DYNAMICS OF SPINNING FLEXIBLE SATELLITES

by

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The author wishes to dedicate this work to the memory of his father who passed away during this research investigation.

TABLE OF CONTENTS

1. Introduction

Spin stabilization has been used as a means of maintaining spacecraft in a fixed orientation with respect to an inertial space. It is well known that rotational motion of a torque-free rigid body is stable if the rotation takes place about the axis of the maximum or minimum moment of inertia and unstable if the body rotates about the axis of intermediate moment of inertia (Ref. 1, Sec. 6.7). The tumbling motion of the Explorer I satellite, which was stabilized about the axis of minimum moment of inertia, revealed that a spacecraft equipped with flexible antennas cannot be idealized as a rigid body. Thomson and Reiter[2] were able to show that the phenomenon can be attributed to energy dissipation resulting from the vibration of the flexible antennas. They used the energy sink method to reach the conclusion that a flexible spacecraft cannot be stabilized about the axis of minimum moment of inertia. In 1961 Meirovitch[3], using a mathematical model consisting of two circular elastic disks connected by a rigid shaft, confirmed the conclusions reached in Ref. 2. As a by-product, he obtained the elastic solution for the vibration of the disks subjected to gyroscopic forces.

With spacecraft increasing in size, flexibility has become an important factor in the spacecraft attitude stability. In 1963 and 1965, Buckens[4,5] examined the influence of elasticity on the attitude stability of a spinning satellite. In a 1966 paper, Meirovitch and Nelson[6] showed the effect of flexible antennas upon the stability of motion of a spin-stabilized satellite which consisted of a symmetric rigid body with two flexible rods extending along the symmetric axis in opposite directions. The flexible antennas are first replaced by viscously damped oscillators and then considered as elastic rods whose displacements are discretized by normal modes. On the basis of a linearized infinitesimal stability analysis they concluded that the motion is unstable for any condition if the system rotates about the axis of minimum moment of inertia, and it is stable if the body rotates about the axis of maximum moment of inertia provided that a minimum requirement of the stiffness of the rods is fulfilled. They also pointed out that the inclusion of an additional mode of elastic motion does not appreciably alter the results obtained by using only one mode. The analysis in Ref. 6 appears to have the first use of the assumed-modes method to study stability of flexible spacecraft.

Dokuchaev[7] used a similar approach to that of Ref. 6 to study the stability of a spinning rigid body with four radial rods in a plane perpendicular to the spin axis and one axial rod along the spin axis. Ref. 7 considers several special cases, one of them being that of Ref. 6. For this case, his results agree with those of Ref. 6. For other cases, he concludes that rotation about the axis of maximum moment of inertia is stable for all spinning rates.

The Liapunov direct method, generally used for stability analysis of discrete systems,[8,9] has been extended to distributed systems.[10-15] In the area of attitude dynamics of flexible spacecrafts, Nelson and Meirovitch[16] used this method to investigate the stability of a rigid satellite with elastically connected moving parts, which is a simplified model of that of Ref. 6. At the same time, Pringle[17] also used this method to study the stability of a body with connected moving parts. Since the distributed elastic members are simulated in Ref. 16 by means of discrete masses, both Ref. 16 and Ref. 17 essentially deal with discrete systems.

In a first attempt to apply the Liapunov direct method to study the stability of hybrid systems, i.e. systems governed by both ordinary and partial differential equations, Meirovitch[18] presented a general formulation for a gravity-gradient stabilized satellite with flexible antennas. The method consists of an extension of the Liapunov direct method by considering a hybrid form for testing purpose, i.e., a form which is a function and functional at the same time. The method also uses the bounding property of Rayleigh quotient[19-22] to place a bound on the elastic potential energy; therefore it involves no series truncation or spatial discretization. As an illustration, he used the continuous model of Ref. 6. The simple form of the integrals in the testing functional enables him to define the so called density function. Results are in agreement with those of Ref. 6.

Hughes and Fung[23] tried to use the hybrid form of system Hamiltonian as Liapunov functional for stability analysis of a spin-stabilized satellite which consists of a rigid body and seven booms in a plane perpendicular to the spin axis, but they could only obtain the "closest" positive definite form for the testing functional. It turns out that they did not incorporate certain motion integrals in the analysis.

Meirovitch[24] overcame this difficulty by an approach similar to that of Ref. 18, but took into account automatically motion integrals resulting from conservation of angular momentum. In doing so, the matrix of moments of inertia must be inverted and terms of order higher than second are dropped. As an application, he considered the problem of a spinning rigid body with two axial thin rods. However, some integrals in the testing functional prevented the definition of a density function. To circumvent this problem, he employed Schwarz's inequality for functions to replace these integrals. The results obtained are more general than those reached in Ref. 6. These are the well-known "greatest moment of inertia" requirement and the maximum spin-rate restriction.

Both Ref. 18 and 24 were extended to hybrid dynamical systems with multi-elastic domains.[25] The example used was an earth-pointing satellite with three pairs of rods extended along the principal axes of the static equilibrium state.

In a 1972 paper Kulla[26] studied the dynamics of spinning bodies containing elastic rods and rigid symmetric rotors. The rods are along the spin axis and perpendicular to it and the rotor is parallel to the same axis. Linearized partial and ordinary differential equations are transformed to the frequency domain. The system becomes unstable when the lowest natural frequency approaches zero. It is shown that the system becomes less stable if the translational motion of the center of mass is suppressed. Part of the results are in good agreement with the parameter plot given in Ref. 6.

Meirovitch and Calico[27] again use the formulation of Ref. 24

to investigate the stability of a spin-stabilized satellite with six booms extended along the principal axes of the rigid body. Note that this example was used in Ref. 25, where the satellite was gravity-gradient and not spin stabilized. The analysis of Ref. 27 shows that the stability criteria obtained by ignoring the motion of the center of mass, although more conservative, can be used to predict stability for cases with arbitrary motion of the mass center. This is in agreement with a similar statement made in Ref. 26. The analysis was performed by the assumed modes method and by the method of integral coordinates. In the latter, the integrals of the continuous coordinates are treated as new generalized coordinates, which depend on time only. In addition to verifying the results of Ref. 24, they showed that a satellite which is stable without radial rods remains stable if radial rods are added.

All previously-cited papers discuss stability of satellites with flexible rods which are moderately long and which are extended along the principal axes of the body. These spacecrafts have negligible static deformation of the flexible parts if the elongations of the rods are neglected. Static deformations of a spinning spacecraft occur when the flexible parts are inclined to the plane perpendicular to the spin axis or when they are excessively long. Barbera and Likins[28] considered the stability of a rigid, spinning body with an arbitrary attached appendage by idealizing the flexible appendage as a collection of elastically interconnected particles and used the Hamiltonian, constrained by the motion integral, as Liapunov function. The condition for positive definiteness of the function could be established in any case by means of numerical procedures. In order to obtain literal closed-form stability criteria, they restricted the flexible appendage model to lie in a plane containing the center of mass and normal to the spin axis. Their work provided a preliminary analysis for stability study of more complex spacecraft. However, the question remains as to what extent the flexible appendages can be simulated by a collection of elastically interconnected particles.

In 1974, Meirovitch[29] published a paper on the Liapunov stability analysis of gravity-gradient stabilized hybrid dynamical systems in the neighborhood of nontrivial equilibrium, where the latter is defined as an equilibrium in which the angular coordinates are nonzero and the elastic members are in deformed state. Assuming small time-dependent displacements from the nontrivial equilibrium, the Liapunov functional is expanded in Taylor series up to the quadratic terms, then the sign property of the Hessian matrix associated with the quadratic form is evaluated by Sylvester's criterion. The theory has been used to test the stability of the RAE/B satellite which is a gravity-gradient stabilized satellite.

In the present study, we shall first formulate the equations of motion of a spin-stabilized spacecraft possessing flexible parts, some of them being parallel to the principal axes of the body (but not in the same plane) and some of them being inclined to the plane normal to the spin axis. The assumed-modes method will be employed to discretize the kinetic energy and potential energy. Then we use the constrained Hamiltonian as a testing function. The system dynamic characteristics are determined both from a stability analysis and the solution of the eigenvalue problem.

2. Mathematical Formulation

2.1 Problem Description

Let us consider a body consisting of n+1 parts, of which one part is rigid and n parts are flexible. The body is defined over a domain of extension D which can be regarded as the sum of the subdomains D_i (i=0,1,...,n) (see Fig. 1), where D_0 is the domain occupied by the rigid part and D_i (i=1,2,...,n) are the domains occupied by the flexible parts when in undeformed state. Correspondingly, the masses associated with the domain D_i are denoted by M_i (i=0,1,...,n) and the total mass is denoted by M, so that $M=\sum_{i=0}^{n} M_i$. The domains D_i (i=1,2,...,n) are rigidly attached to D_0 and have common boundaries only with D_0.

The body is assumed to move around a fixed center in an inertial space XYZ and spin freely in space with constant angular velocity Ω. The present study is concerned with the dynamic characteristics and stability of motion when the body is perturbed slightly from the uniform spin equilibrium state.

In describing the motion of M, it will prove convenient to identify a system of axes xyz (see Fig. 1) with nominal undeformed state, namely the state corresponding to the static equilibrium of M. The origin 0 of xyz is taken to coincide with the mass center of M in the undeformed state and axes xyz themselves are taken to coincide with the principal axes of M in the same state. We note that the system xyz is embedded in the rigid body D_0 but may not be a set of principal axes for that part. Because the center rigid body plays

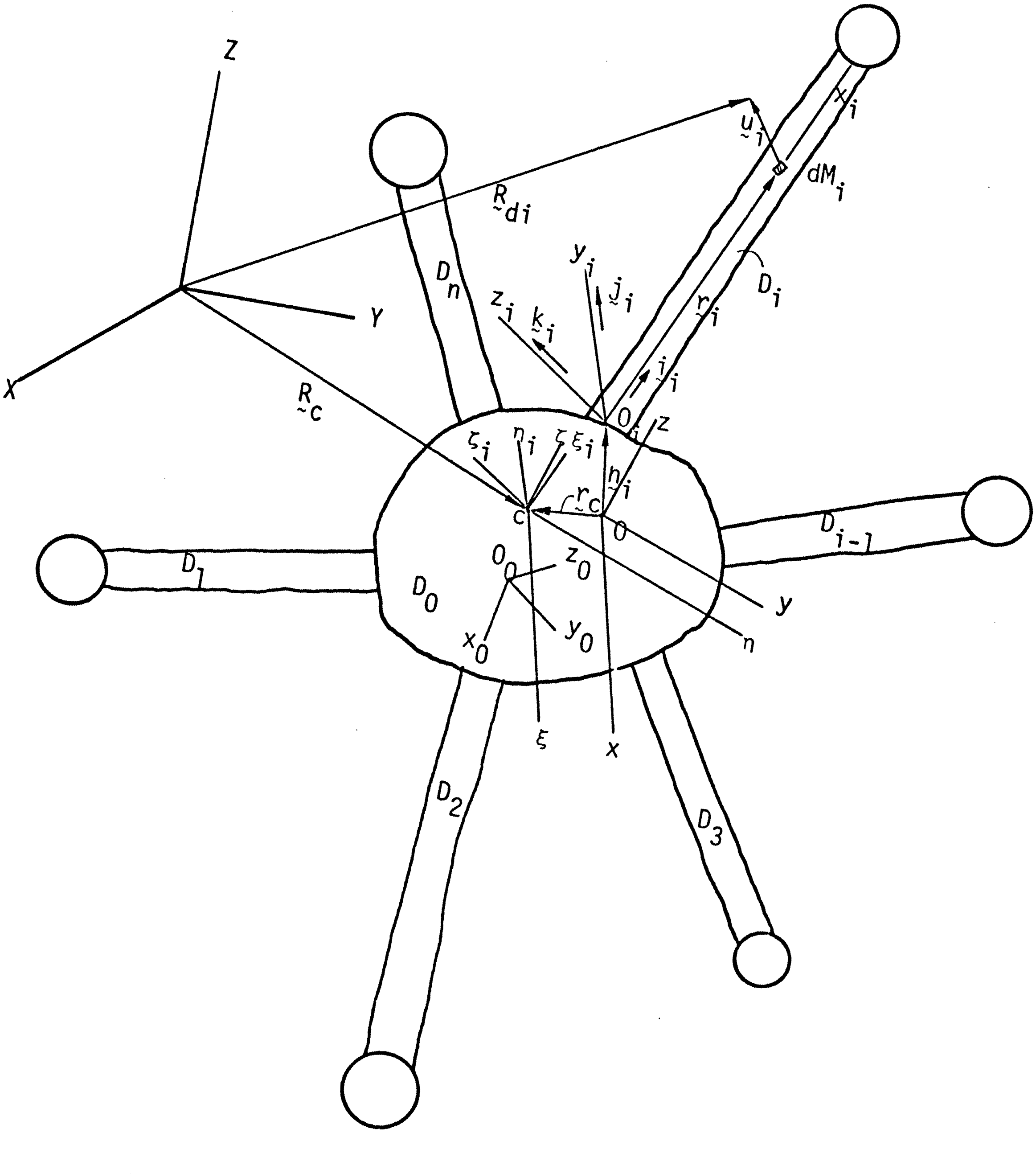

FIGURE 1. MATHEMATICAL MODEL

a significant role in the motion description, we set up a system of axes $x_0y_0z_0$ with origin at the mass center 0_0 of M_0. The directions of $x_0y_0z_0$ are chosen along the principal axes with principal moments of inertia A_0, B_0, C_0 respectively. In measuring elastic deformations, we consider reference frames $x_iy_iz_i$ fixed relative to the flexible domains D_i $(i=1,2,\ldots,n)$. The directions of axes $x_iy_iz_i$ depend on the nature of the flexible deformations. They are chosen so that the displacement components are parallel to these axes. Due to the motion of the flexible parts relative to the undeformed state, the mass center of M does not generally coincide with 0. We shall denote the center of mass of M in the deformed state by c and introduce a system of axes $\xi\eta\zeta$ parallel to xyz with the origin at c. The set $\xi\eta\zeta$ does not form, in general, a principal set of axes for the deformed body. For convenience, we also introduce sets of axes $\xi_i\eta_i\zeta_i$ parallel to $x_iy_iz_i$ $(i=0,1,\ldots,n)$ but with the origins at the mass center c.

Let us denote the radius vector from the mass center 0 to a point in the domain D_i $(i=1,2,\ldots,n)$ by $\underset{\sim}{h}_i+\underset{\sim}{r}_i$, where the point is occupied by an element of mass dM_i when the body is in the undeformed state. The constant-magnitude vector $\underset{\sim}{h}_i$ denotes the position vector from 0 to 0_i while the vector $\underset{\sim}{r}_i$ is the vector from 0_i to the point in question. Introducing the unit vectors $\underset{\sim}{i}_i$, $\underset{\sim}{j}_i$, $\underset{\sim}{k}_i$ along axes x_i, y_i, z_i respectively, the vectors can be written in terms of its components

$$\underset{\sim}{h}_i+\underset{\sim}{r}_i = (h_{xi}+x_i)\underset{\sim}{i}_i + (h_{yi}+y_i)\underset{\sim}{j}_i + (h_{zi}+z_i)\underset{\sim}{k}_i \qquad i = 0,1,\ldots,n \qquad (2.1)$$

We shall assume that the oscillations of the flexible parts about the nontrivial equilibrium are infinitesimally small so that the geometric and material nonlinearities are excluded. This assumption implies that the strain-displacement and stress-strain relations are all linear. The displacement vector $\underset{\sim}{u}_i$ of a mass element dM_i in the deformed state can be expressed as

$$\underset{\sim}{u}_i = u_i(x_i,y_i,z_i,t)\underset{\sim}{i}_i + v_i(x_i,y_i,z_i,t)\underset{\sim}{j}_i + w_i(x_i,y_i,z_i,t)\underset{\sim}{k}_i \tag{2.2}$$

where u_i,v_i and w_i are displacement components measured along x_i,y_i and z_i respectively. They include the steady-state deformations, u_{i0}, v_{i0},w_{i0}, due to the spin of the body, and the oscillatory deformations, u_{i1},v_{i1},w_{i1}, due to the perturbation, so that

$$\underset{\sim}{u}_i = \underset{\sim}{u}_{i0} + \underset{\sim}{u}_{i1} = (u_{i0}+u_{i1})\underset{\sim}{i}_i + (v_{i0}+v_{i1})\underset{\sim}{j}_i + (w_{i0}+w_{i1})\underset{\sim}{k}_i \tag{2.3}$$

The steady-state deformations vanish if the flexible parts are axial. Otherwise, they are determined by static methods because they are constants in time. Denoting the position vector of the mass center c of the deformed state relative to 0 by $\underset{\sim}{r}_c$, we have

$$\underset{\sim}{r}_c = \frac{1}{M}\sum_{i=0}^{n}\int_{M_i}(\underset{\sim}{h}_i+\underset{\sim}{r}_i+\underset{\sim}{u}_i)dM_i = \frac{1}{M}\sum_{i=1}^{n}\int_{M_i}\underset{\sim}{u}_{i0}dM_i + \frac{1}{M}\sum_{i=1}^{n}\int_{M_i}\underset{\sim}{u}_{i1}dM_i \tag{2.4}$$

where $\sum_{i=0}^{n}\int_{M_i}(\underset{\sim}{h}_i+\underset{\sim}{r}_i)dM_i$ is identically zero by the definition of point 0. Note that the first term on the right side of (2.4) is a constant.

The absolute position $\underset{\sim}{R}_{di}$ of dM_i at any time in the inertial space XYZ can be expressed as

$$\underset{\sim}{R}_{di} = \underset{\sim}{R}_c + \underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i - \underset{\sim}{r}_c \qquad (2.5)$$

where $\underset{\sim}{R}_c$ is the radius vector of the mass center in the inertial space. For complete description of the spinning motion, we let $\underset{\sim}{\omega}$ be the angular velocity vector of the body M relative to the same inertial space.

2.2 Kinetic energy

With the various position vectors defined above, the kinetic energy has the form

$$T = \frac{1}{2} \sum_{i=0}^{n} \int_{M_i} \left(\frac{d}{dt}\underset{\sim}{R}_{di}\right) \cdot \left(\frac{d}{dt}\underset{\sim}{R}_{di}\right) dM_i \qquad (2.6)$$

By (2.5), we have

$$T = \frac{1}{2} \sum_{i=0}^{n} \int_{M_i} \Big(\dot{\underset{\sim}{R}}_c + \underset{\sim}{\omega} \times (\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i - \underset{\sim}{r}_c) + (\dot{\underset{\sim}{u}}_i - \dot{\underset{\sim}{r}}_c)\Big) \cdot \Big(\dot{\underset{\sim}{R}}_c$$
$$+ \underset{\sim}{\omega} \times (\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i - \underset{\sim}{r}_c) + (\dot{\underset{\sim}{u}}_i - \dot{\underset{\sim}{r}}_c)\Big) dM_i \qquad (2.7a)$$

which is expanded in the form

$$T = \frac{1}{2} M\, \dot{\underset{\sim}{R}}_c \cdot \dot{\underset{\sim}{R}}_c + \frac{1}{2} \underset{\sim}{\omega} \cdot \underset{\sim}{J}_d \cdot \underset{\sim}{\omega} + \underset{\sim}{\omega} \cdot \left(\sum_{i=0}^{n} \int_{M_i} (\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i - \underset{\sim}{r}_c)\right.$$
$$\left. \times (\dot{\underset{\sim}{u}}_i - \dot{\underset{\sim}{r}}_c) dM_i\right) + \frac{1}{2} \sum_{i=0}^{n} \int_{M_i} (\dot{\underset{\sim}{u}}_i - \dot{\underset{\sim}{r}}_c) \cdot (\dot{\underset{\sim}{u}}_i - \dot{\underset{\sim}{r}}_c) dM_i \qquad (2.7b)$$

where $\underset{\sim}{J}_d$ is the inertia dyadic of the body in deformed state about axes $\xi\eta\zeta$.

The matrix form of (2.7b) will be more convenient than the vector form for later operations. To obtain the matrix form, let us define the moment of inertia of the deformed i-th part with respect to a set of axes which are parallel to $x_iy_iz_i$ but with the origin at 0 by the matrix $[J_i]$ and denote the matrix of direction cosines between coordinate axes $\xi_i\eta_i\zeta_i$ and $\xi\eta\zeta$ by $[\ell_i]$. The total moment of inertia matrix, $[J_d]$, with respect to $\xi\eta\zeta$ axes is

$$[J_d] = \sum_{i=0}^{n} [\ell_i]^T[J_i][\ell_i] - M \begin{bmatrix} y_c^2+z_c^2 & -x_cy_c & -x_cz_c \\ -x_cy_c & x_c^2+z_c^2 & -y_cz_c \\ -x_cz_c & -y_cz_c & x_c^2+y_c^2 \end{bmatrix} \qquad (2.8)$$

where x_c, y_c, z_c are the components of the vector $\underset{\sim}{r}_c$. Written in column vector form, it is $\{r_c\}=\{x_c \;\; y_c \;\; z_c\}^T$. The vectors $\underset{\sim}{R}_c$, $\underset{\sim}{\omega}$, $\underset{\sim}{u}_i$ expressed in matrix form are just $\{R_c\}$, $\{\omega\}$, $\{u_i\}$. Because the components of the vectors $\underset{\sim}{h}_i+\underset{\sim}{r}_i$, $\underset{\sim}{u}_i$, $\underset{\sim}{r}_c$ were in local coordinates $x_iy_iz_i$, if we represent the angular momentum $\int_{M_i}(\underset{\sim}{h}_i+\underset{\sim}{r}_i+\underset{\sim}{u}_i-\underset{\sim}{r}_c)\times(\dot{\underset{\sim}{u}}_i-\dot{\underset{\sim}{r}}_c)dM_i$ by the vector $\{P_i\}=\{P_{i\xi} \;\; P_{i\eta} \;\; P_{i\zeta}\}^T$, which can be interpreted as the angular momentum vector of the i-th part with respect to axes $\xi_i\eta_i\zeta_i$, then the total angular momentum vector $\{P\}=\{P_\xi \;\; P_\eta \;\; P_\zeta\}^T$ with respect to axes $\xi\eta\zeta$ is

$$\{P\} = \sum_{i=1}^{n} [\ell_i]^T\{P_i\} \qquad (2.10)$$

Finally, using (2.4), the last term in (2.7) can be expanded and reduced to the vector form $\frac{1}{2}(\sum_{i=1}^{n}\int_{M_i}\{\dot{u}_i\}^T\{\dot{u}_i\}dM_i - M\{\dot{r}_c\}^T\{\dot{r}_c\})$. The kinetic energy can now be written in matrix form:

$$T = \frac{1}{2} M \{\dot{R}_c\}^T\{\dot{R}_c\} + \frac{1}{2}\{\omega\}^T[J_d]\{\omega\} + \{\omega\}^T\{P\} + \frac{1}{2}\Big(\sum_{i=1}^{n} \int_{M_i} \{\dot{u}_i\}^T\{\dot{u}_i\}dM_i - M\{\dot{r}_c\}^T\{\dot{r}_c\}\Big) \tag{2.11}$$

The interest lies in the spinning motion of the body and the vibration of flexible parts, so that we can ignore the first term in (2.11). Note that this term is constant if the orbit is circular or the motion of c is uniform or zero. Let the system $\xi\eta\zeta$ be obtained from the inertial system XYZ by three consecutive rotations θ_3, θ_1, θ_2 (see Fig. 2). The transformations between system $\xi\eta\zeta$ and the inertial system can be expressed in the matrix form

$$\begin{Bmatrix} \xi' \\ \eta' \\ \zeta' \end{Bmatrix} = [T_1]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad \begin{Bmatrix} \xi'' \\ \eta'' \\ \zeta'' \end{Bmatrix} = [T_2]\begin{Bmatrix} \xi' \\ \eta' \\ \zeta' \end{Bmatrix}, \quad \begin{Bmatrix} \xi \\ \eta \\ \zeta \end{Bmatrix} = [T_3]\begin{Bmatrix} \xi'' \\ \eta'' \\ \zeta'' \end{Bmatrix} \tag{2.12}$$

where the matrices $[T_1]$, $[T_2]$, $[T_3]$ are

$$[T_1] = \begin{bmatrix} \cos\theta_3 & \sin\theta & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$[T_3] = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \tag{2.13}$$

The transformation from XYZ to $\xi\eta\zeta$ is

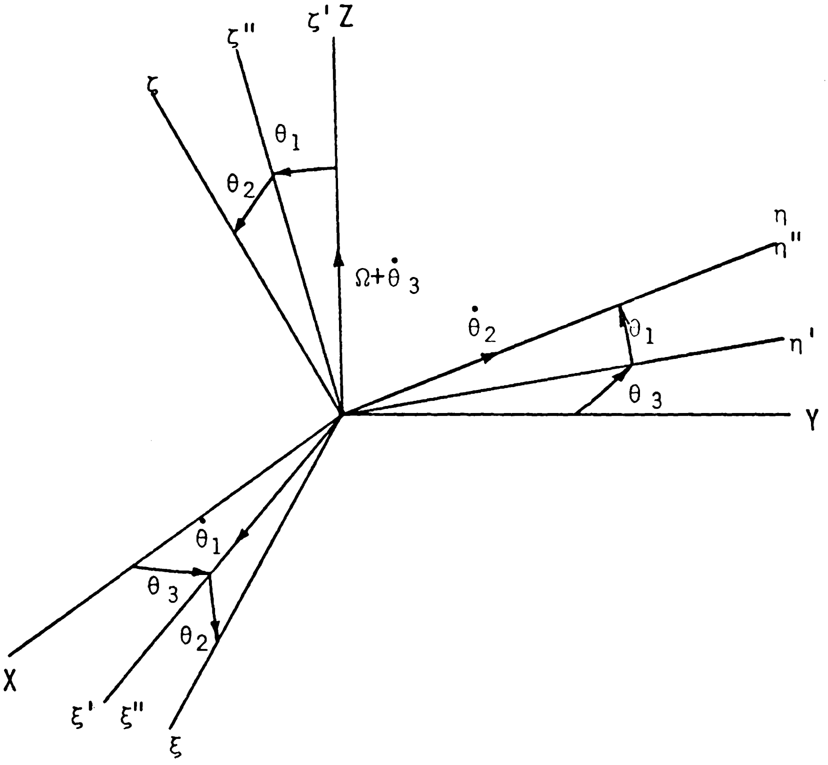

FIGURE 2. ROTATIONAL MOTION

$$\begin{Bmatrix}\xi\\ \eta\\ \zeta\end{Bmatrix} = [T]\begin{Bmatrix}X\\ Y\\ Z\end{Bmatrix} \quad , \qquad [T] = [T_3][T_2][T_1] \tag{2.14}$$

Because the angular velocity components ω_ξ, ω_η, ω_ζ do not represent time rates of change of certain angles, in order to define the motion, they must be related to the angles θ_1, θ_2, θ_3 just defined. It is easily concluded from Fig. 2 that

$$\begin{aligned}
\omega_\xi &= \dot{\theta}_1\cos\theta_2 - (\Omega + \dot{\theta}_3)\cos\theta_1\sin\theta_2\\
\omega_\eta &= \dot{\theta}_2 + (\Omega + \dot{\theta}_3)\sin\theta_1\\
\omega_\zeta &= \dot{\theta}_1\sin\theta_2 + (\Omega + \dot{\theta}_3)\cos\theta_1\cos\theta_2
\end{aligned} \tag{2.15}$$

Assuming that θ_1 and θ_2 are small and of the same order as $\dot{\theta}_i$, i=1,2,3, (2.15) can be approximated by

$$\begin{aligned}
\omega_\xi &= -\Omega\theta_2 + \dot{\theta}_1 - \dot{\theta}_3\theta_2\\
\omega_\eta &= \Omega\theta_1 + \dot{\theta}_2 + \dot{\theta}_3\theta_1\\
\omega_\zeta &= \Omega - \frac{\Omega}{2}(\theta_1^2 + \theta_2^2) + \dot{\theta}_1\theta_2 + \dot{\theta}_3
\end{aligned} \tag{2.16}$$

where we keep terms up to the second order in angular displacements and products of angular displacements and angular velocities. The right side of (2.16) can be separated into two parts, one containing the angular displacements only and the other angular velocities. Hence,

$$\{\omega\} = \{\omega\}_0 + \{\omega\}_1 \tag{2.17}$$

where

$$\{\omega\}_0 = \Omega \begin{Bmatrix} -\theta_2 \\ \theta_1 \\ 1-\frac{1}{2}(\theta_1^2+\theta_2^2) \end{Bmatrix} \quad , \quad \{\omega\}_1 = \begin{Bmatrix} \dot{\theta}_1-\dot{\theta}_3\theta_2 \\ \dot{\theta}_2+\dot{\theta}_3\theta_1 \\ \dot{\theta}_1\theta_2+\dot{\theta}_3 \end{Bmatrix} \tag{2.18}$$

The kinetic energy, (2.11), becomes the sum of three parts

$$T = T_2 + T_1 + T_0 \tag{2.19}$$

where

$$T_2 = \frac{1}{2}\{\omega\}_1^T[J_d]\{\omega\}_1 + \{\omega\}_1^T\{P\} + \frac{1}{2}\left(\sum_{i=1}^{n}\int_{M_i}\{\dot{u}_i\}^T\{\dot{u}_i\}dM_i - M\{\dot{r}_c\}^T\{\dot{r}_c\}\right) \tag{2.20a}$$

$$T_1 = \{\omega\}_0^T[J_d]\{\omega\}_1 + \{\omega\}_0^T\{P\} \tag{2.20b}$$

$$T_0 = \frac{1}{2}\{\omega\}_0^T[J_d]\{\omega\}_0 \tag{2.20c}$$

in which T_2 is quadratic in the generalized velocities, T_1 includes linear terms in the generalized velocities, and T_0 is free from them. Note that the vector $\{P\}$ consists of only first-order terms in the generalized velocities.

For a systematic analysis, further separations of terms in T_2, T_1, T_0 are necessary. Assuming that the generalized velocities are of the same order as the generalized displacements, the vectors $\{\omega\}_0$, $\{\omega\}_1$, $\{P\}$ and the inertia matrices can be separated as follows:

$$\{\omega\}_0 = \{\omega\}_{00} + \{\omega\}_{01} + \{\omega\}_{02} \tag{2.21}$$

$$\{\omega\}_1 = \{\omega\}_{11} + \{\omega\}_{12} \tag{2.22}$$

$$\{P\} = \{P\}_1 + \{P\}_2 \tag{2.23}$$

$$[J_d] = [J_d]_0 + [J_d]_1 + [J_d]_2 \tag{2.24a}$$

$$[J_i] = [J_i]_0 + [J_i]_1 + [J_i]_2 \tag{2.24b}$$

where

$$\{\omega\}_{00} = \Omega \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} , \quad \{\omega\}_{01} = \Omega \begin{Bmatrix} -\theta_2 \\ \theta_1 \\ 0 \end{Bmatrix} , \quad \{\omega\}_{02} = \Omega \begin{Bmatrix} 0 \\ 0 \\ -\frac{1}{2}(\theta_1^2+\theta_2^2) \end{Bmatrix} \tag{2.25a}$$

$$\{\omega\}_{11} = \begin{Bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{Bmatrix} , \quad \{\omega\}_{12} = \begin{Bmatrix} -\theta_2\dot\theta_3 \\ \theta_1\dot\theta_3 \\ \theta_2\dot\theta_1 \end{Bmatrix} \tag{2.25b}$$

$$[J_d]_j = \sum_{i=0}^{n} [\ell_i]^T [J_i]_j [\ell_i] - M \begin{bmatrix} y_c^2+z_c^2 & -x_c y_c & -x_c z_c \\ -x_c y_c & x_c^2+z_c^2 & -y_c z_c \\ -x_c z_c & -y_c z_c & x_c^2+y_c^2 \end{bmatrix}_j \tag{2.26}$$

$$j = 0,\ 1,\ 2$$

On the right side of (2.21) and (2.22), the first of the double subscripts are the same as the ones on the left side while the second subscripts represent the order of the quantities. The single subscripts on the right side of (2.23) and (2.24) also stand for the order of magnitude. Note that $[J_d]_0$ is the moment of inertia matrix of the steady-state deformed body with respect to system $\xi\eta\zeta$. The last matrix in (2.26) has elements which consist of the products of x_c, y_c, z_c. Hence, by (2.4), that matrix can be separated into three orders of matrices.

Substituting Eqs.(2.21) - (2.24) into (2.20) and keeping up to the second order in the generalized velocities, generalized displacements and their products, we have

$$T_2 = \frac{1}{2}\{\omega\}_{11}^T[J_d]_0\{\omega\}_{11} + \{\omega\}_{11}^T\{P\}_1 + \frac{1}{2}\left(\sum_{i=1}^{n}\int_{M_i}\{\dot{u}_i\}^T\{\dot{u}_i\}dM_i - M\{\dot{r}_c\}^T\{\dot{r}_c\}\right) \qquad (2.27a)$$

$$T_1 = \{\omega\}_{00}^T[J_d]_0\{\omega\}_{11} + \{\omega\}_{00}^T[J_d]_0\{\omega\}_{12} + \{\omega\}_{00}^T[J_d]_1\{\omega\}_{11} + \{\omega\}_{01}^T[J_d]_0\{\omega\}_{11} + \{\omega\}_{00}\{P\}_1 + \{\omega\}_{00}\{P\}_2 + \{\omega\}_{01}\{P\}_1 \qquad (2.27b)$$

$$T_0 = \frac{1}{2}\Big(\{\omega\}_{00}[J_d]_0\{\omega\}_{00} + 2\{\omega\}_{00}^T[J_d]_0\{\omega\}_{01} + 2\{\omega\}_{00}^T[J_d]_0\{\omega\}_{02} + 2\{\omega\}_{00}^T[J_d]_1\{\omega\}_{01} + \{\omega\}_{00}[J_d]_1\{\omega\}_{00} + \{\omega\}_{00}^T[J_d]_2\{\omega\}_{00} + \{\omega\}_{01}^T[J_d]_0\{\omega\}_{01}\Big) \qquad (2.27c)$$

where terms $\{\omega\}_{00}^T[J_d]_0\{\omega\}_{11}$, $\{\omega\}_{00}[J_d]_0\{\omega\}_{00}$, $\{\omega\}_{00}^T[J_d]_0\{\omega\}_{01}$, $\{\omega\}_{00}[J_d]_1\{\omega\}_{00}$, $\{\omega\}_{00}\{P\}_1$ are either constant or linear terms and do not contribute to the differential equations. Writing the matrices $[J_d]_j$ as

$$[J_d]_j = \begin{bmatrix} (J_{\xi\xi})_j & (J_{\xi\eta})_j & (J_{\xi\zeta})_j \\ (J_{\xi\eta})_j & (J_{\eta\eta})_j & (J_{\eta\zeta})_j \\ (J_{\xi\zeta})_j & (J_{\eta\zeta})_j & (J_{\zeta\zeta})_j \end{bmatrix} \qquad j = 0,\ 1,\ 2 \qquad (2.28)$$

and considering (2.25) - (2.26), the explicit forms of T_2, T_1, T_0 are

$$T_2 = \frac{1}{2}\left((J_{\xi\xi})_0\dot{\theta}_1^2 + (J_{\eta\eta})_0\dot{\theta}_2^2 + (J_{\zeta\zeta})_0\dot{\theta}_3^2 + 2(J_{\xi\eta})_0\dot{\theta}_1\dot{\theta}_2 + 2(J_{\xi\zeta})_0\dot{\theta}_1\dot{\theta}_3\right.$$
$$+ 2(J_{\eta\zeta})_0\dot{\theta}_2\dot{\theta}_3 + 2(P_\xi)_1\dot{\theta}_1 + 2(P_\eta)_1\dot{\theta}_2 + 2(P_\zeta)_1\dot{\theta}_3$$
$$\left. + \sum_{i=1}^{n} \int_{M_i} \{\dot{u}_i\}^T\{\dot{u}_i\}dM_i - M\{\dot{r}_c\}^T\{\dot{r}_c\}\right) \tag{2.29a}$$

$$T_1 = \Omega\left([(J_{\zeta\zeta})_0 - (J_{\xi\xi})_0]\theta_2\dot{\theta}_1 + (J_{\eta\eta})_0\theta_1\dot{\theta}_2 + 2(J_{\eta\zeta})_0\theta_1\dot{\theta}_3 - 2(J_{\xi\zeta})_0\theta_2\dot{\theta}_3\right.$$
$$+ (J_{\xi\eta})_0\theta_1\dot{\theta}_1 - (J_{\xi\eta})_0\theta_2\dot{\theta}_2 + (J_{\xi\zeta})_1\dot{\theta}_1 + (J_{\eta\zeta})_1\dot{\theta}_2 + (J_{\zeta\zeta})_1\dot{\theta}_3$$
$$\left. + (P_\eta)_1\theta_1 - (P_\xi)_1\theta_2 + (P_\zeta)_2\right) \tag{2.29b}$$

$$T_0 = \frac{1}{2}\Omega^2\left([(J_{\eta\eta})_0 - (J_{\zeta\zeta})_0]\theta_1^2 + [(J_{\xi\xi})_0 - (J_{\zeta\zeta})_0]\theta_2^2 - 2(J_{\xi\eta})_0\theta_1\theta_2\right.$$
$$\left. + 2(J_{\eta\zeta})_1\theta_1 - 2(J_{\xi\zeta})_1\theta_2 + (J_{\zeta\zeta})_2\right) \tag{2.29c}$$

Note that the steady-state deformations render the matrix $[J_d]_0$ a general symmetric one, not a diagonal matrix.

2.3 Potential energy

The potential energy includes gravitational and elastic potential energy. The gravitational potential energy is assumed to be small compared with the kinetic energy or elastic potential energy and will be neglected. It is further assumed that the material of the i-th member is homogeneous, isotropic within domain D_i and obeys Hooke's law

$$\{\sigma_i\} = [E_i]\{\varepsilon_i\} \tag{2.30}$$

where

$$\{\sigma_i\} = \{\sigma_{i11} \quad \sigma_{i22} \quad \sigma_{i33} \quad \sigma_{i12} \quad \sigma_{i13} \quad \sigma_{i23}\}^T \tag{2.31a}$$

$$\{\varepsilon_i\} = \{\varepsilon_{i11} \quad \varepsilon_{i22} \quad \varepsilon_{i33} \quad \varepsilon_{i12} \quad \varepsilon_{i13} \quad \varepsilon_{i23}\}^T \tag{2.31b}$$

$$[E_i] = \frac{E_i}{(1+\nu_i)(1-2\nu_i)} \begin{bmatrix} 1-\nu_i & \nu_i & \nu_i & 0 & 0 & 0 \\ \nu_i & 1-\nu_i & \nu_i & 0 & 0 & 0 \\ \nu_i & \nu_i & 1-\nu_i & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\nu_i & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}-\nu_i & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}-\nu_i \end{bmatrix} \tag{2.31c}$$

E_i denotes Young's modulus and ν_i is Poisson's ratio. Then the elastic strain energy of the flexible parts is the sum of that of n parts V_i, i=1,2,...,n and can be written as

$$V_{EL} = \sum_{i=1}^{n} V_i = \frac{1}{2}\sum_{i=1}^{n}\int_{D_i}\{\varepsilon_i\}^T\{\sigma_i\}dD_i = \frac{1}{2}\sum_{i=1}^{n}\int_{D_i}\{\varepsilon_i\}^T[E_i]\{\varepsilon_i\}dD_i \tag{2.32}$$

Equation (2.32) can be expressed in terms of displacement components u_i, v_i, w_i by virtue of the strain-displacement relations

$$\{\varepsilon_i\} = [L_i]\{u_i\} \tag{2.33}$$

in which $[L_i]$ represents the matrix of the operator that relates the displacements to the strain components. Its general form for linear strain-displacement relations is

$$[L_i] = \begin{bmatrix} \frac{\partial}{\partial x_i} & 0 & 0 \\ 0 & \frac{\partial}{\partial y_i} & 0 \\ 0 & 0 & \frac{\partial}{\partial z_i} \\ \frac{\partial}{\partial y_i} & \frac{\partial}{\partial x_i} & 0 \\ \frac{\partial}{\partial z_i} & 0 & \frac{\partial}{\partial x_i} \\ 0 & \frac{\partial}{\partial z_i} & \frac{\partial}{\partial y_i} \end{bmatrix} \tag{2.33b}$$

Some special forms of $[E_i]$ and $[L_i]$ are of interest in later applications. For a vibrating string they are

$$[E_i] = T_{xi}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad [L_i] = \begin{bmatrix} \frac{\partial}{\partial x_i} & 0 \\ 0 & \frac{\partial}{\partial x_i} \end{bmatrix} \tag{2.34a}$$

where T_{xi} is the tension force. For a simple beam of circular cross section in lateral vibration, we have

$$[E_i] = E_i I_i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad [L_i] = \begin{bmatrix} \frac{\partial^2}{\partial x_i^2} & 0 \\ 0 & \frac{\partial^2}{\partial x_i^2} \end{bmatrix} \tag{2.34b}$$

where $E_i I_i$ is the flexural rigidity. In both cases $\{u_i\}=\{v_i \quad w_i\}^T$.

Finally, substituting (2.33) into (2.32), the potential energy becomes

$$V_{EL} = \frac{1}{2}\sum_{i=1}^{n}\int_{D_i}\{u_i\}^T[L_i]^T[E_i][L_i]\{u_i\}dD_i \tag{2.35}$$

2.4 Nontrivial Equilibrium

The formulation so far is general in the sense that no particular form of flexible parts has been specified thus leaving the elements of $[J_d]$ and components of $\{P\}$ undetermined. For structures possessing flexible booms, we assume that the mass element dM_i undergoes transverse flexural, but not axial vibration. The displacement vector reduces to two components

$$\underset{\sim}{u}_i(x_i,t) = v_i(x_i,t)\underset{\sim}{j}_i + w_i(x_i,t)\underset{\sim}{k}_i \tag{2.36}$$

The Lagrangian is defined by

$$L = T - V_{EL} = L_0(t) + \sum_{i=1}^{n} \int_{M_i} \hat{L}_i(x_i,t)dM_i \tag{2.37}$$

In view of (2.29) and (2.35), $\hat{L}_i$ is a functional of the form

$$\hat{L}_i(x_i,t) = \hat{L}_i\Big(\theta_j(t),\dot{\theta}_j(t),v_i(x_i,t),w_i(x_i,t),\dot{v}_i(x_i,t),\dot{w}_i(x_i,t), v_i'(x_i,t),w_i'(x_i,t),v_i''(x_i,t),w_i''(x_i,t)\Big) \tag{2.38}$$

$$j = 1,2,3 \qquad\qquad i = 1,2,\ldots,n$$

Hamilton's principle can be stated as follows.

$$\int_{t_1}^{t_2}\delta L dt = 0 \quad , \quad \delta\theta_j = \delta v_i = \delta w_i = 0 \ \text{ at } \ t = t_1, t_2 \tag{2.39}$$

Thus, inserting (2.37) and (2.38) into (2.39), we obtain Lagrange's equations for rotational motion

$$\frac{\partial L}{\partial\theta_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_j}\right) = 0 \qquad j = 1,2,3 \tag{2.40}$$

as well as the equations for elastic motion

$$\frac{\partial \hat{L}_i}{\partial v_i} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}_i}{\partial \dot{v}_i}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial v_i'}\right) + \frac{\partial^2}{\partial x_i^2}\left(\frac{\partial \hat{L}_i}{\partial v_i''}\right) = 0 \qquad (2.41)$$

$$0 < x_i < \ell_i \qquad\qquad i = 1,2,\ldots,n$$

where ℓ_i is the length of i-th member. Moreover, v_i is subject to the boundary conditions

$$\left(\frac{\partial \hat{L}_i}{\partial \dot{v}_i} + \frac{\partial \hat{L}_i}{\partial v_i'} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial v_i''}\right)\right)\delta v_i = 0 \quad , \quad \frac{\partial \hat{L}_i}{\partial v_i''}\,\delta v_i' = 0 \quad \text{at } x_i = \ell_i \quad (2.42a)$$

$$-\left(\frac{\partial \hat{L}_i}{\partial v_i'} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}_i}{\partial v_i''}\right)\right)\delta v_i = 0 \quad , \quad -\frac{\partial \hat{L}_i}{\partial v_i''}\,\delta v_i' = 0 \quad \text{at } x_i = 0 \quad (2.42b)$$

$$i = 1,2,\ldots,n$$

Equations similar in structure to (2.41) and (2.42) can be written for w_i by simply replacing v_i by w_i.

The nontrivial equilibrium is defined by a set of dependent variables θ_j, v_i, w_i, constant in time and satisfying equations obtained from (2.40), (2.41), and (2.42) as well as from corresponding equations for w_i by omitting the time derivative terms. These solutions are the static displacements θ_{j0}, v_{i0}, w_{i0}, where the first are constants and the latter are functions of the spatial variables x_i alone.

2.5 The Routhian Function

Since the coordinate θ_3 does not appear in the Lagrangian it is ignorable (Ref. 1, sec. 2.10, 2.11). The generalized momentum associated with an ignorable coordinate is conserved. By (2.19), (2.29), and (2.37), we have

$$\frac{\partial L}{\partial \dot{\theta}_3} = (J_{\xi\zeta})_0\dot{\theta}_1 + (J_{\eta\zeta})_0\dot{\theta}_2 + (J_{\zeta\zeta})_0\dot{\theta}_3 + 2\Omega(J_{\eta\zeta})_0\theta_1 - 2\Omega(J_{\xi\zeta})_0\theta_2$$
$$+ \Omega(J_{\zeta\zeta})_1 + (P_\zeta)_1 = \beta \qquad (2.43)$$

where the constant β is determined by the initial condition. This equation is solved for θ_3

$$\dot{\theta}_3 = \frac{1}{(J_{\zeta\zeta})_0}\Big(\beta - (J_{\xi\zeta})_0\dot{\theta}_1 - (J_{\eta\zeta})_0\dot{\theta}_2 - 2\Omega(J_{\eta\zeta})_0\theta_1 + 2\Omega(J_{\xi\zeta})_0\theta_2$$
$$- \Omega(J_{\zeta\zeta})_1 - (P_\zeta)_1\Big) \qquad (2.44)$$

Next we construct the Routhian function

$$R = L - \frac{\partial L}{\partial \dot{\theta}_3}\dot{\theta}_3 = L - \beta\dot{\theta}_3 = T_2 + T_1 + T_0 - V_{EL} - \beta\dot{\theta}_3 \qquad (2.45)$$

Substituting (2.44) into (2.29) and neglecting terms that do not contribute to the equations of motion, we obtain

$$R = R_2 + R_1 + R_0 - V_{EL} \qquad (2.46)$$

where

$$R_2 = \frac{1}{2}\Big(\big((J_{\xi\xi})_0 - r_\xi^2(J_{\zeta\zeta})_0\big)\dot{\theta}_1^2 + \big((J_{\eta\eta})_0 - r_\eta^2(J_{\zeta\zeta})_0\big)\dot{\theta}_2^2 + 2\big((J_{\xi\eta})_0$$
$$- r_\xi r_\eta(J_{\zeta\zeta})_0\big)\dot{\theta}_1\dot{\theta}_2 + 2\big((P_\xi)_1 - r_\xi(P_\zeta)_1\big)\dot{\theta}_1 + 2\big((P_\eta)_1 - r_\eta(P_\zeta)_1\big)\dot{\theta}_2$$

$$- \frac{1}{(J_{\zeta\zeta})_0}(P_\zeta)_1^2 + \sum_{i=1}^{n} \int_{M_i} \{\dot{u}_i\}^T\{\dot{u}_i\}dM_i - M\{\dot{r}_c\}^T\{\dot{r}_c\}\Big) \tag{2.47a}$$

$$\begin{aligned} R_1 = \Omega\Big(&\big((J_{\xi\eta})_0 - 2r_\xi r_\eta (J_{\zeta\zeta})_0\big)\theta_1\dot{\theta}_1 - \big((J_{\xi\eta})_0 - 2r_\xi r_\eta (J_{\zeta\zeta})_0\big)\theta_2\dot{\theta}_2 \\ &+ \big((J_{\zeta\zeta})_0 - (J_{\xi\xi})_0 + 2r_\xi^2 (J_{\zeta\zeta})_0\big)\theta_2\dot{\theta}_1 + \big((J_{\eta\eta})_0 - 2r_\eta^2 (J_{\zeta\zeta})_0\big)\theta_1\dot{\theta}_2 \\ &+ \big((J_{\xi\zeta})_1 - r_\xi (J_{\zeta\zeta})_1\big)\dot{\theta}_1 + \big((J_{\eta\zeta})_1 - r_\eta (J_{\zeta\zeta})_1\big)\dot{\theta}_2 + \theta_1\big((P_\eta)_1 \\ &- 2r_\eta (P_\zeta)_1\big) - \theta_2\big((P_\xi)_1 - 2r_\xi (P_\zeta)_1\big) + (P_\zeta)_2 - \frac{1}{(J_{\zeta\zeta})_0}(J_{\zeta\zeta})_1 (P_\zeta)_1\Big) \end{aligned} \tag{2.47b}$$

$$\begin{aligned} R_0 = \frac{1}{2}\,\Omega^2\Big(&\big((J_{\eta\eta})_0 - (1 + 4r_\eta^2)(J_{\zeta\zeta})_0\big)\theta_1^2 + \big((J_{\xi\xi})_0 - (1 + 4r_\xi^2)(J_{\zeta\zeta})_0\big)\theta_2^2 \\ &- 2\big((J_{\xi\eta})_0 - 4r_\xi r_\eta (J_{\zeta\zeta})_0\big)\theta_1\theta_2 + 2\big((J_{\eta\zeta})_1 - 2r_\eta (J_{\zeta\zeta})_1\big)\theta_1 - 2\big((J_{\xi\zeta})_1 \\ &- 2r_\xi (J_{\zeta\zeta})_1\big)\theta_2 + (J_{\zeta\zeta})_2 - \frac{1}{(J_{\zeta\zeta})_0}(J_{\zeta\zeta})_1^2\Big) \end{aligned} \tag{2.47c}$$

in which r_ξ, r_η are the ratios of the elements of the matrix $[J_d]_0$ defined by

$$r_\xi = (J_{\xi\zeta})_0 \,/\, (J_{\zeta\zeta})_0 \quad , \quad r_\eta = (J_{\eta\zeta})_0 \,/\, (J_{\zeta\zeta})_0 \tag{2.48}$$

The problem is reduced to the determination of the moment of inertia matrix, $[J_d]$, of the deformed state and the angular momentum vector, $\{P\}$.

A special case of interest is when all the booms and cables are along the principal axes of the body such that no steady-state deflection occurs. It is possible to choose the coordinate axes xyz as the principal axes with principal moments of inertia A, B, C. Then the matrix $[J_d]_0$ reduces to diagonal form with elements A, B, C.

$$[J_d]_0 = \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} \tag{2.49}$$

Therefore the elements of that matrix are

$$(J_{\xi\xi})_0 = A \;,\; (J_{\eta\eta})_0 = B \;,\; (J_{\zeta\zeta})_0 = C \;,\; (J_{\xi\zeta})_0 = (J_{\eta\zeta})_0 = 0 \tag{2.50}$$

from which we see that $r_\xi = r_\eta = 0$. Substituting (2.50) into (2.47), we obtain the special forms of R_2, R_1, R_0.

$$R_2 = \frac{1}{2}\Bigg(A\dot{\theta}_1^2 + B\dot{\theta}_2^2 + 2(P_\xi)_1\dot{\theta}_1 + 2(P_\eta)_1\dot{\theta}_2 - \frac{1}{C}(P_\zeta)_1^2 + \sum_{i=1}^{n}\int_{M_i}\{\dot{u}_i\}^T\{\dot{u}_i\}dM_i$$
$$- M\{\dot{r}_c\}^T\{\dot{r}_c\}\Bigg) \tag{2.51a}$$

$$R_1 = \Omega\,\Bigg((C - A)\theta_2\dot{\theta}_1 + B\theta_1\dot{\theta}_2 + (J_{\xi\zeta})_1\dot{\theta}_1 + (J_{\eta\zeta})_1\dot{\theta}_2 + \theta_1(P_\eta)_1 - \theta_2(P_\xi)_1$$
$$+ (P_\zeta)_2 - \frac{1}{C}(J_{\zeta\zeta})_1(P_\zeta)_1\Bigg) \tag{2.51b}$$

$$R_0 = \frac{1}{2}\,\Omega^2\,\Bigg((B - C)\theta_1 + (A - C)\theta_2 + 2(J_{\eta\zeta})_1\theta_1 - 2(J_{\xi\zeta})_1\theta_2 + (J_{\zeta\zeta})_2$$
$$- \frac{1}{C}\,(J_{\zeta\zeta})_1^2\Bigg) \tag{2.51c}$$

3. Discretization

There are many discretization procedures in common use, such as the lumped-parameter method, the Rayleigh-Ritz method (Ref. 19, Ch. 6), the finite-element method (Ref. 20, Ch. 8), etc. The Rayleigh-Ritz method consists of assuming the displacement field of the flexible parts as a finite series of space-dependent functions multiplied by time-dependent generalized coordinates. The space-dependent functions are eigenfunctions, comparison functions, or admissible functions associated with the flexible member. In general, the eigenfunctions, which must satisfy both the differential equation and the boundary conditions of the problem, are difficult to obtain. It was shown in Ref. 30, however, that there is no particular advantage in using eigenfunctions. The admissible functions, which satisfy geometric boundary conditions only, can yield equally good results. The finite-element method is basically a localized Rayleigh-Ritz method. By this method, the flexible parts are divided into elements, then the displacement field in each element is expressed as the sum of shape functions multiplied by the corresponding element nodal displacements. This is followed by assembly and condensation procedures. The lumped-parameter method is simply a spatial discretization. We shall adopt the Rayleigh-Ritz method and write

$$u_{i1} = \sum_{j=1}^{r_i} \chi_{ij}(x_i,y_i,z_i)q_{ij}(t) = \{\chi_i\}^T\{q_i\}_{r_i} \tag{3.1a}$$

$$v_{i1} = \sum_{j=1}^{s_i} \phi_{ij}(x_i,y_i,z_i)q_{i,r_i+j}(t) = \{\phi_i\}^T\{q_i\}_{s_i} \qquad i=1,2,\ldots,n \tag{3.1b}$$

$$w_{i1} = \sum_{j=1}^{t_i} \psi_{ij}(x_i,y_i,z_i)q_{i,r_i+s_i+j}(t) = \{\psi_i\}^T\{q_i\}_{t_i} \qquad (3.1c)$$

where χ_{ij}, ϕ_{ij}, ψ_{ij} are space-dependent admissible functions and q_{ij} are generalized coordinates. Note that the vectors $\{\chi_i\}_{r_i}$, $\{\phi_i\}_{s_i}$, $\{\psi_i\}_{t_i}$ are r_i-, s_i-, and t_i-vectors, respectively. Equations (3.1) can be written in matrix form

$$\{u_{i1}\} = [N_i]\{q_i\} \qquad i = 1,2,\dots,n \qquad (3.2)$$

where

$$\{u_{i1}\} = \{u_{i1} \quad v_{i1} \quad w_{i1}\}^T \qquad (3.3)$$

$$\{q_i\} = \{q_{i1} \quad q_{i2} \cdots \quad q_{i,r_i} \quad q_{i,r_i+1} \cdots \quad q_{i,r_i+s_i} \quad q_{i,r_i+s_i+1} \cdots \quad q_{i,p_i}\}^T \qquad (3.4)$$

$$[N_i] = \begin{bmatrix} \chi_{i1} & \chi_{i2} & \cdots & \chi_{i,r_i} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \phi_{i1} & \phi_{i2} & \cdots & \phi_{i,s_i} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \psi_{i1} & \psi_{i2} & \cdots & \psi_{i,t_i} \end{bmatrix} \qquad (3.5)$$

in which $p_i=r_i+s_i+t_i$. The velocity is simply

$$\{\dot{u}_{i1}\} = [N_i]\{\dot{q}_i\} \qquad (3.6)$$

Next, let us define the notations:

$$p = \sum_{i=1}^{n} p_i \quad , \quad q_1 = \theta_1 \quad , \quad q_2 = \theta_2 \qquad (3.7)$$

$$\{\bar{q}\} = \{\{q_1\}^T \ \ \{q_2\}^T \ \ldots \ \ \{q_n\}^T\}^T \tag{3.8}$$

$$\{q\} = \{q_1 \ \ q_2 \ \ \{\bar{q}\}^T\}^T \tag{3.9}$$

in which $\{\bar{q}\}$ is a p-vector and $\{q\}$ is a (p+2)-vector.

The elements $(J_{\xi\zeta})_1$, $(J_{\eta\zeta})_1$, $(J_{\zeta\zeta})_1$, and $(J_{\zeta\zeta})_2$ depend on the displacements of the flexible parts from the nontrivial equilibrium and are generally functions of $\{r_c\}$ and $\{u_{i1}\}$, $i=1,2,\ldots,n$. They can be written in the form

$$(J_{\xi\zeta})_1 = \{H_\xi\}^T\{\bar{q}\} \tag{3.10a}$$

$$(J_{\eta\zeta})_1 = \{H_\eta\}^T\{\bar{q}\} \tag{3.10b}$$

$$(J_{\zeta\zeta})_1 = \{H_\zeta\}^T\{\bar{q}\} \tag{3.10c}$$

$$(J_{\zeta\zeta})_2 = \{\bar{q}\}^T[H_{\zeta\zeta}]\{\bar{q}\} \tag{3.11}$$

The components of the vectors $\{H_\xi\}$, $\{H_\eta\}$, $\{H_\zeta\}$ and the elements of the matrix $[H_{\zeta\zeta}]$ consist of combinations of integrals of admissible functions.

The linear momentum of the generalized velocities with respect to axes $\xi\eta\zeta$, $(P_\xi)_1$, $(P_\eta)_1$, $(P_\zeta)_1$ contain the components of $\{\dot{r}_c\}$ and the integrals of spatial coordinates multiplied by generalized velocities. From (2.4) we see that the vector $\{\dot{r}_c\}$ depends on $\{\dot{u}_{i1}\}$, $i=1,2,\ldots,n$, which are related to $\{\dot{\bar{q}}\}$ by (3.2) - (3.5). Hence we can write

$$(P_\xi)_1 = \{F_\xi\}^T\{\dot{\bar{q}}\} \tag{3.12a}$$

$$(P_\eta)_1 = \{F_\eta\}^T\{\dot{\bar{q}}\} \qquad (3.12b)$$

$$(P_\zeta)_1 = \{F_\zeta\}^T\{\dot{\bar{q}}\} \qquad (3.12c)$$

$$(P_\zeta)_2 = \{\bar{q}\}^T[F_{\zeta\zeta}]\{\dot{\bar{q}}\} \qquad (2.12d)$$

Note that the subscripts 1 and 2 in (3.11) and (3.12) indicate linear and quadratic quantities, respectively. Using (2.3) and (3.2), we express the kinetic energy of the flexible parts in the form:

$$\sum_{i=1}^{n} \int_{M_i} \{\dot{u}_i\}^T\{\dot{u}_i\}dM_i = \sum_{i=1}^{n} \{\dot{q}_i\}^T[m_i]\{\dot{q}_i\} = \{\dot{\bar{q}}\}^T[m_s]\{\dot{\bar{q}}\} \qquad (3.13)$$

where

$$[m_i] = \int_{M_i} [N_i]^T[N_i]dM_i \qquad (3.14)$$

$$[m_s] = \begin{bmatrix} [m_1] & & & \\ & [m_2] & & \\ & & \cdots & \\ & & & [m_n] \end{bmatrix} \qquad (3.15)$$

To obtain the corresponding expression for the term $M\{\dot{r}_c\}^T\{\dot{r}_c\}$, we refer to (2.4) and (3.2) and write

$$M\{\dot{r}_c\}^T\{\dot{r}_c\} = \frac{1}{M} \sum_{i=1}^{n} \{\dot{q}_i\}^T\left(\int_{M_i} [N_i]^T dM_i\right) \sum_{j=1}^{n} \left(\int_{M_j} [N_j]dM_j\right)\{\dot{q}_j\}$$

$$= \frac{1}{M} \sum_{i=1}^{n} \sum_{j=1}^{n} \{\dot{q}_i\}^T[m_{ij}]\{\dot{q}_j\}$$

$$= \frac{1}{M} \{\dot{\bar{q}}\}^T[m_c]\{\dot{\bar{q}}\} \qquad (3.16)$$

where

$$[m_{ij}] = \left(\int_{M_i} [N_i]^T dM_i\right)\left(\int_{M_j} [N_j] dM_j\right) \qquad i,\ j = 1,2,\ldots,n \qquad (3.17)$$

and $[m_c]$ is a matrix of blocks with matrix $[m_{ij}]$ as its i,j-block. Both $[m_s]$ and $[m_c]$ are $p \times p$ matrices.

Spatial derivatives were denoted by the operator $[L_i]$ in (2.33). Equations (2.33) and (3.2) give

$$\{\varepsilon_i\} = [L_i]\{u_i\} = [D_i]\{q_i\} \qquad (3.18)$$

where

$$[D_i] = [L_i][N_i] = \begin{bmatrix} \left\{\frac{\partial \chi_i}{\partial x_i}\right\}^T & 0 & 0 \\ 0 & \left\{\frac{\partial \phi_i}{\partial y_i}\right\}^T & 0 \\ 0 & 0 & \left\{\frac{\partial \psi_i}{\partial z_i}\right\}^T \\ \left\{\frac{\partial \chi_i}{\partial y_i}\right\}^T & \left\{\frac{\partial \phi_i}{\partial x_i}\right\}^T & 0 \\ \left\{\frac{\partial \chi_i}{\partial z_i}\right\}^T & 0 & \left\{\frac{\partial \psi_i}{\partial x_i}\right\}^T \\ 0 & \left\{\frac{\partial \phi_i}{\partial z_i}\right\}^T & \left\{\frac{\partial \psi_i}{\partial y_i}\right\}^T \end{bmatrix} \qquad (3.19)$$

The strain energy, given by Eq. (2.35), becomes

$$V_{EL} = \frac{1}{2}\sum_{i=1}^{n}\int_{D_i}\{q_i\}^T[D_i]^T[E_i][D_i]\{q_i\}dD_i \qquad (3.20a)$$

which is further represented by

$$V_{EL} = \frac{1}{2} \sum_{i=1}^{n} \{q_i\}^T[k_i]\{q_i\} = \frac{1}{2} \{\bar{q}\}^T[k_s]\{\bar{q}\} \tag{3.20b}$$

where

$$[k_i] = \int_{D_i} [D_i]^T[E_i][D_i]^T dD_i \tag{3.21}$$

which is the stiffness matrix of the i-th part corresponding to $\theta_1=\theta_2=0$. Although the matrix $[L_i]$ takes special forms for cables and booms, the general expression, (2.34), will be used here. The matrix $[k_s]$, which is a $p \times p$ matrix, is block-diagonal with the submatrices $[k_i]$ $(i=1,2,\ldots n)$ on the diagonal.

$$[k_s] = \begin{bmatrix} [k_1] & & & \\ & [k_2] & & \\ & & \cdots\cdot & \\ & & & [k_n] \end{bmatrix} \tag{3.22}$$

Now we are ready to rewrite the expression for R_2, R_1, R_0 in matrix form. Substituting (3.7) - (3.13) and (3.16) into (2.47) we obtain

$$\begin{aligned} R_2 = \frac{1}{2} \Big(& ((J_{\xi\xi})_0 - r_\xi^2(J_{\zeta\zeta})_0)\dot{q}_1^2 + ((J_{\eta\eta})_0 - r_\eta^2(J_{\zeta\zeta})_0)\dot{q}_2^2 + 2((J_{\xi\eta})_0 \\ & - r_\xi r_\eta(J_{\zeta\zeta})_0)\dot{q}_1\dot{q}_2 + 2\dot{q}_1(\{F_\xi\}^T - r_\xi\{F_\zeta\}^T)\{\dot{\bar{q}}\} + 2\dot{q}_2(\{F_\eta\}^T \\ & - r_\eta\{F_\zeta\}^T)\{\dot{\bar{q}}\} - \frac{1}{(J_{\zeta\zeta})_0}\{\dot{\bar{q}}\}^T\{F_\zeta\}\{F_\zeta\}^T\{\dot{\bar{q}}\} + \{\dot{\bar{q}}\}^T[m_s]\{\dot{\bar{q}}\} \end{aligned}$$

$$- \frac{1}{M} \{\dot{\bar{q}}\}^T[m_c]\{\dot{\bar{q}}\}\Big) \tag{3.23a}$$

$$\begin{aligned} R_1 = \Omega \Big(&\big((J_{\xi\eta})_0 - 2r_\xi r_\eta (J_{\zeta\zeta})_0\big)q_1\dot{q}_1 - \big((J_{\xi\eta})_0 - 2r_\xi r_\eta (J_{\zeta\zeta})_0\big)q_2\dot{q}_2 \\ &+ \big((J_{\zeta\zeta})_0 - (J_{\xi\xi})_0 + 2r_\xi^2(J_{\zeta\zeta})_0\big)q_2\dot{q}_1 + \big((J_{\eta\eta})_0 - 2r_\eta^2(J_{\zeta\zeta})_0\big)q_1\dot{q}_2 \\ &+ \dot{q}_1\big(\{H_\xi\}^T - r_\xi\{H_\zeta\}^T\big)\{\bar{q}\} + \dot{q}_2\big(\{H_\eta\}^T - r_\eta\{H_\zeta\}^T\big)\{\bar{q}\} + q_1\big(\{F_\eta\}^T \\ &- 2r_\eta\{F_\zeta\}^T\big)\{\dot{\bar{q}}\} - q_2\big(\{F_\xi\}^T - 2r_\xi\{F_\zeta\}^T\big)\{\dot{\bar{q}}\} + \{\bar{q}\}^T[F_{\zeta\zeta}]\{\dot{\bar{q}}\} \\ &- \frac{1}{(J_{\zeta\zeta})_0}\{\bar{q}\}^T\{H_\zeta\}\{F_\zeta\}^T\{\dot{\bar{q}}\}\Big) \end{aligned} \tag{3.23b}$$

$$\begin{aligned} R_0 = \frac{1}{2}\Omega^2 \Big(&\big((J_{\eta\eta})_0 - (J_{\zeta\zeta})_0 - 4r_\eta^2(J_{\zeta\zeta})_0\big)q_1^2 + \big((J_{\xi\xi})_0 - (J_{\zeta\zeta})_0 \\ &- 4r_\xi^2(J_{\zeta\zeta})_0\big)q_2^2 - 2\big((J_{\xi\eta})_0 - 4r_\xi r_\eta(J_{\zeta\zeta})_0\big)q_1q_2 + 2q_1\big(\{H_\eta\}^T \\ &- 2r_\eta\{H_\zeta\}^T\big)\{\bar{q}\} - 2q_2\big(\{H_\xi\}^T - 2r_\xi\{H_\zeta\}^T\big)\{\bar{q}\} + \{\bar{q}\}^T[H_{\zeta\zeta}]\{\bar{q}\} \\ &- \frac{1}{(J_{\zeta\zeta})_0}\{\bar{q}\}^T\{H_\zeta\}\{H_\zeta\}^T\{\bar{q}\}\Big) \end{aligned} \tag{3.23c}$$

Using (3.10), (3.20) and (3.23), we can write

$$R_2 = \frac{1}{2}\{\dot{q}\}^T[m]\{\dot{q}\} \tag{3.24a}$$

$$R_1 = \{q\}^T[f]\{\dot{q}\} \tag{3.24b}$$

$$V_{EL} - R_0 = \frac{1}{2}\{q\}^T[k]\{q\} \tag{3.24c}$$

where the matrices [m], [f] and [k] are

$$[m] = \left[\begin{array}{cc|c} (J_{\xi\xi})_0 - r_\xi^2 (J_{\zeta\zeta})_0 & (J_{\xi\eta})_0 - r_\xi r_\eta (J_{\zeta\zeta})_0 & \{F_\xi\}^T - r_\xi \{F_\zeta\}^T \\ (J_{\xi\eta})_0 - r_\xi r_\eta (J_{\zeta\zeta})_0 & (J_{\eta\eta})_0 - r_\eta^2 (J_{\zeta\zeta})_0 & \{F_\eta\}^T - r_\eta \{F_\zeta\}^T \\ \hline \{F_\xi\} - r_\xi \{F_\zeta\} & \{F_\eta\} - r_\eta \{F_\zeta\} & [m_s] - \frac{1}{M}[m_c] - \frac{1}{(J_{\zeta\zeta})_0}\{F_\zeta\}\{F_\zeta\}^T \end{array}\right]$$

$$[f] = \Omega \left[\begin{array}{cc|c} (J_{\xi\eta})_0 - 2r_\xi r_\eta (J_{\zeta\zeta})_0 & (J_{\eta\eta})_0 - 2r_\eta^2 (J_{\zeta\zeta})_0 & \{F_\eta\}^T - 2r_\eta \{F_\zeta\}^T \\ (J_{\zeta\zeta})_0 - (J_{\xi\xi})_0 + 2r_\xi^2 (J_{\zeta\zeta})_0 & -(J_{\xi\eta})_0 + 2r_\xi r_\eta (J_{\zeta\zeta})_0 & -\{F_\xi\}^T + 2r_\xi \{F_\zeta\}^T \\ \hline \{H_\xi\} - r_\xi \{H_\zeta\} & \{H_\eta\} - r_\eta \{H_\zeta\} & [F_{\zeta\zeta}] - \frac{1}{(J_{\zeta\zeta})_0}\{H_\zeta\}\{F_\zeta\}^T \end{array}\right] \qquad (3.25)$$

$$[k] = \Omega^2 \left[\begin{array}{cc|c} (1+4r_\eta^2)(J_{\zeta\zeta})_0 - (J_{\eta\eta})_0 & (J_{\xi\eta})_0 - 4r_\xi r_\eta (J_{\zeta\zeta})_0 & -\{H_\eta\}^T + 2r_\eta \{H_\zeta\}^T \\ (J_{\xi\eta})_0 - 4r_\xi r_\eta (J_{\zeta\zeta})_0 & (1+4r_\xi^2)(J_{\zeta\zeta})_0 - (J_{\xi\xi})_0 & \{H_\xi\}^T - 2r_\xi \{H_\zeta\}^T \\ \hline -\{H_\eta\} + 2r_\eta \{H_\zeta\} & \{H_\xi\} - 2r_\xi \{H_\zeta\} & \frac{1}{\Omega^2}[k_s] - [H_{\zeta\zeta}] + \frac{1}{(J_{\zeta\zeta})_0}\{H_\zeta\}\{H_\zeta\}^T \end{array}\right]$$

[m] and [k] are refered to as the mass and stiffness matrix, respectively. Note that the dimensions of the partioned matrices are 2×2, 2×p, p×2, and p×p, correspondingly.

For the special case when all the rods are along the principal axes such that conditions (2.49) and (2.50) are satisfied, the matrices [m], [f], and [k] reduce to the following.

$$[m] = \left[\begin{array}{cc|c} A & 0 & \{F_\xi\}^T \\ 0 & B & \{F_\eta\}^T \\ \hline \{F_\xi\} & \{F_\eta\} & [m_s] - \frac{1}{M}[m_c] - \frac{1}{C}\{F_\zeta\}\{F_\zeta\}^T \end{array}\right]$$

$$[f] = \Omega \left[\begin{array}{cc|c} 0 & B & \{F_\eta\}^T \\ C-A & 0 & -\{F_\xi\}^T \\ \hline \{H_\xi\} & \{H_\eta\} & [F_{\zeta\zeta}] - \frac{1}{C}\{H_\zeta\}\{F_\zeta\}^T \end{array}\right] \qquad (3.26)$$

$$[k] = \Omega^2 \left[\begin{array}{cc|c} C-B & 0 & -\{H_\eta\}^T \\ 0 & C-A & \{H_\xi\}^T \\ \hline -\{H_\eta\} & \{H_\xi\} & \frac{1}{\Omega^2}[k_s] - [H_{\zeta\zeta}] + \frac{1}{C}\{H_\zeta\}\{H_\zeta\} \end{array}\right]$$

For a vibrating string or a flexurally vibrating rod, both $[E_i]$ and $[L_i]$ are replaced by (2.34). The displacement vector is reduced to $\{v_{i1} \quad w_{i1}\}^T$. Accordingly, the first row in $[N_i]$ is skipped and $r_i=0$.

4. Equations of Motion

Considering (2.46) and (3.24), the Routhian function can be written as

$$R = \frac{1}{2}\{\dot{q}\}^T[m]\{\dot{q}\} + \{q\}^T[f]\{\dot{q}\} - \frac{1}{2}\{q\}^T[k]\{q\} \tag{4.1}$$

The equations of motion have the general form

$$\frac{d}{dt}\left\{\frac{\partial R}{\partial \dot{q}}\right\} - \left\{\frac{\partial R}{\partial q}\right\} = \left\{0\right\} \tag{4.2}$$

which is recognized as the matrix form of the Lagrangian equations of motion. Substituting (4.1) into (4.2), we obtain the equations of motion of the torque-free system

$$[m]\{\ddot{q}\} + [g]\{\dot{q}\} + [k]\{q\} = \{0\} \tag{4.3}$$

where $[g]=[f]^T-[f]$ which is a skew-symmetric matrix, while [m] and [k] are symmetric matrices.

$$[g]^T = -[g] \quad , \quad [m]^T = [m] \quad , \quad [k] = [k] \tag{4.4}$$

By (3.25) and the same partition in that expression, the explicit form of the matrix [g] is

$$[g] = \Omega \begin{bmatrix} [g_{11}] & [g_{12}] \\ [g_{21}] & [g_{22}] \end{bmatrix} \tag{4.5a}$$

where the submatrices are

$$[g_{11}]=\begin{bmatrix} 0 & (1+2r_\xi^2+2r_\eta^2)(J_{\zeta\zeta})_0-(J_{\xi\xi})_0-(J_{\eta\eta})_0 \\ -(1+2r_\xi^2+2r_\eta^2)(J_{\zeta\zeta})_0+(J_{\xi\xi})_0+(J_{\eta\eta})_0 & 0 \end{bmatrix}$$

$$[g_{12}]=\begin{bmatrix} \{H_\xi\}^T - \{F_\eta\}^T - r_\xi\{H_\zeta\}^T + 2r_\eta\{F_\zeta\}^T \\ \{H_\eta\}^T + \{F_\xi\}^T - r_\eta\{H_\zeta\}^T - 2r_\xi\{F_\zeta\}^T \end{bmatrix}$$

$$[g_{21}]=\begin{bmatrix} -\{H_\xi\}+\{F_\eta\}+r_\xi\{H_\zeta\}-2r_\eta\{F_\zeta\} & -\{H_\eta\}-\{F_\xi\}+r_\eta\{H_\zeta\}+2r_\xi\{F_\zeta\} \end{bmatrix} \qquad (4.5b)$$

$$[g_{22}]=\begin{bmatrix} \frac{1}{(J_{\zeta\zeta})_0}(\{H_\zeta\}\{F_\zeta\}^T-\{F_\zeta\}\{H_\zeta\}^T) \end{bmatrix}$$

Since the equations of motion posses the gyroscopic terms, $[g]\{\dot{q}\}$, (4.3) represents a gyroscopic system.

Introducing the 2(p+2)-dimensional state vector

$$\{x(t)\} = \begin{Bmatrix} \{\dot{q}(t) \\ \{q(t)\} \end{Bmatrix} \qquad (4.6)$$

Equation (4.3) can be reduced to a set of 2(p+2) first order equations.

$$[J]\{\dot{x}(t)\} + [G]\{x(t)\} = \{0\} \qquad (4.7)$$

where

$$[J] = \begin{bmatrix} [m] & [0] \\ [0] & [k] \end{bmatrix} , \quad [G] = \begin{bmatrix} [g] & [k] \\ -[k] & [0] \end{bmatrix} \qquad (4.8)$$

Clearly, [J] is symmetric and [G] is skew symmetric. Equation (4.7) is more convenient than (4.3) for latter analysis.

5. Stability

5.1 Preliminary

Consider the autonomous system

$$\dot{\underset{\sim}{x}} = \underset{\sim}{X}\ (\underset{\sim}{x}) \tag{5.1}$$

in which $\underset{\sim}{x}$ represents a vector in a finite dimensional vector space. The vector $\underset{\sim}{X}$ satisfies the relation $\underset{\sim}{X}(\underset{\sim}{0})=\underset{\sim}{0}$, so that the origin is a singular point and the vector $\underset{\sim}{x}=\underset{\sim}{0}$ is the null solution of the system. Denote the integral curve at a given time $t_0>0$ by $\underset{\sim}{x}(t_0)=\underset{\sim}{x}_0$, and introduce the following definitions due to Liapunov (Ref. 1, Sec. 6.7).

Definition 1: The null solution of a system (5.1) is called stable in the sense of Liapunov if for any $\epsilon>0$ and a given $t_0\geq 0$ it is possible to find a $\delta=\delta(\epsilon,t_0)>0$ such that if the inequality

$$\|\underset{\sim}{x}_0\| < \delta \tag{5.2}$$

is satisfied, we shall have

$$\|\underset{\sim}{x}(t)\| < \epsilon \quad , \quad t_0 < t < \infty \tag{5.3}$$

If $\delta=\delta(\epsilon)$, i.e. independent of t_0, the stability is said to be uniform.

Definition 2: The null solution is called asymptotically stable if it is stable in the sense of Liapunov and, furthermore,

$$\lim_{t\to\infty} \|\underset{\sim}{x}(t)\| = 0 \tag{5.4}$$

A uniformly stable solution which satisfies (5.4) is called uniformly asymptotically stable.

Definition 3: The null solution is said to be unstable if there is an $\varepsilon>0$ and a t_0 such that for any $\delta>0$, there is a $t_1>t_0$ with

$$\|x_0\| < \delta \tag{5.5}$$

and

$$\|x(t_1)\| \geqq \varepsilon \tag{5.6}$$

For the autonomous system (5.1) the stability is always uniform.

Next let us define a scalar function $U(\underset{\sim}{x})$ such that $U(\underset{\sim}{0})=0$. The total time derivative of U along a trajectory of the system is defined by

$$\dot{U} = \frac{dU}{dt} = \nabla U \cdot \dot{\underset{\sim}{x}} = \nabla U \cdot \underset{\sim}{X} \tag{5.7}$$

Now consider the following theorems for stability and instability.

Theorem 1: If the system is such that it is possible to find a positive (negative) definite function $U(\underset{\sim}{x})$ whose total time derivative $\dot{U}(\underset{\sim}{x})$ is negative (positive) semidefinite along every trajectory of the system, then the trivial solution $\underset{\sim}{x}=\underset{\sim}{0}$ is stable.

Theorem 2: If the conditions of Theorem 1 are satisfied and furthermore the set of points at which $\dot{U}(\underset{\sim}{x})$ is zero contains no nontrivial positive half-trajectory, then the trivial solution is asymptotically stable.

Theorem 3: If the system is such that it is possible to find a function $U(\underset{\sim}{x})$ whose total time derivative $\dot{U}(\underset{\sim}{x})$ is positive (negative) definite along every trajectory of the system and the function itself can admit positive (negative) values in the neighborhood of the origin, then the trivial solution is unstable.

Theorem 4: Suppose that a function $U(\underset{\sim}{x})$ such as in Theorem 3 exists but for which $\dot{U}(\underset{\sim}{x})$ is only positive (negative) semidefinite and, in addition, the set of points at which $\dot{U}(\underset{\sim}{x})$ is zero contains no nontrivial positive half-trajectory. Furthermore, in every neighborhood of the origin there is a point $\underset{\sim}{x}_0$ such that for arbitrary $t_0>0$ we have $U(\underset{\sim}{x}_0)>0$ (<0). Then the trivial solution is unstable and the trajectories $\underset{\sim}{x}(\underset{\sim}{x}_0,t_0,t)$ for which $U(\underset{\sim}{x}_0)>0$ (<0) must leave the open domain $\|x\|<\varepsilon$ as the time t increases.

A function U satisfying any of the preceeding theorems is referred to as a Liapunov function.

5.2 Stability of the Equilibrium

Because the system Lagrangian L does not contain the time explicitly, the system is scleronomic. For such system there exists a motion integral which is called the Jacobi integral.

$$H = \sum_{k=1}^{p+2} \frac{\partial L}{\partial \dot{q}_3} \dot{q}_k + \frac{\partial L}{\partial \dot{\theta}_3} \dot{\theta}_3 - L = \text{Constant} \qquad (5.8)$$

In general, because the Lagrangian includes the term T_1 containing the gyroscopic effects as well as the term T_0 containing the centrifugal effects, the Jacobi integral, (5.8), becomes

$$H = 2T_2 + T_1 - (T_2 + T_1 + T_0 - V_{EL}) = T_2 - T_0 + V_{EL} = \text{Constant} \quad (5.9)$$

Now we have the condition

$$\frac{dH}{dt} = 0 \quad (5.10)$$

In reality, damping always exists in a flexible structure causing energy to be dissipated so that (5.10) should be replaced by

$$\frac{dH}{dt} \leqq 0 \quad (5.11)$$

If we can prove that $T_2-T_0+V_{EL}$ is positive-definite then, by Theorem 1, $T_2-T_0+V_{EL}$ is a suitable Liapunov function and the system is stable. The conditions that guarantee the sign definiteness of $T_2-T_0+V_{EL}$ constitute the stability criteria. The latter can be used to plot stability boundaries in terms of the system parameters, thus providing useful information for design purpose. For complicated structures with a large number of degrees of freedom, it is not easy to obtain these criteria in closed form, because this procedure involves the evaluation of the leading principal minor determinants of a certain matrix. In this case, we obtain the stability boundaries numerically, by means of a digital computer.

Unfortunately, $T_2-T_0+V_{EL}$ is not sign definite because θ_3 does not appear in $-T_0+V_{EL}$, hence it is ignorable. To remove it from the formulation, we rewrite (5.8) as

$$H = \sum_{k=1}^{p+2} \frac{\partial L}{\partial \dot{q}_k} \dot{q}_k - (L - \frac{\partial L}{\partial \dot{\theta}_3} \dot{\theta}_3) \tag{5.12a}$$

By (2.46) and (Ref. 1, sec. 2.11), this equation can be expressed in terms of Routhian function

$$H = \sum_{k=1}^{p+2} \frac{\partial R}{\partial \dot{q}_k} \dot{q}_k - R \tag{5.12b}$$

Substituting (2.55) into (5.12), we have

$$H = (2R_2 + R_1) - (R_2 + R_1 + R_0 - V_{EL}) = R_2 - R_0 + V_{EL} \tag{5.13}$$

The left side satisfies (5.10) and the right side will be used as a Liapunov function. Let us write

$$H = R_2 + (V_{EL} - R_0) \tag{5.14}$$

and recall(3.24) and (4.6); the Liapunov function can be expressed as

$$H = \frac{1}{2} \{\dot{q}\}^T[m]\{\dot{q}\} + \frac{1}{2} \{q\}^T[k]\{q\} = \frac{1}{2} \{x\}^T[J]\{x\} \tag{5.15a}$$

The system is stable if the testing function is positive definite, which in turn requires that both [m] and [k] are positive definite. A real symmetric matrix is positive definite if and only if all its eigenvalues are positive. This property is being used for testing large-order matrices. To obtain the stability criteria for small order matrices, we rely on the Sylvester's theorem, which can be stated as follows: The quadratic form (5.15) is positive definite if and only if all the principal minor determinants corresponding to the symmetric matrices of the coefficients are positive.

Applying this theorem to the first two leading principal minor determinants of [k] of the special case, (3.26), we immediately have

$$C - B > 0 \qquad \text{and} \qquad C - A > 0 \tag{5.16}$$

These are the conditions for a stable motion of a rigid body, i.e. all the flexible parts are visualized as rigid. These conditions are only necessary as other leading principal minor determinants lead to expressions which require that C-B and C-A be greater than some values.

6. Eigenvalue Problem

When the equilibrium is stable which, from section 5, is equivalent to the condition that both [m] and [k] be positive definite, the eigenvalue problem possesses interesting properties. We would like to demonstrate these properties as well as to show an efficient way of obtaining the solution of the eigenvalue problem.

6.1 Properties

Let us seek a solution of (4.7) in the form

$$\{x(t)\} = \{x\}e^{\lambda t} \tag{6.1}$$

where $\{x\}$ is a 2(p+2)-dimensional constant vector and λ is a constant scalar. Introducing (6.1) into (4.7) and dividing through by $e^{\lambda t}$, we obtain

$$(\lambda[J] + [G])\{x\} = \{0\} \tag{6.2}$$

This is the eigenvalue problem in which [J] is real symmetric and positive definite and [G] is real skew symmetric and nonsingular.

The solution of the eigenvalue problem (6.2) consists of 2(p+2) eigenvalues λ_r and associated eigenvectors $\{x_r\}$ $(r=1,2,\ldots, 2(p+2))$. First we want to show that the eigenvalues are complex conjugate pure imaginary. Let us consider the characteristic equation

$$\det(\lambda[J] + [G]) = 0 \tag{6.3}$$

Since the determinant of a matrix is equal to the determinant of transposed matrix, we must also have

$$\det(\lambda[J] + [G])^T = \det(\lambda[J]^T + [G]^T) = \det(\lambda[J] - [G]) = 0 \tag{6.4}$$

Comparing (6.3) and (6.4), we conclude that if λ is an eigenvalue of the system, then $-\lambda$ is also an eigenvalue. Next, assuming the eigenvector associated with a known eigenvalue λ_r is

$$\{x_r\} = \{y_r\} + i\{z_r\} \tag{6.5}$$

where both $\{y_r\}$ and $\{z_r\}$ are real and $i=\sqrt{-1}$, then the eigenvalue problem corresponding to λ_r is

$$\lambda_r[J]\{x_r\} + [G]\{x_r\} = \{0\} \tag{6.6}$$

Let the complex conjugate of $\{x_r\}$ be $\{\bar{x}_r\}$. Premultiplying (6.6) by $\{\bar{x}_r\}^T$, we have

$$\lambda_r\{\bar{x}_r\}^T[J]\{x_r\} + \{\bar{x}_r\}^T[G]\{x_r\} = 0 \tag{6.7}$$

Substituting (6.5) into (6.7), solving for λ_r, and noting that

$$\{y_r\}^T[G]\{y_r\} = \{z_r\}^T[G]\{z_r\} = 0 \tag{6.8}$$

we obtain

$$\lambda_r = \frac{2i\{y_r\}^T[G]\{z_r\}}{\{y_r\}^T[J]\{y_r\} + \{z_r\}^T[J]\{z_r\}} \tag{6.9}$$

Since the denominator is real and the numerator is pure imaginary, we conclude that λ_r is pure imaginary. Hence, because $-\lambda_r$ is also an eigenvalue, it follows that all the eigenvalues are complex conjugate pure imaginary. We shall denote the system eigenvalues by $\pm i\omega_r$ $(r=1,2,\ldots,2(p+2))$ where ω_r are recognized as the natural frequencies of the system. If we take complex conjugate of both sides of (6.6), we have

$$\bar{\lambda}_r[J]\{\bar{x}_r\} + [G]\{\bar{x}_r\} = \{0\} \tag{6.10}$$

so that if $i\omega_r$ and $\{x_r\}$ represent a solution of the eigenvalue problem, then $-i\omega_r$ and $\{\bar{x}_r\}$ also represent a solution. Hence, the system eigenvectors consist of (p+2) pairs of complex-conjugate eigenvectors $\{x_r\}$ and $\{\bar{x}_r\}$ belonging to the eigenvalues $i\omega_r$ and $-i\omega_r$ respectively.

6.2 Reduction of the Eigenvalue Problem

Since the matrix [J] is symmetric and positive definite, it is possible to decomposite this matrix by Cholesky decomposition.

$$[J] = [L][L]^T \tag{6.11}$$

in which [L] is a nonsingular lower triangular matrix. Substituting (6.11) into (6.2), introducing the transformation

$$\{x'\} = [L]^T\{x\} \tag{6.12}$$

and premultiplying the whole equation by $[L]^{-1}$, we obtain

$$(\lambda[I] + [A])\{x'\} = \{0\} \tag{6.13}$$

where [I] is identy matrix and

$$[A] = [L]^{-1}[G][L]^{-T} \tag{6.14}$$

is a skew symmetric matrix and $[L]^{-T}=([L]^{-1})^{T}$. Hence, the system eigenvalue problem (6.2) is reduced to a standard eigenvalue problem for a skew symmetric matrix. A Jacobi-like algorithm for skew symmetric matrices was presented by Paardekooper.[31]

Meirovitch[32] showed that the eigenvalue problem (6.2) can be solved by working with real quantities. Consider a given solution $\lambda_r = i\omega_r$ and substituting (6.5) into (6.6), we have

$$\omega_r[J](\{y_r\} + i\{z_r\}) + [G](\{y_r\} + i\{z_r\}) = \{0\} \tag{6.15}$$

The real and imaginary parts give two equations in real quantities alone

$$-\omega_r[J]\{z_r\} + [G]\{y_r\} = \{0\} \tag{6.16a}$$

$$\omega_r[J]\{y_r\} + [G]\{z_r\} = \{0\}$$

Equation (6.16a) yields

$$\{z_r\} = \frac{1}{\omega_r}[J]^{-1}[G]\{y_r\} \tag{6.17}$$

Substituting (6.17) into (6.16b), we have

$$\omega_r^2[J]\{y_r\} = [K]\{y_r\} \qquad r = 1,2,\ldots,2(p+2) \qquad (6.18a)$$

where

$$[K] = -[G][J]^{-1}[G] = [G]^T[J]^{-1}[G] \qquad (6.18b)$$

is real symmetric matrix. Similarly

$$\omega_r^2[J]\{z_r\} = [K]\{z_r\} \qquad r = 1,2,\ldots,2(p+2) \qquad (6.18c)$$

Hence both $\{y_r\}$ and $\{z_r\}$ solve the same eigenvalue problem with eigenvalues ω_r^2. The eigenvalue problem (6.2) defined by one real symmetric matrix and one real skew symmetric matrix and possessing complex solutions has been reduced to the eigenvalue problem (6.18) defined by two real symmetric matrices and possessing real solutions.

Introducing the decomposition (6.11) and the notation

$$\{y_r'\} = [L]^T\{y_r\} \qquad , \qquad \{z_r'\} = [L]^T\{z_r\} \qquad (6.19)$$

and premultiplying (6.18) by $[L]^{-1}$, we obtain

$$\omega_r^2\{y_r'\} = [K']\{y_r'\} \quad , \quad \omega_r^2\{z_r'\} = [K']\{z_r'\} \qquad (6.20)$$

where

$$[K'] = [L]^{-1}[K][L]^{-T} = [L]^{-1}[G]^T[J]^{-1}[G][L]^{-T} \qquad (6.21)$$

is a real symmetric matrix. In fact it is also positive definite. this can be shown by substituting

$$[J]^{-1} = [L]^{-T}[L]^{-1} \qquad (6.22)$$

into (6.17) and regrouping in the following way

$$[K'] = ([L]^{-1}[G]^T[L]^{-T})([L]^{-1}[G][L]^{-T}) = [Q]^T[Q] \qquad (6.23)$$

where

$$[Q] = [L]^{-1}[G][L]^{-T} \qquad (6.24)$$

From (6.23) we conclude that [K'] is positive definite.

Now equation (6.20) can be solved by any algorithm for rea symmetric matrices.[33-36] Routines[37-39] are available in different computer languages.

7. Symmetric Structures

Structures symmetric with respect to z-axis are very common in practice. We shall define the symmetric properties via the general expression of the partial differential equation of motion for a non-spin, force-free, vibrating continuous system

$$L\,[u(x,y,z,t)] + M(x,y,z)\,\frac{\partial^2 u}{\partial t^2}(x,y,z,t) = 0 \tag{7.1}$$

where L is a differential operator which defines the stiffness properties and M is a function of spatial variables that defines the inertia properties of the system. A structure is symmetric with respect to z-axis if

$$L\,[u(x,y,z,t)] = L\,[u(-x,-y,z,t)] \tag{7.2a}$$

$$M\,(x,y,z) = M\,(-x,-y,z) \tag{7.2b}$$

For this type of structure it is generally assumed that the motion of the flexible parts is symmetric or antisymmetric. Since displacements of any mass point in domain D_i are expressed in terms of coordinates $x_iy_iz_i$, the origin O_i and axes $x_iy_iz_i$ for corresponding symmetric parts of the structures must be chosen symmetrically with respect to the global z-axis. Denoting the symmetric parts by D_i and D_j and imagining the origin O_i and the frame $x_iy_iz_i$ are fixed in the frame xyz, then a 180 degree of rotation about z axis should bring O_i and $x_iy_iz_i$ into coincidence with O_j and $x_jy_jz_j$, respectively. The two types of motion are written as

$$\text{Symmetric motion}\quad \begin{cases} u_i(x_i,y_i,z_i,t) = -\,u_j(x_j,y_j,z_j,t) \\ v_i(x_i,y_i,z_i,t) = -\,v_j(x_j,y_j,z_j,t) \\ w_i(x_i,y_i,z_i,t) = \,w_j(x_j,y_j,z_j,t) \end{cases} \qquad (7.4a)$$

$$\text{Antisymmetric motion}\quad \begin{cases} u_i(x_i,y_i,z_i,t) = \,u_j(x_j,y_j,z_j,t) \\ v_i(x_i,y_i,z_i,t) = \,v_j(x_j,y_j,z_j,t) \\ w_i(x_i,y_i,z_i,t) = -\,w_j(x_j,y_j,z_j,t) \end{cases} \qquad (7.4b)$$

Utilizing (7.4), the generalized displacements of the flexible parts are reduced to one half of the original set. Similarly, the number of discretized equations of motion are reduced to $2+\sum_{i=1}^{n/2} p_i$, where n is an even number.

Physically, symmetric motion causes the mass center c to wander away from the geometric center 0. On the other hand, in the case of antisymmetric motion the mass center c coincides with 0 at all times. In general, the elastic displacements are linear combinations of both symmetric and antisymmetric motions. If both types of motion about the equilibrium are stable, then the general motion about the same equilibrium point is also stable. Since each type of motion yields different stability criteria, both cases must be considered in the analysis. However, in view of the reduction in the degree of freedom mentioned earlier, the advantage of separating the motion into symmetric and antisymmetric is obvious. This is particularly true in the case of a system with a large number of degrees of freedom, for which a computer storage problem may exist.

8. Application to the GEOS satellite

The preceeding formulation has been applied to the GEOS satellite of the European Space Agency (ESA). The model which consists of a rigid core, one pair of radial booms, one pair of cables, and two pairs of axial booms is shown in Fig. 3. The coordinate systems are chosen such that the x_i-axis ($i=1,2,\ldots,8$) are along the booms or cables and the z_i-axis ($i=1,2,\ldots,8$) are in the x_i-z plane. The stability analysis was coded in a computer program[40] capable of accommodating satellites with more general configurations than the GEOS, in the sense that the number of flexible members and their orientation relative to the rigid core is arbitrary. The spacecraft was found to be stable about the nontrivial equilibrium in four different cases.[40] However, as it was pointed out, the margin of safety may not be very large, and it appeared desirable to investigate the characteristics of the system by examining the spacecraft natural frequencies.

Since the cables are longer and more flexible than the booms their effect is more critical. For the purpose of comparing with the results of another investigator, the model was simplified to a rigid core with all the booms considered as rigid and a pair of flexible cables with tip masses (see Fig. 4). This is the special case of (2.50) in which the equilibrium is trivial. Hence $v_{i0}=w_{i0}=0$, so that we can drop the subscript 1 from v_{i1} and w_{i1}.

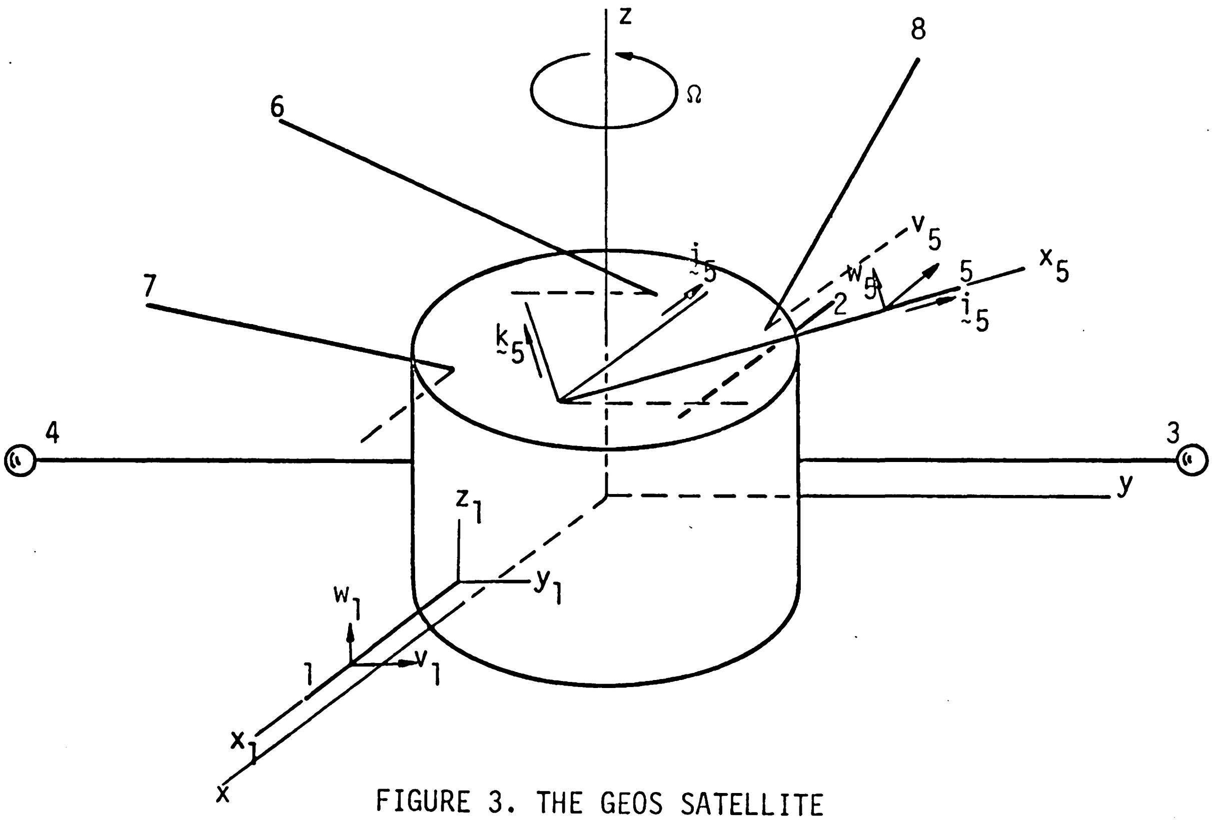

FIGURE 3. THE GEOS SATELLITE

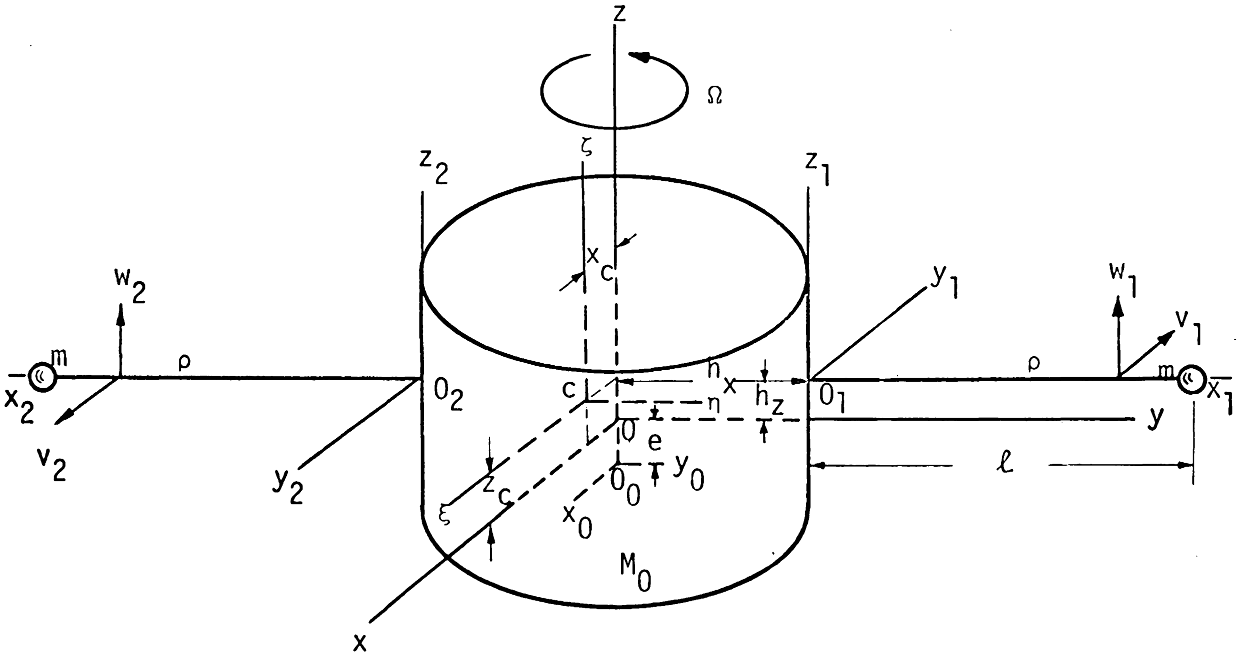

FIGURE 4. SIMPLIFIED MODEL

8.1 Explicit Expressions for Kinetic Energy and Potential Energy

The spacecraft is symmetric with respect to z-axis so that the density, length, tip mass, and geometry of symmetric flexible parts are identical.

$$\begin{aligned} &\rho_1 = \rho_2 = \rho \quad , \quad \ell_1 = \ell_2 = \ell \quad , \quad m_1 = m_2 = m \\ &h_{x1} = h_{x2} = h_x \quad , \quad h_{z1} = h_{z2} = h_z \end{aligned} \tag{8.1}$$

The total mass and the distance between 0 and 0_0 are

$$\begin{aligned} M &= M_0 + M_1 + M_2 = M_0 + 2(\rho\ell + m) \\ e &= \frac{2}{M_0}(\rho\ell + m)h_z \end{aligned} \tag{8.2}$$

The components of the vector $\underset{\sim}{r}_c$ from 0 to c in the deformed state are

$$\begin{aligned} x_c &= \frac{1}{M}\left(-\int_0^\ell \rho v_1 dx_1 - mv_1(\ell) + \int_0^\ell \rho v_2 dx_2 + mv_2(\ell)\right) \\ y_c &= 0 \\ z_c &= \frac{1}{M}\left(\int_0^\ell \rho w_1 dx_1 + mw_1(\ell) + \int_0^\ell \rho w_2 dx_2 + mw_2(\ell)\right) \end{aligned} \tag{8.3}$$

in which the displacement components v_i, w_i are functions of x_i and t only, while $v_i(\ell)$, $w_i(\ell)$ represent functional values evaluated at ℓ, they are also functions of time. The component y_c is approximately zero because the motion is essentially in xz-plane.

The elements of the inertia matrix, $[J_i]$, of the deformed state with respect to 0 are

$$
\begin{aligned}
J_{i11} &= \int_0^{\ell} \rho\left((h_z+w_i)^2 + v_i^2\right) dx_i + m\left((h_z+w_i(\ell))^2 + v_i^2(\ell)\right) \\
J_{i22} &= \int_0^{\ell} \rho\left((h_x+x_i)^2 + (h_z+w_i)^2\right) dx_i + m\left((h_x+\ell)^2 + (h_z+w_i(\ell))^2\right) \\
J_{i33} &= \int_0^{\ell} \rho\left((h_x+x_i)^2 + v_i^2\right) dx_i + m\left((h_x+\ell)^2 + v_i^2(\ell)\right) \\
J_{i12} &= J_{i21} = -\int_0^{\ell} \rho(h_x+x_i)v_i dx_i - m(h_x+\ell)v_i(\ell) \\
J_{i13} &= J_{i31} = -\int_0^{\ell} \rho(h_x+x_i)(h_z+w_i) dx_i - m(h_x+\ell)(h_z+w_i(\ell)) \\
J_{i23} &= J_{i32} = -\int_0^{\ell} \rho(h_z+w_i)v_i dx_i - m(h_z+w_i(\ell))v_i(\ell)
\end{aligned}
\tag{8.4}
$$

$$i = 1,\ 2$$

The coordinate transformation matrices $[\ell_i]$, i=1, 2, are

$$
[\ell_1] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad [\ell_2] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\tag{8.5}
$$

The matrix $[J_d]$ could be obtained by (2.8). However, we wish to obtain (2.26) directly by spliting the matrices $[J_i]$ into three parts, $[J_i]=[J_i]_0+[J_i]_1+[J_i]_2$, according to their order of magnitude, where

$$
[J_i]_0 = \begin{bmatrix}
(\rho\ell+m)h_z^2 & & \text{symmetric} \\
0 & \begin{array}{c}(h_x^2+h_z^2)+\rho\ell^2 h_x+\frac{1}{3}\rho\ell^3 \\ +m\left((h_x+\ell)^2+h_z^2\right)\end{array} & \\
\begin{array}{l}-\left(\rho\ell h_z(h_x+\frac{1}{2}\ell)\right. \\ \left.+mh_z(h_x+\ell)\right)\end{array} & 0 & \begin{array}{c}\rho\ell h_x(h_x+\ell)+\frac{1}{3}\rho\ell^3 \\ +m(h_x+\ell)^3\end{array}
\end{bmatrix}
$$

$$[J_i]_1 = \begin{bmatrix} 2h_z\left(\int_0^{\ell} \rho w_i dx_i + m w_i(\ell)\right) & & \text{symmetric} \\ \begin{matrix} -\left(\int_0^{\ell} \rho(h_x+x_i) v_i dx_i \right. \\ \left. + m(h_x+\ell) v_i(\ell)\right) \end{matrix} & 2h_z\left(\int_0^{\ell} \rho w_i dx_i + m w_i(\ell)\right) & \\ \begin{matrix} -\left(\int_0^{\ell} \rho(h_x+x_i) w_i dx_i \right. \\ \left. + m(h_x+\ell) w_i(\ell)\right) \end{matrix} & -h_z\left(\int_0^{\ell} \rho v_i dx_i + m v_i(\ell)\right) & 0 \end{bmatrix}$$

$$[J_i]_2 = \begin{bmatrix} \begin{matrix} \int_0^{\ell} \rho(v_i^2+w_i^2) dx_i \\ + m\left(v_i^2(\ell)+w_i^2(\ell)\right) \end{matrix} & & \text{symmetric} \\ 0 & \int_0^{\ell} \rho w_i^2 dx_i + m w_i^2(\ell) & \\ 0 & \begin{matrix} -\int_0^{\ell} \rho v_i w_i dx_i \\ - m v_i(\ell) w_i(\ell) \end{matrix} & \int_0^{\ell} \rho v_i^2 dx_i + m v_i^2(\ell) \end{bmatrix}$$

$$i = 1,\ 2 \tag{8.6}$$

Using (8.5) we can obtain the components $[J_d]_0$, $[J_d]_1$, $[J_d]_2$. Hence, we write

$$[J_d]_0 = \sum_{i=0}^{2} [\ell_i][J_i]_0[\ell_i] = \begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix} \tag{8.7}$$

where

$$A = A_0 + 2\left(\left(\rho\ell(h_x^2+h_z^2) + \rho\ell^2 h_x + \tfrac{1}{3}\rho\ell^3\right) + m\left((h_x+\ell)^2 + h_z^2\right)\right) + M_0 e^2 \tag{8.8a}$$

$$B = B_0 + 2h_z^2(\rho\ell+m) + M_0 e^2 \tag{8.8b}$$

$$C = C_0 + 2\left(\rho\ell h_x(h_x+\ell) + \tfrac{1}{3}\rho\ell^3 + m(h_x+\ell)^2\right) \tag{8.8c}$$

and

$$[J_d]_1 = \sum_{i=1}^{2} [\ell_i][J_i]_1[\ell_i] \tag{8.9a}$$

$$[J_d]_2 = \sum_{i=1}^{2} [\ell_i][J_i]_2[\ell_i] - M \begin{bmatrix} z_c^2 & 0 & -x_c z_c \\ 0 & x_c^2+z_c^2 & 0 \\ -x_c z_c & 0 & x_c^2 \end{bmatrix} \tag{8.9b}$$

Next, we whish to determine the elements $(J_{\xi\zeta})_1$, $(J_{\eta\zeta})_1$, $(J_{\zeta\zeta})_1$, of the matrix $[J_d]_1$ and the element $(J_{\zeta\zeta})_2$ of the matrix $[J_d]_2$, as they are the only ones which will be needed. Substituting (8.3), (8.5) and (8.6) into (8.9), we obtain

$$(J_{\xi\zeta})_1 = h_z\left(\int_0^\ell \rho v_1 dx_1 + mv_1(\ell) - \int_0^\ell \rho v_2 dx_2 - mv_2(\ell)\right)$$

$$(J_{\eta\zeta})_1 = -\left(\int_0^\ell \rho(h_x+x_1)w_1 dx_1 + m(h_x+\ell)w_1(\ell) - \int_0^\ell \rho(h_x+x_2)w_2 dx_2 - m(h_x+\ell)w_2(\ell)\right)$$

$$(J_{\zeta\zeta})_1 = 0$$

$$(J_{\zeta\zeta})_2 = \int_0^\ell \rho v_1^2 dx_1 + mv_1^2(\ell) + \int_0^\ell \rho v_2^2 dx_2 + mv_2^2(\ell) - \frac{1}{M}\left(\int_0^\ell \rho v_1 dx_1 + mv_1(\ell) - \int_0^\ell \rho v_2 dx_2 - mv_2(\ell)\right)^2 \tag{8.10}$$

The displacements with respect to axes $\xi_i\eta_i\zeta_i$ (i=0,1,2) are

$$\{h_0+r_0+u_0-r_c\} = \{-x_c \quad 0 \quad -(e+z_c)\}^T \tag{8.11a}$$

$$\{h_1+r_1+u_1-r_c\} = \{h_x+x_1 \quad v_1+x_c \quad h_z+w_1-z_c\}^T \tag{8.11b}$$

$$\{h_2+r_2+u_2-r_c\} = \{h_x+x_2 \quad v_2-x_c \quad h_z+w_2-z_c\}^T \tag{8.11c}$$

from which the velocity components with respect to the same axes are obtained in the form

$$\{\dot{u}_0-\dot{r}_c\} = \{-\dot{x}_c \quad 0 \quad -\dot{z}_c\}^T \tag{8.12a}$$

$$\{\dot{u}_1-\dot{r}_c\} = \{0 \quad \dot{v}_1+\dot{x}_c \quad \dot{w}_1+\dot{z}_c\}^T \tag{8.12b}$$

$$\{\dot{u}_2-\dot{r}_c\} = \{0 \quad \dot{v}_2-\dot{x}_c \quad \dot{w}_2-\dot{z}_c\}^T \tag{8.12c}$$

The moment of momentum with respect to point c for i-th member is

$$\underset{\sim}{P}_i = \int_{M_i} (\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i - \underset{\sim}{r}_c) \times (\dot{\underset{\sim}{u}}_i - \dot{\underset{\sim}{r}}_c)\, dM_i \qquad i = 0,\ 1,\ 2 \tag{8.13}$$

Substituting the components of the vectors, (8.11) and (8.12), into (8.13) and noting that $\int_{M_0}\underset{\sim}{r}_0 dM_0=0$, we have

$$P_{0\xi} = P_{0\zeta} = 0$$

$$P_{0\eta} = M_0\left(-x_c\dot{z}_c + (e+z_c)\dot{x}_c\right) \tag{8.14a}$$

$$P_{1\xi} = \int_0^{\ell}\rho\left((v_1+x_c)(\dot{w}_1-\dot{z}_c)-(h_z+w_1-z_c)(\dot{v}_1+\dot{x}_c)\right)dx_1 + m\left((v_1(\ell)+x_c)(\dot{w}_1(\ell)-\dot{z}_c)-(h_z+w_1(\ell)-z_c)(\dot{v}_1(\ell)+\dot{x}_c)\right)$$

$$P_{1\eta} = -\int_0^{\ell} (h_x+x_1)(\dot{w}_1-\dot{z}_c)dx_1 - m(h_x+\ell)(\dot{w}_1(\ell)-\dot{z}_c) \tag{8.14b}$$

$$P_{1\zeta} = \int_0^{\ell} (h_x+x_1)(\dot{v}_1+\dot{x}_c)dx_1 + m(h_x+\ell)(\dot{v}_1(\ell)+\dot{x}_c)$$

$$P_{2\xi} = \int_0^{\ell}\rho\left((v_2-x_c)(\dot{w}_2-\dot{z}_c)-(h_z+w_2-z_c)(\dot{v}_2-\dot{x}_c)\right)dx_2 + m\left((v_2(\ell)-x_c)(\dot{w}_2(\ell)-\dot{z}_c)-(h_z+w_2(\ell)-z_c)(\dot{v}_2(\ell)-\dot{x}_c)\right)$$

$$P_{2\eta} = -\int_0^{\ell} \rho(h_x+x_2)(\dot{w}_2-\dot{z}_c)dx_2 - m(h_x+\ell)(\dot{w}_2(\ell)-\dot{z}_c) \qquad (8.14c)$$

$$P_{2\zeta} = \int_0^{\ell} \rho(h_x+x_2)(\dot{v}_2-\dot{x}_c)dx_2 + m(h_x+\ell)(\dot{v}_2(\ell)-\dot{x}_c)$$

Note that the components of the moment of momentum vector $\underset{\sim}{P}_i$ was expressed as $\{P_i\}=\{P_{i\xi}\ \ P_{i\eta}\ \ P_{i\zeta}\}^T$. Substituting (8.5) and (8.14) into (2.10), we obtain the components of the vector $\{P\}$.

$$P_{\xi} = \int_0^{\ell} \rho(h_x+x_1)\dot{w}_1dx_1 + m(h_x+\ell)\dot{w}_1(\ell) - \int_0^{\ell} \rho(h_x+x_2)\dot{w}_2dx_2 - m(h_x+\ell)\dot{w}_2(\ell)$$

$$\begin{aligned} P_{\eta} = {} & \int_0^{\ell} \rho(v_1+x_c)(\dot{w}_1-\dot{z}_c)dx_1 - \int_0^{\ell} \rho(h_z+w_1-z_c)(\dot{v}_1+\dot{x}_c)dx_1 + m(v_1(\ell)+x_c)(\\ & \dot{w}_1(\ell)-\dot{z}_c) - m(h_z+w_1(\ell)-z_c)(\dot{v}_1(\ell)+\dot{x}_c) - \int_0^{\ell} \rho(v_2-x_c)(\dot{w}_2-\dot{z}_c)dx_2 \\ & + \int_0^{\ell} \rho(h_z+w_2-z_c)(\dot{v}_2-\dot{x}_c)dx_2 - m(v_2(\ell)-x_c)(\dot{w}_2(\ell)-\dot{z}_c) + m(h_z+w_2(\ell) \\ & -z_c)(\dot{v}_2(\ell)-\dot{x}_c) + M_0(-x_c\dot{z}_c+(e+z_c)\dot{x}_c) \end{aligned}$$

$$P_{\zeta} = \int_0^{\ell} \rho(h_x+x_1)\dot{v}_1dx_1 + m(h_x+\ell)\dot{v}_1(\ell) + \int_0^{\ell} \rho(h_x+x_2)\dot{v}_2dx_2 + m(h_x+\ell)\dot{v}_2(\ell)$$

(8.15)

The first order terms $(P_{\xi})_1$, $(P_{\eta})_1$, $(P_{\zeta})_1$, and the second order term $(P_{\zeta})_2$ in (8.15) are

$$(P_{\xi})_1 = \int_0^{\ell} \rho(h_x+x_1)\dot{w}_1dx_1+m(h_x+\ell)\dot{w}_1(\ell) - \int_0^{\ell} \rho(h_x+x_2)\dot{w}_2dx_2-m(h_x+\ell)\dot{w}_2(\ell)$$

$$(P_\eta)_1 = -\frac{M_0}{M}(h_z+e)\left(\int_0^\ell \rho\dot{v}_1 dx_1 + m\dot{v}_1(\ell) - \int_0^\ell \rho\dot{v}_2 dx_2 - m\dot{v}_2(\ell)\right) \tag{8.16a}$$

$$(P_\zeta)_1 = \int_0^\ell \rho(h_x+x_1)\dot{v}_1 dx_1 + m(h_x+\ell)\dot{v}_1(\ell) + \int_0^\ell \rho(h_x+x_2)\dot{v}_2 dx_2 + m(h_x+\ell)\dot{v}_2(\ell)$$

$$(P_\zeta)_2 = 0 \tag{8.16b}$$

Finally, using (8.12), we can show that the last term in (2.11) has the form

$$\sum_{i=1}^{2} \int_{M_i} \{\dot{u}_i\}^T\{\dot{u}_i\}dM_i - M\{\dot{r}_c\}^T\{\dot{r}_c\}$$

$$= \int_0^\ell \rho \begin{Bmatrix} 0 \\ \dot{v}_1 \\ \dot{w}_1 \end{Bmatrix}^T \begin{Bmatrix} 0 \\ \dot{v}_1 \\ \dot{w}_1 \end{Bmatrix} dx_1 + m \begin{Bmatrix} 0 \\ \dot{v}_1(\ell) \\ \dot{w}_1(\ell) \end{Bmatrix}^T \begin{Bmatrix} 0 \\ \dot{v}_1(\ell) \\ \dot{w}_1(\ell) \end{Bmatrix} + \int_0^\ell \rho \begin{Bmatrix} 0 \\ \dot{v}_2 \\ \dot{w}_2 \end{Bmatrix}^T \begin{Bmatrix} 0 \\ \dot{v}_2 \\ \dot{w}_2 \end{Bmatrix} dx_2$$

$$+ m \begin{Bmatrix} 0 \\ \dot{v}_2(\ell) \\ \dot{w}_2(\ell) \end{Bmatrix}^T \begin{Bmatrix} 0 \\ \dot{v}_2(\ell) \\ \dot{w}_2(\ell) \end{Bmatrix} + M \begin{Bmatrix} 0 \\ \dot{x}_c \\ \dot{z}_c \end{Bmatrix}^T \begin{Bmatrix} 0 \\ \dot{x}_c \\ \dot{z}_c \end{Bmatrix}$$

$$= \int_0^\ell \rho(\dot{v}_1^2+\dot{w}_1^2)dx_1 + m(\dot{v}_1^2(\ell)+\dot{w}_1^2(\ell)) + \int_0^\ell \rho(\dot{v}_2^2+\dot{w}_2^2)dx_2 + m(\dot{v}_2^2(\ell)+\dot{w}_2^2(\ell))$$

$$- \frac{1}{M}\left(\left(- \int_0^\ell \rho\dot{v}_1 dx_1 - m\dot{v}_1(\ell) + \int_0^\ell \rho\dot{v}_2 dx_2 + m\dot{v}_2(\ell)\right)^2 + \left(\int_0^\ell \rho\dot{w}_1 dx_1 + m\dot{w}_1(\ell)\right.\right.$$

$$\left.\left. + \int_0^\ell \rho\dot{w}_2 dx_2 + m\dot{w}_2(\ell)\right)^2\right) \tag{8.17}$$

The potential energy of the cables comes from the tension due to the centrifugal force. Refering to Fig. 5, the position vector of a mass element dM_i is $\underset{\sim}{h}_i+\underset{\sim}{r}_i+\underset{\sim}{u}_i$ which can be written in term of components in axes $x_iy_iz_i$.

$$\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i = (h_x+x_i)\underset{\sim}{i}_i + v_i\underset{\sim}{j}_i + (h_z+w_i)\underset{\sim}{k}_i \qquad i = 1,2 \qquad (8.18)$$

Moreover, the angular velocity is

$$\underset{\sim}{\Omega} = \Omega\underset{\sim}{k}_i \qquad (8.19)$$

so that the centripetal accleration $\underset{\sim}{a}_i$ has the expression

$$\underset{\sim}{a}_i = \underset{\sim}{\Omega} \times \left(\underset{\sim}{\Omega} \times (\underset{\sim}{h}_i + \underset{\sim}{r}_i + \underset{\sim}{u}_i)\right) = \Omega^2\left(-(h_x+x_i)\underset{\sim}{i}_i - v_i\underset{\sim}{j}_i\right) \quad i=1,2 \quad (8.20)$$

The tension along the cables, T_{xi}, is the integral of the inertia force

$$\begin{aligned} T_{xi} &= -\int_{x_i}^{\ell} \rho\Omega^2\left(-(h_x+x_i)\right)dx_i + m\Omega^2(h_x+\ell) \\ &= \Omega^2\left(\frac{1}{2}\rho\left((h_x+\ell)^2 - (h_x+x_i)^2\right) + m(h_x+\ell)\right) \qquad i=1,2 \qquad (8.21) \end{aligned}$$

which is a function of the spatial coordinate x_i. Introducing (8.21) into

$$V_{EL} = \frac{1}{2}\sum_{i=1}^{2}\int_0^{\ell} T_{xi}\left(\left(\frac{\partial v_i}{\partial x_i}\right)^2 + \left(\frac{\partial w_i}{\partial x_i}\right)^2\right)dx_i \qquad (8.22a)$$

we obtain the potential energy

$$V_{EL} = \frac{1}{2}\Omega^2\sum_{i=1}^{2}\int_0^{\ell}\left(\frac{1}{2}\rho\left((h_x+\ell)^2-(h_x+x_i)^2\right)+m(h_x+\ell)\right)\left(\left(\frac{\partial v_i}{\partial x_i}\right)^2+\left(\frac{\partial w_i}{\partial x_i}\right)^2\right)dx_i \quad (8.22b)$$

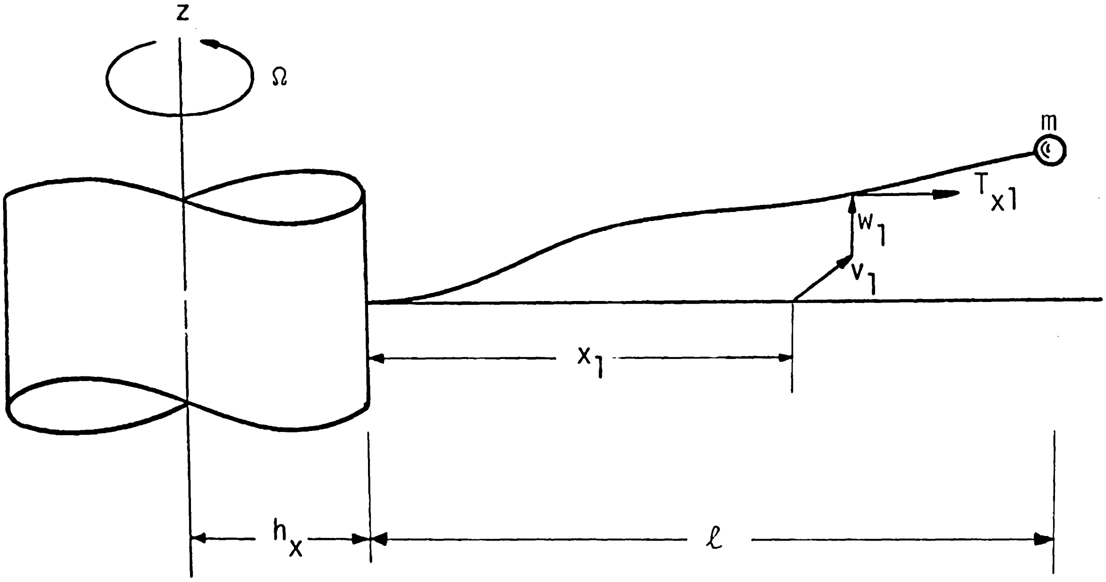

FIGURE 5. ROTATING CABLE

8.2 The Eigenvalue Problem for the Rotating Cable with Tip Mass

To discretize the spatial coordinates v_i, w_i $(i=1,2)$, we need the admissible functions for the vibration of the rotating cables. The eigenvalue problem for the rotating cable with tip mass is defined by the differential equation (see Fig. 5).

$$-\Omega^2\frac{d}{dx_i}\left(\left\{\frac{1}{2}\rho[(h_x+\ell)^2-(h_x+x_i)^2]+m(h_x+\ell)\right\}\frac{d\phi}{dx_i}\right)=\Lambda^2\rho\phi \qquad 0 > x_i > \ell \qquad i = 1,2 \tag{8.23a}$$

and the boundary conditions

$$\phi = 0 \qquad \text{at } x_i = 0 \tag{8.23b}$$

$$-m(h_x+\ell)\Omega^2\frac{d\phi}{dx_i} + m\Lambda^2\phi = 0 \qquad \text{at } x_i = \ell \tag{8.23c}$$

where ϕ and Λ are the eigenfunctions and eigenvalues respectively. The problem (8.23) has no closed-form solution. We observe, however, that the eigenvalue problem of the rotating string with $h_x=0$ and without tip mass is satisfied by the Legendre functions. Hence, the Legendre functions of odd degree can be used as admissible functions in conjunction with a Rayleigh-Ritz solution of the eigenvalue problem (8.23). Note that admissible functions need not satisfy the dynamical boundary condition of the problem. The admissible functions are

$$\begin{aligned}
\phi_1(x_i) &= \frac{x_i}{\ell} \\
\phi_2(x_i) &= \frac{1}{2}\left[5\left(\frac{x_i}{\ell}\right)^3 - 3\left(\frac{x_i}{\ell}\right)\right] \qquad i = 1,2 \\
\phi_3(x_i) &= \frac{1}{8}\left[63\left(\frac{x_i}{\ell}\right)^5 - 70\left(\frac{x_i}{\ell}\right)^3 + 15\left(\frac{x_i}{\ell}\right)\right]
\end{aligned} \tag{8.24}$$

They possess the orthogonality property

$$\int_0^{\ell} \phi_j(x_i)\phi_k(x_i)dx_i = 0 \quad , \qquad j, k = 1, 2, 3, \ldots \qquad (8.25a)$$

and they satisfy the relation

$$\int_0^{\ell} \phi_j^2(x_i)dx_i = \frac{\ell}{2j+1} \qquad j = 1, 2, 3, \ldots \qquad (8.25b)$$

8.3 Equations of Motion and Stability Criteria

The number of degrees of freedom of the whole structure depends upon the series truncation of (3.1). The equations of motion

$$[m]\{\ddot{q}\} + [g]\{\dot{q}\} + [k]\{q\} = \{0\} \qquad (8.26)$$

represent a gyroscopic system. We shall discuss three cases, namely, the rigid-body case (when $v_i=w_i=o$, $i=1,2$), one admissible function approximation for v_i and w_i, and two admissible functions approximation.

(a) $v_i = w_i = 0 \quad , \quad i = 1, 2$

If we consider the cables as rigid, then the equations of motion reduce to

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \Omega\begin{bmatrix} 0 & C-A-B \\ -(C-A-B) & 0 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \Omega^2\begin{bmatrix} C-B & 0 \\ 0 & C-A \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(8.27)

which yields the eigenvalues squared. For convenience we normalize the eigenvalues by dividing them with the spin rate Ω.

$$\begin{aligned} (\lambda_1 / \Omega)^2 &= -1 \\ (\lambda_2 / \Omega)^2 &= -(\tfrac{C}{A} - 1)(\tfrac{C}{B} - 1) \end{aligned} \tag{8.28}$$

The necessary conditions of a stable equilibrium of a rigid body, (5.16), imply that the eigenvalues are complex conjugate pure imaginary.

$$\begin{aligned} \lambda_1 / \Omega &= \pm i \\ \lambda_2 / \Omega &= \pm \left[(\tfrac{C}{A} - 1)(\tfrac{C}{B} - 1)\right]^{1/2} i \end{aligned} \tag{8.29}$$

(b) One admissible function approximation

Letting $r_i=0$, $s_i=t_i=1$ $(i=1, 2)$ in (3.1) we obtain

$$\begin{aligned} v_i &= \phi_{i1}(x_i)q_{i1}(t) \\ w_i &= \psi_{i1}(x_i)q_{i2}(t) \end{aligned} \qquad i = 1, 2 \tag{8.30}$$

The elastic displacement vector $\{u_i\}$ and the generalized coordinate vector $\{q_i\}$ are related by (3.2) where

$$\{u_i\} = \begin{Bmatrix} v_i \\ w_i \end{Bmatrix} \;, \quad \{q_i\} = \begin{Bmatrix} q_{i1} \\ q_{i2} \end{Bmatrix} \;, \quad [N_i] = \begin{bmatrix} \phi_{i1} & 0 \\ 0 & \psi_{i1} \end{bmatrix}$$

$$i = 1, 2 \tag{8.31}$$

The vectors $\{\bar{q}\}$, $\{q\}$ defined by (3.9) and (3.10) are

$$\{\bar{q}\} = \{\{q_1\}^T \ \{q_2\}^T\}^T = \{q_{11} \ q_{12} \ q_{21} \ q_{22}\}^T \tag{8.32a}$$

$$\{q\} = \{q_1 \ q_2 \ \{q\}^T\}^T = \{q_1 \ q_2 \ q_{11} \ q_{12} \ q_{21} \ q_{22}\}^T$$

Substituting (8.30) into (8.10) and (8.16) and referring to Appendix 1 any time integration of admissible functions is required, we have the following vectors in term of parameters defined in Appendix 1.

$$\{H_\xi\} = h_z\{\alpha_1 \quad 0 \quad -\alpha_1 \quad 0\}^T$$

$$\{H_\eta\} = \{0 \quad -\gamma_1 \quad 0 \quad \gamma_1\}^T \tag{8.33a}$$

$$\{H_\zeta\} = \{0 \quad 0 \quad 0 \quad 0\}^T$$

$$[H_{\zeta\zeta}] = \begin{bmatrix} \beta_1-\alpha_1^2/M & 0 & \alpha_1^2/M & 0 \\ & 0 & 0 & 0 \\ & & \beta_1-\alpha_1^2/M & 0 \\ \text{symmetric} & & & 0 \end{bmatrix} \tag{8.33b}$$

$$\{F_\xi\} = \{0 \quad \gamma_1 \quad 0 \quad -\gamma_1\}^T$$

$$\{F_\eta\} = -\frac{M_0}{M}(h_z+e)\{\alpha_1 \quad 0 \quad -\alpha_1 \quad 0\}^T \tag{8.34a}$$

$$\{F_\zeta\} = \{\gamma_1 \quad 0 \quad \gamma_1 \quad 0\}^T$$

$$[F_{\zeta\zeta}] = [0] \tag{8.34b}$$

$$[m_s] = \begin{bmatrix} \beta_1 & & & \\ & \beta_1 & & \\ & & \beta_1 & \\ & & & \beta_1 \end{bmatrix} \qquad (8.35a)$$

$$[m_c] = \begin{bmatrix} \alpha_1^2 & 0 & \alpha_1^2 & 0 \\ & \alpha_1^2 & 0 & \alpha_1^2 \\ & & \alpha_1^2 & 0 \\ \text{symmetric} & & & \alpha_1^2 \end{bmatrix} \qquad (8.35b)$$

Similarly, introducing (8.30) into (8.22) and referring to Appendix 1, we obtain

$$[k_s] = \begin{bmatrix} [k_1] & \\ & [k_2] \end{bmatrix} = \Omega^2 \begin{bmatrix} Q_1 & & & \\ & Q_1 & & \\ & & Q_1 & \\ & & & Q_1 \end{bmatrix} \qquad (8.36)$$

The matrices [m], [g], [k] are obtained from (3.26), (4.5), and (8.33) - (8.36) in the form

$$[m] = \begin{bmatrix} A & 0 & 0 & \gamma_1 & 0 & -\gamma_1 \\ & B & -\hat{M}\bar{h}_z\alpha_1 & 0 & \hat{M}\bar{h}_z\alpha_1 & 0 \\ & & \beta_1-\frac{\gamma_1^2}{C}-\bar{M}\alpha_1^2 & 0 & -\frac{\gamma_1^2}{C}+\bar{M}\alpha_1^2 & 0 \\ & & & \beta_1-\bar{M}\alpha_1^2 & 0 & -\bar{M}\alpha_1^2 \\ & \text{symmetric} & & & \beta_1-\frac{\gamma_1^2}{C}-\bar{M}\alpha_1^2 & 0 \\ & & & & & \beta_1-\bar{M}\alpha_1^2 \end{bmatrix}$$

$$[g]= \Omega\begin{bmatrix} 0 & C-A-B & (h_z+\hat{M}\bar{h}_z)\alpha_1 & 0 & -(h_z+\hat{M}\bar{h}_z)\alpha_1 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & \text{skew-symmetric} & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \tag{8.37}$$

$$[k]=\Omega^2\begin{bmatrix} C-B & 0 & 0 & \gamma_1 & 0 & -\gamma_1 \\ & C-A & h_z\alpha_1 & 0 & -h_z\alpha_1 & 0 \\ & & Q_1-\beta_1+\bar{M}\alpha_1^2 & 0 & -\bar{M}\alpha_1^2 & 0 \\ & & & Q_1 & 0 & 0 \\ & \text{symmetric} & & & Q_1-\beta_1+\bar{M}\alpha_1^2 & 0 \\ & & & & & Q_1 \end{bmatrix}$$

in which $\hat{M}=M_0/M$, $\bar{M}=1/M$, $\bar{h}_z=h_z+e$.

It was pointed out in Sec. 5 that if both [m] and [k] are positive definite then the eigenvalues of (8.26) are complex conjugate pure imaginary. It is not difficult to show that [m] is always positive definite. We proposed to check this property of matrix [k] by means of Sylvester's theorem which, in this case, yields two inequalities

$$Q_1\left((C - B)Q_1 - 2\gamma_1^2\right) > 0 \tag{8.38a}$$

$$(Q_1 - \beta_1)\left((C - A)(Q_1 - \beta_1 + 2\,\frac{\alpha_1^2}{M}) - 2h_z^2\alpha_1^2\right) > 0 \tag{8.38b}$$

Because $Q_1>0$, $Q_1-\beta_1>0$, if we use the system parameters represented by α_1, β_1, γ_1 listed in Appendix 1, then (8.38) can be written as

$$C - B > 2\ell\left(\rho\ell(\tfrac{1}{2}h_x + \tfrac{1}{3}\ell) + m(h_x + \ell)\right) \tag{8.39}$$

$$C - A > 2h_z^2\,\frac{(\frac{1}{2}\rho\ell + m)}{\frac{h_x}{\ell} + \frac{2}{M}(\frac{1}{2}\rho\ell + m)} \tag{8.40}$$

Inequalities (8.39) and (8.40) represent the conditions which must be satisfied by the various structural parameters and the spacecraft geometry for the equilibrium to be stable.

Letting $\ell=0$ and $m=0$ in (8.39) and (8.40), we have

$$C_0 - B_0 > 0 \tag{8.41a}$$

$$C_0 - A_0 > 0 \tag{8.41b}$$

which are the well-known conditions for stable rotation of a rigid body about the axis of maximum moment of inertia.

(c) Two or more admissible functions approximation

Letting $r_i=0$, $s_i=t_i \geqq 2$ $(i=1,2)$ we have two or more admissible functions approximations. The derivation of the equations of motion follows the pattern of Part (b) and will not be repeated here. The matrices [m], [g], and [k] for two admissible functions approximation are listed in Appendix 2, for reference.

8.4 The Motion of the Mass Center

First let us assume that the motion of the center of mass is zero, $x_c=y_c=z_c=0$, so that the term α_1^2/M disappears from all the elements of the matrix [k]. The inequality (8.40) becomes

$$(C - A)\frac{h_x}{\ell} > 2h_z^2(\tfrac{1}{2}\rho\ell + m) \tag{8.42}$$

If $h_x \neq 0$, inequality (8.42) can be rewritten as

$$C - A > 2\frac{\ell}{h_x} h_z^2(\tfrac{1}{2}\rho\ell + m) \tag{8.43}$$

On the other hand, inequality (8.40), which contains the motion of the mass center effect, can be written as

$$C - A > 2\,\frac{\ell}{h_x}\,h_z^2\,(\tfrac{1}{2}\rho\ell + m)\left[\frac{1}{1 + \frac{\ell}{h_x}\,\frac{2}{M}\,(\frac{1}{2}\rho\ell + m)}\right] \quad , \quad h_x \neq 0 \tag{8.44}$$

Clearly, inequality (8.44) implies inequality (8.43) because the last factor in inequality (8.44) is smaller than one. Hence, it is immaterial whether movement of the mass center is neglected as far as the stability criteria are concerned. The stability criteria obtained by neglecting such shifting of the mass center are more conservative than those obtained by keeping it in the analysis. This conclusion confirms the results obtained in Ref. 40 and Ref. 41.

However, for $h_x=0$ and (8.38) reduces

$$C - A > Mh_z \tag{8.45}$$

If this condition is satisfied by the system parameters, the matrix [k] is still positive definite. But inequality (8.42), which is obtained by neglecting the motion of mass center, can never be satisfied such that matrix [k] is not positive definite. Hence, this is the circumstance that motion of the mass center cannot be neglected.

8.5 Symmetric and Antisymmetric Motion

The GEOS satellite is a symmetric structure, satisfying (7.1) and (7.2), so that it is possible to assume symmetric and antisymmetric motions to reduce the order of the problem.

(a) Symmetric motion

By (7.4) we have the equations

$$v_1(x_1,t) = - v_2(x_2,t) \tag{8.46a}$$

$$w_1(x_1,t) = w_2(x_2,t) \tag{8.46b}$$

Introducing (8.46) into (8.10) and (8.16), eliminating the generalized displacements v_2 and w_2, and discretizing v_1 and w_1 by (8.30), we obtain matrices [m], [g], [k]. All these operations can be performed by means of a coordinate transformation. From (8.46) and (8.30), we have the relations for the generalized coordinates

$$q_{11} = -q_{21} \qquad , \qquad q_{12} = q_{22} \tag{8.47}$$

so that the generalized coordinate vector is transformed from $\{q_s\}$ to $\{q\}$ by

$$\{q\} = [T_s]\{q_s\} \tag{8.48}$$

where

$$\{q\} = \{q_1 \;\; q_2 \;\; q_{11} \;\; q_{12} \;\; q_{21} \;\; q_{22}\}^T \tag{8.49a}$$

$$\{q_s\} = \{q_1 \;\; q_2 \;\; q_{11} \;\; q_{12}\}^T \tag{8.49b}$$

$$[T_s] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8.49c}$$

Using transformation (8.48), we have new forms of expressions (3.24).

$$R_2 = \frac{1}{2}\{\dot{q}_s\}^T[m']\{\dot{q}_s\}$$

$$R_1 = \{q_s\}^T[f']\{\dot{q}_s\} \tag{8.50}$$

$$V_{EL} - R_0 = \frac{1}{2}\{q_s\}^T[k']\{q_s\}$$

where

$$[m'] = [T_s]^T[m][T_s] \quad , \quad [f'] = [T_s]^T[f][T_s] \quad , \quad [k'] = [T_s]^T[k][T_s] \qquad (8.51)$$

In addition, the matrix $[g']$ defined by $[g']=[f']^T-[f']$ has the transformation

$$[g'] = [T_s]^T[g][T_s] \qquad (8.52)$$

Expressions (8.51), (8.52) together with (8.49c) yield the matrices

$$[m'] = \begin{bmatrix} A & 0 & 0 & 0 \\ & B & -2\hat{M}\bar{h}_z\alpha_1 & 0\ 0 \\ & & 2(\beta_1-2\bar{M}\alpha_1^2) & 0 \\ \text{symmetric} & & & 2(\beta_1-2\bar{M}\alpha_1^2) \end{bmatrix}$$

$$[g'] = \Omega\begin{bmatrix} 0 & C-A-B & 2(h_z+\hat{M}\bar{h}_z)\alpha_1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ \text{antisymmetric} & & & 0 \end{bmatrix} \qquad (8.53)$$

$$[k'] = \Omega^2\begin{bmatrix} C-B & 0 & 0 & 0 \\ & C-A & 2h_z\alpha_1 & 0 \\ & & 2(Q_1-\beta_1+2\bar{M}\alpha_1^2) & 0 \\ \text{symmetric} & & & 2Q_1 \end{bmatrix}$$

Matrix $[k]$ is positive definite if and only if

$$C - B > 0 \tag{8.54a}$$

$$C - A > \frac{2h_z^2(\frac{1}{2}\rho\ell + m)}{\frac{h_x}{\ell} + \frac{2}{M}(\frac{1}{2}\rho\ell + m)} \tag{8.54b}$$

Note that (8.54b) was obtained in sec. 8.3, but (8.39) cannot be obtained from [k'].

(b) Antisymmetric motion

The constraint equations now are

$$v_1(x_1,t) = v_2(x_2,t) \tag{8.55a}$$

$$w_1(x_1,t) = -\ w_2(x_2,t) \tag{8.55b}$$

so that we can write the transformation

$$\{q\} = [T_a]\{q_a\} \tag{8.56}$$

where

$$\{q_a\} = \{q_s\} \tag{8.57a}$$

$$[T_a] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{8.57b}$$

The matrices [m"], [g"], and [k"] are derived in the same way as [m'], [g'], [k'] and have the form

$$[m''] = \begin{bmatrix} A & 0 & 0 & 2\gamma_1 \\ & B & 0 & 0 \\ & & 2(\beta_1 - 2\frac{\gamma_1^2}{C}) & 0 \\ \text{symmetric} & & & 2\beta_1 \end{bmatrix}$$

$$[g''] = \Omega \begin{bmatrix} 0 & C-A-B & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ \text{antisymmetric} & & & 0 \end{bmatrix} \qquad (8.58)$$

$$[k''] = \Omega^2 \begin{bmatrix} C-B & 0 & 0 & 2\gamma_1 \\ & C-A & 0 & 0 \\ & & 2(Q_1 - \beta_1) & 0 \\ \text{symmetric} & & & 2Q_1 \end{bmatrix}$$

Positiveness of [k"] requires that

$$C - A > 0 \qquad (8.59a)$$

$$C - B > \ell\left(\rho\ell(\tfrac{1}{2}h_x + \tfrac{1}{3}\ell) + m(h_x + \ell)\right) \qquad (8.59b)$$

Of these two criteria, only the second one is a sufficient condition.

Approximations using more admissible functions can be obtained using exactly the same procedure. Matrices corresponding to two- and three-admissible functions approximation are listed in Appendix 3.

8.6 Data and Results

$A_0 = 87.7\ \mathrm{kgm}^2$, $B_0 = 138.9\ \mathrm{kgm}^2$, $C_0 = 137.0\ \mathrm{kgm}^2$

$\rho = 0.5\ \mathrm{kg/m}$, $M_0 = 100.0\ \mathrm{kg}$, $m = 0.1\ \mathrm{kg}$

$h_x = 0.73\ \mathrm{m}$, $h_y = 0$, $h_z = 0$, $\ell = 20\ \mathrm{m}$

$\Omega = 1.04719755\ \mathrm{rad/sec}$

Using the above data, we see that inequalities (8.39) and (8.40) are satisfied so that the equilibrium is stable. The natural frequencies are tabulated in the following in nondimensional form ω_r/Ω.

(a) General Motion.

Rigid	One Admiss.	Two Admiss.
0.58721	0.45417	0.45264
1.00000	0.49731	0.49730
	1.00000	1.00000
	1.09830	1.09767
	1.11312	1.10827
	1.62512	1.61773
		2.31917
		2.33175
		2.53693
		2.53713

(b) Symmetric Motion.

One Admiss.	Two Admiss.	Three Admiss.
0.45417	0.45264	0.45263
0.58721	0.58721	0.58721
1.00000	1.00000	1.00000
1.09830	1.09767	1.09734
	2.33175	2.32626
	2.53713	2.53136
		3.86453
		3.99184

(c) Antisymmetric Motion.

One Admiss.	Two Admiss.	Three Admiss.
0.49730	0.49730	0.49730
1.00000	1.00000	1.00000
1.11312	1.10827	1.10773
1.70183	1.61773	1.61698
	2.31917	2.31890
	2.53693	2.53650
		3.85515
		3.98753

The effect of the flexibility of cables on the natural frequencies can be seen by comparing the fundamental frequency of the rigid body model with that of one admissible function approximation. It drops noticeably from 0.58721 to 0.45417. More admissible functions approximations merely increase the dimensions of the matrices and the amount of work but do not improve the results much.

9. Summary and Conclusions

The equations of motion of a spin-stabilized flexible body were formulated, based on the small motion assumption and the assumed-modes discretization procedure. The equations were obtained by the Lagrangian approach in conjunction with the Routhian ignoration of coordinates. The linearized equations of motion can be defined in terms of a symmetric mass matrix, a skew-symmetric gyroscopic matrix and a symmetric stiffness matrix. The associated eigenvalue problem can be reduced to the standard form by changing the problem from the configuration space to the state space. When the mass matrix and the stiffness matrix are positive definite, the system possesses complex-conjugate, pure-imaginary eigenvalues and complex-conjugate orthogonal eigenvectors. These properties were utilized to solve the eigenvalue problem by working with real quantities.

A stability analysis of the system was performed by the Liapunov direct method. It was shown that the Hamiltonian, written in terms of the Routhian function, is a suitable Liapunov function for stability analysis. It was concluded that the positive definiteness of the Liapunov function, and hence the system stability, requires that the mass matrix and stiffness matrix both be positive definite.

The elastic motion of structures symmetric with respect to the spin axis can be separated into symmetric and antisymmetric parts. It is easy to show that any possible motion can be regarded as combinations of symmetric and antisymmetric motions (see Ref. 19).

As an application, the simplified model of the European Space Agency's GEOS spacecraft was studied. The stability criteria were obtained in terms of structure parameters. It was found that the motion of the mass center is immaterial as far as stability is concerned except for the case in which the flexible part is attached to the spin axis. The natural frequencies furnished the need of quantitative analysis of the stability characteristics as cited in Ref. 40.

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11. Appendices

Appendix 1: Admissible Functions and Their Integrals

(a) Admissible Functions

$$\phi_1(x) = c_1\left(\frac{x}{\ell}\right)$$

$$\phi_2(x) = \frac{1}{2}\,c_2\left[5\left(\frac{x}{\ell}\right)^3 - 3\left(\frac{x}{\ell}\right)\right]$$

$$\phi_3(x) = \frac{1}{8}\,c_3\left[63\left(\frac{x}{\ell}\right)^5 - 70\left(\frac{x}{\ell}\right)^3 + 15\left(\frac{x}{\ell}\right)\right]$$

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where c_j $(j = 1,2,3,\ldots)$ are some constants. They are chosen such that $\int_0^\ell \rho\phi_j^2(x)dx=1$, $(j = 1,2,3,\ldots)$.

(b) Integrals

$$\alpha_j = c_j\left[\int_0^\ell \rho\phi_j(x)dx + m\phi_j(\ell)\right]$$

$$\beta_j = c_j^2\left[\int_0^\ell \rho\phi_j^2(x)dx + m\phi_j^2(\ell)\right]$$

$$\gamma_j = c_j\left[\int_0^\ell \rho(h_x + x)\phi_j(x)dx + m(h_x + \ell)\phi_j(\ell)\right]$$

$$Q_j = c_j^2\int_0^\ell T_x(x)\phi_j'^2(x)dx$$

$$R_{jk} = c_j c_k\int_0^\ell T_x(x)\phi_j'(x)\phi_k'(x)dx$$

$$j,\ k = 1,\ 2,\ 3,\ \ldots$$

where primes represent derivatives with respect to x and T_x has the expression

$$T_x(x) = \Omega^2\left(\frac{1}{2}\rho[(h_x + \ell)^2 - (h_x + x)^2] + m(h_x + \ell)\right)$$

Several integral values used in sec. 8 are list below.

$$\alpha_1 = c_1(\frac{1}{2}\rho\ell + m) \quad , \quad \beta_1 = c_1^2(\frac{1}{3}\rho\ell + m)$$

$$\alpha_2 = c_2(\frac{1}{8}\rho\ell - m) \quad , \quad \beta_2 = c_2^2(\frac{1}{7}\rho\ell + m)$$

$$\alpha_3 = c_3(\frac{1}{16}\rho\ell + m) \quad , \quad \beta_3 = c_3^2(\frac{1}{11}\rho\ell + m)$$

$$\gamma_1 = c_1[\rho\ell(\frac{h_x}{2} + \frac{\ell}{3}) + m(h_x + \ell)]$$

$$\gamma_2 = -c_2[\frac{1}{8}\rho\ell h_x - m(h_x + \ell)]$$

$$\gamma_3 = c_3[\frac{1}{16}\rho\ell h_x + m(h_x + \ell)]$$

$$Q_1 = \frac{c_1^2}{\ell}[\rho\ell(\frac{\ell}{3} + \frac{h_x}{2}) + m(h_x + \ell)]$$

$$Q_2 = \frac{c_2^2}{\ell}[\rho\ell(\frac{6}{7}\ell + \frac{9}{8}h_x) + 6m(h_x + \ell)]$$

$$Q_3 = \frac{c_3^2}{\ell}[\rho\ell(\frac{15}{11}\ell + \frac{225}{128}h_x) + 15m(h_x + \ell)]$$

$$Q_{12} = \frac{c_1c_2}{\ell}[-\frac{1}{8}\rho\ell h_x + m(h_x + \ell)]$$

$$Q_{13} = \frac{c_1c_3}{\ell}[\frac{1}{16}\rho\ell h_x + m(h_x + \ell)]$$

$$Q_{23} = \frac{c_2c_3}{\ell}\left[-\frac{27}{128}\rho\ell h_x + 6m(h_x + \ell)\right]$$

The constants c_1, c_2, c_3 are

$$c_1 = \left(\frac{3}{\rho\ell}\right)^{1/2} \quad , \quad c_2 = \left(\frac{7}{\rho\ell}\right)^{1/2} \quad , \quad c_3 = \left(\frac{11}{\rho\ell}\right)^{1/2}$$

<u>Appendix 2</u>: Matrices [m], [f], [g], and [k] for Two Admissible Functions Approximation.

Additional notations are introduced for convenience. They are

$$c_{12} = c_1c_2 \quad , \quad \alpha_{12} = \alpha_1\alpha_2 \quad , \quad \gamma_{12} = \gamma_1\gamma_2$$

$$c_{13} = c_1c_3 \quad , \quad \alpha_{13} = \alpha_1\alpha_3 \quad , \quad \gamma_{13} = \gamma_1\gamma_3$$

$$c_{23} = c_2c_3 \quad , \quad \alpha_{23} = \alpha_2\alpha_3 \quad , \quad \gamma_{23} = \gamma_2\gamma_3$$

Note that

$$\hat{M} = M_0/M \quad , \quad \bar{M} = 1/M \quad , \quad \bar{h}_z = h_z+e$$

which have been defined before.

$$[m]=\begin{bmatrix}
A & 0 & 0 & 0 & \gamma_1 & \gamma_2 & 0 & 0 & -\gamma_1 & -\gamma_2 \\
 & B & -\hat{M}\bar{h}_z\alpha_1 & -\hat{M}\bar{h}_z\alpha_2 & 0 & 0 & \hat{M}\bar{h}_z\alpha_1 & \hat{M}\bar{h}_z\alpha_2 & 0 & 0 \\
 & & \beta_1-\frac{\gamma_1^2}{C}-\bar{M}\alpha_1^2 & c_{12}m-\frac{\gamma_{12}}{C}-\bar{M}\alpha_{12} & 0 & 0 & -\frac{\gamma_1^2}{C}+\bar{M}\alpha_1^2 & -\frac{\gamma_{12}}{C}+\bar{M}\alpha_{12} & 0 & 0 \\
 & & & \beta_2-\frac{\gamma_2^2}{C}-\bar{M}\alpha_2^2 & 0 & 0 & -\frac{\gamma_{12}}{C}+\bar{M}\alpha_{12} & -\frac{\gamma_2^2}{C}+\bar{M}\alpha_2^2 & 0 & 0 \\
 & & & & \beta_1-\bar{M}\alpha_1^2 & c_{12}m-\bar{M}\alpha_{12} & 0 & 0 & -\bar{M}\alpha_1^2 & -\bar{M}\alpha_{12} \\
 & & & & & \beta_2-\bar{M}\alpha_2^2 & 0 & 0 & -\bar{M}\alpha_{12} & -\bar{M}\alpha_2^2 \\
 & & & & & & \beta_1-\frac{\gamma_1^2}{C}-\bar{M}\alpha_1^2 & c_{12}m-\frac{\gamma_{12}}{C}-\bar{M}\alpha_{12} & 0 & 0 \\
 & \text{symmetric} & & & & & & \beta_2-\frac{\gamma_2^2}{C}-\bar{M}\alpha_2^2 & 0 & 0 \\
 & & & & & & & & \beta_1-\bar{M}\alpha_1^2 & c_{12}m-\bar{M}\alpha_{12} \\
 & & & & & & & & & \beta_2-\bar{M}\alpha_2^2
\end{bmatrix}$$

$$[f]=\Omega\begin{bmatrix}
0 & B & -\hat{M}\bar{h}_z\alpha_1 & -\hat{M}\bar{h}_z\alpha_2 & 0 & 0 & \hat{M}\bar{h}_z\alpha_1 & \hat{M}\bar{h}_z\alpha_2 & 0 & 0 \\
C-A & 0 & 0 & 0 & -\gamma_1 & -\gamma_2 & 0 & 0 & \gamma_1 & \gamma_2 \\
h_z\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_z\alpha_2 & 0 & 0 & 0 & & & & & & \\
0 & -\gamma_1 & 0 & & 0 & & & & & \\
0 & -\gamma_2 & 0 & & & 0 & & & & \\
-h_z\alpha_1 & 0 & 0 & & & & 0 & & & \\
-h_z\alpha_2 & 0 & 0 & & & & & 0 & & \\
0 & \gamma_1 & 0 & & & & & & 0 & \\
0 & \gamma_2 & 0 & & & & & & & 0
\end{bmatrix}$$

$$[g]=\Omega\begin{bmatrix} 0 & C-A-B & (h_z+\hat{M}\bar{h}_z)\alpha_1 & (h_z+\hat{M}\bar{h}_z)\alpha_2 & 0 & 0 & -(h_z+\hat{M}\bar{h}_z)\alpha_1 & -(h_z+\hat{M}\bar{h}_z)\alpha_2 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & & & & & \\ & & & 0 & & & & & & \\ & & & & 0 & & & & & \\ & & & & & 0 & & & & \\ & & & & & & 0 & & & \\ & & & & & & & 0 & & \\ & \text{Antisymmetric} & & & & & & & 0 & \\ & & & & & & & & & 0 \end{bmatrix}$$

$$[k]=\Omega^2\begin{bmatrix}
C-B & 0 & 0 & 0 & \gamma_1 & \gamma_2 & 0 & 0 & -\gamma_1 & -\gamma_2 \\
 & C-A & h_z\alpha_1 & h_z\alpha_2 & 0 & 0 & -h_z\alpha_1 & -h_z\alpha_2 & 0 & 0 \\
 & & Q_1-\beta_1+\frac{\alpha_1^2}{M} & R_{12}+\frac{\alpha_{12}}{M}-c_{12}m & 0 & 0 & -\frac{\alpha_1^2}{M} & -\frac{\alpha_{12}}{M} & 0 & 0 \\
 & & & Q_2-\beta_2+\frac{\alpha_2^2}{M} & 0 & 0 & -\frac{\alpha_{12}}{M} & -\frac{\alpha_2^2}{M} & 0 & 0 \\
 & & & & Q_1 & R_{12} & 0 & 0 & 0 & 0 \\
 & & & & & Q_2 & 0 & 0 & 0 & 0 \\
 & & & & & & Q_1-\beta_1+\frac{\alpha_1^2}{M} & R_{12}+\frac{\alpha_{12}}{M}-c_{12}m & 0 & 0 \\
 & & & & & & & Q_2-\beta_2+\frac{\alpha_2^2}{M} & 0 & 0 \\
 & \text{Symmetric} & & & & & & & Q_1 & R_{12} \\
 & & & & & & & & & Q_2
\end{bmatrix}$$

Appendix 3: Matrices [m], [f], [g], and [k] for Symmetric and Antisymmetric Motion.

(a) Symmetric motion with two admissible functions approximation.

$$[m]=\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ & B & -2\hat{M}\bar{h}_z\alpha_1 & -2\hat{M}\bar{h}_z\alpha_2 & 0 & 0 \\ & & 2(\beta_1-2\bar{M}\alpha_1^2) & 2(c_{12}m-2\bar{M}\alpha_{12}) & 0 & 0 \\ & & & 2(\beta_2-2\bar{M}\alpha_2^2) & 0 & 0 \\ & \text{Symmetric} & & & 2(\beta_1-2\bar{M}\alpha_1^2) & 2(c_{12}m-2\bar{M}\alpha_{12}) \\ & & & & & 2(\beta_2-2\bar{M}\alpha_2^2) \end{bmatrix}$$

$$[f]=\Omega\begin{bmatrix} 0 & B & -2\hat{M}\bar{h}_z\alpha_1 & -2\hat{M}\bar{h}_z\alpha_2 & 0 & 0 \\ C-A & 0 & 0 & 0 & 0 & 0 \\ 2h_z\alpha_1 & 0 & 0 & & & \\ 2h_z\alpha_2 & 0 & & 0 & & \\ 0 & 0 & & & 0 & \\ 0 & 0 & & & & 0 \end{bmatrix}$$

$$[g]=\Omega\begin{bmatrix} 0 & C-A-B & 2(h_z+\hat{M}\bar{h}_z)\alpha_1 & 2(h_z+\hat{M}\bar{h}_z)\alpha_2 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & \\ & \text{Antisymmetric} & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

$$[k]=\Omega^2\begin{bmatrix} C-B & 0 & 0 & 0 & 0 & 0 \\ & C-A & 2h_z\alpha_1 & 2h_z\alpha_2 & 0 & 0 \\ & & 2(Q_1-\beta_1+\frac{2\alpha_1^2}{M}) & 2(R_{12}-c_{12}m+\frac{2\alpha_{12}}{M}) & 0 & 0 \\ & & & 2(Q_2-\beta_2+\frac{2\alpha_2^2}{M}) & 0 & 0 \\ & \text{Symmetric} & & & 2Q_1 & 2R_{12} \\ & & & & & 2Q_2 \end{bmatrix}$$

(b) Symmetric motion with three admissible functions approximation.

$$[m]=\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & B & -2\hat{M}\bar{h}_z\alpha_1 & -2\hat{M}\bar{h}_z\alpha_2 & -2\hat{M}\bar{h}_z\alpha_3 & 0 & 0 & 0 \\ & & 2(\beta_1-2\bar{M}\alpha_1^2) & 2(c_{12}m-2\bar{M}\alpha_{12}) & 2(c_{13}m-2\bar{M}\alpha_{13}) & 0 & 0 & 0 \\ & & & 2(\beta_2-2\bar{M}\alpha_2^2) & 2(c_{23}m-2\bar{M}\alpha_{23}) & 0 & 0 & 0 \\ & & & & 2(\beta_3-2\bar{M}\alpha_3^2) & 0 & 0 & 0 \\ & & & & & 2(\beta_1-2\bar{M}\alpha_1^2) & 2(c_{12}m-2\bar{M}\alpha_{12}) & 2(c_{13}m-2\bar{M}\alpha_{13}) \\ & \text{Symmetric} & & & & & 2(\beta_2-2\bar{M}\alpha_2^2) & 2(c_{23}m-2\bar{M}\alpha_{23}) \\ & & & & & & & 2(\beta_3-2\bar{M}\alpha_3^2) \end{bmatrix}$$

$$[f]=\Omega\begin{bmatrix} 0 & B & -2\hat{M}\bar{h}_z\alpha_1 & -2\hat{M}\bar{h}_z\alpha_2 & -2\hat{M}\bar{h}_z\alpha_3 & 0 & 0 & 0 \\ C-A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2h_z\alpha_1 & 0 & 0 & & & & & \\ 2h_z\alpha_2 & 0 & & 0 & & & & \\ 2h_z\alpha_3 & 0 & & & 0 & & & \\ 0 & 0 & & & & 0 & & \\ 0 & 0 & & & & & 0 & \\ 0 & 0 & & & & & & 0 \end{bmatrix}$$

$$[g]=\Omega\begin{bmatrix} 0 & C-A-B & 2(h_z+\hat{M}\bar{h}_z)\alpha_1 & 2(h_z+\hat{M}\bar{h}_z)\alpha_2 & 2(h_z+\hat{M}\bar{h}_z)\alpha_3 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ \text{Antisymmetric} & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix}$$

$$[k]=\Omega^2\begin{bmatrix} C-B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & C-A & 2h_z\alpha_1 & 2h_z\alpha_2 & 2h_z\alpha_3 & 0 & 0 & 0 \\ & & 2(Q_1-\beta_1+\frac{2\alpha_1^2}{M}) & 2(R_{12}-c_{12}m+\frac{2\alpha_{12}}{M}) & 2(R_{13}-c_{13}m+\frac{2\alpha_{13}}{M}) & 0 & 0 & 0 \\ & & & 2(Q_2-\beta_2+\frac{2\alpha_2^2}{M}) & 2(R_{23}-c_{23}m+\frac{2\alpha_{23}}{M}) & 0 & 0 & 0 \\ & & & & 2(Q_3-\beta_3+\frac{2\alpha_3^2}{M}) & 0 & 0 & 0 \\ & & & & & 2Q_1 & 2R_{12} & 2R_{13} \\ & \text{Symmetric} & & & & & 2Q_2 & 2R_{23} \\ & & & & & & & 2Q_3 \end{bmatrix}$$

(c) Antisymmetric motion with two admissible functions approximation.

$$[m]=\begin{bmatrix} A & 0 & 0 & 0 & 2\gamma_1 & 2\gamma_2 \\ & B & 0 & 0 & 0 & 0 \\ & & 2(\beta_1-\frac{2\gamma_1^2}{C}) & 2(c_{12}m-\frac{2\gamma_{12}}{C}) & 0 & 0 \\ & & & 2(\beta_2-\frac{2\gamma_2^2}{C}) & 0 & 0 \\ & \text{Symmetric} & & & 2\beta_1 & 2c_{12}m \\ & & & & & 2\beta_2 \end{bmatrix}$$

$$[f]=\Omega\begin{bmatrix} 0 & B & 0 & 0 & 0 & 0 \\ C-A & 0 & 0 & 0 & -2\gamma_1 & -2\gamma_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & \\ 0 & -2\gamma_1 & 0 & & 0 & \\ 0 & -2\gamma_2 & 0 & & & 0 \end{bmatrix}$$

$$[g]=\Omega\begin{bmatrix} 0 & C-A-B & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & \\ & & & 0 & & \\ & \text{Antisymmetric} & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

$$[k]=\Omega^2\begin{bmatrix} C-B & 0 & 0 & 0 & 2\gamma_1 & 2\gamma_2 \\ & C-A & 0 & 0 & 0 & 0 \\ & & 2(Q_1-\beta_1) & 2(R_{12}-c_{12}m) & 0 & 0 \\ & & & 2(Q_2-\beta_2) & 0 & 0 \\ & \text{Symmetric} & & & 2Q_1 & 2R_{12} \\ & & & & & 2Q_2 \end{bmatrix}$$

(d) Antisymmetric motion with three admissible functions approximation.

$$[m]=\begin{bmatrix} A & 0 & 0 & 0 & 0 & 2\gamma_1 & 2\gamma_2 & 2\gamma_3 \\ & B & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 2(\beta_1-\frac{2\gamma_1^2}{C}) & 2(c_{12}m-\frac{4\gamma_{12}}{C}) & 2(c_{13}m-\frac{4\gamma_{13}}{C}) & 0 & 0 & 0 \\ & & & 2(\beta_2-\frac{4\gamma_2^2}{C}) & 2(c_{23}m-\frac{2\gamma_{23}}{C}) & 0 & 0 & 0 \\ & & & & 2(\beta_3-\frac{2\gamma_3^2}{C}) & 0 & 0 & 0 \\ & \text{Symmetric} & & & & 2\beta_1 & 2c_{12}m & 2c_{13}m \\ & & & & & & 2\beta_2 & 2c_{23}m \\ & & & & & & & 2\beta_3 \end{bmatrix}$$

$$[f]=\Omega\begin{bmatrix} 0 & B & 0 & 0 & 0 & 0 & 0 & 0 \\ C-A & 0 & 0 & 0 & 0 & -2\gamma_1 & -2\gamma_2 & -2\gamma_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & & 0 & & & \\ 0 & -2\gamma_1 & 0 & & & 0 & & \\ 0 & -2\gamma_2 & 0 & & & & 0 & \\ 0 & -2\gamma_3 & 0 & & & & & 0 \end{bmatrix}$$

$$[g]=\Omega\begin{bmatrix} 0 & C-A-B & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & \text{Antisymmetric} & & & & & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix}$$

$$[k]=\Omega^2\begin{bmatrix} C-B & 0 & 0 & 0 & 0 & 2\gamma_1 & 2\gamma_2 & 2\gamma_3 \\ & C-A & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 2(Q_1-\beta_1) & 2(R_{12}-c_{12}m) & 2(R_{13}-c_{13}m) & 0 & 0 & 0 \\ & & & 2(Q_2-\beta_2) & 2(R_{23}-c_{23}m) & 0 & 0 & 0 \\ & & & & 2(Q_3-\beta_3) & 0 & 0 & 0 \\ & & & & & 2Q_1 & 2R_{12} & 2R_{13} \\ & \text{Symmetric} & & & & & 2Q_2 & 2R_{23} \\ & & & & & & & 2Q_3 \end{bmatrix}$$

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DYNAMICS OF SPINNING FLEXIBLE SATELLITES

by

Ching-Pyng Chang

(ABSTRACT)

This investigation is concerned with the dynamic characteristics and stability of a spinning rigid body with a number of flexible parts. The system is hybrid in the sense that it is described by coordinates depending on time alone and coordinates depending on spatial position and time. The space-dependent coordinates are discretized by the assumed-modes method based on the Rayleigh-Ritz approach. The matrix form of the linearized equations of motion, which is of gyroscopic type, can be reduced to a general matrix multiplied by the state vector. The constrained system Hamiltonian is employed as a Liapunov function for stability analysis. It is shown that for stable nontrivial equilibrium the mass matrix and the stiffness matrix must be positive definite. In this case, the general matrix multiplied the state vector becomes skew-symmetric and its eigenvalues are complex conjugate pure imaginary.

The method has been applied to the simplified model of the European Space Agency's GEOS spacecraft to obtain explicit forms of the equations of motion and stability criteria in terms of system parameters. It is found that the motion about the equilibrium is stable but the fundamental frequency is lower than the spin rate. As a by-product, it is shown that neglecting the motion of the mass center is immaterial as far as the stability is concerned, except in the case in which the points of

attachment of the flexible parts are off-set along the spin axis relative to the mass center of the spacecraft. In the later case, the effect of the motion of the mass center must be examined carefully.