frequency response function measurement at each structural point and then a curve-fit the frequency domain data; this can take several minutes per point if there is a user interface involved. A reduction in the calculations is possible by assuming common poles at all structural points, however there will still be an entire analysis frequency range to fit. A faster technique is desired.

Recent developments in sine testing have been presented as faster and more robust alternatives to traditional sine-dwell modal testing. The Spatial Sine Testing Technique [11 - 14] presented in Chapter 2 will reduce the amount of frequency-domain data needed to obtain parameters, but the technique assumes a large number of reference inputs will be used. Driving several shakers on a structure is not always practical and the hardware to accomplish this can be expensive.

The least squares technique using the direct or iterative approach attempts to solve the problem of rapidly determining mode vectors. Least squares on a sparse data set uses

predetermined global parameters to reduce the calculations necessary for determining mode vectors. The technique finds modal residues using only a few frequency domain measurements as opposed to many measurements over an entire analysis range. Multi-sine excitation can further reduce the data acquisition time by simultaneously exciting all the frequencies needed to find the residues.

A significant advantage in using least squares is that sine excitation can be employed throughout the estimation process. The global parameters are extracted once at the beginning of the test from an FRF which is accurately determined using step-sine excitation. Since the whole FRF is measured only once, the extra time needed to perform a step-sine test can be justified. The rest of the structural points do not require an entire FRF to be measured. Instead, only a few responses are measured at selected frequencies within a range where the modal information is desired. This reduced set of measurements at the structural points of interest is accomplished using sine or multi-sine excitation. Signal/noise ratios of greater than 1000 were demonstrated on an aluminum structure, exceeding the random excitation signal/noise ratio by a factor of 100.

Eliminating random excitation from the modal test also eliminates windowing and leakage problems. As demonstrated in Chapter 4, use of an exponential window is necessary when testing a lightly damped structure with random excitation. This can lead to damping errors if the window function is not precisely known. Random excitation may also lack the energy density needed to excite all frequencies in the range of interest; sine excitation concentrates the energy at one spectral line which usually provides enough energy for large structures.

An improved residual model is presented which accounts for modes that are not modeled within the "in-range" set. The residual model shows much less error when compared in a simulation to the commonly accepted residual model. Using this model, the

out-of-range modal content is removed to improve the in-range least squares fit; an iterative technique is employed to progressively remove the residual content.

Validation of the iterative technique is made using a simulated FRF based on a real structure. Cases of global parameter error, modes which are low in amplitude or close to each other, and the number of frequency-domain measurements are studied to identify trends using the iterative approach. With high signal/noise ratios, the final errors from the iterative technique are close to the errors in using the direct approach. Since only one structure was simulated, this study provided typical trends and not quantitative data for all cases. The trends seen in the simulation are as follows:

For a lightly damped structure,

1. The minimum number of mobility measurements necessary for an over-determined system is adequate based on random errors due to noise.
2. Measurements near resonance result in faster convergence but also cause larger errors in the residues if the global parameters are in error.
3. Measurements near closely spaced modes will decrease the final residue error and increase the rate of convergence for the close modes.
4. Measurements near low amplitude modes will decrease the final residue error and increase the rate of convergence for the low amplitude mode.

For a moderately damped structure,

1. The convergence rate and final errors are similar to the lightly damped case, although cases of global parameter error were not investigated.

Complementing the least squares technique, a curve fitter is presented that assumes structurally damped behavior and allows the fitting of two FRF's simultaneously. The

curve fitter uses Chebechev Polynomials as the fitting basis and a Gram Schmidt Technique for solving the over-determined system. The fitter shows superior results when compared to a commercially available viscously damped fitter, implying that for the particular case, a structurally damped model is more accurate.

A demonstration of using iterative least squares on experimental data from an aluminum structure is presented. The magnitude error in the regenerated FRF from the iterative-derived modal parameters is about an order of magnitude less than the error resulting from conventional FRF curve fitting. Comparisons are made to viscous and structurally damped curve fitting algorithms, step-sine and burst random excitation techniques. In all cases, the least squares technique shows less error in the regenerated FRF. A possible reason for this is the ability of the residual model to accommodate unmodeled dynamic behavior of the system.

The experimentally derived residues using iterative least squares are compared to the residues obtained with a structurally damped FRF curve fitter. There is close agreement in the residues for both techniques which gives assurance the iterative scheme works.

Although not addressed here, the time required to map the mode vectors of a structure using an iterative or direct approach is likely to be much less than using conventional FRF curve-fitting techniques. The QR decomposition and back-substitution required at each iteration is significantly less computation than the common 400 spectral line curve fitting, even if the global parameters are known. If resonance sine-dwell testing is being used, the data acquisition time can be reduced but mode isolation and complexity becomes a problem.

6.2 Recommendations

Although much of the work presented shows that least squares will satisfactorily determine residues quickly, there are issues associated with using the technique which should be addressed in the future.

1. Comparing the flop count to the count for other techniques. A study of computational requirements for available modal techniques should be made to determine the magnitude of the benefit in using an iterative or direct scheme. This would be helpful just for sorting existing techniques in terms of their flop demand.
2. The similarity of the iterative approach to an optimization technique is evident in the iteration scheme: A vector of remaining mobility is used to find a residual vector or a residue vector (depending on the stage of the iteration) through a transformation. The remaining mobility can be considered as a search direction to find the optimal residues and residuals. The iterative technique should be investigated from this standpoint to perhaps arrive at yet a better residual model.
3. The curve fitter of Chapter 4 works well for data with high signal/noise ratios, as was the case presented here. However, a bias error occurs when noise exists on the data. To account for this, an estimate of the noise must be subtracted from a term which contains the bias error. A statistical approach must be taken to determine the noise level, and then the curve fitting should be performed on averaged data. This problem should be approached with the intention of obtaining a curve fitter that works well with noisy data.

4. The regenerated FRF from a curve fit of experimental data in Chapter 4 showed inadequacies in the model when the real part of the mobility was examined. This was attributed to actual behavior that did not conform to a structurally damped or viscously damped model. It is recommended that other damping models be considered in the curve-fitting process to minimize this error.
5. Higher-damped structures should be investigated to see what effect variations in the global parameter estimate has on the residue determination. Also, structures with lower natural frequencies should be investigated. Part of the motivation for this work was the author's experience in the aircraft industry, and applications to aircraft would imply the testing of structures that have low natural frequency, moderate damping and complexity in the mode shape.

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Appendix

The first program listing is an algorithm for converting the vector of Chebechev coefficients in equation (4.15) to ordinary power series coefficients. In addition, the imaginary operator is considered which changes the sign of certain coefficients.

The second program listing is the batch program used on the FFT to perform the step-sine modal test. Amplitude and force signals are monitored and the input signal is adjusted to maintain an acceptable level of force and acceleration during the test.

Chebechev-to-Power Series Conversion Algorithm

```
% this will convert chebechev polynomials to regular power polynomials
% May 27 92
% vector ab is the br,bi,ar,ai vector
% generate transformation matrix tr, then decimate for
% numerator and denominator coefficients

clear tr
clear ttd
clear ttn
clear td
clear tn
clear aaa
clear bbb

lg = length(ab);

order3 = lg/4;

lim = 2*order3+1;

% initialize

tr = zeros(lim,lim);

tr(2,2) = 1;

flip = 1;
cof = 1;
for k = 1:lim;

    if flip == 1;

        tr(1,k) = cof;
        cof = -cof;
    end;
    flip = -flip;
end;

for k2 = 3:lim;

    for k1 = 2:lim;

        tr(k1,k2) = 2*tr(k1-1,k2-1) - tr(k1,k2-2);
    end;
end;
```

```

end;

% generate numerator and denominator transformations

flip = 1;
incl = 1;
inc2 = 1;

for k = 1:lim;

    k
    if flip == 1;

        td(incl,:) = tr(k,:);
        incl = incl + 1;
    end;

    if flip == -1;
        tn(inc2,:) = tr(k,:);
        inc2 = inc2 + 1;
    end;

    flip = -flip;

end;

% decimate for odd and even values only

flip = 1;
incl = 1;
inc2 = 1;

for k = 1:lim;

    if flip == 1;

        ttd(:,incl) = td(:,k);

        if k < lim;

            ttn(:,incl) = tn(:,k+1);
        end;

        incl = incl+1;

    end;

    flip = -flip;

end;

% ttn, ttd is the correct matrix

%determine p from tt

% br is 1st order+1
% bi is 2nd order+1
% ar is 3rd order (highest term is 1)
% ai is 4th order (highest term is 1)

```

```

br = ttd*ab(1:order3+1);
bi = ttd*ab(order3+2:2*order3+2);

% a's are numerator, one less order and highest order is already determined = 1

arb = zeros(1:order3)';
arb(1:order3-1) = ab(2*order3+3:3*order3+1);
arb(order3) = 1;
aib = zeros(1:order3)';
aib(1:order3-1) = ab(3*order3+2:4*order3);
aib(order3) = 1;

ar = ttn*arb;
ai = ttn*aib;

% convert to complex values,
% normalize b's, a's

for k = 1:order3+1

    bbb(k) = (br(k) + i*bi(k));
%    bbb(k) = (br(k) + i*bi(k)) / (br(order+1) + i*bi(order+1));
end;

for k = 1:order3;

    aaa(k) = (ar(k) + i*ai(k));
%    aaa(k) = (ar(k) + i*ai(k)) / (ar(order) + i*ai(order));
%    aaa(k) = (ar(k) + i*ai(k)) / (ar(order));
end;

% convert polynomial coefficients to iw coefficients

flip = 1;
for k = 1:length(aaa);

    if flip == 1;

        aaa(k) = -i*aaa(k);
%        aaa(k) = -aaa(k);
        end;

        if flip == -1;

            aaa(k) = i*aaa(k);
%            aaa(k) = aaa(k);
            end;

        flip = -flip;
    end;

flip = 1;
for k = 1:length(bbb);

    if flip == 1;

        bbb(k) = bbb(k);
        end;

        if flip == -1;

```

```

        bbb(k) = -bbb(k);
    end;

    flip = -flip;
end;

conj(aaa')

% pause;

% save normalization factor
normfacb = bbb(order3+1);
normfaca = aaa(order3);

bbb = bbb / normfacb;

bbb
end;

```

Batch Program for Step-Sine Test with Force and Acceleration Feedback

```
1.000 iactivate 1,2,3,4
2.000 oactivate 1
3.000 defblk b1 512 complex
3.010 defblk b7 512 complex
3.020 defblk b8 512 complex
4.110 defblk b2 512 real
4.120 defblk b3 512 real
4.130 defblk b4 512 real
4.140 defblk b5 512 real
4.150 defblk b6 512 real
4.210 sens 1 1
4.330 sens 2 1
4.400 sens 3 1
4.500 sens 4 1
5.000 oswaveform sine
6.000 window none
7.000 setwin all resp
7.100 i2=1
8.000 i1=17
9.000 f1=100
10.000 f2=.05
10.100 f3=.02
10.200 f4=.150
10.300 f5=.5
10.400 osfr 100
11.000 osamp f2
18.000 sgon
19.000 ifrc internal 2500
20.000 osfr f1
21.000 fsv 1 auto
22.000 fsv 2 auto
22.100 fsv 3 auto
22.200 fsv 4 auto
24.000 arange block 1 1.3
26.000 spec1.2acquire2
26.010 ou 'the frequency is'
26.020 ou f1
26.030 ou 'the signal gen output is'
26.040 ou f2
26.930 ou 'the max accel is'
26.940 ou f5
26.950 ou 'the accel is'
26.951 ou d2[i1]
27.000 if d2[i1] > f5 jump 30
28.200 if d2[i1] < .006 jump 32
```

```

29.000 jump 34.1
30.000 f2=f2*.9
31.000 jump 33
32.000 f2=f2*1.1
32.100 if f2 < f4 jump 33
32.200 jump 33.2
33.000 osamp f2
33.100 jump 20
33.200 f2=f4
34.100 b4[i1]=d1[i1]/d1[i1-5]
34.200 b5[i1]=d2[i1]/d2[i1-5]
34.300 b2[i1]=d1[i1]
34.400 b3[i1]=d2[i1]
34.500 b6[i1]=f2
35.000 frf2,1.frf3,1.frf4,1acquire
37.000 b1[i1]=d1[i1]
37.100 b7[i1]=d2[i1]
37.200 b8[i1]=d3[i1]
40.010 if f2 = f4 jump 40.5
40.020 jump 40.6
40.500 f2=.05
40.550 osamp f2
40.600 i1=i1+1
40.700 f1=f1+6.25
41.000 if i1 > 401 jump 43
41.100 savfunct 'frfplt1' b1
41.110 savfunct 'frfplt2' b7
41.120 savfunct 'frfplt3' b8
41.200 savfunct 'specfp' b2
41.300 savfunct 'specap' b3
41.400 savfunct 'stonf' b4
41.500 savfunct 'stona' b5
41.600 savfunct 'ampp' b6
42.000 jump 20
43.000 sgoff
43.100 redirect 'jul251.dat'
43.200 lblkt b1
43.300 redirect 'jul252.dat'
43.400 lblkt b2
43.500 redirect 'jul253.dat'
43.600 lblkt b3
43.700 redirect 'jul254.dat'
43.800 lblkt b4
43.900 redirect 'jul255.dat'
44.000 lblkt b5
45.000 redirect 'jul256.dat'
46.000 lblkt b6
47.000 redirect 'jul257.dat'
48.000 lblkt b7
49.000 redirect 'jul258.dat'
50.000 lblkt b8
51.000 redirect
52.000 end

```

Vita

Kevin Kochersberger was born on May 25, 1961 in Jamestown, NY on the day that John F. Kennedy announced his plan for a manned mission to the moon. He grew up in a really small lake-effect town called Sinclairville and graduated from Cassadaga Valley Central School in 1979. From there he attended Jamestown Community College in Jamestown, NY and graduated in 1981 with a degree in Engineering Technology. In the fall of 1981, he moved to Blacksburg, VA and entered the Mechanical Engineering program at Virginia Tech. He graduated Summa Cum Laude in 1983 and stayed on to complete a Masters Degree in 1984. From there, Kevin moved to Manassas, VA to work at IBM as a design engineer. Not challenged enough, he moved to St. Louis, MO in 1986 and worked in the Structural Dynamics Department at McDonnell Douglas until 1989. Not challenged enough, he then came back to Blacksburg to complete a Ph. D. in Mechanical Engineering. Sufficiently challenged, Kevin has decided never to get another academic degree. Kevin is presently seeking employment in academics or the aviation industry, and hopes to get a date in the near future.

A handwritten signature in black ink, appearing to read 'K- Vg' followed by a large checkmark.