KINETIC THEORY AND GLOBAL EXISTENCE IN L^1 FOR A DENSE SQUARE-WELL FLUID

by

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Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of requirements for the degree of
DOCTOR OF PHILOSOPHY
in
Mathematical Physics

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April 1995
Blacksburg, Virginia

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

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(ABSTRACT)

In this paper, we consider the kinetic equation for a dense square-well fluid and the geometric factor $Y \equiv 1$, provide the related kinetic theory, and prove a global existence theorem in L^1 for the kinetic equation under rather general initial value condition. An analogue of the classical H-theorem is verified.

ACKNOWLEDGEMENT

I am greatly indebted to Professor William Greenberg, my advisor, for his invaluable guidance and many stimulating discussions on this work.

I would like to express great thanks to Professor Mingzhu Yang for his encouragement of and confidence in the author.

I also wish to express my deep gratitude to Dr. Rongsheng Liu and Dr. Peng Lei for many useful discussions and suggestions.

Finally, I am indebted to the Department of Mathematics and the Center for Transport Theory and Mathematical Physics for support and guidance during my study at Virginia Polytechnic Institute and State University.

Contents

§1 . Introduction

The main purpose of statistical mechanics is to predict and to explain the properties of macroscopic matter from the properties of its microscopic constitutents. This subject can be divided into two parts: *equilibrium* and *nonequilibrium*. Historically, nonequilibrium statistical mechanics has taken two directions: *kinetic theory of gases* and *theory of Brownian motion.* For more than a century the Boltzmann equation has been a mainstay of kinetic theory in nonequilibrium statistical mechanics, because of its proper description of rarefied gases.

Despite its preeminent role in the history of statistical mechanics, the Boltzmann equation is known to be valid only in the dilute gas regime, indeed yielding transport coefficients of an ideal fluid. In an attempt to generalized the Boltzmann equation to higher densities, Enskog, in 1921, first proposed a kinetic equation for the single-particle distribution function by introducing a Boltzman-like collision process with hard core interaction, describing purely the elastic collision of hard spheres. The theory is known as the so-called standard Enskog theory (SET).

Essentially Enskog's derivation is based upon the following two assumptions:

(i) Only elastic collisions, with conservation of momentum and energy, are taken into account for pairs of particles which are the centers of two colliding spheres a distance a apart, equal to the hard-sphere diameter;

(ii) The collision frequency is modeled by a factor Y, taken as a function of the gas density at the contact point of the two colliding spheres. Y represents the so-called pair correlation function corresponding to the system in uniform equilibrium.

These two modifications lead to the Enskog equation, which has the same structure as the Boltzmann equation for the single-particle distribution function, $f(\vec{r}, \vec{v}, t)$, with an external force $\vec{F}$, in the form

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}} f + \vec{F} \cdot \nabla_{\vec{v}} f = E(f, f),$$
$$E = E^{+} - E^{-}. \tag{1.1a}$$

The right-hand side collision term E may be written as the difference of gain and loss terms, E^{+} and E^{-}, given by

$$E^{+}(\vec{r}, \vec{v}, t) = a^2 \iint\limits_{R^3 \times S_+^2} Y^E(n(\vec{r} + \frac{1}{2}a\vec{\sigma}, t))\theta(\vec{\sigma} \cdot \vec{V}) f(\vec{r}, \vec{v}', t) f(\vec{r} + a\vec{\sigma}, \vec{w}', t)(\vec{\sigma} \cdot \vec{V}) d\vec{w} d\vec{\sigma},$$
$$E^{-}(\vec{r}, \vec{v}, t) = a^2 \iint\limits_{R^3 \times S_+^2} Y^E(n(\vec{r} - \frac{1}{2}a\vec{\sigma}, t))\theta(\vec{\sigma} \cdot \vec{V}) f(\vec{r}, \vec{v}, t) f(\vec{r} - a\vec{\sigma}, \vec{w}, t)(\vec{\sigma} \cdot \vec{V}) d\vec{w} d\vec{\sigma}, \tag{1.1b}$$

where $\vec{r}$ is the position of the particle, $\vec{v}'$ and $\vec{w}'$ are the post-collisional velocities, and $\vec{v}$ and $\vec{w}$ are the pre-collisional velocities. The integration over the solid angle $d\vec{\sigma}$ is restricted to a unitary semi-sphere $S_+^2 = \{\vec{\sigma} \in R^3 \mid |\vec{\sigma}| = 1, \vec{\sigma} \cdot \vec{V} \geq 0\}$, where $\vec{V} = \vec{v} - \vec{w}$. The function Y is the equilibrium value of the pair distribution function, evaluated as a function of the local density at the contact point $\vec{r} \pm \frac{1}{2}a\vec{\sigma}$ of two spheres. The function $\theta(x)$ is the step unit function

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{1.2}$$

As known [10], the pre-collisional velocities are related to the post-collisional velocities by

$$\vec{v}' = \vec{v} - (\vec{\sigma} \cdot \vec{V})\vec{\sigma},$$
$$\vec{w}' = \vec{w} + (\vec{\sigma} \cdot \vec{V})\vec{\sigma}. \tag{1.3}$$

Unlike the Boltzmann theory, which describes the behavior of a dilute gas, standard Enskog theory (SET) deals with dense gases, more correctly modeling moderately dense gases. However, SET does not yield correct hydrodynamics. In order to correct this deficiency and to account for more realistic potentials, several *modified*, *revised* and *extended* versions have been proposed over the past thirty years [14],[6],[21],[19],[23],[22],[5],[39],[7],[9],[25],[35],[28],[26].

Kinetic equations for dense classical fluids can be obtained by closure of the BBGKY (Bogoliubov-Born-Green-Kirkwood-Yvon) hierachy equations. One method for obtaining such closure is based on the maximization of entropy subject to constraints on microscopic phase functions [23],[36]. For a fluid of hard-sphere particles, this method yields the revised Enskog theory (RET), an irreversible kinetic equation derivable, for example, on the basis of diagrammatic methods[6].

Let us describe briefly the derivation of RET. Consider a system of N classical identical interacting particles of mass m confined to a box of volume Ω. It can be described by the Hamiltonian

$$H_N = \sum_{i=1}^{N}\{\frac{\vec{p}_i^2}{2m} + U^{box}(\vec{r}_i)\} + \sum_{i<j}^{N}\phi_{ij}, \tag{1.4}$$

where $\phi_{ij} \equiv \phi_{ij}(|\vec{r}_i - \vec{r}_j|)$ indicates that the particles interact through central forces, $(\vec{r}_i, \vec{p}_i)$ denotes the position and momentum of the i-th particle, and U^{box} enforces that particles are confined to the box. For convenience we set $m = 1$ so that $\vec{p} = \vec{v}$, and set $x_i = (\vec{r}_i, \vec{p}_i)$. Define the *N-particle distribution function (density function)* $\rho_N(x_1, ..., x_N, t)$ in such a way that $\rho_N(x_1, ..., x_N, t)dx_1...dx_N$ is the probability of finding the system in the state: particle i within an element dx_i around x_i, $i = 1, ..., N$, at time t. The distribution function $\rho_N(x_1, ..., x_N, t)$ satisfies the *N-particle Liouville equation*:

$$\frac{\partial}{\partial t}\rho_N(x_1, ..., x_N, t) = \{H_N, \rho_N(x_1, ..., x_N, t)\}, \tag{1.5}$$

where $\{\cdot, \cdot\}$ denotes the *poisson brackets*, i.e.

$$\{H_N, \rho_N\} = \sum_{i=1}^{N}\left(\frac{\partial H_N}{\partial \vec{r}_i}\cdot\frac{\partial \rho_N}{\partial \vec{p}_i} - \frac{\partial H_N}{\partial \vec{p}_i}\cdot\frac{\partial \rho_N}{\partial \vec{r}_i}\right). \tag{1.6}$$

Liouville's equation describes the time evolution of $\rho_N(x_1, ..., x_N, t)$, which relates the theory to macroscopic phenomena. Let $S_t^{(n)}$ denote the solution operator of the n-particle mechanical system, i.e., if the system at time $t = 0$ is represented by the state $\{x_1, ..., x_n\}$, then it will be represented by the state $\{x_1', ..., x_n'\} = S_t^{(n)}\{x_1, ..., x_n\}$ at time t. Define $S_t^{(n)}g$ by

$$S_t^{(n)}g(x_1, ..., x_{n+k}, \tau) = g(S_t^{(n)}\{x_1, ..., x_n\}, x_{n+1}, ..., x_{n+k}). \tag{1.7}$$

Then one may express the solution of the initial value problem for Liouville's equation in terms of the solution operator, in the form

$$\rho_N(x_1, ..., x_N, t) = S_{-t}^{(N)}\rho_N(x_1, ..., x_N, 0) \tag{1.8}$$

However, since $S_{-t}^{(N)}$ cannot be calculated for very large N, and since $\rho_N(x_1, ..., x_N, 0)$ is unknown in general, the solution (Eq.(1.8)) is not practical. Forturnately these difficulties can be circumvented by introducing the *s-particle density functions* [27]

$$F_s^N(x_1, ..., x_s, t) = \Omega^s \int \rho_N(x_1, ..., x_N, t) dx_{s+1} ... dx_N; \tag{1.9}$$
$$s = 0, 1, 2, \ldots\ .$$

It follows that F_s^N is symmetric in $(x_1, \ldots, x_s)$, $F_0^N = 1$, and

$$\frac{1}{\Omega^s} \int F_s^N\ dx_1 \ldots dx_s = \int \rho_N\ dx_1 \ldots dx_N = 1; \tag{1.10}$$
$$s = 1, 2, \ldots\ .$$

Taking the *thermodynamic limit* $N \to \infty$ and $|\Omega| \to \infty$ in such a way that $N/|\Omega| = n$ is finite (where Ω is the volume containing the N particles and n is the number of particles per unit volume), one can obtain [27], from (1.5), the infinite system of *hierachy* equations called the **BBGKY hierarchy**:

$$\frac{\partial}{\partial t} F_s(x^s, t) = \{H_s, F_s(x^s, t)\} + n \int dx_{s+1} \sum_{i=1}^{s} \frac{\partial}{\partial \vec{r}_i} \phi_{is+1} \cdot \frac{\partial}{\partial \vec{p}_i} F_{s+1}(x^{s+1}, t), \tag{1.11}$$

where $H_s = \sum_{i=1}^{s} \frac{\vec{p}_i}{2m} + \sum_{1 \le i < j \le s} \phi_{ij}$, $\phi_{ij} = \phi(|\vec{r}_i - \vec{r}_j|)$, and $F_s(x^s, t)$ is defined by

$$F_s(x^s, t) = \lim_{\substack{N \to \infty \\ |\Omega| \to \infty}} F_s^N(x_1, \ldots, x_s, t),$$
$$s = 1, 2, \ldots .$$

A formal solution of (1.11) in powers of n can be obtained [27]:

$$F_s(x^s, t+\tau) = \sum_{k=0}^{\infty} n^k \int dx_{s+1} \ldots dx_{s+k} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!(k-j)!} T_{-\tau}^{(j+s)} F_{k+s}(x^{k+s}, t), \tag{1.12}$$
$$s = 1, 2, \ldots,$$

where the operator $T_\tau^{(s)}$ is defined by

$$T_\tau^{(s)} F_{s+k}(x^{s+k}, t) = F_{s+k}(S_\tau^{(s)} x^s, S_\tau^{(1)} x_{s+1}, \ldots, S_\tau^{(1)} x_{s+k},\ t), \tag{1.13}$$

and $S_{-\tau}^{(j)} = exp\ \tau\{H_j, \cdot\}$.

It should be noted that the BBGKY hierarchy connects the evolution of a s-particle distribution function F_s to the distribution function F_{s+1} of $s+1$ particles. In order to solve F_1, we set $s = 1$ in (1.12), and obtain

$$\begin{aligned} &F_1(x_1, t+\tau) - T_{-\tau}^{(1)} F_1(x_1, t) \\ &= n \int dx_2 [T_{-\tau}^{(2)} F_2(x_1, x_2, t) - T_{-\tau}^{(1)} F_2(x_1, x_2, t)] \\ &\quad + n^2 \int dx_2 dx_3 [\frac{1}{2} T_{-\tau}^{(3)} F_3(x^3, t) - T_{-\tau}^{(2)} F_3(x^3, t) + \frac{1}{2} T_{-\tau}^{(1)} F_3(x^3, t)] + \ldots . \end{aligned} \tag{1.14}$$

Replace x_i by $S_\tau^{(1)} x_i = (\vec{r}_i + \tau \vec{v}_i, \vec{v}_i), (i = 1, 2, \ldots)$ in (1.14), while dx_i remains unchanged. Then (1.14) becomes

$$
\begin{aligned}
&F_1(S_\tau^{(1)}x_1, t+\tau) - F_1(x_1,t) \\
&= n\int dx_2[F_2(S_{-\tau}^{(2)}\{S_\tau^{(1)}x_1, S_\tau^{(1)}x_2\}, t) - F_2(x_1,x_2,t)] \\
&\quad + \frac{n^2}{2}\int dx_2 dx_3[F_3(S_{-\tau}^{(3)}\{S_\tau^{(1)}x_1, S_\tau^{(1)}x_2, S_\tau^{(1)}x_3\}, t) \\
&\quad - 2F_2(S_{-tau}^{(2)}\{S_\tau^{(1)}x_1, S_\tau^{(1)}x_2\}, x_3, t) + F_3(x_1,x_2,x_3,t)] + \dots \, .
\end{aligned}
\tag{1.15}
$$

The left-hand side (LHS) of (1.15) can be rewritten (see [23],[28]) as

$$
\begin{aligned}
&F_1(S_\tau^{(1)}x_1, t+\tau) - F_1(x_1,t) = F_1(\vec{r}_1 + \tau\vec{v}_1, \vec{v}_1, t+\tau) - F_1(\vec{r}_1, \vec{v}_1, t) \\
&= \int_0^\tau ds \frac{d}{ds} F_1(\vec{r}_1 + s\vec{v}_1, \vec{v}_1, t+s) \\
&= \int_0^\tau ds \left[\frac{\partial}{\partial t} + \vec{v}_1\cdot\nabla_{\vec{r}_1}\right] F_1(\vec{r}_1 + s\vec{v}_1, \vec{v}_1, t+s) \\
&= \left[\frac{\partial}{\partial t} + \vec{v}_1\cdot\nabla_{\vec{r}_1}\right]\int_0^\tau ds F_1(S_s^{(1)}x_1, t+s) \\
&= \tau\left[\frac{\partial}{\partial t} + \vec{v}_1\cdot\nabla_{\vec{r}_1}\right]\bar{F}_1(x_1,t)
\end{aligned}
\tag{1.16}
$$

where $\bar{F}_1(x_1,t) = \frac{1}{\tau}\int_0^\tau ds F_1(S_s^{(1)}x_1, t+s)$.

By introducing the notation $F_2(S_{-\tau}^{(2)}\{S_\tau^{(1)}x_1, S_\tau^{(1)}x_2\}, t) \equiv F_2(x_1', x_2', t)$, we can rewrite the binary collision term on the right-hand side (RHS) of (1.15) as

$$
n\int dx_2[F_2(x_1', x_2', t) - F_2(x_1,x_2,t)],
\tag{1.17}
$$

where $(x_1', x_2') = S_{-\tau}^{(2)}\{S_\tau^{(1)}x_1, S_\tau^{(1)}x_2\}$ is related to (x_1, x_2) through the interparticle potential. Similar forms can be obtained for the ternary and high-order terms on the RHS of (1.15). However we can choose n or τ [23] such that these terms will be very small, and $\bar{F}_1(x_1,t) \approx F_1(x_1,t)$. Consequently, combining (1.15), (1.16) and (1.17), and neglecting small corrections, we obtain

$$
\tau\left[\frac{\partial}{\partial t} + \vec{v}_1\cdot\nabla_{\vec{r}_1}\right] F_1(x_1,t) = n\int dx_2[F_2(x_1', x_2', t) - F_2(x_1,x_2,t)].
\tag{1.18}
$$

Nevertheless the exact BBGKY hierarchy is not closed. By defining the correlation function

$$Y(x_1, x_2, t) = \frac{F_2(x_1, x_2, t)}{F_1(x_1, t)F_1(x_2, t)}, \tag{1.19}$$

Eq.(1.18) can be closed.

Applying (1.19) to (1.18) gives

$$\begin{aligned}
&\tau \left[\frac{\partial}{\partial t} + \vec{v}_1 \cdot \nabla_{\vec{r}_1}\right] F_1(x_1, t) \\
&= n \int dx_2 [Y(x_1', x_2', t)F_1(x_1', t)F_1(x_2', t) - Y(x_1, x_2, t)F_1(x_1, t)F_1(x_2, t)].
\end{aligned} \tag{1.20}$$

This method yields the RET. In particular, if $Y(x_1, x_2, t)$ is set to $Y \equiv 1$ (correlations among particles are neglected in this approximate ensemble), (1.20) takes the form of the Boltzmann equation [23],[28].

There are two major approaches to extending the Enskog theory which have so far yielded existence results.

On adding a smooth tail to the hard-core potential one obtains kinetic equations that treat the hard core collision in an irreversible way, but approximate the tail dynamics by a reversible mean-field type term (see [36],[26]). This approach was first studied by Luis de Sobrino for the nonequilibrium problem of a van der Waals gas [35].

In contrast, another approach is to add to the hard core a tail consisting of piecewise-constant steps. Special cases of this kind are the square-well or square shoulder potentials. This direction has been studied by a number of researchers (see [21],[14],[22],[39],[7],[25],[20],[28]).

It should be pointed out that for the latter approach, the kinetic equations have multiple Enskog-like collision terms, as will be shown here in this paper. In this paper, we will take this second approach, considering the related kinetic equation for a dense square-well fluid. For a detailed analysis of the mathematical problem for various versions of the Enskog equation, the reader is referred to the book [8].

For the Enskog equation, the hard-core potential is taken as

$$\phi(r) = \begin{cases} \infty, & r < a, \\ 0, & r > a. \end{cases} \tag{1.21}$$

This potential can be treated as the limit of the sequence $\phi_j(r) = (\frac{a}{r})^j$, $j \to \infty$. With this resolution, (1.20) produces a revised Enskog equation of the form [23],[28]

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \vec{v}_1 \cdot \nabla_{\vec{r}_1}\right] F_1(\vec{r}_1, \vec{v}_1, t) \\ &= a^2 \iint\limits_{R^3 \times S_+^2} d\vec{v}_2 d\vec{\sigma} \theta(\vec{\sigma} \cdot \vec{V}_{21})\, \vec{\sigma} \cdot \vec{V}_{21}\, [Y(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}, t) F_1(\vec{r}_1, \vec{v}_1', t) F_1(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2', t) \\ &\quad - Y(\vec{r}_1, \vec{r}_1 - a\vec{\sigma}, t) F_1(\vec{r}_1, \vec{v}_1, t) F_1(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)], \end{aligned} \tag{1.22}$$

where $\vec{V}_{21} = \vec{v}_2 - \vec{v}_1$, $\vec{v}_1' = \vec{v}_1 + \vec{\sigma}(\vec{\sigma} \cdot \vec{V}_{21})$ and $\vec{v}_2' = \vec{v}_2 - \vec{\sigma}(\vec{\sigma} \cdot \vec{V}_{21})$. The only difference between (1.22) and Enskog's equation (1.1) lies in the form of dependence of Y on density; in the original formulation Y was treated as a uniform-equilibrium function evaluated at the density at the point of contact. As far as the linear transport properties of the one-component hard-core fluid are concerned, the revised and standard Enskog theories are identical in prediction. However, the revised theory appears to be superior when applied to hard-core mixtures [6].

We may consider approximating a finite range potential by a sequence of step functions. The square-well (SW) potential

$$\phi(r) = \begin{cases} \infty, & r < a, \\ -\varepsilon, & a < r < Ra, \\ 0, & Ra < r, \end{cases} \tag{1.23}$$

is the simplest such representation of an intermolecular potential. For this potential, the square-well kinetic equation is obtained [23],[28] from (1.20):

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + \vec{v}_1 \cdot \nabla_{\vec{r}_1}\right] F_1(\vec{r}_1, \vec{v}_1, t) \\
&= a^2 \iint\limits_{R^3 \times S^2_+} d\vec{v}_2 d\vec{\sigma} \theta(\vec{\sigma} \cdot \vec{V}_{21})\ \vec{\sigma} \cdot \vec{V}_{21} \\
&\quad [Y(\vec{r}_1, \vec{r}_1 + a^+\vec{\sigma}|n) F_1(\vec{r}_1, \vec{v}_1', t) F_1(\vec{r}_1 + a\vec{\sigma}, \vec{v}_2', t) \\
&\qquad - Y(\vec{r}_1, \vec{r}_1 - a^+\vec{\sigma}|n) F_1(\vec{r}_1, \vec{v}_1, t) F_1(\vec{r}_1 - a\vec{\sigma}, \vec{v}_2, t)] \\
&+ R^2 a^2 \iint\limits_{R^3 \times S^2_+} d\vec{v}_2 d\vec{\sigma} \theta(\vec{\sigma} \cdot \vec{V}_{21})\ \vec{\sigma} \cdot \vec{V}_{21} \\
&\quad [Y(\vec{r}_1, \vec{r}_1 + Ra^-\vec{\sigma}|n) F_1(\vec{r}_1, \vec{v}_1'', t) F_1(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2'', t) \\
&\qquad - Y(\vec{r}_1, \vec{r}_1 - Ra^+\vec{\sigma}|n) F_1(\vec{r}_1, \vec{v}_1, t) F_1(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2, t)] \\
&+ R^2 a^2 \iint\limits_{R^3 \times S^2_+} d\vec{v}_2 d\vec{\sigma} \theta(\vec{\sigma} \cdot \vec{V}_{21} - \sqrt{4\varepsilon})\ \vec{\sigma} \cdot \vec{V}_{21} \\
&\quad [Y(\vec{r}_1, \vec{r}_1 - Ra^+\vec{\sigma}|n) F_1(\vec{r}_1, \vec{v}_1''', t) F_1(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2''', t) \\
&\qquad - Y(\vec{r}_1, \vec{r}_1 + Ra^-\vec{\sigma}|n) F_1(\vec{r}_1, \vec{v}_1, t) F_1(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)] \\
&+ R^2 a^2 \iint\limits_{R^3 \times S^2_+} d\vec{v}_2 d\vec{\sigma} \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V}_{21})\ \vec{\sigma} \cdot \vec{V}_{21} \\
&\quad [Y(\vec{r}_1, \vec{r}_1 - Ra^-\vec{\sigma}|n) F_1(\vec{r}_1, \vec{v}_1', t) F_1(\vec{r}_1 - Ra\vec{\sigma}, \vec{v}_2', t) \\
&\qquad - Y(\vec{r}_1, \vec{r}_1 + Ra^-\vec{\sigma}|n) F_1(\vec{r}_1, \vec{v}_1, t) F_1(\vec{r}_1 + Ra\vec{\sigma}, \vec{v}_2, t)],
\end{aligned} \tag{1.24}$$

where

$$\begin{aligned}
\vec{v}_1' - \vec{v}_1 &= \vec{\sigma}(\vec{\sigma} \cdot \vec{V}_{21}), \\
\vec{v}_1'' - \vec{v}_1 &= \frac{1}{2}\vec{\sigma}\{\vec{\sigma} \cdot \vec{V}_{21} - [(\vec{\sigma} \cdot \vec{V}_{21})^2 + 4\varepsilon]^{\frac{1}{2}}\}, \\
\vec{v}_1''' - \vec{v}_1 &= \frac{1}{2}\vec{\sigma}\{\vec{\sigma} \cdot \vec{V}_{21} - [(\vec{\sigma} \cdot \vec{V}_{21})^2 - 4\varepsilon]^{\frac{1}{2}}\},
\end{aligned} \tag{1.25}$$

the Enskog geometric factor $Y(\vec{r}_1, \vec{r}_2|n) \equiv Y(n(\vec{r}_1, t, F_1), n(\vec{r}_2, t, F_1))$, and $n(\vec{r}, t, F_1) = \int F_1(\vec{r}, \vec{v}, t) d\vec{v}$.

Let us present a brief review of known existence theorems on Enskog theory. Lachowicz in [24] proved a local-in-time existence theorem, while a global-in-time existence result was proved by Toscani and Bellomo [38] in the case of a perturbation of the vacuum. Polewczak showed [30] that the solution obtained in [38] is actually a classical solution if the initial datum is smooth. Cercignani [13] obtained global-in-time L^1 solutions for small initial data and the Enskog geometric factor $Y \equiv 1$. Furthermore, the same author [12] obtained global-in-time L^1 solutions for large initial data, in the case of data depending on one space variable and $Y \equiv 1$. Arkeryd [1] extended Cercignani's result to two space variables using a weak compactness argument, together with $Y \equiv 1$, however, with the (unphysical) assumption that the range of integration is extended to the whole space S^2; in other words, the collision kernel is symmetrized. In fact, the standard Enskog equation and the revised equation, with integration over S^2_+, distinguish between forward and backward (time-reversed) collisions. When DiPerna and Lions [15] provided their ingenious proof of existence of solutions to the Boltzmann equation, the study of the initial value problem for the Enskog equation underwent an important change. Polewczak [32] proved a global existence theorem with large initial data in L^1 for the modified Enskog equation and extended the results further in [33] for the generalized Enskog equation with unsymmetrized collision kernel; he introduced, however, an assumption on the Enskog geometric factor Y, which essentially amounts to having a collision term dominated by a linear operator. Arkeryd [3] obtained global existence for $Y \equiv 1$ under the assumptions that the initial data is differentiable in $\vec{r}$ in an L^1 sense and has sufficiently high moments. Removing the restrictions of [33] and (unphysical) symmetrized collision kernel, Arkeryd and Cercignani [4] proved global existence for the Enskog equation in a periodic box with $Y \equiv 1$. Recently, Liu [28] obtained global-in-time existence in an L^1 sense for the Enskog equation with square-well potential, under the same assumption on the Enskog geometric factor Y as in [33].

In this paper, we show how to extend Akeryd and Cercignani's method [4] to the kinetic equation with square-well potential considered in [28], and remove the restriction of [33] and [28], proving a global existence theorem in L^1 for the kinetic equation. The difficulty here is to obtain controls (bounds) for the mass, energy, and entropy of the solution. By complicated calculations we find that some terms turn out to be cancelled, which helps us to get controls. These estimates will be shown in Section 4 and Section 5. In order to prove the existence and uniqueness of the solution to truncated equations, we use Arkeryd and Cercignani's method, splitting each function into two parts, free and collision terms, proving the sequence to be Cauchy. After splitting, there would appear 64 terms in our case. Fortunately this complicated situation can be simplified by introducing several lemmas. This will be seen in Section 6. An analogue of the classical H-theorem is also verified here in Section 3 for the kinetic equation.

The organization of the paper is as follows. In Section 2 we introduce the kinetic equation for the square well fluid and the related Cauchy problem, as well as some notation for convenience. Some useful properties of the kinetic equation are discussed and the proof ideas are outlined there. In Section 3 we derive an entropy inequality and show the H-theorem. These results turn out to concide with those of Arkeryd, Cercignani [4] and Polewczak [33], by setting appropriate parameters to zero.

In order to prove global existence for the original kinetic equation, we start with the truncated generalized Enskog equation and discuss its properties. Some useful equalities, inequalities and bounds, such as mass equality, energy equality, transport inequality, entropy inequality, gain-loss estimations, are obtained in Section 4 and Section 5. This sets up the foundation for our theory.

In Section 6, we demonstrate existence and uniqueness of the solution to the truncated equations, based on the contraction mapping theorem and the splitting method used by Arkeryd and Cercignani [3],[13],[4]. By applying velocity-averaging methods [18],[15], the paper concludes with its main result that a global solution in L^1 does exist for the generalized Enskog equation for a dense square-well fluid under rather general initial value conditions.

§2. Properties of the Kinetic Equation

We consider the generalized Enskog equation (GEE) with Square-well (SW) potential:

$$(\partial_t + \vec{v} \cdot \nabla_{\vec{r}}) f = J(f, f), \tag{2.1a}$$

under the initial condition

$$f(\vec{r}, \vec{v}, t = 0) = f_0(\vec{r}, \vec{v}), \tag{2.1b}$$

with the collision operator

$$J(f, f) = \sum_{i=1}^{4} C_i(f, f) = \sum_{i=1}^{4} [G_i(f, f) - f L_i(f, f)]. \tag{2.1c}$$

Here $C_1(f, f)$ is the Enskog term,

$$\begin{aligned} &C_1(f, f) = G_1(f, f) - f L_1(f, f) \\ &= \iint_{R^3 \times S_+^2} Y[f(\vec{r}_1, \vec{v}', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{w}', t) \\ &\qquad - f(\vec{r}_1, \vec{v}, t) f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t)] a^2 \vec{\sigma} \cdot \vec{V} d\vec{w} d\vec{\sigma}, \end{aligned} \tag{2.2.1}$$

$C_2(f, f)$ is the entering Square-Well (SW) term,

$$\begin{aligned} &C_2(f, f) = G_2(f, f) - f L_2(f) \\ &= \iiint_{R^3 \times R^3 \times S_+^2} Y[k_+(\vec{r}_1, \vec{r}_2; \vec{\sigma}) f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t) \\ &\qquad - k_-(\vec{r}_1, \vec{r}_2; \vec{\sigma}) f(\vec{r}_1, \vec{v}, t) f(\vec{r}_2, \vec{w}, t)] \\ &\qquad (Ra)^2 \vec{\sigma} \cdot \vec{V} d\vec{w} d\vec{\sigma} d\vec{r}_2, \end{aligned} \tag{2.2.2}$$

$C_3(f, f)$ is the exiting SW term,

$$\begin{aligned} &C_3(f, f) = G_3(f, f) - f L_3(f) \\ &= \iiint_{R^3 \times R^3 \times S_+^2} Y[k_-(\vec{r}_1, \vec{r}_2; \vec{\sigma}) f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t) \\ &\qquad - k_+(\vec{r}_1, \vec{r}_2; \vec{\sigma}) f(\vec{r}_1, \vec{v}, t) f(\vec{r}_2, \vec{w}, t)] \\ &\qquad (Ra)^2 \vec{\sigma} \cdot \vec{V} \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) d\vec{w} d\vec{\sigma} d\vec{r}_2, \end{aligned} \tag{2.2.3}$$

$C_4(f, f)$ is the bounded state term,

$$C_4(f, f) = G_4(f, f) - f L_4(f)$$

$$= \iiint\limits_{R^3 \times R^3 \times S_+^2} Y[k_-(\vec{r}_1, \vec{r}_2; \vec{\sigma}) f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t)$$
$$- k_+(\vec{r}_1, \vec{r}_2; \vec{\sigma}) f(\vec{r}_1, \vec{v}, t) f(\vec{r}_2, \vec{w}, t)]$$
$$\theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})(Ra)^2 \vec{\sigma} \cdot \vec{V} d\vec{w} d\vec{\sigma} d\vec{r}_2, \tag{2.2.4}$$

with $\vec{V} = \vec{v} - \vec{w}$ and $S_+^2 = \{\vec{\sigma} \epsilon R^3 | \ |\vec{\sigma}| = 1, \vec{\sigma} \cdot \vec{V} \geq 0\}$,

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The parameter a is the hard core diameter, $Ra - a$ is the width of Square-Well, and ε is the depth of the square-well. $G_i(f, f)$ is the *i-th* gain term, and $fL_i(f, f)$ is the *i-th* loss term (i = *1, ..., 4*). $k_+(\vec{r}_1, \vec{r}_2; \vec{\sigma}) = \delta(\vec{r}_2 - \vec{r}_1 - Ra\vec{\sigma})$ and $k_-(\vec{r}_1, \vec{r}_2; \vec{\sigma}) = \delta(\vec{r}_2 - \vec{r}_1 + Ra\vec{\sigma})$. We assume that the geometric factor Y is a constant.

Among the velocities, there are some relations summarized here:

$$\begin{cases} \vec{v}' + \vec{w}' = \vec{v} + \vec{w}; \\ \vec{v}^+ + \vec{w}^+ = \vec{v} + \vec{w}; \\ \vec{v}^- + \vec{w}^- = \vec{v} + \vec{w}; \\ \vec{v}^B + \vec{w}^B = \vec{v} + \vec{w}; \\ (\vec{v}')^2 + (\vec{w}')^2 = \vec{v}^2 + \vec{w}^2; \\ (\vec{v}^+)^2 + (\vec{w}^+)^2 = \vec{v}^2 + \vec{w}^2 + 2\varepsilon; \\ (\vec{v}^-)^2 + (\vec{w}^-)^2 = \vec{v}^2 + \vec{w}^2 - 2\varepsilon; \\ (\vec{v}^B)^2 + (\vec{w}^B)^2 = \vec{v}^2 + \vec{w}^2 \end{cases} \tag{2.3}$$

and

$$\begin{cases} \vec{v}' = \vec{v} - \vec{\sigma}(\vec{\sigma} \cdot \vec{V}), \\ \vec{w}' = \vec{w} + \vec{\sigma}(\vec{\sigma} \cdot \vec{V}); \\ \vec{v}^+ = \vec{v} - \frac{1}{2}\vec{\sigma}\{\vec{\sigma} \cdot \vec{V} - \sqrt{(\vec{\sigma} \cdot \vec{V})^2 + 4\varepsilon}\}, \\ \vec{w}^+ = \vec{w} + \frac{1}{2}\vec{\sigma}\{\vec{\sigma} \cdot \vec{V} - \sqrt{(\vec{\sigma} \cdot \vec{V})^2 + 4\varepsilon}\}; \\ \vec{v}^- = \vec{v} - \frac{1}{2}\vec{\sigma}\{\vec{\sigma} \cdot \vec{V} - \sqrt{(\vec{\sigma} \cdot \vec{V})^2 - 4\varepsilon}\}, \\ \vec{w}^- = \vec{w} + \frac{1}{2}\vec{\sigma}\{\vec{\sigma} \cdot \vec{V} - \sqrt{(\vec{\sigma} \cdot \vec{V})^2 - 4\varepsilon}\}; \\ \vec{v}^B = \vec{v} - \vec{\sigma}(\vec{\sigma} \cdot \vec{V}), \\ \vec{w}^B = \vec{w} + \vec{\sigma}(\vec{\sigma} \cdot \vec{V}). \end{cases} \tag{2.4}$$

(2.3) is obvious by conservation of momentum and energy, and (2.4) can be obtained from (2.3).

Here we list some propositions for later use.

Proposition 2.1:

$$
\begin{aligned}
&(\vec{v}')' = \vec{v}, \qquad && (\vec{w}')' = \vec{w};\\
&(\vec{v}^{+})^{-} = \vec{v}, && (\vec{w}^{+})^{-} = \vec{w};\\
&(\vec{v}^{-})^{+} = \vec{v}, && (\vec{w}^{-})^{+} = \vec{w};\\
&(\vec{v}^{B})^{B} = \vec{v}, && (\vec{w}^{B})^{B} = \vec{w}.
\end{aligned}
\tag{2.5}
$$

Proof: It is easy to obtain (2.5) by using the relation (2.4). Let us prove, for example, the first two expressions. Due to (2.4), one has

$$\vec{V}' = \vec{v}' - \vec{w}' = \vec{V} - 2\vec{\sigma}(\vec{\sigma}\cdot\vec{V}).$$

Taking the scalar product on both sides with $\vec{\sigma}$ yields

$$
\begin{aligned}
\vec{\sigma}\cdot\vec{V}' &= -\vec{\sigma}\cdot\vec{V},\\
(\vec{v}')' &= \vec{v}' - \vec{\sigma}(\vec{\sigma}\cdot\vec{V}')\\
&= \vec{v} - \vec{\sigma}(\vec{\sigma}\cdot\vec{V}) - \vec{\sigma}(\vec{\sigma}\cdot\vec{V}')\\
&= \vec{v},\\
(\vec{w}')' &= \vec{w}' + \vec{\sigma}(\vec{\sigma}\cdot\vec{V}')\\
&= \vec{w} + \vec{\sigma}(\vec{\sigma}\cdot\vec{V}) + \vec{\sigma}(\vec{\sigma}\cdot\vec{V}')\\
&= \vec{w}.
\end{aligned}
$$

We can also prove the other expressions in (2.5) in a similar way.

The following proposition is obvious by the definition of $k_{+}(\vec{r}_1, \vec{r}_2; \vec{\sigma})$ and $k_{-}(\vec{r}_1, \vec{r}_2; \vec{\sigma})$.

Proposition 2.2:

$$
\begin{cases}
k_{+}(\vec{r}_1, \vec{r}_2; -\vec{\sigma}) = k_{-}(\vec{r}_1, \vec{r}_2; \vec{\sigma}),\\
k_{-}(\vec{r}_1, \vec{r}_2; -\vec{\sigma}) = k_{+}(\vec{r}_1, \vec{r}_2; \vec{\sigma}),\\
k_{+}(\vec{r}_2, \vec{r}_1; \vec{\sigma}) = k_{-}(\vec{r}_1, \vec{r}_2; \vec{\sigma}),\\
k_{-}(\vec{r}_2, \vec{r}_1; \vec{\sigma}) = k_{+}(\vec{r}_1, \vec{r}_2; \vec{\sigma}).
\end{cases}
\tag{2.6}
$$

For convenience, we will define the following notation:

$$
\begin{cases}
f_1 \equiv f(\vec{r}_1, \vec{v}, t),\\
f_2 \equiv f(\vec{r}_2, \vec{w}, t),\\
B \equiv a^2(\vec{\sigma}\cdot\vec{V});
\end{cases}
\tag{2.7}
$$

and

$$\begin{cases} M \equiv R^3 \times R^3 \times R^3 \times S_+^2, \\ d\mu \equiv d\vec{r}_1 d\vec{v} d\vec{w} d\vec{\sigma}, \\ M_1 \equiv R^3 \times R^3 \times R^3 \times R^3 \times S_+^2, \\ d\mu_1 \equiv d\vec{r}_1 d\vec{r}_2 d\vec{v} d\vec{w} d\vec{\sigma}. \end{cases} \tag{2.8}$$

By direct calculations and appropriate variable changes, $(\vec{v},\vec{w}) \to (\vec{v}',\vec{w}')$ or $(\vec{v},\vec{w}) \to (\vec{w},\vec{v})$, we have that, for the Enskog term,

$$\begin{aligned} &\iint\limits_{R^3 \times R^3} \psi(\vec{r}_1,\vec{v}) C_1(f,f) d\vec{r}_1 d\vec{v} \\ &\quad = \frac{1}{2} \int \cdots \int\limits_{M} Y\{\psi(\vec{r}_1,\vec{v}') + \psi(\vec{r}_1 + a\vec{\sigma},\vec{w}') - \psi(\vec{r}_1,\vec{v}) - \psi(\vec{r}_1 + a\vec{\sigma},\vec{w})\} \\ &\qquad f(\vec{r}_1,\vec{v},t) f(\vec{r}_1 + a\vec{\sigma},\vec{w},t) B d\mu, \end{aligned} \tag{2.9}$$

which can be found in [33]. For the entering SW term, we get

$$\begin{aligned} &\iint\limits_{R^3 \times R^3} \psi(\vec{r}_1,\vec{v}) C_2(f,f) d\vec{r}_1 d\vec{v} \\ &\quad = \frac{1}{2} \int \cdots \int\limits_{M_1} Y\{[\psi(\vec{r}_1,\vec{v}^-) + \psi(\vec{r}_2,\vec{w}^-)] k_+(\vec{r}_1,\vec{r}_2;\vec{\sigma}) \theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon}) \\ &\qquad - [\psi(\vec{r}_1,\vec{v}) + \psi(\vec{r}_2,\vec{w})] k_-(\vec{r}_1,\vec{r}_2;\vec{\sigma})\} f_1 f_2 R^2 B d\mu_1. \end{aligned} \tag{2.10}$$

Here variable changes $(\vec{v},\vec{w}) \to (\vec{v}^+,\vec{w}^+)$ and $(\vec{v},\vec{w}) \to (\vec{w},\vec{v})$ are used. Also we used the following equalities (refer to [28],[20], see also (2.4) and (2.5)):

$$\vec{V}^+ = \vec{v}^+ - \vec{w}^+, \quad \vec{\sigma}\cdot\vec{V} d\vec{v} d\vec{w} = \vec{\sigma}\cdot\vec{V}^+ d\vec{v}^+ d\vec{w}^+,$$

$$\vec{\sigma}\cdot\vec{V}^+ = \sqrt{(\vec{\sigma}\cdot\vec{V})^2 + 4\varepsilon},$$

$$(\vec{v}^+)^- = \vec{v}, \quad (\vec{w}^+)^- = \vec{w}.$$

In a similar way, we can treat the exiting SW term and bounded state term if we make the variable changes: $(\vec{v},\vec{w}) \to (\vec{v}^-,\vec{w}^-), (\vec{v},\vec{w}) \to (\vec{v}^B,\vec{w}^B)$ and $(\vec{v},\vec{w}) \to (\vec{w},\vec{v})$. Notice that [28]

$$(\vec{\sigma}\cdot\vec{V}) d\vec{v} d\vec{w} = (\vec{\sigma}\cdot\vec{V}^-) d\vec{v}^- d\vec{w}^- \quad \textit{and} \quad (\vec{\sigma}\cdot\vec{V}) d\vec{v} d\vec{w} = (\vec{\sigma}\cdot\vec{V}^B) d\vec{v}^B d\vec{w}^B. \tag{2.11}$$

and that ((2.4) and (2.5))

$$\vec{\sigma}\cdot\vec{V}^- = \sqrt{(\vec{\sigma}\cdot\vec{V})^2 - 4\varepsilon}, \quad \vec{\sigma}\cdot\vec{V}^B = -(\vec{\sigma}\cdot\vec{V}),$$

$$(\vec{v}^-)^+ = \vec{v}, \quad (\vec{w}^-)^+ = \vec{w},$$
$$(\vec{v}^B)^B = \vec{v}, \quad (\vec{w}^B)^B = \vec{w}.$$

Utilizing these equalities, we have

$$\iint\limits_{R^3\times R^3} \psi(\vec{r}_1,\vec{v})C_3(f,f)d\vec{r}_1 d\vec{v}$$
$$= \frac{1}{2}\int\limits_{M_1}\cdots\int Y\{[\psi(\vec{r}_1,\vec{v}^+) + \psi(\vec{r}_2,\vec{w}^+)]k_-(\vec{r}_1,\vec{r}_2;\vec{\sigma})$$
$$- [\psi(\vec{r}_1,\vec{v}) + \psi(\vec{r}_2,\vec{w})]k_+(\vec{r}_1,\vec{r}_2;\vec{\sigma})\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})\}$$
$$f_1 f_2 R^2 B d\mu_1; \tag{2.12}$$

$$\iint\limits_{R^3\times R^3} \psi(\vec{r}_1,\vec{v})C_4(f,f)d\vec{r}_1 d\vec{v}$$
$$= \frac{1}{2}\int\limits_{M_1}\cdots\int Y\{[\psi(\vec{r}_1,\vec{v}^B) + \psi(\vec{r}_2,\vec{w}^B)]k_+(\vec{r}_1,\vec{r}_2;\vec{\sigma})$$
$$- [\psi(\vec{r}_1,\vec{v}) + \psi(\vec{r}_2,\vec{w})]k_+(\vec{r}_1,\vec{r}_2;\vec{\sigma})\}$$
$$\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V})f_1 f_2 R^2 B d\mu_1. \tag{2.13}$$

Collecting these gives

$$\iint\limits_{R^3\times R^3} \psi(\vec{r}_1,\vec{v})J(f,f)d\vec{r}_1 d\vec{v}$$
$$= \frac{1}{2}\int\limits_{M}\cdots\int Y\{\psi(\vec{r}_1,\vec{v}') + \psi(\vec{r}_1 + a\vec{\sigma},\vec{w}') - \psi(\vec{r}_1,\vec{v}) - \psi(\vec{r}_1 + a\vec{\sigma},\vec{w})\}$$
$$f(\vec{r}_1,\vec{v},t)f(\vec{r}_1 + a\vec{\sigma},\vec{w},t)Bd\mu$$
$$+ \frac{1}{2}\int\limits_{M_1}\cdots\int Y\{[\psi(\vec{r}_1,\vec{v}^-) + \psi(\vec{r}_2,\vec{w}^-)]k_+(\vec{r}_1,\vec{r}_2;\vec{\sigma})\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})$$
$$- [\psi(\vec{r}_1,\vec{v}) + \psi(\vec{r}_2,\vec{w})]k_-(\vec{r}_1,\vec{r}_2;\vec{\sigma})\}f_1 f_2 R^2 B d\mu_1$$
$$+ \frac{1}{2}\int\limits_{M_1}\cdots\int Y\{[\psi(\vec{r}_1,\vec{v}^+) + \psi(\vec{r}_2,\vec{w}^+)]k_-(\vec{r}_1,\vec{r}_2;\vec{\sigma})$$
$$- [\psi(\vec{r}_1,\vec{v}) + \psi(\vec{r}_2,\vec{w})]k_+(\vec{r}_1,\vec{r}_2;\vec{\sigma})\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})\}f_1 f_2 R^2 B d\mu_1$$
$$+ \frac{1}{2}\int\limits_{M_1}\cdots\int Y\{[\psi(\vec{r}_1,\vec{v}^B) + \psi(\vec{r}_2,\vec{w}^B)]k_+(\vec{r}_1,\vec{r}_2;\vec{\sigma})$$

$$
\begin{aligned}
&\quad - [\psi(\vec{r}_1, \vec{v}) + \psi(\vec{r}_2, \vec{w})]k_+(\vec{r}_1, \vec{r}_2; \vec{\sigma})\}\theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})f_1 f_2 R^2 B d\mu_1 \\
&= \frac{1}{2} \int_M \cdots \int Y\{\psi(\vec{r}_1, \vec{v}') + \psi(\vec{r}_1 + a\vec{\sigma}, \vec{w}') - \psi(\vec{r}_1, \vec{v}) - \psi(\vec{r}_1 + a\vec{\sigma}, \vec{w})\} \\
&\quad f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&+ \frac{1}{2} \int_{M_1} \cdots \int Y\{[\psi(\vec{r}_1, \vec{v}^-) + \psi(\vec{r}_2, \vec{w}^-)]k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \\
&\quad + [\psi(\vec{r}_1, \vec{v}^+) + \psi(\vec{r}_2, \vec{w}^+)]k_- \\
&\quad + [\psi(\vec{r}_1, \vec{v}^B) + \psi(\vec{r}_2, \vec{w}^B)]k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\} \\
&\quad \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V}) f_1 f_2 R^2 B d\mu_1 \\
&- \frac{1}{2} \int_{M_1} \cdots \int Y\{[\psi(\vec{r}_1, \vec{v}) + \psi(\vec{r}_2, \vec{w})](k_+ + k_-)\} f_1 f_2 R^2 B d\mu_1. \qquad (2.14)
\end{aligned}
$$

By setting $\psi \equiv 1$ in (2.14), we can easily see

Proposition 2.3:

$$
\iint_{R^3 \times R^3} J(f, f)(\vec{r}_1, \vec{v}, t) d\vec{r}_1 d\vec{v} = 0. \qquad (2.15)
$$

Proposition 2.4:

$$
\begin{aligned}
&\iint_{R^3 \times R^3} \vec{v}^2 J(f, f)(\vec{r}_1, \vec{v}, t) d\vec{r}_1 d\vec{v} \\
&\quad = \int_{M_1} \cdots \int Y\varepsilon[k_- - k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})] f_1 f_2 R^2 B d\mu_1. \qquad (2.16)
\end{aligned}
$$

Proof: Using (2.14) with $\psi \equiv \vec{v}^2$ and also (2.3) yields

$$
\begin{aligned}
&\iint_{R^3 \times R^3} \vec{v}^2 J(f, f)(\vec{r}_1, \vec{v}, t) d\vec{r}_1 d\vec{v} \\
&= \frac{1}{2} \int_M \cdots \int Y\{(\vec{v}')^2 + (\vec{w}')^2 - \vec{v}^2 - \vec{w}^2\} \\
&\quad f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\quad + \frac{1}{2} \int_{M_1} \cdots \int Y\{[(\vec{v}^+)^2 + (\vec{w}^+)^2]k_-
\end{aligned}
$$

$$
\begin{aligned}
&+[(\vec{v}^{-})^2 + (\vec{w}^{-})^2]k_+\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon}) \\
&+[(\vec{v}^{B})^2 + (\vec{w}^{B})^2]k_+\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V})\}f_1f_2R^2Bd\mu_1 \\
&-\frac{1}{2}\underset{M_1}{\int\cdots\int} Y\{[\vec{v}^2 + \vec{w}^2](k_+ + k_-)\}f_1f_2R^2Bd\mu_1 \\
&= \frac{1}{2}\underset{M_1}{\int\cdots\int} Y\{2\varepsilon k_- - 2\varepsilon k_+\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})\}f_1f_2R^2Bd\mu_1 \\
&= \varepsilon\underset{M_1}{\int\cdots\int} Y\{k_- - k_+\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})\}f_1f_2R^2Bd\mu_1
\end{aligned}
$$

The proof is completed.

§3 . Entropy inequality and H-theorem

Let us begin with the equation

$$\begin{cases} (\partial_t + \vec{v}\cdot\nabla_{\vec{r}})f = & J(f,f), \\ f(\vec{r},\vec{v},t=0) = f_0(\vec{r},\vec{v}). \end{cases} \tag{3.1}$$

Define

$$f^{\#}(\vec{r},\vec{v},t) = \quad f(\vec{r}+t\vec{v},\vec{v},t). \tag{3.2}$$

Then (3.1) can be rewritten as

$$\frac{d}{dt}f^{\#}(\vec{r},\vec{v},t) = \quad J(f,f)^{\#}.$$

Thus one has

$$f^{\#}(\vec{r},\vec{v},t) - f^{\#}(\vec{r},\vec{v},s) \quad = \quad \int_s^t J(f,f)^{\#}d\tau, \tag{3.3}$$

or

$$f(\vec{r}+\vec{v}t,\vec{v},t) \quad = \quad f_0(\vec{r},\vec{v}) + \int_0^t J(f,f)(\vec{r}+\vec{v}s,\vec{v},s)ds, \tag{3.4}$$

or

$$f(\vec{r},\vec{v},t) \quad = \quad f_0(\vec{r}-\vec{v}t,\vec{v}) + \int_0^t J(f,f)(\vec{r}-\vec{v}t+\vec{v}s,\vec{v},s)ds. \tag{3.5}$$

Due to Proposition 2.3 and variable changes between $\vec{r}_1$ and $\vec{r}_1+\vec{v}t$, one gets that

$$\begin{aligned}
&\frac{d}{dt}\iint_{R^3\times R^3} f(\log f)d\vec{r}_1 d\vec{v} \\
&= \frac{d}{dt}\iint_{R^3\times R^3}(f \log f)^{\#}d\vec{r}_1 d\vec{v} \quad = \quad \iint_{R^3\times R^3}(1+\log f^{\#})\frac{d}{dt}f^{\#}d\vec{r}_1 d\vec{v} \\
&= \iint_{R^3\times R^3}(1+\log f^{\#})J(f,f)^{\#}d\vec{r}_1 d\vec{v} \\
&= \iint_{R^3\times R^3}(\log f^{\#})J(f,f)^{\#}d\vec{r}_1 d\vec{v} \\
&= \iint_{R^3\times R^3}(\log f)J(f,f)(\vec{r}_1,\vec{v},t)d\vec{r}_1 d\vec{v}.
\end{aligned} \tag{3.6}$$

Using (2.14) with $\psi = log\ f$ gives

$$\iint\limits_{R^3 \times R^3} (log\ f) J(f,f) d\vec{r}_1 d\vec{v}$$

$$= \frac{1}{2} \int_M \cdots \int Y\{log\ f(\vec{r}_1, \vec{v}', t) + log\ f(\vec{r}_1 + a\vec{\sigma}, \vec{w}', t) - log\ f(\vec{r}_1, \vec{v}, t)$$

$$- log\ f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)\} f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu$$

$$+ \frac{1}{2} \int_{M_1} \cdots \int Y\{[log\ f(\vec{r}_1, \vec{v}^+, t) + log\ f(\vec{r}_2, \vec{w}^+, t)] k_-$$

$$+ [log\ f(\vec{r}_1, \vec{v}^-, t) + log\ f(\vec{r}_2, \vec{w}^-, t)] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})$$

$$+ [log\ f(\vec{r}_1, \vec{v}^B, t) + log\ f(\vec{r}_2, \vec{w}^B, t)] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\}$$

$$f_1 f_2 R^2 B d\mu_1$$

$$+ \frac{1}{2} \int_{M_1} \cdots \int Y\{[log\ f_1 + log\ f_2](k_+ + k_-)\} f_1 f_2 R^2 B d\mu_1$$

$$= -\frac{1}{2} \int_M \cdots \int Y \left[log \frac{f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)}{f(\vec{r}_1, \vec{v}', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{w}', t)} \right] f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu$$

$$- \frac{1}{2} \int_{M_1} \cdots \int Y\{k_- \left[log \frac{f_1 f_2}{f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t)} \right]$$

$$+ k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \left[log \frac{f_1 f_2}{f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t)} \right]$$

$$+ k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V}) \left[log \frac{f_1 f_2}{f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t)} \right] \} f_1 f_2 R^2 B d\mu_1. \quad (3.7)$$

After recombination of terms in Proposition 2.3 and appropriate variable changes, (2.15) can be written as:

$$0 = \iint\limits_{R^3 \times R^3} J(f,f) d\vec{r}_1 d\vec{v}$$

$$= \int_M \cdots \int Y[f(\vec{r}_1, \vec{v}', t) f(\vec{r}_1 - a\vec{\sigma}, \vec{w}', t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] B d\mu$$

$$+ \int_{M_1} \cdots \int Y\{[k_+ f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t)$$

$$+ k_- \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t)$$

$$+ k_- \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V}) f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t)]$$

$$
\begin{aligned}
&- [k_+ + k_-] f_1 f_2\} R^2 B d\mu_1 \\
&= \int_M \cdots \int Y[f(\vec{r}_1, \vec{v}', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{w}', t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] B d\mu \\
&\quad + \int_{M_1} \cdots \int Y\{k_-[f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t) - f_1 f_2] \\
&\quad + k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})[f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t) - f_1 f_2] \\
&\quad + k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})[f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t) - f_1 f_2]\} \\
&\quad R^2 B d\mu_1 \\
&= \int_M \cdots \int Y \left[\frac{f(\vec{r}_1, \vec{v}', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{w}', t)}{f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)} - 1 \right] \\
&\quad f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\quad + \int_{M_1} \cdots \int Y\{k_- \left[\frac{f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t)}{f_1 f_2} - 1 \right] f_1 f_2 \\
&\quad + k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \left[\frac{f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t)}{f_1 f_2} - 1 \right] f_1 f_2 \\
&\quad + k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V}) \left[\frac{f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t)}{f_1 f_2} - 1 \right] f_1 f_2\} R^2 B d\mu_1.
\end{aligned} \tag{3.8}
$$

Define the function

$$
\ell(x) = log\ x + \frac{1}{x} - 1 \quad for \quad x > 0. \tag{3.9}
$$

It is easy to see that the function $\ell(x)$ is non-negative, and that $\ell(x)$ is decreasing for $0 < x < 1$ and increasing for $x > 1$.

Adding the right hand side of (3.8) to the right hand side of (3.7) yields

$$
\begin{aligned}
&\iint_{R^3 \times R^3} (log\ f) J(f, f) d\vec{r}_1 d\vec{v} \\
&= -\frac{1}{2} \int_M \cdots \int Y \ell \left(\frac{f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)}{f(\vec{r}_1, \vec{v}', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{w}', t)} \right) \\
&\quad f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\quad - \frac{1}{2} \int_{M_1} \cdots \int Y\{k_- \ell \left(\frac{f_1 f_2}{f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t)} \right) \\
&\quad + k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \ell \left(\frac{f_1 f_2}{f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t)} \right)
\end{aligned}
$$

$$+ k_+\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V})\ell\left(\frac{f_1 f_2}{f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)}\right)\}f_1 f_2 R^2 B d\mu_1$$

$$\leq \quad 0. \tag{3.10}$$

Furthermore, with the inequality $g(log\ g - log\ h) \geq g - h$ for $g, h > 0$ (i.e. $-g\ log\ \frac{g}{h} \leq h - g$), (3.7) produces

$$\iint\limits_{R^3\times R^3} (log\ f)J(f,f)d\vec{r}_1 d\vec{v}$$

$$\leq \frac{1}{2}\int\limits_{M}\cdots\int Y[f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1 + a\vec{\sigma},\vec{w}',t) - f_1 f(\vec{r}_1 + a\vec{\sigma},\vec{w},t)]B\mu$$

$$+\frac{1}{2}\int\limits_{M_1}\cdots\int Y\{k_-[f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t) - f_1 f_2]$$

$$+ k_+\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})[f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t) - f_1 f_2]$$

$$+ k_+\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V})[f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t) - f_1 f_2]\}R^2 B d\mu_1. \tag{3.11}$$

Notice that only the first term on the right hand side of (3.11) survives (only the Enskog term), and all the other terms vanish. This can be seen by the following:

$$\int\limits_{M_1}\cdots\int k_+\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)R^2 B d\mu_1$$

$$= \int\limits_{M_1}\cdots\int k_+ f_1 f_2 R^2 B d\mu_1;$$

$$\int\limits_{M_1}\cdots\int k_- f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)R^2 B d\mu_1$$

$$= \int\limits_{M_1}\cdots\int k_-\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})f_1 f_2 R^2 B d\mu_1;$$

$$\int\limits_{M_1}\cdots\int k_+\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V})f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)R^2 B d\mu_1$$

$$= \int\limits_{M_1}\cdots\int k_-\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V})f_1 f_2 R^2 B d\mu_1.$$

Thus we have

$$\int\limits_{M_1}\cdots\int Y\{k_-[f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t) - f_1 f_2]$$

$$
\begin{aligned}
&+ k_+\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})[f(\vec{r}_1, \vec{v}^-, t)f(\vec{r}_2, \vec{w}^-, t) - f_1 f_2] \\
&+ k_+\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V})[f(\vec{r}_1, \vec{v}^B, t)f(\vec{r}_2, \vec{w}^B, t) - f_1 f_2]\} R^2 B d\mu_1 \\
= \;& 0.
\end{aligned} \tag{3.12}
$$

Let us define particle density and flows:

$$
\rho(\vec{r}, t) = \int\limits_{R^3} f(\vec{r}, \vec{v}, t) d\vec{v}, \qquad \vec{j}(\vec{r}, \vec{v}, t) = \int\limits_{R^3} \vec{v}\; f(\vec{r}, \vec{v}, t) d\vec{v}, \tag{3.13}
$$

and use the fact

$$
\int\limits_{S} \vec{F}\cdot\vec{n}\; d\vec{n} = \int\limits_{V} div\; \vec{F}\; dx,
$$

and

$$
div\; \vec{j} = -\frac{\partial}{\partial t}\rho.
$$

We obtain that

$$
\begin{aligned}
&\iint\limits_{R^3 \times R^3} (log\; f) J(f, f) d\vec{r}_1 d\vec{v} \\
&\le \frac{1}{2} \int \cdots \int\limits_{M} Y[f(\vec{r}_1, \vec{v}', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{w}', t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] B d\mu \\
&= \frac{1}{2} \int \cdots \int\limits_{M} Y[f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] B d\mu \\
&= \frac{1}{2} \iiiint\limits_{R^3 \times R^3 \times R^3 \times \{S^2_+ \cup S^2_-\}} Y f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) B d\mu \\
&= \frac{1}{2} \iiiint\limits_{R^3 \times R^3 \times R^3 \times S^2} Y f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) a^2 (\vec{\sigma}\cdot\vec{V}) d\mu \\
&= \frac{1}{2} Y a^2 \iint\limits_{R^3 \times S^2} \{ \left(\int\limits_{R^3} \vec{v} f(\vec{r}_1, \vec{v}, t) d\vec{v} \right) \left(\int\limits_{R^3} f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) d\vec{w} \right) \\
&\quad - \left(\int\limits_{R^3} f(\vec{r}_1, \vec{v}, t) d\vec{v} \right) \left(\int\limits_{R^3} \vec{w} f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) d\vec{w} \right) \} \cdot \vec{\sigma} d\vec{r}_1 d\vec{\sigma} \\
&= \frac{1}{2} Y a^2 \iint\limits_{R^3 \times S^2} [\vec{j}(\vec{r}_1, t)\rho(\vec{r}_1 - a\vec{\sigma}, t) - \rho(\vec{r}_1, t)\vec{j}(\vec{r}_1 - a\vec{\sigma}, t)] \cdot \vec{\sigma} d\vec{r}_1 d\vec{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} Y a^2 \iint\limits_{R^3 \times S^2} [\vec{j}(\vec{r}_1 + a\vec{\sigma}, t)\rho(\vec{r}_1, t) - \rho(\vec{r}_1, t)\vec{j}(\vec{r}_1 - a\vec{\sigma}, t)] \cdot \vec{\sigma} d\vec{r}_1 d\vec{\sigma} \\
&= \frac{1}{2} Y a^2 \iint\limits_{R^3 \times S^2} \rho(\vec{r}_1, t)[\vec{j}(\vec{r}_1 + a\vec{\sigma}, t) - \vec{j}(\vec{r}_1 - a\vec{\sigma}, t)] \cdot \vec{\sigma} d\vec{r}_1 d\vec{\sigma} \\
&= Y a^2 \iint\limits_{R^3 \times S^2} \rho(\vec{r}_1, t)\vec{j}(\vec{r}_1 + a\vec{\sigma}, t) \cdot \vec{\sigma} d\vec{r}_1 d\vec{\sigma} \\
&= Y a^2 \int\limits_{R^3} \rho(\vec{r}_1, t) \left(\int\limits_{S^2} \vec{j}(\vec{r}_1 + a\vec{\sigma}, t) \cdot \vec{\sigma} d\vec{\sigma} \right) d\vec{r}_1 \\
&= Y a^2 \int\limits_{R^3} \rho(\vec{r}_1, t) \left(\int\limits_{B_a(\vec{r}_1)} div\ \vec{j}(\vec{r}_2)\ d\vec{r}_2 \right) d\vec{r}_1 \\
&= -Y a^2 \int\limits_{R^3} \rho(\vec{r}_1, t) \left(\int\limits_{B_a(\vec{r}_1)} \frac{\partial}{\partial t}\rho(\vec{r}_2)\ d\vec{r}_2 \right) d\vec{r}_1 \\
&= -\frac{1}{2} Y a^2 \frac{d}{dt} \iint\limits_{R^3 \times B_a} \rho(\vec{r}_1, t)\rho(\vec{r}_2, t) d\vec{r}_1 d\vec{r}_2,
\end{aligned}
\tag{3.14}
$$

where $\quad B_a(\vec{r}_1) = \{\vec{r}_2 \quad | \quad |\vec{r}_1 - \vec{r}_2| \leq a\}$.

Thus far we have shown that

$$
\begin{aligned}
&\iint\limits_{R^3 \times R^3} (log\ f) J(f, f) d\vec{r}_1 d\vec{v} \\
&\leq \frac{1}{2} \int \cdots \int\limits_{M} Y[f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] B d\mu \\
&= \frac{1}{2} \iiiint\limits_{R^3 \times R^3 \times R^3 \times S^2} Y f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) B d\mu \\
&= -\frac{1}{2} Y a^2 \frac{d}{dt} \iint\limits_{R^3 \times B_a} \rho(\vec{r}_1, t)\rho(\vec{r}_2, t) d\vec{r}_2 d\vec{r}_1.
\end{aligned}
\tag{3.15}
$$

Note that if we set the appropriate parameters to zero, for example, $\varepsilon = 0$, we remove the terms related to Square-Well potential, and our original equation (3.1) (or (2.1)) becomes the Enskog Equation. Under this limit the results shown here concide with both those of Arkeryd and Cercignani [4] and Polewczak [33].

From the above we have

Theorem 3.1 (H-Theorem): Let us define the H-function

$$\begin{aligned}
H &\equiv \iint\limits_{R^3 \times R^3} (f \ log \ f) d\vec{r}_1 d\vec{v} \\
&\quad + \frac{1}{2} Y a^2 \iint\limits_{R^3 \times B_a} \rho(\vec{r}_1, t) \rho(\vec{r}_2, t) d\vec{r}_2 d\vec{r}_1 \\
&= \iint\limits_{R^3 \times R^3} (f \ log \ f) d\vec{r}_1 d\vec{v} \\
&\quad - \frac{1}{2} \int\limits_0^t \int\limits_M \cdots \int Y [f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) - f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] f_1 B d\mu.
\end{aligned}$$

Then

$$\frac{d}{dt} H \quad \leq \quad 0.$$

§4 . Truncated equation

We consider the original equations (2.1a)-(2.1c):

$$\begin{cases} (\partial_t + \vec{v}\cdot\nabla_{\vec{r}})f = J(f,f), \\ f(\vec{r},\vec{v},t=0) = f_0(\vec{r},\vec{v}), \\ J(f,f) = \sum_{i=1}^4 C_i(f,f). \end{cases}$$

Make the substitutions:

$$\begin{array}{ll} Y \to Y W_n^{\pm} X_n^{\pm} & \textit{in gain terms}, \\ Y \to Y W_n^{\pm} X_n^{\pm} + \eta & \textit{in loss terms}. \end{array}$$

with a perturbation parameter $\eta > 0$, where

$$W(x) = X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ \textit{between}, & 1 < x < 2 \\ 0, & x \ge 2, \end{cases} \tag{4.1}$$

$W(x)$ and $X(x)$ are smooth monotonic functions, and

$$W_n(\vec{v},\vec{w}) = W\left(\frac{\vec{v}^2+\vec{w}^2}{n^2}\right), \tag{4.2}$$

$$X_n(\vec{r}_1,\vec{r}_2) = X\left(\frac{\vec{r}_1^2+\vec{r}_2^2}{n^2}\right), \tag{4.3}$$

$$X_n^{\pm}(\vec{r}) = X_n(\vec{r},\vec{r}\pm a\vec{\sigma}), \tag{4.4}$$

$$\begin{aligned} W_n^{\pm}(\vec{v},\vec{w}) &= W_n(\vec{v}^{\pm},\vec{w}^{\pm}) \\ &= W\left(\frac{\vec{v}^2+\vec{w}^2\pm 2\varepsilon}{n^2}\right). \end{aligned} \tag{4.5}$$

For convenience, let $k_\pm$ denote $k_\pm(\vec{r}_1,\vec{r}_2;\vec{\sigma})$, W_n denote $W_n(\vec{v},\vec{w})$ and X_n denote $X_n(\vec{r}_1,\vec{r}_2)$.

The truncated equation is

$$\begin{cases} (\partial_t + \vec{v}\cdot\nabla_{\vec{r}})f = J_n(f,f) \\ f(\vec{r},\vec{v},t=0) = f_0(\vec{r},\vec{v}), \end{cases} \tag{4.6}$$

where

$$J_n(f,f) = \sum_{i=1}^4 C_{in}(f,f), \tag{4.7}$$

with

$$C_{1n}(f,f) = G_{1n}(f,f) - fL_{1n}(f)$$

$$= \iint_{R^3\times S^2_+} \{YW_n(\vec{v},\vec{w})X_n^-(\vec{r}_1)f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1 - a\vec{\sigma},\vec{w}',t)$$
$$- [YW_n(\vec{v},\vec{w})X_n^+(\vec{r}_1) + \eta]f(\vec{r}_1,\vec{v},t)f(\vec{r}_1 + a\vec{\sigma},\vec{w},t)\}Bd\vec{\sigma}d\vec{w}, \tag{4.8.1}$$

$$C_{2n}(f,f) = G_{2n}(f,f) - fL_{2n}(f)$$
$$= \iiint_{R^3\times R^3\times S^2_+} \{k_+YW_n^+(\vec{v},\vec{w})X_n(\vec{r}_1,\vec{r}_2)f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)$$
$$- k_-[YW_n(\vec{v},\vec{w})X_n(\vec{r}_1,\vec{r}_2) + \eta]f_1f_2\}R^2Bd\vec{\sigma}d\vec{w}d\vec{r}_2, \tag{4.8.2}$$

$$C_{3n}(f,f) = G_{3n}(f,f) - fL_{3n}(f)$$
$$= \iiint_{R^3\times R^3\times S^2_+} \{k_-YW_n^+(\vec{v},\vec{w})X_n(\vec{r}_1,\vec{r}_2)f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)$$
$$- k_+[YW_n(\vec{v},\vec{w})X_n(\vec{r}_1,\vec{r}_2) + \eta]f_1f_2\}\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})$$
$$R^2Bd\vec{\sigma}d\vec{w}d\vec{r}_2, \tag{4.8.3}$$

$$C_{4n}(f,f) = G_{4n}(f,f) - fL_{4n}(f)$$
$$= \iiint_{R^3\times R^3\times S^2_+} \{k_-YW_n(\vec{v},\vec{w})X_n(\vec{r}_1,\vec{r}_2)f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)$$
$$- k_+[YW_n(\vec{v},\vec{w})X_n(\vec{r}_1,\vec{r}_2) + \eta]f_1f_2\}\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V})$$
$$R^2Bd\vec{\sigma}d\vec{w}d\vec{r}_2. \tag{4.8.4}$$

Analogous to Section 2, we have similar formulas for the truncated terms:

$$\int_M \cdots \int \psi(\vec{r}_1,\vec{v})YW_nX_n^- f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1 - a\vec{\sigma},\vec{w}',t)Bd\mu$$
$$= \frac{1}{2}\int_M \cdots \int [\psi(\vec{r}_1,\vec{v}') + \psi(\vec{r}_1 + a\vec{\sigma},\vec{w}')]$$
$$\times YW_nX_n^+ f_1 f(\vec{r}_1 + a\vec{\sigma},\vec{w},t)Bd\mu; \tag{4.9.1}$$

$$\int_{M_1} \cdots \int \psi(\vec{r}_1,\vec{v})YW_n^+X_nk_+f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)R^2Bd\mu_1$$

$$= \frac{1}{2} \int_{M_1} \cdots \int [\psi(\vec{r}_1, \vec{v}^-) + \psi(\vec{r}_2, \vec{w}^-)] Y W_n X_n k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) f_1 f_2 R^2 B d\mu_1; \tag{4.9.2}$$

$$\int_{M_1} \cdots \int \psi(\vec{r}_1, \vec{v}) Y W_n^- X_n k_- \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t) R^2 B d\mu_1$$

$$= \frac{1}{2} \int_{M_1} \cdots \int [\psi(\vec{r}_1, \vec{v}^+) + \psi(\vec{r}_2, \vec{w}^+)] Y W_n X_n k_- f_1 f_2 R^2 B d\mu_1; \tag{4.9.3}$$

$$\int_{M_1} \cdots \int \psi(\vec{r}_1, \vec{v}) Y W_n X_n k_- \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V}) f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t) R^2 B d\mu_1$$

$$= \frac{1}{2} \int_{M_1} \cdots \int [\psi(\vec{r}_1, \vec{v}^B) + \psi(\vec{r}_2, \vec{w}^B)] Y W_n X_n k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V}) f_1 f_2 R^2 B d\mu_1. \tag{4.9.4}$$

Using the above formulas, we obtain that

$$\iint_{R^3 \times R^3} \psi(\vec{r}_1, \vec{v}) J_n(f, f) d\vec{r}_1, d\vec{v}$$

$$= \int_M \cdots \int [\psi(\vec{r}_1, \vec{v}') + \psi(\vec{r}_1 + a\vec{\sigma}, \vec{w}') - \psi(\vec{r}_1, \vec{v}) - \psi(\vec{r}_1 + a\vec{\sigma}, \vec{w})]$$

$$Y W_n(\vec{v}, \vec{w}) X_n(\vec{r}_1, \vec{r}_1 + a\vec{\sigma}) f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu$$

$$- \frac{1}{2}\eta \int_M \cdots \int [\psi(\vec{r}_1, \vec{v}) + \psi(\vec{r}_1 + a\vec{\sigma}, \vec{w})] f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu$$

$$+ \frac{1}{2} \int_{M_1} \cdots \int \{[\psi(\vec{r}_1, \vec{v}^-) + \psi(\vec{r}_2, \vec{w}^-)] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})$$

$$- [\psi(\vec{r}_1, \vec{v}) + \psi(\vec{r}_2, \vec{w})] k_- \} Y W_n X_n f_1 f_2 R^2 B d\mu_1$$

$$+ \frac{1}{2} \int_{M_1} \cdots \int \{[\psi(\vec{r}_1, \vec{v}^+) + \psi(\vec{r}_2, \vec{w}^+)] k_-$$

$$- [\psi(\vec{r}_1, \vec{v}) + \psi(\vec{r}_2, \vec{w})] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \} Y W_n X_n f_1 f_2 R^2 B d\mu_1$$

$$+ \frac{1}{2} \int_{M_1} \cdots \int \{\psi(\vec{r}_1, \vec{v}^B) + \psi(\vec{r}_2, \vec{w}^B) - \psi(\vec{r}_1, \vec{v}) - \psi(\vec{r}_2, \vec{w})\}$$

$$k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) Y W_n X_n f_1 f_2 R^2 B d\mu_1$$

$$- \frac{1}{2}\eta \int_{M_1} \cdots \int \{\psi(\vec{r}_1, \vec{v}) + \psi(\vec{r}_2, \vec{w})\} (k_+ + k_-) f_1 f_2 R^2 B d\mu_1$$

$$
\begin{aligned}
&= \frac{1}{2} \int_M \cdots \int [\psi(\vec{r}_1, \vec{v}') + \psi(\vec{r}_1 + a\vec{\sigma}, \vec{w}') - \psi(\vec{r}_1, \vec{v}) - \psi(\vec{r}_1 + a\vec{\sigma}, \vec{w})] \\
&\quad YW_n X_n^+ f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\quad - \frac{1}{2}\eta \int_M \cdots \int [\psi(\vec{r}_1, \vec{v}) + \psi(\vec{r}_1 + a\vec{\sigma}, \vec{w})] f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\quad + \frac{1}{2} \int_{M_1} \cdots \int \{[\psi(\vec{r}_1, \vec{v}^+) + \psi(\vec{r}_2, \vec{w}^+)] k_- \\
&\qquad + [\psi(\vec{r}_1, \vec{v}^-) + \psi(\vec{r}_2, \vec{w}^-)] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \\
&\qquad + [\psi(\vec{r}_1, \vec{v}^B) + \psi(\vec{r}_2, \vec{w}^B)] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V}) \\
&\qquad - [\psi(\vec{r}_1, \vec{v}) + \psi(\vec{r}_2, \vec{w})](k_+ + k_-)\} \\
&\qquad YW_n(\vec{v}, \vec{w}) X_n(\vec{r}_1, \vec{r}_2) f_1 f_2 R^2 B d\mu_1 \\
&\quad - \frac{1}{2} \int_{M_1} \cdots \int [\psi(\vec{r}_1, \vec{v}) + \psi(\vec{r}_2, \vec{w})](k_+ + k_-) f_1 f_2 R^2 B d\mu_1. \qquad (4.10)
\end{aligned}
$$

(1) Mass Equality:

Setting $\psi = 1$ in (4.10) gives

$$
\begin{aligned}
&\iint_{R^3 \times R^3} J_n(f, f) d\vec{r}_1 d\vec{v} \\
&\quad = -\eta \int_M \cdots \int f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\qquad - \eta \int_{M_1} \cdots \int (k_+ + k_-) f_1 f_2 R^2 B d\mu_1,
\end{aligned}
$$

and the mass equality:

$$
\begin{aligned}
\iint_{R^3 \times R^3} f(t) d\vec{r}_1 d\vec{v} \quad &= \quad \iint_{R^3 \times R^3} f_0 d\vec{r}_1 d\vec{v} \\
&\quad - \eta \int_0^t \{ \int_M \cdots \int f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, s) B d\mu \\
&\quad + \int_{M_1} \cdots \int (k_+ + k_-) f_1 f_2 R^2 B d\mu_1 \} ds. \qquad (4.11)
\end{aligned}
$$

If we assume that the solution f is nonnegative, then we have that:

$$\int_0^t ds\{\int \cdots \int_M f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu$$

$$+ \int \cdots \int_{M_1} (k_+ + k_-) f_1 f_2 R^2 B d\mu_1\}$$

$$\leq \frac{1}{\eta} \iint_{R^3 \times R^3} f_0 d\vec{r}_1 d\vec{v}. \tag{4.12}$$

(2) Energy equality:
Using (4.10) with $\psi = v^2$ together with (2.3) yields

$$\int_{R^3 \times R^3} \vec{v}^2 J_n(f, f) d\vec{r}_1 d\vec{v}$$

$$= \frac{1}{2} \int \cdots \int_M [(\vec{v}')^2 + (\vec{w}')^2 - \vec{v}^2 - \vec{w}^2]$$

$$Y W_n X_n^+ f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu$$

$$+ \frac{1}{2} \int \cdots \int_{M_1} \{[(\vec{v}^+)^2 + (\vec{w}^+)^2] k_-$$

$$+ [(\vec{v}^-)^2 + (\vec{w}^-)^2] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})$$

$$+ [(\vec{v}^B)^2 + (\vec{w}^B)^2] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})$$

$$- [\vec{v}^2 + \vec{w}^2](k_+ + k_-)\} Y W_n X_n f_1 f_2 R^2 B d\mu_1$$

$$- \frac{1}{2}\eta \int \cdots \int_M (\vec{v}^2 + \vec{w}^2) f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu$$

$$- \frac{1}{2}\eta \int \cdots \int_{M_1} (\vec{v}^2 + \vec{w}^2)(k_+ + k_-) f_1 f_2 R^2 B d\mu_1$$

$$= \frac{1}{2} \int \cdots \int_{M_1} \{[2\varepsilon] k_- + [-2\varepsilon] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})\}$$

$$Y W_n X_n f_1 f_2 R^2 B d\mu_1$$

$$- \frac{1}{2}\eta \int \cdots \int_M (\vec{v}^2 + \vec{w}^2) f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu$$

$$- \frac{1}{2}\eta \int \cdots \int_{M_1} (\vec{v}^2 + \vec{w}^2)(k_+ + k_-) f_1 f_2 R^2 B d\mu_1. \tag{4.13}$$

Thus we have the energy equality:

$$
\begin{aligned}
&\int\limits_{R^3\times R^3} \vec{v}^2 f(t) d\vec{r}_1 d\vec{v} \\
&= \int\limits_{R^3\times R^3} \vec{v}^2 f_0 d\vec{r}_1 d\vec{v} \\
&\quad + \varepsilon \int\limits_0^t \int\limits_{M_1} \cdots \int \{k_- - k_+\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})\} Y W_n X_n f_1 f_2 R^2 B d\mu_1 ds \\
&\quad - \frac{1}{2}\eta \int\limits_0^t \{ \int\limits_M \cdots \int (\vec{v}^2 + \vec{w}^2) f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\qquad + \int\limits_{M_1} \cdots \int (\vec{v}^2 + \vec{w}^2)(k_+ + k_-) f_1 f_2 R^2 B d\mu_1 \} ds. \qquad (4.14)
\end{aligned}
$$

Thus if we assume that f is non-negative, we see that the second term on the right hand side is less than $(\varepsilon/\eta)\int f_0$, so for the last term of (4.14) we have that:

$$
\begin{aligned}
&\frac{1}{2}\eta \int\limits_0^t \{ \int\limits_M \cdots \int (\vec{v}^2 + \vec{w}^2) f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\quad + \int\limits_{M_1} \cdots \int (\vec{v}^2 + \vec{w}^2)(k_+ + k_-) f_1 f_2 R^2 B d\mu_1 \} ds \\
&\le \int\limits_{R^3\times R^3} \vec{v}^2 f_0 d\vec{r}_1 d\vec{v} + \frac{\varepsilon}{\eta} \int\limits_{R^3\times R^3} f_0 d\vec{r}_1 d\vec{v}. \qquad (4.15)
\end{aligned}
$$

(3) Transport inequalities:

If we take some variable changes between $\vec{r}_1$ and $\vec{r}_1 + \vec{v}t$, we see that

$$
\begin{aligned}
&\frac{d}{dt}\int (\vec{r}_1 - \vec{v}t)^2 f(t) d\vec{r}_1 d\vec{v} \\
&= \frac{d}{dt}\int (\vec{r}_1)^2 f^{\#}(t) d\vec{r}_1 d\vec{v} \\
&= \int (\vec{r}_1)^2 J_n^{\#}(f,f) d\vec{r}_1 d\vec{v} \\
&= \int (\vec{r} - \vec{v}t)^2 J_n(f,f) d\vec{r} d\vec{v}.
\end{aligned}
$$

Due to (4.10), setting $\psi = (\vec{r} - \vec{v}t)^2$ gives

$$
\begin{aligned}
&\iint\limits_{R^3\times R^3} (\vec{r} - \vec{v}t)^2 J_n(f,f) d\vec{r}_1 d\vec{v} \\
&= \frac{1}{2}\int\limits_{M}\cdots\int [(\vec{r}_1 - \vec{v}'t)^2 + (\vec{r}_1 + a\vec{\sigma} - \vec{w}'t)^2 - (\vec{r}_1 - \vec{v}t)^2 - (\vec{r}_1 + a\vec{\sigma} - \vec{w}t)^2] \\
&\quad YW_n X_n^+ f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&+ \frac{1}{2}\int\limits_{M_1}\cdots\int \{[(\vec{r}_1 - \vec{v}^+t)^2 + (\vec{r}_2 - \vec{w}^+t)^2]k_- \\
&\quad + [(\vec{r}_1 - \vec{v}^-t)^2 + (\vec{r}_2 - \vec{w}^-t)^2]k_+\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon}) \\
&\quad + [(\vec{r}_1 - \vec{v}^Bt)^2 + (\vec{r}_2 - \vec{w}^Bt)^2]k_+\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V}) \\
&\quad - [(\vec{r}_1 - \vec{v}t)^2 + (\vec{r}_2 - \vec{w}t)^2](k_+ + k_-)\} \\
&\quad\quad YW_n X_n f_1 f_2 R^2 B d\mu_1 \\
&- \frac{1}{2}\eta \int\limits_{M}\cdots\int [(\vec{r}_1 - \vec{v}t)^2 + (\vec{r}_1 + a\vec{\sigma} - \vec{w}t)^2] f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&- \frac{1}{2}\eta \int\limits_{M_1}\cdots\int [(\vec{r}_1 - \vec{v}t)^2 + (\vec{r}_2 - \vec{w}t)^2](k_+ + k_-) f_1 f_2 R^2 B d\mu_1. \qquad (4.16)
\end{aligned}
$$

By basic calculations and (2.3), one has

$$
\begin{aligned}
&(\vec{r}_1 - \vec{v}'t)^2 + (\vec{r}_1 + a\vec{\sigma} - \vec{w}'t)^2 - (\vec{r}_1 - \vec{v}t)^2 - (\vec{r}_1 + a\vec{\sigma} - \vec{w}t)^2 \\
&\quad = (\vec{r}_1)^2 - 2\vec{r}_1\cdot\vec{v}'t + t^2(\vec{v}')^2 + (\vec{r}_1 + a\vec{\sigma})^2 - 2t(\vec{r}_1 + a\vec{\sigma})\cdot\vec{w}' + t^2(\vec{w}')^2 \\
&\quad\quad - [(\vec{r}_1)^2 - 2\vec{r}_1\cdot\vec{v}t + t^2\vec{v}^2 + (\vec{r}_1 + a\vec{\sigma})^2 - 2t(\vec{r}_1 + a\vec{\sigma})\cdot\vec{w} + t^2\vec{w}^2] \\
&\quad = -2t\vec{r}_1\cdot[\vec{v}' + \vec{w}' - \vec{v} - \vec{w}] - 2ta\sigma\cdot(\vec{w}' - \vec{w}) \\
&\quad\quad + t^2[(\vec{v}')^2 + (\vec{w}')^2 - \vec{v}^2 - \vec{w}^2] \\
&\quad = -2ta\sigma\cdot(\vec{w}' - \vec{w}) \\
&\quad = -2ta\sigma\cdot\vec{V};
\end{aligned}
$$

$$
\begin{aligned}
&(\vec{r}_1 - \vec{v}^+t)^2 + (\vec{r}_2 - \vec{w}^+t)^2 - (\vec{r}_1 - \vec{v}t)^2 - (\vec{r}_2 - \vec{w}t)^2 \\
&\quad = -2t[\vec{r}_1\cdot(\vec{v}^+ - \vec{v}) + \vec{r}_2\cdot(\vec{w}^+ - \vec{w})] + t^2[(\vec{v}^+)^2 + (\vec{w}^+)^2 - \vec{v}^2 - \vec{w}^2] \\
&\quad = -2t[\vec{r}_1\cdot(\vec{v}^+ - \vec{v}) + \vec{r}_2\cdot(\vec{w}^+ - \vec{w})] + t^2[2\varepsilon];
\end{aligned}
$$

$$
(\vec{r}_1 - \vec{v}^-t)^2 + (\vec{r}_2 - \vec{w}^-t)^2 - (\vec{r}_1 - \vec{v}t)^2 - (\vec{r}_2 - \vec{w}t)^2
$$

$$= -2t[\vec{r}_1 \cdot (\vec{v}^- - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^- - \vec{w})] + t^2[(\vec{v}^-)^2 + (\vec{w}^-)^2 - \vec{v}^2 - \vec{w}^2]$$
$$= -2t[\vec{r}_1 \cdot (\vec{v}^- - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^- - \vec{w})] + t^2[-2\varepsilon];$$

$$(\vec{r}_1 - \vec{v}^B t)^2 + (\vec{r}_2 - \vec{w}^B t)^2 - (\vec{r}_1 - \vec{v}t)^2 - (\vec{r}_2 - \vec{w}t)^2$$
$$= -2t[\vec{r}_1 \cdot (\vec{v}^B - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^B - \vec{w})] + t^2[(\vec{v}^B)^2 + (\vec{w}^B)^2 - \vec{v}^2 - \vec{w}^2]$$
$$= -2t[\vec{r}_1 \cdot (\vec{v}^B - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^B - \vec{w})].$$

The 2nd term on the right hand side of (4.16) is

$$= \frac{1}{2} \int_{M_1} \cdots \int \{[(\vec{r}_1 - \vec{v}^+ t)^2 + (\vec{r}_2 - \vec{w}^+ t)^2 - (\vec{r}_1 - \vec{v}t)^2 - (\vec{r}_2 - \vec{w}t)^2] k_-$$
$$+ [(\vec{r}_1 - \vec{v}^- t)^2 + (\vec{r}_2 - \vec{w}^-)^2 - (\vec{r}_1 - \vec{v}t)^2 - (\vec{r}_2 - \vec{w}t)^2] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})$$
$$+ [(\vec{r}_1 - \vec{v}^B t)^2 + (\vec{r}_2 - \vec{w}^B t)^2 - (\vec{r}_1 - \vec{v}t)^2 - (\vec{r}_2 - \vec{w}t)^2] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\}$$
$$Y W_n X_n f_1 f_2 R^2 B d\mu_1$$
$$= \frac{1}{2} \int_{M_1} \cdots \int \{-2t[\vec{r}_1 \cdot (\vec{v}^+ - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^+ - \vec{w})] k_- + 2\varepsilon t^2 k_-$$
$$- 2t[\vec{r}_1 \cdot (\vec{v}^- - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^- - \vec{w})] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) - 2\varepsilon t^2 k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})$$
$$- 2t[\vec{r}_1 \cdot (\vec{v}^B - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^B - \vec{w})] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\}$$
$$Y W_n X_n f_1 f_2 R^2 b d\mu_1$$
$$= -t \int_{M_1} \cdots \int \{[\vec{r}_1 \cdot (\vec{v}^+ - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^+ - \vec{w})] k_-$$
$$+ [\vec{r}_1 \cdot (\vec{v}^- - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^- - \vec{w})] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})$$
$$+ [\vec{r}_1 \cdot (\vec{v}^B - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^B - \vec{w})] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\}$$
$$Y W_n X_n f_1 f_2 R^2 B d\mu_1$$
$$+ t^2 \varepsilon \int_{M_1} \cdots \int [k_- - k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})] Y W_n X_n f_1 f_2 R^2 B d\mu_1.$$

To deal with this kind of problem, we have the following lemma:

Lemma 4.1:

$$\int_{M_1} \cdots \int \{[\vec{r}_1 \cdot (\vec{v}^+ - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^+ - \vec{w})] k_-$$
$$+ [\vec{r}_1 \cdot (\vec{v}^- - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^- - \vec{w})] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})$$

$$
\begin{aligned}
&+ [\vec{r}_1 \cdot (\vec{v}^B - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^B - \vec{w})] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\} \\
&\quad Y W_n X_n f_1 f_2 R^2 B d\mu_1 \\
\geq \quad & 0.
\end{aligned}
$$

Proof: Because of (2.4), we have the following:

$$
\begin{aligned}
\vec{\sigma} \cdot (\vec{w}^+ - \vec{w}) &= \frac{1}{2}\{\vec{\sigma} \cdot \vec{V} - \sqrt{(\vec{\sigma} \cdot \vec{V})^2 + 4\varepsilon}\} \quad \leq \quad 0 \\
\vec{\sigma} \cdot (\vec{w}^- - \vec{w}) &= \frac{1}{2}\{\vec{\sigma} \cdot \vec{V} - \sqrt{(\vec{\sigma} \cdot \vec{V})^2 - 4\varepsilon}\} \quad \geq \quad 0 \\
\vec{\sigma} \cdot (\vec{w}^B - \vec{w}) &= \quad \vec{\sigma} \cdot \vec{V} \quad \geq \quad 0.
\end{aligned}
$$

After integration with respect to $\vec{r}_2$, one has that

$$
\begin{aligned}
& \int \cdots \int_{M_1} \{[\vec{r}_1 \cdot (\vec{v}^+ - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^+ - \vec{w})] k_- \\
&\quad + [\vec{r}_1 \cdot (\vec{v}^- - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^- - \vec{w})] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \\
&\quad + [\vec{r}_1 \cdot (\vec{v}^B - \vec{v}) + \vec{r}_2 \cdot (\vec{w}^B - \vec{w})] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\} \\
&\quad Y W_n X_n f_1 f_2 R^2 B d\mu_1 \\
= & \int \cdots \int_M \{[\vec{r}_1 \cdot (\vec{v}^+ - \vec{v}) + (\vec{r}_1 - Ra\vec{\sigma}) \cdot (\vec{w}^+ - \vec{w})] \\
&\quad + [\vec{r}_1 \cdot (\vec{v}^- - \vec{v}) + (\vec{r}_1 + Ra\vec{\sigma}) \cdot (\vec{w}^- - \vec{w})] \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \\
&\quad + [\vec{r}_1 \cdot (\vec{v}^B - \vec{v}) + (\vec{r}_1 + Ra\vec{\sigma}) \cdot (\vec{w}^B - \vec{w})] \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\} \\
&\quad Y W_n X_n f_1 f_2 R^2 B d\mu_1 \\
= & \int \cdots \int_M \{-Ra\vec{\sigma} \cdot (\vec{w}^+ - \vec{w}) \\
&\quad + Ra\vec{\sigma} \cdot (\vec{w}^- - \vec{w}) \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \\
&\quad + Ra\vec{\sigma} \cdot (\vec{w}^B - \vec{w}) \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\} \\
&\quad Y W_n X_n f_1 f_2 R^2 B d\mu \\
\geq \quad & 0.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \iint_{R^3 \times R^3} (\vec{r}_1 - \vec{v}t)^2 J_n(f, f) d\vec{r}_1 d\vec{v} \\
= & -\frac{1}{2}\eta \int \cdots \int_M [(\vec{r}_1 - \vec{v}t)^2 + (\vec{r}_1 + a\vec{\sigma} - \vec{w}t)^2] f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\eta\int_{M_1}\cdots\int[(\vec{r}_1-\vec{v}t)^2+(\vec{r}_2-\vec{w}t)^2](k_++k_-)f_1f_2R^2Bd\mu_1\\
&-2ta\int_{M}\cdots\int\vec{\sigma}\cdot\vec{V}YW_nX_n^+f_1f(\vec{r}_1+a\vec{\sigma},\vec{w},t)Bd\mu\\
&-t\int_{M_1}\cdots\int\{[\vec{r}_1\cdot(\vec{v}^+-\vec{v})+\vec{r}_2\cdot(\vec{w}^+-\vec{w})]k_-\\
&\quad+[\vec{r}_1\cdot(\vec{v}^--\vec{v})+\vec{r}_2\cdot(\vec{w}^--\vec{w})]k_+\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})\\
&\quad+[\vec{r}_1\cdot(\vec{v}^B-\vec{v})+\vec{r}_2\cdot(\vec{w}^B-\vec{w})]k_+\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})\}\\
&\quad YW_nX_nf_1f_2R^2Bd\mu_1\\
&+t^2\varepsilon\int_{M_1}\cdots\int[k_--k_+\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})]YW_nX_nf_1f_2R^2Bd\mu_1. \qquad (4.17)
\end{aligned}
$$

Notice that the first four terms of (4.17) on the right hand side are negative, based on the previous discussion. For the last term, one has

$$
\begin{aligned}
&t^2\varepsilon\int_{M_1}\cdots\int[k_--k_+\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})]YW_nX_nf_1f_2R^2Bd\mu_1\\
\leq\quad & T^2Y\frac{\varepsilon}{\eta}\iint_{R^3\times R^3}f_0d\vec{r}_1d\vec{v}.
\end{aligned}
$$

Thus we have the following transport inequalities:

$$
\begin{aligned}
&\sup_{t\epsilon[0,T]}\iint_{R^3\times R^3}(\vec{r}_1-\vec{v}t)^2f(t)d\vec{r}_1d\vec{v}\\
&\qquad\leq\iint_{R^3\times R^3}(\vec{r}_1)^2f_0d\vec{r}_1d\vec{v}+T^2Y\frac{\varepsilon}{\eta}\iint_{R^3\times R^3}f_0d\vec{r}_1d\vec{v}; \qquad (4.18)
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{t\epsilon[0,T]}\iint_{R^3\times R^3}(\vec{r}_1)^2f^{\#}(t)d\vec{r}_1d\vec{v}\\
&\qquad\leq\iint_{R^3\times R^3}(\vec{r}_1)^2f_0d\vec{r}_1d\vec{v}+T^2Y\frac{\varepsilon}{\eta}\iint_{R^3\times R^3}f_0d\vec{r}_1d\vec{v}; \qquad (4.19)
\end{aligned}
$$

$$
\frac{\eta}{2}\int_0^t\int_M\cdots\int(\vec{r}_1)^2f_1^{\#}f^{\#}(\vec{r}_1+a\vec{\sigma},\vec{w},t)Bd\mu ds
$$

$$+\frac{\eta}{2}\int_0^t \int \cdots \int_{M_1} ((\vec{r}_1)^2+(\vec{r}_2)^2)(k_+ + k_-) f_1^\# f_2^\# R^2 B d\mu_1 ds$$

$$\leq \quad 2\left(\iint_{R^3\times R^3} (\vec{r}_1)^2 f_0 d\vec{r}_1 d\vec{v} + T^2 Y \frac{\varepsilon}{\eta} \iint_{R^3\times R^3} f_0 d\vec{r}_1 d\vec{v} \right). \tag{4.20}$$

These results are essential. They will be used to control the entropy.

(4) Entropy inequality for the truncated equation:

$$\begin{aligned}
\frac{d}{dt}\int f(t)\; log\; f(t) d\vec{r}_1 d\vec{v} \quad &= \quad \frac{d}{dt}\int f^\#\; log\; f^\# \\
= \int (1 + log\; f^\#) \frac{d}{dt} f^\# \quad &= \quad \int (1 + log\; f^\#) J_n^\#(f,f) d\vec{r}_1 d\vec{v} \\
= \int (1 + log\; f^\#) J_n(f,f) d\vec{r}_1 d\vec{v} & \\
= \iint_{R^3\times R^3} J_n(f,f) d\vec{r}_1 d\vec{v} + \iint_{R^3\times R^3} & log\; f\; J_n(f,f) d\vec{r}_1 d\vec{v}.
\end{aligned} \tag{4.21}$$

According to (4.11), one has

$$\int J_n(f,f) d\vec{r}_1 d\vec{v} \quad \leq \quad 0. \tag{4.22}$$

Also it is easy to see from (4.10) that

$$\begin{aligned}
&\iint_{R^3\times R^3} log\; f\; J_n(f,f) d\vec{r}_1 d\vec{v} \\
&= \frac{1}{2}\int \cdots \int_M \left(log \frac{f(\vec{r}_1, \vec{v}', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{w}', t)}{f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)} \right) Y W_n X_n^+ f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\quad - \frac{1}{2}\eta \int \cdots \int_M [log\,(f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t))] f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&\quad + \frac{1}{2}\int \cdots \int_{M_1} \{ \left[log \frac{f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t)}{f_1 f_2} \right] k_- \\
&\quad + \left[log \frac{f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t)}{f_1 f_2} \right] k_+ \theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})
\end{aligned}$$

$$
\begin{aligned}
&+\left[log\frac{f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)}{f_1f_2}\right]k_+\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})\}\\
&\quad f_1f_2YW_nX_nR^2Bd\mu_1\\
&-\frac{1}{2}\eta\underset{M_1}{\int\cdots\int}[log(f_1f_2)](k_++k_-)f_1f_2R^2Bd\mu_1\\
&= \quad A_1(f)+h(f),
\end{aligned}
\tag{4.23}
$$

where

$$
A_1(f)=-\frac{1}{2}\eta\underset{M}{\int\cdots\int}[log\ (f_1f(\vec{r}_1+a\vec{\sigma},\vec{w},t))]f_1f(\vec{r}_1+a\vec{\sigma},\vec{w},t)Bd\mu
$$

$$
-\frac{1}{2}\eta\underset{M_1}{\int\cdots\int}[log\ (f_1f_2)](k_++k_-)f_1f_2R^2Bd\mu_1, \tag{4.24}
$$

$$
\begin{aligned}
h(f)&=\frac{1}{2}\underset{M}{\int\cdots\int}\left(log\frac{f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1+a\vec{\sigma},\vec{w}',t)}{f_1f(\vec{r}_1+a\vec{\sigma},\vec{w},t)}\right)YW_nX_n^+f_1f(\vec{r}_1+a\vec{\sigma},\vec{w},t)Bd\mu\\
&+\frac{1}{2}\underset{M_1}{\int\cdots\int}\{\left[log\frac{f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)}{f_1f_2}\right]k_-\\
&+\left[log\frac{f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)}{f_1f_2}\right]k_+\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})\\
&+\left[log\frac{f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)}{f_1f_2}\right]k_+\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})\}\\
&\quad f_1f_2YW_nX_nR^2Bd\mu_1.
\end{aligned}
\tag{4.25}
$$

Let $h^+(f)=max\{h(f),0\}$, $h^-(f)=max\{-h(f),0\}$. Then $h(f)=h^+(f)-h^-(f)$. Thus $h(f)\leq h^+(f)$ and for any function F, $log\ F=log^+\ F-log^-\ F\leq log^+F$. Also notice that $z\cdot log\ \frac{y}{z}\leq y-z$ if $y,z\geq 0$. We have the following:

$$
\begin{aligned}
h(f)\quad&\leq\quad h^+(f)\\
&\leq\frac{1}{2}\underset{M}{\int\cdots\int}\left(log^+\frac{f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1+a\vec{\sigma},\vec{w}',t)}{f_1f(\vec{r}_1+a\vec{\sigma},\vec{w},t)}\right)YW_nX_n^+f_1f(\vec{r}_1+a\vec{\sigma},\vec{w},t)Bd\mu\\
&+\frac{1}{2}\underset{M_1}{\int\cdots\int}\{\left[log^+\frac{f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)}{f_1f_2}\right]k_-\\
&+\left[log^+\frac{f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)}{f_1f_2}\right]k_+\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})\\
&+\left[log^+\frac{f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)}{f_1f_2}\right]k_+\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})\}
\end{aligned}
$$

$$
\begin{aligned}
& f_1 f_2 Y W_n X_n R^2 B d\mu_1 \\
\leq & \frac{1}{2} \int_M \cdots \int [f(\vec{r}_1, \vec{v}', t) f(\vec{r}_1 + a\vec{\sigma}, \vec{w}', t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] Y W_n X_n^+ B d\mu \\
& + \frac{1}{2} \int_{M_1} \cdots \int \{[f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t) - f_1 f_2] k_- \\
& \quad + [f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t) - f_1 f_2] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \\
& \quad + [f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t) - f_1 f_2] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\} \\
& \quad Y W_n X_n R^2 B d\mu_1 \\
\leq & \frac{1}{2} \int_M \cdots \int [f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] Y B d\mu \\
& + \frac{1}{2} \int_{M_1} \cdots \int \{[f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t) - f_1 f_2] k_- \\
& \quad + [f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t) - f_1 f_2] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \\
& \quad + [f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t) - f_1 f_2] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\} \\
& \quad Y R^2 B d\mu_1 \\
& + \frac{1}{2} \int_M \cdots \int (1 - W_n X_n^+) f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) Y B d\mu \\
& + \frac{1}{2} \int_{M_1} \cdots \int (1 - W_n X_n) \{k_- + k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \\
& \quad + k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\} f_1 f_2 Y R^2 B d\mu_1 \\
= & \quad A_2 + A_3.
\end{aligned}
$$

Here A_2 represents the first two integrals on the right hand side of the last expression, and A_3 stands for the last two integrals, i.e.:

$$
\begin{aligned}
A_2 = & \frac{1}{2} \int_M \cdots \int [f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] Y B d\mu \\
& + \frac{1}{2} \int_{M_1} \cdots \int \{[f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t) - f_1 f_2] k_- \\
& \quad + [f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t) - f_1 f_2] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon}) \\
& \quad + [f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t) - f_1 f_2] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V})\} \\
& \quad Y R^2 B d\mu_1 \qquad (4.26)
\end{aligned}
$$

$$A_3 = \frac{1}{2}\int_M \cdots \int (1 - W_n X_n^+) f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) Y B d\mu$$
$$+ \frac{1}{2}\int_{M_1} \cdots \int (1 - W_n X_n)(k_- + k_+) f_1 f_2 Y R^2 B d\mu_1. \tag{4.27}$$

Therefore

$$\iint_{R^3 \times R^3} log f J_n(f,f) d\vec{r}_1 d\vec{v} \quad \leq \quad A_1 + A_2 + A_3. \tag{4.28}$$

Next, to control these three terms, we must find upper bounds for them. It is easy to get the bound for A_3 because of (4.12):

$$\int_0^t A_3 ds \leq \frac{Y}{2}\int_0^t \{ \int_M \cdots \int f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu$$
$$+ \int_{M_1} \cdots \int (k_+ + k_-) f_1 f_2 R^2 B d\mu_1 \} ds$$
$$\leq \frac{Y}{2\eta} \iint_{R^3 \times R^3} f_0 d\vec{r}_1 d\vec{v} \overset{newdef.}{==} C_3(T, f_0) \tag{4.29}$$

Applying (3.12) and (3.14) to A_2 yields

$$A_2 = \frac{1}{2}\int_M \cdots \int [f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] Y B d\mu$$
$$+ \frac{1}{2}\int_{M_1} \cdots \int \{ [f(\vec{r}_1, \vec{v}^+, t) f(\vec{r}_2, \vec{w}^+, t) - f_1 f_2] k_-$$
$$+ [f(\vec{r}_1, \vec{v}^-, t) f(\vec{r}_2, \vec{w}^-, t) - f_1 f_2] k_+ \theta(\vec{\sigma} \cdot \vec{V} - \sqrt{4\varepsilon})$$
$$+ [f(\vec{r}_1, \vec{v}^B, t) f(\vec{r}_2, \vec{w}^B, t) - f_1 f_2] k_+ \theta(\sqrt{4\varepsilon} - \vec{\sigma} \cdot \vec{V}) \} Y R^2 B d\mu_1$$
$$= \frac{1}{2}\int_M \cdots \int [f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) - f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)] Y B d\mu + 0$$

(Polewczak [33])

$$= \frac{1}{2} \iiiint_{R^3 \times R^3 \times R^3 \times S^2} f_1 f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) Y B d\mu$$
$$= -\frac{1}{2} Y a^2 \frac{d}{dt} \int_{R^3 \times B_a(\vec{r}_1)} \rho(\vec{r}_1, t) \rho(\vec{r}_2, t) d\vec{r}_1 d\vec{r}_2,$$

(Arkeryd & Cercignani [4])

where $B_a(\vec{r}_1) = \{\vec{r}_2 \epsilon R^3 \quad | \quad |\vec{r}_2 - \vec{r}_1| \leq a\}$.
This implies that

$$\int_0^t A_2 ds \quad \leq \quad \frac{1}{2} Y a^2 \left(\iint\limits_{R^3 \times R^3} f_0 d\vec{r}_1 d\vec{v} \right)^2 \overset{newdef.}{==} C_2(T, f_0). \tag{4.30}$$

In order to get the bound for A_1, we use inequality $-y \ log \ y \leq y\psi + e^{-(\psi+1)}$ if $y, \psi > 0$.

$$\begin{aligned}
A_1 = &-\frac{1}{2}\eta \int \cdots \int\limits_M log\left[f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)\right] f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&- \frac{1}{2}\eta \int \cdots \int\limits_{M_1} [log(f_1 f_2)](k_+ + k_-) f_1 f_2 R^2 B d\mu_1 \\
\leq &\frac{\eta}{2}\{ \int \cdots \int\limits_M [(\vec{v})^2 + (\vec{w})^2 + (\vec{r}_1)^2] f_1 f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\mu \\
&+ \int \cdots \int\limits_M exp(-[(\vec{v})^2 + (\vec{w})^2 + (\vec{r}_1)^2 + 1]) B d\mu\} \\
&+ \frac{\eta}{2}\{ \int \cdots \int\limits_{M_1} (\vec{v}^2 + \vec{w}^2 + (\vec{r}_1)^2 + (\vec{r}_2)^2)(k_+ + k_-) f_1 f_2 R^2 B d\mu_1 \\
&\int \cdots \int\limits_{M_1} exp(-[(\vec{v})^2 + (\vec{w})^2 + (\vec{r}_1)^2 + (\vec{r}_2)^2])(k_+ + k_-) R^2 B d\mu_1\}.
\end{aligned} \tag{4.31}$$

Notice that $B = a^2 \vec{\sigma} \cdot \vec{V}$, and $|B| \leq a^2(|\vec{v}| + |\vec{w}|) \leq a^2(1 + |\vec{v}|)(1 + |\vec{w}|)$. After fundamental calculations, we obtain that

$$\begin{aligned}
&\int \cdots \int\limits_M exp(-[\vec{v}^2 + \vec{w}^2 + (\vec{r}_1)^2 + 1]) B d\mu \\
&\leq \frac{a^2}{e} \left[\int\limits_{R^3} (1 + |\vec{v}|) e^{-\vec{v}^2} d\vec{v} \right]^2 \cdot \left(\int\limits_{R^3} e^{-\vec{r}^2} d\vec{r} \right) \cdot \left(\int\limits_{S_+^2} d\vec{\sigma} \right) \\
&= \frac{a^2}{e} \left[\int\limits_{R^3} (1 + |\vec{v}|) e^{-\vec{v}^2} d\vec{v} \right]^2 \cdot 2\pi^{3/2} \cdot 2\pi
\end{aligned}$$

$$= \frac{4\pi^{5/2}a^2}{e}\left[\int\limits_{R^3}(1+|\vec{v}|)e^{-\vec{v}^2}d\vec{v}\right]^2, \tag{4.32}$$

$$\int\limits_{M_1}\cdots\int exp(-[\vec{v}^2+\vec{w}^2+(\vec{r}_1)^2+(\vec{r}_2)^2+1])(k_+ + k_-)R^2Bd\mu_1$$

$$\leq \frac{R^2a^2}{e}\left[\int\limits_{R^3}(1+|\vec{v}|)e^{-\vec{v}^2}d\vec{v}\right]^2\cdot\iiint\limits_{R^3\times R^3\times S^2_+} e^{-[(\vec{r}_1)^2+(\vec{r}_2)^2]}(k_+ + k_-)d\vec{r}_1d\vec{r}_2d\vec{\sigma},$$

and

$$\iiint\limits_{R^3\times R^3\times S^2_+} e^{-[(\vec{r}_1)^2+(\vec{r}_2)^2]}(k_+ + k_-)d\vec{r}_1d\vec{r}_2d\vec{\sigma}$$

$$= \iint\limits_{R^3\times S^2_+} e^{-(\vec{r}_1)^2}\left[e^{-(\vec{r}_1+Ra\vec{\sigma})^2}+e^{-(\vec{r}_1-Ra\vec{\sigma})^2}\right]d\vec{r}_1d\vec{\sigma}$$

$$= \iint\limits_{R^3\times S^2_+} e^{-2(\vec{r}_1)^2}e^{-R^2a^2}\left[e^{-2Ra\vec{\sigma}\cdot\vec{r}_1}+e^{2Ra\vec{\sigma}\cdot\vec{r}_1}\right]d\vec{r}_1d\vec{\sigma}$$

$$= e^{-R^2a^2}\int\limits_{S^2}d\vec{\sigma}\int\limits_{R^3}e^{-2r^2}e^{2Ra\vec{\sigma}\cdot\vec{r}}d\vec{r}$$

$$= \frac{4\pi^2}{Ra}e^{-R^2a^2}\int\limits_0^\infty e^{-2x^2}x\ sh(2Rax)\ dx.$$

Consequently

$$\int\limits_{M_1}\cdots\int exp(-[\vec{v}^2+\vec{w}^2+(\vec{r}_1)^2+(\vec{r}_2)^2+1])(k_+ + k_-)R^2Bd\mu_1$$

$$\leq \frac{4\pi^2Ra}{e}e^{-R^2a^2}\left[\int\limits_{R^3}(1+|\vec{v}|)e^{-\vec{v}^2}d\vec{v}\right]^2\cdot\int\limits_0^\infty e^{-2x^2}x\ sh(2Rax)\ dx. \tag{4.33}$$

Referring to (4.15) and (4.18)-(4.20) and previous discussions, one has

$$\int\limits_0^t A_1ds$$

$$\leq 4\pi^{5/2}a^2e^{-1}T\left[\int\limits_{R^3}(1+|\vec{v}|)e^{-\vec{v}^2}d\vec{v}\right]^2$$

$$+ 4\pi^2 Rae^{-1}e^{-R^2a^2}T\left[\int\limits_{R^3}(1+|\vec{v}|)e^{-\vec{v}^2}d\vec{v}\right]^2\cdot\int\limits_0^\infty e^{-2x^2}x\ sh(2Rax)\ dx$$

$$+\iint\limits_{R^3\times R^3}\vec{v}^2 f_0 d\vec{r}_1 d\vec{v}+\frac{\varepsilon}{\eta}\iint\limits_{R^3\times R^3} f_0 d\vec{r}_1 d\vec{v}$$

$$+2\left(\iint\limits_{R^3\times R^3}(\vec{r}_1)^2 f_0 d\vec{r}_1 d\vec{v}+T^2Y\frac{\varepsilon}{\eta}\iint\limits_{R^3\times R^3} f_0 d\vec{r}_1 d\vec{v}\right)$$

$$\stackrel{newdef.}{==} C_1(T,f_0). \tag{4.34}$$

Let

$$H_n = \int f\ log\ f-\int\limits_0^t(A_1+A_2+A_3)ds. \tag{4.35}$$

Then we have the H-theorem for the truncated equations:

$$\frac{dH_n}{dt} \leq 0. \tag{4.36}$$

Thus

$$\int f\ log\ f \leq \int f_0\ log\ f_0+\int\limits_0^t(A_1+A_2+A_3)ds,$$

and

$$\int f\ log^+\ f \leq \int f_0\ log\ f_0+\int f\ log^-\ f+\int\limits_0^t(A_1+A_2+A_3)ds.$$

On the other hand,

$$\iint\limits_{R^3\times R^3} f\ log^-\ fd\vec{r}_1 d\vec{v}=-\iint\limits_{\{(\vec{r}_1,\vec{v})|0\leq f\leq 1\}} f\ log\ fd\vec{r}_1 d\vec{v}$$

$$\leq\iint\limits_{\{(\vec{r}_1,\vec{v})|0\leq f\leq 1\}}[((\vec{r}_1)^2+\vec{v}^2)f+e^{-((\vec{r}_1)^2+\vec{v}^2+1)}]d\vec{r}_1 d\vec{v}$$

$$\leq \iint\limits_{R^3\times R^3} [((\vec{r}_1)^2+\vec{v}^2)f + e^{-((\vec{r}_1)^2+\vec{v}^2+1)}] d\vec{r}_1 d\vec{v}$$

$$\leq \left[\ \iint\limits_{R^3\times R^3} \vec{v}^2 f_0 d\vec{r}_1 d\vec{v}\right] + \left[\ \iint\limits_{R^3\times R^3} (\vec{r}_1)^2 f_0 d\vec{r}_1 d\vec{v} + T^2 Y \frac{\varepsilon}{\eta} \iint\limits_{R^3\times R^3} f_0 d\vec{r}_1 d\vec{v}\right] + 4\pi^3. \tag{4.37}$$

Finally we have that:

$$\iint\limits_{R^3\times R^3} f\ |log\ f|\ d\vec{r}_1 d\vec{v}$$

$$\leq \iint\limits_{R^3\times R^3} f_0\ log\ f_0\ d\vec{r}_1 d\vec{v} + 2 \iint\limits_{R^3\times R^3} f\ log^-\ f\ d\vec{r}_1 d\vec{v} + \int_0^t (A_1+A_2+A_3) ds$$

$$\leq \iint\limits_{R^3\times R^3} f_0\ log\ f_0\ d\vec{r}_1 d\vec{v} + 2\left[\ \iint\limits_{R^3\times R^3} \vec{v}^2 f_0 d\vec{r}_1 d\vec{v}\right] + 2\left[\ \iint\limits_{R^3\times R^3} (\vec{r}_1)^2 f_0 d\vec{r}_1 d\vec{v}\right]$$

$$+ 2T^2 Y \frac{\varepsilon}{\eta} \iint\limits_{R^3\times R^3} f_0 d\vec{r}_1 d\vec{v} \quad + \quad 8\pi^3$$

$$+ C_1(T, f_0)$$

$$+ \frac{1}{2} Y a^2 \left(\ \iint\limits_{R^3\times R^3} f_0\ d\vec{r}_1 d\vec{v}\right)^2 + \frac{Y}{2\eta}\left(\ \iint\limits_{R^3\times R^3} f_0\ d\vec{r}_1 d\vec{v}\right)$$

$$\overset{newdef.}{==} C(T, f_0), \tag{4.38}$$

where $C(T, f_0)$ is a constant depending on T and f_0.

§5. Gain-loss estimations

In order to get a gain-loss estimation, let us look back to (4.23). After making some variable changes, we have

$$
\begin{aligned}
&\iint\limits_{R^3\times R^3} J_n(f,f) log f d\vec{r}_1 d\vec{v}\\
&=\frac{1}{2}\int\cdots\int\limits_{M}\left[log\frac{f(\vec{r}_1,\vec{v},t)f(\vec{r}_1-a\vec{\sigma},\vec{w},t)}{f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1-a\vec{\sigma},\vec{w}',t)}\right]f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1-a\vec{\sigma},\vec{w}',t)YW_nX_n^- Bd\mu\\
&\quad-\frac{1}{2}\eta\int\cdots\int\limits_{M}[log(f_1f(\vec{r}_1+a\vec{\sigma},\vec{w},t))]f_1f(\vec{r}_1+a\vec{\sigma},\vec{w},t)Bd\mu\\
&\quad+\frac{1}{2}\int\cdots\int\limits_{M_1}\{\left[log\frac{f_1f_2}{f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)}\right]f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)k_+YW_n^+X_n\\
&\quad+\left[log\frac{f_1f_2}{f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)}\right]f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})YW_n^-X_n\\
&\quad+\left[log\frac{f_1f_2}{f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)}\right]f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)k_-\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})YW_nX_n\\
&\quad\}R^2Bd\mu_1-\frac{1}{2}\eta\int\cdots\int\limits_{M_1}[log(f_1f_2)]f_1f_2(k_++k_-)R^2Bd\mu_1\\
&=-\frac{1}{2}\int\cdots\int\limits_{M}\left[log\frac{f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1-a\vec{\sigma},\vec{w}',t)}{f(\vec{r}_1,\vec{v},t)f(\vec{r}_1-a\vec{\sigma},\vec{w},t)}\right]\times\\
&\quad f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1-a\vec{\sigma},\vec{w}',t)YW_nX_n^-Bd\mu\\
&\quad-\frac{1}{2}\int\cdots\int\limits_{M_1}\{\left[log\frac{f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)}{f_1f_2}\right]f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)k_+YW_n^+X_n\\
&\quad+\left[log\frac{f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)}{f_1f_2}\right]f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})YW_n^-X_n\\
&\quad+\left[log\frac{f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)}{f_1f_2}\right]f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)k_-\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})YW_nX_n\\
&\quad\}R^2Bd\mu_1\quad+\quad A_1\\
&=\quad A_1+h(f),
\end{aligned}
\tag{5.1}
$$

where A_1 is defined in (4.24).

Recalling the function $l(x)$ defined by (3.9),

$$
l(x)=log x+\frac{1}{x}-1\quad for\quad x>1,
$$

we have

$$l\left(\frac{a}{b}\right) = log\frac{a}{b} + \left(\frac{b}{a} - 1\right),$$

$$-alog\frac{a}{b} = -al\left(\frac{a}{b}\right) + (b - a).$$

Thus (5.1) can be written as

$$\begin{aligned}
&\iint\limits_{R^3\times R^3} J_n(f,f) log f d\vec{r}_1 d\vec{v} \\
&= -\frac{1}{2}\int\limits_{M}\cdots\int l\left(\frac{f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1 - a\vec{\sigma},\vec{w}',t)}{f(\vec{r}_1,\vec{v},t)f(\vec{r}_1 - a\vec{\sigma},\vec{w},t)}\right)\times \\
&\quad f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1 - a\vec{\sigma},\vec{w}',t)YW_nX_n^- Bd\mu \\
&\quad -\frac{1}{2}\int\limits_{M_1}\cdots\int \{l\left(\frac{f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)}{f_1f_2}\right) f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)k_+YW_n^+X_n \\
&\quad + l\left(\frac{f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)}{f_1f_2}\right) f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)k_-\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})YW_n^-X_n \\
&\quad + l\left(\frac{f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)}{f_1f_2}\right) f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)k_-\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V})YW_nX_n \\
&\quad \}R^2Bd\mu_1 \quad + \quad A_1 \\
&\quad +\frac{1}{2}\int\limits_{M}\cdots\int [f_1f(\vec{r}_1 - a\vec{\sigma},\vec{w},t) - f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1 - a\vec{\sigma},\vec{w}',t)]YW_nX_n^- Bd\mu \\
&\quad +\frac{1}{2}\int\limits_{M_1}\cdots\int \{[f_1f_2 - f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)]k_+YW_n^+X_n \\
&\quad + [f_1f_2 - f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)]k_-\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})YW_n^-X_n \\
&\quad + [f_1f_2 - f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)]k_-YW_nX_n\}R^2Bd\mu_1. \qquad (5.2)
\end{aligned}$$

By making appropriate variable changes, we have following two equations for the last two terms on the right hand side of (5.2):

$$\begin{aligned}
&\int\limits_{M}\cdots\int [f_1\ f(\vec{r}_1 - a\vec{\sigma},\vec{w},t) - f(\vec{r}_1,\vec{v}',t)\ f(\vec{r}_1 - a\vec{\sigma},\vec{w}',t)]YW_nX_n^- Bd\mu \\
&= \int\limits_{M}\cdots\int f_1[f(\vec{r}_1 - a\vec{\sigma},\vec{w},t)X_n^- - f(\vec{r}_1 + a\vec{\sigma},\vec{w},t)X_n^+]YW_nBd\mu; \qquad (5.3)
\end{aligned}$$

$$\int_{M_1}\cdots\int\{[f_1f_2-f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)]k_+YW_n^+X_n$$
$$+[f_1f_2-f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)]k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})YW_n^-X_n$$
$$+[f_1f_2-f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)]k_-\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})YW_nX_n\}R^2Bd\mu_1$$
$$=\int_{M_1}\cdots\int[k_+(W_n-W_n^+)-k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})(W_n^- -W_n)]f_1f_2YX_nR^2Bd\mu_1. \tag{5.4}$$

Therefore (5.2) becomes

$$\iint_{R^3\times R^3} J_n(f,f)log f d\vec{r}_1d\vec{v}$$
$$= \quad A_1+h(f)$$
$$=-\frac{1}{2}\int_M\cdots\int l\left(\frac{f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1-a\vec{\sigma},\vec{w}',t)}{f(\vec{r}_1,\vec{v},t)f(\vec{r}_1-a\vec{\sigma},\vec{w},t)}\right)\times$$
$$f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1-a\vec{\sigma},\vec{w}',t)YW_nX_n^-Bd\mu$$
$$-\frac{1}{2}\int_{M_1}\cdots\int\{l\left(\frac{f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)}{f_1f_2}\right)f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)k_+YW_n^+X_n$$
$$+l\left(\frac{f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)}{f_1f_2}\right)f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})YW_n^-X_n$$
$$+l\left(\frac{f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)}{f_1f_2}\right)f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)k_-\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})YW_nX_n$$
$$\}R^2Bd\mu_1 \quad + \quad A_1$$
$$+\int_M\cdots\int f_1[f(\vec{r}_1-a\vec{\sigma},\vec{w},t)X_n^- - f(\vec{r}_1+a\vec{\sigma},\vec{w},t)X_n^+]YW_nBd\mu$$
$$+\int_{M_1}\cdots\int[k_+(W_n-W_n^+)-k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})(W_n^- -W_n)]f_1f_2YX_nR^2Bd\mu_1. \tag{5.5}$$

For each $j>1$, any $O\subset \mathbb{R}^3\times\mathbb{R}^3$, one has

$$\int_{O\cap(R^3\times R^3)} J_n^+(f,f)d\vec{r}_1d\vec{v}$$
$$\overset{def.}{=\!=}\int_{O\cap M} f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1-a\vec{\sigma},\vec{w}',t)YW_nX_n^-Bd\mu$$

$$+\int\limits_{O\cap M_1}\{[f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)k_+YW_n^+X_n]$$
$$+[f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})YW_n^-X_n]$$
$$+[f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)k_-\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})YW_nX_n]\}R^2Bd\mu_1$$

$$\leq j\int\limits_{O\cap M} f(\vec{r}_1,\vec{v},t)f(\vec{r}_1-a\vec{\sigma},\vec{w},t)YW_nX_n^-Bd\mu$$
$$+\frac{1}{l(j)}\int\cdots\int\limits_{M} l\left(\frac{f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1-a\vec{\sigma},\vec{w}',t)}{f(\vec{r}_1,\vec{v},t)f(\vec{r}_1-a\vec{\sigma},\vec{w},t)}\right)\times$$
$$f(\vec{r}_1,\vec{v}',t)f(\vec{r}_1-a\vec{\sigma},\vec{w}',t)YW_nX_n^-R^2Bd\mu_1$$
$$+j\int\limits_{O\cap M_1} f_1f_2k_+YW_n^+X_nR^2Bd\mu_1$$
$$+\frac{1}{l(j)}\int\cdots\int\limits_{M_1} l\left(\frac{f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)}{f_1f_2}\right)f(\vec{r}_1,\vec{v}^+,t)f(\vec{r}_2,\vec{w}^+,t)\times$$
$$k_+YW_n^+X_nR^2Bd\mu_1$$
$$+j\int\limits_{O\cap M_1} f_1f_2k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})YW_n^-X_nR^2Bd\mu_1$$
$$+\frac{1}{l(j)}\int\cdots\int\limits_{M_1} l\left(\frac{f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)}{f_1f_2}\right)f(\vec{r}_1,\vec{v}^-,t)f(\vec{r}_2,\vec{w}^-,t)\times$$
$$k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})YW_n^-X_nR^2Bd\mu_1$$
$$+j\int\limits_{O\cap M_1} f_1f_2k_-\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})YW_nX_nR^2Bd\mu_1$$
$$+\frac{1}{l(j)}\int\cdots\int\limits_{M_1} l\left(\frac{f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)}{f_1f_2}\right)f(\vec{r}_1,\vec{v}^B,t)f(\vec{r}_2,\vec{w}^B,t)\times$$
$$k_-\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})YW_nX_nR^2Bd\mu_1$$

$$=j\int\limits_{O\cap M} f(\vec{r}_1,\vec{v},t)f(\vec{r}_1-a\vec{\sigma},\vec{w},t)YW_nX_n^-Bd\mu$$
$$+j\int\limits_{O\cap M_1}\{k_+W_n^++k_-\theta(\vec{\sigma}\cdot\vec{V}-\sqrt{4\varepsilon})W_-+k_-\theta(\sqrt{4\varepsilon}-\vec{\sigma}\cdot\vec{V})W_n\}$$

$$
\begin{aligned}
&f_1 f_2 Y X_n R^2 B d\mu_1 \\
&+\frac{1}{l(j)} \int_M \cdots \int l\left(\frac{f(\vec{r}_1,\vec{v}',t) f(\vec{r}_1 - a\vec{\sigma},\vec{w}',t)}{f(\vec{r}_1,\vec{v},t) f(\vec{r}_1 - a\vec{\sigma},\vec{w},t)}\right) \times \\
&f(\vec{r}_1,\vec{v}',t) f(\vec{r}_1 - a\vec{\sigma},\vec{w}',t) Y W_n X_n^- R^2 B d\mu_1 \\
&+\frac{1}{l(j)} \int_{M_1} \cdots \int \{ l\left(\frac{f(\vec{r}_1,\vec{v}^+,t) f(\vec{r}_2,\vec{w}^+,t)}{f_1 f_2}\right) f(\vec{r}_1,\vec{v}^+,t) f(\vec{r}_2,\vec{w}^+,t) k_+ Y W_n^+ X_n \\
&+ l\left(\frac{f(\vec{r}_1,\vec{v}^-,t) f(\vec{r}_2,\vec{w}^-,t)}{f_1 f_2}\right) f(\vec{r}_1,\vec{v}^-,t) f(\vec{r}_2,\vec{w}^-,t) k_- \theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon}) Y W_n^- X_n \\
&+ l\left(\frac{f(\vec{r}_1,\vec{v}^B,t) f(\vec{r}_2,\vec{w}^B,t)}{f_1 f_2}\right) f(\vec{r}_1,\vec{v}^B,t) f(\vec{r}_2,\vec{w}^B,t) k_- \theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V}) Y W_n X_n \\
&\} R^2 B d\mu_1
\end{aligned}
$$

$$
\begin{aligned}
\le\ & j \int_{O\cap M} f(\vec{r}_1,\vec{v},t) f(\vec{r}_1 - a\vec{\sigma},\vec{w},t) Y W_n X_n^- B d\mu \\
&+ j \int_{O\cap M_1} \{ k_+ W_n^+ + k_- \theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon}) W_- + k_- \theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V}) W_n \} \\
&f_1 f_2 Y X_n R^2 B d\mu_1 \\
&+ \frac{1}{l(j)} \{ -2h(f) + \\
&+ 2 \int_M \cdots \int f_1 [f(\vec{r}_1 - a\vec{\sigma},\vec{w},t) X_n^- - f(\vec{r}_1 + a\vec{\sigma},\vec{w},t) X_n^+] Y W_n B d\mu \\
&+ 2 \int_{M_1} \cdots \int [k_+ (W_n - W_n^+) - k_- \theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})(W_n^- - W_n)] f_1 f_2 Y X_n R^2 B d\mu_1 \} \\
=\ & j \int_0^t ds \int_{O\cap(R^3\times R^3)} J_n^-(f,f) d\vec{r}_1 d\vec{v} + \frac{1}{l(j)} \{ -2h(f) + \\
&+ 2 \int_M \cdots \int f_1 [f(\vec{r}_1 - a\vec{\sigma},\vec{w},t) X_n^- - f(\vec{r}_1 + a\vec{\sigma},\vec{w},t) X_n^+] Y W_n B d\mu \\
&+ 2 \int_{M_1} \cdots \int [k_+ (W_n - W_n^+) - k_- \theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon})(W_n^- - W_n)] \\
&\times f_1 f_2 Y X_n R^2 B d\mu_1 \}. \qquad (5.6)
\end{aligned}
$$

As discussed in Section 4, one has $-h(f) = h^-(f) - h^+(f) \le h^-(f) \le A_2 + A_3$, and thus,

$$\int_0^t |h(f)|ds \le 2\int_0^t A_2 ds + 2\int_0^t A_3 ds. \tag{5.7}$$

However, the sum of the last two terms in (5.6) is less than $\frac{4}{\eta}\int f_0 d\vec{r}_1 d\vec{v}$. Consequently we obtain

$$\begin{aligned}
&\int_0^t ds \int_{O\cap(R^3 \times R^3)} J_n^+(f,f) d\vec{r}_1 d\vec{v} \\
&\le j\int_0^t ds \int_{O\cap(R^3 \times R^3)} J_n^-(f,f) d\vec{r}_1 d\vec{v} \\
&\quad + \frac{1}{l(j)}\{4\left[\int_0^t A_2 ds + \int_0^t A_3 ds\right] + \frac{4}{\eta}\iint_{R^3 \times R^3} f_0 d\vec{r}_1 d\vec{v}\} \\
&\le j\int_0^t ds \int_{O\cap(R^3 \times R^3)} J_n^-(f,f) d\vec{r}_1 d\vec{v} \\
&\quad + \frac{1}{l(j)}\{4[\ C_2(T,f_0) + C_3(T,f_0)] + \frac{4}{\eta}\iint_{R^3 \times R^3} f_0 d\vec{r}_1 d\vec{v}\}.
\end{aligned} \tag{5.8}$$

§6. Solution of the truncated equation

Let us consider the existence of solutions to the following equations:

$$\begin{cases} (\partial_t + \vec{v} \cdot \nabla_{\vec{r}})f = J_n(f,f), \\ f(\vec{r}, \vec{v}, t=0) = f_0(\vec{r}, \vec{v}), \\ J_n(f,f) = \sum_{i=1}^4 C_{in}(f,f). \end{cases} \tag{6.1}$$

$J_n(f,f)$ in (6.1) can be written as:

$$\begin{aligned} J_n(f,f) &= J_n^+(f,f)^\# - J_n^-(f,f)^\# \\ &= \sum_{i=1}^4 G_{in}(f,f)^\# - f \sum_{i=1}^4 L_{in}(f)^\#. \end{aligned} \tag{6.2}$$

Let us define

$$L_n(f) \overset{newdef}{=\!=} \sum_{i=1}^4 L_{in}(f). \tag{6.3}$$

Then the first equation of (6.1) becomes

$$D_t f^\# = J_n^+(f,f)^\# - f L_n(f)^\#. \tag{6.4}$$

Taking the integration with respect to t on both sides of (6.4) under the initial condition $f(\vec{r}, \vec{v}, t=0) = f_0(\vec{r}, \vec{v})$, we have an equivalent integral equation:

$$(f)^\#(t) = f_0 + \int_0^t [J_n^+(f,f)^\#(s) - (f)^\# L_n(f)^\#] ds. \tag{6.5.1}$$

On the other hand, if we rewrite (6.4) as

$$D_t f^\# + f L_n(f)^\# = J_n^+(f,f)^\#,$$

and take the integration with respect to t, we get the following equivalent equation of (6.5.1):

$$(f)^\#(t) = f_0 e^{-\int_0^t L_n^\#(f) d\xi} + \int_0^t e^{-\int_s^t L_n^\#(f) d\xi} J_n^+(f,f)^\# ds. \tag{6.5.2}$$

This gives us a motivation to set the iteration scheme:

$$(f^{j+1})^\#(t) = f_0 e^{-\int_0^t L_n^\#(f^j) d\xi} + \int_0^t e^{-\int_s^t L_n^\#(f^j) d\xi} J_n^+(f^j, f^j)^\# ds, \tag{6.6.1}$$

which is equivalent to the following:

$$(f^{j+1})^{\#}(t) = f_0 + \int_0^t [J_n^+(f^j, f^j)^{\#}(s) - (f^{j+1})^{\#} L_n(f^j)^{\#}] ds. \tag{6.6.2}$$

It is easy to see from (6.6.1) that f^j is nonnegative if we assume that $f_0 \geq 0$.

Next we need to prove that $\{(f^j)^{\#}\}_{j \in \mathbf{N}}$ is a Cauchy sequence. Notice that for the transport equation of free particles

$$\begin{cases} (\partial_t + \vec{v} \cdot \nabla_{\vec{r}}) f = 0 \\ f(\vec{r}, \vec{v}, t = 0) = f_0(\vec{r}, \vec{v}), \end{cases} \tag{6.7}$$

the solution is

$$f^{\#}(\vec{r}, \vec{v}, t) = f_0(\vec{r}, \vec{v}), \tag{6.8.1}$$

i.e.,

$$f(\vec{r}, \vec{v}, t) = f_0(\vec{r} - t\vec{v}, \vec{v}) = e^{-At} f_0(\vec{r}, \vec{v}) = U(t) f_0(\vec{r}, \vec{v}), \tag{6.8.2}$$

where $A \stackrel{def.}{==} \vec{v} \cdot \nabla$ and $U(t) \stackrel{def.}{==} e^{-At}$ is the strongly continuous semi-group generated by A.

We will apply Arkeryd & Cercignani's splitting method (see [3],[13],[4]) to our discussion. Let us introduce some lemmas first. Without loss of generality, we set $Y \equiv 1$ in the following. If $Y = constant \neq 1$, we can adjust appropriate control constants. We will explain more about this later on. Define

$$\|f\|_{T'} = \iint_{R^3 \times R^3} \sup_{0 \leq t \leq T'} |f^{\#}(\vec{r}, \vec{v}, t)| d\vec{r} d\vec{v}, \tag{6.9}$$

$$f_i^0(\vec{r}, \vec{v}) = min(f_0(\vec{r}, \vec{v}), \omega) = f_0(\vec{r}, \vec{v}) \chi(\frac{f_0}{\omega}), \quad \omega > n, \tag{6.10}$$

$$f_i^{\#}(\vec{r}, \vec{v}, t) = f_i^0(\vec{r}, \vec{v}). \tag{6.11}$$

Without any further comment we will write $\|f\|_{T'}$ as $\|f\|$. We will split each function f^j into two terms, the free particle term and the collision term:

$$(f^j)^{\#}(\vec{r}, \vec{v}, t) = (f_i)^{\#}(\vec{r}, \vec{v}, t) + (f_e^j)^{\#}(\vec{r}, \vec{v}, t) = f_i^0(\vec{r}, \vec{v}) + f_e^j(\vec{r} + t\vec{v}, \vec{v}, t). \tag{6.12}$$

Lemma 6.1([3],[13]): If $|\vec{v} - \vec{w}| \leq n$, then

$$\left| \int_0^{T'} \iint_{R^3 \times S_+^2} F(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}) B W_n d\vec{w} d\vec{\sigma} \right| \leq \varphi(\delta, F), \tag{6.13}$$

with $\delta = a^2\pi nT'$, where $\varphi(\delta, F)$ is defined by

$$\varphi(\delta, F) = \sup_{M(\delta)} \int\limits_M |F(\vec{x}, \vec{v})| d\vec{x} d\vec{v}, \tag{6.14}$$

$M_{\vec{v}} = \{\vec{x} \mid (\vec{x}, \vec{v}) \in M \subseteq \mathbb{R}^6\}$, and $M(\delta) = \{\ M \subseteq \mathbb{R}^6 \mid$ for almost every $\vec{v} \in \mathbb{R}^3$, measure of $M_{\vec{v}} < \delta\ \}$.

Lemma 6.2:

$$\begin{aligned}
&\int\limits_0^{T'} ds \iint\limits_{R^3 \times R^3} \iint\limits_{R^3 \times S^2_+} (f_i)^{\#}(\vec{r}_1, \vec{v}, s)(f_i)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, s) \\
&\qquad W_n X_n^+ B d\vec{r}_1 d\vec{v} d\vec{w} d\vec{\sigma} \\
&\leq \|f_i^0\| \cdot \varphi(\pi a^2 2nT', f_i^0)
\end{aligned} \tag{6.15}$$

Proof: For the left-hand side (LHS) of (6.15),

$$\begin{aligned}
LHS \leq &\iint\limits_{R^3 \times R^3} d\vec{r}_1 d\vec{v} \sup_{0 \leq s \leq T'} |(f_i)^{\#}(\vec{r}_1, \vec{v}, s)| \times \\
&\int\limits_0^{T'} ds \iint\limits_{R^3 \times S^2_+} (f_i)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, s) W_n X_n^+ B d\vec{w} d\vec{\sigma}
\end{aligned}$$

Viewing $\vec{r}_1, \vec{v}, \vec{w}$ as fixed, the variable change

$$\vec{y} = \vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w})$$

leads to the Jacobian $d\vec{y} = a^2\vec{\sigma} \cdot (\vec{v} - \vec{w}) ds d\sigma = B ds d\vec{\sigma}$. Now

$$\begin{aligned}
LHS \leq &\iint\limits_{R^3 \times R^3} d\vec{r}_1 d\vec{v} \sup_{0 \leq s \leq T'} |(f_i)^{\#}(\vec{r}_1, \vec{v}, s)| \times \\
&\iint\limits_{\mathbb{R} \times M_{\vec{y}}} (f_i)^{\#}(\vec{y}, \vec{w}, s) d\vec{w} d\vec{y},
\end{aligned}$$

with measure $\{M_{\vec{y}}\} \leq \pi a^2 2nT'$ and $(f_i)^{\#}(\vec{y}, \vec{w}, s) = f_i^0(\vec{y}, \vec{w})$. Thus

$$\begin{aligned}
LHS &\leq \iint\limits_{R^3 \times R^3} \sup_{0 \leq s \leq T'} |(f_i)^{\#}(\vec{r}_1, \vec{v}, t)| \cdot \varphi(\delta, f_i^0) d\vec{r}_1 d\vec{v} \\
&= \|f_i^0\| \cdot \varphi(\delta, f_i^0),
\end{aligned}$$

with $\delta = \pi a^2 2nT'$. The proof is completed.

Lemma 6.3:

$$\int_0^{T'} ds \iint_{R^3\times R^3} \iint_{R^3\times S^2_+} (f_i)^{\#}(\vec{r}_1, \vec{v}, s)\times$$
$$(f_e^j)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, s) W_n B d\vec{r}_1 d\vec{v} d\vec{w} d\vec{\sigma}$$
$$\leq \|f_e^j\| \cdot \varphi(\pi a^2 2nT', f_i^0). \tag{6.16}$$

Proof: Let $\vec{r}_2 = \vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w})$, and view $\vec{v}, \vec{w}, \vec{\sigma}$ as fixed. Then $\vec{r}_1 = \vec{r}_2 - a\vec{\sigma} - s(\vec{v} - \vec{w})$, and $d\vec{r}_2 = d\vec{r}_1$. We have

$$LHS = \int_0^{T'} ds \iint_{R^3\times R^3} (f_i)^{\#}(\vec{r}_2 - a\vec{\sigma} - s(\vec{v} - \vec{w}), \vec{v}, s)$$
$$\times \iint_{R^3\times S^2_+} (f_e^j)^{\#}(\vec{r}_2, \vec{w}, s) W_n B d\vec{r}_2 d\vec{v} d\vec{w} d\vec{\sigma}.$$

Take the integration with respect to $\vec{r}_2$ and then change variable in the remained integral: $\vec{y} = \vec{r}_2 - a\vec{\sigma} - s(\vec{v} - \vec{w})$. $d\vec{y} = a^2\vec{\sigma} \cdot (\vec{v} - \vec{w}) ds d\vec{\sigma} = B ds d\vec{\sigma}$. Thus

$$LHS \leq \|f_e^j\| \cdot \varphi(\delta, f_i^0),$$

with $\delta = \pi a^2 2nT'$.

Similar to Lemma 6.3, one has the following two lemmas:

Lemma 6.4:

$$\int_0^{T'} ds \iint_{R^3\times R^3} \iint_{R^3\times S^2_+} (f_i)^{\#}(\vec{r}_1, \vec{v}, s)\times$$
$$(f_e^j)^{\#}(\vec{r}_1 - a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, s) W_n B d\vec{r}_1 d\vec{v} d\vec{w} d\vec{\sigma}$$
$$\leq \|f_e^j\| \cdot \varphi(\pi a^2 2nT', f_i^0). \tag{6.17}$$

Lemma 6.5:

$$\int_0^{T'} ds \iint_{R^3\times R^3} \iint_{R^3\times S^2_+} (f_e^{j+1})^{\#}(\vec{r}_1, \vec{v}, s)$$

$$
\begin{aligned}
&\times (f_e^j)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v}-\vec{w}), \vec{w}, s) W_n X_n B d\vec{r}_1 d\vec{v} d\vec{w} d\vec{\sigma} \\
&\leq \|f_e^{j+1}\| \cdot \|f_e^j\|.
\end{aligned} \tag{6.18}
$$

The proofs are straightforward.

In order to prove $\{(f^j)^{\#}\}_{j\in\mathbb{N}}$ to be a Cauchy sequence, we split $(f^j)^{\#}$ into two parts as in (6.12), $(f_i)^{\#} + (f_e^j)^{\#}$ for each j:

$$
(f^j)^{\#} = (f_i)^{\#} + (f_e^j)^{\#} = f_i^0 + (f_e^j)^{\#}.
$$

(6.6.2) can be rewritten in the following form:

$$
\begin{aligned}
(f_e^{j+1})^{\#}(t) = f_e^0 + \int_0^t &\{J_n^+((f_i)^{\#} + (f_e^j)^{\#}, (f_i)^{\#} + (f_e^j)^{\#}) \\
&- ((f_i)^{\#} + (f_e^{j+1})^{\#}) L_n((f_i)^{\#} + (f_e^j)^{\#})\} ds
\end{aligned} \tag{6.19}
$$

Writing each term of (6.19) in detail, we have

$$
\begin{aligned}
&\int_0^t [(f_i)^{\#} + (f_e^{j+1})^{\#}] L_n((f_i)^{\#} + (f_e^j)^{\#}) ds \\
&= \int_0^t ds \iint_{R^3 \times S_+^2} (W_n X_n^+ + \eta)[(f_i)^{\#}(f_i)^{\#} + \\
&\quad (f_i)^{\#}(f_e^j)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v}-\vec{w}), \vec{w}, s) + (f_i)^{\#}(f_e^{j+1})^{\#}(\vec{r}_1 + s(\vec{v}-\vec{w}), \vec{w}, s) \\
&\quad + (f_e^{j+1})^{\#}(\vec{r}_1 + s(\vec{v}-\vec{w}), \vec{w}, s)(f_e^j)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v}-\vec{w}), \vec{w}, s)] B d\vec{w} d\vec{\sigma} \\
&\quad + \int_0^t ds \iiint_{R^3 \times R^3 \times S_+^2} (k_+ + k_-)(W_n X_n + \eta) \\
&\quad \times [(f_i)^{\#}(\vec{r}_1, \vec{v}) \cdot (f_i)^{\#}(\vec{r}_2 + s(\vec{v}-\vec{w}), \vec{w}) + (f_i)^{\#}(\vec{r}_1, \vec{v}) \cdot (f_e^j)^{\#}(\vec{r}_2 + s(\vec{v}-\vec{w}), \vec{w}, s) \\
&\quad + (f_e^{j+1})^{\#}(\vec{r}_1, \vec{v}, s) \cdot (f_i)^{\#}(\vec{r}_2 + s(\vec{v}-\vec{w}), \vec{w}, s) \\
&\quad + (f_e^{j+1})^{\#}(\vec{r}_1, \vec{v}, s) \cdot (f_e^j)^{\#}(\vec{r}_2 + s(\vec{v}-\vec{w}), \vec{w}, s)] \\
&\quad R^2 B d\vec{r}_1 d\vec{r}_2 d\vec{v} d\vec{w} d\vec{\sigma},
\end{aligned} \tag{6.20}
$$

and

$$
\int_0^t ds J_n^+((f_i)^{\#} + (f_e^j)^{\#}, (f_i)^{\#} + (f_e^j)^{\#})
$$

$$
\begin{aligned}
&= \int_0^t ds \iint\limits_{R^3\times S_+^2} [(f_i)^\# + (f_e^j)^\#(\vec{r}_1 + s(\vec{v}-\vec{v}'), \vec{v}', s)] \\
&\quad \times [(f_i)^\# + (f_e^j)^\#(\vec{r}_1 - a\vec{\sigma} + s(\vec{v}-\vec{w}'), \vec{w}', s)] W_n X_n^- B d\vec{w} d\vec{\sigma} \\
&\quad + \int_0^t ds \iiint\limits_{R^3\times R^3\times S_+^2} [(f_i)^\# + (f_e^j)^\#(\vec{r}_1 + s(\vec{v}-\vec{v}^+), \vec{v}^+, s)] \\
&\qquad \times [(f_i)^\# + (f_e^j)^\#(\vec{r}_2 - s\vec{w}^+, \vec{w}^+, s)] k_+ W_n^+ X_n R^2 B d\vec{r}_2 d\vec{w} d\vec{\sigma} \\
&\quad + \int_0^t ds \iiint\limits_{R^3\times R^3\times S_+^2} [(f_i)^\# + (f_e^j)^\#(\vec{r}_1 + s(\vec{v}-\vec{v}^-), \vec{v}^-, s)] \\
&\qquad \times [(f_i)^\# + (f_e^j)^\#(\vec{r}_2 - s\vec{w}^-, \vec{w}^-, s)] \\
&\qquad \times k_-\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon}) W_n^- X_n R^2 B d\vec{r}_2 d\vec{w} d\vec{\sigma} \\
&\quad + \int_0^t ds \iiint\limits_{R^3\times R^3\times S_+^2} [(f_i)^\# + (f_e^j)^\#(\vec{r}_1 + s(\vec{v}-\vec{v}^B), \vec{v}^B, s)] \\
&\qquad \times [(f_i)^\# + (f_e^j)^\#(\vec{r}_2 - s\vec{w}^B, \vec{w}^B, s)] \\
&\qquad \times k_-\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V}) W_n X_n R^2 B d\vec{r}_2 d\vec{w} d\vec{\sigma}. \qquad (6.21)
\end{aligned}
$$

Let us work out some of terms completely. The Enskog loss term is

$$
L_{1n}(f) = \iint\limits_{R^3\times S_+^2} [W_n X_n + \eta] f(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) B d\vec{w} d\vec{\sigma},
$$

and thus

$$
\begin{aligned}
(L_{1n})^\#(f)(\vec{r}_1, \vec{v}, s) &= \iint\limits_{R^3\times S_+^2} [W_n X_n + \eta] f(\vec{r}_1 + a\vec{\sigma} + s\vec{v}, \vec{w}, s) B\vec{w} d\vec{\sigma} \\
&= \iint\limits_{R^3\times S_+^2} [W_n X_n + \eta] f^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v}-\vec{w}), \vec{w}, s) B\vec{w} d\vec{\sigma}.
\end{aligned}
$$

Multiplying the function $(f_e^{j+1})^\#$ and taking the splitting of (6.12), we have

$$
\begin{aligned}
&(f_e^{j+1})^\#(\vec{r}_1, \vec{v}, s)(L_{1n})^\#(f^j)(\vec{r}_1, \vec{v}, s) \\
&= (f_e^{j+1})^\#(\vec{r}_1, \vec{v}, s) \iint\limits_{R^3\times S_+^2} [W_n X_n^+ + \eta](f^j)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v}-\vec{w}), \vec{w}, s) B d\vec{w} d\vec{\sigma}
\end{aligned}
$$

$$= [(f_i)^\#(\vec{r}_1, \vec{v}, s) + (f_e^{j+1})^\#(\vec{r}_1, \vec{v}, s)] \cdot \iint\limits_{R^3 \times S^2_+} [W_n X_n^+ + \eta]$$

$$\times [(f_i)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, s) + (f_e^j)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, s)] B d\vec{w} d\vec{\sigma}.$$

By the Lemmas 6.1 — 6.5, we get

$$\begin{aligned}
&\| \int_0^t (f_e^{j+1})^\# (\mathrm{L}_{1n})^\# (f^j) ds \| \\
&\le \int_0^{T'} \iint\limits_{R^3 \times R^3} d\vec{r}_1 d\vec{v} \sup_{0 \le s \le T'} \iint\limits_{R^3 \times S^2_+} [W_n X_n^+ + \eta] \\
&\quad \times [(f_i)^\#(\vec{r}_1, \vec{v}, s) + (f_e^{j+1})^\#(\vec{r}_1, \vec{v}, s)] \\
&\quad \times [(f_i)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, s) + (f_e^j)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, s)] B d\vec{w} d\vec{\sigma} \\
&\le 2[\|f_i^0\| \cdot \varphi(\delta, f_0^i) + \|f_e^j\| \cdot \varphi(\delta, f_i^0) \\
&\qquad + \|f_e^{j+1}\| \cdot \varphi(\delta, f_i^0) + \|f_e^{j+1}\| \cdot \|f_e^j\|]
\end{aligned} \tag{6.22}$$

with $\delta = \pi a^2 2nT$. By similar treatment of the other loss terms, one has

$$\begin{aligned}
&\| \int_0^t (f^{j+1})^\# \sum_{i=2}^4 (L_{in})^\# (f^j) ds \| \\
&\le \int_0^{T'} \iint\limits_{R^3 \times R^3} \iiint\limits_{R^3 \times R^3 \times S^2_+} (k_+ + k_-)(W_n X_n + \eta) \\
&\quad \times [(f_i)^\#(\vec{r}_1, \vec{v}) \cdot (f_i)^\#(\vec{r}_2 + s(\vec{v} - \vec{w}), \vec{w}) + (f_i)^\#(\vec{r}_1, \vec{v}) \cdot (f_e^j)^\#(\vec{r}_2 + s(\vec{v} - \vec{w}), \vec{w}, s) \\
&\qquad + (f_e^{j+1})^\#(\vec{r}_1, \vec{v}, s) \cdot (f_i)^\#(\vec{r}_2 + s(\vec{v} - \vec{w}), \vec{w}, s) \\
&\qquad + (f_e^{j+1})^\#(\vec{r}_1, \vec{v}, s) \cdot (f_e^j)^\#(\vec{r}_2 + s(\vec{v} - \vec{w}), \vec{w}, s)] \\
&\quad R^2 B d\vec{r}_1 d\vec{r}_2 d\vec{v} d\vec{w} d\vec{\sigma} \\
&\le 4\{\|f_i^0\| \cdot \varphi(\delta_1, f_i^0) + \|f_e^j\| \cdot \varphi(\delta_1, f_i^0) + \|f_e^{j+1}\| \cdot [\varphi(\delta_1, f_i^0) + \|f_e^j\|]\},
\end{aligned}$$

with $\delta_1 = \pi R^2 a^2 2nT$. Therefore,

$$\| \int_0^t (f^{j+1})^\# L_n (f^j)^\# \| \le 6\{\varphi(\delta_1, f_i^0) \cdot [\|f_i^0\| + \|f_e^j\|] + \|f_e^{j+1}\| \cdot [\varphi(\delta_1, f_i^0) + \|f_e^j\|]\}. \tag{6.23}$$

Here the following fact is used: $\varphi(\delta, f_i^0) \le \varphi(\delta_1, f_i^0)$, where $\delta = \pi a^2 2nT$ and $\delta_1 = \pi R^2 a^2 2nT$.

Let us turn to the Enskog gain term.

$$G_{1n)}(f,f)(\vec{r}_1,\vec{v},s) = \iint\limits_{R^3\times S^2_+} W_n X_n^- f(\vec{r}_1,\vec{v}',s) f(\vec{r}_1 - a\vec{\sigma},\vec{w}',s),$$

and thus

$$\begin{aligned}
&(G_{1n})^{\#}(f,f)(\vec{r}_1,\vec{v},s) = G_{1n}(f,f)(\vec{r}_1 + s\vec{v},\vec{v},s)\\
&= \iint\limits_{R^3\times S^2_+} W_n X_n^- f(\vec{r}_1 + s\vec{v},\vec{v}',s) f(\vec{r}_1 - a\vec{\sigma} + s\vec{v},\vec{w}',s)\\
&= \iint\limits_{R^3\times S^2_+} W_n X_n^- f^{\#}(\vec{r}_1 + s(\vec{v}-\vec{v}'),\vec{v}',s) f^{\#}(\vec{r}_1 - a\vec{\sigma} + s(\vec{v}-\vec{w}'),\vec{w}',s).
\end{aligned} \tag{6.24}$$

Similar arguments apply to the other gain terms.

$$\begin{aligned}
&(J_n^+)^{\#}(f,f)(\vec{r}_1,\vec{v},s) = J_n^+(f,f)(\vec{r}_1 + s\vec{v},\vec{v},s)\\
&= \iint\limits_{R^3\times S^2_+} W_n X_n^- f^{\#}(\vec{r}_1 + s(\vec{v}-\vec{v}'),\vec{v}',s) f^{\#}(\vec{r}_1 - a\vec{\sigma} + s(\vec{v}-\vec{w}'),\vec{w}',s)\\
&\quad + \iiint\limits_{R^3\times R^3\times S^2_+} \{ f^{\#}(\vec{r}_1 + s(\vec{v}-\vec{v}^+),\vec{v}^+,s)\cdot f^{\#}(\vec{r}_2 - s\vec{w}^+,\vec{w}^+,s) k_+(\vec{r}_1 + s\vec{v},\vec{r}_2;\vec{\sigma}) W_n^+ X_n\\
&\qquad + f^{\#}(\vec{r}_1 + s(\vec{v}-\vec{v}^-),\vec{v}^-,s)\cdot f^{\#}(\vec{r}_2 - s\vec{w}^-,\vec{w}^-,s)\\
&\qquad \times k_-(\vec{r}_1 + s\vec{v},\vec{r}_2;\vec{\sigma})\theta(\vec{\sigma}\cdot\vec{V} - \sqrt{4\varepsilon}) W_n^- X_n\\
&\qquad + f^{\#}(\vec{r}_1 + s(\vec{v}-\vec{v}^B),\vec{v}^B,s)\cdot f^{\#}(\vec{r}_2 - s\vec{w}^B,\vec{w}^B,s)\}\\
&\qquad \times k_-(\vec{r}_1 + s\vec{v},\vec{r}_2;\vec{\sigma})\theta(\sqrt{4\varepsilon} - \vec{\sigma}\cdot\vec{V}) W_n X_n\\
&\qquad \times R^2 B d\vec{r}_2 d\vec{w} d\vec{\sigma}
\end{aligned}$$

Plugging in $(f^j)^{\#} = (f_i)^{\#} + (f_e^j)^{\#}$ to the above expression, we will obtain (6.21).

Next we deduce the bounds for the gain terms of (6.21). Due to (6.24), we have

$$\begin{aligned}
&(G_{1n})^{\#}(f^j,f^j)(\vec{r}_1,\vec{v},s)\\
&= \iint\limits_{R^3\times S^2_+} W_n X_n^- [(f_i)^{\#}(\vec{r}_1 + s(\vec{v}-\vec{v}'),\vec{v}',s) + (f_e^j)^{\#}(\vec{r}_1 + s(\vec{v}-\vec{v}'),\vec{v}',s)]\\
&\quad \times [(f_i)^{\#}(\vec{r}_1 - a\vec{\sigma} + s(\vec{v}-\vec{w}'),\vec{w}',s) + (f_e^j)^{\#}(\vec{r}_1 - a\vec{\sigma} + s(\vec{v}-\vec{w}'),\vec{w}',s)] B d\vec{w} d\vec{\sigma}.
\end{aligned}$$

Again by Lemmas 6.1 — 6.5, we get

$$\| \int_0^T (G_{1n})^{\#}(f^j, f^j) \| \leq \varphi(\delta, f_i^0) \cdot \|f_i^0\| + 2\varphi(\delta, f_i^0) \cdot \|f_e^j\| + \|f_e^j\|^2. \tag{6.25}$$

Take the integration with respect to $\vec{r}_2$ of the second gain term, apply properties of k_+ to it, and then split f^j. We have the following:

$$\begin{aligned}
&(G_{2n})^{\#}(f^j, f^j)(\vec{r}_1, \vec{v}, s) \\
&= \iiint\limits_{R^3 \times R^3 \times S_+^2} (f^j)^{\#}(\vec{r}_1 + s(\vec{v} - \vec{v}^+), \vec{v}^+, s) \cdot (f^j)^{\#}(\vec{r}_2 - s\vec{w}^+, \vec{w}^+, s) \\
&\quad \times k_+(\vec{r}_1 + s\vec{v}, \vec{r}_2, \vec{\sigma}) W_n^+ X_n R^2 B d\vec{r}_2 d\vec{w} d\vec{\sigma} \\
&= \iint\limits_{R^3 \times S_+^2} (f^j)^{\#}(\vec{r}_1 + s(\vec{v} - \vec{v}^+), \vec{v}^+, s) \cdot (f^j)^{\#}(\vec{r}_1 + Ra\vec{\sigma} + s(\vec{v} - \vec{w}^+), \vec{w}^+, s) \\
&\quad \times W_n^+ X_n^+ R^2 B d\vec{w} d\vec{\sigma} \\
&= \iint\limits_{R^3 \times S_+^2} [(f_i)^{\#}(\vec{r}_1 + s(\vec{v} - \vec{v}^+), \vec{v}^+, s) + (f_e^j)^{\#}(\vec{r}_1 + s(\vec{v} - \vec{v}^+), \vec{v}^+, s)] \\
&\qquad \times [(f_i)^{\#}(\vec{r}_1 + Ra\vec{\sigma} + s(\vec{v} - \vec{w}^+), \vec{w}^+, s) + (f_e^j)^{\#}(\vec{r}_1 + Ra\vec{\sigma} + s(\vec{v} - \vec{w}^+), \vec{w}^+, s)] \\
&\qquad \times W_n^+ X_n^+ R^2 B d\vec{w} d\vec{\sigma},
\end{aligned}$$

which yields

$$\| \int_0^T G_{2n}^{\#}(f^j, f^j) \| \leq \varphi(\delta_1, f_i^0) \cdot \|f_i^0\| + 2\varphi(\delta_1, f_i^0) \cdot \|f_e^j\| + \|f_e^j\|^2. \tag{6.26}$$

Similar results are true for all other gain terms. Consequently,

$$\| \int_0^T J_n^+(f^j, f^j) \| \leq 4\{\varphi(\delta_1, f_i^0) \cdot \|f_i^0\| + 2\varphi(\delta_1, f_i^0) \cdot \|f_e^j\| + \|f_e^j\|^2\}. \tag{6.27}$$

Combining these results together, one has

$$\begin{aligned}
\|f_e^{j+1}\| &\leq \|f_e^0\| + 6\{\varphi(\delta_1, f_i^0) \cdot \|f_i^0\| + \varphi(\delta_1, f_i^0) \cdot \|f_e^j\| \\
&\quad + \|f_e^{j+1}\| \cdot [\|f_e^j\| + \varphi(\delta_1, f_i^0)]\} \\
&\quad + 4\{\varphi(\delta_1, f_i^0) \cdot \|f_i^0\| + 2\varphi(\delta_1, f_i^0) \cdot \|f_e^j\| + \|f_e^j\|^2\} \\
&= \|f_e^0\| + \varphi(\delta_1, f_i^0) \cdot [10\|f_i^0\| + 12\|f_e^j\| + 6\|f_e^{j+1}\|]
\end{aligned}$$

$$
\begin{aligned}
&\quad + \|f_e^j\| \cdot [4\|f_e^j\| + 6\|f_e^{j+1}\|] \\
&\leq \|f_e^0\| + 16\varphi(\delta_1, f_i^0) \cdot [\|f_i^0\| + \|f_e^j\| + \|f_e^{j+1}\|] \\
&\quad + 8\|f_e^j\| \cdot [\|f_e^j\| + \|f_e^{j+1}\|].
\end{aligned} \tag{6.28}
$$

Choose $w > n$, and T' such that

$$
\left\{
\begin{array}{l}
\|f_e^0\| \leq \frac{1}{8 \cdot 128}, \\
\varphi(\delta_1, f_i^0) \leq \frac{1}{8 \cdot 128 \cdot 16}, \\
\varphi(\delta_1, \beta)\beta \leq \frac{1}{8 \cdot 128 \cdot 16}.
\end{array}
\right. \tag{6.29}
$$

Then we will have

$$
\|f_e^j\| \leq \frac{1}{128}, \quad j \in \mathbb{N}. \tag{6.30}
$$

Now we are ready to prove that the sequence $\{(f^j)^\#\}_{j \in \mathbb{N}}$ is Cauchy. Looking back to (6.6.2) and taking an appropriate splitting, one has

$$
\begin{aligned}
&(f_e^{j+1})^\#(\vec{r}_1, \vec{v}, t) = (f_e^0)^\#(\vec{r}_1, \vec{v}, t) \\
&\quad + \int_0^t [J_n^+(f^j, f^j)^\#(\vec{r}_1, \vec{v}, s) - (f^{j+1})^\#(\vec{r}_1, \vec{v}, s) L_n(f^j)^\#(\vec{r}_1, \vec{v}, t)] ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&(f_e^{m+1})^\#(\vec{r}_1, \vec{v}, t) = (f_e^0)^\#(\vec{r}_1, \vec{v}, t) \\
&\quad + \int_0^t [J_n^+(f^m, f^m)^\#(\vec{r}_1, \vec{v}, s) - (f^{m+1})^\#(\vec{r}_1, \vec{v}, s) L_n(f^m)^\#(\vec{r}_1, \vec{v}, t)] ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&(f_e^{j+1})^\#(\vec{r}_1, \vec{v}, t) - (f_e^{m+1})^\#(\vec{r}_1, \vec{v}, t) \\
&= \int_0^t [J_n^+(f^j, f^j)^\#(\vec{r}_1, \vec{v}, s) - (f^{j+1})^\#(\vec{r}_1, \vec{v}, s) L_n(f^j)^\#(\vec{r}_1, \vec{v}, t)] ds \\
&\quad - \int_0^t [J_n^+(f^m, f^m)^\#(\vec{r}_1, \vec{v}, s) - (f^{m+1})^\#(\vec{r}_1, \vec{v}, s) L_n(f^m)^\#(\vec{r}_1, \vec{v}, t)] ds.
\end{aligned} \tag{6.31}
$$

Notice that $J_n^+(f, g)$ is bilinear, and $L_n(f)$ is linear. Then $(f^j)^\# - (f^m)^\# = (f_e^j)^\# - (f_e^m)^\#$. Now we come to the following equality:

$$
(f_e^{j+1})^\#(\vec{r}_1, \vec{v}, t) - (f_e^{m+1})^\#(\vec{r}_1, \vec{v}, t)
$$

$$= \int_0^t [J_n^+(f^j, f^j)^\# - J_n^+(f^m, f^m)^\# ds + \int_0^t (f^{j+1})^\# L_n(f^j)^\# - (f^{m+1})^\# L_n(f^m)^\#$$

$$= \int_0^t [J_n^+(f^j, f^j - f^m)^\#(\vec{r}_1, \vec{v}, s) + J_n^+(f^m, f^j - f^m)^\#(\vec{r}_1, \vec{v}, s)]ds$$

$$+ \int_0^t \{-[(f^{j+1})^\#(\vec{r}_1, \vec{v}, s) - (f^{m+1})^\#(\vec{r}_1, \vec{v}, s)] \cdot L_n(f^j)^\#(\vec{r}_1, \vec{v}, t)$$

$$- (f^{m+1})^\#(\vec{r}_1, \vec{v}, s) \cdot L_n(f^j - f^m)^\#(\vec{r}_1, \vec{v}, t)\}ds$$

$$= \int_0^t [J_n^+(f^j, f_e^j - f_e^m)^\#(\vec{r}_1, \vec{v}, s) + J_n^+(f^m, f_e^j - f_e^m)^\#(\vec{r}_1, \vec{v}, s)]ds$$

$$+ \int_0^t \{-[(f_e^{j+1})^\#(\vec{r}_1, \vec{v}, s) - (f_e^{m+1})^\#(\vec{r}_1, \vec{v}, s)] \cdot L_n(f^j)^\#(\vec{r}_1, \vec{v}, t)$$

$$- (f^{m+1})^\#(\vec{r}_1, \vec{v}, s) \cdot L_n(f_e^j - f_e^m)^\#(\vec{r}_1, \vec{v}, t)\}ds. \tag{6.32}$$

Here we used the fact : $f^j f^j - f^m f^m = f^j(f^j - f^m) + (f^j - f^m)f^m$. In order to get bounds for each term in the above expression, let us check the Enskog gain term first,

$$G_{1n}^\#(f^j, f^j) - G_{1n}^\#(f^m, f^m)$$

$$= G_{1n}^\#(f^j, f^j - f^m) - G_{1n}^\#(f^m - f^j, f^m)$$

$$= \iint_{R^3 \times S_+^2} \{(f^j)^\#(\vec{r}_1, \vec{v}, t)$$

$$\times [(f^j)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, t) - (f^m)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, t)]$$

$$+ [(f^j)^\#(\vec{r}_1, \vec{v}, t) - (f^m)^\#(\vec{r}_1, \vec{v}, t)] \cdot (f^m)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, t)$$

$$\}W_n X_n^+ d\vec{w} d\vec{\sigma}$$

$$= \iint_{R^3 \times S_+^2} \{(f^j)^\#(\vec{r}_1, \vec{v}, t)$$

$$\times [(f_e^j)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, t) - (f_e^m)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, t)]$$

$$+ [(f_e^j)^\#(\vec{r}_1, \vec{v}, t) - (f_e^m)^\#(\vec{r}_1, \vec{v}, t)] \cdot (f^m)^\#(\vec{r}_1 + a\vec{\sigma} + s(\vec{v} - \vec{w}), \vec{w}, t)$$

$$\}W_n X_n^+ d\vec{w} d\vec{\sigma}.$$

And thus, by Lemmas 6.1 — 6.5, we have

$$\| \int_0^t [G_{1n}^\#(f^j, f^j) - G_{1n}^\#(f^m, f^m)]ds \|$$

$$
\begin{aligned}
&\leq \int_0^t \iint_{R^3\times R^3} \iint_{R^3\times S^2_+} \{[(f_i)^{\#}(\vec{r}_1,\vec{v},t) + (f_e^j)^{\#}(\vec{r}_1,\vec{v},t)] \\
&\quad \times [(f_e^j)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v}-\vec{w}),\vec{w},s) - (f_e^m)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v}-\vec{w}),\vec{w},t)] \\
&\quad + [(f_e^j)^{\#}(\vec{r}_1,\vec{v},t) - (f_e^m)^{\#}(\vec{r}_1,\vec{v},t)] \\
&\quad \times [(f_i)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v}-\vec{w}),\vec{w},s) + (f_e^m)^{\#}(\vec{r}_1 + a\vec{\sigma} + s(\vec{v}-\vec{w}),\vec{w},s)]\} \\
&\qquad W_n X_n^+ d\vec{r}_1 d\vec{v} d\vec{w} d\vec{\sigma} ds \\
&\leq [\varphi(\delta, f_i^0) + \|f_e^j\|] \cdot \|f_e^j - f_e^m\| \\
&\qquad + \|f_e^j - f_e^m\| \cdot [\varphi(\delta, f_i^0) + \|f_e^m\|] \\
&= [\|f_e^j\| + \|f_e^m\| + 2\varphi(\delta, f_i^0)] \cdot \|f_e^j - f_e^m\|.
\end{aligned}
$$

Similar results are true for the other gain terms due to Lemmas 6.1 — 6.5. We have

$$
\begin{aligned}
&\| \int_0^t [G_{in}^{\#}(f^j, f^j) - G_{in}^{\#}(f^m, f^m)] ds \| \\
&\leq [\|f_e^j\| + \|f_e^m\| + 2\varphi(\delta, f_i^0)] \cdot \|f_e^j - f_e^m\|, \qquad (i = 1,2,3,4).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\| \int_0^t [J_n^+(f^j, f^j)^{\#} - J_n^+(f^m, f^m)^{\#} ds \| \\
&\leq 4[\|f_e^j\| + \|f_e^m\| + 2\varphi(\delta, f_i^0)] \cdot \|f_e^j - f_e^m\|. \qquad (6.33)
\end{aligned}
$$

With a little bit modification, the discussion can be applied to the loss terms. Because $(f^{j+1}f^j - f^{m+1}f^m)^{\#} = (f^{j+1})^{\#}(f^j - f^m)^{\#} + (f^{j+1} - f^{m+1})^{\#}(f^m)^{\#} = [(f_i)^{\#} + (f_e^{j+1})^{\#}] \cdot [(f_e^j)^{\#} - (f_e^m)^{\#}] + [(f_e^{j+1})^{\#} - (f_e^{m+1})^{\#}] \cdot [(f_i)^{\#} + (f_e^m)^{\#}]$, it follows by using Lemmas 6.1 — 6.5 that

$$
\begin{aligned}
&\left| \int_0^t \iint_{R^3\times S^2_+} [f^{j+1}f^j - f^{m+1}f^m]^{\#} ds \right| \\
&\leq [\varphi(\delta, f_i^0) + \|f_e^{j+1}\|] \cdot \|f_e^j - f_e^m\| + \|f_e^{j+1} - f_e^{m+1}\| \cdot [\varphi(\delta, f_i^0) + \|f_e^m\|],
\end{aligned}
$$

and thus

$$
\| \int_0^t (f^{j+1})^{\#} L_{in}(f^j)^{\#} - (f^{m+1})^{\#} L_{in}(f^m)^{\#} \|
$$

$$\leq [\varphi(\delta, f_i^0) + \|f_e^{j+1}\|] \cdot \|f_e^j - f_e^m\| + \|f_e^{j+1} - f_e^{m+1}\| \cdot [\varphi(\delta, f_i^0) + \|f_e^m\|],$$
$$(i = 1, 2, 3, 4).$$

Furthermore,

$$\|\int_0^t (f^{j+1})^{\#} L_n(f^j)^{\#} - (f^{m+1})^{\#} L_n(f^m)^{\#}\|$$
$$\leq 8\{[\varphi(\delta, f_i^0) + \|f_e^{j+1}\|] \cdot \|f_e^j - f_e^m\| + \|f_e^{j+1} - f_e^{m+1}\| \cdot [\varphi(\delta, f_i^0) + \|f_e^m\|]\}. \tag{6.34}$$

Consequently it follows from (6.32)—(6.34), (6.29) and (6.30) that

$$\begin{aligned}
&\|f_e^{j+1} - f_e^{m+1}\| \\
&\leq 4[\|f_e^j\| + \|f_e^m\| + 2\varphi(\delta_1, f_i^0)] \cdot \|f_e^j - f_e^m\| \\
&\quad + 8\{[\varphi(\delta, f_i^0) + \|f_e^{j+1}\|] \cdot \|f_e^j - f_e^m\| + \|f_e^{j+1} - f_e^{m+1}\| \cdot [\varphi(\delta, f_i^0) + \|f_e^m\|]\} \\
&\leq \frac{1}{4}\|f_e^j - f_e^m\| + \frac{1}{8}\|f_e^{j+1} - f_e^{m+1}\|.
\end{aligned} \tag{6.35}$$

Hence $\{f_e^j\}_{j\in \mathbb{N}}$ is Cauchy in the $\|\cdot\|_{T'}$-norm. So is $\{f^j\}_{j\in\mathbb{N}}$. Denote the limit by f_n. It follows by the contraction mapping theorem that Eq.(6.1) has a unique nonnegative solution on [0, T'] with $\|f_n\|_{T'} < \infty$. Eqs. (4.11), (4.14), (4.19), and (4.38) hold in a strict sense for $f^j, j \in \mathbb{N}$. Hence f_n satisfies (4.11), (4.14),(4.19) and (4.38). The facts discussed above imply the global well-posedness for Eq.(6.1) under the indicated initial condition on f_0. Let us note that if $Y \equiv constant \neq 1, and\, Y > 0$, then we may adjust $w > n$, and T' such that (6.29) and (6.30) are satisfied with $\frac{1}{Y}$ multiplying the right hand sides. All the above results are true. Now we have the following theorem.

Theorem 6.1: Suppose $f_0 \in L_+^1(R^3 \times R^3)$ with $f_0 log f_0 \in L_+^1(R^3 \times R^3)$ and $f_0 = 0$ for $|\vec{v}| \geq 2n$. Then there is a unique solution to the Eq.(6.1) with initial value f_0 for $t > 0$. The solution satisfies Eqs. (4.11) , (4.14), (4.19) and (4.38).

Proof : Given any time interval $[0, T]]$, ω and T' can be chosen in such a way that (6.29) is satisfied. Then Eq.(6.1) has a unique solution f on $[0, T']$. If f_0 is chosen to be smooth enough, then the solution is smooth enough on $[0, T'] \times R^3 \times R^3$ for the formal computations. The solution f satisfies (4.11), (4.14), (4.19) and (4.38) on $[0, T']$. The argument can next be applied to $[T', 2T']$ with initial value $f(\vec{r}_1, \vec{v}, T')$ and then successively on subintervals of length T' covering $[0, T]$. The proof is completed.

§7. Existence theorem

In the last section, we proved the existence and uniqueness of the solution to the trucated equation (6.1). Denote it by f_n for each n. We shall now take the limit $n \to \infty$, and show that the limit of the truncated solution functions is the solution to the equation (2.1), which provides the existence of solution of the equation (2.1). Let us discuss some weak limit results first.

Lemma 7.1: (Dunford-Pettis Theorem [16]) If $X = L^1(\Omega, \mu)$ for a positive Radon measure μ on a locally compact space Ω, then $\{f_n\}_{n\in\mathbb{N}} \subset X$ is weakly compact if and only if

(i) $\|f_n\| \leq M < \infty, \forall n$;
(ii) $\forall \epsilon > 0, \exists \delta > 0$ such that $\int\limits_B |f_n| d\mu < \epsilon$ for all $B \subset \Omega$ with $\mu(B) < \delta$;
(iii) $\forall \epsilon > 0, \exists$ compact set K such that

$$\int\limits_{\Omega\setminus K} |f_n| d\mu < \epsilon, \forall n.$$

Define

$$f_n^\delta = \frac{1}{\delta} log(1 + \delta f_n). \tag{7.1}$$

Then

$$(\partial_t + \vec{v} \cdot \nabla_{\vec{r}}) f_n^\delta = \frac{1}{1 + \delta f_n} J_n(f_n, f_n). \tag{7.2}$$

Lemma 7.2: The sequences $\{J_{in}^+(f_n, f_n)/(1 + \delta f_n)\}$ and $\{f_n L_{in}(f_n)/(1 + \delta f_n)\}$ are weakly compact in $L^1(R^3 \times B_K \times (0, T))$ for $\delta > 0$ and $K > 0$ $(i = 1, 2, 3, 4)$, where $B_K = \{\vec{v} \in R^3 \quad | \quad |\vec{v}| \leq K\}$.

Proof: Let us begin with the Enskog loss term. It is easy to see that for any set $E \subseteq R^3 \times B_K \times (0, T)$, and $\delta > 0$,

$$\begin{aligned}
&\iiint\limits_E \frac{f_n L_{En}(f_n)}{1 + \delta f_n} d\vec{r}_1 d\vec{v} dt \\
&\leq \iiint\limits_E \iint\limits_{R^3 \times S_+^2} \frac{f_n(\vec{r}_1, \vec{v}, t) f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)}{1 + \delta f_n} [Y W_n X_n^+ + \eta] B d\vec{r}_1 d\vec{v} dt d\vec{w} d\vec{\sigma} \\
&\leq \frac{2Y}{\delta} \iiint\limits_E \iint\limits_{R^3 \times S_+^2} f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) \cdot (|\vec{v}| + |\vec{w}|) d\vec{r}_1 d\vec{v} dt d\vec{w} d\vec{\sigma}.
\end{aligned} \tag{7.3}$$

Let $A_K = \{(\vec{r}_1, \vec{v}, t, \vec{w}, \vec{\sigma}) \quad | \quad |\vec{w}| \leq K\}$. For any measurable set $B_1 \subseteq R^3 \times B_K \times (0, T) \times R^3 \times S^2$,

$$\int \cdots \int_{B_1} f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)(|\vec{v}| + |\vec{w}|) d\vec{r}_1 d\vec{v} dt d\vec{w} d\vec{\sigma}$$

$$\begin{aligned}
&\leq K \int \cdots \int_{B_1} f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) d\vec{r}_1 d\vec{v} dt d\vec{w} d\vec{\sigma} \\
&\quad + \int \cdots \int_{B_1 \cap \{|\vec{w}| \leq K\}} f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)|\vec{w}| d\vec{r}_1 \vec{v} dt d\vec{w} d\vec{\sigma} \\
&\quad + \int \cdots \int_{B_1 \cap \{|\vec{w}| \geq K\}} f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)|\vec{w}| \cdot \frac{|\vec{w}|}{K} d\vec{r}_1 d\vec{v} dt d\vec{w} d\vec{\sigma}. \qquad (7.4)
\end{aligned}$$

As discussed in previous sections, $\{f_n\}_{n \in \mathbb{N}}$ satisfy the Eqs. (4.11), (4.14), (4.19), (4.38), and (5.8). If the right hand of (7.4) is weakly compact, then the left hand side of (7.4) is weakly compact in $L^1(R^3 \times B_K \times (0,T) \times R^3 \times S_+^2)$. Let us check the conditions of Lemma 7.1 (Dunford-Pettis Theorem) for the sequence $f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)(|\vec{v}| + |\vec{w}|)$. (i) is obvious. If we choose K big enough, and let the measure B_1 be small enough, we will have (ii) easily from (7.4). (iii) of lemma 7.1, can be obtained by the following. Let $E_K^1 = \{(\vec{r}_1, \vec{v}, t, \vec{w}, \vec{\sigma}) \mid |\vec{r}_1| + |\vec{v}| + |\vec{w}| \leq K\}$, then

$$\begin{aligned}
&\int \cdots \int_{(R^3 \times B_K \times (0,T) \times R^3 \times S_+^2) \cap E_K^{1C}} f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) \cdot (|\vec{v}| + |\vec{w}|) d\vec{r}_1 d\vec{v} dt d\vec{w} d\vec{\sigma} \\
&\leq \int \cdots \int_{(R^3 \times B_K \times (0,T) \times R^3 \times S_+^2)} f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t) \cdot (|\vec{v}| + |\vec{w}|) \cdot \frac{|\vec{r}_1| + |\vec{v}| + |\vec{w}|}{K} d\vec{r}_1 d\vec{v} dt d\vec{w} d\vec{\sigma}. \qquad (7.5)
\end{aligned}$$

Thus $f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t)(|\vec{v}| + |\vec{w}|)$ is weakly compact in $L^1(R^3 \times B_K \times (0,T) \times R^3 \times S_+^2)$. Therefore the sequence $\{f_n L_{En}(f_n)/(1 + \delta f_n)\}$ is weakly compact due to (7.3) and the boundedness of the operator $\iint_{R^3 \times S_+^2} \cdot B d\vec{w} d\vec{\sigma}$ from $L^1(R^3 \times B_K \times (0,T) \times R^3 \times S_+^2)$ to $L^2(R^3 \times B_K \times (0,T))$. For the weak compactness of other loss terms, the proofs are largely the same as the above for the Enskog loss term. Hence $\{f_n L_{in}(f_n)/(1 + \delta f_n)\}$ $(i = 1,2,3,4)$ is weakly compact. This, together with the gain-loss estimation Eq.(5.8), implies the same weak L^1-compactness for $\{J_{in}^+(f_n, f_n)/(1 + \delta f_n)\}$ when $\delta > 0$. ∎

Lemma 7.3 (Golse Lemma [18],[15]): If f_n, $g_n \in L^1_{loc}(R^3 \times R^3 \times (0,T))$ satisfy $(\partial_t + \vec{v} \cdot \nabla_{\vec{r}}) f_n = g_n$ in the distributional sense, and if for each compact set $K \subset R^3 \times R^3 \times (0,T)$, the sequences $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ are weakly compact in $L^1(R^3 \times R^3 \times (0,T))$ and $L^1(K)$, respectively, then the set $\{\int_{R^3} f_n \psi d\vec{v}\}_{n \in \mathbb{N}}$ is compact in $L^1(R^3 \times (0,T))$ for all $\psi \in L^\infty(R^3 \times R^3 \times (0,T))$.

In order to prove our main therem, we introduce sevearal lemmas first.

Lemma 7.4 (see [16]): Let g_n *and* h_n be measurable on $\Sigma = R^3 \times R^3 \times (0,T)$. Then

(i) $\int\limits_{\Sigma} h_n g_n \to \int\limits_{\Sigma} hg$ *if* $g_n \to g$ *weakly in* L^1, $\sup\limits_{n\geq 1} \|h_n\|_{L^1} < \infty$, *and* $h_n \to h$ *a.e.*

(ii) $\int\limits_{\Sigma} h_n g_n \to \int\limits_{\Sigma} hg$ *if* $h_n \to h$ *weakly in* L^1, $\sup\limits_{n\geq 1} \|h_n\|_{L^\infty} < \infty$, *and* $g_n \to g$ *strongly in* L^1.

Lemma 7.5: For any ψ with $(1 + |\vec{r}_1|^k + |\vec{v}|^k)^{-1}\psi \in L^\infty(R^3 \times R^3 \times (0,T))$ and $0 \leq k < 2$,

$$\lim_{n\to\infty} \int\limits_{R^3} f_n \, \psi \, d\vec{v} \quad = \quad \int\limits_{R^3} f \, \psi \, d\vec{v} \quad in \ L^1(R^3 \times (0,T)), \tag{7.6}$$

and

$$L_{in}(f_n) \to (1+\eta) L_i(f) \quad in \quad L^1(R^3 \times B_K \times (0,T)) \quad for \ any \ K > 0, \tag{7.7}$$

$i = 1,2,3,4.$

Proof of lemma 7.5: Since $0 \leq f_n^\delta \leq f_n$, $\{f_n\}$ is weakly compact in $L^1(R^3 \times R^3 \times (0,T))$, and so is $\{f_n^\delta\}$. Without loss of generality, we may assume that $f_n \to f$, and $f_n^\delta \to f^\delta$ for some $f, f^\delta \in L^1(R^3 \times R^3 \times (0,T))$. Lemma 7.3 and Lemma 7.4 imply that, after passing to a subsequence if necessary, the average velocity sequence $\{\int_{R^3} f_n^\delta \psi d\vec{v}\}$ is compact for any $\psi \in L^\infty(R^3 \times R^3 \times (0,T))$. We may assume that $\int_{R^3} f_n^\delta \psi d\vec{v} \underset{n\to\infty}{\to} \int_{R^3} f^\delta \psi d\vec{v}$ in $L^1(R^3 \times (0,T))$, for some weak limit function f^δ.

Notice

$$\begin{aligned} 0 \leq f_n - f_n^\delta &= f_n - \frac{1}{\delta} log(1 + \delta f_n) \\ &\leq f_n \left[\left(1 - \frac{log(1+\delta f_n)}{\delta f_n} \right) \chi_{\{f_n \leq K\}} \right] + f_n \cdot \chi_{\{f_n \geq K\}}. \end{aligned} \tag{7.8}$$

To see the limit of two terms on the right hand side of (7.8) as $n \to \infty$, we have

$$\left[\left(1 - \frac{log(1+\delta f_n)}{\delta f_n} \right) \chi_{\{f_n \leq K\}} \right] \overset{\delta \to 0+}{\to} 0 \quad uniformly, \tag{7.9}$$

and

$$\sup_{n\geq 1} \sup_{t\in[0,T]} \iint\limits_{R^3 \times R^3} f_n \chi_{\{f_n \geq K\}} d\vec{r}_1 d\vec{v} \overset{K\to\infty}{\to} 0, \tag{7.10}$$

because

$$\mu(f_n \geq K) = \iint\limits_{f_n \geq K} f_n d\vec{r}_1 d\vec{v} \leq \frac{1}{K} \iint\limits_{R^3 \times R^3} f_n d\vec{r}_1 d\vec{v} \overset{K\to\infty}{\to} 0. \tag{7.11}$$

It follows from (7.8)—(7.11) that (This method was used by Arkeryd [1], DiPerna and Lions [15], and more recently by Liu [28])

$$\sup_{n\geq 1} \sup_{t\in[0,T]} \|f_n - f_n^\delta\|_{L^1(R^3\times R^3)} \underset{\delta\to 0+}{\to} 0. \tag{7.12}$$

On the other hand, we can write, for any $\psi \in L^\infty(R^3 \times R^3 \times (0,T))$,

$$\int_{R^3} f_n \psi d\vec{v} = \int_{R^3} (f_n - f_n^\delta)\psi d\vec{v} + \int_{R^3} (f_n^\delta - f^\delta)\psi d\vec{v} + \int_{R^3} f^\delta \psi d\vec{v}. \tag{7.13}$$

By the lower weak semicontinuity of the norm and (7.12), it follows that

$$\|f - f^\delta\|_{L^1(R^3\times R^3\times(0,T))} \leq \sup_{t\in[0,T]} \ lim\ inf \|f_n - f_n^\delta\|_{L^1(R^3\times R^3)} \to 0. \tag{7.14}$$

Combining these results, and taking the limits $n \to \infty$ and $\delta \to 0$ in (7.13), we get (7.6). In the above proof, ψ can be replaced by any ψ satisfying

$$(1 + |\vec{r}|^k + |\vec{v}|^k)^{-1}\psi \in L^\infty(R^3 \times R^3 \times (0,T)), \tag{7.15}$$

with $0 \leq k < 2$. Because the operator $\int_{S_+^2} \cdot d\vec{\sigma}$ is bounded. $\{L_{in}(f_n)\}$ and $\{L_i(f_n)\}$ are compact in $L^1(R^3 \times B_K \times (0,T))$. The same argument as in lemma 7.2 is true for $\{L_{in}(f_n)\}$ and $L_{in}(f_n) \to (1+\eta)L_i(f)$ weakly in $L^1(R^3 \times B_K \times (0,T))$. Consequently $L_{in}(f_n) \to (1+\eta)L_i(f)$ in the norm of $L^1(R^3 \times B_K \times (0,T))$. ∎

Lemma 7.6 ([28]):

1) $$\int_{R^3} G_{in}(f_n, f_n)\psi d\vec{v} \underset{n\to\infty}{\to} \int_{R^3} G_i(f,f)\psi d\vec{v} \qquad \textit{in measure on} \quad B_K \times (0,T); \tag{7.16}$$

2) $$\int_{R^3} f_n L_n(f_n)\psi d\vec{v} \underset{n\to\infty}{\to} (1+\eta)\int_{R^3} f L(f)\psi d\vec{v} \qquad \textit{in measure on} \quad B_K \times (0,T). \tag{7.17}$$

Proof: Lemma 7.5 implies that

$$\int_{R^3} f_n \psi_n d\vec{v} \quad \underset{n\to\infty}{\to} \quad \int_{R^3} f\psi d\vec{v}$$

in $L^1(R^3 \times (0,T))$ for $\psi_n \to \psi$, *a.e. in* $(\vec{r}, \vec{v}, t)$ and

$$\sup_{n\geq 1} \quad \|\frac{\psi_n}{1+|\vec{v}|}\|_{L^\infty} < \infty. \tag{7.18}$$

Indeed

$$\int_{R^3} f_n \psi_n d\vec{v} = \int_{R^3} f_n (1+|\vec{v}|) \frac{\psi_n - \psi}{1+|\vec{v}|} d\vec{v} + \int_{R^3} f_n \psi d\vec{v},$$

and

$$L_{in}(f_n) \underset{a.e.}{\rightarrow} (1+\eta) L_i(f), \quad (i = 1,2,3,4).$$

Thus we have, i = 1, 2, 3, 4,

$$\frac{L_{in}(f_n)}{1 + \iint\limits_{R^3 \times S_+^2} (1+|w|) f_n(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) d\vec{w} d\vec{\sigma}} \rightarrow \frac{(1+\eta) L_i(f)}{1 + \iint\limits_{R^3 \times S_+^2} (1+|w|) f_n(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) d\vec{w} d\vec{\sigma}}.$$

Let

$$\psi_n = \frac{L_{1n}(f_n)}{1 + \iint\limits_{R^3 \times S_+^2} (1+|w|) f_n(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) d\vec{w} d\vec{\sigma}} \cdot \varphi.$$

Then ψ_n satisfies (7.18). Thus Lemma 7.5 leads to

$$\frac{\int\limits_{R^3} f_n L_{1n}(f_n) \varphi d\vec{v}}{1 + \iint\limits_{R^3 \times S_+^2} (1+|w|) f_n(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) d\vec{w} d\vec{\sigma}} \rightarrow \frac{(1+\eta) \int\limits_{R^3} f L_1(f) \varphi d\vec{v}}{1 + \iint\limits_{R^3 \times S_+^2} (1+|w|) f(\vec{r}_1 - a\vec{\sigma}, \vec{w}, t) d\vec{w} d\vec{\sigma}}.$$

Since

$$\begin{aligned} &\| \iint\limits_{R^3 \times S_+^2} (1+|w|) f_n d\vec{w} d\vec{\sigma} - \iint\limits_{R^3 \times S_+^2} (1+|w|) f d\vec{w} d\vec{\sigma} \|_{L^1(R^3 \times (0,T))} \\ &\leq \iint\limits_{R^3 \times (0,T)} \int_0^T | \int\limits_{R^3} (1+|w|)(f_n - f) d\vec{w} | d\vec{r}_1 d\vec{\sigma} dt \quad \underset{n \to \infty}{\longrightarrow} \quad 0, \end{aligned}$$

we have

$$\iint\limits_{R^3 \times S_+^2} (1+|w|) f_n d\vec{w} d\vec{\sigma} \overset{a.e.}{\rightarrow} \iint\limits_{R^3 \times S_+^2} (1+|w|) f d\vec{w} d\vec{\sigma} \quad in\ (\vec{r}_1, \vec{v}, t).$$

Therefore

$$\int_{R^3} f_n \cdot L_{1n}(f_n) \varphi d\vec{v} \rightarrow (1+\eta) \int_{R^3} f \cdot L_1(f) \varphi d\vec{v}.$$

The same method applies to $i = 2,3,4$ for L_{in}. This proves (7.17).

By making appropriate variable changes, $J_{in}^{+}(f_n, f_n)$ will be reduced to the case for $f_n \cdot L_{in}(f_n)$ (i=1,2,3,4), i.e., gain terms can be reduced to the case of lost terms. For example, after making variable change $(\vec{v}, \vec{w}) \rightarrow (\vec{v}', \vec{w}')$,

$$\iint\limits_{R^3 \times S_+^2} Y W_n X_n^- f_n(\vec{r}_1, \vec{v}', t)\ f_n(\vec{r}_1 - a\vec{\sigma}, \vec{w}', t)$$
$$= \iint\limits_{R^3 \times S_+^2} Y W_n X_n^+ f_n(\vec{r}_1, \vec{v}, t)\ f_n(\vec{r}_1 + a\vec{\sigma}, \vec{w}, t).$$

We can treat this case the same way as we did the case of loss terms. Taking the limit $n \rightarrow \infty$, we have (7.16). The proof is completed. ∎

Lemma 7.7:

$$f^{\#}(\vec{r}_1, \vec{v}, t) - f^{\#}(\vec{r}_1, \vec{v}, s) \cdot e^{-(1+\eta)\sum_{i=E}^{4}(F_i^{\#}(t) - F_i^{\#}(s))}$$
$$\geq \int_s^t \sum_{i=E}^{4} G_i^{\#}(f, f)(\vec{r}_1, \vec{v}, \tau) \cdot e^{-(1+\eta)\sum_{i=E}^{4}(F_i^{\#}(t) - F_i^{\#}(\tau))} d\tau \tag{7.19}$$

almost everywhere in $(\vec{r}_1, \vec{v}) \in R^3 \times R^3$, where

$$F_i^{\#}(\vec{r}_1, \vec{v}, t) = \int_0^t L_i^{\#}(f)(\vec{r}_1, \vec{v}, \tau) d\tau. \tag{7.20}$$

Proof: Let

$$F_{in}^{\#}(\vec{r}_1, \vec{v}, t) = \int_0^t L_{in}(f_n)^{\#}(\vec{r}_1, \vec{v}, s) ds. \tag{7.21}$$

Then (7.20) and (7.21) leads to

$$\iint\limits_{R^3 \times R^3} |F_{in}^{\#}(\vec{r}_1, \vec{v}, t) - F_i^{\#}(\vec{r}_1, \vec{v}, t)| d\vec{r}_1 d\vec{v}$$
$$\leq \int_0^t ds \iint\limits_{R^3 \times R^3} |L_{in}(f_n)^{\#} - L_i(f)^{\#}| d\vec{r}_1 d\vec{v}. \tag{7.22}$$

Lemma 7.5 and (7.22) imply

$$F_{in}^{\#}(\vec{r}_1, \vec{v}, t) \underset{n \rightarrow \infty}{\longrightarrow} (1+\eta) F_i^{\#} \equiv (1+\eta) \int_0^t L_i(f)^{\#}(\vec{r}_1, \vec{v}, s) ds,$$

in $C([0,T], L^1_{loc}(R^3 \times R^3))$, and thus

$$exp\{-\sum(F^{\#}_{in}(t) - F^{\#}_{in}(s))\} \underset{n\to\infty}{\longrightarrow} exp\{-(1+\eta)\sum(F^{\#}_{i}(t) - F^{\#}_{i}(s))\}$$

almost everywhere in $(\vec{r}_1, \vec{v})$, since $exp\{-\sum(F^{\#}_{in}(t) - F^{\#}_{in}(s))\} \leq 1$.

In the previous section we have already proved that the solution sequence f_n satisfies

$$\begin{aligned} & f^{\#}_n(\vec{r}_1, \vec{v}, t) - f^{\#}_n(\vec{r}_1, \vec{v}, s) \times e^{-\int_t^s L_n(f_n)d\xi} \\ & = \int_t^s d\tau J^+_n(f_n, f_n)^{\#} \times e^{-\int_\tau^t L_n(f_n)d\xi} \end{aligned}$$

for any $0 \leq s \leq t \leq T$, and $a.e.\ (\vec{r}_1, \vec{v}) \in R^3 \times R^3$.

To prove Lemma 7.7, it is enough to show that

$$\begin{aligned} & \int_0^T\int_0^T dsdt \iint_{R^3\times R^3} \varphi \int_s^t d\tau J^+(f,f)^{\#} \times e^{-(1+\eta)\int_\tau^t L(f)d\xi} d\vec{r}_1 d\vec{v} \\ & \leq \liminf_{n\to\infty} \int_0^T\int_0^T dsdt \iint_{R^3\times R^3} \varphi \int_s^t d\tau J^+_n(f_n,f_n)^{\#} \times e^{-\int_\tau^t L_n(f_n)d\xi} d\vec{r}_1 d\vec{v}. \end{aligned}$$

Define $f^\omega_n = min\{f_n, \omega\}$ for $0 < \omega < \infty$. Since $f^\omega_n \leq f_n$, f^ω_n is weakly compact in $L^1(R^3 \times R^3 \times (0,T))$. Denoted by f^ω the weak limit of f^ω_n.

Because $0 \leq f_n - f^\omega_n \leq f_n \cdot \chi_{\{f_n \geq \omega\}}$, $\{f_n\}$ is weakly compact, and

$$\mu(f_n \geq \omega) = \iint_{f_n\geq\omega} f_n d\vec{r}_1 d\vec{v} \leq \frac{1}{\omega} \iint_{R^3\times R^3} f_n d\vec{r}_1 d\vec{v} \to 0 \quad uniformly,$$

we get by the same argument as in the proof of (7.12) that

$$\sup_{n\geq 1} \sup_{t\in[0,T]} \|f_n - f^\omega_n\|_{L^1(R^3\times R^3)} \underset{\omega\to\infty}{\longrightarrow} 0,$$

and

$$\|f - f^\omega\|_{L^1(R^3\times R^3\times(0,T))} \leq T \cdot \sup_{t\in[0,T]} \liminf_{n\to\infty} \|f_n - f^\omega_n\|_{L^1(R^3\times R^3)} \underset{n\to\infty}{\longrightarrow} 0.$$

Referring to Eqs.(4.8.1)—(4.8.4), define $G^\omega_{in}(f_n, f_n) = G_{in}(f^\omega_n, f), (i = 1,2,3,4)$, e.g.:

$$G^\omega_{1n}(f_n, f_n) = \iint_{R^3\times S^2_+} YW_nX^-_n f^\omega_n(\vec{r}_1, \vec{v}', t) \cdot f_n(\vec{r}_1 - a\vec{\sigma}, \vec{w}', t) d\vec{r}_1 d\vec{\sigma}.$$

Thus for fixed $\omega < \infty$, and $0 \le s \le t \le T$

$$\begin{aligned}
&\lim_{n\to\infty} \int_s^t d\tau \iint\limits_{R^3 \times R^3} d\vec{r}_1 d\vec{v} \varphi e^{-\int_\tau^t L_n(f_n) d\xi} \cdot G_{En}(f_n, f_n) \\
&\ge \lim_{n\to\infty} \int_s^t d\tau \iint\limits_{R^3 \times R^3} d\vec{r}_1 d\vec{v} \varphi e^{-\int_\tau^t L_n(f_n) d\xi} \cdot G^{\omega}_{En}(f_n, f_n) \\
&= \lim_{n\to\infty} \int_s^t d\tau \iint\limits_{R^3 \times R^3} d\vec{r}_1 d\vec{v} \varphi \iint\limits_{R^3 \times S^2_+} e^{-\int_\tau^t L_n(f_n) d\xi} f^{\omega}_n(\vec{r}_1, \vec{v}', \tau) \cdot f_n(\vec{r}_1 - a\vec{\sigma}, \vec{w}', \tau) d\vec{w} d\vec{\sigma} \\
&= \int_s^t d\tau \iint\limits_{R^3 \times R^3} d\vec{r}_1 d\vec{v} \varphi \iint\limits_{R^3 \times S^2_+} e^{-(1+\eta)\int_\tau^t L(f) d\xi} f^{\omega}(\vec{r}_1, \vec{v}', \tau) \cdot f(\vec{r}_1 - a\vec{\sigma}, \vec{w}', \tau) d\vec{w} d\vec{\sigma}.
\end{aligned}$$

Finally we have that $f^\omega \nearrow f$, $G^{\omega}_{in}(f_n, f_n) \nearrow G_i(f, f)$ pointwise as $\omega \nearrow \infty$. Therefore Lemma 7.7 follows by the monotone convergence theorem and Fatou's lemma. ∎

Lemma 7.8:

$$f^{\#}(\vec{r}_1, \vec{v}, t) - f^{\#}(\vec{r}_1, \vec{v}, s) \le \int_s^t [J^{\#}(f, f) - \eta f^{\#} L^{\#}(f)] d\tau. \tag{7.23}$$

Proof: Looking back to the iteration scheme (6.6.2), we have that the solution f_n of (6.1) satisfies

$$(f_n)^{\#}(\vec{r}_1, \vec{v}, t) = f_0 + \int_0^t [J^+_n(f_n, f_n)^{\#}(s) - (f_n)^{\#} \cdot L_n(f_n)^{\#}(s)] ds. \tag{7.24}$$

Thus the function f^δ_n defined by (7.1) satisfies

$$\begin{aligned}
&f^{\delta\#}_n(\vec{r}_1, \vec{v}, t) - f^{\delta\#}_n(\vec{r}_1, \vec{v}, s) \\
&= \int_s^t \sum_{i=1}^4 \left[\frac{J^+_{in}(f_n, f_n)^{\#}}{1 + \delta f_n} - \left(\frac{f_n}{1 + \delta f_n} \right)^{\#} \cdot L^{\#}_{in}(f_n) \right] d\tau.
\end{aligned} \tag{7.25}$$

Define

$$h^\delta_n \equiv \frac{f_n}{1 + \delta f_n}, \quad and \quad h^\delta \equiv \frac{f}{1 + \delta f}. \tag{7.26}$$

As $n \to \infty$, $f^\delta_n \to f^\delta$ *weakly*, where f^δ is defined by (7.1). By Lemmas 7.2 — 7.6, we have $\frac{J^+_{in}(f_n, f_n)}{1+\delta f_n} \to \frac{J^+_i(f,f)}{1+\delta f}$ *weakly*, and $L_{in}(f_n) \to (1+\eta) L_i(f)$. Thus $h^\delta_n \to h^\delta$.

Letting $n \to \infty$ in (7.25) yields

$$f^{\delta}(\vec{r}_1, \vec{v}, t) - f^{\delta}(\vec{r}_1, \vec{v}, s) = \int_s^t \sum_{i=E}^{4} \left[\frac{J_i^+(f,f)}{1+\delta f} - (1+\eta)\left(\frac{f}{1+\delta f}\right) L_i(f) \right] d\tau, \quad (7.27)$$

for $0 \le s \le t \le T$ and $a.e.\ (\vec{r}_1, \vec{v}) \in R^3 \times R^3$.
Referring to (7.12), we have

$$\|f - f^{\delta}\|_{L^1(R^3 \times R^3} \le T \sup_{t \in [0,T]} \ lim\ inf \|f_n - f_n^{\delta}\| \overset{\delta \to 0+}{\to} 0, \quad (7.28)$$

because of the lower weak semicontinuity of the norm. Thus $f^{\delta\#} \to f^{\#}$ in $L^1(R^3 \times R^3)$ uniformly in $t \in [0,T]$ *as* $\delta \to 0+$.

Notice that $h^{\delta} \nearrow f$ *as* $\delta \searrow 0+$. The inequalities $0 \le f_n - \frac{f_n}{1+\delta f_n} \le f_n[(1 - \frac{1}{1+\delta f_n})\chi_{\{f_n \le K\}}] + f_n \cdot \chi_{\{f_n \ge K\}}$ lead to

$$\sup_{n \ge 0} \sup_{t \in [0,T]} \|f_n - \frac{f_n}{1+\delta f_n}\|_{L^1(R^3 \times R^3)} \underset{n \to \infty}{\to} 0. \quad (7.29)$$

Referring to the lower weak semicontinuity of the norm, we have that

$$\sup_{t \in [0,T]} \|f - h^{\delta}\|_{L^1(R^3 \times R^3)} \le \sup_{t \in [0,T]} \ \lim_{n \to \infty} inf \|f_n - h_n^{\delta}\|_{L^1(R^3 \times R^3} \underset{\delta \to 0+}{\to} 0. \quad (7.30)$$

Since $\frac{J_i^+(f,f)}{1+\delta f} \le J_i^+(f,f)$, $h^{\delta} \nearrow f$ and $\frac{J_i^+(f,f)}{1+\delta f} \nearrow J_i^+(f,f)$, as $\delta \searrow 0+$, Lemma 7.8 follows by the monotone convergence theorem. ■

Now we may demonstrate the following theorem.

Theorem 7.1: Let f_0 satisfy

$$\iint_{R^3 \times R^3} (1 + |\vec{r}_1|^2 + |\vec{v}|^2 + log f_0) f_0 \quad < \quad \infty. \quad (7.31)$$

Then there exists a function $f \in C((0,\infty); L^1(R^3 \times R^3))$ satisfying

$$(\partial_t + \vec{v} \cdot \nabla_{\vec{r}}) f + \eta f L(f) = J(f,f). \quad (7.32)$$

Proof: $F_i^{\#}(\vec{r}_1, \vec{v}, t) = \int_0^t L_i^{\#}(f)(\vec{r}_1, \vec{v}, \tau) d\tau$ is absolutely continuous with respect to t for almost all $(\vec{r}_1, \vec{v})$, and $\frac{d}{dt} F_i^{\#} = L_i^{\#}(f)$ a.e. in t. Lemma 7.7 and Lemma 7.8

imply that $f^{\#}$ is an absolutely continuous function of t for almost all $(\vec{r}_1, \vec{v})$, so is $f^{\#} e^{(1+\eta)\sum F_i^{\#}}$. Therefore

$$\frac{d}{dt}\left[f^{\#} e^{(1+\eta)\sum_{i=E}^{4} F_i^{\#}}\right] \geq \sum_{i=E}^{4} J_i^{+}(f,f)^{\#} \cdot e^{(1+\eta)\sum_{i=E}^{4} F_i^{\#}}. \tag{7.33}$$

It follows that

$$\frac{d}{dt} f^{\#} \geq J^{+}(f,f)^{\#} - (1+\eta) f^{\#} L^{\#}(f) = J(f,f)^{\#} - \eta f^{\#} L^{\#}(f). \tag{7.34}$$

This, together with Lemma 7.8, implies (7.32). This completes the proof of Theorem 7.1.

Beginning with a sequence of solutions $\{f_{\eta_\nu}\}$ of Eq.(8.20) with $\eta_\nu \to 0$ as $\nu \to \infty$, all with initial value $f_{\eta_\nu}(0) = f_0$, we may mimic the proof of Theorem 7.1 to obtain the main result of this work.

Theorem 7.2: Let f_0 satisfy (7.31). Then there exists a function f satisfying (2.1) with $f \in C([0,\infty), L^1(R^3 \times R^3))$.

§8. Conclusions

We have treated the Enskog equation with attractive square-well intermolecular potential and geometric factor $Y \equiv 1$. The main result is the existence Theorem 7.2. This extends earlier work of Liu and Greenberg [28],[20] on this problem with a (high density) cutoff geometric factor. The case $Y \equiv 1$ is important for two reasons. First, it is a better model of a moderately low density gas than the cutoff model. In the second place, $Y \equiv 1$ is the zeroth order term in a density expression of the full geometric factor Y for a dense gas. Thus, this represents the first step in treating the case of general Y.

The Enskog equation with repulsive square well potential can be treated in the same manner as the model studied here. Indeed, the estimates for this case are somewhat simpler. Thus, the results of Therem 7.2 can be taken over *mutatis mutandis* for the repulsive square-well potential.

Actually, more can be said. The Enskog equation with arbitrary finite range piecewise constant potential consisting of N steps can be written:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\vec{r}_1} f = C_1(f,f) + \iint\limits_{R^3 \times S^2_+} \sum_{j=1}^{3N} Y[k^{\pm}_{R_j a}(\vec{r}_1, \vec{r}_2; \vec{\sigma}) f(\vec{r}_1, \vec{v}_j^{\pm}, t) f(\vec{r}_2, \vec{w}_j^{\pm}, t)$$
$$- k^{\mp}_{R_j a}(\vec{r}_1, \vec{r}_2; \vec{\sigma}) f(\vec{r}_1, \vec{v}, t) f(\vec{r}_2, \vec{w}, t)](R_j a)^2 (\vec{\sigma} \cdot \vec{V}) \theta_j d\vec{w} d\vec{\sigma} \tag{8.1}$$

where C_1 is the Enskog collision term, and velocities $\vec{v}_j^{\pm}$, $\vec{w}_j^{\pm}$ as well as the Heaviside function θ_j are determined by the kinetics of the collision process at the N well interfaces. The results of Theorem 7.2 extend immediately to this case as well. Thus we may consider existence to have been proved for an arbitrary step function approximation to a true van der Waals gas.

The results herein have been stated for the configuration space with $\vec{r}$ contained in all R^3. There is no difficulty in carrying the same arguments through for $\vec{r}$ in a periodic box, with the obvious modification of (7.31), of course. The case of $\vec{r}$ confined to a bounded region with diffuse or specular reflection at the boundary is an important problem which is still open.

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1990–95 Virginia Tech.:

Ph.D. (Mathematical Physics), May, 1995

1982–84 China Institute of Atomic Energy (Beijing):

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1978–81 Wuhan University (Wuhan, China):

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EMPLOYMENT AND EXPERIENCE:

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PUBLICATIONS

1. Aixiang Yao, *Neumann Series Solution for the Integral Boltzmann Transport Equation*, Acta Math. Sci., English Ed., **6**(1986), no.3, 279-285. (Mathematical Review 89a: 82064).
2. Aixiang Yao and Mingzhu Yang, *Neumann Series Solution for General Stationary Transport Equation*, (in Chinese), J. Engineering Mathematics **4**(1987), no.3, 27-32.
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