A Mathematical Programming-Based Analysis of

A Two-Stage Model of Interacting Producers

by

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(ABSTRACT)

This dissertation is concerned with the characterization, existence and computation of equilibrium solutions in a two-stage model of interacting producers. The model represents an industry involved with two major stages of production. On the production side there exist some (upstream) firms which perform the first stage of production and manufacture a semi-finished product, and there exist some other (downstream) firms which perform the second stage of production and convert this semi-finished product to a final commodity. There also exist some (vertically integrated) firms which handle the entire production process themselves.

In this research, the final commodity market is an oligopoly which may exhibit one of two possible behavioral patterns: follower-follower or multiple leader-follower. In both cases, the downstream firms are assumed to be price takers in purchasing the intermediate product. For the upstream stage, we consider two situations: a Cournot oligopoly or a perfectly competitive market.

An equilibrium analysis of the model is conducted with output quantities as decision variables. The defined equilibrium solutions employ an inverse derived demand function for the semi-finished product. This function is derived and characterized through the use of mathematical programming problems which represent the equilibrating process in the final commodity market. Based on this analysis, we provide sufficient conditions for the existence (and uniqueness) of an equilibrium solution, under various market assumptions. These conditions are formulated in terms of properties of the cost functions and the final product demand function.

Next, we propose some computational techniques for determining an equilibrium solution. The algorithms presented herein are based on structural properties of the inverse derived demand function and its local approximation. Both convex as well as nonconvex cases are considered.

We also investigate in detail the effects of various integrations among the producers on firms' profits, and on industry outputs and prices at equilibrium. This sensitivity analysis provides rich information and insights for industrial analysts and policy makers into how the foregoing quantities are effected by mergers and collusions and the entry or exit of various types of firms, as well as by differences in market behavior.

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Chapter 1

Introduction

1.1. Model description

The model which is presented and analyzed herein can be described in the following way. A homogeneous product is being supplied to a market with a perfectly competitive demand side. This product must undergo two major production stages before entering the consumer market. There are some n_1 firms (upstream producers) which handle only the first stage of production and sell a semi-finished version of the product to some other n_2 firms (downstream producers) which in turn perform the second stage of production and then market the final product. There also exist some n_3 firms (vertically integrated) which handle the entire production process themselves.

For example, in the copper industry there are some firms which engage only in the mining operations (upstream stage), and there are other firms which perform only the smelting and refinery processes (downstream stage). However, there also exist firms which are engaged in the entire operation of mining, smelting and refining. Also, the petroleum industry can be viewed as a two-stage structure (Greenhut and Ohta [G3]), where the upstream stage involves petroleum refining and the downstream one involves gasoline distribution to the consumer market.

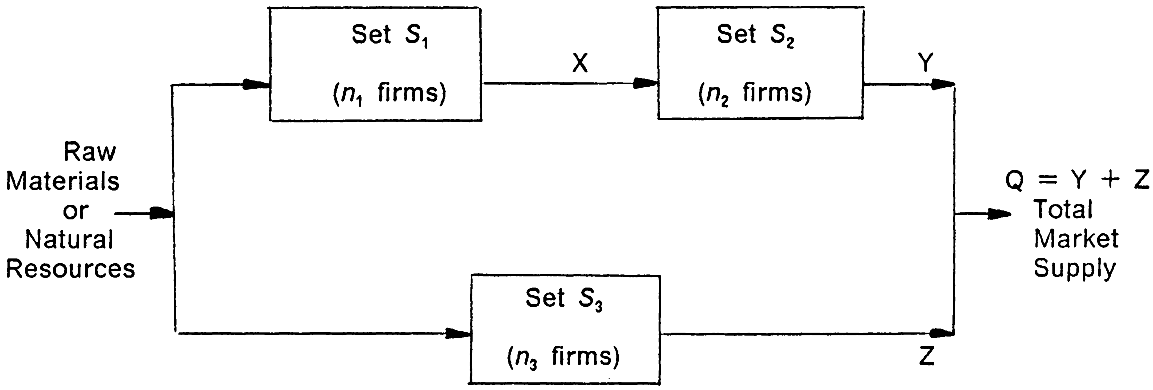


Figure 1. Graphical Illustration of a Two-Stage Model

In the considered model, the input and output of the n_2 firms performing the second production stage is assumed to be in a fixed proportion. For simplicity and without loss of generality, this proportion is taken to be one-to-one.

Figure 1 illustrates the situation and motivates our description of the model as a "two-stage network". Here, the above n_1, n_2 and n_3 firms constitute sets S_1, S_2 and S_3 respectively, and the quantities X (or Y) and Z denote the total outputs of the firms in S_1, (or S_2), and S_3 respectively, with Q being the total market supply.

We assume that the demand function for the final product and the cost functions, for each firm in $S_1 \cup S_2 \cup S_3$, are known. Furthermore, the downstream producers (S_2) are assumed to be perfect competitors (price takers) in purchasing the intermediate product.

The described model is viewed as a two-market structure in which an equilibrating process in the final commodity market generates a demand function for the semi-finished product, faced by the upstream producers. This demand function, being referred to as the derived demand function, is used by the firms in S_1 to determine their equilibrating output levels.

For a fixed price of the intermediate product, various behavioral assumptions for the firms in $S_2 \cup S_3$ produce various equilibrating outputs for these firms, and in particular, various outputs for the downstream producers. Consequently, the derived demand functions vary, depending on the form of the final commodity market. Also, given a derived demand function, an equilibrium solution in the semi-finished product market is associated with a certain type of behavioral interaction among the upstream producers. Therefore, various behavioral assumptions for the final product suppliers, as well as for the upstream firms produce various definitions of an equilibrium solution.

In our analysis, the final commodity market is an oligopoly which may exhibit one of two possible behavioral patterns. In the first case this market represents a Cournot model of oligopoly, and consequently, each firm in $S_2 \cup S_3$ chooses its profit maximizing output level assuming that its decision does not precipitate any reaction from the remaining firms in $S_2 \cup S_3$. That is, each firm is satisfied by producing at a level which maximizes its own profit, given the semi-finished product price and the output levels of its rivals as fixed. This behavioral pattern is also referred to as "follower-follower", thus emphasizing that all the final product suppliers behave in the same manner, i.e. as followers. However, since the firms in S_3 can be viewed as resulting from a vertical integration of some upstream and downstream producers, they are likely to identify the reaction of the firms in S_2, and adopt a leadership role in the final commodity market. Therefore, we also consider a "leader-follower" type of behavior, with firms in S_2 as followers and those in S_3 as leaders. In this case, the final commodity market is modelled as a multiple leader-follower extension of Stackelberg's oligopoly, in the spirit introduced first by Sherali [S3]. Each follower is assumed to behave like a Cournot oligopolist. Each leader is assumed to be aware of the aggregate follower reaction curve, which for a fixed semi-finished product price determines how the total equilibrating output of the firms in S_2 changes depending on variations in the total leader output, and considers this curve explicitly in selecting its profit maximizing decision. That is, each leader seeks to maximize its profit by manipulating the outputs of the downstream producers. However, it is assumed to make no effort in manipulating the outputs of the remaining leader firms, so that interactions among the leaders themselves are of Cournot type. One may justify that this behavior is relevant in situations when the downstream firms, being in some sense dependent on the upstream producers, are relatively smaller or weaker than the vertically integrated firms in S_3 . It is assumed that each producer knows who is the leader and who is the follower, and furthermore, knows the demand function and the cost functions of all its opponents. Such a situation is quite plausible when either the firms in S_2 or S_3 are identical. As in a Stackelberg leader-follower oligopoly, sequential decision making is imposed [F2]. First, the leader firms, being aware of the aggregate follower reaction, simultaneously choose their output levels, and announce their decisions to the followers. After that, the followers simultaneously choose their output quantities. When $n_3 = 1$, the above model reduces to the usual Stackelberg leader-follower oligopoly, and when $n_3 = 0$ or $n_2 = 0$, it becomes the Cournot oligopoly.

Given a derived demand function, an equilibrium solution in the semi-finished product market is characterized in two cases, namely, when all the upstream producers are Cournot firms, and also when they exhibit a perfectly competitive behavior. In the case of the Cournot oligopoly, interactions among the firms are conceptually the same as described earlier, for the final commodity market. In the case of perfectly competitive behavior, each firm in S_1 knows that it has no unilateral impact on the market price of its product, and therefore, considers it as fixed. Consequently, each firm is satisfied to produce at that level which equates its marginal cost with the market price.

1.2. Summary of Results

We begin our analysis of the two-stage model by providing conditions for the existence of an equilibrium solution. The crucial task here lies in demonstrating the existence and properties of the (inverse) derived demand function to be used by the upstream producers in selecting their profit maximizing output levels. Attention is focused on identifying those properties of the market demand function and the cost functions of the firms in $S_2 \cup S_3$ which guarantee that the inverse demand function for the semi-finished product exists and is well defined and continuous, being strictly decreasing and concave over its positive range. The first of these properties are useful in demonstrating the existence of an equilibrium solution in the semi-finished product market when the firms in S_1 are perfect competitors, while all four properties are required to analyze the case when they are Cournot firms.

For the purpose of derivation and characterization of the demand function for the intermediate product, the equilibrating process in the final commodity market is cast in terms of mathematical programming problems, following the approach of Murphy, Sherali and Soyster, [M3] , with an additional parameter being the semi-finished product price.

When considering the follower-follower behavior of the firms in $S_2 \cup S_3$, the inverse demand function for the intermediate product is shown to exist, be continuous and strictly decreasing over its positive range under quite general assumptions for the market demand function and the cost functions in $S_2 \cup S_3$. Concavity of the inverse derived demand function is demonstrated under assumptions which involve a particular form of the market demand function and convexity and concavity of the marginal cost functions for the firms in S_2 and in S_3, respectively. Moreover, the downstream producers are required to have identical cost functions, since otherwise, the derived demand function can be only locally concave.

The assumption of identical firms in S_2 is even more crucial when the firms in S_3 behave as leaders. It is shown that if this requirement is relaxed, then various prices for the semi-finished product can yield the same, positive total (equilibrating) output for the firms in S_2 . Consequently, for that value of output, the inverse image set contains more than one element and hence, the inverse mapping is a correspondence (point-to-set map) but not a function.

For the leader-follower model we show that the inverse demand function for the semi-finished product is continuous and strictly decreasing over its positive range under certain conditions which are more restrictive than in the follower-follower case. In contrast to the follower-follower model, the cost functions are assumed to be identical, and moreover, quadratic or linear. Furthermore, the market demand function is assumed to exhibit certain properties which involve the first, second and third order derivatives. Concavity of this derived inverse demand function is shown under additional requirements regarding the nature of the cost functions of the firms in S_3.

As far as the computation of an equilibrium solution is concerned, it may be worth mentioning that except for some simple cases, the derived demand function is available only implicitly, and hence the determination of an equilibrium solution is not a trivial task. The algorithm presented herein is based on the structural properties of this function and its local approximation, under both leader-follower and follower-follower behavior. It is fashioned to perform an iterative bisection search on the semi-finished product price interval until the equilibrating price (and hence, an equilibrium solution) is determined with a required accuracy. In the case when the upstream producers are Cournot firms the search is conducted with the use of tangential approximations to the derived inverse demand function. On the other hand, when the upstream stage is perfectly competitive, the algorithm simply performs a bisection search to determine the intersection point of the derived demand function and the supply function of the firms in S_1.

Finally, we proceed to investigate issues concerning integrations among producers when both upstream and downstream markets are oligopolistic. By way of terminology, the conversion of firms from the set S_1 or S_2 to firms of the type in S_3, perhaps by merging from S_1 and S_2 is referred to as vertical integration. On the other hand, mergers and collusion of firms within any single set itself is referred to as horizontal integration.

In this part of the analysis, we seek to determine who benefits and who loses if the producers integrate in various ways. Hence, this analysis provides insights into incentives to merge or collude, and for making various policy decisions. In general, there does not exist a straightforward answer to the question of whether or not a given type of integration is profitable, particularly, when the integration involves the firms in S_2 and/or those in S_3. The difficulty arises in that the resulting configuration of the new firms in the final product market imposes a new derived demand function faced by the firms in S_1, and a comparative analysis becomes quite complex. Therefore, to give some insight into such integration issues, we carefully examine a simpler model in which the equilibrium solution is unique and is available in closed form. For this case, regardless of the assumed market behavior for the firms in $S_2 \cup S_3$, a decrease in the total industry output (and therefore, a higher price paid by consumers) is observed as a result of horizontal integrations within any type of producers. On the other hand, a vertical integration may provide more industry output at lower prices. However, as more and more firms integrate, a decrease in the output may result and therefore give higher prices. As far as the profits of the individual producers are concerned, it is demonstrated that except for some extreme situations, horizontal integration is not beneficial for those firms which merge. On the other hand, the producers who abstain from merging always gain additional profits. This result cannot be extended for the case of vertical integrations, where changes in firms' profits depend intimately on the relative values of n_1, n_2 and n_3. Here, it may or may not be profitable to integrate, and we derive conditions on when such situations arise.

Also, by comparing the outcomes of the follower-follower model and the multiple leader-follower model for the situation analyzed, one can conclude that the latter provides more total industry output. However, the general nature of the consequences of various types of integrations are similar for the two types of models.

1.3. Literature Survey

The purpose of this section is to present a brief survey of papers which have considered similar concepts and issues that arise in the context of our analysis. The topics discussed here include the Cournot oligopoly, the Stackelberg type of oligopoly and models related to the two-stage network of our interest.

The Cournot oligopoly [C1] is the most famous and classical model of imperfect competition. In an n-firm setting, it describes an industry in which all n firms produce a homogeneous product. Each firm knows the market demand function, and seeks to maximize its profit with respect to its own output level assuming that its rivals' outputs are fixed. The firms act in a noncooperative fashion so that any agreements are excluded. The Cournot equilibrium solution is then defined as a set of n output levels having the property that none of the firms can be better off given the equilibrating outputs of the others.

In modern studies the n-firm Cournot oligopoly is cast in terms of a noncooperative n-person game. The concept of an equilibrium point in such games was introduced by Nash [N1], and therefore, the Cournot equilibrium is often referred to as the Nash-Cournot (NC) equilibrium.

Existence of a NC equilibrium solution received much attention in the literature and was approached either by means of fixed point theorems [F1, F3, N2, O1, S2] or by means of mathematical programming concepts [M3, S7, S8] . As a result, various sets of conditions are known to guarantee the existence of a NC equilibrium. Regardless of the employed methodology, investigations have been aimed at identifying conditions which ensure concavity and boundedness of each firm's profit function with respect to its own output level [F1, M3, S1, S7, S8]. Friedman [F2, F4], in the context of noncooperative games, has relaxed the concavity requirement and based his proof on quasiconcavity and continuity. In the work by Nishimura and Friedman [N2], even quasiconcavity was relaxed and was replaced by weaker conditions which involve certain properties of the best reply mappings. Although this result is of an unarguable theoretical value, its practical usefulness is somehow limited because the conditions required therein can hardly be verified a priori.

The weakest conditions for the uniqueness of the NC equilibrium point were obtained by Szidarovszky and Yakowitz [S8], and also by Murphy et al. [M3]. The significance of their results lies in that, in contrast to earlier works [F4, O1, S7], they do not require the existence of second order derivatives of the demand function and all the cost functions. However, as pointed out by Szidarovszky and Yakowitz [S8], differentiability of the demand function is crucial for the uniqueness of the NC equilibrium solution, but not for the uniqueness of the total industry output at equilibrium.

Mathematical programming-based analysis by Murphy et al. [M3] also provides an interesting procedure for finding NC equilibrium solutions. The algorithm employs singly constrained convex programs parameterized by the total industry output. The authors have demonstrated that in order to find an equilibrium solution one simply needs to determine that value of the total industry output parameter for which the optimal Lagrange multiplier associated with the single constraint is zero. Then, the corresponding optimal solution represents a NC equilibrium. The search is supported by continuity and monotonicity of the Lagrange multiplier with respect to this parameter. Similar concepts, though under more restrictive assumptions, have been presented by Szidarovsky and Yakowitz [S7], where a NC equilibrium solution for an n-firm oligopoly is cast in terms of optimal solutions to n unconstrained convex programs.

Fixed point algorithms (e.g. Scarf [S2]) can also be used to approximate a NC equilibrium point. However, in this case, they are likely to be computationally expensive due to the relative inefficiency of the complementary pivoting operations involved in each iteration. Some computational results concerning general economic equilibrium have been reported by Scarf [S2] , and by MacKinnon [M1].

Another question posed in the context of Cournot models is how the total industry output depends on the number of oligopolists. Cournot [C1] himself expected that an increase in the number of firms results in a higher output and hence, a lower market price. Although in general, this assertion is not true [F1, M2], there exists a large class of models which exhibit such a property [F2, M2, O1, R2, S8] . On the other hand, a decrease in the number of oligopolists, perhaps by the collusion of firms, may lead to a lower industry output and a higher market price. However, as demonstrated in [S1, S8], the collusion of firms may also bring unexpected changes in firms' profits. Salant, Switzer and Reynolds [S1] have investigated losses from horizontal mergers in Cournot oligopoly. Using a linear demand function and a linear cost function, identical for all the firms, they have demonstrated that if less than eighty percent of producers merge into one firm, then integration brings losses to the new firm. However, in spite of the losses incurred by the merging producers and by the consumers, social effects may be positive when all the firms have a positive fixed cost. Szidarovszky and Yakowitz [S8] have analyzed the impacts of horizontal integration under more general assumptions. The demand function is assumed to be concave and decreasing while the cost functions are assumed to be convex and increasing, not necessarily identical. They have shown that if the demand function is strictly decreasing, or if all the cost functions are strictly convex, then the integration does not increase the total industry output. Simultaneously, producers who abstain from merging do not lose in their profits and in fact, they may gain. This assertion, however, requires differentiability of the demand function so that the equilibrium solution is unique regardless of the number of firms. As far as the profits of the merging firms are concerned, Szidarovszky and Yakowitz indicate that these may decrease.

Also, Sherali and Rajan [S4] have investigated, in the context of cooperative games, the effects on profits resulting from oligopolistic firm collusions. Therein, the authors focus on determining what kind of coalition structure would emerge, under various types of behavior for the producers, using Shapley value allocations based on profits as the characteristic function.

Models incorporating a leader type of behavior were first introduced by Stackelberg [S6], in the context of a duopoly. In a leader-follower duopoly, the leader firm assumes that the other is a follower which will react to its own strategies, and consequently, uses the follower's reaction function in selecting its profit maximizing output level. In this manner, the leader firm in fact manipulates its rival's decision. At the Stackelberg equilibrium neither the follower nor the leader has incentives to depart from their output levels.

In static models with one leader and at least one follower, the Stackelberg equilibrium is relevant only when a sequential decision making is imposed [F2, S5]: first, the leader announces its decision and after that the followers simultaneously announce theirs. The Stackelberg equilibrium can then be viewed as a Nash noncooperative equilibrium.

In Stackelberg's leader-leader duopoly, each firm assumes that the other is a follower and hence, uses an invalid reaction curve. The model therefore exhibits serious inconsistencies that lead to a disequilibrium or chaos [F2, O1] . Few attempts [F5, 01, S3] have been made to formulate a multiple leader model in which an equilibrium rather than chaos or warfare is the outcome. In the context of static, quantity models, Sherali [S3] has presented a multiple leader model which is a generalization of a one-leader-several-followers case [S6], and which provides a consistent extension of Stackelberg's leader-follower duopoly. Here, each firm knows who is the leader and who is the follower. Each leader treats the other leaders as Cournot oligopolists. However, each leader is aware of the aggregate reaction curve of the followers in response to variations in the total leader firms output, and uses this information while making its own decisions. In this formulation, each leader uses true follower reaction curves, so that the previously mentioned inconsistencies are not encountered. Existence of a Stackelberg type of equilibrium has been demonstrated in cases which yield a convex aggregate reaction curve. Sherali [S3] has also shown that at the unique (interior) equilibrium, each leader earns more profit than a follower, provided they have identical cost functions. However, except for the case with one leader [S5], it may be more profitable for each individual firm to act as a follower, that is belong to the pure Cournot oligopoly.

In the context of dynamic processes, the stability of different multiple leader types of models have been investigated by Okuguchi [O1] and by Furth [F5]. Okuguchi [O1] has considered a leader-leader duopoly with incomplete information on the rival's cost function, and has derived conditions for stability when the demand function and the perceived (rival's) marginal cost functions are linear. In Furth's [F5] oligopoly model (with complete information), a follower is allowed to actively participate in price formation, in the way described by Okuguchi [O2]. That is, each follower predicts some linear price reaction function for the remaining firms in response to its own price, and maximizes its expected profit on that knowledge. On the other hand, Furth's leader assumes that all the other firms are followers, but surprisingly, makes no real use of their reaction function so that in fact, the leaders have been degraded to the Cournot firms.

Models closely related to our two-stage network were first studied by Greenhut and Ohta [G1, G3], mostly from the viewpoint of studying benefits resulting from the integration of firms. These models have been critiqued by Haring and Kaserman [H1] and by Perry [P1]. Greenhut and Ohta [G1] have considered a situation in which the intermediate product market is monopolistic, while the final commodity market is either perfectly competitive, or oligopolistic, or monopolistic, with the firms in S_3 being absent. It is assumed that the demand function is decreasing and that all the cost functions are linear, being identical in the case of the downstream producers. Under certain additional assumptions concerning the demand curve, Greenhut and Ohta demonstrate that at an equilibrium, the price charged by the monopolist (S_1) is independent of the behavioral conditions in the final commodity market, and furthermore, is independent of the number of firms n_2 in the Cournot oligopoly S_2. However, Haring and Kaserman [H1] showed that this result does not hold if the monopolist has an increasing and quadratic cost function. Greenhut and Ohta also assert that vertical integration of firms in $S_1 \cup S_2$ brings about higher industry output and hence lower market prices. As pointed out by Perry [P1] the vertical integration in [G1] was stated improperly and moreover, in the case of total collusion, the claimed result follows from the elimination of the upstream monopolist, rather than from the vertical integration itself. Robustness of the above conclusions on pricing was questioned by Waterson [W1], who investigated the effects of vertical integration of a monopolist into an oligopolistic industry in the case when substitution (with constant elasticity) between the inputs is possible, and the demand function exhibits constant price elasticity. Based on a simulation analysis of the model, he concludes that if the elasticity of substitution is fairly low, the integration is likely to result in higher prices for the final product.

Subsequently, Greenhut and Ohta [G3] have considered a two-stage model with Cournot oligopolies, in which the firms in S_1 have identical, linear costs, while those in S_2 have zero costs. The demand function is assumed to be "well behaved" in the sense that it results in concave revenue functions for all stages. They show that if some ℓ Cournot firms in each of S_1 and S_2 vertically integrate and result in $n_3 = \ell$ Cournot firms in S_3, while maintaining the linear costs costs of the firms in S_3 (i.e., ignoring any economies of scale other than market structural implications), then industry output increases and prices fall. Using a linear demand curve they point out however, that the individual firm profits may not favor this integration when $\ell > 1$. Next, Greenhut and Ohta [G3] compare the above new structure with ℓ Cournot firms in S_3 with one in which the ℓ vertically integrated firms adopt a leadership role, and assert that the latter provides even more industry output. Although these results are true under certain circumstances, Greenhut and Ohta derive them erroneously. First, in comparing the above three models, they use the same equilibrating price for the semi-finished product. Second, in writing the optimality conditions for the upstream firms, they neglect to include the reaction curve of the (integrated) producers in S_3.

In comparison with the work by Greenhut and Ohta [G1, G3] , this dissertation gives a far more thorough insight into the nature of the equilibrating process in a two-stage industry, and into the issues and consequences of various types of integrations of firms. First of all, the present analysis is concerned with a derivation of conditions under which an equilibrium exists, and is unique, including the case of a perfectly competitive market for the intermediate product. In particular, we accomplish this by deriving properties for the implicitly determined intermediate product demand function, rather than a priori assume its structure. The analysis therefore provides a basis for a rigorous evaluation and comparison of two-stage oligopolistic models with follower-follower and leader-follower behavior of the final commodity suppliers, based on available cost function and end-product demand function characterization. Moreover, we offer some computational techniques for determining an equilibrium solution, under various market assumptions. The framework of the model itself allows for more general assumptions on the cost functions for all three types of producers. In particular, the downstream stage is not restricted to operate at zero cost, and moreover, the vertically integrated firms are allowed to set their costs independently from those of the upstream and downstream stages. Furthermore, in investigating the firms integration issues, we allow for various types of mergers or integrations, and examine a wide range of effects, including changes in profits faced by each producer.

1.4. Outline of the Thesis

Chapter 2 is concerned with the existence of an equilibrium solution in the network when the final commodity suppliers exhibit the follower-follower behavior. The multiple leader-follower case is analyzed in Chapter 3. Algorithms for finding (approximating) the four types of equilibrium solutions discussed in Chapters 2 and 3 are presented in Chapter 4. Chapter 5 deals with issues concerning horizontal and vertical integration of firms. Chapter 6 provides some insight into a comparison of total industry outputs associated with oligopolistic equilibria. Also, conclusions and suggestions for the future studies are included in this Chapter. Appendix A contains the proof of Theorem 3.8 (ii) stated in Chapter 3, Section 3.3 and also a brief derivation of results presented in Chapter 5. Finally, Appendix B contains some graphical illustration of the effects of vertical integration.

Chapter 2

The Two-Stage Model with Cournot Oligopoly

in the Final Product Market

The purpose of this chapter is to present an analysis of the two-stage model in the case when the final commodity market is a Cournot oligopoly. Relevant notation, assumptions and definitions are introduced in Section 2.1. Section 2.2 deals with the characterization of derived (inverse) demand function faced by the upstream producers. Existence and uniqueness of an equilibrium solution when the upstream firms are perfect competitors or Cournot firms is established in Section 2.3. Finally, Section 2.4 gives a brief summary of results concerning the analyzed model.

2.1. Notation, Assumptions and Definitions

In this section, we state our main assumptions, introduce the relevant notation and formally define what we mean by an equilibirum solution under stated market behavioral conditions.

Toward this end, consider the two-stage network introduced in Section 1.1, with the sets S_1, S_2 and S_3 being comprised of n_1 ,n_2 and n_3 firms respectively. Notationally, we will use symbols $x = (x_1, \dots , x_{n_1})$, $y = (y_1, \dots, y_{n_2})$, and $z = (z_1, \dots, z_{n_3})$ to denote the outputs of the firms in S_1, S_2 and S_3 respectively. Additionally, the upper case symbols X, Y and Z will be used to respectively denote the total outputs of the firms in the sets S_1, S_2 and S_3.

Next, consider the following set of assumptions on the demand and cost functions. Let $p(Q)$, $Q \geq 0$, denote the market (inverse) demand function for the final commodity, i.e., $p(Q)$ is the price at which the consumers will purchase Q units of the final product. We assume that

(A1) $p(Q)$ is twice differentiable, strictly decreasing, with $p'(Q) < 0$ for $Q > 0$ and moreover,

(A2) $p'(Q) + Qp''(Q) \leq 0$ for all $Q \geq 0$.

As proven by Murphy et al. [M3], assumptions (A1) and (A2) imply that for each $K \geq 0$, $Qp(Q + K)$ is a strictly concave function of $Q \geq 0$

Further, let $f_i(\cdot)$, $i = 1,\dots,n_1$, $g_i(\cdot), i = 1,\dots,n_2$, $h_i(\cdot)$, $i = 1,\dots,n_3$, denote the cost functions for the firms in S_1, S_2 and S_3 respectively. We assume throughout that

(A3) $f_i(\cdot)$, $i = 1,\dots,n_1$, $g_i(\cdot), i = 1,\dots,n_2$, and $h_i(\cdot)$, $i = 1,\dots,n_3$ are nondecreasing, twice differentiable and convex over the nonnegative real line.

Moreover, we assume that

(A4) there exists a quantity $q_u \geq 0$ such that

$$g_i'(q_u) \geq p(q_u), \; i = 1,\dots,n_2$$

$$h_i'(q_u) \geq p(q_u), \; i = 1,\dots,n_3$$

Assumption (A4), along with (A1) and (A3), simply guarantees the existence of an upper bound of q_u on the outputs of the profit maximizing firms in $S_2 \cup S_3$.

For a fixed price $P \geq 0$ for the semi-finished product, the effective total cost functions of the firms in S_2 are given by $g_i(y_i) + Py_i$, $y_i \geq 0$, for $i = 1,\dots,n_2$. Hence, given an input price $P \geq 0$, the firms in $S_2 \cup S_3$ behaving in a Nash-Cournot manner, will produce at equilibrium $y(P) = (y_1(P), \dots, y_{n_2}(P))$ with total $Y(P)$, and $z(P) = (z_1(P), \dots, z_{n_3}(P))$ with total $Z(P)$, where $y(P)$, $z(P)$ satisfy (2.1) and (2.2) below:

$$y_i(P) \quad \text{solves} \quad \underset{q \geq 0}{\text{maximize}} \; \{ qp[q + \sum_{\substack{j=1 \\ j \neq i}}^{n_2} y_j(P) + Z(P)] - Pq - g_i(q) \} \tag{2.1}$$

$$z_i(P) \quad \text{solves} \quad \underset{q \geq 0}{\text{maximize}} \; \{ qp[q + \sum_{\substack{j=1 \\ j \neq i}} z_j(P) + Y(P)] - h_i(q) \} \tag{2.2}$$

From the development in Murphy et al. [M3], we can conclude that under assumptions (A1)-(A4), if a certain price P prevails for the semi-finished product purchased by the firms in S_2 , then this elicits a unique equilibrating response $y(P)$ and $z(P)$ from the firms in S_2 and S_3, respectively. In fact, as shown in the sequel, under the foregoing assumptions, $y(P)$ and $z(P)$ are continuous functions of $P \geq 0$. In view of this, we will assume the reasonable property that the right-hand derivatives of $y(P)$ and $z(P)$ with respect to P exist, and include this in the above set of assumptions, henceforth referred to as **Assumptions A1.**

Now, since the total output of the firms in S_1 and S_2 must match, when the price established for the semi-finished product is P the demand faced by the firms in S_1 is $Y(P)$. Hence, as P varies, it generates a derived demand curve for the firms in S_1 according to the price reaction curve $Y(P)$. Under Assumptions A1, as shown in the sequel, $Y(P)$ is a nonnegative, continuous function, strictly decreasing over its positive range, say, $[0, P_0)$. Denote $Y_0 = Y(0)$ and define

$$\Phi(\theta) = \begin{cases} P_0 & \text{for} \quad \theta = 0 \\ Y^{-1}(\theta) & \text{for} \quad 0 < \theta < Y_0 \\ 0 & \text{for} \quad \theta \geq Y_0 \end{cases} \tag{2.3}$$

$\Phi(\theta)$, is the (inverse) **derived demand function** faced by the upstream producers, for the semi-finished product. This function will be used to define the overall equilibrium solutions in the two-stage model.

Definition 2.1. A set of nonnegative output quantities $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n_1})$ with total $\bar{X}$, $\bar{y} = (\bar{y}_1, \dots, \bar{y}_{n_2})$ with total $\bar{Y}$, $\bar{z} = (\bar{z}_1, \dots, \bar{z}_{n_3})$ with total $\bar{Z}$, for the firms $1,\dots, n_1$ in S_1, firms $1,\dots, n_2$ in S_2, and firms $1,\dots,n_3$ in S_3, is said to be a **competitive equilibrium solution**, provided

$$\bar{x}_i \text{ solves } \underset{q \geq 0}{\text{maximize}} \{q\Phi(\bar{X}) - f_i(q)\} \text{ for each } i = 1,\dots, n_1 \tag{2.4}$$

and moreover,

$$\bar{y} = y[\Phi(\bar{X})] \quad \text{and} \quad \bar{z} = z[\Phi(\bar{X})].$$

Hence, the above equilibrium corresponds to an equilibriation of supply and demand at a certain semi-finished product price.

On the other hand, when the firms in S_1 behave as Cournot firms, we obtain the following equilibrium solution definition:

Definition 2.2. A set of nonnegative output quantities $x^o = (x_1^o, \dots, x_{n_1}^o)$ with total X^o, $y^o = (y_1^o, \dots, y_{n_2}^o)$ with total Y^o, $z^o = (z_1^o, \dots, z_{n_3}^o)$ with total Z^o, for the firms $1,\dots, n_1$ in S_1, firms $1,\dots, n_2$ in S_2, and firms $1,\dots, n_3$ in S_3, is said to be an **oligopolistic equilibrium solution**, provided

$$x_i^o \text{ solves } \underset{q \geq 0}{\text{maximize}} \{ qp(q + \sum_{\substack{j=1 \\ j \neq i}}^{n_1} x_j^o) - f_i(q)\} \tag{2.5}$$

and moreover, with $P^o = \Phi(X^o)$,

$$y^o = y^o(P^o) \quad \text{and} \quad z^o = z^o(P^o) \tag{2.6}$$

Note, that in the above definitions, the firms in S_1 are assumed to know and use the functional form of the price reaction of the final product suppliers (or its inverse $\Phi(\cdot)$) in maximizing their profits. At an oligopolistic equilibrium, when it exists, each firm in S_1 will have maximized its profit, given the outputs of the other firms in S_1, while considering the price reaction of the firms in $S_2 \cup S_3$ oligopoly. Let X^o be the total output of the firms in S_1. Similarly, each firm in S_2 will have maximized its profit by purchasing the semi-finished product at the price $P^o = \Phi(X^o)$, given the outputs of the other firms in $S_2 \cup S_3$, with $Y^o \equiv Y(P^o) \equiv X^o$ being the total output of the firms in S_2. Finally, each firm in S_3 will have maximized its profit, given the outputs of the other firms in $S_2 \cup S_3$, with the total production level being $Z^o = Z(P^o)$. The prevailing price for the final product is therefore $p(Q^o)$ where $Q^o = Y^o + Z^o$.

2.2. Characterization of the Derived Demand Curve

In order to provide facility for establishing the existence of the derived demand curve $\Phi(\cdot)$ and studying its nature, we define below a family of convex programming problems in the spirit of Murphy et al. [M3]. For a fixed value of $P \geq 0$ representing the price for the semi-finished product, and for a fixed value of $Q \geq 0$ representing the total market supply, consider the following equilibrating program parameterized by P and Q

EP(P,Q):

$$\text{maximize}\{\, p(Q)\Big(\sum_{i=1}^{n_2} y_i + \sum_{i=1}^{n_3} z_i\Big) + \frac{1}{2}p'(Q)\Big(\sum_{i=1}^{n_2} y_i^2 + \sum_{i=1}^{n_3} z_i^2\Big) - \sum_{i=1}^{n_2} g_i(y_i) + Py_i - \sum_{i=1}^{n_3} h_i(z_i)\}$$

subject to

$$\sum_{i=1}^{n_2} y_i + \sum_{i=1}^{n_3} z_i = Q \tag{2.7}$$

$$y_i \geq 0\,, \quad i = 1,\dots, n_2\,, \quad z_i \geq 0\,, \quad i = 1,\dots, n_3. \tag{2.8}$$

The program EP(P,Q) involves the maximization of a strictly concave function over a nonempty polytope, and consequently a unique optimum exists. Hence, the Karush-Kuhn-Tucker conditions for EP(P,Q), which are both necessary and sufficient for this problem, have a unique solution. These conditions are rewritten below, where λ is the Lagrange multiplier associated with constraint (2.7), and $v = (v_1, \dots, y_{n_2})$ and $w = (w_1, \dots, z_{n_3})$ are the Lagrange multipliers associated with constraints (2.8).

$$p(Q) + y_i p'(Q) - g_i'(y_i) - P + v_i - \lambda = 0, \quad i = 1,\dots, n_2 \tag{2.9}$$

$$p(Q) + z_i p'(Q) - h_i'(z_i) + w_i - \lambda = 0, \quad i = 1,\dots, n_3 \tag{2.10}$$

$$\sum_{i=1}^{n_2} y_i + \sum_{i=1}^{n_3} z_i = Q \tag{2.11}$$

$$y_i \geq 0, \ v_i \geq 0, \ i = 1,\dots, n_2 \ \text{ and } \ z_i \geq 0, \ w_i \geq 0, \ i = 1,\dots, n_3 \tag{2.12}$$

$$y_i v_i = 0, \ i = 1,\dots, n_2 \ \text{ and } \ z_i w_i = 0, \ i = 1,\dots, n_3. \tag{2.13}$$

Now, given $(P,Q) \geq 0$, denote by $y_E(P,Q)$ and $z_E(P,Q)$ the unique optimal solution to EP(P,Q), and let $\lambda_E(P,Q)$ be the corresponding Lagrange multiplier associated with constraint (2.7). (In case $Q = 0$, whence alternative optimal values exist for this multiplier, define $\lambda_E(P,0)$ to be the shadow price of Q, i.e., to be the smallest nonnegative optimal multiplier, which is readily seen to exist via (2.9)-(2.13)). The following result is in the spirit of the analysis in Murphy et al. [M3], and relates to the equilibrating nature of program EP(P,Q).

Theorem 2.1. For a fixed $P \geq 0$, let $\hat{y}$ and $\hat{z}$ represent some nonnegative output vectors of the firms in S_2 and S_3 respectively, and denote $\hat{Q} = \sum_{i=1}^{n_2} \hat{y}_i + \sum_{i=1}^{n_3} \hat{z}_i$. Then, the quantities $\hat{y}$ and $\hat{z}$ with total output $\hat{Q}$ constitute a Nash-Cournot equilibrium for the firms in $S_2 \cup S_3$ (hence yielding $y(P) = \hat{y}$, $z(P) = \hat{z}$ and $Q(P) = \hat{Q}$) if and only if $\hat{y}$ and $\hat{z}$ solve EP(P, $\hat{Q}$) and moreover, $\lambda_E(P, \hat{Q}) = 0$.

Proof. The proof follows by simply comparing the necessary and sufficient Karush-Kuhn-Tucker conditions for problems (2.1) and (2.2) with the corresponding conditions (2.9)-(2.13) for EP(P, $\hat{Q}$). ■

Hence, with $P \geq 0$ fixed, if Q is adjusted to a value $\hat{Q}$ such that in the problem EP(P, $\hat{Q}$), the optimal Lagrange multiplier associated with (2.7) vanishes, then by Theorem 2.1, the resulting optimal solution $\hat{y} = y_E(P, \hat{Q})$ ands $\hat{z} = z_E(P, \hat{Q})$ represents a Nash-Cournot equilibrium for the $S_2 \cup S_3$ oligopoly. In fact, the following result implies that a unique value of such a $\hat{Q}$ exists in the interval $[0, (n_2 + n_3)q_u)$, and may be determined via a bisection search based on the sign of $\lambda_E(P,Q)$ for $Q \in [0, (n_2 + n_3)q_u)$.

Lemma 2.1. Let $P \geq 0$ be given and fixed, and let $\lambda_E(P,Q)$, $Q \geq 0$ be as defined above. Then $\lambda_E(P,0) \geq 0 > \lambda_E(P, (n_2 + n_3)q_u)$. Moreover, $\lambda_E(P,Q)$ is a continuous, strictly decreasing function of $Q > 0$, and

(i) $\lambda_E(P,Q) < 0$ for $Q > 0$ if $\lambda_E(P,0) = 0$,

(ii) $\lambda_E(P,Q)$ is continuous at $Q = 0$ if $\lambda_E(P,0) > 0$.

Proof. By definition, $\lambda_E(P,0) \geq 0$. Define $Q_u = (n_2 + n_3)q_u$ and consider $\lambda_E(P,Q_u)$. Since at least one firm must produce at least q_u in EP(P,Q_u), letting this firm's cost function $g_i(\cdot)$ or $h_i(\cdot)$ be denoted by $G(\cdot)$, we obtain from (2.9)-(2.13) that $\lambda_E(P,Q_u) < p(Q_u) - G'(q_u)$, since $P \geq 0$, $G(\cdot)$ is convex and $p'(Q_u) < 0$. From (A4) this implies that $\lambda_E(P,Q_u) < p(Q_u) - p(q_u) \leq 0$.

The assertion that for any fixed $P \geq 0$, $\lambda_E(P,Q)$ is continuous and strictly decreasing in $Q > 0$ follows directly from the development in Murphy et al. [M3]. Hence, let us prove parts (i) and (ii). For notational convenience, let $G_i(q) = Pq + g_i(q)$, $q \geq 0$ for $i = 1,..., n_2$ and $G_i(q) = h_{i-n_2}(q)$, $q \geq 0$ for $i = n_2 + 1,..., n_2 + n_3$ be the cost functions for the firms in S_2 and S_3, respectively. Similarly, let $g_i(Q)$ be $y_{Ei}(P,Q)$ for $i = 1,..., n_2$ and $z_{E(i-n_2)}(P,Q)$ for $i = n_2 + 1,..., n_2 + n_3$. Let $G_k'(0) = \min\{G_i'(0),\ i = 1,..., n_2 + n_3\}$, and note from (2.9)-(2.13) that $\lambda_E(P,0) = \max\{p(0) - G'_k(0),\ 0\}$.

If $\lambda_E(P,0) = 0$ then $p(0) - G_i'(0) \leq 0$ for all $i = 1,..., n_2 + n_3$ and since for each $Q > 0$ from (2.9)-(2.13), $\lambda_E(P,Q) = p(Q) + q_i(Q)p'(Q) - G'_i[q_i(Q)]$ for some $i \in \{1,..., n_2 + n_3\}$, where $q_i(Q) > 0$, we have using $p(Q) < p(0)$, $p'(Q) < 0$ and the convexity of $G_i(\cdot)$ that $\lambda_E(P,Q) < 0$. This proves part (i).

On the other hand, suppose that $\lambda_E(P,Q) > 0$. If $q_k(Q) = 0$ for any $Q > 0$, then from (2.9)-(2.13) and the fact that $p'(Q) < 0$, and that $G_i(\cdot)$ is convex for all $i = 1,\ldots, n_2 + n_3$, we get that $\lambda_E(P,Q) \geq p(Q) - G_k'(0) \geq p(Q) - g_i'(0) > p(Q) + q_i(Q)p'(Q) - G_i'[q_i(Q)]$ for any $i \in \{1,\ldots, n_2 + n_3\}$ with $q_i(Q) > 0$, a contradiction to (2.9), (2.10) and (2.13). Hence, $q_k(Q) > 0$ for all $Q > 0$, and so from (2.9), (2.10) and (2.13), we get that

$$\lambda_E(P,Q) = p(Q) + q_k(Q)p'(Q) - G_k'[q_k(Q)], \quad \text{for each } Q > 0. \tag{2.14}$$

Now, as $Q \to 0^+$, $q_k(Q)$ must tend to zero by (2.11) and (2.12). Hence, we get from (2.14) that $\lambda_E(P,Q)$ approaches $p(0) - G_k'(0)$, which equals $\lambda_E(P,0)$ since $\lambda_E(P,0) > 0$ This proves part (ii) and completes the proof. ■

Hence, given $P \geq 0$, once such a unique value of $\hat{Q}$ is determined for which $\lambda_E(P, \hat{Q}) = 0$ we have

$$Q(P) = \hat{Q}, \quad y(P) = y_E(P,Q(P)), \quad z(P) = z_E(P,Q(P)) \tag{2.15}$$

Corollary 2.1. For any fixed $P \geq 0$ there exists a unique set of Nash-Cournot equilibrating outputs $y(P)$, $z(P)$ with totals $Y(P)$, $Z(P)$, for the firms in S_2 and S_3 respectively. Moreover, $Q(P) = Y(P) + Z(P)$ satisfies $0 \leq Q(P) < (n_2 + n_3)q_u$.

Proof. Follows directly from Theorem 2.1 and Lemma 2.1. ■

Based on the foregoing results, $y(P)$ and $z(P)$ are well defined functions of $P \geq 0$, and hence so are $Y(P)$, $Z(P)$ and $Q(P)$. The following result establishes continuity of $Q(P)$, $y(P)$ and $z(P)$.

Lemma 2.2. $Q(P)$, $y(P)$ and $z(P)$ are all continuous functions of $P \geq 0$.

Proof. First of all, note from Lemma 2.1 and Theorem 2.1 that given any $P \geq 0$, $Q(P)$ is given by that unique value of $\hat{Q}$ for which $\lambda_E(P, \hat{Q}) = 0$ in EP(P, $\hat{Q}$), and as in (2.15), $y(P)$ and $z(P)$ are accordingly obtained as the unique solutions to EP(P,$Q(P)$). Hence, from (2.9)-(2.13) and (2.15), given any $P \geq 0$, $Q(P)$, $y(P)$ and $z(P)$ satisfy the following system of equations, where $v_i(P)$, $i = 1,\ldots, n_2$ and $w_i(P)$, $i = 1,\ldots, n_3$ are again unique slacks in (2.9) and (2.10) respectively, for problem EP(P, $Q(P)$).

$$p[Q(P)] + y_i(P)p'[Q(P)] - g_i'[y_i(P)] - P + v_i(P) = 0, \quad i = 1,\ldots, n_2 \tag{2.16}$$

$$p[Q(P)] + z_i(P)p'[Q(P)] - h_i'[z_i(P)] + w_i(P) = 0, \quad i = 1,\ldots, n_3 \tag{2.17}$$

$$\sum_{i=1}^{n_2} y_i(P) + \sum_{i=1}^{n_3} z_i(P) = Q(P) \tag{2.18}$$

$$y_i(P) \geq 0, \quad v_i(P) \geq 0, \quad i = 1,\ldots, n_2 \quad \text{and} \quad z_i(P) \geq 0, \quad w_i(P) \geq 0, \quad i = 1,\ldots, n_3 \tag{2.19}$$

$$y_i(P)v_i(P) = 0, \quad i = 1,\ldots, n_2 \quad \text{and} \quad z_i(P)w_i(P) = 0, \quad i = 1,\ldots, n_3 \tag{2.20}$$

For the sake of notation, denote $\xi(P) = (Q(P), y(P), z(P), v(P), w(P))$. Now, consider any $P = P_o \geq 0$, and let $\{P_n\}$ be any nonnegative sequence convergent to P_o . Accordingly, define $\xi_n = (Q(P_n), y(P_n), z(P_n), v(P_n), w(P_n))$, and observe that from Corollary 2.1 and (2.16)-(2.20) that $\{\xi_n\}$ is a bounded sequence, and therefore has a convergent subsequence. Without loss of generality, assume that $\{\xi_n\}$ is itself convergent to some limit point ξ_o. We will show that $\xi_o = \xi(P_o)$.

Toward this end, observe that for each n, ξ_n solves (2.16)-(2.20) with $P = P_n$. Taking limits as $n \to \infty$ in this system of equations, and noting that $p(\cdot)$, $p'(\cdot)$, $g_i'(\cdot)$, $i = 1,\ldots, n_2$ and $h_i'(\cdot)$, $i = 1,\ldots, n_3$ are continuous functions, we obtain

$$p(Q_o) + y_{oi}\, p'(Q_o) - g_i'(y_{oi}) - P_o + v_{oi} = 0, \quad i = 1,\ldots, n_2$$

$$p(Q_o) + z_{oi}\, p'(Q_o) - h_i'(z_{oi}) + w_{oi} = 0, \quad i = 1,\ldots, n_3$$

$$\sum_{i=1}^{n_2} y_{oi} + \sum_{i=1}^{n_3} z_{oi} = Q_o$$

$$y_{oi} \geq 0,\ v_{oi} \geq 0,\ i = 1,\dots, n_2 \text{ and } z_{oi} \geq 0,\ w_{oi} \geq 0,\ i = 1,\dots, n_3$$

$$y_{oi}v_{oi} = 0,\ i = 1,\dots, n_2 \text{ and } z_{oi}w_{oi} = 0,\ i = 1,\dots, n_3 .$$

But this means that (y_o, z_o) is the (unique) optimal solution to EP(P_o,Q_o) with $\lambda_E(P_o,Q_o) = 0$. Hence, by Corollary 2.1 and Theorem 2.1, we obtain $\xi_o = \xi(P_o)$, and the proof is complete. ■

Corollary 2.2. $v_i(P)$, $i = 1,\dots, n_2$ and $w_i(P)$, $i = 1,\dots, n_3$ defined in (2.16)-(2.20) are continuous functions of $P \geq 0$.

Proof. Evident from the proof of Lemma 2.2. ■

By Lemma 2.2, $Y(P)$, $Z(P)$ and $Q(P)$ are all continuous functions of $P \geq 0$. The next result addresses monotonicity of these functions and plays a fundamental role in establishing the existence of the inverse demand function $\Phi(\cdot)$ defined in (2.3).

Theorem 2.2. $Y(P)$ and $Q(P)$ are nonincreasing continuous functions of $P \geq 0$, and each $z_i(P)$, $i = 1,\dots, n_3$ and hence $Z(P)$ are nondecreasing, continuous functions of $P \geq 0$. Moreover, $Y(P)$ is strictly decreasing (with a negative right-hand derivative) over its positive range.

Proof. From Lemma 2.2, $Y(P)$, $Z(P)$ and $Q(P)$ are all continuous functions of $P \geq 0$. Now, let a subscript "+" denote the right-hand derivative. Since Assumptions A1 state that $y^+(\cdot)$ and $z^+(\cdot)$ exist, we have from (2.16)-(2.18) that $Q^+(\cdot)$, $v^+(\cdot)$ and $w^+(\cdot)$ also exist. Denote

$$I_2(P) = \{ i : y_i(P) \neq 0 \text{ or } y_i^+(P) \neq 0\} \tag{2.21}$$

and

$$I_3(P) = \{ i : z_i(P) \neq 0 \text{ or } z_i^+(P) \neq 0\} . \tag{2.22}$$

Note that the continuity of $y_i(P)$, $z_i(P)$, $v_i(P)$ and $w_i(P)$, $P \geq 0$, along with (2.20) imply that

$$v_i^+(P) = 0 \text{ for all } i \in I_2(P) \quad \text{and} \quad w_i^+(P) = 0 \text{ for all } i \in I_3(P). \tag{2.23}$$

Taking right-hand derivatives in (2.16)-(2.18) and (2.20), we obtain,

$$Q^+(P)\{p'[Q(P)] + y_i(P)p''[Q(P)]\} + y_i^+\{p'[Q(P)] - g_i''[y_i(P)]\} = 1 - v_i^+(P)$$

$$i = 1,\dots, n_2, \tag{2.24}$$

$$Q^+(P)\{p'[Q(P)] + z_i(P)p''[Q(P)]\} + z_i^+\{p'[Q(P)] - h_i''[z_i(P)]\} = - w_i^+(P)$$

$$i = 1,\dots, n_3, \tag{2.25}$$

$$\sum_{i \in I_2(P)} y_i^+(P) + \sum_{i \in I_3(P)} z_i^+(P) = Q^+(P) \tag{2.26}$$

$$y_i^+(P)v_i(P) + y_i(P)v_i^+(P) = 0 \quad \text{for} \quad i = 1,\dots, n_2, \tag{2.27}$$

$$z_i^+(P)w_i(P) + z_i(P)w_i^+(P) = 0 \quad \text{for} \quad i = 1,\dots, n_3. \tag{2.28}$$

For notational convenience, define

$$\alpha_i(P) = p'[Q(P)] + y_i(P)p''[Q(P)] \quad \text{for} \quad i \in I_2(P). \tag{2.29}$$

$$\beta_i(P) = 1 / \{p'[Q(P)] - g_i''[y_i(P)]\} \quad \text{for} \quad i \in I_2(P), \tag{2.30}$$

and similarly, let

$$\gamma_i(P) = p'[Q(P)] + z_i(P)p''[Q(P)] \quad \text{for} \quad i \in I_3(P) \tag{2.31}$$

$$\delta_i(P) = 1 / \{p'[Q(P)] - h_i''[z_i(P)]\} \quad \text{for} \quad i \in I_3(P). \tag{2.32}$$

We will show that by Assumptions A1,

$\alpha_i(P) \le 0$ and $\beta_i(P) < 0$, with $\alpha_i(P) < 0$ if $y_i(P) < Q(P)$ for $i \in I_2(P)$ (2.33)

$\gamma_i(P) \le 0$ and $\delta_i(P) < 0$, with $\gamma_i(P) < 0$ if $z_i(P) < Q(P)$ for $i \in I_3(P)$. (2.34)

First, consider $\alpha_i(P)$, $i \in I_2(P)$. If $p''[Q(P)] \le 0$ then $\alpha_i(P) < 0$. On the other hand, if $p''[Q(P)] > 0$, then $\alpha_i(P) \le p'[Q(P)] + Q(P)p''[Q(P)] \le 0$ by (A2). Moreover, $\alpha_i(P) < 0$ if $y_i(P) < Q(P)$. The sign of $\beta_i(P)$ follows directly from Assumptions A1. Similarly, (2.34) holds. Now, using (2.23) along with (2.24) and (2.25), and the notation (2.29)-(2.32), we get

$$y_i^+(P) = -\beta_i(P)[\alpha_i(P)Q^+(P) - 1] \quad \text{for} \quad i \in I_2(P) \tag{2.35}$$

$$z_i^+(P) = -\gamma_i(P)\delta_i(P)Q^+(P) \quad \text{for} \quad i \in I_3(P). \tag{2.36}$$

Summarizing (2.35) and (2.36) and using (2.26), we get

$$Q^+(P) = \frac{\sum_{i \in I_2(P)} \beta_i(P)}{1 + \sum_{i \in I_2(P)} \alpha_i(P)\beta_i(P) + \sum_{i \in I_3(P)} \gamma_i(P)\delta_i(P)} \tag{2.37}$$

Now, from (2.33) and (2.34), we get that $Q^+(P) \le 0$, with $Q^+(P) < 0$ if $I_2(P) \ne \emptyset$. Hence, (2.36) implies that $z_i^+ \ge 0$ for all $i \in I_3(P)$, which means that $Z^+(P) \ge 0$. Further, from (2.26), $Y^+(P) = Q^+(P) - Z^+(P)$ implies that $Y^+(P) \le 0$, with $Y^+(P) < 0$ whenever $I_2(P) \ne \emptyset$, i.e., by (2.21), whenever $Y(P) > 0$. This completes the proof. ■

Corollary 2.3. The inverse demand function $\Phi(\theta)$ defined in (2.3) exists and is continuous and strictly decreasing on $0 \le \theta \le Y_0$, where $Y_0 = Y(0)$.

Proof. From Theorem 2.2, $Y(P)$ is a continuous and nonincreasing function of $P \ge 0$, being strictly decreasing over its positive range. Hence, for the proof it is sufficient to demonstrate that there exists price P_0 such that $Y(P) > 0$ for $0 \le P < P_0$ and $Y(P) = 0$ for all $P \ge P_0$. Let

$z_{NC} = (z_{NC1}, ..., z_{NCn_3})$ denote the unique Nash-Cournot equilibrium solution for the firms in S_3, given that $S_2 = \emptyset$. That is, for each $i = 1,..., n_3$

$$z_{NCi} \text{ solves } \underset{q \geq 0}{\text{maximize}} \{ qp(q + \sum_{\substack{j=1 \\ j \neq i}}^{n_3} z_{NCj}) - h_i(q)\} \tag{2.38}$$

Furthermore, denoting $Z_{NC} = \sum_{i=1}^{n_3} z_{NCi}$ we obtain, from (2.16) and (2.21), that $Y(P) = 0$ if and only if $p(Z_{NC}) - g_i'(0) - P \leq 0$ for $i = 1,..., n_2$. Define

$$P_{NC} = \max\{0, p(Z_{NC}) - \min\{g_i'(0), \ i = 1,..., n_2\}\}. \tag{2.39}$$

and note that if $P_{NC} = 0$ then $Y(P) = 0$ for all $P \geq 0$, and if $P_{NC} > 0$ then $Y(P) > 0$ for $0 \leq P < P_{NC}$, and $Y(P) = 0$ for $P \geq P_{NC}$. Letting $P_0 = P_{NC}$ competes the proof. ■

Corollary 2.4. Suppose that $g_i(\cdot)$, $i = 1,..., n_2$ are identical. If $z(0) > 0$ then $\Phi(\cdot)$ is continuously differentiable over its positive range.

Proof. Let $g_i(\cdot) = g(\cdot)$, $i = 1,..., n_2$. First, we will show that for identical firms in S_2, whenever $Y(P) > 0$ for any $P \geq 0$, we have that $I_2(P) \equiv \{1,..., n_2\}$ and that each of $y_i(P)$, $i = 1,..., n_2$, are identical. On the contrary, if for any $P \geq 0$, we have $y_k(P) > y_\ell(P)$ for some $k, \ell \in \{1,..., n_2\}$, then from (2.16), (2.19) and (2.20) we get that $p[Q(P)] + y_k(P)p'[Q(P)] - g'[y_k(P)] - P = 0 \geq p[Q(P)] + y_\ell(P)p'[Q(P)] - g'[y_\ell(P)] - P$. Hence, using $p' < 0$ and the convexity of $g(\cdot)$, this means that $0 > p'[Q(P)][y_k(P) - y_\ell(P)] \geq g'[y_k(P)] - g'[y_\ell(P)] \geq 0$, a contradiction. Therefore, $y_i(P)$ are identical and so are $y_i^+(P)$, and $I_2(P) = \{1,..., n_2\}$ for all $P \geq 0$ such that $Y(P) > 0$.

Now, suppressing the index i, we get from (2.35) that

$$Y^+(P) = -n_2\beta(P)[\alpha(P)Q^+(P) - 1]. \tag{2.40}$$

Further, observe that if $z(0) > 0$ so that $I_3(0) = \{1,..., n_3\}$, then we also have $I_3(P) = \{1,..., n_3\}$ for all $P \geq 0$, since by Theorem 2.2, $z_i(P)$ are continuous and nondecreasing in $P \geq 0$. Consequently, from (2.29)-(2.32) and (2.37), the terms on the right-hand side in (2.40) are all continuous functions, and so $Y(P)$ and hence $\Phi(\cdot)$ are continuously differentiable over their positive ranges. This completes the proof. ■

Now we can proceed to characterize the revenue function $\theta\Phi(\theta)$, for $0 \le \theta \le Y_0$. (Note that $\Phi(\theta) \equiv 0$ for $\theta \ge Y_0$.) Our interest in this curve obviously stems from the role it plays in the optimization problem (2.5) for the upstream producers in S_1, being the Cournot firms. In particular, we would like this revenue curve to be concave so that these problems are linearly constrained convex programs. In such a case, the existence of a solution to the (simultaneous) Karush-Kuhn-Tucker system for (2.5) would imply the existence of an oligopolistic equilibrium, and one would need to find a solution to this system in order to compute the equilibrium solution. The next result gives some additional assumptions under which this desirable property of a concave revenue function follows. These assumptions employ a particular decreasing, concave demand function with a decreasing price elasticity, and require that the firms in S_2 be identical, while nonidentical firms in S_3 may be permissible in some cases. The case of nonidentical firms in S_2 is addressed subsequently.

Theorem 2.3. Let $p(Q) = a - bQ^k$, where $a > 0,\ b > 0,\ k \ge 1$. Assume that $g_i'(\cdot),\ i = 1,\dots, n_2$ are identical and convex, and that $h_i'(\cdot),\ i = 1,\dots, n_3$ are identical and concave. Then the function $\Phi(\theta)$, given by (2.3), is concave in θ, $0 \le \theta \le Y_0$. Moreover, this property of $\Phi(\theta)$ holds under the alternative modifications of the above assumptions:

(i) $k = 1$ or 2, and $h_i'(\cdot)$ are concave, $i = 1,\dots, n_3$ (not necessarily identical)

(ii) $p(Q) = a - bQ^2 - cQ,\ a > 0, b > 0, c > 0$ and $h_i(\cdot)$ are linear (and identical) for $i = 1,\dots, n_3$.

Proof. Note from (2.3), Theorem 2.2 and Corollary 2.3, that it is sufficient to show that $Y(P)$ is concave over its positive range. Namely, we need to show that

$$Y^+(P + \Delta) \le Y^+(P) \text{ for all } P \ge 0,\ \Delta > 0 \text{ such that } Y^+(P + \Delta) > 0. \tag{2.41}$$

Let $g(\cdot)$ and $h(\cdot)$ denote the cost functions for the firms in S_2 and in S_3, respectively. By the arguments used in the proof of Corollary 2.4, we can conclude that for any $P \geq 0$, $y_i(P)$ are identical and so are $z_i(P)$. Hence, from (2.40), eliminating subscripts and suppressing all function arguments for notational ease, we get by using (2.37), along with (2.29)-(2.32), that

$$Y^+ = \frac{-n_2[h'' - p' - N_3(p' + zp'')]}{(p' - h'')[(p' - g'') + n_2(p' + yp'')] + N_3(p' + zp'')(p' - g'')}$$

where $N_3 = |I_3(P)|$. For $p(Q) = a - bQ^k$, $k \geq 1$, this reduces to, upon some simplification,

$$Y^+ = \frac{-n_2(h'' + A)}{(bkQ^{k-1})^2(n_2 + N_3 + k) + Ag'' + bkQ^{k-2}h''[(n_2 + 1)Q + n_2(k - 1)y] + h''g''} \tag{2.42}$$

where $A = bkQ^{k-2}[(N_3 + 1)Q + N_3(k - 1)z]$. (When $k = 1$, the terms involving p'' above are automatically zero.) Similarly, when $Y(P + \Delta) > 0$, we have $Y^+(P + \Delta) = Y_\Delta^+$, say, given by (2.42) with h'', A, Q, g'', y, z and N_3 replaced by $h_\Delta'', A_\Delta, Q_\Delta, g_\Delta'', y_\Delta, z_\Delta$ and $N_{3\Delta}$ respectively, where the subscript Δ denotes the evaluation of all functions at $P + \Delta$. Noting that $Y(P + \Delta) > 0$ implies $Y(P) > Y(P + \Delta) > 0$ by Theorem 2.2, and that the denominator in (2.42) is positive, in order to show (2.41), we need to show that the numerator, say, D , of the difference $Y_\Delta^+ - Y^+$ is nonpositive, where,

$$\begin{aligned}\frac{D}{n_2} = {} & (A + h'')\Big\{b^2k^2Q_\Delta^{2(k-1)}(n_2 + N_{3\Delta} + k) + A_\Delta g_\Delta'' \\ & + h_\Delta''[(n_2 + 1)bkQ_\Delta^{k-1} + n_2bk(k - 1)y_\Delta Q_\Delta^{k-2} + g_\Delta'']\Big\} \\ & - (A_\Delta + h_\Delta'')\Big\{b^2k^2Q^{2(k-1)}(n_2 + N_3 + k) + Ag'' \\ & + h''[(n_2 + 1)bkQ^{k-1} + n_2bk(k - 1)yQ^{k-2} + g'']\Big\}.\end{aligned}$$

Noting that $Q = n_2y + N_3z = Y + Z$ and $Q_\Delta = n_2y_\Delta + N_3z_\Delta = Y_\Delta + Z_\Delta$, we get,

$$\begin{aligned}\frac{D}{n_2} = {} & b^2k^2[AQ_\Delta^{2(k-1)}(n_2 + N_{3\Delta} + k) - A_\Delta Q^{2(k-1)}(n_2 + N_3 + k)] \\ & + (A + h'')(A_\Delta + h_\Delta'')(g_\Delta'' - g'') \\ & + bkh''h_\Delta''[(n_2 + 1)(Q_\Delta^{k-1} - Q^{k-1}) + (k - 1)(Y_\Delta Q_\Delta^{k-2} - YQ^{k-2})]\end{aligned}$$

$$+ b^2k^2[h_\Delta''Q^{k-1}(n_2 + N_3 + k) + h''Q_\Delta^{k-1}(n_2 + N_{3\Delta} + k)](Q_\Delta^{k-1} - Q^{k-1})$$

$$+ n_2b^2k^2(QQ_\Delta)^{k-1}(N_3h_\Delta'' - N_{3\Delta}h'')$$

$$+ (k - 1)b^2k^2(QQ_\Delta)^{k-2}[h_\Delta''(N_3YY_\Delta + n_2ZZ_\Delta) - h''(N_{3\Delta}YY_\Delta + n_2ZZ_\Delta)]$$

$$+ (k - 1)b^2k^2(QQ_\Delta)^{k-2}[(n_2 + N_3 + k - 1)ZY_\Delta h_\Delta'' - (n_2 + N_{3\Delta} + k - 1)Z_\Delta Yh'']$$

$$+ (k - 1)b^2k^2(QQ_\Delta)^{k-2}(h_\Delta'' + h'')(ZY_\Delta - YZ_\Delta).$$

Now, from Theorem 2.2., note that $y_\Delta < y$, $z_\Delta \geq z$, $Q_\Delta < Q$, and $N_{3\Delta} \geq N_3$. Moreover, since $g'(\cdot)$ is convex, we have $g_\Delta'' \leq g''$ and since $h'(\cdot)$ is concave, we have $h_\Delta'' \leq h''$. Using these inequalities, it can be verified that the first term $[\cdot]$ is negative, whereas the other terms in $[\cdot]$ above are all nonpositive. Hence, $D < 0$. This proves the main part of the theorem, actually demonstrating that $\Phi(\theta)$, $0 < \theta \leq Y_0$ is strictly concave in this case.

To prove modification (i) of Theorem 2.3, we obtain in lieu of (2.42) under the stated conditions when $k = 1$ that

$$Y^+ = \frac{-n_2(1 + A)}{(b + g'')(1 + A) + n_2b}, \quad \text{where } A = \sum_{i \in I_3(P)} \frac{b}{b + h''_i}. \tag{2.43}$$

Hence, to demonstrate (2.41), we need to show as before that the numerator D of the difference $Y_\Delta^+ - Y^+$ is nonpositive, where from (2.43)

$$\frac{D}{n_2} = (1 + A)(1 + A_\Delta)(g_\Delta'' - g'') + n_2b(A - A_\Delta).$$

Note that $g_\Delta'' \leq g''$. Further, $h_i'' \geq h_{i\Delta}''$ and so $\frac{b}{(b + h_i'')} \leq \frac{b}{(b + h_{i\Delta}'')}$. Since $I_3(P)$ $I_3(P + \Delta)$, this therefore implies that $A \leq A_\Delta$, and hence, $D \leq 0$ above.

Likewise, when $k = 2$ in part (i) of Theorem 2.3, we obtain in lieu of (2.42) that

$$Y^+ = \frac{-n_2(1 + A)}{2n_2bQ + (2bQ + g'')(1 + A) + 2n_2by} \tag{2.44}$$

where

$$A = \sum_{i \in I_3(P)} \frac{2b(Q + z_i)}{2bQ + h_i''} \tag{2.45}$$

Again, to demonstrate (2.41), we need to show that $D \le 0$, where from (2.44)

$$D = 2n_2b[(1 + A)Q_\Delta - (1 + A_\Delta)Q] + 2b(1 + A)(1 + A_\Delta)(Q_\Delta - Q)$$
$$+ (1 + A)(1 + A_\Delta)(g_\Delta'' - g'') + 2bn_2[y_\Delta(1 + A) - y(1 + A_\Delta)]$$

Using $y_\Delta < y$, $g_\Delta'' \le g''$, and $Q_\Delta < Q$, this gives

$$D \le 2n_2b(AQ_\Delta - A_\Delta Q) + 2n_2(y_\Delta A - yA_\Delta). \tag{2.46}$$

Now, observe from (2.45) that

$$A = 2b(BQ + C), \text{ where } B = \sum_{i \in I_3(P)} \frac{1}{2bQ + h_i''}, \text{ and } C = \sum_{i \in I_3(P)} \frac{z_i}{2bQ + h_i''}.$$

With this notation, (2.46) may be rewritten as

$$\frac{D}{4n_2b^2} \le QQ_\Delta(B - B_\Delta) + (CQ_\Delta - QC_\Delta) + (y_\Delta C - yC_\Delta) + (BQy_\Delta - B_\Delta Q_\Delta y).$$

Since $Q > Q_\Delta \ge 0$, $y > y_\Delta \ge 0$, $h_i'' \ge h_\Delta'' \ge 0$, $0 \le z_i \le z_{i\Delta}$ and $I_3(P) \subseteq I_3(P + \Delta)$, we get $0 \le B \le B_\Delta$ and $0 \le C \le C_\Delta$. Furthermore, $Qy_\Delta - Q_\Delta y = y_\Delta(n_2y + Z) - y(n_2y_\Delta + Z_\Delta) \le 0$. Consequently, each term in brackets above is nonpositive, giving $D \le 0$. This proves part (i). The proof of part (ii) requires similar manipulations, and hence is omitted. ■

There is one additional situation under which we would like to characterize the revenue function associated with the derived demand function. This situation is interesting in that it assumes non-identical firms in both S_2 and S_3, and hence permits one to empirically study any differential effects on firms profits due to mergers or integrations. Note that Theorem 2.3 (i) admits non-identical firms in S_3, but requires the firms in S_2 to be identical. However, in the following case, the revenue function turns out to be only piecewise concave (i.e., composed of a finite number of concave pieces), and one has to deal with nonconvex problems in (2.5). Nonetheless, as shown in the sequel, this situation is probably one of the most amenable cases of this type.

Theorem 2.4. Let $p(Q) = a - bQ$, where $a > 0$, $b > 0$. Assume that $g_i'(\cdot)$, $i = 1,..., n_2$ are convex functions, while $h_i'(\cdot)$, $i = 1,..., n_3$ are concave. Then

(i) $y_i(P), i = 1,..., n_2$ are all continuous nonincreasing functions of $P \geq 0$, strictly decreasing over their positive ranges

(ii) $\Phi(\cdot)$ is continuous, strictly decreasing and piecewise concave over its positive range.

The following two lemmas are useful in establishing Theorem 2.4.

Lemma 2.3. For any $P \geq 0$ such that $I_2(P) \neq \emptyset$, the set $I_3(P)$ defined in (2.21) is alternatively given by

$$I_3(P) = \{i: \; w_i(P) = 0\} \tag{2.47}$$

Proof. First of all note that when $I_2(P) \neq \emptyset$, (2.22) may be rewritten as

$$I_3(P) = \{i: \; z_i^+(P) > 0\} \tag{2.48}$$

since from (2.33) and (2.34), all the quantities in (2.29)-(2.32) are negative in this case. To show that (2.48) is identical to (2.47), suppose that $z_i^+(P) > 0$. If $z_i(P) > 0$, then $w_i(P) = 0$ from (2.20), and if $z_i(P) = 0$, then again $w_i(P) = 0$ from (2.28). Conversely, suppose that $z_i^+(P) = 0$. Then we must have $z_i(P) = 0$ and therefore from (2.25), we have $w_i^+(P) = -Q^+(P)p'[Q(P)]$ which is negative, since $I_2(P) \neq \emptyset$. This implies that $w_i(P)$ must be positive, and hence the proof is complete. ■

Observe by the above result that for any P such that $I_2(P) \neq \emptyset$,

$$\bar{I}_3(P) = \{i: \; z_i^+(P) = 0\} = \{i: \; w_i^+(P) < 0\} = \{i: \; w_i(P) > 0\} \tag{2.49}$$

Lemma 2.4. Suppose that $p(Q)$ is a convex function. Then for any fixed $\hat{P} \geq 0$ such that $I_2(\hat{P}) \neq \emptyset$, there exists a $\Delta > 0$ such that

$$I_2(P) = I_2(\hat{P} + \Delta), \quad I_2(P) = I_2(\hat{P} + \Delta), \quad \bar{I}_3(P) = \bar{I}_3(\hat{P} + \Delta) \tag{2.50}$$

for all $P \in \left[\hat{P}, \hat{P} + \Delta\right)$.

Proof. Let us first show that by the convexity of $p(\cdot)$, the set $I_2(P)$, for any P such that $I_2(P) \neq \emptyset$, may be equivalently rewritten as

$$I_2(P) \equiv \{i\colon y_i^+(P) < 0\} \equiv \{i\colon y_i(P) > 0\} \tag{2.51}$$

Note that it is sufficient to show that for any $i \in I_2(P)$, we must have $y_i^+(P) < 0$, i.e., by (2.35) to show that $1 - \alpha_i(P)Q^+(P) > 0$ for any $i \in I_2(P)$. Using (2.37) , this amounts to demonstrating that

$$1 + \sum_{j \in I_2(P)} \alpha_j(P)\beta_j(P) - \alpha_i(P) \sum_{j \in I_2(P)} \beta_j(P) + \sum_{j \in I_3(P)} \gamma_j(P)\delta_j(P) > 0.$$

Since the last term is nonnegative (actually positive if $I_3(P) \neq \emptyset$), we need to show , using (2.29) on the remaining terms, say, R_i, that $R_i > 0$ for $i \in I_2(P)$, where

$$R_i = 1 + p''[Q(P)] \sum_{j \in I_2(P)} y_j(P)\beta_j(P) - p''[Q(P)]y_i(P) \sum_{j \in I_2(P)} \beta_j(P). \tag{2.52}$$

Note from (2.30) that $\beta_j(P) \geq \dfrac{1}{p'[Q(P)]}$. Using this in the second term in (2.52) along with the convexity of $p(\cdot)$ and the identity $Y(P) = \sum_{j \in I_2(P)} y_j(P)$, we obtain,

$$R_i \geq \frac{p'[Q(P)] + p''[Q(P)]Y(P)}{p'[Q(P)]} - p''[Q(P)]y_i(P) \sum_{j \in I_2(P)} \beta_j(P).$$

By the convexity of $p(\cdot)$ and from Assumptions A1 and (2.33), observe that $R_i \geq 0$, and so $y_i^+(P) \leq 0$. If $p''[Q(P)] = 0$, then $R_i = 1 > 0$. Hence, suppose that $p''[Q(P)] > 0$, since $p(\cdot)$ is assumed convex. In this case, if $y_i(P) > 0$, then the second signed term is positive by (2.33), and since the first term is nonnegative under Assumptions A1, we again have $R_i > 0$. Finally, suppose $p''[Q(P)] > 0$ and $y_i(P) = 0$. Since $y_i^+(P) \leq 0$, and since $i \in I_2(P)$, we must have $y_i^+(P) < 0$, and therefore (2.51) holds.

Now, from (2.51) and (2.49), and using the continuity of $y_i(\cdot)$ and $w_i(\cdot)$ through Lemma 2.2 (along with (2.16) and (2.17)), and using the fact that $I_3(P) \equiv \{1,\ldots, n_3\} - \bar{I}_3(P)$, we obtain (2.50) for some $\Delta > 0$, and the proof is complete. ■

Proof of Theorem 2.4. By Lemma 2.2 $y_i(P)$ are all continuous functions, and by (2.51) of Lemma 2.4, since $p(Q)$ is convex, they are also strictly decreasing over their positive ranges. This proves part (i). For the proof of part (ii) we will show that $Y(P)$ is piecewise concave on $P \in [0, P_0]$. Observe that for any $P \geq 0$ and $\Delta > 0$, $I_2(P) \supseteq I_2(P + \Delta)$ and $I_3(P) \subseteq I_3(P + \Delta)$, since $y_i(P)$ are nonincreasing, and $z_i(P)$, by (2.48) of Lemma 2.3, are nondecreasing in $P \geq 0$. This, along with Lemma 2.4 implies that the interval $[0, P_0)$ is a sum of a finite number (at most $n_2 + n_3$) of adjacent intervals, say $[P_k, P_{k+1})$ such that

$$I_i(P) = I_i(P_k),\ i = 2,\ 3 \text{ for } P \in [P_k,\ P_{k+1}). \tag{2.53}$$

We will show that in any such interval, $Y^+(P)$ is nonincreasing in P . Toward this end, consider any $P \in [P_k, P_{k+1})$ and any $\Delta > 0$ such that $P + \Delta \in [P_k, P_{k+1})$. Then by equations (2.35), (2.37) and (2.29)-(2.32), with $p(Q) = a - bQ$, we obtain

$$Y^+(P) = -A\frac{1 + bB}{1 + bA + bB} \quad \text{for} \quad P \in [P_k,\ P_{k+1}) \tag{2.54}$$

where $A = \sum_{i \in I_2(P_k)} [b + g_i''(y_i(P))]^{-1}$ and $B = \sum_{i \in I_3(P_k)} [b + h_i''(z_i(P))]^{-1}$. Readily, the sign of the difference $Y^+(P + \Delta) - Y^+(P)$ is the same as that of $A(1 + bB)(1 + bA_\Delta + bB_\Delta) - A_\Delta(1 + bB_\Delta)(1 + bA + bB)$, where subscript Δ denotes evaluation at $P + \Delta$. The last expression reduces to $(1 + bB)(1 + bB_\Delta)(A - A_\Delta) + b^2AA_\Delta(B - B_\Delta)$. But, by the convexity of $g_i'(\cdot)$ and concavity of $h_i'(\cdot)$ we have $g_{i\Delta}'' \leq g_i''$ and $h_{i\Delta}'' \leq h_i''$, hence $0 < A \leq A_\Delta$ and $0 < B \leq B_\Delta$. Therefore, the considered expression is nonpositive and so $Y^+(P + \Delta) \leq Y^+(P)$ thus completing the proof of part (ii). ■

Corollary 2.5. Under assumptions of Theorem 2.4, suppose that the cost functions $g_i(\cdot)$, $i = 1,..., n_2$, and $h_i(\cdot)$, $i = 1,..., n_3$ are quadratic or linear. Then

(i) $y_i(P)$, $i = 1,..., n_2$ and $z_i(P)$, $i = 1,..., n_3$ are all continuous, picewise linear functions of $P \geq 0$. Moreover, $\Phi(\theta)$ is continuous, piecewise linear and strictly decreasing over its positive range.

(ii) If $z(0) > 0$ then $\Phi(\theta)$ is a convex and piecewise linear function of $\theta \geq 0$.

Proof. Part (i). For any $P \in [P_k, P_{k+1})$ defined by (2.53), the right-hand derivative $Q^+(P)$ given by (2.37) is a constant and so, by (2.35) and (2.36), are $y_i^+(P)$ and $z_i^+(P)$. Therefore, $y_i(P)$ and $z_i(P)$ are all piecewise linear and the proof of part (i) is complete.

Part (ii). Given Theorem 2.4, Corollary 2.3 and the above part (i), it is sufficient to show that $Y^+(P_k) \leq Y^+(P_{k+1})$, where P_k and P_{k+1} satisfy equation (2.53). From (2.54) we have

$$Y^+(P_k) = -A_k \frac{1 + bB_k}{1 + bA_k + bB_k},$$

where A_k, B_k denote the evaluation of expressions A and B at $P = P_k$ respectively. Observe that $B_k = B_{k+1}$, since $z(0) > 0$ implies $I_3(P) = I_3(0) = \{1,..., n_3\}$ for any $P \geq 0$. Simultaneously, $A_k \geq A_{k+1}$ since $I_2(P_k) \supseteq I_2(P_{k+1})$. From this, it can be easily verified that $Y^+(P_k) \leq Y^+(P_{k+1})$, thus completing the proof. ∎

Note from (2.24) and (2.25) that under assumptions of Corollary 2.6, $v_i(P)$, $i = 1,..., n_2$ and $w_i(P)$, $i = 1,..., n_3$ are also continuous, piecewise linear functions of $P \geq 0$, and moreover, that $w_i(P)$ are nonincreasing in $P \geq 0$, being strictly decreasing whenever $I_2(P) \neq \emptyset$ and $i \notin I_3(P)$.

The proof of Theorem 2.4 actually facilitates the construction of $Y(P)$ function and hence, its inverse. Namely, as P is increased continuously from zero, the slopes of the various concave segments of $Y(P)$ via closed form expressions for the individual $y_i(P)$ curves, $i = 1,...,, n_2$, and the breakpoints for the piecewise concave function $Y(P)$ are given by prices at which some firms in S_2 leave the market and/or some firms in S_3 enter it.

Finally, we remark that the piecewise concave, but not concave, behavior of $\theta\Phi(\theta)$, $\theta \geq 0$, persists even when the nonidentical cost functions of the firms in $S_2 \cup S_3$ are all linear. In fact, as apparent from Theorem 2.3 (i) and Corollary 2.5 , this behavior is in particular due to the non-identical nature of the firms in S_2 rather than in S_3.

2.3. Existence and Uniqueness of Equilibrium Solution

Based on the properties stated in the foregoing sections, we now construct sufficient conditions for the existence and uniqueness of competitive and oligopolistic solutions. First, consider the competitive equilibrium solution.

Theorem 2.5. Assume that the cost functions $f_i(x)$ are strictly convex for $x \geq 0$. Then there exists a unique competitive equilibrium solution $(\bar{x}, \bar{y}, \bar{z})$ to the two-stage model, where $\bar{x}$, $\bar{y}$ and $\bar{z}$ are vectors of outputs of firms in S_1, S_2 and S_3 respectively. Moreover,

(i) $\bar{X} > 0$ if and only if $Y(P_\ell) > 0$

(ii) if $Y(P_\ell) > 0$ then the equilibrating price $\bar{P} \in (P_\ell, P_U)$, where

$$P_\ell = \min\{f_i'(0), i = 1,\ldots, n_1\}, \tag{2.55}$$

$$P_U = \min\{P_{NC}, P_u\}, \tag{2.56}$$

$$P_u = \min\{f_j'(Y_0): j \in \operatorname{argmin}\{f_i'(0), i = 1,\ldots, n_1\}\}, \tag{2.57}$$

and P_{NC} is given in (2.39).

Proof. Note from problem (2.4), writing $\Phi(\bar{X})$ as P, that under the strict convexity of $f_i'(\cdot)$, $i = 1,\ldots, n_1$, the optimal solution is given by

$$x_i(P) = \begin{cases} f_i'^{-1}(P) & \text{if } P \geq f_i'(0) \\ 0 & \text{otherwise} \end{cases} \qquad i = 1,\ldots, n_1 \tag{2.58}$$

Define the aggregate supply curve of the competitive market as $X(P) = \sum_{i=1}^{n_1} x_i(P)$. Note that for each i, $x_i(P)$ is a well defined, continuous and nondecreasing function, being strictly increasing over its positive range. Consequently, so is $X(P)$, and in fact, it is strictly increasing for $P \geq P_\ell$, since $f_m'^{-1}(P)$, where $m \in \text{argmin}\{f_i'(0), i = 1,..., n_1\}$, is strictly increasing for $P \geq P_\ell$. Also note that $X(P) = 0$ for $0 \leq P \leq P_\ell$.

Consider the curve $Y(P)$, and note that P corresponds to a competitive equilibrium price if and only if $Y(P) = X(P)$. From Theorem 2.2, $Y(P)$ is a continuous, nonincreasing function of $P \geq 0$, being strictly decreasing over its positive range. Hence, the equilibrating output $\overline{X} = \overline{Y} = 0$ if and only if $Y(P_\ell) = 0$. Hence, suppose that $Y(P_\ell) > 0$ and recall that $X(P_\ell) = 0$. Then readily, $\overline{P} \leq P_{NC}$, since from the proof of Corollary 2.3, $Y(P) = 0$ for $P \geq P_{NC}$. Further, from (2.57) let $P_u = f_k'(Y_0)$. Then we have $X(P_u) \geq x_k(P_u) = f_k'^{-1}(P_u) = Y(0) \geq Y(P_\ell)$. Consequently, a unique point of intersection of the curves $X(P)$ and $Y(P)$ exists, and occurs in the interval (P_ℓ, P_U). This completes the proof. ■

Theorem 2.6. Suppose that the assumptions of Theorem 2.3 hold. Then there exists an oligopolistic equilibrium solution (x^o, y^o, z^o) to the two-stage model. Moreover,

(i) for any equilibrium solution, the total supply X^o of the firms in S_1 is the same, and y^o, z^o are unique,

(ii) if $\Phi(\theta)$ is differentiable at X^o then (x^o, y^o, z^o) is unique ,

(iii) $X^o > 0$ if and only if $Y(P_\ell) > 0$, where P_ℓ is defined in (2.55),

(iv) if there exists a competitive equilibrium solution, with a total output of $\overline{X}$ for the firms in S_1, then $X^o \leq \overline{X}$.

Proof. Under the assumptions of Theorem 2.3, $\Phi(\theta)$ is a continuous, strictly decreasing and concave function of s, $0 \leq \theta \leq Y_0$. Further, by Assumptions A1, each function $f_i(\cdot)$, $i = 1,..., n_1$, is continuous, nondecreasing and convex. Hence, from Szidarovszky and Yakowitz [S8], an equilibrium solution x^o exists for (2.5), and moreover, the total equilibrating output for the firms in S_1 is the same for any such solution. Therefore, by Corollary 2.1, an oligopolistic equilibrium (x^o, y^o, z^o) exists, and y^o, z^o are unique. If $\Phi(\theta)$ is differentiable at X^o, then, again by the development in Szidarovszky and Yakowitz [S8], the solution to problems (2.5) is unique, so that (x^o, y^o, z^o) is unique too. Thus the proof of parts (i) and (ii) is complete.

To prove part (iii), we will show that $X^o = 0$ if and only if $Y(P_\ell) = 0$. First, suppose that $X^o = 0$. Then $x^o = 0$ and from the optimality of x^o to problems (2.5), we necessarily have $-f_i(0) \geq q\Phi(q) - f_i(q)$ for any $q > 0$, $i = 1,..., n_1$. Dividing each of these inequalities by $q > 0$ results in $\Phi(q) \leq [f_i(q) - f_i(0)]/q$ for any $q > 0$, which along with the differentiablity of $f_i(\cdot)$ and the continuity of $\Phi(\cdot)$, implies that $\Phi(0) \leq f_i'(0)$, $i = 1,..., n_1$, and hence, that $Y(P_\ell) = 0$. Now, for the reverse implication, suppose that $Y(P_\ell) = 0$. Then, readily $\Phi(0) \leq f_i'(0)$, $i = 1,..., n_1$, and further, for any $q > 0$ we have $q\Phi(q) \leq q\Phi(0) \leq qf_i'(0)$, $i = 1,..., n_1$. But by the convexity of $f_i(\cdot)$, this gives $-f_i(0) \geq q\Phi(q) - f_i(q)$ for any $q > 0$, $i = 1,..., n_1$. This means that $x^o = 0$ is an optimal solution to problems (2.5). Hence, $X^o = 0$, and the proof of part (iii) is complete.

Finally, part (iv) follows from the development in Rovinsky, Shoemaker and Todd [R1] and from Murphy et al. [M3], and this completes the proof. ∎

Corollary 2.6. Under assumptions of Theorem 2.6, if $Y(P_\ell) > 0$, then $P^o \in (P_\ell, P_{NC})$, where P_{NC} is given by (2.39).

Proof. If $Y(P_\ell) > 0$ then by Theorem 2.6 (iii), $Y^o > 0$, which implies that $P^o < P_{NC}$, since from the proof of Corollary 2.3, $Y(P) = 0$ for all $P \geq P_{NC}$. If $P^o \leq P_\ell$ then problems (2.5) yield $X^o = 0$ thus contradicting $X^o > 0$. This completes the proof. ∎

As suggested by Theorem 2.6, one may note that if $\Phi(\cdot)$ is not differentiable at X^o, then the uniqueness of an oligopolistic equilibrium cannot be claimed, even if the revenue function $\theta\Phi(\theta)$ is strictly concave. For example, when $\Phi(\theta) = 7 - \theta$ for $0 \leq \theta \leq 3$, and $\Phi(\theta) = 10 - 2\theta$ for $3 \leq \theta \leq 5$, and there are $n_1 = 2$ firms with identical cost functions $f_i(x) = 2x$, $i = 1, 2$, one can verify that $x^o = (1.5, 1.5)$ or $(2, 1)$ or $(1, 2)$ for example, all correspond to oligipolistic equilibrium solutions. However, as noted in Theorem 2.6 (i), the total output $X^o = 3$ in each case, with the semi-finished product price at equilibrium being $P^o = \Phi(3) = 4$. Consequently, from Corollary 2.1, the equilibrating outputs y^o and z^o are uniquely determined.

Finally, let us consider the existence of an oligopolistic equilibrium solution when the derived demand function is obtained under the conditions of Theorem 2.4. Because of nonconcavity of the revenue function, it is difficult to make any global statements regarding the existence of a simultaneous solution to problems (2.5). However, suppose that we modify Definition 2.2 to require x_i^o to be a strict local maximizer in (2.5) for each $i = 1, \ldots, n_1$, and accordingly, call the solution (x^o, y^o, z^o) satisfying this condition along with (2.6) as a local oligopolistic equilibrium solution. Then we can assert the following:

Theorem 2.7. If $n_1 = 1$, then there exists an oligopolistic equilibrium solution (x^o, y^o, z^o). If $n_1 \geq 2$, suppose that in addition, the assumptions of Theorem 2.4 hold. Then there exists a local oligopolistic equilibrium solution (x^o, y^o, z^o). In either case, if a competitive equilibrium $(\bar{x}, \bar{y}, \bar{z})$ exists, then $0 \leq Y^o \leq \bar{Y}$.

Proof. In the case of a monopoly $(n_1 = 1)$, since the problem (2.5) involves the maximization of a continuous function essentially over the compact set $[0, Y_0]$ under Assumptions A1, there clearly exists an oligopolistic equilibrium solution (x^o, y^o, z^o). Hence, consider the case of $\Phi(\cdot)$ available through Theorem 2.4, and denote its breakpoints by $\theta_0 = 0, \theta_1, \ldots, \theta_{n-1}, \theta_n = Y_0$. Accordingly, let $\Phi(\theta)$, $0 \leq \theta \leq Y_0$ be represented as

$$\Phi(\theta) = \Phi_t(\theta) \quad \text{for } \theta_{t-1} \leq \theta \leq \theta_t, \quad t = 1, \ldots, n,$$

where $\Phi_t(\theta)$ is the t^{th} segment defining $\Phi(\cdot)$ on $[0, Y_0]$.

Now, for each $t = 1, \ldots, n$ consider the determination of an oligopolistic equilibrium solution through (2.5), with $\Phi(\theta)$ replaced by $\Phi_t(\theta)$, $\theta \geq 0$. Since $\Phi_t(\theta)$, $\theta \geq 0$ is a strictly decreasing, concave demand curve and the associated revenue function is strictly concave, there exists a unique oligopolistic equilibrium solution (x^t, y^t, z^t), say. If $X^t = \sum_{i=1}^{n_1} x_i^t$ lies in the interval (θ_{t-1}, θ_t), then (x^t, y^t, z^t) is evidently a local oligopolistic equilibrium solution. Hence, suppose that this condition does not hold for any $t = 1, \ldots, n$. Then, there must exist a $k \in \{1, \ldots, n\}$ for which $X^k \geq \theta_k$ and $X^{k+1} \leq \theta_k$, since $X^1 > \theta_0$ and $X^n < \theta_n$. Now, for any such k, denoting the slopes of $\Phi_k(\cdot)$ and $\Phi_{k+1}(\cdot)$ by $s_k(\cdot) \leq 0$ and by $s_{k+1}(\cdot) < 0$ repectively, define as a function of $\theta \geq 0$, for each $i = 1, \ldots, n_1$, following the approach of Szidarovszky and Yakowitz [S7],

$$x_i^k(\theta) = \begin{cases} x, & \text{such that } x \geq 0 \text{ and } xs_k(\theta) + \Phi_k(\theta) = f_i'(x) \text{ if it exists} \\ 0, & \text{if no such } x \text{ exists,} \end{cases} \tag{2.59}$$

$$x_i^{k+1}(\theta) = \begin{cases} x, & \text{such that } x \geq 0 \text{ and } xs_{k+1}(\theta) + \Phi_{k+1}(\theta) = f_i'(x) \text{ if it exists} \\ 0, & \text{if no such } x \text{ exists.} \end{cases} \tag{2.60}$$

Note that since $f_i'(x) - xs_k(\theta)$ is strictly decreasing in x, $x_i^k(\theta)$, $\theta \geq 0$, is a well defined function. Moreover, as in Szidarovszky and Yakowitz [S7], this function is readily verified to be continuous and strictly decreasing in θ over its positive range. Hence, $X_k(\theta) = \sum_{i=1}^{n_1} x_i^k(\theta)$ is continuous in $\theta \geq 0$, and is strictly decreasing over its positive range. Furthermore, the solution to $X_k(\theta) = \theta$ gives the unique oligopoly equilibrium total output X^k using the demand curve $\Phi_k(\cdot)$. Hence, since $X^k \geq \theta_k$, we have $X_k(\theta_k) \geq X_k(X^k) = X^k \geq \theta_k$. Similar remarks hold for $x_i^{k+1}(\theta)$, $\theta \geq 0$, $i = 1, \ldots, n_1$ and for $X_{k+1}(\theta) = \sum_{i=1}^{n_1} x_i^{k+1}(\theta)$, with $X_{k+1}(\theta_k) \leq \theta_k$. Consequently, let $0 \leq \mu_o \leq 1$ be such that

$$\mu_o X_k(\theta_k) + (1 - \mu_o) X_{k+1}(\theta_k) = \theta_k,$$

and accordingly, define

$$x^o = \mu_o x^k(\theta_k) + (1 - \mu_o) x^{k+1}(\theta_k). \tag{2.61}$$

We will show that x_i^o is a strict local maximizer in (2.5) for each $i = 1, \ldots, n_1$.

Toward this end, observe from (2.59) and (2.60) that at $\theta = \theta_k$, we have, for each $i = 1, \ldots, n_1$,

$$x_i^k(\theta_k) s_k(\theta_k) + \Phi_k(\theta_k) - f_i'[x_i^k(\theta_k)] \begin{cases} = & 0 \text{ if } x_i^k(\theta_k) > 0 \\ \leq & 0 \text{ otherwise} \end{cases} \tag{2.62}$$

$$x_i^{k+1}(\theta_k)s_{k+1}(\theta_k) + \Phi_{k+1}(\theta_k) - f_i'[x_i^{k+1}(\theta_k)] \begin{cases} = & 0 \text{ if } x_i^{k+1}(\theta_k) > 0 \\ \leq & 0 \text{ otherwise} \end{cases} \tag{2.63}$$

Let us consider two cases, namely, $s_k(\theta_k) > s_{k+1}(\theta_k)$ and $s_k(\theta_k) < s_{k+1}(\theta_k)$. First, suppose that $s_k(\theta_k) > s_{k+1}(\theta_k)$ so that the objective function in (2.5) is strictly concave in a neighborhood of x^o . In this case, from (2.62) and (2.63), we have that $x_i^k(\theta_k) > 0$, then $x_i^{k+1}(\theta_k) < x_i^k(\theta_k)$, and if $x_i^k(\theta_k) = 0$, then $\Phi_k(\theta_k) - f_i'(0) \leq 0$, and so $x_i^{k+1}(\theta_k) = 0$ as well. Therefore, if $x_i^k(\theta_k) > 0$, then from (2.61), $x_i^o \leq x_i^k(\theta_k)$ and so from (2.62), we get that

$$-\left[x_i^o s_k(\theta_k) + \Phi_k(\theta_k) - f_i'(x_i^o)\right] \leq 0 \tag{2.64}$$

or that the left-hand directional derivative of the objective function in (2.5) at x_i^o is nonpositive, which by the strict concavity of the objective function in the neighborhood of x^o, implies that the objective value strictly decreases if x_i is reduced below x_i^o. Similarly, since $x_i^o \geq x_i^{k+1}(\theta_k)$ for all $i = 1,..., n_1$, we get from (2.63) that

$$x_i^o s_{k+1}(\theta_k) + \Phi_{k+1}(\theta_k) - f_i'(x_i^o) \leq 0 \quad \text{for all } i = 1,..., n_1, \tag{2.65}$$

or that the right-hand directional derivative of the objective function in (2.5) at x_i^o is nonpositive. Therefore, this establishes that x^o defined by (2.61), with the associated y^o and z^o from (2.6) represents a local oligopolistic equilibrium solution.

In case $s_k(\theta_k) < s_{k+1}(\theta_k)$ since $X_k(\theta_k) \geq \theta_k > 0$, we obtain by proceeding as above that $X_{k+1}(\theta_k) > X_k(\theta_k)$ which contadicts our hypothesis that $X_{k+1}(\theta_k) \leq \theta_k \leq X_k(\theta_k)$. Hence, this case cannot arise.

To complete the proof, by denoting $P^o = \Phi(X^o)$, $I_+ = \{i : x_i^o > 0\}$, we get $Y(P^o) = X^o = \sum_{i \in I_+} f_i'^{-1}(P^o + \hat{s}x_i^o)$,where $\hat{s}$ is the slope of $\Phi_i(\theta)$ at X^i in case $x^o = x^i$ for some i for which $\theta_{i-1} < X^i < \theta_i$, or otherwise as in the second case above $\hat{s}$ is suitably chosen from (2.64) and (2.65) in the interval $[s_{k+1}(\theta_k), s_k(\theta_k)]$, noting that $\Phi_k(\theta_k) = \Phi_{k+1}(\theta_k) = P^o$ from (2.61). Also, observe that $P^o > f_i'(0)$ for $i \in I_+$ and $P^o \leq f_i'(0)$ for $i \notin I_+$ Therefore, letting $X(P), P \geq 0$ be as defined in the proof of Theorem 2.5, we obtain $X(P^o) = \sum_{i \in I_+} f_i'^{-1}(P^o)$. Hence, $Y(P^o) < X(P^o)$, and so the competitive equilibrium occurs at a price lesser than P^o which gives $\overline{Y} > Y^o$ and this completes the proof. ■

2.4. Summary of Results

In this chapter a two-stage model was analyzed in which the final commodity suppliers are Cournot firms. For the purpose of characterizing the equilibrating process among the firms in $S_2 \cup S_3$, a mathematical programming-based approach was employed in the manner introduced by Murphy et al. [M3]. Under general assumptions (A1), we showed that the derived demand function for the semi-finished product is well behaved, i.e., is continuous and decreasing (Theorem 2.2 and Corollary 2.3). Given this derived demand function, we obtained sufficient conditions for the existence of an equilibrium solution in the model given that the upstream stage is either perfectly competitive or oligopolistic (Theorems 2.5 and 2.6). For the latter case, we required that the derived demand function be concave, which in turn imposed some additional properties of the market demand function and the cost functions for the firms in $S_2 \cup S_3$. In particular, we required that the downstream producers have identical cost functions, since otherwise the derived demand function is piecewise concave only, or sometimes even convex (Theorem 2.4, Corollary 2.5). The issue of computing the equilibrium solutions discussed in this chapter is presented later, in Chapter 4 (Section 4.1). The next chapter discusses the two-stage model, with an alternate behavior for the firms In $S_2 \cup S_3$, namely, a leader-follower type of behavior.

Chapter 3

The Two-Stage Model with a Multiple Leader-Follower Oligopoly

in the Final Product Market

This chapter contains an analysis of the two-stage model, given that the firms in S_2 act as followers while those in S_3 act as leaders, in the final commodity market. Assumptions, relevant notation, and definitions are introduced in Section 3.1. Section 3.2 studies the properties of the aggregate reaction curve of the firms in S_2 in response to the input semi-finished product price and to the output of the firms in S_3. Section 3.3 characterizes the demand curve perceived by the leader firms in S_3. Based on this, Section 3.4 examines the existence and uniqueness of the multiple leader-follower equilibrium in the final commodity market, characterizing its behavior as a function of the established price for the semi-finished product. The issues concerning the existence and uniqueness of the overall equilibrium solutions are presented in Section 3.5. Finally, Section 3.6 contains a summary of results.

3.1.Notation, Assumptions and Definitions

For the considered model we assume that the assumptions introduced in Section 2.1 are in force. For ease in reading we briefly restate them below.

Let $p(Q)$, $Q \geq 0$ denote the market (inverse) demand function for the final commodity. Further let $x = (x_1, \dots, x_{n_1})$ denote the outputs of the n_1 firms in S_1 with total X, and similarly, let $y = (y_1, \dots, y_{n_2})$, with total Y and $z = (z_1, \dots, z_{n_3})$ with total Z denote the otputs of the n_2 and n_3 firms in S_2 and S_3 , respectively. Denote the total industry output by $Q = Y + Z$. Let the total cost functions for the firms in S_1, S_2 and S_3 be given by $f_i(x_i), x_i \geq 0$, for $i = 1,\dots, n_1$, $g_i(y_i)$, $y_i \geq 0$, for $i = 1,\dots, n_2$ and $h_i(z_i), z_i \geq 0$ for $i = 1,\dots, n_3$, respectively. We assume throughout that these functions have the following properties:

(A1) $p(Q)$ is twice differentiable, strictly decreasing, with $p'(Q) < 0$ for $Q > 0$,

(A2) $p'(Q) + Qp''(Q) \leq 0$ for all $Q \geq 0$.

(A3) $f_i(\cdot)$, $i = 1,\dots,n_1$, $g_i(\cdot), i = 1,\dots,n_2$, and $h_i(\cdot)$, $i = 1,\dots,n_3$ are nondecreasing, twice differentiable and convex over the nonnegative real line.

(A4) There exists a quantity $q_u \geq 0$ such that

$$g_i'(q_u) \geq p(q_u),\ i = 1,\dots,n_2$$

$$h_i'(q_u) \geq p(q_u),\ i = 1,\dots,n_3$$

Recall that as proven by Murphy et al. [M3], assumptions (A1) and (A2) imply that for each $K \geq 0$, $Qp(Q + K)$ is a strictly concave function of $Q \geq 0$. Also, assumption (A4), along with (A1) and (A3) simply guarantees the existence of an upper bound of q_u on the outputs of the profit maximizing firms in $S_2 \cup S_3$.

Now, let us present the assumed leader-follower Stackelberg [S6] - Nash [N1] - Cornout [C1] (SNC) equilibrating process for the firms in $S_2 \cup S_3$, given some prevailing price $P \geq 0$ for the semi-finished product. For a fixed input price $P \geq 0$, the effective total cost functions for the firms in S_2 are given by $g_i(y_i) + Py_i$, $y_i \geq 0$, for $i = 1,\dots, n_2$. Hence, for a given total output $Z \geq 0$ of the leader firms in S_3, the follower firms in S_2 , behaving in a Nash-Cournot manner, will produce at equilibrium a set of quantities $y_R(Z,P) = (y_{R1}(Z,P), \dots, y_{Rn_2}(Z,P))$, where for each $i = 1,\dots, n_2$, $y_{Ri}(Z,P)$ solves the problem:

$$\underset{q \geq 0}{\text{maximize}} \; \{ qp[q + \sum_{\substack{j=1 \\ j \neq i}}^{n_2} y_{Rj}(Z,P) + Z] - Pq - g_i(q)\} \tag{3.1}$$

Let us indicate that given an input price $P \geq 0$, the quantities $y_{R1}(Z,P),..., y_{Rn_2}(Z,P)$ represent a Nash-Cournot equilibrium solution for the firms in S_2, in reaction or response to a given extraneous market supply of $Z \geq 0$ units. Consequently, the quantities $y_{Ri}(Z,P)$, $i = 1,..., n_2$, are referred to as the **joint reaction (or response) curves** of the follower firms with respect to the total supply $Z \geq 0$ of the leader firms, and the total reaction $Y_R(Z,P) = \sum_{i=1}^{n_2} y_{Ri}(Z,P)$ is referred to as the **aggregate reaction curve**.

Now, being aware of this aggregate reaction curve of the followers, the leader firms effectively face the following so called **perceived demand curve** $F(Z, P)$ $Z \geq 0$, for any given $P \geq 0$:

$$F(Z, P) = p[Z + Y_R(Z,P)]. \tag{3.2}$$

Hence, at equilibrium, if one exists, the leader firms produce a set of quantities $z(P) = (z_1(P),..., z_{n_3}(P))$, with a total $Z(P) = \sum_{i=1}^{n_3} z_i(P)$, where for each $i = 1,..., n_3$, $z_i(P)$ solves the problem:

$$\underset{q \geq 0}{\text{maximize}} \; \{ qF[q + \sum_{\substack{j=1 \\ j \neq i}}^{n_3} z_j(P), \; P] - h_i(q)\} \tag{3.3}$$

Consequently, we have the following definition.

Definition 3.1 For a fixed input price $P \geq 0$, a set of quantities $y(P) = (y_1(P),..., y_{n_2}(P))$ with total $Y(P)$, and $z(P) = (z_1(P),..., z_{n_3}(P))$ with total $Z(P)$, for the follower firms in S_2 and the leader firms in S_3, respectively is said to constitute a **Stackelberg-Nash-Cournot (SNC) equilibrium solution** in the final product market if $z_i(P)$ solves problem (3.3) for each $i = 1,..., n_3$, and

$$y_i(P) = y_{Ri}[Z(P), P], \quad i = 1,\dots, n_3, \qquad Y(P) = Y_R[Z(P), P]. \tag{3.4}$$

At equilibrium, the total industry output will be denoted by $Q(P) = Z(P) + Y(P)$, for any given $P \geq 0$.

In the sequel, we will show that under the foregoing assumptions, $y_{Ri}(Z,P), Z \geq 0, P \geq 0$, $i = 1,\dots, n_2$ are continuous functions, and we will provide some sufficient conditions under which $y(P)$ and $z(P)$ are also continuous functions of $P \geq 0$. Whenever such properties hold, we will make the reasonable assumption (A5) that the right-hand derivatives of these functions $y_{Ri}(\cdot, \cdot)$, $y(\cdot)$ and $z(\cdot)$ exist. Notationally, for a function $\psi(s_1, s_2) : R^2 \to R$, we will use the symbol $D^+_{s_i}(\psi)$ to denote the partial right-hand derivative of $\psi(\cdot)$ with respect to s_i, $i = 1, 2$, and for a univariate function $\psi(s) : R \to R$, we will denote its right-hand derivative by $\psi^+(s)$, as in Chapter 2. Henceforth, assumptions (A1)-(A5) will be referred to as **Assumptions A2**, and will be assumed to hold throughout this chapter.

We now begin our analysis by first characterizing continuity, monotonicity, differentiabilty and convexity properties of the aggregate reaction curve $Y_R(Z,P)$ $Z \geq 0$ under various conditions. This will permit us to characterize certain properties of the perceived demand function $F(Z, P)$, $Z \geq 0$, for any $P \geq 0$, and further to specify sufficient conditions for the existence and uniqueness of a SNC equilibrium $y(P), z(P)$ for any $P \geq 0$. Furthermore, continuity, monotonicity, differentiablity and convexity properties of $Y(P)$ and $Z(P)$ as functions of $P \geq 0$ will be investigated. In particular, under some sufficient conditions which guarantee that $Y(P), P \geq 0$ is a continuous, nonincreasing function, which is strictly decreasing over its positive range, say, $[0, P_0)$, we define the inverse **derived demand function** $\Phi(\theta)$, $\theta \geq 0$ in a similar way as in Chapter 2 (eq. (2.3)):

$$\Phi(\theta) = \begin{cases} P_0 & \text{for } \theta = 0 \\ Y^{-1}(\theta) & \text{for } 0 < \theta < Y_0 \\ 0 & \text{for } \theta \geq Y_0 \end{cases} \tag{3.5}$$

Based on this definition of the inverse derived demand curve, the overall competitive and oligopolistic equilibria are defined in the following way.

Definition 3.2. A set of nonnegative output quantities $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{n_1})$ with total $\bar{X}$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_{n_2})$ with total $\bar{Y}$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_{n_3})$ with total $\bar{Z}$, for the firms $1, \ldots, n_1$ in S_1, firms $1, \ldots, n_2$ in S_2, and firms $1, \ldots, n_3$ in S_3, is said to be a **competitive equilibrium solution**, provided

(i) for each $i = 1, \ldots, n_1$, $\bar{x}_i$ solves

$$\underset{q \geq 0}{\text{maximize}} \{q\Phi(\bar{X}) - f_i(q)\} \tag{3.6}$$

and

(ii) $(\bar{y}, \bar{z}) = [y(P^o), z(P^o)]$ if $\bar{X} > 0$, where $P^o = \Phi(\bar{X})$, and $(\bar{y}, \bar{z}) = (0, z_{NC})$ if $\bar{X} = 0$, where z_{NC} denotes a Nash-Cournot equilibrium for the firms in S_3 under the demand curve $p(\cdot)$. That is, $z_{NC} = (z_{NC1}, \ldots, z_{NCn_3})$, where for each $i = 1, \ldots, n_3$, z_{NCi} solves

$$\underset{q \geq 0}{\text{maximize}} \{ qp(q + \sum_{\substack{j=1 \\ j \neq i}}^{n_3} z_{NCj}) - h_i(q)\}.$$

Definition 3.3. A set of nonnegative output quantities $x^o = (x_1^o, \ldots, x_{n_1}^o)$ with total X^o, $y^o = (y_1^o, \ldots, y_{n_2}^o)$ with total Y^o, $z^o = (z_1^o, \ldots, z_{n_3}^o)$ with total Z^o, for the firms $1, \ldots, n_1$ in S_1, firms $1, \ldots, n_2$ in S_2, and firms $1, \ldots, n_3$ in S_3, is said to be an **oligopolistic equilibrium solution**, provided

(i) for each $i = 1, \ldots, n_1$, x_i^o solves

$$\underset{q \geq 0}{\text{maximize}} \{ q\Phi(q + \sum_{\substack{j=1 \\ j \neq i}}^{n_1} x_j^o) - f_i(q)\} \tag{3.7}$$

and the above (ii) in Definition 3.2 with $\bar{X}$, $(\bar{y}, \bar{z})$ replaced by X^o, (y^o, z^o) respectively holds.

Observe that the above definitions of competitive and oligopolistic equilibrium solutions respectively derive from an assumed market behavior of either perfect competition or a Nash-

Cournot oligopoly on the part of the firms in S_1, given the leader-follower Stackelberg-Nash-Cournot behavior of the firms in S_2 and S_3 as embodied by Definition 3.1.

3.2. Characterization of the Aggregate Reaction Curve

The purpose of this section is to present some important properties of the aggregate reaction curve $Y_R(Z,P)$, $Z \geq 0$, $P \geq 0$, which will play a central role in characterizing the perceived demand curve $F(Z, P)$, $Z \geq 0$, defined in (3.2) for any given $P \geq 0$, and in characterizing a SNC equilibrium. First, we demonstrate that $Y_R(Z,P)$ is a continuous function of $Z \geq 0$. Next, using the right-hand derivatives, we study monotonicity and concavity. In particular, we show that $Y_R(Z,P)$ is strictly decreasing in $Z \geq 0$ and in $P \geq 0$ over its positive range, while $Z + Y_R(Z,P)$ is strictly increasing in $Z \geq 0$. Finally, we provide conditions under which $Y_R(Z,P)$ is convex in $Z \geq 0$ and concave in $P \geq 0$ over its positive range. The former property is important due to problems (3.3), while the latter one is important in establishing concavity of $Y(P)$. Under Assumptions A2, the reaction curves $y_{Ri}(Z,P)$, $i = 1,\dots, n_2$ and $Y_R(Z,P)$ are defined by the following system obtained via the necessary and sufficient optimality conditions for problems (3.1):

$$p[Z + Y_R(Z,P)] + y_{Ri}(Z,P)p'[Z + Y_R(Z,P)] - g_i'[y_{Ri}(Z,P)] - P + v_i(Z, P) = 0,$$

$$i = 1,\dots, n_2 \tag{3.8}$$

$$\sum_{i=1}^{n_2} y_{Ri}(Z,P) = Y_R(Z,P) \tag{3.9}$$

$$y_{Ri}(Z,P) \geq 0, \quad v_i(Z, P) \geq 0, \quad i = 1,\dots, n_2 \tag{3.10}$$

$$y_{Ri}(Z,P)v_i(Z,P) = 0, \quad i = 1,\dots, n_2 \tag{3.11}$$

Moreover, for any fixed $Z \geq 0$ the solution to the above system is unique (see Sherali, Soyster and Murphy [S5] and Corollary 2.1), so that the reaction curves $y_{Ri}(Z,P)$, $i = 1,\dots, n_2$, and

$Y_R(Z,P)$ are well defined functions for any $Z \geq 0$. Furthermore, the following continuity and monotonicity properties hold.

Theorem 3.1. The functions $y_{Ri}(Z,P)$, $i = 1,..., n_2$, and hence the aggregate function $Y_R(Z,P)$, are continuous functions of $Z \geq 0$.

Proof. Consider any fixed $(Z_o, P_o) \geq 0$ and any nonnegative sequence $\{(Z_k, P_k)\}$ convergent to (Z_o, P_o). Define $Y_{Rk} = Y_R(Z_k, P_k)$, $y_{Rk} = y_R(Z_k, P_k)$, and $v_{Rk} = v_R(Z_k, P_k)$ in (3.8)-(3.11). Note that by Assumptions A2 and equations (3.8)-(3.11), the sequnce $\{(y_{Rk}, v_k, Y_{Rk})\}$ is bounded, and therefore has a convergent subsequence. For simplicity, assume that $\{(y_{Rk}, v_k, Y_{Rk})\}$ is itself convergent and let (y_{Ro}, v_o, Y_{Ro}) denote its limit. We will show that $(y_R(Z_o, P_o), v(Z_o, P_o), Y_R(Z_o, P_o)) = (y_{Ro}, v_o, Y_{Ro})$. Observe that for each k we have $p(Z_k + Y_{Rk}) + y_{Rki}p'(Z_k + Y_{Rk}) - g_i'(y_{Rki}) - P_k + v_{ki} = 0$ for $i = 1,..., n_2$, $\sum_{i=1}^{n_2} y_{Rki} = Y_{Rk}$ and $y_{Rki} \geq 0$, $v_{ki} \geq 0$, $y_{Rki}v_{ki} = 0$ for $i = 1,..., n_2$. Noting that the functions $p(\cdot)$, $p'(\cdot)$ and $g_i'(\cdot)$ are continuous, and taking limits as $k \to \infty$, we obtain $p(Z_o + Y_{Ro}) + y_{Roi}p'(Z_o + Y_{Ro}) - g_i'(y_{Roi}) - P_o + v_{oi} = 0$ for $i = 1,..., n_2$, $\sum_{i=1}^{n_2} y_{Roi} = Y_{Ro}$ and $y_{Roi} \geq 0$, $v_{oi} \geq 0$, $y_{Roi}v_{oi} = 0$ for $i = 1,..., n_2$. Since the solution to the system (3.8)-(3.11) is unique for any $Z \geq 0$ (see Sherali et al. [S5] and Corollary 2.1), we necessarily have $(y_R(Z_o, P_o), v(Z_o, P_o), Y_R(Z_o, P_o)) = (y_{Ro}, v_o, Y_{Ro})$. This completes the proof.■

Theorem 3.2. The aggregate reaction curve $Y_R(Z,P)$ is strictly decreasing in $Z \geq 0$ and in $P \geq 0$ over its positive range. Moreover, $Z + Y_R(Z,P)$ is strictly increasing in $Z \geq 0$. Furthermore. for each $i = 1,..., n_2$, the reaction curve $y_{Ri}(Z,P)$ is strictly decreasing in $Z \geq 0$ over its positive range, and if $p(\cdot)$ is convex, then $y_{Ri}(Z,P)$ is also strictly decreasing in $P \geq 0$ over its positive range.

Proof. From the development in Sherali et al. [S5] it follows that for any $Z \geq 0$ the right-hand (partial) derivative $D_Z^+(Y_R)$ of $Y_R(Z,P)$ with respect to Z satisfies $-1 < D_Z^+(Y_R) \leq 0$, where the right inequality is strict whenever $Y_R(Z,P) > 0$. Also, $D_Z^+(y_{Ri}) \leq 0$ for each $i = 1,..., n_2$. However, if $y_{Ri}(Z,P) > 0$, then let us show that $D_Z^+(y_{Ri}) < 0$. Consider some $(Z_o, P_o) \geq 0$. If

$y_{Ri}(Z_o, P_o) = Y_R(Z_o, P_o)$, then the assertion is immediate from above. Hence, assume that $0 < y_{Ri}(Z_o, P_o) < Y_R(Z_o, P_o)$, and suppose that $D_Z^+(y_{Ri}) = 0$. Then, by equations (2.15), (2.16) in Sherali et al. [S5], we obtain (using our notation) $[1 + D_Z^+(Y_R)][p' + y_{Ri}(Z_o, P_o)p''] = 0$, where p' and p'' are evaluated at $Z_o + Y_R(Z_o, P_o)$. But this means $p' + y_{Ri}(Z_o, P_o)p'' = 0$. If $p'' \leq 0$ then a contradiction is evident. Hence, assume that $p'' > 0$. Then we get $p' + y_{Ri}(Z_o, P_o)p'' < p' + [Z_o + Y_R(Z_o, P_o)]p'' \leq 0$ by assumption (A2), which again results in a contradiction. Hence, $y_{Ri}(Z,P)$ is strictly decreasing in $Z \geq 0$ over its positive range.

The monotonicity of $Y_R(Z,P)$, and $y_{Ri}(Z,P)$ in $P \geq 0$ follows readily from Theorem 2.1 and proof of Lemma 2.4, by noting that for any fixed $Z \geq 0$, $y_{Ri}(Z,P)$, $i = 1,\ldots, n_2$, is the unique Nash-Cournot equilibrium solution with an extraneous market supply of Z units. This completes the proof. ■

Before proceeding, let us point out that one can actually derive expressions for the right-hand partial derivatives $D_Z^+(Y_R)$ and $D_P^+(Y_R)$ of $Y_R(Z,P)$ with respect to Z and P, respectively. In fact, by differentating equations (3.8)-(3.11) with respect to an increase in Z or P, and noticing by Theorem 3.1 and (3.8)-(3.11) that for $i = 1,\ldots, n_2$, $v_i(\cdot,\cdot)$ are continuous and right-differentiable by Assumptions A2, and moreover, $D_Z^+(v_i) = 0$ whenever $y_{Ri}(Z,P) > 0$, and $D_P^+(v_i) = 0$ whenever $y_{Ri}(Z,P) > 0$ or $D_P^+(y_{Ri}) > 0$, we readily obtain

$$D_Z^+(Y_R) = -\frac{\sum_{i\in J_1(Z,P)} R_i(Z,P)T_i(Z,P)}{1 + \sum_{i\in J_1(Z,P)} R_i(Z,P)T_i(Z,P)} \tag{3.12}$$

$$D_P^+(Y_R) = -\frac{\sum_{i\in J_2(Z,P)} T_i(Z,P)}{1 + \sum_{i\in J_2(Z,P)} R_i(Z,P)T_i(Z,P)} \tag{3.13}$$

where

$J_1(Z, P) = \{i: y_{Ri}(Z,P) > 0\}$,

$J_2(Z, P) = \{i: y_{Ri}(Z,P) > 0 \text{ or } D_P^+(y_{Ri}(Z,P)) > 0\}$,

$R_i(Z, P) = p'[Z + Y_R(Z,P)] + y_{Ri}(Z,P)p''[Z + Y_R(Z,P)]$,

$T_i(Z, P) = 1/\{p'[Z + Y_R(Z,P)] - g_i''[y_{Ri}(Z,P)]\}$, for $i = 1,\ldots, n_2$.

Based on this, we can establish the following results related to the differentiability of $Y_R(\cdot, \cdot)$, the convexity of $Y_R(\cdot, P)$ for a given $P \geq 0$ and the concavity of $Y_R(Z, \cdot)$ for a given $Z \geq 0$.

Lemma 3.1. Assume that the firms in S_2 are identical, i.e., $g_i(\cdot) = g(\cdot)$ for $i = 1,\ldots, n_2$. Then $Y_R(Z,P)$ is differentiable in $Z \geq 0$ over its positive range.

Proof. Since $g_i(\cdot) = g(\cdot)$ for $i = 1,\ldots, n_2$, it follows that for any $Z \geq 0$ the unique solution to (3.8)-(3.11) yields $y_{Ri}(Z,P) = Y_R(Z,P)/n_2$, $i = 1,\ldots, n_2$. Therefore, over the positive range of $Y_R(Z,P)$, we get $J_1(Z, P) = J_2(Z, P) = \{1,\ldots, n_2\}$ as defined in (3.12) and (3.13). Upon simplifying and dropping the arguments in equations (3.12) and (3.13) we obtain

$$D_Z^+(Y_R) = -\frac{n_2 p' + Y_R p''}{(n_2 + 1)p' + Y_R p'' - g''} \qquad (3.14)$$

$$D_P^+(Y_R) = \frac{n_2}{(n_2 + 1)p' + Y_R p'' - g''} \qquad (3.15)$$

where p', and p'' are evaluated at $Z + Y_R(Z,P)$, while g'' is evaluated at $Y_R(Z,P)/n_2$. Notice that $Y_R(Z,P)$, p', and g'' are all continuous functions, and that $p' + Y_R p'' - g'' < 0$, since it is negative if $p'' \leq 0$ and if $p'' > 0$, it is strictly less than $p' + (Z + Y_R)p'' - g'' \leq 0$ by Assumptions (A2) and (A3). Hence, the partial derivatives of $Y_R(\cdot, \cdot)$ exist and are continuous over its positive range, being given by (3.14) and (3.15) and this completes the proof. ■

Lemma 3.2. Suppose that the industry demand function is given by $p(Q) = a - bQ^k$, where $a > 0$, $b > 0$, $k \geq 1$. Assume that

(i) $k = 1$ and $g_i'(\cdot)$ are concave for $i = 1,\ldots, n_2$,

or

(ii) $k > 1$ and $g_i(\cdot)$ are linear for $i = 1,\ldots, n_2$.

Then the aggregate reaction function $Y_R(Z,P)$ is convex in $Z \geq 0$ for any given $P \geq 0$.

Proof. Part (i) is established in Sherali [S3]. Therefore, consider part (ii). For this purpose we will demonstrate that for any $Z \geq 0$ and any $\Delta > 0$, we have $D^+_{Z\Delta}(Y_R) - D^+_Z(Y_R) \geq 0$, where the subscript Δ denotes evaluation at $Z + \Delta$, both here and below. Using the particular form of the demand and cost functions in equation (3.12), we obtain

$$D_Z^+(Y_R) = -\frac{\sum_{i \in J_1} [Q + (k-1)y_{Ri}]}{Q + \sum_{i \in J_1} [Q + (k-1)y_{Ri}]}$$

where $Q = Z + Y_R(Z,P)$, and $J_1 = J_1(Z,P)$. Therefore, the sign of the difference $D^+_{Z\Delta}(Y_R) - D^+_Z(Y_R)$ is the same as that of B, where

$$B = -Q \sum_{i \in J_{1\Delta}} [Q_\Delta + (k-1)y_{Ri\Delta}] + Q_\Delta \sum_{i \in J_1} [Q + (k-1)y_{Ri}].$$

Here, $J_{1\Delta} = J_1(Z + \Delta, P)$, the functions Q and y_{Ri} are evaluated at (Z, P), while Q_Δ and $y_{Ri\Delta}$ are evaluated at $(Z + \Delta, P)$. By Theorem 3.2 we have $J_{1\Delta} \subseteq J_1$. Hence, taking the second sum in B only over $J_{1\Delta}$, we get $B \geq (k-1) \sum_{i \in J_{1\Delta}} (-Qy_{Ri\Delta} + Q_\Delta y_{Ri})$. Since, $Q_\Delta > Q$ and $y_{Ri} \geq y_{Ri\Delta}$ by Theorem 3.2, this implies that $B \geq 0$ and the proof is complete. ∎

Lemma 3.3. Let the industry demand function be given by $p(Q) = a - bQ^k$, where $a > 0$, $b > 0$, $k \geq 1$. Assume that the marginal cost functions $g_i'(\cdot)$ for $i = 1,\ldots, n_2$, are identical and convex. Then for any fixed $Z \geq 0$, $Y_R(Z,P)$ is concave in $P \geq 0$ over its positive range, being strictly concave over its positive range whenever $k > 1$.

Proof. It is sufficient to show that the denominator in (3.15) is nondecreasing in $P \geq 0$, being strictly increasing if $k > 1$. By assumption, g'' is a nondecreasing function while $Y_R(Z,P)$ is strictly decreasing in $P \geq 0$ over its positive range. Hence, for any $\Delta \geq 0$, $-g''[Y_R(Z, P + \Delta)/n_2] \geq -g''[Y_R(Z,P)/n_2]$. Therefore, let us focus on the terms involving p' and p''. If $k = 1$ the result is trivial. For $k > 1$, letting $Q = Z + Y_R(Z,P)$ and using the function $p(\cdot)$ specified above, we get that $(n_2 + 1)p' + Y_R p'' = -bkQ^{k-2}[(n_2 + 1)Q + (k - 1)Y_R]$, where Q and Y_R are strictly decreasing in $P \geq 0$. If $k \geq 2$ then the lemma is readily true. For the case of $1 < k < 2$ note that $(n_2 + 1)Q + (k - 1)y_R = (n_2 + k)Q - (k - 1)Z$. Thus, alternatively, $(n_2 + 1)p' + Y_R p'' = -(n_2 + k)bkQ^{k-1} + bk(k - 1)Q^{k-2}Z$ which is again strictly increasing in $P \geq 0$, and this completes the proof. ■

We remark here that the role played in Lemma 3.1 by the assumption of identical firms in S_2 is crucial in avoiding kinks in the aggregate reaction curve which may appear when some, but not all, of the firms in S_2 drop off from production or begin producing under variations in the parameters Z and P, as governed by Theorem 3.2. This assumption is also necessary in Lemma 3.3, without which one may only have concavity between the consecutive kink points in the aggreate reaction curve. Furthermore, the requirement that the demand function $p(\cdot)$ or the cost functions $g_i(\cdot)$, $i = 1,..., n_2$ be linear as in Lemma 3.2 plays an important role in establishing the convexity of $Y_R(\cdot, P)$, given $P \geq 0$. For example, if $n_2 = 1$, $g(y) = 2.5y^2$ and $p(Q) = 20 - 0.25Q^2$, then for $P = 1$, problem (3.1) yields $1.5Y_R(Z, 1) = -(Z + 5) + \sqrt{0.25Z^2 + 10Z + 82}$, for $Z \geq 0$. One can easily verify that $Y_R(Z, 1)$ is in fact concave, and not convex, in $Z \geq 0$. Finally, we remark that the particular polynomial form of the demand function $p(Q) = a - bQ^k$, $a > 0$, $b > 0$, $k \geq 1$, is a direct generalization of a linear demand curve, and it is strictly decreasing, concave, and has a decreasing price elasticity. Moreover, it satisfies Assumptions A2. We will be employing this form of the demand function frequently in our analysis.

3.3. Characterization of the Perceived Demand Function

Recall that the demand function $F(Z, P)$, $Z \geq 0$, perceived by by the firms in S_3, for any fixed $P \geq 0$, is given by $F(Z, P) = p[Z + Y_R(Z,P)]$, $Z \geq 0$, as in equation (3.2). Hence, observe that from Theorem 3.2, as $Z \geq 0$, increases for any fixed $P \geq 0$, the function $Y_R(Z,P)$ decreases, and if $Y_R(Z,P)$ gets driven to zero for some $Z \geq 0$, the function $F(Z, P)$ will then coincide with the demand function $p(Z)$ for further increasing values of Z. We are therefore interested in such a critical value of Z, if it exists. Hence, for any fixed $P \geq 0$, define

$$Z_0(P) = \begin{cases} \min\{Z \geq 0:\ Y_R(Z,P) = 0\} & \text{if } Y_R(Z,P) = 0 \text{ for some } Z \geq 0 \\ \infty & \text{otherwise} \end{cases} \tag{3.16}$$

The following result characterizes $Z_0(P)$ and provides an analytical expression for this treshold value of Z, when it exists.

Lemma 3.4. Let $C_0 = \{P \geq 0:\ Y_R(Z,P) = 0 \text{ for some } Z \geq 0\}$. Then C_0 is a nonempty connected set. In particular, if $p(Q) = 0$ for some finite $Q \geq 0$, then $C_0 \equiv \{P:\ P \geq 0\}$. Moreover, $Z_0(P)$ is a continuous and nonincreasing function of P on C_0, being strictly decreasing over its positive range, and for $P \in C_0$ is in fact given by

$$Z_0(P) = \begin{cases} p^{-1}[P + g'_{\min}(0)] & \text{if } P \leq p(0) - g'_{\min}(0) \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

where $g'_{\min}(0) = \text{minimum}\{g_i'(0),\ i \in i = 1,\ldots, n_2\}$.

Proof. Notice from (3.8)-(3.11) that

$$Y_R(Z,P) = 0 \text{ if and only if } p(Z) \leq g'_{\min}(0) + P, \tag{3.18}$$

where $g'_{\min}(0)$ is defined in the lemma. Thus $C_0 \neq \emptyset$ as certainly $p(0) \in C_0$, and C_0 is connected since $P_o \in C_0$ implies $P \in C_0$ for all $P \geq P_o$. Furthermore, if $p(Q_o) = 0$ for some $Q_o < \infty$, then

$0 \in C_0$ because $g'_{\min}(0) \geq 0$ implies that $Y_R(Q_o, 0) = 0$, and so $C_0 = \{P\colon P \geq 0\}$, i.e., a half-line, in this case. Finally, for any $P \in C_0$, since $p(\cdot)$ is strictly decreasing, $Z_0(P)$ is readily given by (3.17) from (3.16) and (3.18). Hence, from (3.17), $Z_0(P)$ is continuous on C_0, being strictly decreasing over its positive range, and the proof is complete. ■

Based on this characterization, we obtain the following result relating to the continuity, monotonicity and differentiability of the perceived demand function.

Theorem 3.3. Let $P \geq 0$ be fixed. Then the perceived demand function $F(Z, P)$, $Z \geq 0$, is continuous and strictly decreasing in $Z \geq 0$. Moreover, if the firms in S_2 are identical, then $F(Z, P)$, $Z \geq 0$ is differentiable in Z for $Z < Z_0(P)$ and is given by $p[Z + Y_R(Z,P)]$, and is differentiable for $Z > Z_0(P)$ (in case $Z_0(P) < \infty$) and is then simply given by $p(Z)$.

Proof. From equation (3.2) and Theorem 3.2, it follows that for any fixed $P \geq 0$, $F(Z, P)$ is continuous and strictly decreasing in $Z \geq 0$. If $Z_0(P)$ defined in (3.16) is positive, then for $Z < Z_0(P)$, $F(Z, P)$ given by (3.2) is differentiable in Z by Lemma 3.1. Finally, for $Z > Z_0(P)$ (where $Z_0(P) < \infty$), since $Y_R(\cdot, P) = 0$ in a neighborhood of Z, we have $F(Z, P) \equiv p(Z)$ and this completes the proof. ■

The role played by the identical firms in S_2 is to assure that there is at most one point of nondifferentiability, namely at $Z = Z_0(P)$, for the perceived demand function $F(Z, P)$, $Z \geq 0$. Furthermore, note that in addition to continuity and monotonicity, we also need some conditions which will guarantee the concavity of $F(\cdot, P)$ in order to make assertions about the existence of SNC equilibrium solutions, and to facilitate their computation (in the spirit of Murphy et al. [M3] or Szidarovszky and Yakowitz [S7, S8]). The next result addresses this issue, and concludes the characterization of $F(\cdot, P)$, given $P \geq 0$.

Theorem 3.4. Let the industry demand function be given by $p(Q) = a - bQ^k$, where $a > 0$, $b > 0$, $k \geq 1$, and suppose that the marginal cost functions $g_i'(\cdot)$ are concave for each $i = 1,..., n_2$. Then for any fixed $P \geq 0$, the function $F(Z, P)$ is concave in $Z \geq 0$.

Proof. If $k = 1$ then by Lemma 3.2 and equation (3.2), since $F(Z, P)$ is a concave (linear) and decreasing function of a convex function, it is concave in $Z \geq 0$. Hence, suppose that $k > 1$ and note that the reaction function $Y_R(Z,P)$ need not be convex in Z in this case. We will show concavity of $F(Z, P)$ in $Z \geq 0$ by demonstrating that $D_Z^+(F)$ is nonincreasing in Z. By (3.2) and (3.12) we have:

$$D_Z^+(F) - D_{Z\Delta}^+(F) = \frac{p'(1 + \sum_{i \in J_{1\Delta}} R_{i\Delta}T_{i\Delta}) - p'_\Delta(1 + \sum_{i \in J_1} R_iT_i)}{(1 + \sum_{i \in J_{1\Delta}} R_{i\Delta}T_{i\Delta})(1 + \sum_{i \in J_1} R_iT_i)}$$

where as before subscript Δ denotes an evaluation at $Z + \Delta$. The denominator in this expression is positive so we need to show that that for any $\Delta > 0$ its numerator, denote it by β, is negative. Toward this end observe that since $p(\cdot)$ is strictly concave and decreasing, and since $J_{1\Delta} \subseteq J_1$ and $R_iT_i > 0$, one obtains $\beta > \sum_{i \in J_{1\Delta}} (p'R_{i\Delta}T_{i\Delta} - p'_\Delta R_iT_i)$. Further, by the concavity of $g'(\cdot)$, and since $y_{Ri\Delta} \leq y_{Ri}$, we have $g_i'' \leq g_{i\Delta}''$. Hence, $0 > T_{i\Delta} > T_i$, which along with $p'R_{i\Delta} > 0$ gives $\beta > \sum_{i \in J_{1\Delta}} T_i(p'R_{i\Delta} - p'_\Delta R_i)$. By the definition of R_i we get $p'R_{i\Delta} - p'_\Delta R_i = p'p_\Delta''y_{Ri\Delta} - p'_\Delta p''y_{Ri}$, which in case of the demand function specified here, gives $p'R_{i\Delta} - p'_\Delta R_i = b^2k^2(k-1)(QQ_\Delta)^{k-2}(Qy_{Ri\Delta} - Q_\Delta y_{Ri})$. By Theorem 3.2, we have $Q < Q_\Delta$ and $y_{Ri\Delta} \leq y_{Ri}$. Hence, for $k > 1$, $p'R_{i\Delta} - p'_\Delta R_i \leq 0$, which along with $T_i < 0$ implies that $\beta > 0$, thus completing the proof. ■

3.4. Existence and Characterization of a SNC Equilibrium as a Function of the Semi-Finished Product Price

The purpose of this section is to examine the existence and properties of the SNC equilibrium solution as a function of $P \geq 0$, as embodied by the functions $y(P) = \{y_i(P), i = 1,..., n_2\}$ and $z(P) = \{z_i(P), i = 1,..., n_3\}$, and their respective totals $Y(P)$ and $Z(P)$ via Definition 3.1. In particular, under concavity and differentiability properties of $F(\cdot, P)$, as may be assured by the results of the foregoing section, and using some other sufficient conditions, we examine the existence, continuity, differentiability, monotonicity and convexity properties of the functions $Y(\cdot)$ and $Z(\cdot)$. First, let us address the existence and uniqueness of a SNC equilibium solution.

Theorem 3.5. Let $P \geq 0$ be fixed, and assume that the derived demand curve $F(Z, P)$ given by (3.2) is concave in $Z \geq 0$. Then there exists a SNC equilibrium solution $(y(P), z(P))$. Moreover, $y(P)$ is unique, while the total $Z(P) = \sum_{i=1}^{n_3} z_i(P)$ is unique over the set of SNC equilibrium solutions.

Proof. By Theorems 3.1 and 3.2, for a fixed $P \geq 0$, the demand function $F(Z, P)$, $Z \geq 0$ perceived by the firms in S_3 is continuous and strictly decreasing. Therefore, if $F(Z, P)$ is concave in $Z \geq 0$ then from the development in Szidarovszky and Yakowitz [S8] it follows that for any fixed $P \geq 0$ the set of solutions to problems (3.3) is nonempty, and moreover, the total output $Z(P)$ is the same for any such solution. This further implies that $y(P)$ defined in (3.4) exists and is unique , and the proof is complete. ■

Corollary 3.1. In addition to assumptions of Theorem 3.5, suppose that $F(Z, P)$ is differentiable in Z at $Z = Z(P)$. Then there exists a unique SNC equilibrium solution $(y(P), z(P))$.

Proof. Follows from Szidarovszky and Yakowitz [S8], using the assertion of Theorem 3.5. ■

Corollary 3.2. In addition to assumptions of Theorem 3.5, suppose that the firms in S_2 are identical with $g_i(\cdot) = g(\cdot)$ for $i = 1,..., n_2$, and suppose that $Y(P) > 0$. Then there exists a unique SNC equilibrium solution $(y(P), z(P))$.

Proof. Let $P = P_o \geq 0$ be fixed such that $Y(P_o) = Y_R[Z(P_o, P_o)] > 0$. If $Z(P_o) = 0$ then the result follows from Theorem 3.5. Hence, suppose that $Z(P_o) > 0$. By Lemma 3.1, $Y_R(Z, P_o)$ is differentiable in Z at $Z = Z(P_o) = 0$, and hence so is $F(Z, P_o)$. Using Corollary 3.1, this completes the proof. ∎

Observe that Theorem 3.5 asserts that when $F(\cdot, P)$ is concave, the equilibrating quantities $y(P)$, $Y(P)$ and $Z(P)$ are uniquely determined, and hence $y(P)$, $Y(P)$ and $Z(P)$ are well defined functions of $P \geq 0$. The two corollaries to the theorem provide conditions under which $z(P)$ is also uniquely determined. The next theorem and its corollaries address the behavior of these quantities as functions of P.

Theorem 3.6. Assume that $F(Z, P)$ is concave in $Z \geq 0$ for each fixed $P \geq 0$. Then $y(P) = \{y_i(P), i = 1,..., n_2\}$, $Y(P)$, $Z(P)$ and $Q(P) = Y(P) + Z(P)$ are all continuous functions of $P \geq 0$.

Proof. First, note by Theorem 3.5 that $Z(P)$, $y(P)$, $Y(P)$ and $Q(P)$ are all well defined functions. Secondly, if $Z(P)$ is a continuous function then by Theorem 3.1, $y_i(P) = y_{Ri}[Z(P), P]$, $i = 1,..., n_2$ and $Y(P) = Y_R[Z(P), P]$, $P \geq 0$ are also continuous, and hence so is $Q(P) = Z(P) + Y(P)$. Therefore, it is sufficient to show continuity of $Z(P)$, $P \geq 0$. Toward this end, consider a nonnegative sequence $\{P_k\} \to P_o$ and let $Z_k = Z(P_k)$. The sequence $\{Z_k\}$ being nonegative and bounded (by Assumption (A4)) has a convergent subsequence. Without loss of generality assume that $\{Z_k\}$ itself converges, and let Z_o denote its limit. We must show that $Z_o = Z(P_o)$. Now, for each k let z_k be a vector of equilibrating outputs for the firms in S_3 with total Z_k. From Theorem 3.5, although z_k is not unique, we have the same total $\sum_{i=1}^{n_3} z_{ki} = Z_k$. Since

$\{z_k\}$ is contained in a compact set, let $\{z_k\}_K \rightarrow z_o$ over an index set K, and note that $\sum_{i=1}^{n_3} z_{oi} = Z_o$. Then, from (3.3), we have from each $i = 1,..., n_3$ and any $q > 0$ that

$$z_{ki}F(Z_k, P_k) - h_i(z_{ki}) \geq qF(q + \sum_{j \neq i} z_{ki}, P_k) - h_i(q)$$

for each k. Taking limits as $k \rightarrow \infty$, $k \in K$ we obtain

$$z_{oi}F(Z_o, P_o) - h_i(z_{oi}) \geq qF(q + \sum_{j \neq i} z_{ki}, P_k) - h_i(q)$$

for all $q \geq 0$, $i = 1,..., n_3$. But this means that at $P = P_o$, z_o is an equilibrium solution with total Z_o, which implies that $Z(P_o) = Z_o$, and hence the proof is complete. ■

Corollary 3.3. Under assumptions of Theorem 3.6, $z(P)$ is a closed point-to-set map.

Proof. Evident from the proof of Theorem 3.6.■

Corollary 3.4. In addition to assumptions of Theorem 3.6, suppose that for a given P_o, $F(Z, P)$ is differentiable in Z at $Z = Z(P)$ for each P in some neighborhood of P_o. Then $z(P)$ is a continuous function of P in some neighborhood of P_o.

Proof. Follows from Corollaries 3.1 and 3.3. ■

From Theorems 3.5 and 3.6, although the total equilibrating output $Z(P)$ of the firms in S_3 is a continuous function of $P \geq 0$, the individual equilibrating output vector $z(P)$ constitutes only a closed point-to-set map in general as in Corollary 3.3, unless some additional conditions such as the one in Corollary 3.4 hold. The following result addresses the monotonicity and differentiability of the total equilibrating quantities $Y(P)$, $Z(P)$ and $Q(P)$ as functions of $P \geq 0$. These conditions will be useful in establishing the existence and uniqueness of equilibrium solutions in the two-stage model.

Theorem 3.7. Let the industry demand function be given by $p(Q) = a - bQ^k$, where $a > 0$, $b > 0$, $k \geq 1$, and assume that the firms in S_2 are identical, with $g_i(\cdot) = g(\cdot)$ for $i = 1,..., n_2$, where $g(\cdot)$ is a quadratic or a linear function. Then for any $P \geq 0$ such that $Y(P) > 0$

(i) $Z^+(P) \geq 0$, with $Z^+(P) > 0$ if $Z(P) > 0$

(ii) $Y^+(P) < 0$, and $Q^+(P) < 0$.

Proof. First of all, note from Theorems 3.4 and 3.6 that $Z(P)$, $Y(P)$ and $Q(P)$ are all continuous functions of $P \geq 0$. Now, consider some $P = P_o$ such that $Y(P_o) > 0$. If $Z(P_o) = 0$ then evidently $Z^+(P_o) \geq 0$. Hence, suppose that $Z(P_o) > 0$. By assumption, $g_i(\cdot) = g(\cdot)$ for $i = 1,..., n_2$, so that by Lemma 3.1, $Y_R(Z,P)$ is differentiable in Z and in P over its positive range. Furthermore, since $Y(P_o) > 0$, where recall that $Y(P) = Y_R[Z(P), P]$, we can conclude that $F(Z, P)$ is differentiable in Z at $Z = Z(P)$ for all P in some neighborhood of P_o, which by Corollary 3.4 implies that $z(P)$ is a continuous function of P in some neighborhood of P_o. Now, define $J_3(P) = \{i\colon z_i(P) > 0 \text{ or } z_i^+(P) > 0\}$ and observe that by the necessary and sufficient conditions for problems (3.3), we get

$$p[Z(P) + Y(P)] + p'[Z(P) + Y(P)][1 + \frac{\partial}{\partial Z}Y_R(Z(P),P)]z_i(P) - h_i'[z_i(P)] = 0$$

$$\text{for } i \in J_3(P) \tag{3.19}$$

and $\sum_{i \in J_3(P)} z_i(P) = Z(P)$ for all P in some neighborhood of P_o. (Note that the optimal Lagrange multiplier associated with nonnegativity constraint in (3.3) is zero for all $P \in [P_o, P_o + \varepsilon]$ for some $\varepsilon > 0$, for any $i \in J_3(P_o)$.) Let us now take the right-hand derivative of (3.19) with respect to (an increase in) P and evaluate it at P_o. Noticing that whenever $Y(P) > 0$, we have $Y^+(P) = \frac{\partial}{\partial Z}Y_R[Z(P), P]Z^+(P) + \frac{\partial}{\partial P}Y_R[Z(P), P]$, and using equations (3.14) and (3.15), we obtain upon some simplification (for the sake of notation the arguments of the functions used below are omitted):

$$(V - g'')^2 z_i^+(P_o) = -[\eta_i + \xi_i Z^+(P_o)]/\alpha_i \quad \text{for all } i \in J_3(P_o) \tag{3.20}$$

where

$$V = (n_2 + 1)p' + p''Y_R,$$

$$\alpha_i = p'(1 + \frac{\partial}{\partial Z}Y_R) - h_i'',$$

$$\eta_i = (\frac{\partial}{\partial P}Y_R)A_i$$

$$\xi_i = (1 + \frac{\partial}{\partial Z}Y_R)[A_i + p'p''z_i(V - g'')]$$

and where

$$A_i = p'(V - g'')^2 + (p' - g'')[(p'')^2 - p'p''']z_iY_R + p'p''z_i(V - p' - g'') + p''(g'')^2z_i.$$

Summing up equations (3.20) for $i \in J_3(P_o)$, we obtain

$$Z^+(P_o)[(V - g'')^2 + \sum_{i\in J_3} \xi_i/\alpha_i] = - \sum_{i\in J_3} \eta_i/\alpha_i. \tag{3.21}$$

Observe that $\alpha_i < 0$ and $A_i < 0$, and hence, $\xi_i < 0$ while $\eta_i > 0$. Therefore, equation (3.21) implies that $Z^+(P_o) > 0$, thus completing the proof of part (i).

Now, let us prove part (ii). Recall that $Y^+(P_o) = \frac{\partial}{\partial Z}Y_R[Z(P_o), P_0]Z^+(P_o) + \frac{\partial}{\partial P}Y_R[Z(P_o), P_o]$. Since $Z^+(P_o) \geq 0$ from above, and since by Theorem 3.2 both the partial derivatives of $Y_R(\cdot, \cdot)$ are negative, we get $Y^+(P_o) < 0$. Finally, we will show that $Q^+(P_o) < 0$. By definition, we have $Q^+(P_o) = Z^+(P_o) + Y^+(P_o)$. If $Z^+(P_o) = 0$ then readily $Q^+(P_o) < 0$. Hence, suppose that $Z^+(P_o) > 0$. Then, by equations (3.14), (3.15) and (3.21), and denoting $W = (V - g'')^2 + \sum_{i\in J_3} \xi_i/\alpha_i$, we obtain $Q^+(P_o) = \frac{\partial}{\partial P}Y_R[(V - g'')^2 + p'p''(p' - g'')\sum_{i\in I_3} z_i/\alpha_i]/W$. The term in $[\cdot]$ is positive which along with $\frac{\partial}{\partial P}Y_R < 0$ and $W > 0$ implies that $Q^+(P_o) < 0$, thus completing the proof. ■

Corollary 3.5. Under assumptions of Theorem 3.7, there exists price $P_0 \geq 0$ such that $Y(P) > 0$ for $0 \leq P < P_0$ and $Y(P) = 0$ for $P \geq P_0$.

Proof. If $Y_0 = 0$, then $P_0 = 0$ since from the proof of Theorem 3.7, we obtain $Y(P) = 0$ for all $P \geq 0$. On the other hand, if $Y_0 > 0$, then from (3.4) and (3.16), we necessarily have $0 \leq Z(0) < Z_0(0)$. Moreover, $Y(P) > 0$ for all $P \geq 0$ such that $Z(P) < Z_0(P)$, and $Y(P) = 0$ for all $P \geq 0$ such that $Z(P) \geq Z_0(P)$. But from Lemma 3.4, $Z_0(P)$ is continuous in $P \geq 0$ and strictly decreasing over its positive range, while $Z(P)$ is continuous in $P \geq 0$, being nondecreasing over the positive range of $Y(P)$. Hence, there exists a price $P_0 > 0$ such that $Z(P) < Z_0(P)$ for $0 \leq P < P_0$ and $Z(P) \geq Z_0(P)$for $P \geq P_0$. Therefore, $Y(P) > 0$ for $P \in [0, P_0)$ and $Y(P) = 0$ for $P \geq P_0$, and this completes the proof. ■

Remark 3.1. From the proof of Theorem 3.7 and Corollary 3.5, it follows that if $Y_0 = Y(0) > 0$, then $Z(P) > 0$ for all $P \in [0, P_0]$ if $Z(0) > 0$, and if $Z(0) = 0$ while $Z(P_0) > 0$, then there exists a P_Z, where $0 < P_Z < P_0$, such that $Z(P) = 0$ for all $P \in [0, P_Z]$, and $Z(P) > 0$ for $P \in (P_Z, P_0]$. Finally, if $Z(P_0) = 0$, then $Z(P) = 0$ for all $P \in [0, P_0]$.

Corollary 3.6. In addition to the assumptions of Theorem 3.7, suppose that the cost functions $h_i(\cdot)$, $i = 1,\ldots, n_3$, are identical, and assume that $P_0 > 0$. If $Z(0) = 0$ and $Z(P_0) > 0$, let P_Z be as defined above, and otherwise, arbitrarily let $P_Z = 0$. Then $Y(P)$ and $Z(P)$ are differentiable for $P \in (0, P_Z)$ and for $P \in (P_Z, P_0)$.

Proof. From Theorem 3.4 and Corollary 3.2 it follows that for all P such that $Y(P) > 0$, since the firms in S_2 as well as in S_3 are identical, $z_i(P)$ is given by $Z(P)/n_3$ for $i = 1,\ldots, n_3$ via (3.19). But this implies that for any $\hat{P} \in (P_Z, P_0)$, we have $Y(\hat{P}) > 0$ and $Z(\hat{P}) > 0$, so that $I_3(\hat{P}) \equiv \{1,\ldots, n_3\}$, and from (3.21), (3.14) and (3.15), we can claim that $Z^+(P)$ exists and is continuous in a neighborhood of $\hat{P}$ and so $Z(P)$ is differentiable at $\hat{P}$. Since $Y(P) = Y_R[Z(P), P]$, we have from Lemma 3.1 that $Y(P)$ is also differentiable at $\hat{P}$. Hence, $Y(P)$ and $Z(P)$ are both differentiable on (P_Z, P_0).

If $P_Z > 0$, then $Z(\cdot)$ is also differentiable on $(0, P_Z)$ since $Z(P) \equiv 0$ for $P \in [0, P_Z]$. Furthermore, since $Y(P) = Y_R[Z(P), P]$ and $Y(P) > 0$ for $P \in (0, P_Z)$ as $P_Z \leq P_0$, it follows from Lemma 3.1 that $Y(\cdot)$ is also differentiable on $(0, P_Z)$, and this completes the proof. ■

The ingredients of Theorem 3.7 are constructed from Lemma 3.1, Theorems 3.3 and 3.4, and the consequences of these results given in Theorem 3.6 and Corollary 3.4. Actually, in Theorem 3.7, if one assumes that $Y(P)$, $Z(P)$, and $Q(P)$ are all continuous in $P \geq 0$, and replaces the particular form of $p(Q)$ by the assumption that $p(Q)$ is concave and that $[p'']^2 \geq p'(Q)p'''(Q)$ for all $Q \geq 0$, then from the proof of theorem, it can be readily shown that the result continues to hold. On the other hand, the assumption of identical firms in S_2 is crucial for the assertion in Theorem 3.7. In fact, if one relaxes this assumption, then even with linear demand and cost functions, and with $n_3 = 1$, the function $Z(P)$ need not be monotone, and the function $Y(P)$ need not be strictly decreasing over its positive range. For example, when $p(Q) = 20 - Q$, and $n_2 = 2$ with $g_1(y) = g_2(y) = 8y$, and $n_3 = 1$ with $h(z) = 10z$, we obtain from the solution to (3.1) that

$$Y_R(Z,P) = \begin{cases} 2(14 - Z - P)/3 & \text{for } (Z + P) \in [0, 8] \\ (16 - Z - P)/2 & \text{for } (Z + P) \in [8, 16] \\ 0 & \text{for } (Z + P) \in [16, \infty) \end{cases}$$

Using the above reaction function in problems (3.3) we get

$$Z(P) = \begin{cases} 1 + P & \text{for } P \in [0, 3.5] \\ 8 - P & \text{for } P \in [3.5, 4] \\ 2 + 0.5P & \text{for } P \in [4, 28/3] \\ 16 - P & \text{for } P \in [28/3, 11] \\ 5 & \text{for } P \in [11, \infty) \end{cases}$$

which gives

$$Y(P) = Y_R[Z(P),P] = \begin{cases} 2(13 - 2P)/3 & \text{for } P \in [0, 3.5] \\ 4 & \text{for } P \in [3.5, 4] \\ 7 - 3P/4 & \text{for } P \in [4, 28/3] \\ 0 & \text{for } P \in [28/3, \infty) \end{cases}$$

Hence, for $P \in [3.5, 4]$, $Z(P)$ is strictly decreasing and $Y(P)$ remains constant, while still being positive.

Corollary 3.7. Under assumptions of Theorem 3.7, suppose that $Y_0 > 0$. Then, the derived demand function $\Phi(\theta)$ is continuous , being strictly decreasing on $(0, Y_0)$. If in addition, the firms in S_3 are identical, then $\Phi(\theta)$ is differentiable in this interval, except perhaps at $\theta = Y(P_Z)$.

Proof. Follows directly from (3.5) and Corollaries 3.5 and 3.6. ■

Remark 3.2. Characterization of $Z(P)$, $P \geq 0$ under the assumptions of Theorem 3.7.

Suppose that $P_0 > 0$. If $Z(0) > 0$, then $Z(P)$ is continuous and strictly increasing on $[0, P_0]$, and if the firms in S_3 are identical, then it is also differentiable on $(0, P_0)$. If $Z(0) = 0$, then $Z(P) = 0$ for $P \in [0, P_Z]$, and $Z(P)$ is strictly increasing in P for $P \in (P_Z, P_0)$. Furthermore, $Z(P)$ is also differentiable on $(0, P_Z)$, and on (P_Z, P_0) if the firms in S_3 are identical.

Next, as in Definition 3.2, let $z_{NC} = (z_{NC1}, ..., z_{NCn_3})$ denote the Nash-Cournot equilibrium solution for the firms in S_3, given that they face the demand function $p(Z)$ (with the firms in S_2 being absent), and let $Z_{NC} = \sum_{i=1}^{n_3} z_{NCi}$. From Szidarovszky and Yakowitz [S8], z_{NC} is unique. Furthermore, from Lemma 3.4 and under the assumptions of Theorem 3.7, $C_0 = \{P\colon P \geq 0\}$ and so $0 \leq Z_0(P) < \infty$ in (3.16) for all $P \geq 0$. Now, observe that minimum$\{P \geq 0\colon Z_0(P) \leq Z_{NC}\}$ is well defined by Lemma 3.4. Moreover, from (2.39) and (3.17) we have

$$P_{NC} = \text{minimum}\,\{P \geq 0\colon\ Z_0(P) \leq Z_{NC}\}. \tag{3.22}$$

Therefore, if $Z_{NC} = 0$, then for $P > P_{NC}$, $Z_0(P) \equiv 0$ from (3.16) and (3.22), and since by Theorem 3.3, the perceived demand function coincides with $p(\cdot)$, we have $Z(P_{NC}) = Z_{NC} = 0$. On the other hand, if $Z_{NC} > 0$, then for $P > P_{NC}$, since $Z_0(P) < Z_{NC}$ by (3.22) and Lemma 3.4, it follows that z_{NC} solves problems (3.3) (uniquely by Theorem 3.3 and Corollary 3.1), because Z_{NC} lies on the differentiable segment of the concave perceived demand curve $F(\cdot, P)$, which coincides with $p(\cdot)$ by Theorem 3.3. Hence, in either case, by continuity of $Z(P)$, we get $Z(P) = Z_{NC}$ for $P \geq P_{NC}$. By Theorem 3.7, therefore, we must have $P_{NC} \geq P_0$. Summarizing,

$Z(P) = Z_{NC}$ for $P \geq P_{NC}$, where $P_{NC} \geq P_0$.

Finally, consider $P \in [P_0, P_{NC}]$, and note that if $Z(P) < Z_0(P)$, then $Y(P)$ is positive by (3.16) which is a contradiction because $P > P_0$. Furthermore, if $Z(P) > Z_0(P)$, then since $Z(P)$ lies on the differentiable segment of the perceived demand function $F(\cdot, P)$ which coincides with $p(\cdot)$ by Theorem 3.3, it must be that $Z(P) = Z_{NC}$, which contradicts $P < P_{NC}$ by (3.22). Therefore,

$Z(P) = Z_0(P)$ given by (3.17) for $P \in [P_0, P_{NC}]$.

We will now proceed to examine the convexity properties of $Z(P)$ and $Y(P)$ over the positive range of $Y(P)$. In particular, as in Chapter 2, attention is focused on establishing sufficient conditions under which $\Phi(\cdot)$ is concave over its positive range, i.e., when it coincides with the inverse of $Y(\cdot)$. This will imply that the optimization problems (3.7) for the upstream producers are convex programming problems, and will be useful in establishing the existence and uniqueness of an overall equilibrium solution. The following results address these convexity properties.

Lemma 3.5. Suppose that the industry demand function $p(\cdot)$ is linear, the firms in S_2 are identical with $g_i(\cdot) = g(\cdot)$ for $i = 1, \ldots, n_2$, where $g(\cdot)$ is quadratic (or linear), and the marginal cost functions $h_i'(\cdot)$, $i = 1, \ldots, n_3$ are concave. Then $Z(P)$ is convex for $0 \leq P \leq P_0$.

Proof. If $P_0 = 0$, the proof is trivial. Hence, suppose that $P_0 > 0$ and consider some $P_o \in [0, P_0)$ so that $Y(P_o) > 0$. By the linearity of $p(\cdot)$ we obtain in the proof of Theorem 3.7 that A_i are identical and negative, and so, ξ_i are also identical and negative, and η_i are identical and positive. Therefore, from (3.20), $z_i^+(P_o)$ has the same sign for all $i \in J_3(P_o)$, which along with Theorem 3.7 (i) gives $z_i^+(P_o) \geq 0$ for all $i \in J_3(P_o)$. Hence, $J_3(P_o) \subseteq J_3(P_o + \Delta)$ for any $\Delta > 0$ such that $Y(P_o + \Delta) > 0$, and by the concavity of $h_i'(\cdot)$, $h_i''(z_{i\Delta}) \leq h_i''(z_i)$, where the subscript Δ denotes an evaluation at $P_o + \Delta$. From (3.17), $1 + \frac{\partial}{\partial Z} Y_R$ is a positive constant, and so $\sum_{i \in J_{3\Delta}} (-1/\alpha_{i\Delta}) \geq \sum_{i \in J_3} (-1/\alpha_i) > 0$. Hence, from (3.22), since $V - g''$ is a constant, we obtain $Z^+(P_o) \leq Z^+(P_o + \Delta)$, which means that $Z(P)$ is convex for $0 \leq P < P_0$, and hence on $[0, P_0]$ by continuity. This completes the proof. ■

Theorem 3.8. Let $p(Q) = a - bQ^k$, where $a > 0$, $b > 0$, $k \geq 1$. Suppose that $g_i'(\cdot) = g'(\cdot)$ for $i = 1,\ldots, n_2$, and assume that the marginal cost functions $h_i'(\cdot)$, for $i = 1,\ldots, n_3$ are concave. Then $\Phi(\theta)$ defined by (3.5) is concave in θ, $0 \leq \theta \leq Y_0$ if one of the following two conditions holds:

(i) $k = 1$ and $g(\cdot)$ is quadratic or linear

(ii) $k > 1$, $g(\cdot)$ is linear, and the firms in S_3 are identical with $h_i(\cdot) = h(\cdot)$ being thrice differentiable for $i = 1,\ldots, n_3$.

Moreover, under condition (ii), $\Phi(\theta)$ is strictly concave for $0 \leq \theta \leq Y_0$.

Proof. Below part (i) is established. The proof of part (ii) is very long and therefore, is relegated to Appendix A. If $Y_0 = 0$, the result is trivial. Hence, suppose that $Y_0 > 0$. As in the proof of Theorem 2.3, it is sufficient to show that $Y(P)$ is concave over its positive range. To establish this we will demonstrate that the right-hand derivative $Y^+(P)$ is nonincreasing in $P \geq 0$ over the positive range of $Y(P)$, i.e., for $P \in [0, P_0)$,by Corollary 3.5. By assumption all the firms in S_2 are identical and so $Y_R(Z,P)$ is differentiable in Z and in P over its positive range, by Lemma 3.1. Therefore,

$$Y^+(P) = (\frac{\partial}{\partial Z}Y_R)Z^+(P) + \frac{\partial}{\partial P}Y_R, \tag{3.23}$$

where the partial derivatives are given by (3.14) and (3.15). From these equations, since $g''(\cdot)$ is a nonnegative constant, while $p'(\cdot)$ is a a negative constant, it follows that both the partial derivatives are negative constants. Furthermore, by the linearity of $p(\cdot)$, and the concavity of $h_i'(\cdot)$, $i = 1,\ldots, n_3$, we obtain from Lemma 3.5 that $Z^+(P)$ is nondecreasing in $P \in [0, P_0)$, and so from (3.23) $Y^+(P)$ is nonincreasing in $P \in [0, P_0)$, and this proves case (i). ∎

3.5. Existence and Uniqueness of Equilibrium Solutions

The purpose of this section is to establish the existence and uniqueness of equilibrium solutions defined in Definitions 3.2 and 3.3, under the assumptions of Theorem 3.7, which ensure the existence of $\Phi(\theta)$ defined in (3.5). Before proceeding, one noteworthy comment regarding Definitions 3.2 and 3.3 in light of Remark 3.2 is that whenever at equilibrium $\overline{X} = 0$ or $X^{o} = 0$, so that $\overline{Y} = 0$ or $Y^{o} = 0$, respectively, the equilibrium price for the semi-finished product can be taken as any value not smaller that P_0. However, noting the behavior of $Z(P)$ for $P \in [P_0, P_{NC}]$, since the firms are effectively nonexistent in this case, we may take $\overline{P} = P_{NC}$ $(P^{o} = P_{NC})$ or greater so that Z_{NC} is the unique equilibrating output for the firms in S_3. Hence, we have

$$\overline{P} = \begin{cases} \Phi(\overline{X}) & \text{if } \overline{X} > 0 \\ P_{NC} & \text{otherwise} \end{cases} \qquad P^{o} = \begin{cases} \Phi(X^{o}) & \text{if } X^{o} > 0 \\ P_{NC} & \text{otherwise} \end{cases} \tag{3.24}$$

The following results provide sufficient conditions for the existence and uniqueness of the overall equilibrium solutions and are the counterparts of Theorems 2.5 and 2.6, established in the case of the follower-follower oligopoly in the final commodity market.

Theorem 3.9. Suppose that the assumptions of Theorem 3.7 hold, and let the cost functions $f_i(\cdot)$, $i = 1, \ldots, n_1$ for the firms in S_1 be strictly convex and increasing over the nonnegative real line. Then there exists a unique competitive equilibrium solution $(\overline{x}, \overline{y}, \overline{z})$ to the two-stage model. Moreover,

(i) $\overline{X} > 0$ if and only if $Y(P_\ell) > 0$

(ii) if $Y(P_\ell) > 0$ then $\overline{P} \in (P_\ell, P_u)$ where

$$P_\ell = \min\{f_i'(0), i = 1, \ldots, n_1\} \tag{3.25}$$

$$P_U = \min\{P_{NC}, P_u\} \tag{3.26}$$

$$P_u = \min\{f_j'(Y_0): j \in \operatorname{argmin}\{f_i'(0), i = 1, \ldots, n_1\}\}. \tag{3.27}$$

Proof. From the proof of Theorem 2.5, it follows that there exists a unique solution $\overline{x}$ to problems (3.6), and moreover, (i) and (ii) hold. Thus, since $\overline{X}$ is unique, (3.24) defines $\overline{P}$ uniquely by Corollary 3.7. By Theorems 3.4 and 3.5 it remains to show that $\bar{z} = z(\bar{P})$ is unique. If $\bar{X} = 0$, then $\bar{y} \equiv 0$ and $\bar{z} = z(\bar{P})$ is unique (see Szidarovszky and Yakowitz [S8]). On the other hand, if $\bar{X} > 0$ then $Y(\bar{P}) > 0$ which by Corollary 3.2 implies that $(\bar{y}, \bar{z})$ is uniquely determined, and this completes the proof. ■

Theorem 3.10. Suppose that the assumptions of Theorem 3.8 hold. Then there exists an oligopolistic equilibrium solution (x^o, y^o, z^o) to the two-stage model. Moreover,

(i) for any equilibrium solution, the total supply X^o of the firms in S_1 is the same, and y^o, z^o are unique,

(ii) if $\Phi(\theta)$ is differentiable at X^o then (x^o, y^o, z^o) is unique ,

(iii) $X^o > 0$ if and only if $Y(P_\ell) > 0$, where P_ℓ is defined in (3.25),

(iv) if there exists a competitive equilibrium solution, with a total output of $\bar{X}$ for the firms in S_1, then $X^o \leq \bar{X}$.

Proof. From Corollary 3.7 and Theorem 3.8, there exists a solution x^o to the problems (3.7) with a unigue total output value X^o by the results in Szidarovszky and Yakowitz [S8]. Furthermore, by the same argument as in the end of the proof of Theorem 3.9, it follows that the accompanying equilibrating outputs (y^o, z^o) are unique since X^o is unique. This establishes the existence of an oligopolistic equilibrium and also proves part (i). If $\Phi(\theta)$ is differentiable at X^o, then x^o is unique since $\Phi(\theta)$ satisfies the conditions in Szidarovszky and Yakowitz [S8] by Corollary 3.7 and Theorem 3.8, and so part (ii) follows from (i). Parts (iii) and (iv) follow directly from the proof of Theorem 2.6. ■

Corollary 3.8. Under assumptions of Theorem 3.10, if $Y(P_\ell) > 0$, then $P^o \in (P_\ell,\ P_{NC})$, where P_{NC} is given by (3.22).

Proof. Result follows from Theorem 3.10 and the proof of Corollary 2.6. ■

Corollary 3.9 Let the assumptions of Theorem 3.8 hold, and let the firms in S_3 be identical with $h_i(\cdot) = h(\cdot)$ for $i = 1, \ldots, n_3$. If (x^o, y^o, z^o) is an oligopolistic equilibrium with $z^o > 0$, then this is a unique equilibrium solution.

Proof. If $X^o = 0$, then the assertion is evident from Theorem 3.8. Hence, suppose that $X^o > 0$, so that $Y^o > 0$. Since all the firms in S_3 are identical and since $Z^o > 0$, $Y(P)$ is differentiable at $P = P^o$ by Corollary 3.5, and so $\Phi(\theta)$ is differentiable at X^o. Therefore, by Theorem 3.10 (ii), the proof is complete. ■

3.6. Summary of Results

This chapter was concerned with a two-stage model in which the final commodity market is assumed to be a multiple leader-follower oligopoly, with the firms in S_2 being followers and those in S_3 being leaders. This oligopoly was modelled in the manner presented in Sherali [S3] and is a consistent extension of the Stackelberg leader-follower duopoly. The analysis of this model was conducted in the same spirit as in the case of the alternate follower-follower behavior presented in Chapter 2. However, the situation is more complicated, due to a more complex nature of interactions among the firms in $S_2 \cup S_3$. In particular, properties of the aggregate follower reaction curve have to be examined for the purpose of characterizing the (net) market demand function perceived by the leader firms. This function, in turn, plays a central role in deriving sufficient conditions for the existence of equilibrating outputs in the final commodity market. The equilibrating process among the firms in $S_2 \cup S_3$ was analyzed with the use of mathematical programming concepts, following the work by Murphy et al. [M3] and also by Sherali et al. [S5].

The main results, concerning the derived demand function are embodied in Theorems 3.7 and 3.8. Existence of an equilibrium solution in the model with either a competitive or an oligopolistic (Cournot) behavior of the upstream firms is addressed in Theorems 3.9 and 3.10. In these theorems, the required properties of the market demand function and the cost functions for the firms in $S_2 \cup S_3$ are not as general as in their counterparts in Chapter 2, but they are not excessively restrictive.

Some comparative results for the follower-follower and leader-follower models, with the Cournot firms in S_1 are included in Chapter 6. A computational approach for determining equilibrium solutions defined in this chapter is presented next, in Chapter 4.

Chapter 4

Computation of Equilibrium Solutions

This chapter deals with the computation of overall equilibrium solutions defined in Chapters 2 and 3, given that they exist. Computational techniques for approximating the competitive and oligopolistic equilibria when the final commodity market is the Cournot oligopoly are discussed in Section 4.1, and those concerning the multiple leader-follower oligopoly are presented in Section 4.2. Section 4.3 gives a brief summary of results.

4.1. Computation of Equilibrium Solutions Given the Cournot Oligopoly in the Final Product Market

In this section the computation of the competitive and the oligopolistic equilibrium solutions defined in Chapter 2 is discussed, given that they exist. Note that in either case, as indicated by Theorems 2.5 and 2.6, if $Y(P_\ell) = 0$, where P_ℓ is given by (2.55), the firms in S_1 and in S_2 produce at zero level, so that $\bar{x} = 0$, $\bar{y} = 0$ and $\bar{z} = z_{NC}$. Hence, suppose that $Y(P_\ell) > 0$. As far as the competitive equilibrium (Definition 2.1) computation is concerned, under the assumptions of Theorem 2.5, one may simply perform a bisection search on the interval (P_ℓ, P_U) defined in (2.55)-(2.57) in order to find the unique intersection point of the curves $Y(P)$ and $X(P)$. $X(P)$ may be evaluated numerically through (2.58) and $Y(P)$ may be evaluated by computing the associated Nash-Cournot equilibrium for the firms in $S_2 \cup S_3$ using the method of Murphy et al. [M3], for instance.

Next, consider the derivation of an oligopolistic equilibrium solution (Definition 2.2) under the assumptions of Theorem 2.6. As pointed out by Novshek [N3], unless the demand and cost functions have some nice properties, there is no efficient algorithm for determining all equilibria for an oligopoly with at least three firms. Of course, if the situation is simple enough (as in the illustrative example of the next chapter) such that $\Phi(\cdot)$ is available in closed form, then using the method of Szidarovszky and Yakowitz [S8] or Murphy et al. [M3], one may obtain a solution x^o to (2.5) and then compute y^o and z^o from (2.6) in order to derive an oligopolistic equilibrium solution (x^o, y^o, z^o). In the more general case, when one can only evaluate $Y(P)$ via some $S_2 \cup S_3$ oligopoly problem, the following procedure based on a sequential approximation of $\Phi(\cdot)$ may be adopted. (This procedure may be used in situations other than that of Theorem 2.3, so long as the derived demand curve $\Phi(\theta)$ is strictly decreasing and concave in s over $[0, Y_0]$.)

To describe the proposed method, consider some $\bar{P}$ for which the solution $(\bar{x}, \bar{y}, \bar{z})$ to EP$[\bar{P}, Q(\bar{P})]$ yields a positive total output $\bar{Y}$, say, for the firms in S_2. Using $Y^+(P)$ as given through (2.29)-(2.32), (2.35) and (2.37), or in the special cases of Theorem 2.3, as given by (2.42) or (2.44) for example, obtain a tangential supporting functional $\bar{a} - \bar{b}\theta$ to $\Phi(\theta)$ at $\theta = \bar{Y}$. (Note that $\bar{b} = -1/Y^+(\bar{P})$ and $\bar{a} = \bar{P} + \bar{b}\bar{Y}$.) Using this tangential approximation $\bar{a} - \bar{b}Y$ for $\Phi(\cdot)$, let $\bar{x}$ denote the corresponding (unique) equilibrium solution for the firms in S_1 with a total output of $\bar{X}$, and consider the following result.

Theorem 4.1 Suppose that the assumptions of Theorem 2.5 hold, and let $(\bar{x}, \bar{y}, \bar{z})$ and the total outputs $\bar{X}$, $\bar{Y}$ be as defined above. If $\bar{X} = \bar{Y}$, then $(\bar{x}, \bar{y}, \bar{z})$ represents an oligopolistic equilibrium solution. Otherwise,

$$\bar{X} > \bar{Y} \quad \text{implies} \quad X^o \geq \bar{Y} \quad \text{and} \quad \bar{X} < \bar{Y} \quad \text{implies} \quad X^o \leq \bar{Y}, \tag{4.1}$$

where $X^o = \sum_{i=1}^{n_1} x_i^o$ is the total output of the firms in S_1 at an oligopolistic equilibrium.

Proof. By the definition of $\bar{x}$, we have for each $i = 1,\ldots, n_1$,

$$-\bar{x}_i\bar{b} + (\bar{a} - \bar{b}\bar{X}) - f_i'(\bar{x}_i) = 0 \text{ if } \bar{x}_i > 0 \text{ and } \bar{a} - \bar{b}\bar{X} \le f_i'(0) \text{ if } \bar{x}_i = 0. \tag{4.2}$$

If $\bar{X} = \bar{Y}$, then $\bar{a} - \bar{b}\bar{X} = \Phi(\bar{X})$, and since $\Phi(\cdot)$ is concave, the right-hand derivative of $\Phi(\cdot)$ at $\bar{Y}$ is not greater than $-\bar{b}$ and the left-hand derivative of $\Phi(\cdot)$ at $\bar{Y}$ is not less that $-\bar{b}$. Consequently, from (4.2), the directional derivative of the objective function in (2.5) at $\bar{x}_i$ (with x_j^o replaced by $\bar{x}_i$ for $j \neq i$) is nonpositive in either feasible directions, and so by strict concavity of this objective function, we get that $(\bar{x}, \bar{y}, \bar{z})$ is an oligopolistic equilibrium solution. If $\bar{X} \neq \bar{Y}$, then consider a sequence $\{\Phi_n(\theta)\}$, $0 < \theta \le \bar{Y}_0$ of arbitrarily close, concave, continuously differentiable approximating functions to $\Phi(\theta)$, where $\{\Phi_n(\theta)\} \to \Phi(\theta)$, $0 < \theta \le \bar{Y}_0$. Correspondingly, there exists a sequence $\{\bar{Y}_n\} \to \bar{Y}$, such that $\Phi'_n(\bar{Y}_n) = -\bar{b}$ for all n. Furthermore, let x^{on} and X_n^o be the unique equilibrium output vector and its sum associated with $\bar{\Phi}_n(\cdot)$, and note that over some appropriately chosen subsequence, we have $x^{on} \to x^o$ and $X_n^o \to X^o$, a set of equilibrium outputs with respect to $\Phi(\cdot)$. Now, for each n, define for $i = 1,\ldots, n_1$,

$$x_i^n(\theta) = \begin{cases} x, & \text{such that } x \ge 0 \text{ and } x\bar{\Phi}'_n(\theta) + \bar{\Phi}_n(\theta) - f_i'(x) = 0 \text{ if it exists} \\ 0, & \text{if no such } x \text{ exists} \end{cases} \tag{4.3}$$

As in Szidarovszky and Yakowitz [S7], observe that $x_i^n(\theta)$ is a continuous, nonincreasing function of θ, being strictly decreasing over its positive range. Hence, $X^n(\theta) = \sum_{i=1}^{n_1} x_i^n(\theta)$ is also continuous and strictly decreasing over its positive range. Now, similar to (4.3) define for $i = 1,\ldots, n_1$,

$$x_i^L(\theta) = \begin{cases} x, & \text{such that } x \ge 0 \text{ and } -x\bar{b} + \bar{a} - \bar{b}\theta - f_i'(x) = 0 \text{ if it exists} \\ 0, & \text{if no such } x \text{ exists} \end{cases} \tag{4.4}$$

and denote $X^L(\theta) = \sum_{i=1}^{n_1} x_i^L(\theta)$. By definition, we have $X^L(\bar{X}) = \bar{X}$.

Suppose that $\bar{X} > \bar{Y}$ in (4.1). Then, necessarily $X^L(\bar{Y}) > \bar{Y}$ since $X^L(\bar{X}) = \bar{X}$. However, since $\bar{\Phi}'_n(\bar{Y}_n) = -\bar{b}$, it follows by comparing (4.3) and (4.4) that $X^n(\bar{Y}_n)$ and $\bar{Y}_n$ can be arbitrarily close to $X^L(\bar{Y})$ and $\bar{Y}$, respectively, and so, we obtain $X^n(\bar{Y}_n) > \bar{Y}_n$ for n large enough. This implies that $X_n^o > \bar{Y}_n$ for n large enough, and by letting $n \to \infty$, we get $X^o \geq \bar{Y}$. By similar arguments, $\bar{X} < \bar{Y}$ implies that $X^o \leq \bar{Y}$, and the proof is complete. ■

Corollary 4.1 Let the assumptions of Theorem 4.1 hold. Let $\hat{x}_i$, $i = 1,\ldots, n_1$, be a unique optimal solution to the following problem:

$$\underset{q \geq 0}{\text{maximize}}\{\alpha q + \frac{1}{2}\beta q^2 - f_i(q)\}, \tag{4.5}$$

where $\alpha = \Phi(\bar{Y})\ (= \bar{P})$, $\beta = 1/Y^+(\bar{P})$. Denote $\hat{X} = \sum_{i=1}^{n_1}\hat{x}_i$. If $\hat{X} = \bar{Y}$, then $\bar{x} = \hat{x}$, and hence $(\hat{x}, \bar{y}, \bar{z})$ is an oligopolistic equilibrium solution. Otherwise, if $\hat{X} > \bar{Y}$ then $X^o \geq \bar{Y}$ and if $\hat{X} < \bar{Y}$ then $X^o \leq \bar{Y}$.

Proof. The corollary follows readily from Theorem 4.1 by noticing that $x_i^L(\bar{Y})$ given by (4.4), and $\hat{x}_i$ defined above satisfy $x_i^L(\bar{Y}) = \hat{x}_i$.■

Hence, an algorithm is evident through Theorem 4.1 and Corollary 4.1.One can initialize with a price interval (P_ε, P_{NC}), defined via (2.55) and (2.39). Now, a bisection search may be performed on this interval, with the use of Theorem 4.1 and Corollary 4.1 in an obvious manner to reduce the interval, until X^o is known with a required accuracy. Note that $\bar{X}$ need not be computed. All that one needs to compute is $\hat{X}$ via (4.5), since $\hat{X} = \bar{Y}$ implies that $\bar{X} = \bar{Y}$, and $\hat{X} > \bar{Y}$ if and only if $\bar{X} > \bar{Y}$, and similarly, for the reverse strict inequality. Of course, given X^o, the quantities y^o and z^o are available through (2.6). It is important to note that the condition $\bar{X} = \bar{Y}$ of Theorem 4.1 may never hold. For example, if $\Phi(\cdot)$ is not differentiable at X^o, then with $\bar{Y} = X^o$ and $\bar{P} = \Phi(X^o)$ above, it is not necessary that the corresponding $\bar{X}$ turns out to be equal to $\bar{Y}$. The reason is that a particular suitable supporting functional must be used at $\bar{Y}$ so that the associated $\bar{b}$ reproduces $\bar{Y}$ as $\bar{X}$ in (4.2). Hence, once X^o is cornered in a sufficiently small interval, and if x^o is still not available, then one may determine x^o in the spirit of (2.61) as in the proof of Theorem 2.7, by using a two segment linear approximation of $\Phi(\cdot)$, one segment of each side in the vicinity of the estimated X^o. Then by Theorems 2.7 and 4.1, the intersection of these two segments would yield the estimate of X^o, with (2.61) giving the individual outputs x_i^o, $i = 1,..., n_1$ of the firms in S_1. The algorithm for approximating an oligopolistic equilibrium solution is summarized below.

Determination of an Oligopolistic Equilibrium Solution (Definition 2.2) Under the Assumptions of Theorem 2.5

Initialization. Calculate $P_\ell = \min\{f_i'(0),\ i = 1,..., n_1\}$ and determine $Y(P_\ell)$ via the Nash-Cournot equilibrium solution $(y(P_\ell), z(P_\ell))$ for the firms in $S_2 \cup S_3$. If $Y(P_\ell) = 0$ then $(x^o, y^o, z^o) = (0, 0, z(P_\ell))$ is the unique oligopolistic equilibrium solution. (Note that then $z(P_\ell) = z_{NC}$, where z_{NC} is defined in (2.38).) On the other hand, if $Y(P_\ell) > 0$, then by Corollary 2.6, $P^o \in (P_\ell, P_{NC})$, where P_{NC} is given in (2.39). Calculate P_{NC} and set lower and upper bounds on P^o as $P_L = P_\ell$ and $P_U = P_{NC}$.

Step 1. Set $P_o = (P_L + P_U)/2$ and calculate $(y(P_o), z(P_o))$ and the total output $Y(P_o)$ for the firms in S_2 via problems (2.1), (2.2). Compute $\hat{\beta} = 1/Y^+(P_o)$ from (2.29)-(2.32), (2.35), (2.37) and set $\hat{\alpha} = P_o$.

Step 2. For each $i = 1,..., n_1$ determine the unique optimal solution $\hat{x}_i$ to the problem $\underset{q \geq 0}{\text{maximize}}\{\hat{\alpha}q + \frac{1}{2}\hat{\beta}q^2 - f_i(q)\}$, and set $\hat{X} = \sum_{i=1}^{n_1} \hat{x}_i$.

(i) If $|\hat{X} - Y(P_o)| \leq \varepsilon$, where $\varepsilon > 0$ is some tolerance, then **STOP** with $(\hat{x}, y(P_o), z(P_o))$ as an approximation of the oligopolistic equilibrium solution.

(ii) If $\hat{X} > Y(P_o)$ ($\hat{X} < Y(P_o)$) then set $P_L = P_o$ ($P_U = P_o$) and go to Step 1.

Finally, consider the determination of the total outputs of the firms in S_1 in all local equilibrium solutions under the assumptions of Theorem 2.7. First, as evident through Theorem 2.4, the piecewise concave segments of the derived demand curve may be traced. Then, as in the proof of Theorem 2.7, for each concave segment, the oligopolistic equilibrium solution may be computed using the algorithm above and the condition $\theta_{i-1} < X_i < \theta_i$ may be checked. If this holds, then each such X_i corresponds to a local oligopolistic equilibrium solution. In addition, by the proof of Theorem 2.7 and by the virtue of the argument ivolving directional derivatives as in (2.64) and (2.65), it is straightforward to show that the only other total outputs of the firms in S_1 in local oligopolistic equilibrium solutions correspond to those breakpoints θ_k for which $X^k \geq \theta_k \geq X^{k+1}$. As shown in the proof of Theorem 2.7, this can occur only if the derived demand curve $\Phi(\cdot)$ is concave about θ_k, and so, this case cannot arise under the assumptions of Corollary 2.5, for example. Thus by determining X^i for $i = 1,..., n$, the total outputs X^o in all local oligopolistic equilibrium solutions may be detected. Of course, whether or not at least one of these corresponds to an oligopolistic equilibrium solution is an open question at this point.

4.2.Computation of Equilibrium Solutions Given the Multiple Leader - Follower Oligopoly in the Final Commodity Market

In this section we briefly discuss the computation of competitive and oligopolistic equilibrium solutions defined in Chapter 3 (Definitions 3.2 and 3.3). As in the case of the follower-follower model in Section 4.1, one is faced with the situation when the derived (inverse) demand function $\Phi(\cdot)$ is not given explicitly. Moreover, in a general case, both the functions $Y_R(Z,P)$ and $Z(P)$ are not available in closed form, so that the evaluation of $\Phi(\cdot)$ via $Y(P)$, where recall $Y(P) = Y_R[Z(P), P]$ becomes more complex that in the follower-follower model. However, the results presented in Section 4.1 provide a useful tool for evaluating $Y(P)$ as well as for evaluating the overall equilibria in the multiple leader-follower model.

In order first of all to evaluate $Y(P)$ for any fixed $P_o \geq 0$, one can use the following procedure based on the development in Szidarovszky and Yakowitz [S7], and on Theorem 4.1, provided $F(Z, P_o)$ is known to be strictly decreasing and concave in $Z \geq 0$ over its positive range (as under assumptions of Theorems 3.4 and 3.2). To begin with, consider the determination of P_{NC} defined in (3.22).

Determination of P_{NC}.

First find Z_{NC} defined in Definition 3.2, using the method of Murphy et al. [M3] or Szidarovszky and Yakowitz [S7]. From (3.17) and (3.22), if $Z_0(0) \le Z_{NC}$, then $P_{NC} = 0$, and otherwise, as in Remark 3.2, P_{NC} is given by that value of P for which $Z_0(P) = Z_{NC}$, i.e., by (3.17), $P_{NC} = p(Z_{NC}) - g'_{\min}(0)$.

Evaluation of $Y(P)$ for $P = P_o \ge 0$.

Initialization. If $P_o \ge P_{NC}$, then by Remark 3.2, $P_o \ge P_0$ and so $Y(P_o) = 0$. Otherwise, $Z(P_o) \le Z_0(P_o)$ as in Remark 3.2. Hence, set lower and upper bounds on $Z(P_o)$ as $Z_L = 0$ and $Z_U = Z_0(P_o)$, respectively.

Step 1. Choose $Z_o = (Z_L + Z_U)/2$, and calculate $y_R(Z_o, P_o)$ and the total output $Y_R(Z_o, P_o)$ for the firms in S_2 via problems (3.1), using the method of Murphy et al. [M3] or Szidarovszky and Yakowitz [S7]. If $Z_U - Z_L \le \varepsilon$, where $\varepsilon > 0$ is some tolerance, **STOP** with $Z(P_o) = Z_o$ and $Y(P_o) = Y_R(Z_o, P_o)$. Otherwise, compute $\alpha = F(Z_o, P_o)$ and $\beta = D_Z^+[F(Z_o, P_o)]$ from (3.2) and (3.12).

Step 2. For each $i = 1,\dots, n_3$ determine the unique optimal solution $\hat{z}_i$ to the problem $\underset{q \ge 0}{\text{maximize}} \ \{ \ \alpha q + \frac{1}{2}\beta q^2 - h_i(q) \ \}$. Let $\hat{Z} = \sum_{i=1}^{n_3} \hat{z}_i$.

(i) If $\hat{Z} = Z_o$ then $(y_R(Z_o, P_o), \hat{z})$ is a SNC equilibrium solution in the final commodity market so that $Z(P_o) = Z_o$ and $Y(P_o) = Y_R(Z_o, P_o)$. **STOP.**

(ii) If $\hat{Z} > Z_o$ $(\hat{Z} < Z_o)$ then set $Z_L = Z_o$ ($Z_U = Z_o$) and go to Step 1.

Remark 4.1. Note that given a guess Z_o for $Z(P_o)$, the above algorithm first constructs a linear support $\alpha' + \beta Z$ for the perceived demand function $F(\cdot, P_o)$ at $Z = Z_o$, where $\alpha' = \alpha - \beta Z_o$, and α and β are defined in Step 1. Let $\bar{Z}$ be the total output for the firms in S_3 when $\alpha' + \beta Z$, $Z \ge 0$ is used as the (linear) perceived demand function. As in Section 4.1, one is interested in ascertaining whether $\bar{Z} \le Z_o$ or $\bar{Z} \ge Z_o$. By Theorem 4.1 and Corollary 4.1, if $\hat{Z} = Z_o$ at Step 2 (i), then $\bar{Z} = \hat{Z} = Z_o$, and this can be readily verified to coincide with $Z(P_o)$. On the other hand, if $\hat{Z} > Z_o$, then it may be verified that $\tilde{Z} > Z_o$, and then one can assert that $Z(P_o) \geq Z_o$. The case of $\hat{Z} < Z_o$ is similar.

Determination of a Competitive Equilibrium Under the Assumptions of Theorem 3.9..

Let P_ℓ be as defined in Theorem 3.9. If $Y(P_\ell) = 0$, then $(x^o, y^o, z^o) = (0, 0, z_{NC})$ is an equilibrium solution. Hence, suppose that $Y(P_\ell) > 0$. By Theorem 3.9 (ii), the equilibrating price lies in the interval (P_ℓ, P_U), where P_ℓ and P_U are defined by (3.25)-(3.27). For a given $P_o \geq 0$, let $X(P_o)$ be the sum of optimal solutions to problems (3.6), solved for $i = 1,..., n_1$ with $\Phi(\cdot)$ replaced with P_o. Then $X(P)$, $P \geq 0$ is the supply curve for the firms in S_1. As shown in the proof of Theorem 2.5, for example, this is continuous and strictly increasing over its positive range. Hence, P^o is the unique intersection point of $X(P)$ and $Y(P)$, and may be determined by a bisection search. This readily yields x^o and P^o, and then (y^o, z^o) may be determined as $[y(P^o), z(P^o)]$.

Determination of an Oligopolistic Equilibrium Under the Assumptions of Theorem 3.10..

Again, if $Y(P_\ell) = 0$, then $(x^o, y^o, z^o) = (0, 0, z_{NC})$ is an equilibrium solution. Hence, suppose that $Y(P_\ell) > 0$. From Corollary 3.8, we then have $P^o \in (P_\ell, P_{NC})$. By Theorem 4.1 and Corollary 4.1, a bisection search may be performed on (P_ℓ, P_{NC}) in the following way. Given an iterate P_o, compute $\alpha = P_o$, $\beta = 1/Y^+(P_o)$ via (3.14), (3.15), (3.20), (3.21) and (3.23), and find the unique optimal solution $\hat{x}_i$ to the problem: maximize$_{q \geq 0}$ $\{ \alpha q + \frac{1}{2}\beta q^2 - f_i(q) \}$. Let $\hat{X} = \sum_{i=1}^{n_1} \hat{x}_i$. If $|\hat{X} - Y(P_o)| \leq \varepsilon$, then $(\hat{x}, y(P_o), z(P_o))$ is an approximation of an oligopolistic equilibrium solution. Otherwise, if $\hat{X} > Y(P_o)$ $(\hat{X} < Y(P_o))$ then $P^o \leq P_o$ $(P^o \geq P_o)$, and the bisection search may be accordingly continued.

4.3.Summary of Results

In this chapter we presented algorithms for finding the competitive and oligopolistic equilibrium solutions arising in the follower-follower and in the multiple leader-follower models, given that these equilibria exist. Attention was focused on situations when the derived demand function is only implicitly available. The numerical procedures presented in this chapter are fashioned to perform an iterative bisection search on the intermediate price interval until an equilibrium solution is determined with a required accuracy. In the multiple leader-follower model, the computation of $Y(P)$ itself is based on an iterative bisection search on the total leader output interval.

In the next chapter, simpler two-stage models are presented, in which the derived demand function and oligopolistic equilibrium solutions are available in closed form.

Chapter 5

Illustrative Examples and Collusion Considerations

for the Two-Stage Oligopolistic Models

The purpose of this chapter is to illustrate the foregoing analysis of the two-stage models presented in Chapters 2 and 3, when the upstream stage is a Cournot oligopoly. Given the follower-follower or multiple leader-follower behavior for the final product suppliers, a simple model is analyzed, in which an oligopolistic equilibrium solution is unique, and is available in closed form. The development provides a basis to investigate the effects of various mergers or integrations on individual firm profits and on industry outputs and prices at equilibrium. The two-stage oligopolistic model with Cournot firms in $S_2 \cup S_3$ is discussed in Section 5.1. Section 5.2 presents the model with the multiple leader-follower oligopoly in the final commodity market. The main results are summarized in Section 5.3.

5.1.The Follower-Follower model

In order to illustrate how the analysis in Chapter 2, and further to investigate the effects of mergers and integrations on the semi-finished product price P^o, the industry output Q^o, the price $p(Q^o)$ and the profits of the firms at an oligopolistic equilibrium, consider the following example.

Let $p(Q) = a - bQ$, and let us assume that the firms within each set S_1, S_2 and S_3 are identical, with $f_i(x) = \frac{1}{2}d_1x^2 + c_1x$ for $i = 1,\dots, n_1$, $g_i(y) = \frac{1}{2}d_2y^2 + c_2y$ for $i = 1,\dots, n_2$ and $h_i(z) = \frac{1}{2}d_3z^2 + c_3z$ for $i = 1,\dots, n_3$, where $a > 0$, $b > 0$, $c_1 \geq 0$, $d_1 \geq 0$, $c_1 + d_1 > 0$, $c_2 \geq 0$, and $d_2 \geq 0$. Further, since the firms in S_3 are supposed to be vertically integrated across the two production stages, we assume that $c_3 = c_1 + c_2$ and $d_3 = d_1 + d_2$. Also, we assume that $a > c_3$, since $a \leq c_3$ results in zero equilibrating outputs for all the firms.

Note that the conditions of Theorem 2.3 hold, and therefore by Theorem 2.5 (i), an oligopolistic equilibrium solution exists. We will show that the conditions of Theorem 2.5 (ii) hold, so that an oligopolistic equilibrium is unique.

The optimality conditions (2.16)-(2.20) for the equilibrating problems EP(P, $Q(P)$) for this example are given below, noting as in the proof of Theorem 2.3 that the identical firms produce identical equilibrating outputs. For convenience, we have rewritten $v_i(P)$ as $v(P)$, $i = 1,\dots, n_2$ and $w_i(P)$ as $w(P)$, $i = 1,\dots, n_3$.

$$n_2[a - bQ(P) - P - c_2] - (b + d_2)Y(P) + n_2v(P) = 0 \tag{5.1}$$

$$n_3[a - bQ(P) - c_3] - (b + d_3)Z(P) + n_3w(P) = 0 \tag{5.2}$$

$$Y(P) + Z(P) = Q(P), \quad Y(P) \geq 0, \quad Z(P) \geq 0, \tag{5.3a}$$

$$v(P) \geq 0, \quad w(P) \geq 0, \quad Y(P)v(P) = 0, \quad Z(P)w(P) = 0. \tag{5.3b}$$

Recall (from Theorem 2.2 and Corollary 2.2) that given any $P \geq 0$, there exists a unique solution to the above system, and this is precisely the $S_2 \cup S_3$ oligopoly equilibrium solution. Now, when $P \geq 0$ is small enough, since $a - c_2 > a - c_3 > 0$, we must have $Q(P) > 0$ from (5.1)-(5.3). However, it is possible for $Z(P)$ to be zero. To see this, let us put $Z(P) = 0$, $Y(P) = Q(P)$ and $v(P) = 0$ in (5.1)-(5.3). From (5.1), this gives

$$Q(P) = Y(P) = (\alpha_1 - P)/\beta_1, \quad \text{where } \alpha_1 = a - c_2 \text{ and } \beta_1 = (n_2b + b + d_2)/n_2. \tag{5.4}$$

Furthermore, from (5.2) and (5.3), this is valid so long as $a - bQ(P) - c_3 \leq 0$, i.e., so long as

$$0 \le P \le P_1, \quad \text{where} \quad P_1 = [n_2 b(c_3 - c_2) - (b + d_2)(a - c_3)]/n_2 b. \tag{5.5}$$

Hence, if $P_1 \ge 0$, we have $Y(P) = Q(P)$ given by (5.4), and $Z(P) = 0$ for $0 \le P \le P_1$.

As P increases beyond P_1 (or beyond 0 if $P_1 \le 0$), we have $Y(P) > 0$ and $Z(P) > 0$ again so long as $P \le P_{NC}$, where from (2.39),

$$P_{NC} = a - c_2 - \frac{n_3 b(a - c_3)}{n_3 b + b + d_3}. \tag{5.6}$$

Putting $Q(P) = Y(P) + Z(P)$ and $v(P) = w(P) = 0$ in (5.1)-(5.3), we obtain $Y(P) = n_2\alpha_2(P_{NC} - P)/\beta_2$ and $Z(P) = n_2 n_3 b(P - P_1)/\beta_2$, where

$$\alpha_2 = n_3 b + b + d_3, \quad \beta_2 = (b + d_2)\alpha_2 + n_2 b(b + d_3). \tag{5.7}$$

Note that the expression for $Z(P)$ above holds for $P \ge P_1$, even when $P_1 \le 0$. Hence, $Z(P) \ge 0$ whenever, $P \ge P_1$, and $Y(P)$ remains positive so long as $P < P_{NC}$. (It may be easily verified that $P_{NC} > P_1$ since $a > c_3$.) For $P \ge P_{NC}$, we get $Y(P) = 0$ and $Z(P) = Z_{NC}$, and $Q(P) = Z_{NC} > 0$. Substituting this into (5.2) along with $w(P) = 0$ gives $Q(P) = Z(P) = n_3(a - c_3)/(n_3 b + b + d_3)$ $= Z_{NC}$ whenever $P \ge P_{NC}$. Note that then $a - bQ(P) - P - c_2 = \alpha_2 - P$, so that $v(P) = P - P_{NC}$ in (5.1) is indeed nonnegative.

Summarizing we obtain

$$Y(P) = \frac{\alpha_1 - P}{\beta_1}, \quad Z(P) = 0, \quad \text{if } 0 \le P \le P_1 \text{ and } P_1 \ge 0$$

$$Y(P) = \frac{n_2\alpha_2(P_{NC} - P)}{\beta_2}, \quad Z(P) = \frac{n_2 n_3 b(P - P_1)}{\beta_2}, \quad \text{if } \max\{0, P_1\} \le P \le P_{NC} \tag{5.8}$$

$$Y(P) = 0, \quad Z(P) = n_3(a - c_3)/(n_3 b + b + d_3) \quad \text{if } P \ge P_{NC},$$

where α_1, β_1, α_2, β_2 P_1 and P_{NC} are given by (5.4)- (5.7). Note that when $P_1 \geq 0$, we have $Y(P_1) = (a - c_3)/b > 0$. Therefore, from (5.8) we obtain the derived demand curve faced by the firms in S_1 in closed form as given below.

$$\Phi(\theta) = \begin{cases} P_{NC} - \beta_2\theta/n_2\alpha_2 & \text{for} \quad 0 \leq \theta \leq n_2\alpha_2 P_{NC}/\beta_2 \\ 0 & \text{for} \quad \theta \geq n_2\alpha_2 P_{NC}/\beta_2 \end{cases} \quad \text{if} \quad P_1 \leq 0, \tag{5.9}$$

and

$$\Phi(\theta) = \begin{cases} P_{NC} - \beta_2\theta/n_2\alpha_2 & \text{for} \quad 0 \leq \theta \leq (a - c_3)/b \\ \alpha_1 - \beta_1\theta & \text{for} \quad (a - c_3)/b \leq \theta \leq \alpha_1/\beta_1 \\ 0 & \text{for} \quad \theta \geq \alpha_1/\beta_1 \end{cases} \quad \text{if} \quad P_1 > 0. \tag{5.10}$$

Now, let us determine the oligopolistic equilibrium solution. First note that from (2.55), $P_\ell = c_1 \geq 0$, and further from (5.6), (5.7), that $P_1 < P_\ell < P_{NC}$, and hence by Corollary 2.3, $Y(P_\ell) > 0$. Therefore, by Theorem 2.6 (iii), we can conclude that the total equilibrating output X^o of the firms in S_1 is positive, and moreover, by Corollary 2.6, that $P^o \in (P_\ell, P_{NC})$.

If $P_1 \leq 0$, then as evident from (5.9), $\Phi(\theta)$ is differentiable over its positive range, and hence by Theorem 2.6 (iii), an equilibrium solution (x^o, y^o, z^o) is unique. Furthermore, since the firms in S_1 are identical, we necessarily have $x_i^o = X^o/n_1$, $i = 1,\ldots, n_1$. Using this result in the necessary and sufficient conditions for optimality in (2.5), and in (2.6) we obtain

$$X^o = n_1 n_2 (b + d_3)(a - c_3)/W, \quad \text{with} \quad x_i^o = X^o/n_1 \quad \text{for} \quad i = 1,\ldots, n_1,$$

$$Y^o = X^o, \quad \text{with} \quad y_i^o = Y^o/n_2 \quad \text{for} \quad i = 1,\ldots, n_2, \tag{5.11}$$

$$Z^o = n_2 n_3 b(P^o - P_1)/\beta_2 \quad \text{with} \quad z_i^o = Z^o/n_3 \quad \text{for} \quad i = 1,\ldots, n_3,$$

where $W = (n_1 + 1)\beta_2 + n_2 d_1 \alpha_2$, $P^o = \Phi(X^o)$, and where α_2, β_2 are given in (5.7).

On the other hand, if $P_1 > 0$, then $\Phi(\cdot)$ is given by (5.10). In this case, observe first of all that first linear segment of $\Phi(\cdot)$ in (5.10) coincides with $\Phi(\cdot)$ in (5.9) for $\theta < (a - c_3)/b$. Moreover, from (5.11), $Y^o < Y(P_1) = (a - c_3)/b$, and so $P^o > P_1$. Consequently from Theorem 2.6 (i), X^o, y^o and z^o are unique equilibrating quantities. Moreover, since $\Phi(\cdot)$ is differentiable at X^o, and the firms in S_1 are identical, we also have from Theorem 2.6 (ii), we also have that (x^o, y^o, z^o) given in (5.11) represents a unique equilibrium in this case as well.

Now, let us investigate the variation in the equilibrating quantities defined in (5.11), along with the associated prices and individual firm profits, as firms merge or integrate, thereby changing n_1, n_2 and n_3 accordingly.

First, consider the effects of **horizontal integration**. Notice that the above equilibrating quantities depend on the number of firms in each of the sets S_1, S_2 and S_3. In particular, these expressions yield $\partial P^o/\partial n_1 < 0$, $\partial P^o/\partial n_2 \geq 0$ with $\partial P^o/\partial n_2 = 0$ only if $d_1 = 0$, $\partial P^o/\partial n_3 < 0$, and $\partial Q/\partial n_i > 0$ for $i = 1, 2, 3$. These partial derivatives indicate that whereas the semi-finished product price may fall if the firms in S_2 collude or may increase if the firms in S_1 or S_3 collude, the net effect of any such merger on the consumers is a smaller industry output at higher prices. This is in contrast with the results of Greenhut and Ohta [G1] , however this generalizes another observation made by Greenhut and Ohta [G1] that in the case of $n_1 = 1$, $S_2 = \emptyset$ and linear total cost functions, the price of the semi-finished product at equilibrium is invariant with respect to the number of firms in S_2. The above derivation shows that this invariance is due in particular to the linearity of the cost curves of firms in S_1 and moreover, it is uneffected by the number of firms in any other set as well.

Next, consider the issue relating to firm profits upon mergers. In order to facilitate closed form computations, let us further assume linear cost functions, i.e., $d_1 = d_2 = d_3 = 0$. Then, denoting π_i^o as the profit of an individual firm in set S_i, $i = 1, 2, 3$ at equilibrium, we obtain

$$\pi_1^o = D_O \frac{n_2}{(n_1 + 1)^2 (n_3 + 1)(n_2 + n_3 + 1)}, \quad \pi_2^o = D_O \left[\frac{n_1}{(n_1 + 1)(n_2 + n_3 + 1)}\right]^2,$$

$$\pi_3^o = D_O \left[\frac{(n_1 + 1)(n_3 + 1) + n_2}{(n_1 + 1)(n_3 + 1)(n_2 + n_3 + 1)}\right]^2, \quad \text{where } D_O = \frac{(a - c_3)^2}{b}. \tag{5.12}$$

This yields

$\partial\pi_1^o/\partial n_1 < 0, \quad \partial\pi_1^o/\partial n_2 > 0, \quad \partial\pi_1^o/\partial n_3 < 0,$

$\partial\pi_2^o/\partial n_1 > 0, \quad \partial\pi_2^o/\partial n_2 < 0, \quad \partial\pi_2^o/\partial n_3 < 0,$

$\partial\pi_3^o/\partial n_1 < 0, \quad \partial\pi_3^o/\partial n_2 < 0, \quad \partial\pi_3^o/\partial n_3 < 0,$

From these partial derivatives, it is evident that horizontal mergers within any set of firms improve the individual firm profits for that set, thus giving a generalization of the result by Szidarovszky and Yakowitz [S8] for a single market oligopoly. In addition, horizontal mergers within S_3 are beneficial to all firms, and conversely, any horizontal merger of firms brings benefits to the producers in S_3. However, consider the case when two firms merge to form a new firm. Letting the subscripts A and B refer to before and after any integrations, we obtain $\pi_{1A}^o \geq 2\pi_{1B}^o$ if and only if $n_1 \leq 2$ when two firms in S_1 merge, $\pi_{2A}^o < 2\pi_{2B}^o$ when two firms in S_2 merge, and $\pi_{3A}^o < 2\pi_{3B}^o$ when two firms in S_3 merge. Hence, as far as the benefits to new firms are concerned, the joint profit after the merger falls short of the sum of the individual profits before the merger, unless if a duoploy in S_1 transforms into a monopoly.

Now, let us investigate the effects of **vertical integration** . For simplicity, using linear cost functions, let us suppose that one firm in S_1 and one firm in S_2 merge to result in a new firm in S_3. Again, let the subscripts A and B relate to the market before and after this integration. Then, one may easily verify that $P_B^o \geq P_A^o$ if and only if $n_1 \geq n_3 + 1$. Furthermore, $n_1 \geq n_3 + 1$ implies that $Q_B^o < Q_A^o$. Hence, if $n_1 \geq n_3 + 1$, then this integration will lower the semi-finished product price and benefit the consumers as well, though not necessarily otherwise (see Appendix B). This result is again at variance with the total collusion model of Greenhut and Ohta [G1] in which integration was shown to be always beneficial. In addition, the new firm does not stand to make more profit either in this case. For example, if $n_1 = 2$, $n_2 = 8$ and $n_3 = 1$, the joint profit $\pi_{1B}^o + \pi_{2B}^o$ is smaller that the profit π_{3B}^o thereby inducing the firms to consider an integration. However, it turns out that $\pi_{1B}^o + \pi_{2B}^o > \pi_{3A}^o$, and so the joint profits would in fact decrease after integration. In general, it can be verified that if $n_2 \leq \min\{n_1, n_3 + 1\}$, then $\pi_{1B}^o + \pi_{2B}^o < \pi_{3A}^o$ and hence, the two firms would benefit from the merger in this case. Furthermore, in all cases, the profit of any firm surviving in S_2 always decreases after such a vertical integration, while that of a firm surviving in S_1 decreases if $n_1 \geq \min\{2n_2 - 1, 2n_3 + 3\}$ and may increase otherwise. Finally, we present a tabulation of results with the above assumptions of linear demand and cost functions. Various scenarios identified by each column in Table 1 are investigated, with each row representing a particular effect on the industry. Here, for example, a column designated $(n_1 - 1, n_2, n_3)$ denotes a scenario in which the number of firms in S_1 reduces by one via horizontal merger, while $(n_1, n_2 - 1, n_3 + 1)$ designates a scenario in which a firm in S_2 upgrades its production facility to become a firm in S_3 In all cases, n_1, n_2 and n_3 are assumed to be positive integers. As far as the rows are concerned, the subscripts B and A, as before refer to the corresponding quantites before and after the change, with P^o, Y^o, Q^o and π_i^o respectively denoting the price, the total output of the downstream producers, the total industry output, and the profits of an individual firm in S_i, $i = 1, 2, 3$, at the oligopolistic equilibrium solution. The cases of particular interest are those of partial collusion in which it is possible for $Q_A^o < Q_B^o$ to occur, which wolud mean lower industry output at higher prices for the consumers. While all the examples studied by Greenhut and Ohta [G1] imply that mergers benefit the consumers, the cases when Q_A^o turns out to be less than Q_B^o illustrate that even with linear demand and cost functions, this need not always be true. In general, this outcome depends upon the market configuration and the type of merger considered. A similar conclusion can be derived in the case of a multiple leader-follower oligopoly in the final commodity market, discussed in next section.

5.2. The Multiple Leader-Follower Model

The purpose of this section is to illustrate the analysis presented in Chapter 3 with a simple linear example, as in the previous section. For convenience, we restate all the assumptions below.

Table 1. Summary of Integration Results in Follower-Follower Model

Effects	Scenarios					
	$n_1 - k, n_2, n_3$ $n_1 - 1 \ge k \ge 1$ $n_2 \ge 1$	$n_1, n_2 - k, n_3$ $n_2 - 1 \ge k \ge 1$ $n_1 \ge 1$	$n_1, n_2, n_3 - k$ $n_3 - 1 \ge k \ge 1$ $(n_1 \ge 1, n_2 \ge 1)$	$n_1 - 1, n_2, n_3 + 1$ $n_1 \ge 2, n_2 \ge 1$	$n_1, n_2 - 1, n_3 + 1$ $n_1 \ge 1, n_2 \ge 2$	$n_1 - 1, n_2 - 1, n_3 + 1$ $n_1 \ge 2, n_2 \ge 2$
$P^o_A - P^o_B$	> 0	$= 0$	> 0	≤ 0 *iff* $n_1 \ge n_3 + 1$	< 0	≤ 0 *iff* $n_1 \ge n_3 + 1$
$Y^o_A - Y^o_B$	< 0	< 0	> 0	< 0	< 0	< 0
$Q^o_A - Q^o_B$	< 0	< 0	< 0	> 0 *if* $n_1 \ge$ $\min\{n_2 - 1, n_3 + 1\}$	> 0	> 0 *if* $n_1 \ge$ $\min\{n_2 - 1, n_3 + 1\}$
$\pi^o_{1A} - \pi^o_{1B}$	> 0	< 0	> 0	< 0 *if* $n_1 \ge 2n_3 + 3$	< 0	< 0 *if* $n_1 \ge$ $\min\{2n_2 - 1, 2n_3 + 3\}$
$\pi^o_{2A} - \pi^o_{2B}$	< 0	> 0	> 0	< 0	$= 0$	< 0
$\pi^o_{3A} - \pi^o_{3B}$	> 0	> 0	> 0	< 0 *if* $n_1 \ge$ $\min\{n_2, n_3 + 1\}$	< 0	< 0 *if* $n_1 \ge$ $\min\{n_2 - 1, n_3 + 1\}$
Net profit benefits for the new firm	$\pi^o_{1A} \ge 2\pi^o_{1B}$ *iff* $n_1 \le 2$	$\pi^o_{2A} < 2\pi^o_{2B}$	$\pi^o_{3A} < 2\pi^o_{3B}$	$\pi^o_{3A} > \pi^o_{1B}$	$\pi^o_{3A} > \pi^o_{2B}$	$\pi^o_{3A} > \pi^o_{1B} + \pi^o_{2B}$ $n_2 \le \min\{n_1, n_3 + 1\}$

Let the demand function be given by $p(Q) = a - bQ$, and let the cost functions for the firms in S_1, S_2 and S_3 be given by $f_i(x) = c_1 x$ for $i = 1,..., n_1$, $g_i(y) = c_2 y$ for $i = 1,..., n_2$ and $h_i(z) = c_3 z$ for $i = 1,..., n_3$, where $a > 0$, $b > 0$, $c_1 \geq 0$, $c_1 > 0$, $c_2 \geq 0$, and $c_3 = c_1 + c_2 > 0$. Also, we assume that $a > c_3$, or else as can be verified, all firms produce zero at equilibrium. Recall that in the illustrated model the downstream producers act as followers and the vertically integrated firms in S_3 act as leaders.

Now, the aggreagate reaction function $Y_R(Z,P)$ of the firms in S_2 is given by the sum of the (joint) optimal solutions $y_{Ri}(Z,P)$, $i = 1,..., n_2$ to problem (3.1). Since the firms in S_2 are identical and the solution to (3.1) is unique, each $y_{Ri}(Z,P)$ is the same, equal to $\hat{y}_R$, say, where $\hat{y}_R$ solves the problem: $\underset{q \geq 0}{\text{maximize}} \; \{ qp[q + (n_2 + 1)\hat{y}_R + Z] - Pq - c_2 q\}$, we readily obtain

$$Y_R(Z,P) = \begin{cases} n_2(a - c_2 - P - bZ)/[(n_2 + 1)b] & \text{if} \quad P + bZ \leq a - c_2 \\ 0 & \text{otherwise} \end{cases} \tag{5.13}$$

Observe from Lemma 3.4 that $C_0 = \{P\colon P \geq 0\}$, and by (3.17) that

$$Z_0(P) = \begin{cases} (a - c_2 - P)/b & \text{for} \quad 0 \leq P \leq a - c_2 \\ 0 & \text{for} \quad P \geq a - c_2 \end{cases} . \tag{5.14}$$

Hence, for a fixed $P \geq 0$, $Y_R(Z,P)$ is given by the first expression in (5.13) provided $Z < Z_0(P)$ (and $Z_0(P) > 0$), and is zero otherwise. Therefore, by (3.2), for a fixed $P \geq 0$,

$$F(Z, P) = \begin{cases} a' - b'Z & \text{for} \quad 0 \leq Z < Z_0(P) \\ a - bZ & \text{for} \quad Z \geq Z_0(P) \end{cases} \tag{5.15}$$

where $a' = [a + n_2(c_2 + P)]/(n_2 + 1)$, and $b' = b/(n_2 + 1)$. Next, given $P \geq 0$, consider the determination of $Z(P)$ by using the above perceived demand function in the equilibrating problems (3.3) for the firms in S_3. Let us derive this in segments. First, note from (3.3) and (3.29) that $Z(P) = 0$ whenever $a' \leq c_3$, i.e., $P \leq c_3 - c_2 - (a - c_3)/n_2 = P_Z$. Hence, if $P_Z > 0$, then $Z(P) = 0$ for $0 \leq P \leq P_Z$. Second, using the linear demand curve $a' - b'Z$ defined above, in the problems (3.3) we obtain uniquely that

$$Z(P) = n_2 n_3 (P - P_Z)/(n_3 + 1)b \tag{5.16}$$

where P_Z is given as above. By (3.29), this is valid so long as $Z_0(P) > 0$ and the expression in (5.16) is not greater than $Z_0(P)$. Using (5.14), this condition translates to $P < a - c_2$ and $P \leq c_3 - c_2 + a - c_3/(n_2 n_3 + n_3 + 1) = P_0$. Note that $P_0 > \max\{0, P_Z\}$ and $P_0 < a - c_2$. Hence, Z (P) is given by (5.16) whenever $\max\{0, P_Z\} \leq P \leq P_0$.

Third, using the linear demand curve $a - bZ$ from (5.15) in (3.3) gives the unique equilibrating output $Z(P) = n_3(a - c_3)/(n_3 + 1)b = Z_{NC}$. This is valid so long as $Z_{NC} \geq Z_0(P)$. From (5.14), this holds whenever $P \geq c_3 - c_2 + (a - c_3)/(n_3 + 1) = P_{NC}$. Finally, as in Remark 3.2, $Z(P) = Z_0(P)$ for $P_0 \leq P \leq P_{NC}$. Summarizing, we obtain

$$Z(P) = \begin{cases} 0 & \text{for } 0 \leq P \leq P_Z, \text{ if } P_Z > 0 \\ n_2 n_3 (P - P_Z)/(n_3 + 1)b & \text{for } \max\{0, P_Z\} \leq P \leq P_0 \\ (a - c_2 - P)/b & \text{for } P_0 \leq P \leq P_{NC} \\ n_3(a - c_3)/(n_3 + 1)b = Z_{NC} & \text{for } P \geq P_{NC} \end{cases} \tag{5.17}$$

where

$$P_Z = c_3 - c_2 - (a - c_3)/n_2,$$

$$P_0 = c_3 - c_2 + (a - c_3)/(n_2 n_3 + n_3 + 1), \tag{5.18}$$

$$P_{NC} = c_3 - c_2 + (a - c_3)/(n_3 + 1).$$

Next, consider determining $Y(P)$, $P \geq 0$. Observe from above that since for $0 \leq P < P_0$, we have $Z(P) < Z_0(P)$ so that $Y(P) = Y_R[Z(P), P] > 0$ and hence, $Y(P)$ is given by the first expression in (5.13). Otherwise, for $P \geq P_0$, we have $Y(P) \equiv 0$. Hence, from (5.13) and (5.17), letting $N = n_2 n_3 + n_3 + 1$ for notational convenience, we get

$$Y(P) = \begin{cases} n_2(a - c_2 - P)/(n_2 + 1)b & \text{for} \quad 0 \le P \le P_Z, \text{ if } P_Z > 0 \\ n_2N(P_0 - P)/(N + n_2)b & \text{for} \quad \max\{0, P_Z\} \le P \le P_0 \\ 0 & \text{for} \quad P \ge P_0 \end{cases} \tag{5.19}$$

Consequently, from (3.5) and (5.19), the (inverse) derived demand function $\Phi(\theta)$ is given as follows. If $P_Z > 0$, then

$$\Phi(\theta) = \begin{cases} P_0 - (N + n_2)b\theta/n_2N & \text{for} \quad 0 \le \theta \le Y(P_Z) \\ a - c_2 - (n_2 + 1)b\theta/n_2 & \text{for} \quad Y(P_Z) \le \theta \le Y(0) \\ 0 & \text{for} \quad \theta \ge Y(0) \end{cases} \tag{5.20}$$

If $P_Z \le 0$, then

$$\Phi(\theta) = \begin{cases} P_0 - (N + n_2)b\theta/n_2N & \text{for} \quad 0 \le \theta \le Y(0) \\ 0 & \text{for} \quad \theta \ge Y(0) \end{cases} \tag{5.21}$$

Note that the value of $Y(0)$ in (5.20)-(5.21) is computed from (5.19) depending on the particular case $P_Z > 0$ or $P_Z \le 0$. Observe also that $\Phi(\theta)$ is strictly decreasing, concave and has a singe kink over its positive range in the first case (5.20) at $Y = Y(P_Z)$, and is strictly decreasing and linear over its positive range in the second case (5.21) (see Corollary 3.6). Furthermore, by Theorem 3.10, an oligopolistic equilibrium exists. Taking the first linear segment of $\Phi(\cdot)$ defined in (5.20) or (5.21), we obtain from problems (3.7) for the firms in S_3,

$$X^o = \frac{n_1 n_2}{(n_1 + 1)(n_2 + 1)(n_3 + 1)} \frac{a - c_3}{b} \tag{5.22}$$

as the unique total equilibrating output of the firms in S_1. The corresponding equilibrium price on the chosen linear segment of $\Phi(\cdot)$ is $P^o = c_3 - c_2 + (a - c_3)/N(n_1 + 1)$. Note that $P^o \in (\max\{0, P_Z\},\ P_0)$ from (5.18). Consequently, in either case (5.20) or (5.21), X^o lies on the first segment of $\Phi(\cdot)$ and moreover, $\Phi(\cdot)$ is differentiable at X^o. Therefore, from Theorem 3.10 (ii), a unique equilibrium solution exists. Evidently, by symmetry, $x_i^o = X^o/n_1$ for $i = 1,\ldots, n_1$ at the equilibrium, and similarly, $y_i^o = Y^o/n_2$ for $i = 1,\ldots, n_2$ and $z_i^o = Z^o/n_3$ for $i = 1,\ldots, n_3$, where $Y^o = Y(P^o) = X^o$ and $Z^o = Z(P^o)$ is given by

$$Z^o = n_3 \frac{n_2 + (n_1 + 1)N}{(n_1 + 1)(n_3 + 1)N} \frac{a - c_3}{b}. \tag{5.23}$$

The total industry output $Q^o = Y^o + Z^o$ is given by

$$Q^o = \frac{n_1 n_2 + n_3(n_1 + 1)(n_2 + 1)^2(n_3 + 1)}{(n_1 + 1)(n_2 + 1)(n_3 + 1)N} \frac{a - c_3}{b}. \tag{5.24}$$

Note that $Z^o > 0$, which by Corollary 3.7 also supports the uniqueness of the equilibrium solution (x^o, y^o, z^o). Furthermore, in the follower-follower two-stage model discussed in Section 5.1, the unique equilibrium turns out to yield a larger value of X^o(and hence Y^o) as well as P^o and a smaller value of Z^o. However, the net consequence of this is that Q^o turns out to be always larger in the leader-follower model, and thereby, the consumer benefits through lower prices.

Now, let us consider, as in Section 5.1, the effects of various types of firms integration on the equilibrating quantities defined in (5.22), (5.23) and (5.24) and on the associated prices and individual firm profits. Table 2 summarizes these sensitivity results. Following the notation in Section 5.1, the firms profits have been denoted by π_i^o , $i = 1, 2, 3$ for the firms in S_i , and the subscripts A and B respectively have been used to denote the models after and before the merger or integration. The heading of Table 2 gives the various scenarios. The first three columns include the results of horizontal integration within each set of firms, while the fourth and fifth columns respectively refer to a unilateral upgrading of a firm in S_1 and S_2 to a firm in S_3. The final column contains the results of a merger of one firm in S_1 and one firm in S_2 to into a new vertically integrated firm in S_3. The proofs of the various results in Table 2 involve long and tedious algebra, and are omitted. Appendix A contains the derivation of the results in the last column of Table 2.

First, consider the horizontal integrations. What is perhaps most noteworthy here is that the final commodity consumers pay higher prices at a lower industry output level, and the firms in S_2 also pay a higher price for the semi-finished product whenever any horizontal integration takes place. Furthermore, whenever any horizontal integration occurs, the nonintegrating firms benefit, except for the case of the firms in S_2 when some of the upstream producers merge. The more important question of whether or not the integration is profitable for the merging firms cannot be answered uniquely, and depends on how many firms integrate to form, say, one firm. For example, in the case of the firms in S_1, one can verify the following result. If $n_1 = 9$ and eight firms in S_1 decide to merge, the the new firm's profits are larger than the joint profits of the eight firms before integration. However, if $n_1 = 10$, the corresponding result is just opposite. Similar examples can be constructed in the case of integration within the sets S_2 or S_3. In particular, the last row in Table 2 gives the profitability conditions for horizontal integrations of two firms from S_i, $i = 1, 2, 3$.

In the case of vertical integrations, including the case of firms in S_1 or S_2 upgrading their production processes to become a firm of the type in S_3, the sensivities of the various quantities are strongly related to the number of firms n_1 , n_2 and n_3. Here, although the conversion of a firm in S_1 to a firm of the type in S_3 always benefits the consumers in the final product market, this is not so if a firm in S_2 becomes one in S_3, or if a firm in each of S_1 and S_2 merge to become a firm in S_3 , unless if $n_2 > n_3$. However, the second type of integration is always profitable for the upgrading firm, while the first and third types of integration are profitable to the upgrading or merging firms if at least one firm in S_3 existed before the integration, and may not be so otherwise.

In order to provide some further insights into the successive vertical integrations, suppose that $n_1 = 34$, $n_2 = 39$ and $n_3 = 1$, and assume that $a - c_3 = b = 1$. Let us consider the effect of some k firms in each of S_1 and S_2 merging to produce k firms in S_3, thus resulting in $n_1 = 34 - k$, $n_2 = 39 - k$ and $n_3 = 1 + k$, as k varies from 0, ..., 33. (For graphical illustration see Appendix B.)

Table 2. Summary of Integration Results in Leader-Follower Model

Effects	Scenarios					
	$n_1 - k, n_2, n_3$ $n_1 - 1 \geq k \geq 1$ $n_2 \geq 1$	$n_1, n_2 - k, n_3$ $n_2 - 1 \geq k \geq 1$ $n_1 \geq 1$	$n_1, n_2, n_3 - k$ $n_3 - 1 \geq k \geq 1$ $(n_1 \geq 1, n_2 \geq 1)$	$n_1 - 1, n_2, n_3 + 1$ $n_1 \geq 2, n_2 \geq 1$	$n_1, n_2 - 1, n_3 + 1$ $n_1 \geq 1, n_2 \geq 2$	$n_1 - 1, n_2 - 1, n_3 + 1$ $n_1 \geq 2, n_2 \geq 2$
$P_A^o - P_B^o$	> 0	> 0	> 0	> 0 *if* $n_3 \geq n_1$ < 0 *if* $n_3 < n_1$	≥ 0 *iff* $n_2 \leq n_3$	> 0 *if* $n_3 \geq \min\{n_1, n_2\}$
$Y_A^o - Y_B^o$	< 0	< 0	> 0	< 0	< 0	< 0
$Q_A^o - Q_B^o$	< 0	< 0	< 0	> 0	< 0 *if* $n_2 \leq n_3$ > 0 *if* $n_2 > n_3$	< 0 *if* $n_2 \leq n_3$ > 0 *if* $n_2 > n_3$
$\pi_{1A}^o - \pi_{1B}^o$	> 0	≥ 0	> 0	> 0 *if* $n_3 \geq n_1$ < 0 *if* $n_3 < n_1$	< 0 *if* $n_2 \geq n_3$	> 0 *if* $n_3 \geq n_1$ < 0 *if* $2n_3 \leq \min\{2n_2 - 2, n_1 - 3\}$
$\pi_{2A}^o - \pi_{2B}^o$	< 0	> 0	> 0	< 0	≥ 0 *iff* $n_2 \leq n_3 + 1$	< 0 *if* $n_2 \geq n_3 + 1$
$\pi_{3A}^o - \pi_{3B}^o$	> 0	> 0	> 0	< 0	< 0 *if* $n_2 \geq n_3 \geq 1$	< 0 *if* $1 \leq n_3 \leq \min\{n_1, n_2\}$
Net profit benefits for the new firm	$\pi_{1A}^o > 2\pi_{1B}^o$ *iff* $n_1 \leq 2$	$\pi_{2A}^o > 2\pi_{2B}^o$ *iff* $n_2 \leq 2$	$\pi_{3A}^o < 2\pi_{3B}^o$ *if* $n_3 \geq 4$	$\pi_{3A}^o > \pi_{1B}^o$ *if* $n_3 \geq 1$	$\pi_{3A}^o > \pi_{2B}^o$	$\pi_{3A}^o > \pi_{1B}^o + \pi_{2B}^o$ *if* $n_3 \geq 1$

As n_3 increases, the total output Y^o (= X^o) of the follower firms keeps decreasing while that of the leader firms keeps increasing. The total industry output, however, increases at first, until n_3 = 20, after which it begins to drop due to an increase in the competition among the leader firms. Hence, the equilibrium price in the final product market decreases as n_3 increases to 20, and begins to rise thereafter. While this is happening, the individual firm output for the leader firms in S_3 keeps falling, and so does the individual firm profits, π_3^o , even when prices begin to rise after n_3 = 20 . However, beyond n_3 = 27, the rise in the prices is sufficient to offset this phenomenon, and π_3^o begins to increase.

As a consequence of this, as n_3 initially increases, the follower firms in S_2 are left with a smaller share of the market along with a reduced total (and individual firm) output, and face a lower price for the final product. Therefore, although their demand for the semi-finished good is dropping, their reduced purchasing capabilities causes them the semi-finished product price P^o to fall, and hence, the individual S_1 firm profit π_1^o also falls. However, as n_3 increases to 13, the increase in Q^o and Z^o slows down, which allows the firms in S_2 to accept an increase of P^o forced by the firms in S_1. Consequently, P^o rises beyond n_3 = 13. The associated increase in π_1^o is reflected a little later, beyond n_3 = 15, as the individual firm output in S_1 continues to fall until n_3 = 17 and then begins to increase thereafter. As far as the individual follower firm outputs y_i^o and profits π_2^o are concerned, as n_3 increases initially, y_i^o and π_2^o both drop until n_3 becomes 22, after which they both begin to increase as the competition among the leader firms in S_3 increases, and begins to therefore diminish the influence of their leadership role. However, when n_3 becomes 32 and n_2 falls to 8, the diminished role of the firms in S_2 causes both y_i^o and π_2^o to fall again as n_3 increases further.

Let us point out that as shown in the final cell in Table 2, the above step by step integration always bring additional profit to the merging firms at each step. However, since the firms act independently, this incentive may turn out to be an illusion. For example, consider the case when n_1 = 34, n_2 = 39 and n_3 = 1 , and simultaneously k = 15 pairs of firms from S_1 and S_2 decide to integrate, thereby resulting in n_1 = 19, n_2 = 24 and n_3 = 16. Then, each of the new firms in S_3 has a profit of 1392 units, while before integration, the individual firm profits were

$\pi_1^o = 97$ units and $\pi_2^o = 1474$ units. Similar results can be obtained in the case when follower firms in S_2 upgrade to enter the leader firm set.

5.3. Summary of Results

In this chapter simple two-stage oligopolistic models were discussed to illustrate the analyses presented in Chapters 2 and 3, and further to provide some insight into various issues concerning firm integrations. Given the follower-follower or the multiple leader-follower oligopoly in the final commodity market, and simultaneously the Cournot oligopoly in the upstream stage, an equilibrium solution was shown to exist and to be unique. Also, a closed form expression for such an equilibrium solution was derived for each of the models above, thus facilitating the analysis of various types of collusions among the firms. A horizontal integration within any set of producers was shown to yield a decrease in the total industry output and hence, a higher price paid by consumer. On the other hand, it was shown that a vertical integration may be beneficial to the consumer, since it may lead to an increase in the total industry output and therefore, yield a lower price. However, as more and more firms integrate, a decrease in this output may result, thus reversing the situation.

By examining the variation in individual firm profits due to firm integrations, it was demonstrated that except for some extreme situations, horizontal integration is not beneficial for those firms which merge. However, it brings about more profits for those producers who do not merge. The above conclusions support, and in some sense extend, the results obtained by Salant et al. [S1] and also by Szidarovszky and Yakowitz [S8]. The effects of vertical integration on firm profits were shown to depend intimately on relative values of the number of firms within each set. Conditions were derived to ascertain when it is or it is not profitable for the firms to integrate. We note that for the examples considered herein, we were able to derive closed form results through some algebraic manipulations. Appendix A provides a sample of the calculations by deriving the last column in Tables 1 and 2, and Appendix B contains figures illustrating the effects of vertical integration. In general, for more complex situations in which the results of Chapters 2 and 3 are still applicable, one would need to use some computational techniques of Chapter 4 in order to empirically study the outcome of various scenarios.

The results obtained in this chapter show that, when the demand function is linear and so are all the cost functions, being identical for each type of producer, the two-stage oligopolistic model with the multiple leader-follower behavior on the part of the firms in $S_2 \cup S_3$ provides more total industry output than the one with the follower-follower oligopoly, given that the numbers n_1, n_2 and n_3 are the same in both cases. When the linearity of the demand function is relaxed, a similar conclusion can be derived, as shown in the next chapter.

Chapter 6

Some Comparative Results and Conclusions

The first section of this chapter presents a rigorous comparison of three models considered by Greenhut and Ohta [G3]. The second section contains some concluding remarks and suggestions for future research.

6.1. Some Comparative Results for the Two-Stage Oligopolistc Models

In this section three models which were investigated by Greenhut and Ohta [G3] are compared with respect to the total industry output associated with an equilibrium solution. These models are referred to as models (a), (b) and (c). In model (a) there are n identical upstream producers (S_1) and m identical downstream firms (S_2). The marginal costs for both stages are assumed constant and are denoted by c_1 and c_2 respectively. The set S_3 is empty, and all the firms are assumed to be Cournot firms. Models (b) and (c) are viewed as resulting from (a) by allowing for vertical integration of some upstream and downstream producers. In both cases, $|S_1| = n - \ell$, $|S_2| = m - \ell$, and $|S_3| = \ell$, where $1 \leq \ell < \min\{m, n\}$. The vertically integrated firms are assumed to be identical, with the marginal cost being c_3, where $c_3 = c_1 + c_2$. The difference between models (b) and (c) lies in the type of interactions among the final commodity suppliers. In (b) all the firms in $S_2 \cup S_3$ are Cournot firms, while in (c) the firms in S_3 are assumed to be leader firms, as discussed in Chapter 3. Notationally, all the quantities and functions used in the sequel are subscripted by a, b or c.

Observe that model (a) is a particular case of model (b), so that the results presented in Chapter 2 in the context of an oligopolistic equilibrium, can be applied in analyzing both these models.

First, consider **model (a)**. Note that if the market demand function $p(Q)$ is strictly decreasing, twice differentiable and satisfies $p'(Q) + Qp''(Q) \leq 0$ for all $Q \geq 0$, as in Assumptions A1, then by Corollary 2.1 we can assert that for any fixed price $P \geq 0$, the firms in S_2 produce at (the unique) equilibrium $Y_a(P) = my_a(P)$, where

$$y_a(P) \text{ solves } \underset{q \geq 0}{\text{maximize}}\{qp[q + (m-1)y_a(P)] - (c_2 + P)q\}. \tag{6.1}$$

Further, by Corollary 2.3, the (inverse) demand function $\Phi(\theta)$ for the intermediate product exists, is continuous and strictly decreasing over its positive range, say, $[0, Y_a(0))$. By the necessary and sufficient conditions for optimality in (6.1), we obtain for $\theta \in [0, Y_a(0)]$

$$\Phi_a(\theta) = p(\theta) + \frac{1}{m}\theta p'(\theta) - c_2. \tag{6.2}$$

Observe that in this case, $\Phi_a(\theta)$ is given explicitly, and as evident from equation (6.2), it is differentiable over its positive range. Next, if $\Phi_a(\theta)$ turns out to be concave for $\theta \in [0, Y_a(0)]$, as for example when $p(Q) = a - bQ^k$, $a > 0$, $b > 0$, $k \geq 1$, then by Theorem 2.6 (ii) we can assert that there exists a unique equilibrium solution for the firms in S_1, with $X_a^o = nx_a^o$, where

$$x_a(P) \text{ solves } \underset{q \geq 0}{\text{maximize}}\{q\Phi_a[q + (n-1)x_a^o] - c_1 q\}. \tag{6.3}$$

Furthermore, by Theorem 2.6 (iii), X_a^o ($= Y_a^o$) is positive if and only if $p(0) > c_3$. Summarizing, under the stated assumptions, the total industry output Q_a^o ($= X_a^o = Y_a^o$) exists, is unique and if $p(0) > c_3$, then it is determined by the equation:

$$mn[p(Q_a^o) - c_3] + (m + n + 1)Q_a^o p'(Q_a^o) + (Q_a^o)^2 p''(Q_a^o) = 0 \tag{6.4}$$

which is derived from the necessary and sufficient conditions for optimality in (6.3), where $\Phi_a(\theta)$ is given by (6.2). Moreover, $P_a^o = \Phi_a(X_a^o) > c_1$.

Let us proceed to investigate **model (b)**, with the same market demand function $p(Q)$ as in (a). By similar arguments we can claim that for any $P \geq 0$, the unique equilibrium solution for the firms in $S_2 \cup S_3$ gives $Y_b(P) = (m - \ell)y_b(P)$, and $Z_b = \ell z_b(P)$, where

$$y_b(P) \text{ solves } \underset{q \geq 0}{\text{maximize}}\{qp[q + (m - \ell - 1)y_b(P) + Z_b(P)] - (c_2 + P)q\} \tag{6.5}$$

$$z_b(P) \text{ solves } \underset{q \geq 0}{\text{maximize}}\{qp[q + (\ell - 1)z_b(P) + Y_b(P)] - c_3 q\}. \tag{6.6}$$

Therefore, the total industry output $Q_b(P) = Y_b(P) + Z_b(P)$ is uniquely defined for any $P \geq 0$. Again, by Corollary 2.3, the (inverse) demand function $\Phi_b(\theta)$ exists, is continuous and is strictly decreasing over its positive range, say, $[0, Y_b(0)]$. However, unlike as in model (a), the function $\Phi_b(\theta)$ cannot be given explicitly in a form similar to (6.2), as suggested by Greenhut and Ohta [G3], who used $\Phi_b(\theta)$ as the one generated by the necessary and sufficient conditions for optimality in (6.5) and therefore, neglected to consider those for (6.6). By Theorem 2.6, parts (ii) and (iii), if $\Phi_b(\theta)$ is concave for $\theta \in [0, Y_b(0)]$ then there exists a unique total equilibrating output X_b^o for the firms in S_1, being positive if $p(0) > c_3$. Also in this case, $P_b^o > c_1$, where $P_b^o = \Phi_b(X_b^o)$.

Lemma 6.1. Assume that the market demand function $p(Q)$ is strictly decreasing and concave, with $p(0) > c_3$. If $\Phi_b(\cdot)$ is concave then $Q_a^o < Q_b^o$.

Proof. First observe that from the problems (6.5) and (6.6), with $P = c_1$, we necessarily have $Q_b(c_1) > 0$, and moreover $Y_b(c_1) > 0$, $Z_b(c_1) > 0$. Therefore, since $p(0) > c_1$ and $Z_b(P)$ is nondecreasing in $P \geq 0$, one can assert that $Y_b^o > 0$ and $Z_b^o > 0$. Next, notice that $Q_b(0) > Q_a^o$. To show that this result holds, let us compare the necessary and sufficient conditions for optimality in (6.1), using $P_a^o > c_1$ with those for (6.5) and (6.6), using $P = 0$. In the first case we obtain

$$mp(Q_a^o) + Q_a^o p'(Q_a^o) - m(P_a^o + c_2) = 0 \qquad (6.7)$$

and in the second case

$$mp[Q_b(0)] + Q_b(0)p'[Q_b(0)] - (m - \ell)c_2 - \ell c_3 \le 0, \qquad (6.8)$$

where equality holds if $Z_b(0) > 0$. Since $P_a^o > c_1$, $c_3 = c_1 + c_2$, $p(\cdot)$ is decreasing and concave, the above equations yield $Q_b(0) > Q_a^o$.

To demonstrate that $Q_a^o < Q_b^o$, suppose that $Q_a^o \le Z_{NC}$, where Z_{NC} is defined via (2.38). If $Q_a^o \le Z_{NC}$, then since $Q_b(0) > Q_a^o$ and $Y_b^o > 0$, we necessarily have $Q_b^o > Z_{NC} \ge Q_a^o$. On the other hand, if $Q_a^o > Z_{NC}$, then, using again $Q_b(0) > Q_a^o$ and Theorem 2.1, one can verify that there exists price $\bar{P}$ such that $Q_b(\bar{P}) = Q_a^o$. Denote $\bar{Y} = Y_b(\bar{P})$ and $\bar{Z} = Z_b(\bar{P})$, and observe that $Q_a^o > Z_{NC}$ implies $\bar{Y} > 0$. Now, to prove that $Q_b^o > Q_a^o$ it suffices to demonstrate that $P_b^o < \bar{P}$. Toward this end, note that if $\bar{Z} = 0$, then since $Z_b^o > 0$ and $Z_b(P)$ is nondecreasing, the above result is trivial. Hence, suppose that $\bar{Z} > 0$. Then, since $\bar{Y} > 0$, the demand function $\Phi_b(\theta)$ is differentiable at $\theta = \bar{Y}$. Consider a function $\Psi_b(\theta)$ defined in the following way:

$$\Psi_b(\theta) = (n - \ell)[\Phi_b(\theta) - c_1] + \theta\Phi'_b(\theta).$$

By Corollary 2.3, $\Phi_b(\theta)$ is strictly decreasing over its positive range, which along with the assumed concavity implies that $\Psi_b(\theta)$ is strictly decreasing. Moreover, one can easily verify that $\Psi_b(Y_b^o) = 0$. We will demonstrate now that $\Psi_b(\bar{Y}) > 0$. Observe that by the choice of $\bar{P}$ above, we have from (2.37) and (2.35) that

$$\Phi'_b(\bar{Y}) = \frac{p'(Q_a^o)}{m - \ell} \, \frac{(m + 1)p'(Q_a^o) + Q_a^o p''(Q_a^o)}{(\ell + 1)p'(Q_a^o) + \bar{Z}p''(Q_a^o)}. \qquad (6.9)$$

Since $p'(\cdot) < 0$ and $p''(\cdot) \le 0$, equation (6.9) implies that

$$(m - \ell)(\ell + 1)\Phi'_b(\bar{Y}) \ge [(m + 1)p'(Q_a^o) + Q_a^o p''(Q_a^o)].$$

Further, from the necessary and sufficient conditions for problems (6.5) and (6.6), we obtain

$$(m - \ell)[\Phi_b(\tilde{Y}) - c_1] = m[p(Q_a^o) - c_3] + Q_a^o p'(Q_a^o),$$

which along with equation (6.4) gives

$$n(m - \ell)[\Phi_b(\tilde{Y}) - c_1] = - Q_a^o[(m + 1)p'(Q_a^o) + Q_a^o p''(Q_a^o)].$$

Since, $Q_a^o > \tilde{Y}$, we obtain

$$(m - \ell)\Psi_b(\tilde{Y}) > \tilde{Y}[(m + 1)p'(Q_a^o) + Q_a^o p''(Q_a^o)]\left(- \frac{n - \ell}{n} + \frac{1}{\ell + 1}\right).$$

The expression on the right-hand side of this inequality is nonnegative because $p'(\cdot) < 0$ and $p''(\cdot) \leq 0$ and $n > \ell$. Therefore, $\Psi_b(\tilde{Y}) > 0$ so that $Y_b^o > \tilde{Y}$ and hence $P_b^o < \tilde{P}$, thus completing the proof.■

As far as the relation between P_a^o and P_b^o is concerned, we can conclude that $mP_a^o > (m - \ell)P_b^o + \ell c_1$. In deriving this result we used Lemma 6.1 and equations (6.7) and (6.8), the latter one being an equality since $Z_b^o > 0$. The above relation contrasts with the work by Greenhut and Ohta [G3] , who assumed that the equilibrating prices P_a^o and P_b^o are equal, and based on that claimed that $Q_a^o < Q_b^o$.

Finally, let us notice that $Q_a^o < Q_b^o$ also in the case of total collusion, that is, when $\ell = \min\{m, n\}$. This model eliminates the producers in $S_1 \cup S_2$, so that it results in a single-market structure, with ℓ final commodity suppliers. To show that $Q_a^o < Q_b^o$, assume that $\ell = m < n$. Then, equation (6.4) along with the necessary and sufficient conditions for problems (6.6), where $Y_b(0) = 0$ gives

$$mn[p(Q_b^o) - p(Q_a^o)] + n[Q_b^o p'(Q_b^o) - Q_a^o p'(Q_a^o)] = (m + 1)Q_a^o p'(Q_a^o) + (Q_a^o)^2 p''(Q_a^o).$$

The right-hand side of this equation is negative, since $p'(\cdot) < 0$ and $p''(\cdot) \le 0$, and therefore, the resulting inequality yields $Q_b^o > Q_a^o$. By similar arguments, one can demonstrate that $Q_b^o > Q_a^o$ when $\ell = n \le m$.

Now, let us compare models (b) and (c). In **model (c)**, let us recall that the ℓ vertically integrated firms behave as leaders, while the downstream producers preserve their Cournot behavior. Here, to ensure that the demand function for the semi-finished product exists and has the required properties, we will assume that $p(Q) = a - bQ^k$, where $a > 0$, $b > 0$, $k \ge 1$. The relation between Q_b^o and Q_c^o is addressed in lemma below.

Lemma 6.2. Suppose that $p(Q) = a - bQ^k$, where $a > 0$, $b > 0$, $k \ge 1$. If $a > c_3$, then $Q_b^o < Q_c^o$.

Proof. First note that under the assumptions of this lemma, Theorem 3.10 can be employed to state the existence of an equilibrium solution (x_c^o, y_c^o, z_c^o) with (y_c^o, z_c^o) being unique, and with $X_c^o\,(= Y_c^o) > 0$. Therefore, in this model as well, we have $P_c^o > c_1$. Now, we will show that $Z_c(c_1) > 0$, so that we can claim that $Z_c^o > 0$, since $Z_c(P)$ is nondecreasing in P over the positive range of $Y_c(P)$. For this purpose note that $Q_c^o > 0$ and $P_c^o > c_1$ imply $Q_c(c_1) > 0$, because $Q_c(P)$ is nonincreasing in $P \ge 0$. Further, suppose that $Y_c(c_1) > 0$ and $Z_c(c_1) = 0$. Then, by the necessary and sufficient conditions for problems (3.1) and (3.3) with $P = c_1$, we obtain $(m - \ell)[p[Y_c(c_1)] - c_3] + Y_c(c_1)p'[Y_c(c_1)] = 0$ and $p[Y_c(c_1)] - c_3 \le 0$, which yields a contradiction. Therefore, in fact $Z_c(c_1) > 0$ and also $Z_c^o > 0$. A similar comparison of the necessary and sufficient conditions for (3.1) and (3.3), using $P = 0$, with those for problems (6.5) and (6.6), where $P = P_b^o$, allows us to conclude that $Q_c(0) > Q_b^o$. Now, since $Q_b^o > Z_{NC}$ we can assert that there exists a $\bar{P}$ such that $Q_c(\bar{P}) = Q_b^o$. Let $\bar{Y} = Y_c(\bar{P})$ and $\bar{Z} = Z_c(\bar{P})$, and note that $\bar{Y} > 0$. We will show that $\bar{P} > P_c^o$. If $\bar{Z} = 0$ then the result is readily verified. In case $\bar{Z} > 0$, let us consider the following function $\Psi_c(\theta)$:

$$\Psi_c(\theta) = (n - \ell)[\Phi_c(\theta) - c_1] + \theta\Phi'_c(\theta)$$

This function is well defined for those $\theta > 0$, for which the corresponding output of the firms in S_3 is positive. Therefore, $\Psi_c(\tilde{Y})$ is well defined, and so is $\Psi_c(Y_c^o)$. Furthermore, observe that $\Psi_c(\theta)$ is strictly decreasing, and moreover, $\Phi_c(Y_c^o) = 0$. Following the approach employed in the proof of Lemma 6.1, we will show that $\Psi_c(\tilde{Y}) > 0$, which suffices for asserting $\tilde{P} > P_c^o$. Toward this end, let us observe that whenever $Y_c(P) > 0$ and $Z_c(P) > 0$ then the necessary and sufficient conditions for optimality in (3.1) and (3.3) give

$$(m - \ell)\{p[Q_c(P)] - c_2 - P\} + Y_c(P)p'[Q_c(P)] = 0 \tag{6.10}$$

$$\ell\{p[Q_c(P)] - c_3\} + Z_c(P)p'[Q_c(P)][1 + \frac{\partial}{\partial Z}Y_R] = 0, \tag{6.11}$$

where by equation (3.14)

$$1 + \frac{\partial}{\partial Z}Y_R = \frac{p'[Q_c(P)]}{(m - \ell + 1)p'[Q_c(P)] + Y_c(P)p''[Q_c(P)]}. \tag{6.12}$$

From the above equations, it can be verified that $(m - \ell)\Phi'_c(\tilde{Y}) = p'[Q_c(\tilde{P})][1 + \frac{A}{B}]$, where $A = [(m - \ell)p' + \tilde{Y}p''][(m - \ell + 1)p' + Q_b^o p'']$, and $B = \ell[(m - \ell + 1)p' + \tilde{Y}p'']^2 + [(m - \ell + 1)p' + \tilde{Y}p''](p' + \tilde{Z}p'') + \tilde{Z}\tilde{Y}[(p'')^2 - p'p''']$, and where p', p'' and p''' are all evaluated at $Q_c(\tilde{P}) = Q_b^o$. For any strictly decreasing and concave function $p(\cdot)$, we have $\ell[(m - \ell + 1)p' + \tilde{Y}p''] + p' + \tilde{Z}p'' < (\ell + 1)p' + \tilde{Z}p''$, and also, $\ell[(m - \ell + 1)p' + \tilde{Y}p''] + p' + \tilde{Z}p'' < \tilde{Y}p''$. Moreover, for the specified demand function $p(Q) = a - bQ^k$, we have $(p'')^2 - p'p''' \geq 0$. Using the above inequalities in the expression for $\Phi'_c(\tilde{Y})$ results in

$$\Phi'_c(\tilde{Y}) > \frac{p'}{m - \ell}\left[\frac{(m + 1)p' + Q_b^o p''}{(\ell + 1)p' + \tilde{Z}p''} + \frac{(m - \ell)p' + \tilde{Y}p''}{(m - \ell + 1)p' + \tilde{Y}p''}\frac{\tilde{Z}}{\tilde{Y}}\right]. \tag{6.13}$$

In order to determine $\tilde{P} - c_1$, note that equations (6.10) and (6.11) imply

$$(m - \ell)(\tilde{P} - c_1) = m[p(Q_b^o) - c_3] + Q_b^o p'(Q_b^o) + \tilde{Z}p'(Q_b^o)\frac{\partial}{\partial Z}Y_R,$$

which along with (6.12) and the equation determining Q_b^o from (6.7) and (6.8) gives

$$(n - \ell)(\bar{P} - c_1) = - \frac{p'}{m - \ell}[Y_b^o \frac{(m + 1)p' + Q_b^o p''}{(\ell + 1)p' + Z_b^o p''} + (n - \ell)\bar{Z}\frac{(m - \ell)p' + \bar{Y}p''}{(m - \ell + 1)p' + \bar{Y}p''}], \quad (6.14)$$

where as before p' and p'' are evaluated at Q_b^o. Finally, notice that $Z_b^o < \bar{Z}$, since $\ell[p(Q_b^o) - c_3] + Z_b^o p'(Q_b^o) = 0$ and from (6.11), $\ell[p(Q_b^o) - c_3] + \bar{Z}p'(Q_b^o) < 0$. Therefore, since $Q_b^o = \bar{Q}$, we also have $\bar{Y} < Y_b^o$. Including these results in (6.13) and (6.14), we obtain $(m - \ell)\Psi_c(\bar{Y}) > - p'(C + D)$, where $D = (n - \ell + 1)\bar{Z}\frac{(m - \ell)p' + \bar{Y}p''}{(m - \ell + 1)p' + \bar{Y}p''}$ and $C = \bar{Y}\frac{(\ell + 1)p'}{(\ell + 1)p' + Z_b^o p''}$. Expressions C and D are positive, while $p' < 0$. Hence, $\Psi_c(\bar{Y}) > 0$ and this completes the proof.■

For a comparison of the equilibrating prices P_b^o and P_c^o , note that the optimality conditions in models (b) and (c) give $mp(Q_b^o) - (m - \ell)c_2 - \ell c_3 + Q_b^o p'(Q_b^o) = (m - \ell)P_b^o$, and $mp(Q_c^o) - (m - \ell)c_2 - \ell c_3 + Q_c^o p'(Q_c^o) = (m - \ell)P_c^o - Z_c^o p'(Q_c^o)\frac{\partial}{\partial Z}Y_R$, which along with $Q_b^o < Q_c^o$ leads to

$$(m - \ell)(P_b^o - P_c^o) > - \frac{1}{m - \ell}Z_c^o p'(Q_b^o)\frac{\partial}{\partial Z}Y_R.$$

This again contrasts with the statement $P_b^o = P_c^o$, as presented in [G3]. The foregoing results, embodied in Lemmas 6.1 and 6.2, are summarized in the corollary below.

Corollary 6.1. Assume that $p(Q) = a - bQ^k$, where $a > 0$, $b > 0$, $k \geq 1$. If $a > c_3$, then $Q_a^o < Q_b^o < Q_c^o$.

Corollary 6.1 states that model (c), with S_3 comprised of leader firms, gives the biggest total industry output, provided that the market demand function $p(Q)$ is as specified above. Consequently, the consumers will benefit from the leader-follower model, since they would pay a lower market price. As far as the equilibrating prices for the intermediate product are concerned, such a comparison is not straightforward. However, when the demand function $p(Q)$ is linear, i.e., $k = 1$, then using the results presented in Chapter 5, one can easily verify that

$$P_a^o = c_1 + \frac{a - c_3}{n+1},\ P_b^o = c_1 + \frac{a - c_3}{(n - \ell + 1)(\ell + 1)},\ P_c^o = c_1 + \frac{a - c_3}{(n - \ell + 1)[1 + \ell(m - \ell + 1)]}.$$

By comparing the above prices, we obtain $P_a^o > P_b^o > P_c^o$. Hence, in this case, the downstream stage also benefits by virtue of lower prices paid to the firms in S_1, given that the vertically integrated firms act as leaders. However, as shown in Table 2, their profits decrease, which may be due to the final commodity price decline.

To summarize, in this section three models were compared with respect to the total equilibrating industry output. For a more thorough comparison, it would be required to also consider profitability issues, which might give an interesting insight into benefits and losses resulting from vertical integration and the types of interactions among the final commodity suppliers. Such problems may be considered in terms of cooperative games, as suggested in the next section.

6.2. Some Comments and Suggestions for Further Research

Models representing a two-stage industry were first considered by Greenhut and Ohta [G1],[G3] in the context of benefits stemming from the vertical integration of firms in the petroleum industry. Their analysis is sketchy and erroneous. However, they realize an alternate way of approaching equilibrium concepts in such models. Greenhut and Ohta ([G1],p. 276) write:

> "Given a demand function for a good, one may deduce a derived demand function for, let us say, the intermediate good or service such as transportation. This function would yield, in turn, another derived demand function, for example, for the tires used by the carrier, and so forth."

The concept of the derived demand function has been extensively employed in the present analysis, for the purpose of examining the existence of an equilibrium solution in the model under various behavioral assumptions for the firms. In its derivation and characterization, a mathematical programming-based approach for determining oligopolistic equilibria, plays a key role. It was first introduced by Murphy et al. [M3], in the context of a Nash-Cournot equilibrium, and carried further by Sherali et al. [S3] in the analysis of a Stackelberg type of oligopoly.

The existence of an equilibrium solution, in the case of the follower-follower and the multiple leader-follower behavior among the final product suppliers, and a perfect competition or oligopolistic behavior in the upstream stage received much attention in our analysis. The development was aimed at identifying those properties of the market demand function and the cost functions of the firms in $S_1 \cup S_2 \cup S_3$ which ensure the existence and uniqueness of an equilibrium solution in the network. Furthermore, some computational techniques were presented, for determining an equilibrium solution, given the above market assumptions. Also, some issues concerning various mergers and integrations among the producers are discussed herein, including changes in market price paid by the consumer and changes in profits faced by each type of the producer.

Several issues remain that are not addressed in this research. One relates to the power of the downstream producers in purchasing the semi-finished product. In our model, it is assumed that the firms in S_2 are price-takers, which may be justified in situations when there are many downstream firms. However, in case of few downstream producers, they are likely to actively participate in price formation for their input, and adopt oligopsonic behavior. Analysis of a model which incorporates this type of behavior for the firms in S_2, being a novel study, would contribute to a more thorough insight into two-stage industries.

Another problem that might be interesting to look into, relates to the collusion of firms. In this research, we were concerned with investigating changes in various quantities (e.g., total industry output, firms profits) that would take place if some firms decide to collude or integrate, while the remaining producers continue to operate in the same manner as before such integration. In particular, the problem of the most profitable configuration for the firms was not addressed here. Recently, these types of problems have been investigated by Sherali and Rajan [S4] in the context of cooperative games, with players being some homogeneous product suppliers. Sherali and Rajan attempt to determine what coalition, if any, would emerge under three types of firms behavior. Accordingly, they analyze three games. In the first game, each coalition assumes that the remaining firms will decide to do what is worst for it. Two other games employ the concept of a Nash-Cournot equilibrium. In the second one, each coalition assumes that its rivals will coalesce, so that a Cournot type of a duopoly will result, while in the third one, they are assumed to remain separate Cournot firms. Conceptually, the third game is closely related to our collusion considerations. Using a linear demand function and quadratic cost functions, identical for all, say n, producers, Sherali and Rajan [S4] demonstrate that the grand coalition (i.e., total collusion) emerges for the first game. A similar result is established for the third game, in case when $n \leq 4$. In contrast with this, the grand coalition will not emerge in the second game, unless $n = 2$, and in particular, if $n = 3$ or $n = 4$, the best the firms can do is to remain separate. Similar questions can be addressed in the context of a two-stage model. Such a study would not be easy, since as was mentioned earlier, any change in the number of firms in S_2 or in S_3 imposes a new derived demand function. Consequently, the task of determining the value of a coalition which involves the final commodity suppliers is not trivial. However, by analyzing the simplified case of a linear demand function and identical firms within each set S_1, S_2 and S_3, one may be able to gain some further insights into the problem of what types of configurations of firms would emerge in two-stage models. Also, it would be interesting to conduct a similar analysis for a model which incorporates oligopsonic behavior on the part of the downstream producers.

In the present analysis, the oligopolistic nature of the firms was based on either the Cournot type of behavior or on Stackelberg's leader behavior. However, various definitions of an equilibrium solution are conceivable, under various appropriate behavioral assumptions. One such possibility incorporates the notion of consistent conjectures as addressed below. The concept of consistent conjectures equilibrium is due to Bresnaham [B1] who also derived conditions for its existence and uniqueness in the case of a duopoly. Here, the issue is to ensure that at an equilibrium, the firm's behavior is consistent with its assumptions about the other's behavior. In other words, at an equilibrium the slope of the reaction function should be equal to its conjectural variation. The Cournot oligopoly yields inconsistencies. The firms are right for wrong reasons, as concluded by Fellner. Each firm assumes zero conjectural variations; however, its reaction function embodies a nontrivial relation between its equilibrating output level and the equilibrating outputs of the remaining firms. In contrast with this, the Stackelberg firm in a leader-follower duopoly, is known to be right for the right reasons. Bresnaham [B1] demonstrates that under some assumptions on the demand function and the cost functions, the Bertrand duopoly also produces a consistent conjectures equilibrium.

Turning to our two-stage model, questions may be posed about the formulation and subsequently, about the the existence of a consistent conjectures equilibrium. First, what type of conjectural variations would ensure that the final commodity suppliers are consistent in their behavior, at least at an equilibrium ? Further, what conditions should be imposed to guarantee that the equilibrating process among the firms in $S_2 \cup S_3$ produces a well behaved demand function for the intermediate product? Would the final commodity consumers benefit from the correctness of the firms' conjectures, as is the case of a (single market) duopoly? It can be expected that mathematical programming concepts would facilitate the analysis in this case as well.

APPENDIX A

A.1. Proof of Theorem 3.8 (ii).

Below we establish strict concavity of $\Phi(\theta)$, for $s \in [0, Y_0)$. Note that for this purpose, it is sufficient to demonstrate that $Y^+(P)$ is strictly decreasing in P for $P \in [0, P_0)$, that is over its positive range. Recall that equation (3.23) states that

$$Y^+(P) = (\frac{\partial}{\partial Z}Y_R)Z^+(P) + \frac{\partial}{\partial P}Y_R \qquad (A.1)$$

From Theorem 3.2 and 3.7, $\frac{\partial}{\partial Z}Y_R < 0$, while $Z^+(P) > 0$. Furthermore, since $k > 1$, we obtain from Lemma 3.3 that $\frac{\partial}{\partial P}Y_R$ is strictly decreasing in P for $P \in [0, P_0)$. This implies that for any $0 \le P_1 < P_2 < P_0$ we have $Y^+(P_2) < Y^+(P_1)$ whenever $Z^+(P_1) = 0$. Hence, in order to complete the proof we need to show that $Y^+(P)$ is strictly decreasing for those $P < P_0$ for which $Z^+(P) > 0$. Now, following the argument in Corollary 3.6, we conclude that $z_i(P) = Z(P)/n_3$, $i = 1,..., n_3$, for all $P \in [0, P_0)$, and that if $Z^+(P) > 0$ for some $P = \tilde{P} < P_0$, then in (3.22), $J_3(\tilde{P}) = \{1,..., n_3\}$. This along with (3.14), (3.15) and (3.23), gives for $P \in [0, P_0)$. (for the sake of notation we henceforth omit all the arguments):

$$Z^+(P) = -n_2\frac{p'C}{(p')^2M - D^2Vh''} \qquad (A.2)$$

$$Y^+(P) = n_2\frac{p'L - D^2h''}{(p')^2M - D^2Vh''} \qquad (A.3)$$

$$Q^{+}(P) = n_2 \frac{(n_2 + k)p'Q^2 - D^2h''}{(p')^2M - D^2Vh''} \tag{A.4}$$

where

$$V = (n_2 + 1)p' + p''Y \tag{A.5}$$

$$L = C + (n_2 + k)Q^2 \tag{A.6}$$

$$M = C + (n_2 + k)DQ \tag{A.7}$$

$$C = n_3D^2 + (k - 1)Z(D - Z) \tag{A.8}$$

$$D = (n_2 + k)Q - (k - 1)Z \tag{A.9}$$

and where $Q=Y+Z$.

Letting $G(P)$ denote the expression on the right-hand side of (A.3) we need to show that $G^{+}(P) < 0$. Toward this end, denote the denominator in (A.3) by $\omega(P)$. Then, by taking the right-hand derivative of $G(P)$ we get

$$\frac{\omega^2}{n_2}G^{+}(P) = (p')^2M(p''Q^{+}L + p'L^{+}) - p'L[2p'p''Q^{+}M + (p')^2M^{+}]$$

$$- D^2Vh''(p''Q^{+}L + p'L^{+}) + p'L(2DD^{+}Vh'' + D^2V^{+}h'' + \frac{1}{n_3}D^2Vh'''Z^{+})$$

$$- [(p')^2M - D^2Vh''](2DD^{+}h'' + \frac{1}{n_3}D^2h'''Z^{+})$$

$$+ D^2h''[2p'p''Q^{+}M + (p')^2M^{+} - 2DD^{+}Vh'' + D^2V^{+}h'' - \frac{1}{n_3}D^2Vh'''Z^{+}]$$

Denote by K the sum of the first two terms on the right-hand side above, that is

$K = (p')^2[M(p''Q^+L + p'L^+) - L(2p''Q^+M + p'M^+)]$,

and by H the sum of the remaining terms. In order to demonstrate that $G^+(P) < 0$ it suffices to show that $H \leq 0$ and $K < 0$. In the proof we will employ the equations below, which readily follow from the assumed form of the demand function $p(Q) = a - bQ^k$ and expressions for V and D introduced in (A.5) and (A.9):

$$(k - 1)p' = p''Q \tag{A.10}$$

$$(k - 1)V = p''D \tag{A.11}$$

$$(k - 1)V^+ = p''D^+ + p'''Q^+D \tag{A.12}$$

$$V^+ = - bk(k - 1)Q^{k-3}[(n_2 + k - 1)Q^+Y + (n_2 + 1)Q^+Z + QY^+] \tag{5.12a}$$

$$(k - 1)^2p'V^+ = (p'')^2[(k - 2)Q^+D + QD^+]. \tag{A.13}$$

First, we will show that $H < 0$. By rearranging the terms we get

$$H = - D^4(h'')^2V^+ + \frac{1}{n_3}p'D^2h'''Z^+(LV - p'M) + D^2h''(- p''Q^+LV + p'LV + 2p'p''Q^+M)$$

$$- p'Dh''(DVL^+ - 2LD^+V + 2p'D^+M - p'DM^+).$$

From (A.12a), since $k > 1$, $, Q^+ < 0$, $Y^+ < 0$, we have $V^+ > 0$. Hence, the first term in the above expression of H is nonpositive. Therefore, since $p' < 0$, $Z^+ > 0$, $h'' \geq 0$, $h''' \leq 0$ (the last result follows from the assumed concavity of $h'(\cdot)$, it remains to show that the bracketed terms, denote them by B_1, B_2, B_3 are all nonpositive. Consider B_1 first. By (A.10) and (A.11) we get $B_1 = LV - p'M = \frac{1}{k-1}p''(LD - QM)$, and further, by (A.6), (A.7), $B_1 = p''C(D - Q)$. Readily, $B_1 < 0$, since $p'' < 0$ and $D > Q$. Let us now proceed to B_2. From equations (A.10), (A.11) and (A.13) we obtain that $B_2 = - p''Q^+LV^+ + p'LV + 2p'p''Q^+M = \frac{1}{(k-1)^2}(p'')^2$ $[- (k - 1)Q^+LD + (k - 2)Q^+DL + QD^+L + 2(k - 1)QQ^+M]$. But, $QQ^+M < 0$, since

$Q^+ < 0$, and therefore, $(k-1)^2 B_2 < (p'')^2 L(QD^+ - Q^+D)$. From equation (A.9) we get $QD^+ - Q^+D = (k-1)(ZQ^+ - QZ^+)$ which along with $Q^+ < 0$ and $Z^+ > 0$ gives

$$QD^+ - Q^+D < 0 \tag{A.14}$$

thus establishing $B_2 < 0$. Now, let us show that $B_3 < 0$. Again, using (A.10) and (A.11) one obtains

$$B_3 = DVL^+ - 2LD^+V + 2p'D^+M - p'DM^+ = \frac{1}{k-1}p''(D^2L^+ - 2LD^+D^2 + 2QD^+M - QDM^+) ,$$

and further, by rearranging the terms, $(k-1)B_3 = p''[2D^+(MQ - LD) + D(DL^+ - QM^+)]$. From (A.16) and (A.7), $MQ - LD = C(Q - D)$, and $DL^+ - QM^+ = C^+(D - Q) + (n_2 + k)Q(DQ^+ - D^+Q)$, which along with (A.14) and $p'' < 0$ gives $(k-1)B_3 < p''(D - Q)(C^+D - 2D^+C)$. Observe that $C^+D - 2D^+C = (k-1)(D - 2Z)(DZ^+ - D^+Z)$. But, $D - 2Z > (n_2 + 1)Q - 2Z > 0$ since $n_2 \geq 1$, and $DZ^+ - D^+Z > 0$ since $Z^+ > 0$ and $D^+ < 0$. Therefore, $C^+D - 2D^+C > 0$ which along with $p'' < 0$ and $D > Q$ implies $B_3 < 0$, and hence $H < 0$.

Now, we proceed to establishing $K < 0$. First, notice that from (A.10) the expression K can be rewritten in the following way:

$$K = \frac{1}{k-1}(p')^2 p''[Q(L^+M - LM^+) - (k-1)Q^+LM].$$

But, $p'' < 0$ and therefore, $K < 0$ if and only if $Q(L^+M - LM^+) - (k-1)Q^+LM$ is positive. Toward this end, observe that from (A.8) and (A.9) we have

$$D^+ = (n_2 + k)Q^+ - (k-1)Z^+$$

$$C^+ = Q^+[2n_3(n_2 + k)D + (k-1)(n_2 + k)Z] + (k-1)Z^+[-2n_3D + D - 2Z - (k-1)Z].$$

By employing the above equalities in the expression for $L^+M - LM^+$ we obtain $Q(L^+M - LM^+) - (k-1)Q^+LM = K_1Q^+ + (n_2+k)K_2Q^2Z^+$, where

$$K_1 = (n_2+k)Q[2CQ + 2n_3(n_2+k)D^2Q + (k-1)(n_2+k)DQZ + 2(n_2+k)DQ^2$$

$$-CD - (n_2+k)DQ^2 - (n_2+k)CQ - (n_2+k)^2Q^3 - 2n_3(n_2+k)DQ^2$$

$$-(k-1)(n_2+k)Q^2Z]$$

$$-(k-1)[C^2 + (n_2+k)CQ^2 + (n_2+k)CDQ + (n_2+k)^2DQ^3],$$

and

$$K_2 = (k-1)[-2n_3D^2 + D(D-2Z) - (k-1)DZ + C + (n_2+k)Q^2 + 2n_3DQ$$

$$-Q(D-2Z) + (k-1)QZ].$$

Further, by equations (A.1) and (A.4) we can rewrite K in the following way:

$$K = \frac{n_3}{\omega}[(n_2+k)p'Q \geq 0(K_1 - CK_2) - K_1D^2h''].$$

Below we show that $K_1 - CK_2 < 0$ and $K_2 < 0$. Then, we will be able to claim that $K < 0$ since $\omega > 0$ and $p' < 0$, $h'' \leq 0$. Toward this end,

$$K_1 = (n_2+k)Q\{2CQ + (n_2+k)Q[2n_3D^2 + (k-1)DZ] - (n_2+k)Q^2[2n_3D + (k-1)Z]$$

$$+ (n_2+k)DQ^2 - CD - (n_2+k)CQ - (n_2+k)^2Q^3\}$$

$$- (k-1)[C^2 + (n_2+k)CQ^2 + (n_2+k)CDQ + (n_2+k)^2DQ^3] <$$

$$(n_2+k)(k-1)CQ(Z-Q) - (k-1)[C^2 + (n_2+k)CDQ + (n_2+k)^2DQ^3].$$

The last inequality is valid through the following derivation:

$$2CQ + (n_2+k)Q[2n_3D^2 + (k-1)DZ] - (n_2+k)Q^2[2n_3D + (k-1)Z]$$

$$+ (n_2+k)DQ^2 - CD - (n_2+k)CQ - (n_2+k)^2Q^3 =$$

$$2CQ + (n_2+k)Q[2C - (k-1)Z(D-2Z)] - (n_2+k)Q^2[2n_3D + (k-1)Z]$$

$$+ (n_2+k)Q^2[D - (n_2+k)Q] - CD - (n_2+k)CQ =$$

$$2CQ + (k-1)CZ - (n_2+k)(k-1)QZ(D-2Z) - 2n_3(n_2+k)Q^2D - 2(n_2+k)(k-1)Q^2Z$$

$$< (k-1)CZ + 2Q[C - n_3(n_2+k)DQ]$$

$$= (k-1)CZ + 2Q[-n_3(k-1)ZD + (k-1)Z(D-Z)]$$

$$= (k-1)CZ + 2(k-1)QZ(-n_3D + D - Z) < (k-1)CZ.$$

The above inequality for K_1 demonstrates that $K_1 < 0$ and moreover, since $Z < Q$ that $K_1 < -(k-1)C^2$. Therefore, in order to show that $K_1 - CK_2 < 0$ it suffices to show that $K_2 + (k-1)C > 0$. But,

$$K_2 + (k-1)C = (k-1)\{2C + (D-2Z)(D-Q) - [2n_3D + (k-1)Z]D$$

$+ (n_2 + k)Q^2 + [2n_3D + (k - 1)Z]Q\}$

$= (k - 1)\{2C + (D - 2Z)(D - Q) - [2C - 2(k - 1)Z(D - Z) + (k - 1)ZD]$

$+ (n_2 + k)Q^2 + [2n_3D + (k - 1)Z]Q\}$

$> (k - 1)Z(2D - 2Z - D) > 0.$

Thus, $K_2 + (k - 1)C > 0$ which gives $K_1 - CK_2 < 0$ and the proof is complete. ■

A.2. Derivation od Results in the Last Column of Table 1 and of Table 2

In this section results in the last column of Table 1 and that of Table 2 are derived. Let us recall that subsrcipts A and B refer to "after" and "before" integration, respectively. That is, for the considered here vertical integration, B refers to n_1, n_2 and n_3 firms in S_1, S_2 and S_3, while A refers to $n_1 - 1$, $n_2 - 1$ and $n_3 + 1$ firms in S_1, S_2 and S_3. All n_1, n_2 and n_3 are positive integers.

A.2.1.Derivation of Results in the Last Column of Table 1

Recall that in this case we are using $d_1 = d_2 = d_3 = 0$.

ROW 1. From (5.11), using (5.7), (5.9) and $d_1 = d_2 = d_3 = 0$, we obtain

$$P_A^o - P_B^o = (a - c_3)\frac{(n_1 + 1)(n_3 + 1) - n_1(n_3 + 2)}{n_1(n_1 + 1)(n_3 + 1)(n_3 + 2)} \leq 0 \text{ if and only if } n_1 \geq n_3 + 1.$$

ROW 2. From (5.11), again using $d_1 = d_2 = d_3 = 0$, we get

$$Y_A^o - Y_B^o = \frac{a - c_3}{b(n_2 + n_3 + 1)}[\frac{(n_1 - 1)(n_2 - 1)}{n_1} - \frac{n_1n_2}{(n_1 + 1)}] = \frac{(a - c_3)(1 - n_1^2 - n_2)}{bn_1(n_1 + 1)(n_2 + n_3 + 1)}$$

which readily yields $Y_A^o - Y_B^o < 0$.

ROW 3. From (5.11), using (5.5), (5.7) and (5.9) and $d_1 = d_2 = d_3 = 0$, we get

$$Q_A^o - Q_B^o = \frac{a - c_3}{b(n_2 + n_3 + 1)}\left[\frac{(n_1 - 1)(n_2 - 1) + n_1(n_3 + 1)(n_2 + n_3 + 1)}{n_1(n_3 + 2)}\right.$$

$$\left. - \frac{n_1 n_2 + n_3(n_1 + 1)(n_2 + n_3 + 1)}{(n_1 + 1)(n_3 + 1)}\right].$$

The numerator of the term in $[\cdot]$ above simplifies to $n_1(n_2 + n_3 + 1) - (n_3 + 1)(n_2 - 1)$. Hence, in particular, if $n_1 \geq \min\{n_2 - 1, n_3 + 1\}$, then $Q_A^o - Q_B^o > 0$.

ROW 4. From (5.12), we obtain

$$\pi_{1A}^o - \pi_{1B}^o = \frac{D_O}{(n_2 + n_3 + 1)}\left[\frac{(n_2 - 1)}{n_1^2(n_3 + 2)} - \frac{n_2}{(n_1 + 1)^2(n_3 + 1)}\right]$$

$$= D_O\frac{n_2(2n_1 + 1)(n_3 + 1) - n_2 n_1^2 - (n_1 + 1)^2(n_3 + 1)}{(n_2 + n_3 + 1)n_1^2(n_1 + 1)^2(n_3 + 1)(n_3 + 2)}$$

Now, from the numerator in $[\cdot]$ above, if $n_1 \geq 2n_2 - 1$, then $n_2(2n_1 + 1)(n_3 + 1) - (n_1 + 1)^2(n_3 + 1) \leq (n_3 + 1)[n_2(2n_1 + 1) - 2n_2(n_1 + 1)] = - n_2(n_3 + 1) < 0$, and so $\pi_{1A}^o - \pi_{1B}^o < 0$. Similarly, if $n_1 \geq 2n_3 + 3$, then $n_2(2n_1 + 1)(n_3 + 1) - n_2 n_1^2 \leq n_2(2n_1 + 1)(n_3 + 1) - n_1 n_2(2n_3 + 3) = n_2[(2n_3 + 3 - n_1) - (n_3 + 2)] < 0$. Hence, if $n_1 \geq \min\{2n_2 - 1, 2n_3 + 3\}$, then $\pi_{1A}^o - \pi_{1B}^o < 0$.

ROW 5. Again from (5.12), we get

$$\pi_{2A}^o - \pi_{2B}^o = \frac{D_O}{(n_2 + n_3 + 1)^2}\left[\frac{(n_1 - 1)^2}{n_1^2} - \frac{n_1^2}{(n_1 + 1)^2}\right] = \frac{D_O(1 - 2n_1^2)}{n_1^2(n_1 + 1)^2(n_2 + n_3 + 1)^2} < 0.$$

ROW 6. For the firms in S_3, we get from (5.12) that

$$\pi_{3A}^o - \pi_{3B}^o = D_O\frac{n_1 n_2(n_3 + 2) + (n_2 - 1)(n_1 + 1)(n_3 + 1)}{n_1(n_2 + n_3 + 1)^2(n_1 + 1)(n_3 + 1)(n_3 + 2)}\left[\frac{n_2 - 1}{n_1(n_3 + 2)} - \frac{n_2}{(n_1 + 1)(n_3 + 1)}\right].$$

The numerator of the term in [·] above simplifies to $n_2(n_3 + 1) - (n_1 + 1)(n_3 + 1) - n_1n_2$ which is negative from the first two terms if $n_1 + 1 \geq n_2$, or from the first and third terms if $n_1 \geq n_3 + 1$. Hence, if $n_1 \geq \min\{n_2 - 1, n_3 + 1\}$, we have $\pi^o_{3A} - \pi^o_{3B} < 0$.

ROW 7. From (5.12), we obtain

$$\pi^o_{3A} - (\pi^o_{1A} + \pi^o_{2A}) =$$

$$D_O\frac{(n_1 + 1)^2(n_3 + 1)[n_1(n_3 + 2) + n_2 - 1]^2 - n_1^2n_2(n_3 + 2)^2(n_2 + n_3 + 1) - n_1^4(n_3 + 1)(n_3 + 2)^2}{n_1^2(n_1 + 1)^2(n_2 + n_3 + 1)^2(n_3 + 1)(n_3 + 2)^2}.$$

Since $n_1(n_3 + 2) + n_2 - 1 \geq n_1(n_3 + 2)$, the numerator above, T, say, satisfies $T \geq n_1^2(n_1 + 1)^2(n_3 + 1)(n_3 + 2)^2 - n_1^2(n_3 + 2)^2[n_2(n_2 + n_3 + 1) + n_1^2(n_3 + 1)]$, and further,

$$T \geq n_1^2(n_3 + 2)^2[(n_3 + 1)(2n_1 + 1) - n_2(n_2 + n_3 + 1)]$$

and hence, $T \geq n_1^2(n_3 + 2)^2\ [(n_3 + 1)(n_1 - n_2) + n_1(n_3 + 1) - n_2^2 + (n_3 + 1)] > 0$ whenever, $n_2 \leq n_1$ and $n_2 \leq n_3 + 1$. Therefore, if $n_2 \leq \min\{n_1, n_3 + 1\}$, then $\pi^o_{3A} - (\pi^o_{1A} + \pi^o_{2A}) > 0$, and this completes the derivation of results in the last column of Table 1. ∎

A.2.2.Derivation of Results in the Last Column of Table 2

ROW 1. From the development in Section 5.2, the equilibrating price P^o is given by $P^o = c_3 - c_2 + (a - c_3)/N(n_1 + 1)$ where $N = n_2n_3 + n_3 + 1$. Therefore, the sign of the difference $P^o_A - P^o_B$ is the same as that of $(n_1 + 1)N - n_1[1 + n_2(n_3 + 1)]$. From this, it can be easily verified that if $n_3 \geq n_1$ or $n_3 \geq n_2$ then $P^o_A > P^o_B$ thus completing this proof.

ROW 2. From (5.22) we have

$$Y^o_A - Y^o_B = \frac{a - c_3}{b}\left[\frac{(n_1 - 1)(n_2 - 1)}{n_1n_2(n_3 + 2)} - \frac{n_1n_2}{(n_1 + 1)(n_2 + 1)(n_3 + 1)}\right].$$

The numerator of the expression in $[\cdot]$ simplifies to $-n_1n_2^2 - (n_3 + 1)(n_1^2 + n_2^2 - 1)$ which is negative since $n_1 \geq 2$ and $n_2 \geq 2$. Hence, $Y_A^o - Y_B^o < 0$.

ROW 3. From (5.24), we get

$$Q_A^o - Q_B^o = \frac{a - c_3}{b}\left[\frac{(n_1 - 1)(n_2 - 1) + n_1n_2^2(n_3 + 1)(n_3 + 2)}{n_1n_2(n_3 + 2)[1 + n_2(n_3 + 1)]}\right.$$

$$\left. - \frac{n_1n_2 + n_3(n_1 + 1)(n_2 + 1)^2(n_3 + 1)}{(n_1 + 1)(n_2 + 1)(n_3 + 1)[1 + n_3(n_2 + 1)]}\right].$$

The numerator T of the term in $[\cdot]$ can be presented in the following way:

$$T = M[-n_1^2n_2^2 - (n_3 + 1)(n_1^2 + n_2^2 - 1) + n_1n_2(n_1 + 1)(n_2 + 1)(n_3 + 1)(n_3 + 2)(n_2 - n_3)]$$

$$+ (n_2 - n_3)(n_1 + 1)(n_2 + 1)(n_3 + 1)[(n_1 - 1)(n_2 - 1) + n_1n_2^2(n_3 + 1)(n_3 + 2)],$$

where $M = 1 + n_2(n_3 + 1)$. Using the equation above it can be easily verified that if $n_2 \leq n_3$ then $T < 0$ and so $Q_A^o - Q_B^o < 0$. On the other hand, if $n_2 > n_3$ then the second term in $[\cdot]$ in the expression for T is positive, and furthermore, since $n_2 \geq n_3 + 1$, we obtain $T > (n_3 + 1)M[n_1n_2(n_1 + 1)(n_2 + 1)(n_3 + 2) - (n_1^2 + 1)(n_2^2 + 1)] > 0$, and hence $Q_A^o - Q_B^o > 0$ if $n_2 > n_3$.

For the purpose of demonstrating the results in rows 4-7, let us present the expressions for individual firm profits π_i^o for the firms in S_i, $i = 1, 2, 3$:

$$\pi_1^o = D_O\frac{n_2}{(n_1 + 1)^2(n_2 + 1)(n_3 + 1)N} \tag{A.15}$$

$$\pi_2^o = D_O[\frac{n_1}{(n_1 + 1)(n_2 + 1)(n_3 + 1)}]^2 \tag{A.16}$$

$$\pi_3^o = D_O\frac{1}{(n_2 + 1)}[\frac{n_2 + N(n_1 + 1)}{(n_1 + 1)(n_3 + 1)N}]^2 \tag{A.17}$$

where $D_O = (a - c_3)^2/b$, and $N = 1 + n_3(n_2 + 1)$.

ROW 4. From (A.15) we have that

$$\pi^o_{1A} - \pi^o_{1B} = D_O[\frac{n_2 - 1}{n_1^2 n_2(n_3 + 1)[1 + n_2(n_3 + 1)]} - \frac{n_2}{(n_1 + 1)^2(n_2 + 1)^2(n_3 + 1)N}].$$

Let T denote the numerator of the expression in $[\cdot]$. Upon simplification we get

$$T = (n_1 + 1)^2(n_2^2 - 1)(n_3 + 1) - n_1^2 n_2^2(n_3 + 1) - n_1^2 n_2^2 - 2n_1^2 n_2^3(n_3 + 1)$$

$$+ n_3(n_3 + 1)[n_1^2(n_2^2 - n_2 - 1) + (2n_1 + 1)(n_2^2 - 1)(n_2 + 1)].$$

Since $n_2 \geq 2$, we have $n_2^2 - n_2 - 1 \geq 1$, which upon simple reduction gives

$$T \geq (n_3 + 1)[n_1^2 + 2(n_1 + 1)(n_2^2 - 1) + n_1^2 n_3 + n_3(n_2^2 - 1)(n_2 + 1)] - n_1^2 n_2^2,$$

and further, with $n_3 \geq n_1$, $T > n_1^2(n_2^2 - n_2 - 1)$. The last result yields $T > 0$, since by assumption $n_2 \geq 2$. Thus indeed, $n_3 \geq n_1$ implies that $\pi^o_{1A} - \pi^o_{1B} > 0$. For the case of $2n_3 \leq \min\{2(n_2 - 1), n_1 - 3\}$, let us rewrite the expression T in the following way:

$$T = M\{(n_1 + 1)[(n_1 + 1)(n_2^2 - 1)(n_3 + 1) - n_2^2(n_1 - 1)(n_3 + 2)] - n_2^2(n_3 + 2)\}$$

$$+ (n_3 - n_2)(n_1 + 1)^2(n_2^2 - 1)(n_3 + 1),$$

where $M = 1 + n_2(n_3 + 1)$. If $n_2 \geq n_3 + 1$, then from the equation above, we get

$$T \leq M\{(n_1 + 1)[(n_1 + 1)(n_2^2 - 1)(n_3 + 1) - n_2^2(n_1 - 1)(n_3 + 2) - (n_1 + 1)(n_2 - 1)] - n_2^2(n_3 + 2)\}$$

$$- n_3(n_1 + 1)^2(n_2 - 1).$$

In order to show that $T < 0$ it suffices to demonstrate that the term in $[\cdot]$ on the right-hand side of the last inequality is nonpositive. Toward this end, observe that this term is less that $n_2^2[(n_1 + 1)(n_3 + 1) - (n_1 - 1)(n_3 + 2)]$, which in turn is negative if $n_1 \geq 2n_3 + 3$. Summarizing, we showed that if $n_2 \geq n_3 + 1$ and simultaneously, if $n_1 \geq 2n_3 + 3$, then $\pi^o_{1A} - \pi^o_{1B} < 0$. This completes the derivation of the results in row 4.

ROW 5. From (A.16) the sign of $\pi^o_{2A} - \pi^o_{2B}$ is the same as that of $(n_2^2 - 1)(n_2 + 1)(n_3 + 1) - n_1^2 n_2(n_3 + 2)$, being negative if $n_2 \geq n_3 + 1$.

ROW 6. From (A.17), it can be easily verified that the sign of $\pi^o_{3A} - \pi^o_{3B}$ is the same as that T, where

$$T = (n_1 + 1)^2(n_2^2 - 1)(n_3 + 1)^2 N^2[(n_2 - 1) + 2n_1 M] - [n_1 n_2(n_3 + 2)M]^2[n_2 + 2(n_1 + 1)N]$$

$$+ [n_1(n_1 + 1)NM]^2[(n_1 + 1)(n_3 + 1) - n_2(n_3 + 2)^2],$$

and where, as before, $N = n_3(n_2 + 1) + 1$, $M = 1 + n_2(n_3 + 1)$. Observe that if $n_2 \geq n_3 \geq 1$ then $(n_2 + 1)(n_3 + 1) - n_2(n_3 + 2)^2 < 0$ and also $M \geq N$. Therefore,

$$T < 2n_1(n_1 + 1)N^2 M[(n_1 + 1)(n_2^2 - 1)(n_3 + 1)^2 - n_1 n_2^2(n_3 + 2)^2]$$

$$+ N^2[(n_1 + 1)^2(n_2^2 - 1)(n_2 - 1)(n_3 + 1)^2 - n_1^2 n_3^3(n_3 + 2)^2].$$

Consider the expression T_1, say, in the first term $[\cdot]$. Upon simplification, $T_1 = - n_1 n_2^2(2n_3 + 3) + (n_2^2 - n_1 - 1)(n_3 + 1)^2$, and furthermore, if $n_1 \geq n_3$, then we obtain $T_1 \leq n_2^2(-n_3^2 - n_3 + 1) - (n_1 + 1)(n_3 + 1)^2$, and hence, $T_1 < - n_2^2 n_3^2$ if only $n_3 \geq 1$. Next, consider the second term $[\cdot]$, say, T_2, in the last inequality for T. Again, by rearranging the terms, we get $T_2 = n_2^3[(n_1 + 1)(n_3 + 1) + n_1(n_3 + 2)]$ $[(n_1 + 1)(n_3 + 1) - n_1(n_3 + 2)]$ $- (n_2^2 + n_2 - 1)(n_3 + 1)^2$. Moreover, $n_1 \geq n_3$ implies that $T_2 < n_2^3[(n_1 + 1)(n_3 + 1) + n_1(n_3 + 2)]$. Using the above bounds for T_1 and T_2, we get the following inequality for T:

$$T < - 2n_1 n_2^2 n_3^2(n_1 + 1)N^2 M + n_2^3 N^2[(n_1 + 1)(n_3 + 1) + n_1(n_3 + 2)],$$

which along with $M > n_2(n_3 + 1)$ gives

$$T < - n_2^3 N^2[2n_1 n_3^2(n_1 + 1)(n_3 + 1) - (n_1 + 1)(n_3 + 1) - n_1(n_3 + 2)].$$

From this one can easily verify that $T < 0$. Thus we showed that $1 \leq n_3 \leq \min\{n_1, n_2\}$ implies $\pi^o_{3A} - \pi^o_{3B} < 0$.

ROW 7. From (A.15). (A.16) and (A.17) we have

$$\pi^o_{3A} - (\pi^o_{1A} + \pi^o_{2A}) = D_O \;\frac{1}{n_2}\Big[\frac{n_2 - 1 + n_1 M}{n_1(n_3 + 2)M}\Big]^2 - \frac{n_2}{(n_1 + 1)^2(n_2 + 1)(n_3 + 1)N}$$

$$- \Big[\frac{n_1}{(n_1 + 1)(n_2 + 1)(n_3 + 1)}\Big]^2 ,$$

where $N = n_3(n_2 + 1) + 1$, $M = 1 + n_2(n_3 + 1)$. Denoting by T the numerator of the expression in $\{\cdot\}$, we get that $T = (n_1 + 1)^2(n_2 + 1)^2(n_2 - 1)(n_3 + 1)(n_2 - 1 + 2n_1M)N + n_1^2M^2\{N[(n_1 + 1)^2(n_2 + 1)^2(n_3 + 1)^2 - n_1^2n_2(n_3 + 2)^2] - n_2^2(n_2 + 1)(n_3 + 1)(n_3 + 2)^2\}$. To show that $T > 0$ it suffices demonstrate that the expression in $\{\cdot\}$, say, T_1 is positive. Observe that the term in $[\cdot]$ upon simplification is equal to $n_1^2[(n_2^2 + 1)(n_3 + 1)^2 + n_2(n_3^2 - 2)] + (2n_1 + 1)(n_2 + 1)^2(n_3 + 1)^2$, and hence for $n_3 \geq 2$ we get $T_1 > (n_3 + 1)^2N[(n_1 + 1)^2(n_2^2 + 1) + 2n_2(2n_1 + 1)] - n_2^2(n_2 + 1)(n_3 + 1)(n_3 + 2)^2$, which along with $N > n_3(n_2 + 1)$ and $(n_1 + 1)^2 \geq 9$, since $n_1 \geq 2$, gives $T_1 > (n_2 + 1)(n_3 + 1)[n_2^2(8n_3^2 + 5n_3 - 4) + 9n_3(n_3 + 1)] > 0$. Hence, if $n_3 \geq 2$, then $T_1 > 0$. On the other hand, if $n_3 = 1$ then $T_1 = N\{n_1^2[4(n_2^2 + 1) - n_2] + 4(2n_1 + 1)(n_1 + 1)^2\} - 18n_2^2(n_2 + 1)$, and from this, since $4(n_2^2 + 1) - n_2 \geq 3n_2^2 + 4$, and $N > n_2 + 1$, we obtain that $T_1 > (n_1 + 1)[4(8n_2^2 + 10n_2 + 9) - 18n_2^2] > 0$. Summarizing, $T_1 > 0$ if only $n_3 \geq 1$, and hence $n_3 \geq 1$ implies $\pi^o_{3A} > \pi^o_{1A} + \pi^o_{2A}$. Let us remark that if $n_3 = 0$, then it may turn out that $\pi^o_{3A} < \pi^o_{1A} + \pi^o_{2A}$, as for example, if $n_1 = 3$ and $n_2 = 10$. In contrast to this example, $n_1 = 3$, $n_2 = 3$, $n_3 = 0$ yields $\pi^o_{3A} > \pi^o_{1A} + \pi^o_{2A}$. This completes the derivation of results in the last column of Table 2. ■

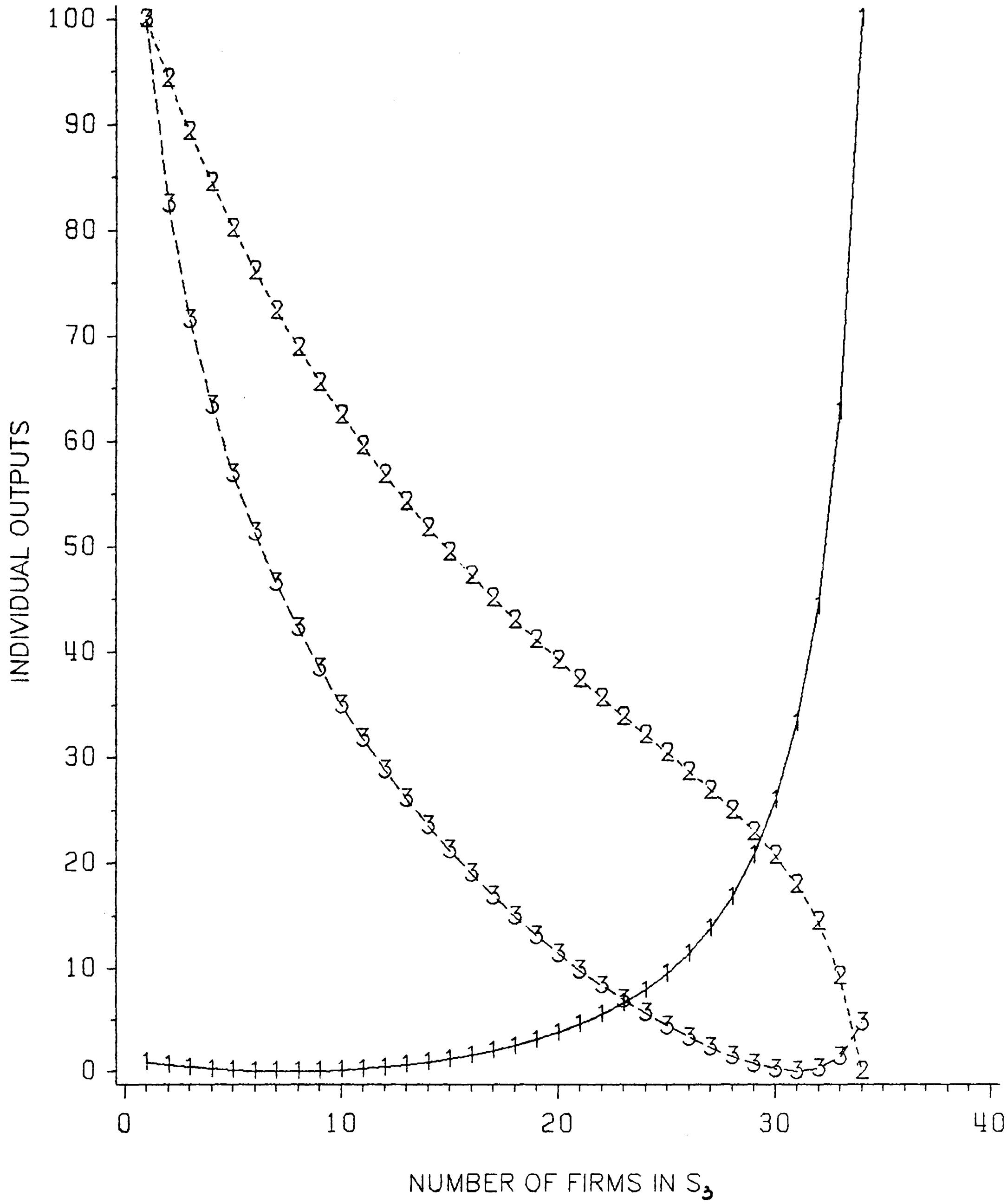
Appendix B

This Appendix contains figures illustrating various effects of vertical integrations in the follower-follower and leader-follower models discussed in Chapter 5. The market demand funcfion is linear, and so are all the cost functions, being identical within each set S_i, $i = 1, 2, 3$.

In comparison with results in Tables 1 and 2 (Chapter 5), the figures presented herein provide additional information on variations in individual firms' outputs and profits, total outputs and the input price at equilibrium. In particular, they demonstrate that vertical integration may result in a decrease of the total industry output, thus bringing losses to the final product consumers.

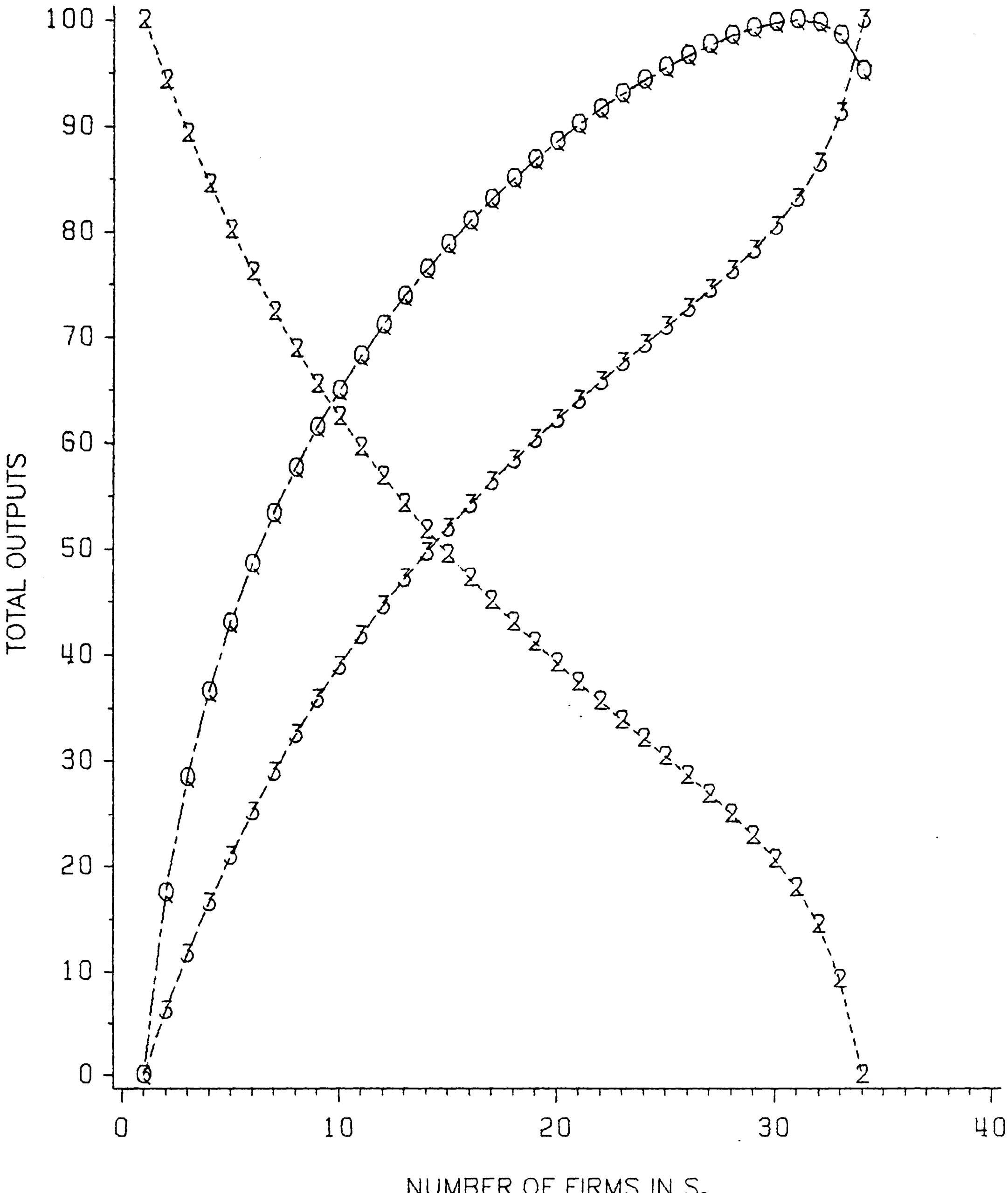
Each curve shows changes in a given quantity based on an individually scaled set of data.

Notationally, Case 4 (5) denotes a scenario of successive conversion of firms from the set S_1 (S_2) to firms of the type in S_3, and corresponds to the fourth (fifth) column in Tables 1 and 2. Case 6 relates to a scenario of successive merger of one firm in S_1 and one firm in S_2 to form a new vertically integrated firm in S_3. This case corresponds to the sixth column in Tables 1 and 2.



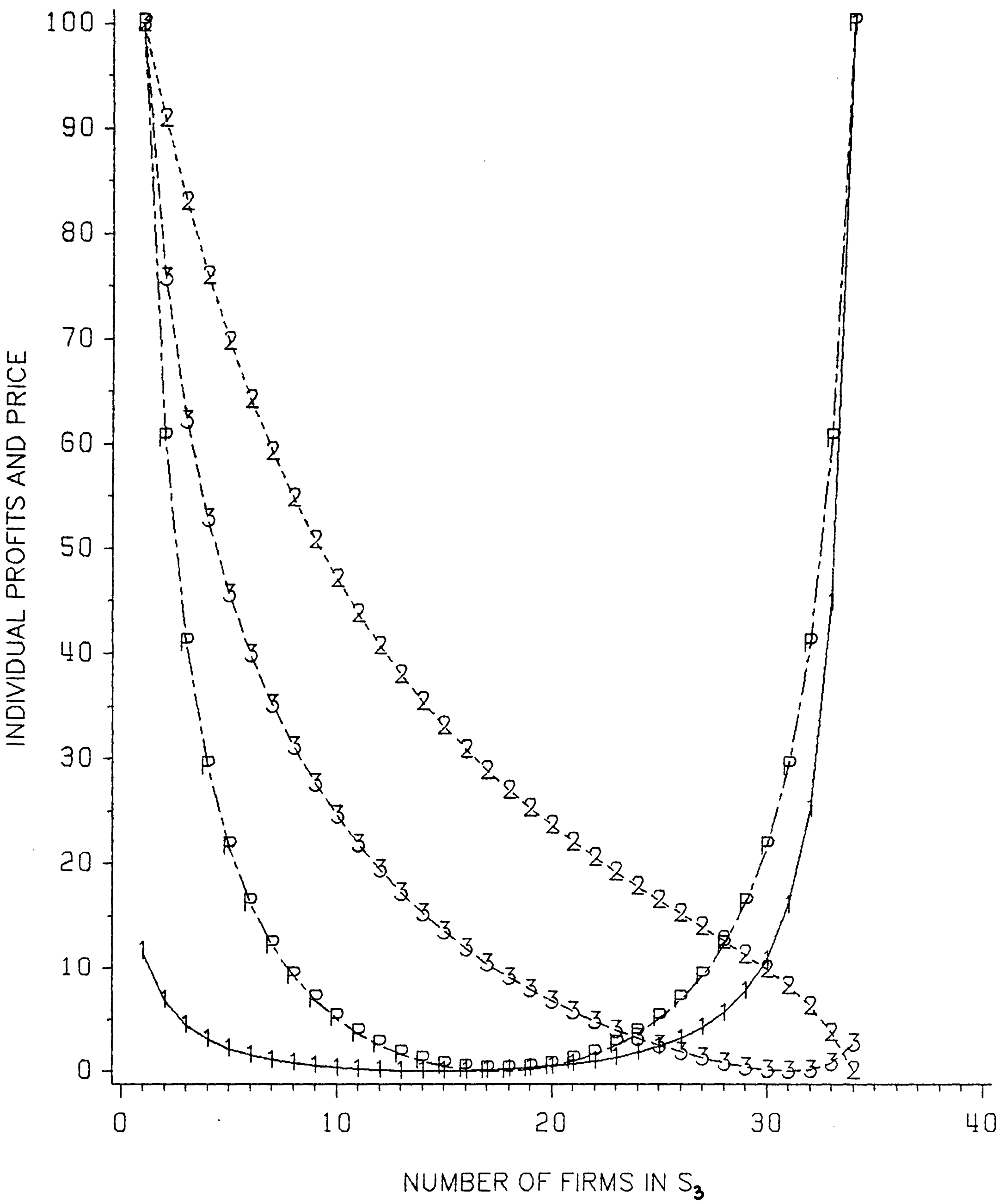
Legend: Symbol i refers to the firms in S_i, i = 1,2,3.

Figure 2. Vertical Integration in Follower-Follower Model, Case 4.: Changes in Firms' Individual Outputs. $n_1 = 35 - n_3$, $n_2 = 20$, $n_3 = 1, \ldots, 34$.



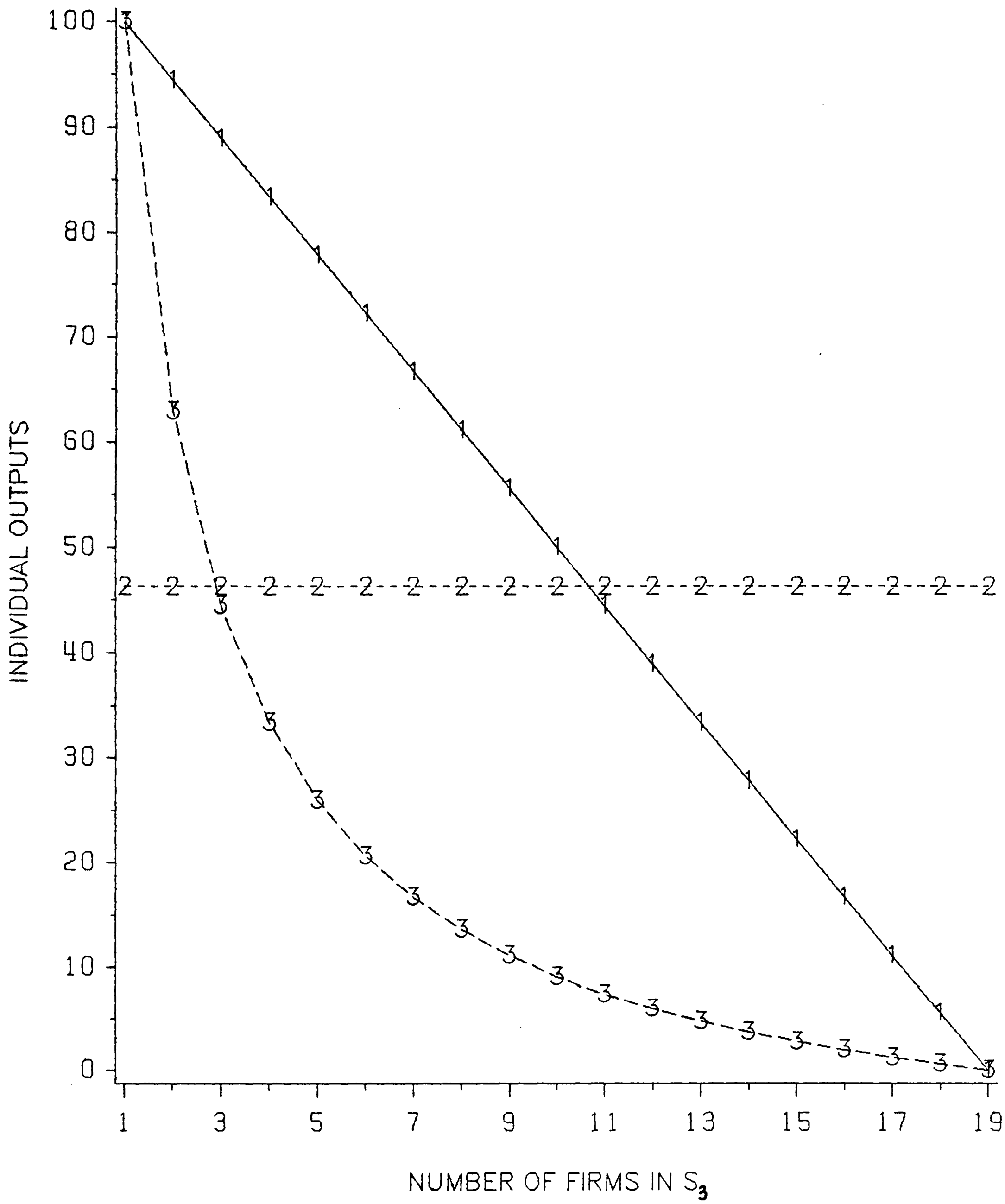
Legend: Symbol *i* refers to the firms in S_i, $i = 2,3$, *Q* refers to the total industry output.

Figure 3. Vertical Integration in Follower-Follower Model, Case 4.: Changes in Firms' Total Outputs. $n_1 = 35 - n_3$, $n_2 = 20$, $n_3 = 1, ..., 34$.



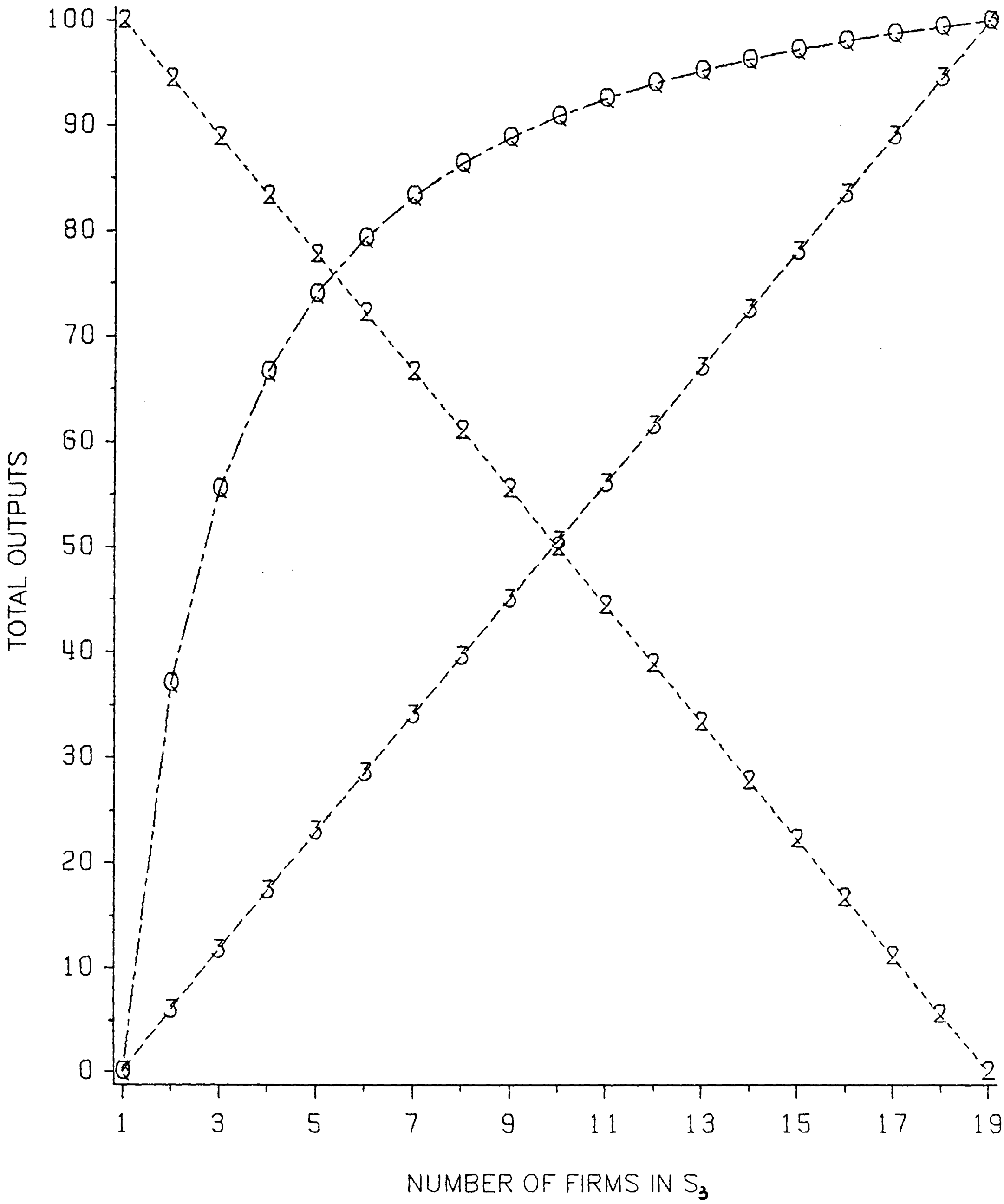
Legend: Symbol *i* refers to the firms in S_i, i = 1,2,3, symbol P refers to the input price.

Figure 4. Vertical Integration in Follower-Follower Model, Case 4.: Changes in Firms' Individual Profits and the Input Price. $n_1 = 35 - n_3$, $n_2 = 20$, $n_3 = 1, \ldots, 34$.



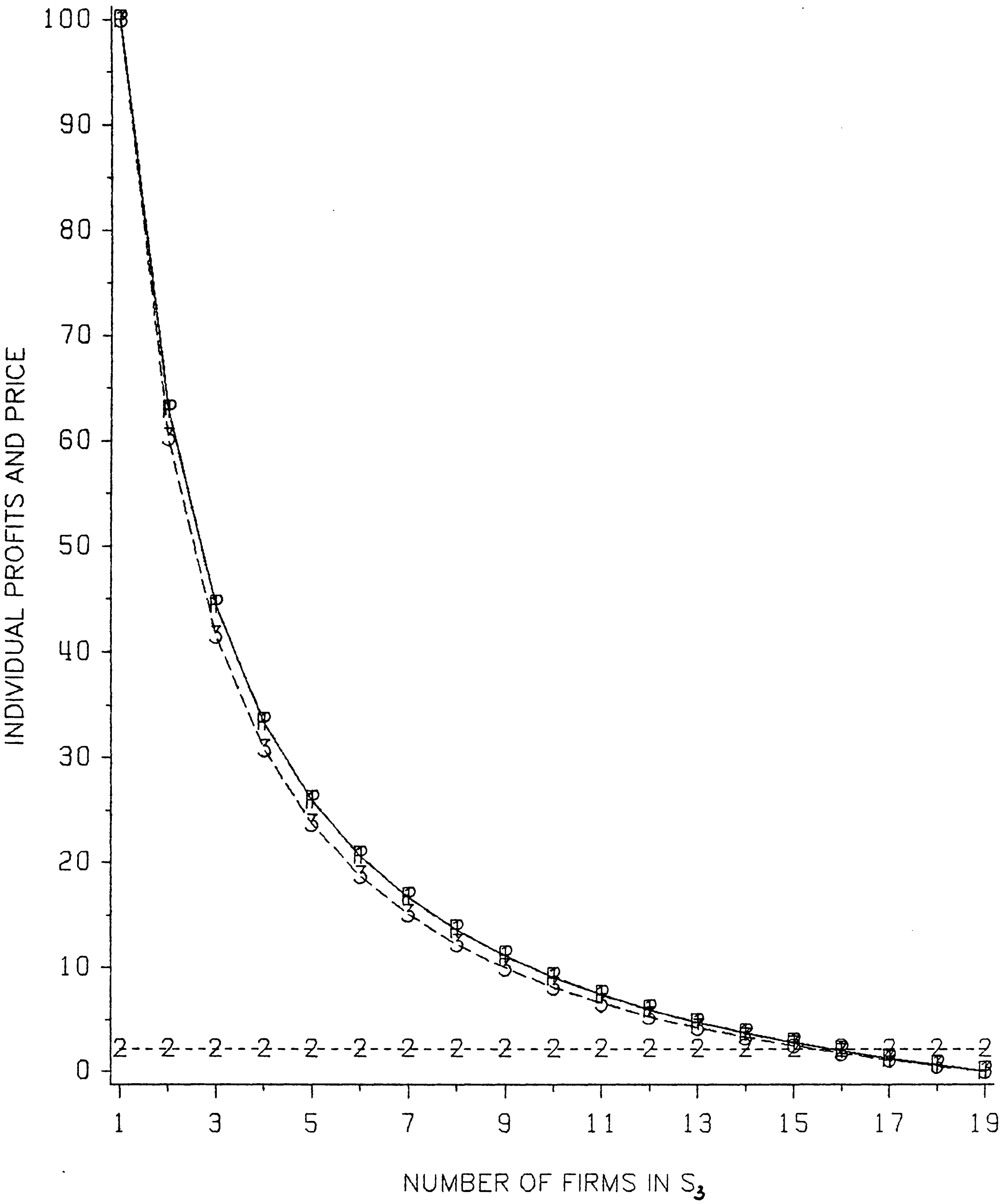
Legend: Symbol i refers to the firms in S_i, i = 1,2,3.

Figure 5. Vertical Integration in Follower-Follower Model, Case 5.: Changes in Firms' Individual Outputs. $n_1 = 35$, $n_2 = 20 - n_3$, $n_3 = 1, ..., 19$.



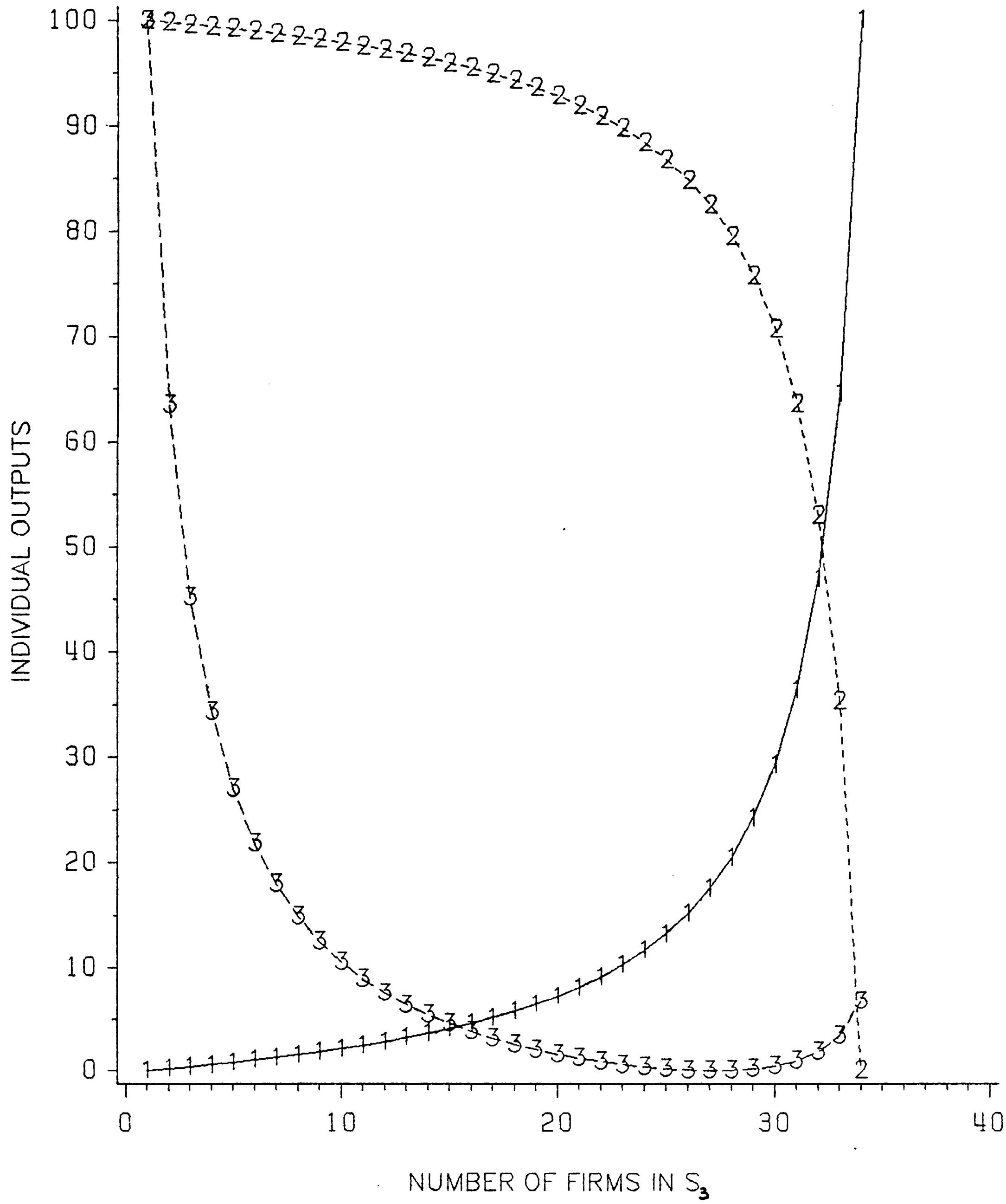
Legend: Symbol *i* refers to the firms in S_i, $i = 2,3$, Q refers to the total industry output.

Figure 6. Vertical Integration in Follower-Follower Model, Case 5.: Changes in Firms' Total Outputs. $n_1 = 35$, $n_2 = 20 - n_3$, $n_3 = 1, ..., 19$.



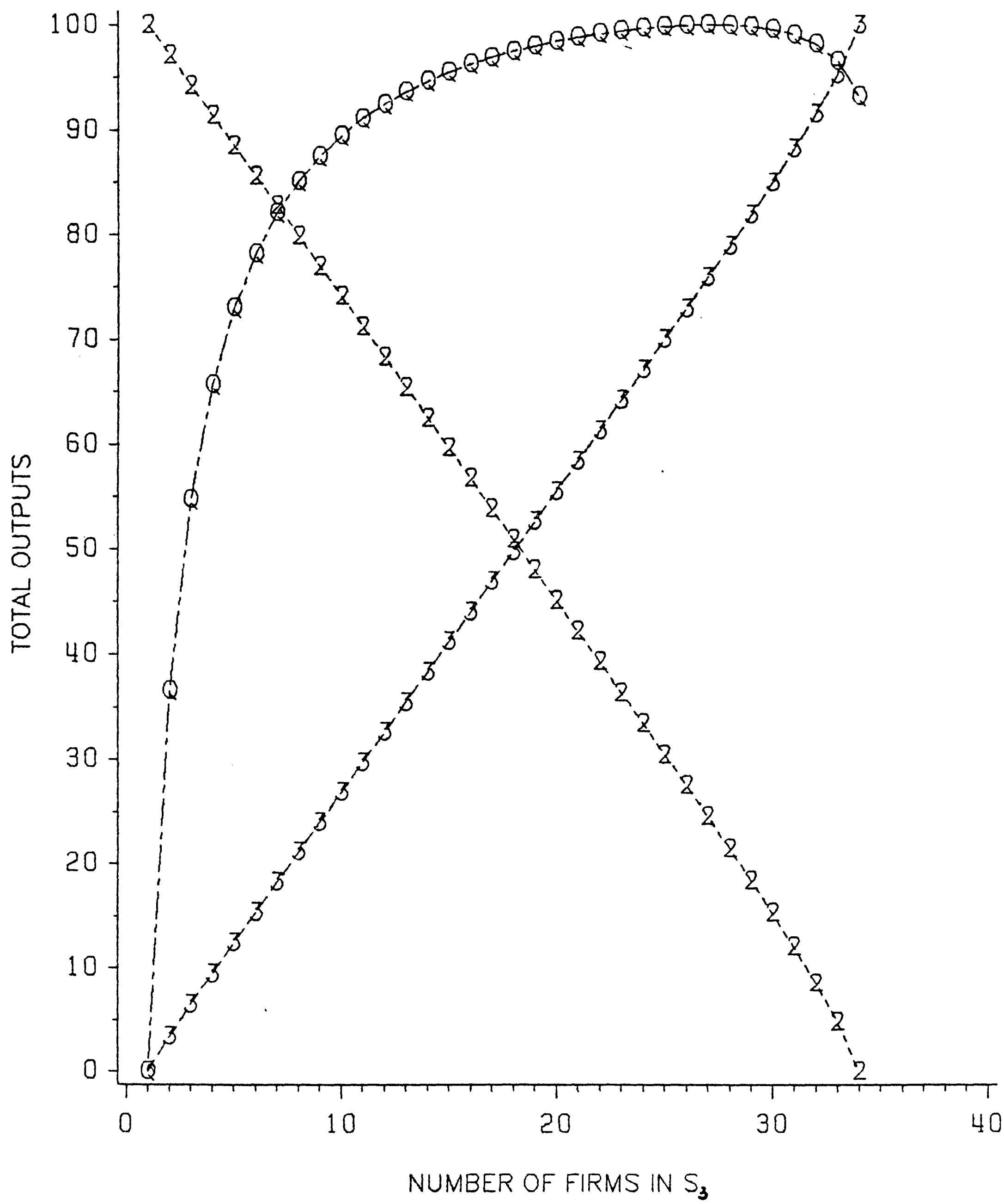
Legend: Symbol i refers to the firms in S_i, i = 1,2,3, symbol P refers to the input price.

Figure 7. Vertical Integration in Follower-Follower Model, Case 5.: Changes in Firms' Individual Profits and the Input Price. $n_1 = 35$, $n_2 = 20 - n_3$, $n_3 = 1, \ldots, 19$.



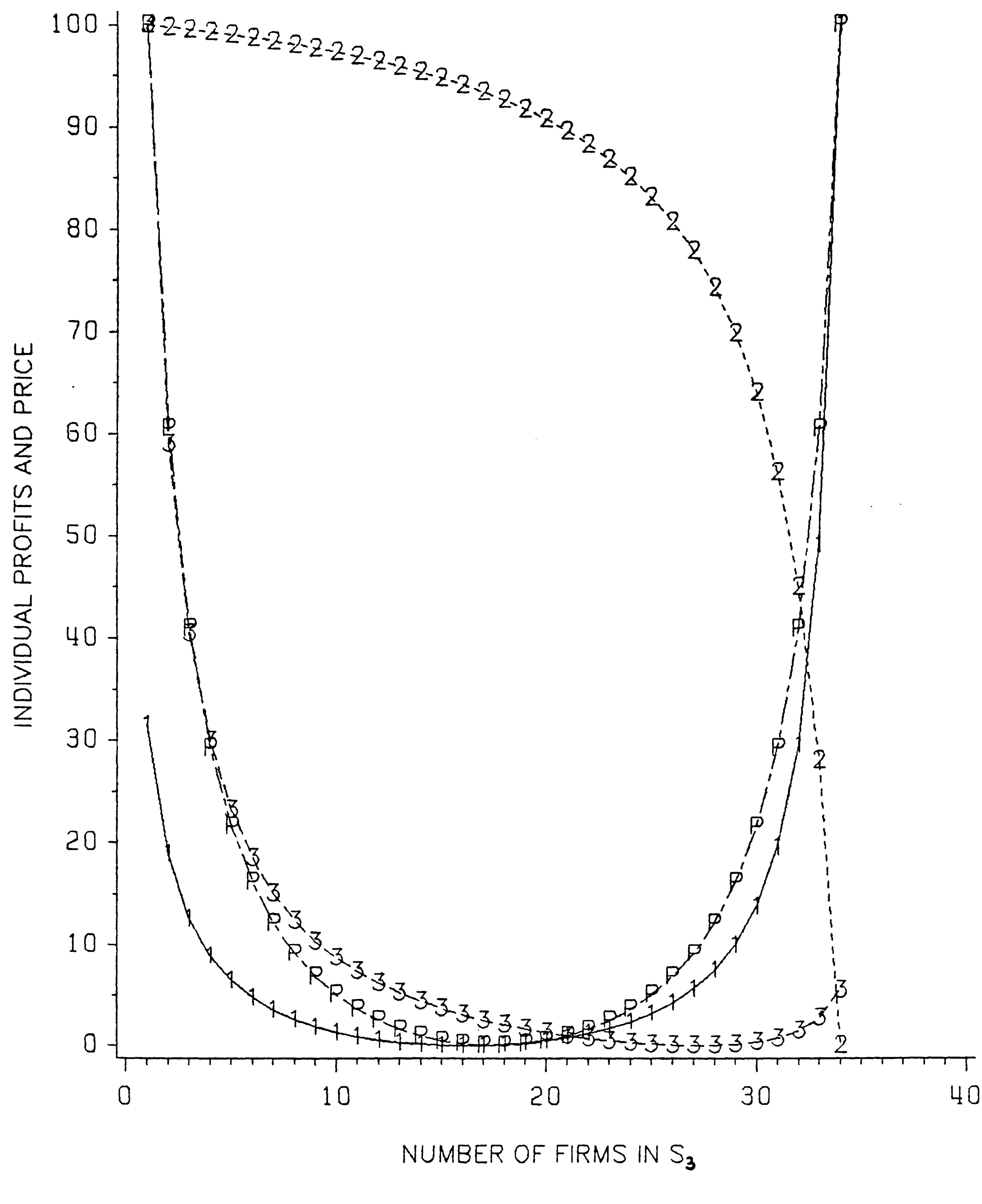
Legend: Symbol i refers to the firms in S_i, i = 1,2,3.

Figure 8. Vertical Integration in Follower-Follower Model, Case 6.: Changes in Firms' Individual Outputs. $n_1 = 35 - n_3$, $n_2 = 40 - n_3$, $n_3 = 1, ..., 34$.



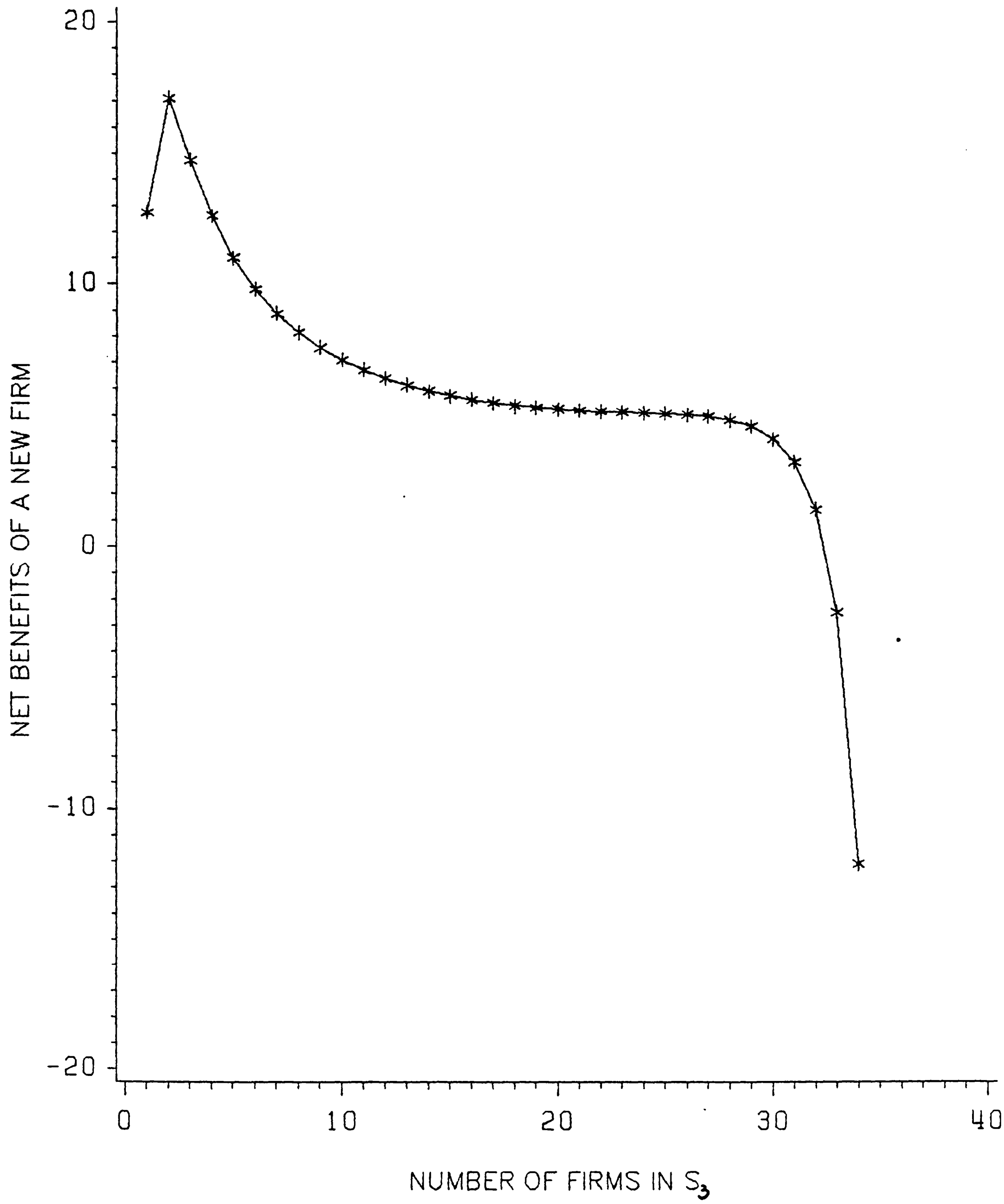
Legend: Symbol *i* refers to the firms in S_i, i = 2,3, Q refers to the total industry output.

Figure 9. Vertical Integration in Follower-Follower Model, Case 6.: Changes in Firms' Total Outputs. $n_1 = 35 - n_3$, $n_2 = 40 - n_3$, $n_3 = 1, ..., 34$.



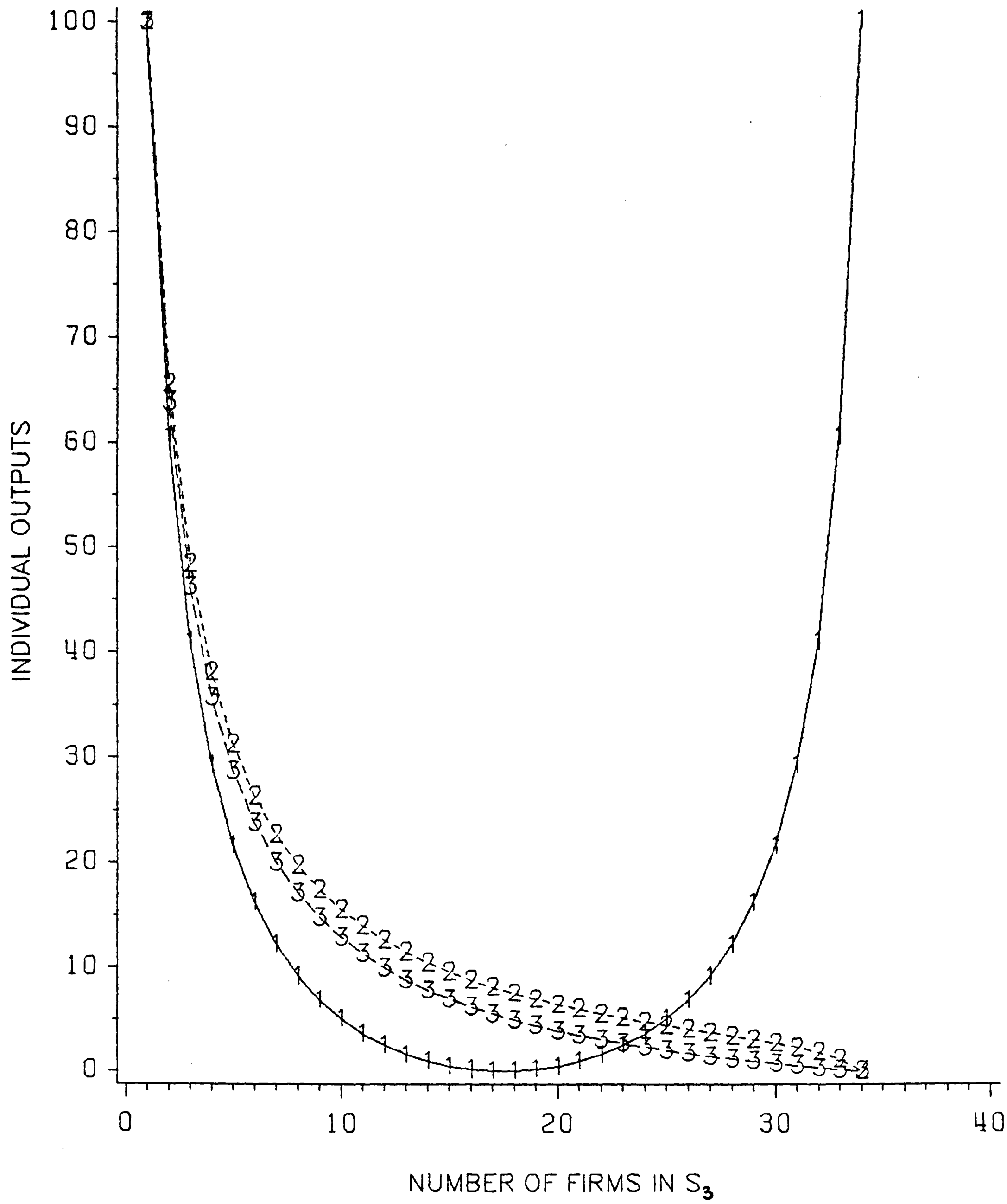
Legend: Symbol i refers to the firms in S_i, i = 1,2,3, symbol P refers to the input price.

Figure 10. Vertical Integration in Follower-Follower Model, Case 6.: Changes in Firms' Individual Profits and the Input Price. $n_1 = 35 - n_3$, $n_2 = 40 - n_3$, $n_3 = 1, \ldots, 34$.



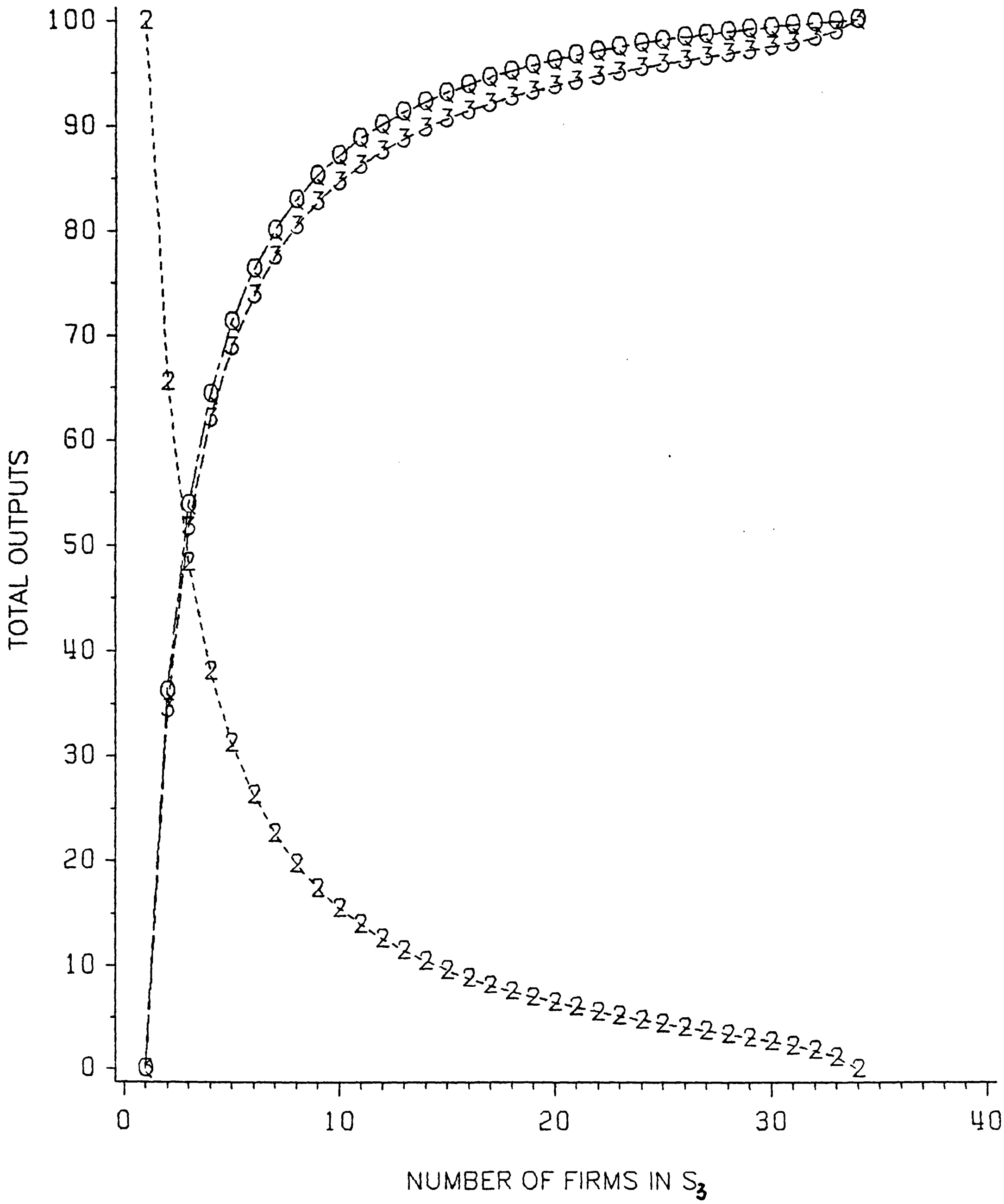
Legend: Symbol * refers to the net profit benefits of a new firm.

Figure 11. Vertical Integration in Follower-Follower Model, Case 6.: Net Profit Benefits of a New Firm. $n_1 = 35 - n_3$, $n_2 = 40 - n_3$, $n_3 = 1, ..., 34$.



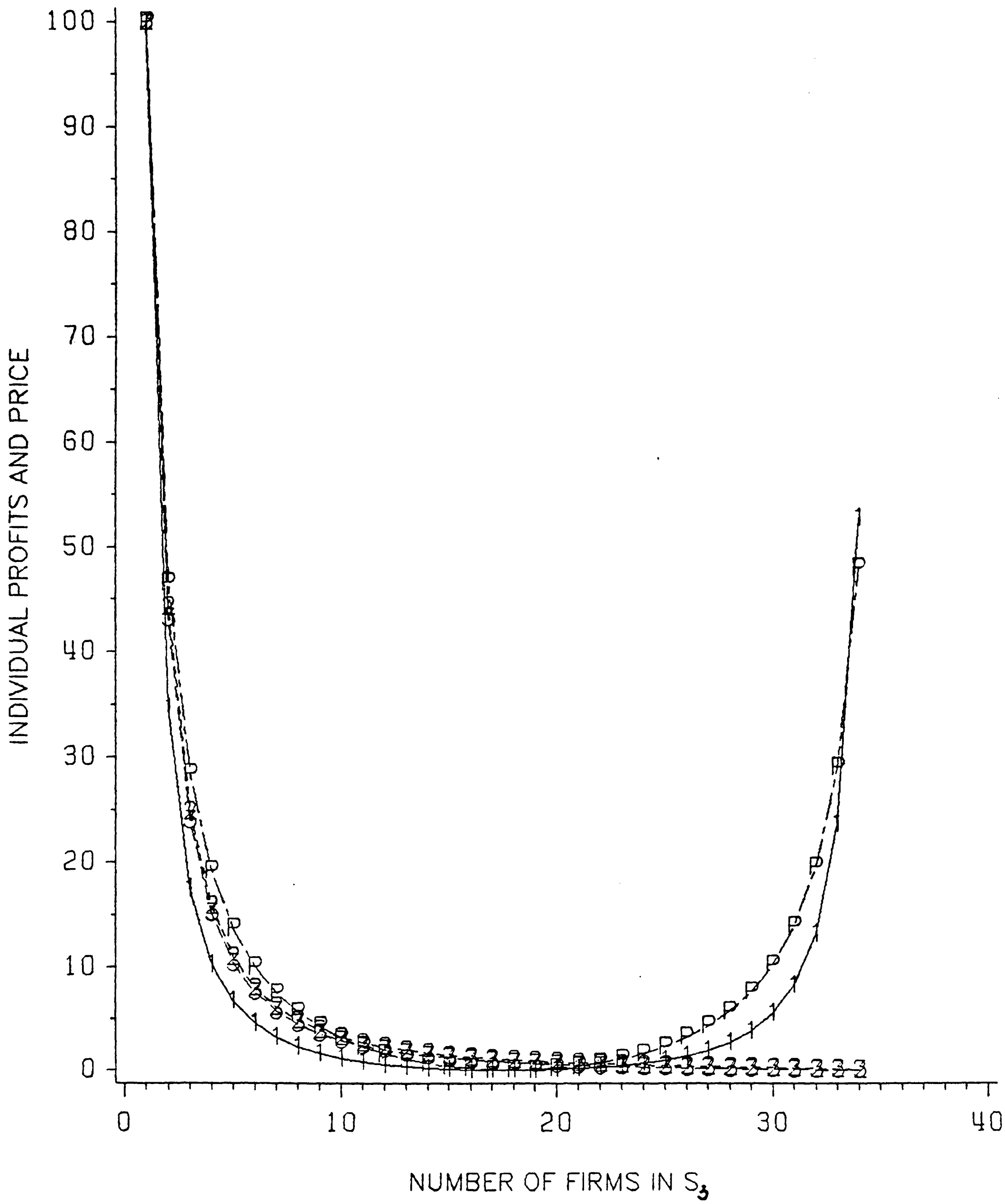
Legend: Symbol i refers to the firms in S_i, i = 1,2,3.

Figure 12. Vertical Integration in Leader-Follower Model, Case 4.: Changes in Firms' Individual Outputs. $n_1 = 35 - n_3$, $n_2 = 20$, $n_3 = 1, ..., 34$.



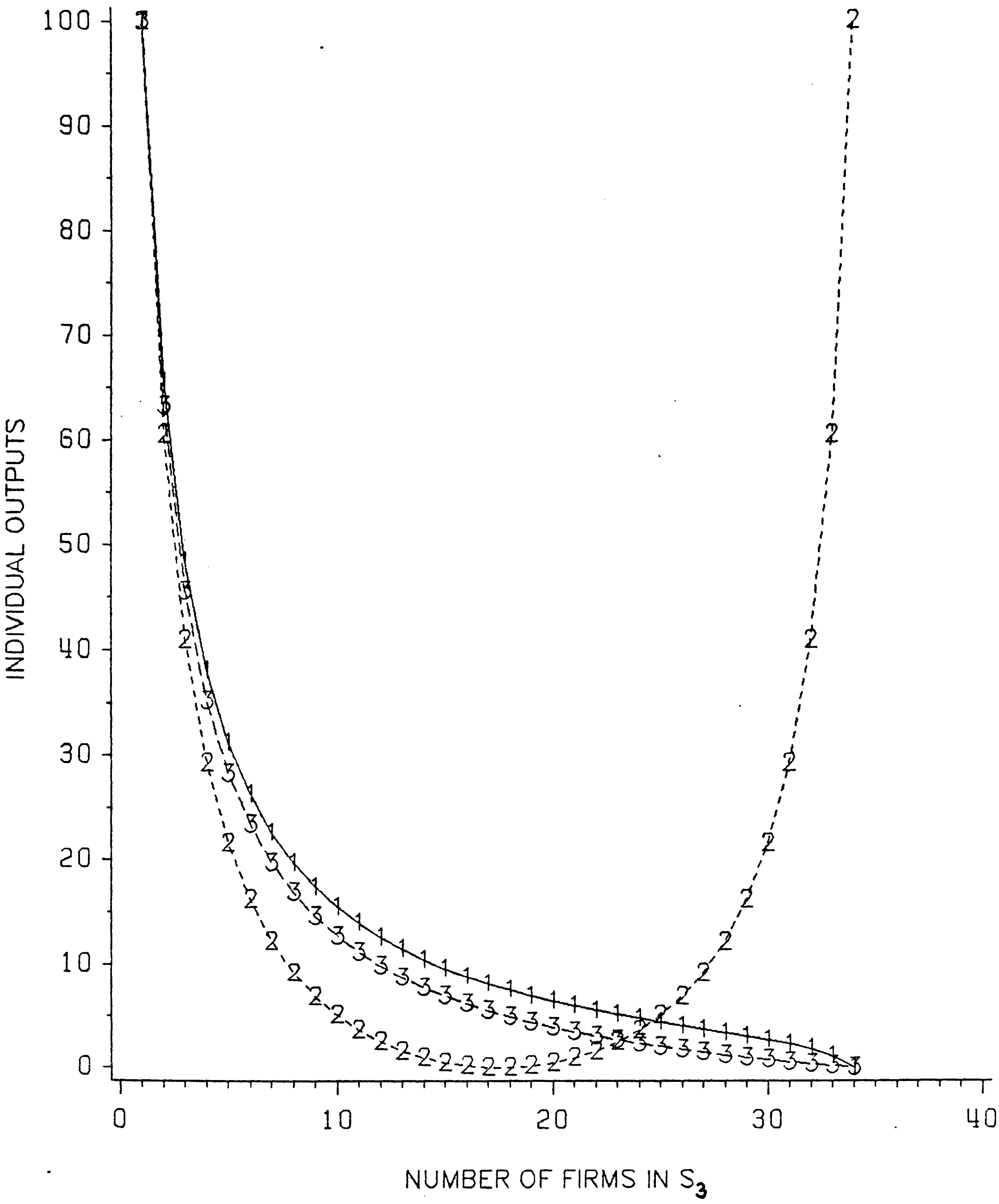
Legend: Symbol i refers to the firms in S_i, i = 2,3, Q refers to the total industry output.

Figure 13. Vertical Integration in Leader-Follower Model, Case 4.: Changes in Firms' Total Outputs. $n_1 = 35 - n_3$, $n_2 = 20$, $n_3 = 1, ..., 34$.



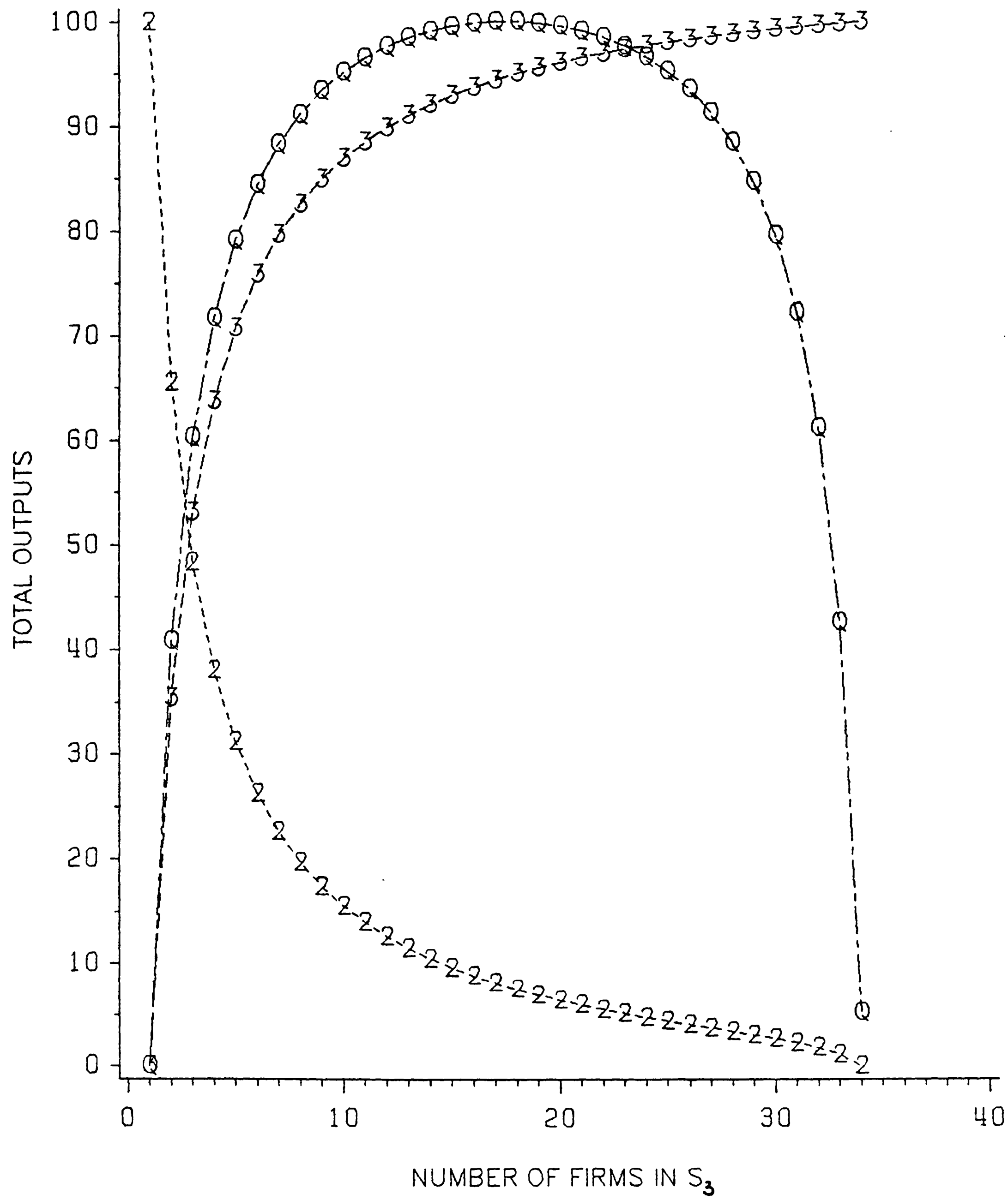
Legend: Symbol *i* refers to the firms in S_i, i = 1,2,3, symbol P refers to the input price.

Figure 14. Vertical Integration in Leader-Follower Model, Case 4.: Changes in Firms' Individual Profits and the Input Price. $n_1 = 35 - n_3$, $n_2 = 20$, $n_3 = 1, ..., 34$.



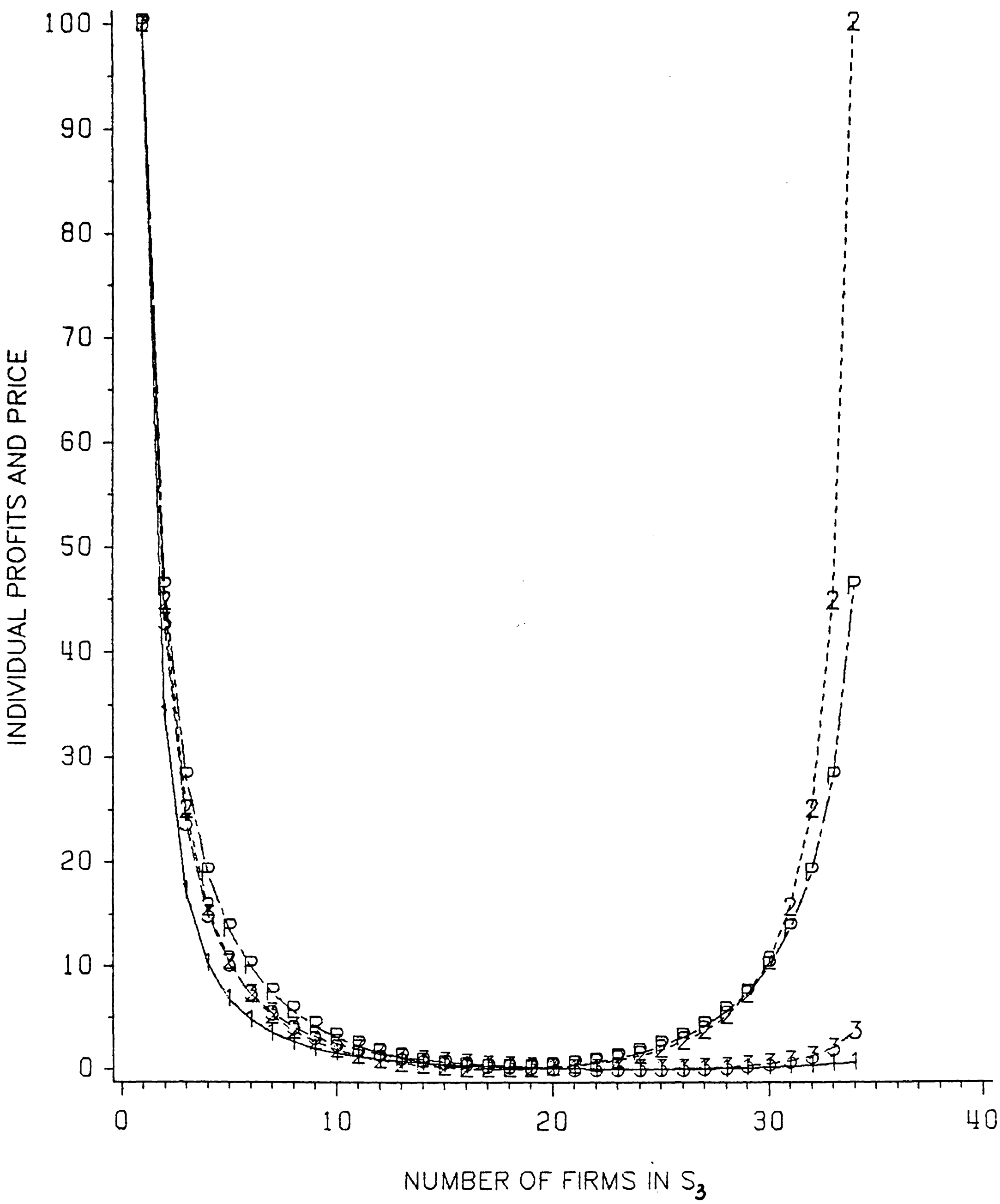
Legend: Symbol i refers to the firms in S_i, i = 1,2,3.

Figure 15. Vertical Integration in Leader-Follower Model, Case 5.: Changes in Firms' Individual Outputs. n_1 = 20, n_2 = 35 − n_3 , n_3 = 1, ...,34.



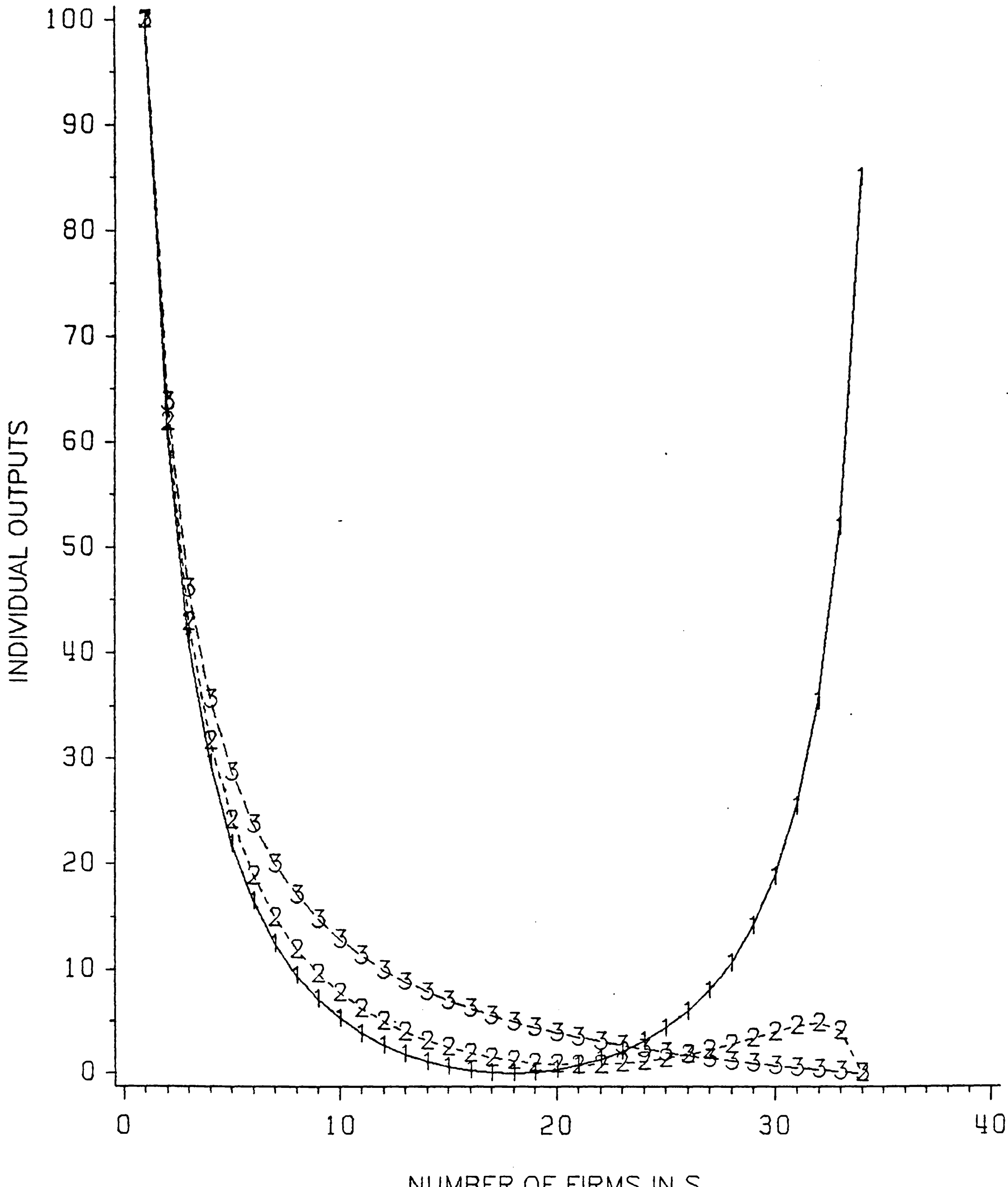
Legend: Symbol i refers to the firms in S_i, i = 2,3, Q refers to the total industry output.

Figure 16. Vertical Integration in Leader-Follower Model, Case 5.: Changes in Firms' Total Outputs. $n_1 = 20$, $n_2 = 35 - n_3$, $n_3 = 1, \ldots, 34$.



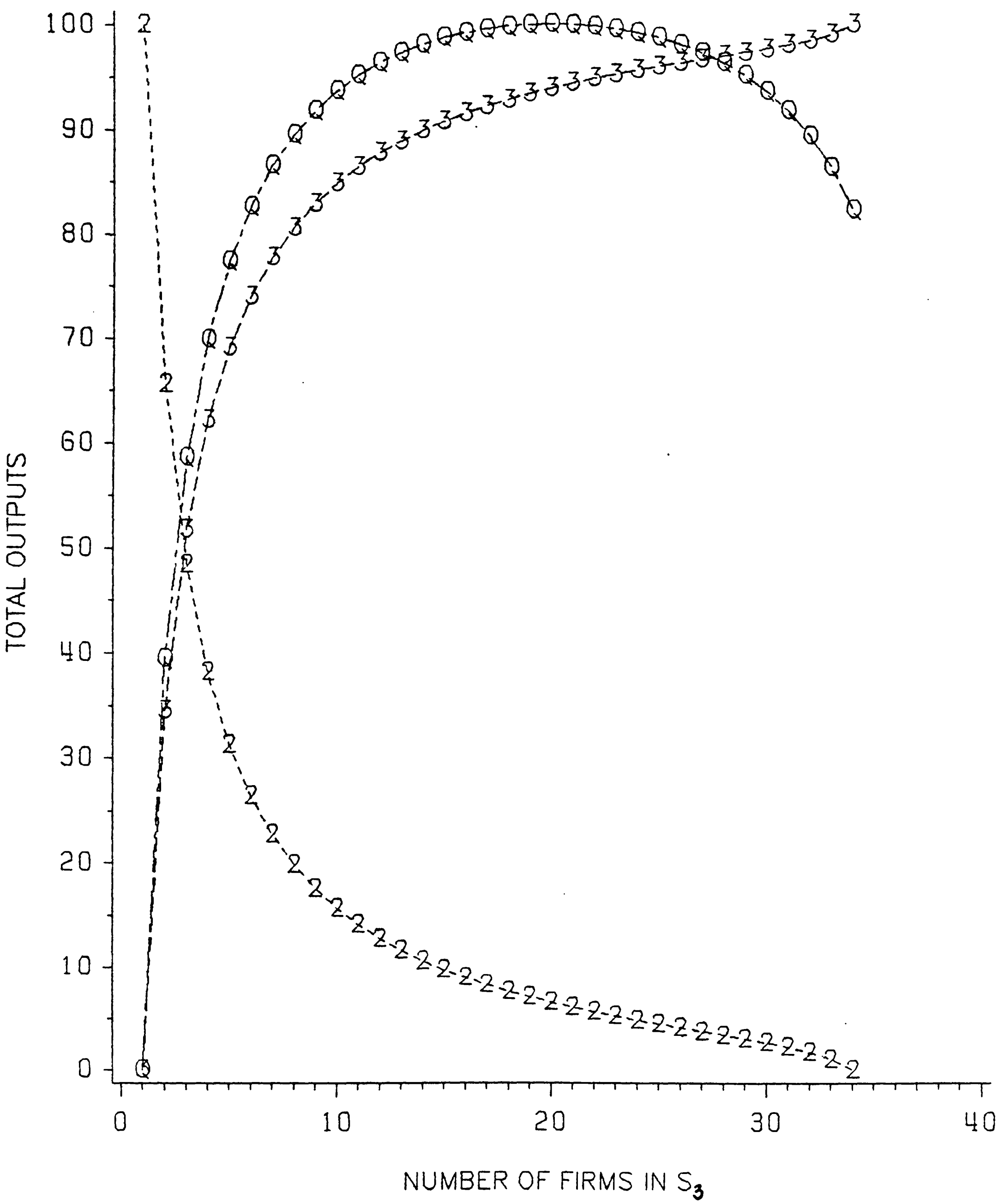
Legend: Symbol i refers to the firms in S_i, i = 1,2,3, symbol P refers to the input price.

Figure 17. Vertical Integration in Leader-Follower Model, Case 5.: Changes in Firms' Individual Profits and the Input Price. $n_1 = 20$, $n_2 = 35 - n_3$, $n_3 = 1, ...,34$.



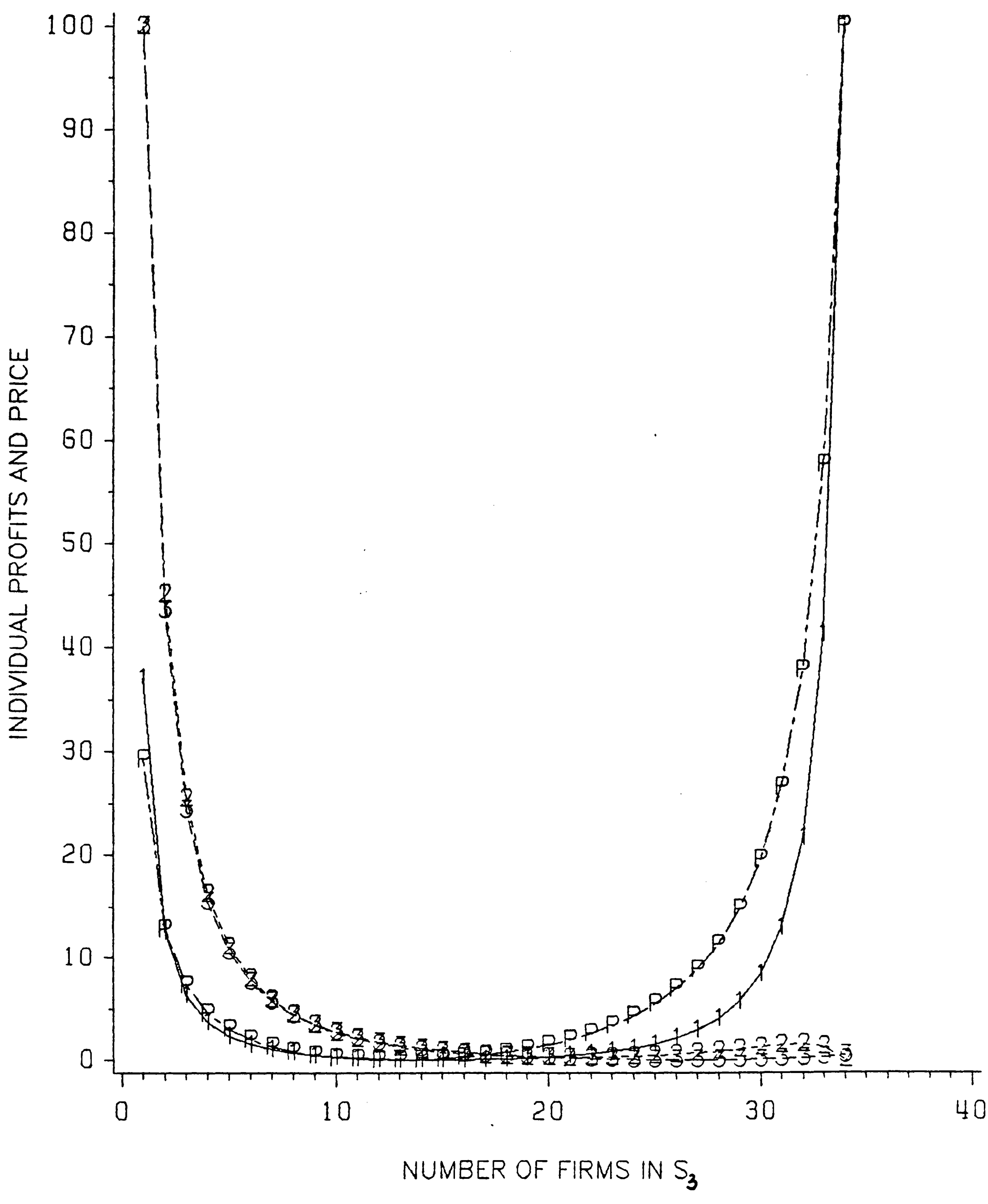
Legend: Symbol i refers to the firms in S_i, i = 1,2,3.

Figure 18. Vertical Integration in Leader-Follower Model, Case 6.: Changes in Firms' Individual Outputs. $n_1 = 35 - n_3$, $n_2 = 40 - n_3$, $n_3 = 1, \ldots, 34$.



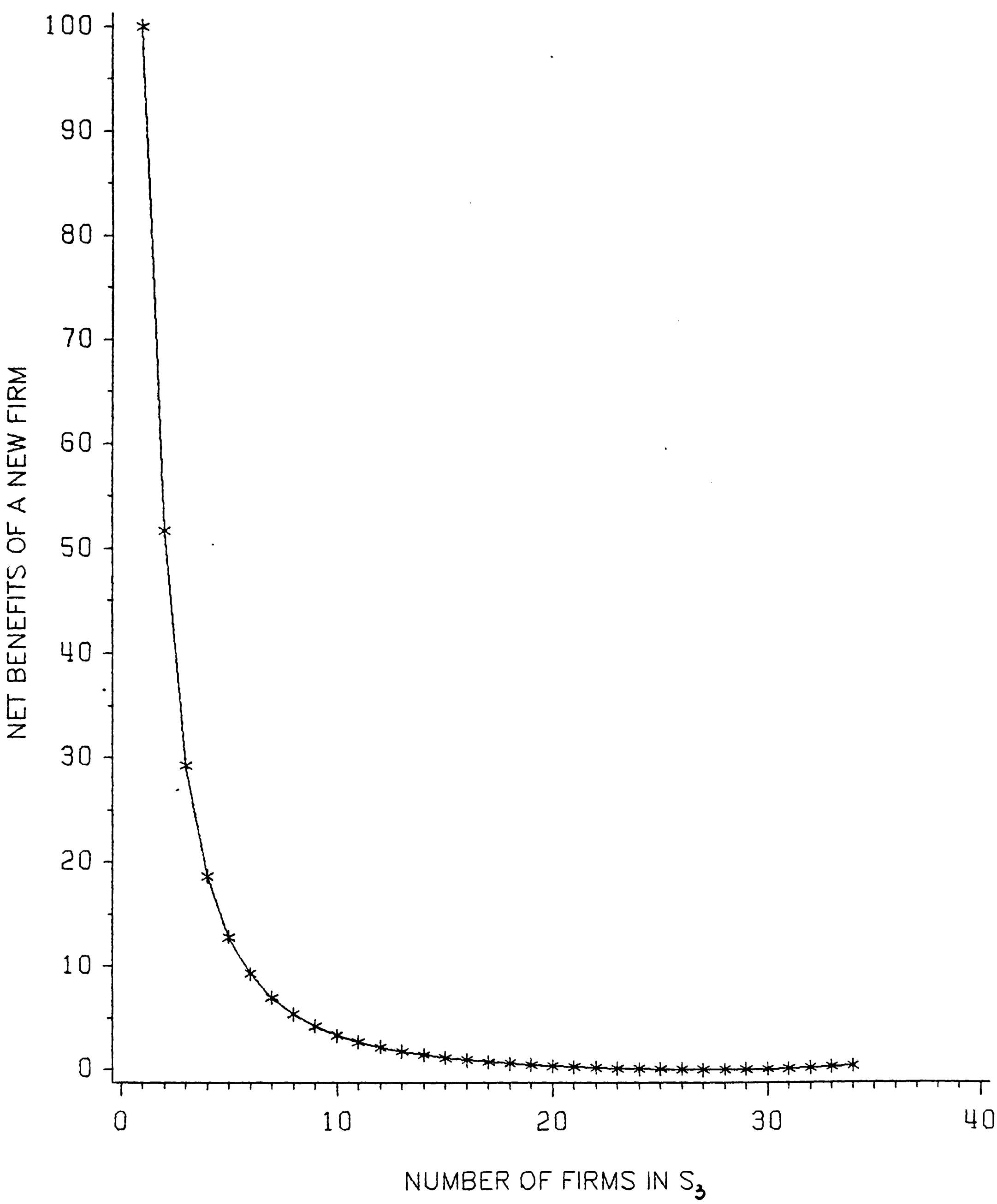
Legend: Symbol *i* refers to the firms in S_i, $i = 2,3$, Q refers to the total industry output.

Figure 19. Vertical Integration in Leader-Follower Model, Case 6.: Changes in Firms' Total Outputs. $n_1 = 35 - n_3$, $n_2 = 40 - n_3$, $n_3 = 1, ..., 34$.



Legend: Symbol i refers to the firms in S_i, i = 1,2,3, symbol P refers to the input price.

Figure 20. Vertical Integration in Leader-Follower Model, Case 6.: Changes in Firms' Individual Profits and the Input Price. $n_1 = 35 - n_3$, $n_2 = 40 - n_3$, $n_3 = 1, \ldots, 34$.



Legend: Symbol * refers to the net profit benefits of a new firm.

Figure 21. Vertical Integration in Leader-Follower Model, Case 6.: Net Profit Benefits of a New Firm. $n_1 = 35 - n_3$, $n_2 = 40 - n_3$, $n_3 = 1, \ldots, 34$.

Bibliography

B1. Bresnahan, T. F. 1981. Duopoly with Consistent Conjectures. *American Economic Review* 71, 934 - 945.

C1. Cournot, A. A. 1838. *Researches into the Mathematical Princeples of the Theory of Wealth.* Translated by Nathaniel T. Bacon. 1927. New York: Macmilan.

F1. Frank, Ch. R. Jr. and R. E. Quandt. 1963. On the Existence of Cournot Equilibrium. *International Economic Review* 4 , 92-96.

F2. Friedman, J. W. 1977. *Oligopoly and the Theory of Games.* Amsterdam :North-Holland.

F3. Friedman, J. W. 1983. *Oligopoly Theory.* New York: Cambridge University Press.

F4. Friedman, J. W. 1986. *Game Theory with Applications to Economics.* New York: Oxford University Press.

F5. Furth, D. 1979. The Stability of Generalized Stackelberg Equilibria in Heterogeneous Oligopoly. *Journal of Economics* 39, 315-322.

G1. Greenhut, M. L. and H. Ohta. 1976. Related Market Conditions and Interindustrial Mergers. *American Economic Review.* 66, 267-277

G2. Greenhut, M. L. and H. Ohta. 1978. Related Market Conditions and Interindustrial Mergers: Reply. *American Economic Review.* 68, 228-230.

G3. Greenhut, M. L. and H. Ohta. 1979. Vertical Integration of Successive Oligopolies. *American Economic Review.* 69, 137-141.

H1. Haring, J. R. and D. L. Kaserman. 1978. Related Market Conditions and Interindustrial Mergers: Comment. *American Economic Review.* 68, 225-227.

M1. MacKinnon, J. 1977. Solving Economic General Equilibrium Models by the Sandwich Method. In *Fixed Point. Algorithms and Applications.* Ed. S Karamardian in collaboration with C.B. Garcia. Academic Press, Inc., 367-402.

M2. McManus, M. 1962. Numbers and Size in Cournot Oligopoly. *Yorkshire Bulletin of Social and Economic Studies* 29, 14-22.

M3. Murphy, F. H., H. D. Sherali and A. L. Soyster. 1982. A Mathematical Programming Approach for Determining Oligopolistic Market Equilibrium. *Mathematical Programming* 24, 92-106.

N1. Nash, J. 1951. Non-cooperative Games. *Annals of Mathematics* 54, 286-295.

N2. Nishimura, K. and J. W. Friedman. 1981. Existence of Nash Equilibrium in n Person Games without Quasiconcavity. *International Economic Review* 22, 637-648.

N3. Novshek, W. 1984. Finding of All n-Firm Cournot Equilibria. *International Economic Review* 25, 61-70.

O1. Okuguchi, K. 1976. Expectations and Stability in Oligopoly Models. (Lecture Notes in Economics and Mathematical Systems, No. 138). Springer-Verlag, New York.

O2. Okuguchi, K. 1978. The Stability of Price Adjusting Oligopoly with Conjectural Variations. *Journal of Economics* 38, 50-60.

P1. Perry, M. K. 1978. Related Market Conditions and Interindustrial Mergers: Comment. *American Economic Review.* 68, 221-224.

R1. Rovinsky, R. B., C. A. Shoemaker and M. J. Todd. 1980. Determining Optimal Use of Resources Among Regional Producers using Different Levels of Cooperation. *Operations Research* 28, 859-866.

R2. Ruffin, R. J. 1971. Cournot Oligopoly and Competitive Behavior. *Review of Economic Studies* 38, 492 - 502.

S1. Salant, S. W., S. Switzer and R. J. Reynolds. 1983. Losses from Horizontal Merger: The Effects of an Exogeneous Change in Industry Structure on Cournot-Nash Equilibrium. *The Quarterly Journal of Economics* 98, 185-199.

S2. Scarf, H. 1973. *The Computation of Economic Equilibria* . Yale, New HAven.

S3. Sherali, H. D. 1984. A Multiple Leader Stackelberg Model and Analysis. *Operations Research* 32, 390-400.

S4. Sherali, H. D. and R. Rajan. 1986. A Game Theoretic-Mathematical Programming Analysis of Cooperative Phenomena in Oligopolistic Markets. *Operations Research* 34, 683-697.

S5. Sherali, H. D., A. L. Soyster and F. H. Murphy. 1983. Stackelberg-Nash-Cournot Equilibria: Characterizations and Computations. *Operations Research* 31, 253-276.

S6. Stackelberg, H. 1934. *Marktform und Gleichgewicht.* Julius Springer, Vienna.

S7. Szidarovszky, F. and S. Yakowitz. 1977. A New Proof of Existence and Uniqueness of the Cournot Equilibrium. *International Economic Review* 18, 787-789.

S8. Szidarovszky, F. and S. Yakowitz. 1982. Contributions to Cournot Oligopoly Theory. *Journal of Economic Theory* 28, 51-70.

W1. Waterson, M. 1982. Vertical Integration, Variable Proportions and Oligopoly. *The Economic Journal* 92, 129-144.

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