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Charge Assignments in Multiple–$U(1)$ Gauge Theories

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Abstract

We discuss the choice of gauge field basis in multiple–$U(1)$ gauge theories. We find that there is a preferred basis, specified by the *charge orthogonality condition*, in which the $U(1)$ gauge fields do not mix under one–loop renormalization group running.

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1. Introduction

In Refs. [1, 2], Lane and Eichten discuss topcolor–assisted technicolor [3] models which include two $U(1)$'s as part of their gauge groups. The charges of the fermions under the two $U(1)$'s were restricted by anomaly cancellation conditions and phenomenological requirements, but beyond that they were free to move in a relatively large parameter space.

In a general model with multiple $U(1)$'s, including those of Lane and Eichten, radiative corrections will mix the $U(1)$'s at different scales even if the gauge fields were orthogonal at the initial scale. As a result, the meaning of the $U(1)$ charges assigned to the matter fields becomes ambiguous and the phenomenology is difficult to decipher or control.

In this letter, we argue that for any multiple–$U(1)$ gauge theory, one can always find a basis for the gauge fields in which this scale–dependent mixing does not occur (at one–loop order). In this basis, the $U(1)$ charges of the matter fields satisfy a constraint which we call the *charge orthogonality condition* (COC), and both the $U(1)$ fields and charges become scale–independent concepts.

Since any multiple–$U(1)$ gauge theory is equivalent to another in which the COC is satisfied, one can impose the COC on the matter–field charges without loss of generality. Not only does this simplify the study of the model at different scales, but also places a welcome additional restriction on the allowed charge parameter space.

2. $U(1)$ Gauge Boson Mixing

Consider a theory consisting of the gauge group $U_1(1) \times U_2(1) \times \cdots \times U_N(1)$ coupled to fermions ψ_a.[‡] The Lagrangian one might naively write down for such a theory is:

$$\mathcal{L} = -\sum_{i=1}^{N} \frac{1}{4g_i^2} F_{\mu\nu}^i F_i^{\mu\nu} + \sum_a \bar{\psi}_a \left(\partial_\mu - \sum_{i=1}^{N} Y_i^a A_\mu^i \right) \gamma^\mu \psi_a.$$

Here, g_i is the gauge coupling of the i–th $U(1)$ gauge group and Y_i^a is the i–th $U(1)$ charge of the a–th fermion. However, for $U(1)$ gauge fields the field strength tensors

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i$$

are by themselves gauge invariant, so terms of the form

$$F_{\mu\nu}^i F_j^{\mu\nu}, \qquad (i \neq j)$$

are also permitted. So, the most general Lagrangian allowed by the symmetry is actually

$$\mathcal{L} = -\frac{1}{4} \sum_{i,j} F_{\mu\nu}^i K_i^j F_j^{\mu\nu} + \sum_a \bar{\psi}_a \left(\partial_\mu - \sum_{i=1}^{N} Y_i^a A_\mu^i \right) \gamma^\mu \psi_a,$$

[‡]Our results apply equally to models with scalars coupled to multiple $U(1)$ fields, but for simplicity we restrict our attention to a model with only fermions.

where the $N \times N$ matrix K_i^j can be assumed to be real, symmetric, and positive definite. Therefore, in general, the $U(1)$ gauge fields will mix, and the diagonal elements of K_i^j do not have a simple interpretation as coupling constants.

Of course, it is always possible to choose a basis which diagonalizes K_i^j and to identify its diagonal elements as the inverse coupling constants squared in that particular basis. However, even if K_i^j is diagonalized at one scale, renormalization reintroduces the off–diagonal mixing terms at other scales, and the mutually orthogonal $U(1)$ fields (and the charges to which they couple) must be redefined at every new scale. For a general $U(1)_1 \times U(1)_2 \times \cdots \times U(1)_N$ theory (as might be embedded in models such as topcolor–assisted technicolor) this scale–dependent mixing of the gauge fields, the gauge couplings, and the charges of the matter fields will likely make the analysis of its phenomenology confusing. The natural question arises whether it is possible to find a basis in which K_i^j is diagonal regardless of scale. The answer, at least at one loop, is *yes*.

To construct such a basis, first diagonalize K_i^j at some initial scale μ:

$$
K_i^j(\mu) = \begin{bmatrix} \frac{1}{\kappa_1^2} & & & 0 \\ & \frac{1}{\kappa_2^2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\kappa_N^2} \end{bmatrix}
$$

Next, rescale the rotated gauge fields and charges:

$$
\begin{aligned}
\frac{1}{\kappa_i} A_\mu^i &\longrightarrow A_\mu^i, \\
\kappa_i Y_i^a &\longrightarrow Y_i^a.
\end{aligned}
$$

This rescaling leaves the coupling term between the gauge bosons and fermions invariant while reducing $K_i^j(\mu)$ to a unit matrix:

$$
K_i^j(\mu) \longrightarrow \delta_i^j.
$$

In other words, we can set all the gauge coupling constants to one by absorbing them into the definition of the charges.

Under renormalization, fermion loops induce scale–dependent mixing of the gauge fields which were diagonalized at scale μ. That is, generically K_i^j receives both diagonal and off–diagonal contributions. At one loop, one finds at scale μ':

$$
K_i^j(\mu') = \delta_i^j - M_i^j \ln\left(\frac{\mu'}{\mu}\right)^2,
$$

where

$$
M_i^j = \frac{1}{24\pi^2} \sum_a Y_i^a Y_j^a
$$

in which the index a runs over the *chiral* fermions.

If the $U(1)$ charges satisfy the condition

$$\sum_a Y_i^a Y_j^a = 0 \qquad \text{if } i \neq j, \tag{1}$$

then M_i^j is diagonal. If it is not, one can always diagonalize M_i^j by an orthonormal rotation of the gauge fields and charges which leaves both $\sum_{i=1}^N Y_i^a A_\mu^i$ and $K_i^j(\mu)$ (the unit matrix) invariant. In this new basis, both $K_i^j(\mu)$ and M_i^j are diagonal and as a result, $K_i^j(\mu')$ remains diagonal for all scales μ'. Note that the new rotated charges now satisfy Eq. (1).

This discussion illustrates that it is always possible to find a basis in which $K_i^j(\mu)$ and M_i^j are simultaneously diagonalized through *rescaling* and *orthonormal rotations*. (For $K_i^j(\mu)$ and M_i^j to be simultaneously diagonalizable by orthonormal rotations alone, they must commute.) In this basis, which is characterized by the fact that Eq. (1) is satisfied, the $U(1)$ fields and charges do not mix under one loop running. Hence, this should be the basis of choice for analyzing the phenomenology of the theory.

However, starting with a general multiple–$U(1)$ theory with arbitrary (anomaly–free) charge assignments and then finding the preferred basis in which Eq. (1) is satisfied is tantamount to imposing Eq. (1) on the charges to begin with. Therefore, without loss of generality, one can impose Eq. (1) as an additional condition to anomaly cancellations when making the initial charge assignments. We call Eq. (1) the *charge orthogonality condition* (COC).

Since M_i^j is real, symmetric, and positive semi–definite, all of its eigenvalues are real and non–negative. Let us denote them by:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

After diagonalization

$$K_i^j(\mu') = \delta_i^j \left[1 - \lambda_i \ln\left(\frac{\mu'}{\mu}\right)^2 \right],$$

and we can make the identification

$$\frac{1}{g_i^2(\mu')} \equiv 1 - \lambda_i \ln\left(\frac{\mu'}{\mu}\right)^2.$$

We make the following observations:

1. The ratios of the $U(1)$ coupling constants $g_i(\mu)$ flow towards infrared fixed points determined by the ratios of the eigenvalues of M_i^j:

$$\frac{g_i^2(\mu)}{g_j^2(\mu)} \overset{\mu \to 0}{\Longrightarrow} \frac{\lambda_j}{\lambda_i}.$$

2. If the fermions are massive and decouple in the course of renormalization group running, the effective theory will change as each threshold is crossed. Consequently, M_i^j will differ across each threshold, and it will be necessary to rediagonalize M_i^j in each new effective theory.

3. The position of the Landau pole (Λ) is determined by the largest eigenvalue, λ_N, of M_i^j:

$$\log\left(\frac{\Lambda^2}{\mu^2}\right) = \frac{1}{\lambda_N}.$$

In a general basis (i.e. one in which M_i^j is not diagonal) one might naively calculate the Landau pole from the diagonal elements alone. However, the largest eigenvalue of a positive semi–definite matrix is always at least as large as its largest diagonal element. Thus, the *actual* Landau pole of the model is always lower than one would estimate neglecting gauge field mixing [4].

3. Charge Orthogonality in GUT's

If the multiple $U(1)$'s under consideration are unbroken subgroups of a simple Lie Group, as they would be in a Grand Unified Theory (GUT), then it is possible to show that the charge orthogonality condition Eq. (1) is automatically satisfied. If the unbroken $U(1)$'s result from the breaking of a simple Lie group, the charge Y_i^a of the a–th fermion under the i–th $U(1)$ is proportional to the a–th diagonal element of the i–th unbroken diagonal generator of the simple Lie group in some irreducible representation. The diagonal generators H_i $(i = 1, 2, \cdots, N)$ of a simple Lie group constitute the Cartan subalgebra of the generating Lie algebra and can always be normalized to the form

$$\mathrm{tr}(H_i H_j) = \delta_{ij} k_D$$

where k_D is a representation–dependent constant [5]. Since $\sum_a Y_i^a Y_j^a$ is proportional to $\mathrm{tr}(H_i H_j)$, it immediately follows that, for $U(1)$'s arising from the breaking of simple Lie groups, the COC is always satisfied.

Of course, if any of the fermions decouple in the course of renormalization group running before the $U(1)$'s themselves break, then a re–diagonalization of the fields and charges would be necessary every time a particle threshold is crossed even for GUT's.

4. Application to Topcolor–Assisted Technicolor Models

To see the consequence of the COC in model–building, we consider its application in the context of topcolor–assisted technicolor models [3]. As a concrete example, we consider the model proposed by Lane and Eichten in Ref. [1], which has two U(1) factors in its gauge group. The particle content and charge assignments of the model are shown in Table 1, where the $U(1)$ charges of technifermions are expressed in terms of six parameters $x_{1,2}$, $y_{1,2}$, and $z_{1,2}$. (Note that the Standard Model particles are charged under only one of the $U(1)$'s.)

Anomaly cancellation requirements lead to the following five equations for the six unknowns:

$$0 = x_1 + y_1 + z_1,$$

Particle	Y_1	Y_2	$Q = T_3 + Y_1 + Y_2$
q_L^l	0	$\frac{1}{6}$	$\frac{2}{3}, -\frac{1}{3}$
c_R, u_R	0	$\frac{2}{3}$	$\frac{2}{3}$
d_R, s_R	0	$-\frac{1}{3}$	$-\frac{1}{3}$
q_L^h	$\frac{1}{6}$	0	$\frac{2}{3}, -\frac{1}{3}$
t_R	$\frac{2}{3}$	0	$\frac{2}{3}$
b_R	$-\frac{1}{3}$	0	$-\frac{1}{3}$
T_L^l	x_1	x_2	$\pm\frac{1}{2} + x_1 + x_2$
U_R^l	x_1	$x_2 + \frac{1}{2}$	$\frac{1}{2} + x_1 + x_2$
D_R^l	x_1	$x_2 - \frac{1}{2}$	$-\frac{1}{2} + x_1 + x_2$
T_L^t	y_1	y_2	$\pm\frac{1}{2} + y_1 + y_2$
U_R^t	$y_1 + \frac{1}{2}$	y_2	$\frac{1}{2} + y_1 + y_2$
D_R^t	$y_1 + \frac{1}{2}$	$y_2 - 1$	$-\frac{1}{2} + y_1 + y_2$
T_L^b	z_1	z_2	$\pm\frac{1}{2} + z_1 + z_2$
U_R^b	$z_1 - \frac{1}{2}$	$z_2 + 1$	$\frac{1}{2} + z_1 + z_2$
D_R^b	$z_1 - \frac{1}{2}$	z_2	$-\frac{1}{2} + z_1 + z_2$

Table 1: Quark and technifermion hypercharges and electromagnetic charges in the model of Ref. [1]. Leptons are not shown.

$$
\begin{aligned}
0 &= x_2 + y_2 + z_2, \\
0 &= x_1\left(y_1 - z_1 + \frac{1}{2}\right), \\
0 &= x_2\left(y_2 - z_2 - \frac{1}{2}\right), \\
0 &= x_1\left(y_2 - z_2 - \frac{1}{2}\right) + x_2\left(y_1 - z_1 + \frac{1}{2}\right).
\end{aligned}
$$

There are two classes of solutions to these equations: the first with $x_1 = x_2 = 0$, and the other with $(y_1 - z_1 + \frac{1}{2}) = (y_2 - z_2 - \frac{1}{2}) = 0$. Each of these classes has two free parameters remaining. The COC provides an additional constraint

$$
4(x_1 x_2 + y_1 y_2 + z_1 z_2) - \left(y_1 - z_1 + \frac{1}{2}\right) + \left(y_2 - z_2 - \frac{1}{2}\right) = 0,
$$

which reduces the number of free parameters to one in each class.

Instead of applying the COC, however, Lane and Eichten require the existence of the four–technifermion operator

$$
\bar{T}_L^l \gamma^\mu T_L^t \, \bar{D}_R^t \gamma_\mu D_R^b
$$

to provide mixing between the third and first two fermion generations. This leads to two additional constraints

$$
0 = x_1 - z_1 + 1,
$$

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$$0 = x_2 - z_2 - 1,$$

which are sufficient to fix one solution in each of the two classes. However, these conditions, and hence the two solutions, are incompatible with the COC.

This means that the charge assignments of Ref. [1] characterize a basis for the $U(1)$ fields which is *not* charge–orthogonal, and in this basis the $U(1)$ fields will necessarily mix with a scale–dependent mixing angle under RG running. To avoid this, one must go to the preferred basis in which the COC is satisfied, but in that basis the Standard Model particles will carry two $U(1)$ charges instead of just one. We are therefore forced to consider more general charge assignments, such as those considered by Lane in Ref. [2].

The model constructed by Lane in Ref. [2] is similar to that of Ref. [1], but the $U(1)$ charge assignments of all the particles, including those in the Standard Model, are kept generic resulting in 26 parameters which must be specified. Anomaly cancellations provide a set of five linear and three cubic equations. Thirteen additional linear equations are imposed from phenomenological considerations. With 21 conditions on 26 parameters, any solution will have at least five free parameters. The COC will provide an additional quadratic equation which will again reduce the number of free parameters by removing the (unphysical) rescaling and rotational degree of freedom in the $U(1)$ field and charge definitions.

5. Conclusion

We have argued that in multiple–$U(1)$ gauge theories, there exists a particular choice of basis for the $U(1)$ fields which is to be preferred over other rescaling and orthonormal rotations of the $U(1)$ fields. In this basis, characterized by the fact that the COC is satisfied, the $U(1)$ gauge bosons do not mix under one–loop renormalization group running. Choosing this basis in model–building avoids the ambiguity in charge definitions associated with rescaling and rotations of the gauge fields and expedites study of the models at different scales.

Since a theory with arbitrary $U(1)$ charge assignments can always be mapped onto another in which the COC is satisfied with a simple change of basis, one can impose the COC as an additional constraint on the charges without loss of generality. This will reduce the dimension of the available charge parameter space and simplify the analysis of models considerably.

Acknowledgement

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