GROWTH SERIES FOR EXPANSION COMPLEXES

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ABSTRACT. This paper is concerned with growth series for expansion complexes for finite subdivision rules. Suppose X is an expansion complex for a finite subdivision rule $\mathcal{R}$ with bounded valence and mesh approaching 0, and let S be a seed for X. One can define a growth series for (X, S) by giving the tiles in the seed norm 0 and then using either the skinny path norm or the fat path norm to recursively define norms for the other tiles. The main theorem is that, with respect to either of these norms, the growth series for (X, S) has polynomial growth. Furthermore, the degrees of the growth rates of hyperbolic expansion complexes are dense in the ray $[2, \infty)$.

Suppose T is the set of tiles in a tiling of a plane and S is a nonempty, finite subset of T (often a single tile). We give T a metric by defining the distance $d(s, t)$ between two tiles s and t to be the minimum nonnegative integer n, such that there is a finite sequence $s_0, s_1, \ldots, s_n$ of tiles such that $s = s_0$, $s_n = t$, and for $i \in \{1, \ldots, n\}$ $s_{i-1} \cap s_i$ contains an edge. For each nonnegative integer n, let a_n be the number of tiles whose distance from an element of S is n. The growth series for (T, S) is the power series $\sum_{n=0}^{\infty} a_n z^n$.

We are interested in the growth series that arise from tilings. Unless one imposes additional structure on the tiling, the only requirement for the growth series is that each $a_n \geq 0$. One can see this by starting with the collection of circles in the plane with center the origin and radius a positive integer. These circle decompose the plane into the union of a disk and a countable family of annuli. If $\sum_{n=0}^{\infty} a_n z^n$ is a power series with each $a_n > 0$, then one can subdivide the central disk radially into a_0 tiles and for each $n > 0$ one can radially subdivide the annuli bounded by circles of radii n and $n + 1$ into a_n tiles. This produces a pair (T, S) with growth series $\sum_{n=0}^{\infty} a_n z^n$.

By contrast, consider a tiling T of the Euclidean or hyperbolic plane coming from the images, under a cocompact group G of isometries of the plane, of a Dirichlet region D for the action of D. Let the seed S be the single tile D. Then the growth series for (T, S) is the growth series for the group G with respect to the geometric generating set $\Sigma = \{g \in G \colon g(D) \cap D \text{ is an edge of } D\}$. Cannon shows in [Can84] that if G is a

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cocompact discrete group of isometries of $\mathbb{H}^n$, then with respect to a finite generating set for G the growth series is rational. In [Ben83], Benson proves the analogous result for groups of Euclidean isometries. In [CanW92], Cannon and Wagreich consider (1) the case that D is a hyperbolic triangle whose angles are submultiples of π and G is the associated group. They explicitly compute the rational growth function f and prove that $f(1) = 1/\chi(G)$ and that all of the poles of f lie in the unit circle except for a pair of positive reciprocal poles. They also consider (2) the case that D is a hyperbolic polygon with $4g$ sides and G is the fundamental group of a closed orientable surface with genus $g \geq 2$, and prove that the rational growth function f has the same properties. There is extensive literature on the special properties of the rational grouwth functions of hyperbolic surface groups. See for example the papers of Bartholdi-Ceccherini-Silberstein [BarC02], Floyd [Flo92], Floyd-Plotnick [FloP87, FloP88, FloP94], and Parry [Par93].

Inspired by the growth functions for surface groups, and motivated by a question from Maria Ramirez Solano about the growth rate for the pentagonal subdivision rule, we decided to consider growth series for expansion complexes. Like the tilings coming from surface groups, expansion complexes are essentially determined by a finite amount of combinatoiral data. But the growth series are very different. In Theorem 1 we prove that the associated growth functions all have polynomial growth, so they rarely have rational growth.

1. FINITE SUBDIVISION RULES AND EXPANSION COMPLEXES

While a finite subdivison rule is defined dynamically, in essence it is a finite combinatorial procedure for recursively subdividing appropriate 2-complexes. A finite subdivision rule $\mathcal{R}$ consists of (1) a finite 2-complex $S_{\mathcal{R}}$, (2) a subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$, and (3) a continuous cellular map $\sigma_{\mathcal{R}} \colon \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ whose restriction to every open cell is a homeomorphism. We further require that $S_{\mathcal{R}}$ is the union of its closed 2-cells and each closed 2-cell is the image of a polygon (called its tile type) with at least three edges by a continuous cellular map whose restriction to each open cell is a homeomorphism. An $\mathcal{R}$-complex is a 2-complex which is the union of its closed 2-cells together with a structure map $f \colon X \to S_{\mathcal{R}}$; we require that f is a continuous cellular map whose restriction to each open cell is a homeomorphism. The subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$ pulls back under f to a subdivision $\mathcal{R}(X)$ of X; $\mathcal{R}(X)$ is an $\mathcal{R}$-complex with structure map $\sigma_{\mathcal{R}} \circ f \colon \mathcal{R}(X) \to S_{\mathcal{R}}$. Since $\mathcal{R}(X)$ is an $\mathcal{R}$-complex, one can subdivide it; this is how one can recursively subdivide complexes with a finite subdivision rule . See [CanFP01] for the basic theory of finite subdivision rules.

As a simple example, consider the pentagonal subdivision rule $\mathcal{P}$ which was first described in [CanFP01]. The subdivision complex $S_{\mathcal{P}}$ has a single vertex, a single edge, and a single face (which is the image of a pentagon). The subdivision of the tile type is shown in Figure 1.
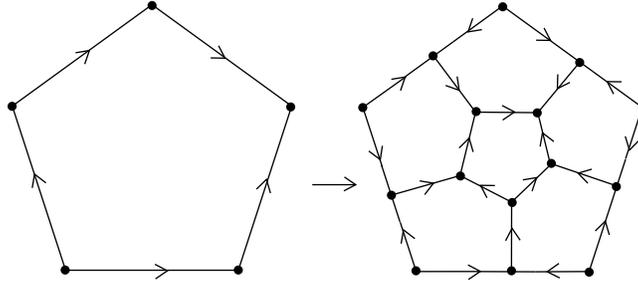
FIGURE 1. The subdivision of the tile type for the pentagonal subdivision rule $\mathcal{P}$
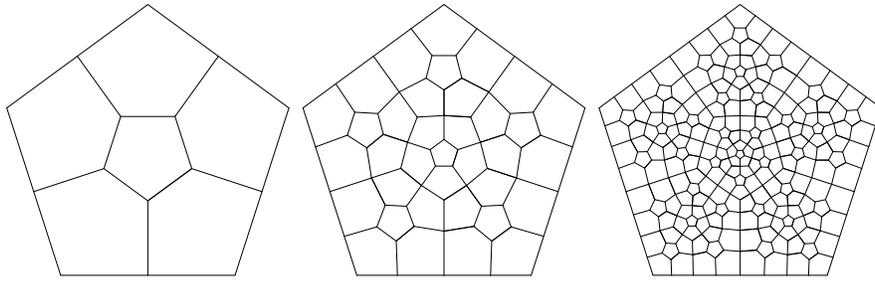

FIGURE 2. $\mathcal{P}(t)$, $\mathcal{P}^2(t)$, and $\mathcal{P}^3(t)$

Bowers and Stephenson created the pentagonal expansion complex in [BowS97] as part of their analysis of the pentagonal subdivision rule. Figure 2 shows the first three subdivisions of the tile type t, drawn with Stephenson's program CirclePack [Ste]. We can identify t with the central pentagon in its first subdivision $\mathcal{P}(t)$, and for each positive integer n this induces an inclusion of the nth subdivision $\mathcal{P}^n(t)$ in $\mathcal{P}^{n+1}(t)$. The direct limit of the sequence of inclusions $\mathcal{P}^n(t) \to \mathcal{P}^{n+1}(t)$ is the pentagonal expansion complex. Bowers and Stephenson put a conformal structure on the pentagonal expansion complex by making each open pentagon conformally a regular pentagon, using butterfiles to give charts for open edges, and using power maps to give charts for vertices. They showed that the expansion complex is conformally equivalent to the plane, and the expansion map (which takes each $\mathcal{P}^n(t)$ to $\mathcal{P}^{n+1}(t)$) is conformal.

In our papers [CanFP06a, CanFP06b] we gave the general definition of an expansion complex for a finite subdivision rule and developed some of the theory. An expansion complex for a finite subdivision rule $\mathcal{R}$ is an $\mathcal{R}$-complex X which is homeomorphic to $\mathbb{R}^2$ such that there is an orientation-preserving homeomorphism $\varphi \colon X \to X$ such that $\sigma_{\mathcal{R}} \circ f = f \circ \varphi$, where f is the structure map for X. If X is an expansion complex and S is a subcomplex of X, then S is a **seed** of X if S is a closed topological disk, $S \subset \varphi(S)$, and $X = \cup_{n=0}^{\infty} \varphi^n(S)$. For the pentagonal expansion complex, one can take the tile type t to be a seed. It is possible for an expansion

complex not to have a seed. For example, if the subdivision map is the identity map then no expansion complex can have a seed. But if $\mathcal{R}$ is a finite subdivision rule with bounded valence and mesh approaching 0, then it follows from [CanFP06a, Lemma 2.5] that every expansion complex for $\mathcal{R}$ has a seed for some iterate of $\mathcal{R}$. As in [BowS97] one can put a conformal structure on an expansion complex by taking each open tile to be conformally regular, using "butterflies" as charts in neighborhoods of open edges, and using power maps to extend the conformal structure over the vertices. We call an expansion complex *parabolic* if with this conformal structure it is conformally equivalent to the plane, and *hyperbolic* if with this conformal structure it is conformally equivalent to the open unit disk.

2. GROWTH SERIES FOR EXPANSION COMPLEXES

Let $\mathcal{R}$ be a finite subdivision rule, let X be an expansion complex for $\mathcal{R}$, and let S be a seed for X. We define the *skinny path norm* $|\cdot|$ on the tiles of X by setting norm $|t| = 0$ if t is in S and if t is not in S then $|t|$ is the minimal positive integer n such that there exist tiles $t_0, \ldots, t_n$ such that t_0 is in S, $t_n = t$, and $t_i \cap t_{i-1} \neq \emptyset$ for $1 \leq i \leq n$. For a nonnegative integer n, let $s_n = \#\{\text{tiles } t \subset X \colon |t| = n\}$ and let $b_n = \#\{\text{tiles } t \subset X \colon |t| \leq n\}$ (so s_n is the number of tiles in the combinatorial sphere of radius n and b_n is the number of tiles in the combinatorial ball of radius n). The *growth series* for (X, S) is the power series $\sum_{n=0}^{\infty} b_n z^n$. The growth series has *exponential growth* if $\limsup_{n \to \infty} \sqrt[n]{b_n} > 1$ and has *subexponential growth* if $\limsup_{n \to \infty} \sqrt[n]{b_n} = 1$. The growth series has *polynomial growth of degree d* if $d = \limsup_{n \to \infty} \frac{\ln(b_n)}{\ln(n)}$. The growth series has *intermediate growth* if the growth is neither exponential nor polynomial.

Theorem 1. *Let $\mathcal{R}$ be a finite subdivision rule with bounded valence and mesh approaching 0, let X be a $\mathcal{R}$-expansion complex, and let S be a seed for X. Then the growth series for (X, S) with respect to the skinny path norm has polynomial growth.*

Proof. We recall the skinny path distance from [CanFP06a]. If $x, y \in X$, the *skinny path distance* $d(x, y)$ is the minimum integer n such that there is a finite sequence $t_0, \ldots, t_n$ of tiles such that $x \in t_0$, $y \in t_n$, and $t_{i-1} \cap t_n \neq \emptyset$ for $i \in \{1, \ldots, n\}$. The skinny path distance does not define a distance function on X since two points in the same tile will have skinny path distance 0, but it does define a pseudometric.

Since $\mathcal{R}$ has mesh approaching 0, there is a positive integer n_0 such that the skinny path distance in X from S to $\partial \varphi^{n_0}(S)$ is at least 2. By [CanFP06a, Lemma 2.7], there is a postive integer n_1 such that if $x, y \in X$ and $d(x, y) \geq 2$ then $d(\varphi^{n_1}(x), \varphi^{n_1}(y)) \geq 2d(x, y)$. It easily follows that there are a positive real number a and a real number $b > 1$ such that for every positive integer n the skinny path distance from S to $\partial \varphi^n(S)$ is greater than ab^n. Since S is compact and there is an upper bound, d, on the number

of subtiles in the first subdivision of a tile type of $\mathcal{R}$, for any positive integer n the number of tiles in $\varphi^n(S)$ is at most cd^n, where $c = \#S$. Suppose $k \geq 2$ is an integer such that $\ln(k) > \ln(a)$. Then there is a unique positive integer n such that $n - 1 < \frac{\ln(k) - \ln(a)}{\ln(b)} \leq n$. Then $ab^{n-1} < k \leq ab^n$ and so

$$\frac{\ln(b_k)}{\ln(k)} < \frac{\ln(cd^n)}{\ln(ab^{n-1})} = \frac{\ln(c) + n\ln(d)}{\ln(a/b) + n\ln(b)}$$

and so $\limsup_{k \to \infty} \frac{\ln(b_k)}{\ln(k)} \leq \frac{\ln(d)}{\ln(b)}$ and the growth series has polynomial growth. $\qquad\square$

One can also consider a growth series for (X, S) with repect to the fat path norm. As above, let $\mathcal{R}$ be a finite subdivision rule, let X be an expansion complex for $\mathcal{R}$, and let S be a seed for X. We define the *fat path norm* $|\cdot|$ on the tiles of X by setting the norm $|t| = 0$ if t is in S and if t is not in S then $|t|$ is the minimal positive integer n such that there exist tiles $t_0, \ldots, t_n$ such that t_0 is in S, $t_n = t$, and $t_i \cap t_{i-1}$ contains an edge for $1 \leq i \leq n$. The other definitions in the first paragraph of this section follow exactly as before. Since for every nonnegative integer n the number of tiles of fat path norm at most n is at most the number of tiles of skinny path norm at most n, one gets the immediate corollary.

Corollary 2. *Let $\mathcal{R}$ be a finite subdivision rule with bounded valence and mesh approaching 0, let X be a $\mathcal{R}$-expansion complex, and let S be a seed for X. Then the growth series for (X, S) with respect to the fat path norm has polynomial growth.*

3. A FAMILY OF EXAMPLES

In all of the examples we consider in this section, the skinny path norms and the fat path norms are the same, so we won't name the norm.

We start with a simple example of an expansion complex for a finite subdivision rule $\mathcal{R}_1$. The subdivisions of the two tile types are shown in Figure 3. The subdivision $\mathcal{R}_1(t_1)$ of the tile type t_1 contains a tile in its interior which is labeled t_1, so the tile type t_1 is the seed of an expansion complex X. Let $\varphi \colon X \to X$ be the expansion map. For convenience we denote the seed by S. Figure 4 shows part of the expansion complex, with the seed in the center. Then $s_0 = 1$, $s_n = 2^{n+1}$ if $n > 0$, and $b_n = 2^{n+2} - 3$ for all $n \geq 0$. The growth series has exponential growth, but this doesn't violate Theorem 1 since $\mathcal{R}_1$ doesn't have mesh approaching 0.

For each positive integer n, let $R_n = \varphi^n(S) \setminus \mathrm{int}(S)$. By [CanFP06b, Theorem 5.5] X is hyperbolic if $\lim_{n \to \infty} M(R_n, \mathcal{S}(X)) \neq \infty$, where $\mathcal{S}(X)$ is the shingling of X by tiles.

Let n be a positive integer. Define a weight function w on R_n as follows. If t is a tile of R_n, then for some $k \in \{1, \ldots, n\}$, $t \in \varphi^k(S) \setminus \mathrm{int}(\varphi^{k-1}(S))$; we give t weight 2^{n-k}. The height curves for R_n have height $H(R_n, w) = \sum_{i=0}^{n-1} 2^i = 2^n - 1$, and w is the sum of the weights associated to the height

curves. Hence by [CanFP94, 2.3.6] w is the optimal weight function for R_n for fat flow modulus. For each $i \in \{0, \dots, n-1\}$, there are 2^{n+1-i} tiles in R_n with w-weight 2^i. Hence $A(R_n, w) = \sum_{i=0}^{n-1} 2^{n+1-i} \cdot (2^i)^2 = 2^{n+1}(2^n - 1)$ and

$$M(R_n, \mathcal{S}(X)) = M(R_n, w) = \frac{H(R_n, w)^2}{A(R_n, w)} = \frac{(2^n - 1)}{2^{n+1}}.$$

Hence $\lim_{n \to \infty} M(R_n, \mathcal{S}(X)) = \frac{1}{2} < \infty$ and X is hyperbolic.
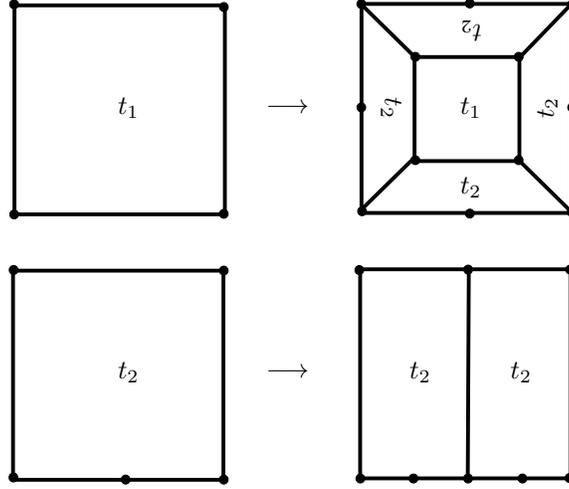

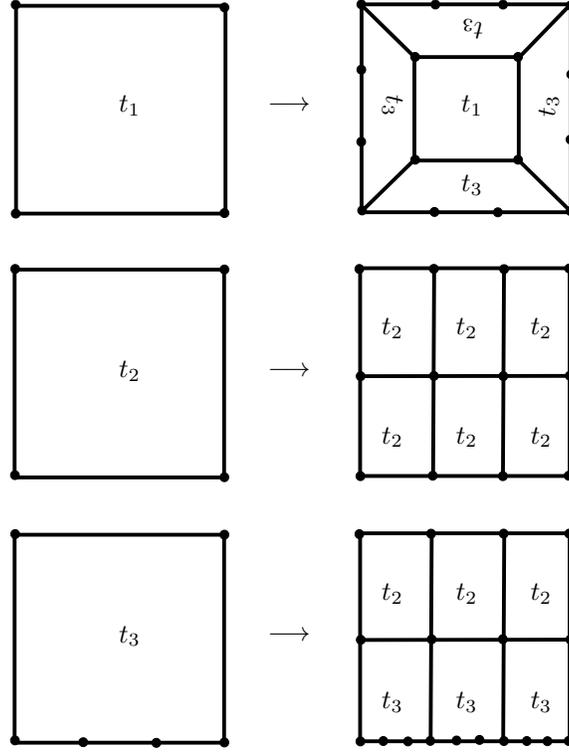FIGURE 3. The subdivisions of the tile types for $\mathcal{R}_1$

The finite subdivision rule $\mathcal{R}_2$ is similar to $\mathcal{R}_1$ but it has been modified to have mesh approaching 0. This time there are three tile types, and the subdivisions are shown in Figure 5. Tile type t_1 is a seed for an expansion complex X; part of this expansion complex is shown in Figure 6. One can show as we did for the previous example that X is hyperbolic. The hyperbolicity of X also follows from the proof of [CanFP06b, Lemma 5.1]; this example is simpler than the example being analyzed there but the approach there fits this example as well. Let S be the seed of X consisting of a single tile labeled t_1, and for each positive integer n let $R_n = \varphi^n(S) \setminus \text{int}(S)$. Given n, define a weight function w on R_n by giving a tile t in R_n weight 3^{n-k} if $t \subset \varphi^k(S) \setminus \text{int}\varphi^{k-1}(S)$. The height $H(R_n, w) = \sum_{k=0}^{n-1} 2^{n-1-k}3^k = 3^n - 2^n$. The weight function w is a sum of weight functions corresponding to height curves, so it is an optimal weight function. The area $A(R_n, w) = 4\sum_{k=0}^{n-1} 3^{2k}6^{n-1-k} = 4 \cdot 3^{n-1} \cdot (3^n - 2^n)$, so $M(R_n, \mathcal{S}(X)) = M(R_n, w) = \frac{H(R_n, w)^2}{A(R_n, w)} = \frac{3^n - 2^n}{4 \cdot 3^{n-1}}$, $\lim_{n \to \infty} M(R_n, \mathcal{S}(X)) = \frac{3}{4} < \infty$, and X is hyperbolic.

The finite subdivision rule $\mathcal{R}_2$ is a special case $(\mathcal{R}_{2,3})$ of a two-parameter family of finite subdivision rules $\mathcal{R}_{p,q}$ for integers $p, q \geq 2$. For a given p and q, $\mathcal{R}_{p,q}$ has three tile types, t_1 (a quadrilateral), t_2 (a quadrilateral), and
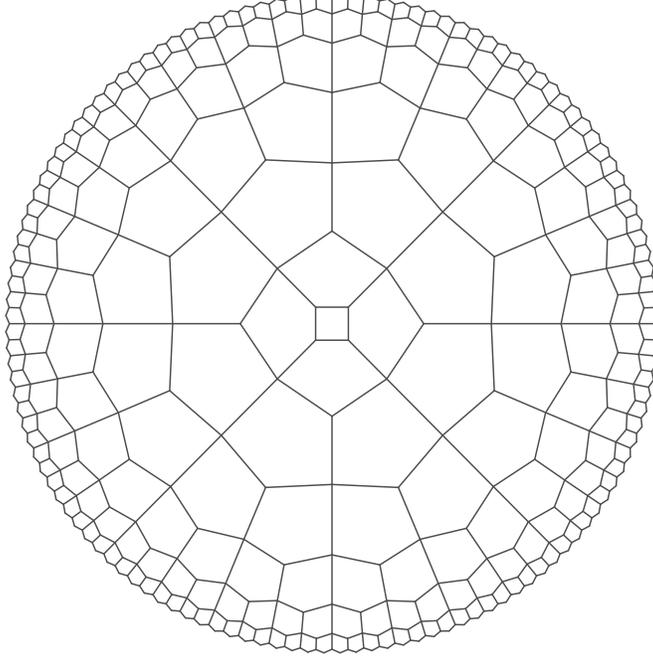
FIGURE 4. Part of the expansion complex for $\mathcal{R}_1$

t_3 (a $(q+3)$-gon) which is viewed as a quadrilateral with the bottom edge subdivided into q subedges. The tile type t_1 is subdivided into 5 subtiles, a central tile of type t_1 surrounded by four tiles of type t_3. The quadrilateral t_2 is subdivided into pq-subtiles, all of type t_2, arranged in p rows and q columns. The tile type t_3 is also subdivided into pq subtiles arranged in p rows and q columns, with each column in the first $p-1$ rows containing a tile of type t_2 and each column in the last row containing a tile of type t_3. As for the previous two examples, there is an expansion complex $X_{p,q}$ for $\mathcal{R}_{p,q}$ whose seed S is a single tile of type t_1. As before, we denote the expansion map by φ. For each positive integer n we let $R_n = \varphi^n(S) \setminus \text{int}(S)$, and we put a weight function on R_n as follows: if t is a tile of R_n and $t \subset \varphi^k(S) \setminus \text{int}(\varphi^{k-1}(S))$, then the weight of t is q^{n-k}. It follows as for $\mathcal{R}_1$ and $\mathcal{R}_2$ that w is an optimal weight function for $M(R_n, \mathcal{S}(X_{p,q}))$. If $p \neq q$, the height of R_n with respect to w is $H(R_n, w) = \sum_{k=0}^{n-1} q^k p^{n-1-k} = \frac{q^n - p^n}{q - p}$, the area is $A(R_n, w) = 4 \sum_{k=0}^{n-1} (pq)^{n-1-k} q^{2k} = \frac{4q^{n-1}(q^n - p^n)}{q - p}$, and the fat flow modulus of R_n is

$$\frac{H(R_n, w)^2}{A(R_n, w)} = \frac{(q^n - p^n)^2}{(q - p)^2} \frac{q - p}{4q^{n-1}(q^n - p^n)} = \frac{q^n - p^n}{4q^{n-1}(q - p)} = \frac{1 - (p/q)^n}{4(1 - p/q)}.$$

FIGURE 5. The subdivisions of the tile types for $\mathcal{R}_2$

If $p = q$, then $H(R_n, w) = n \cdot p^{n-1}$, $A(R_n, w) = 4n \cdot p^{2n-2}$, and $M(R_n, w) = \frac{n}{4}$. If $p < q$, then $\limsup_{n \to \infty} M(R_n, \mathcal{S}(X_{p,q})) = \frac{q}{4(q-p)} < \infty$ and $X_{p,q}$ is hyperbolic.

We next look in more detail at the growth series for $X_{p,q}$. Suppose $p, q \geq 2$, and consider $X_{p,q}$ as an expansion complex for $\mathcal{R}_{p,q}$ with seed S a single tile of type t_1. For looking at the finer detail of the growth series, it is more convenient to look at the growth series $g(z) = \sum_{n=0}^{\infty} s_n z^n$ for spheres instead of the growth series $f(z) = \sum_{n=0}^{\infty} b_n z^n$ for balls. As we saw above,

$$g(z) = 1+4z+4qz^2+4qz^3+\cdots+4qz^{p+1}+4q^2z^{p+2}+\ldots 4q^2 z^{p^2+p+1}+4q^3 z^{p^2+p+2}+\ldots,$$

where each coefficient $4q^k$ appears p^k consecutive times. For example, when $p = 2$ and $q = 3$ (the example $\mathcal{R}_2$),

$$g(z) = 1+4z+12z^2+12z^3+36z^4+\cdots+36z^7+108z^8+\cdots+108z^{15}+324z^{16}+\ldots.$$

Since the sequence $\{s_n\}$ has no upper bound on the number of consecutive terms which are constant, $g(z)$ cannot be rational or even D-finite. However, $g(z)$ does satisfy a functional equation. Note that

$$q \cdot g(z^p) = q + 4qz^p + 4q^2(z^{2p} + z^{3p} + \cdots + z^{p^2+p}) + 4q^3 z^{p^2+2p} + \ldots,$$
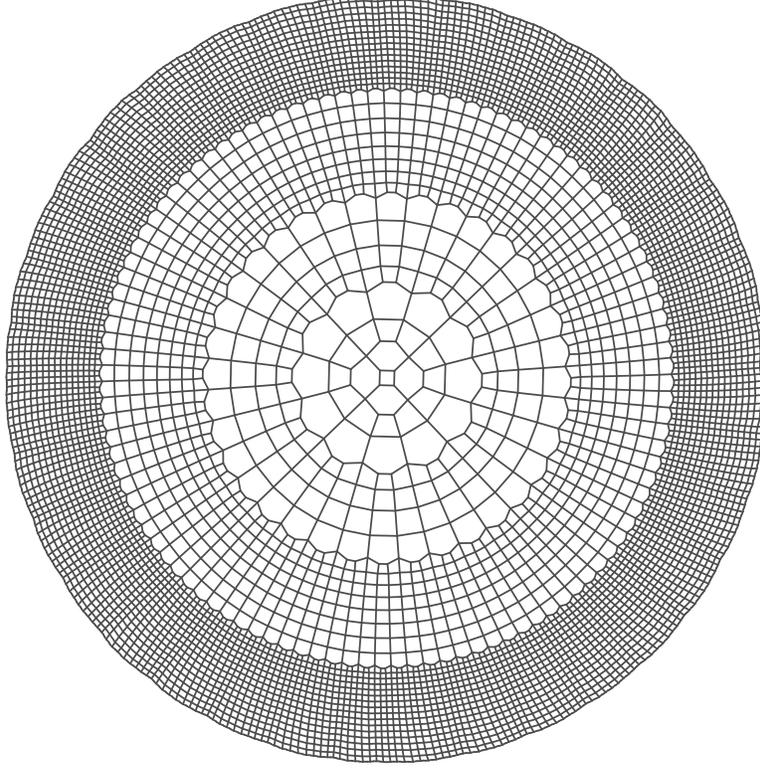
FIGURE 6. Part of the expansion complex for $\mathcal{R}_2$

so

$$q \cdot [g(z^p) - 1](1 + z + \cdots + z^{p-1}) = z^{p-2}(g(z) - 1 - 4z)$$

and g satisfies the functional equation

$$q \cdot (g(z^p) - 1)\frac{(z^p - 1)}{(z - 1)} = z^{p-2}(g(z) - 1 - 4z).$$

We now consider the growth series $\sum_{n=0}^{\infty} b_n z^n$ for $X_{p,q}$ with respect to the seed S consisting of a single tile labeled t_1. For convenience we assume that $p, q \geq 2$. Let n be a nonnegative integer. Then there is a nonnegative integer k such that $\frac{p^k - 1}{p - 1} \leq n < \frac{p^{k+1} - 1}{p - 1}$. Let $m = n - \frac{p^k - 1}{p - 1}$. Then $0 \leq m \leq p^k - 1$. Then

$$n = \frac{p^k - 1}{p - 1} + m \quad \text{and} \quad b_n = 1 + 4\frac{(pq)^k - 1}{pq - 1} + m(4q^k).$$

When $m = 0$,

$$\frac{\ln(b_n)}{\ln(n)} = \frac{\ln(pq + 4(pq)^k - 5) - \ln(pq - 1)}{\ln(p^k - 1) - \ln(p - 1)}$$

and in general

$$\frac{\ln(b_n)}{\ln(n)} < \frac{\ln(pq + 4(pq)^{k+1} - 5) - \ln(pq - 1)}{\ln(p^{k+1} - 1) - \ln(p - 1)}.$$

It follows that the growth series has polynomial growth of degree

$$\limsup_{n \to \infty} \frac{\ln(b_n)}{\ln(n)} = \frac{\ln(pq)}{\ln(p)} = 1 + \frac{\ln(q)}{\ln(p)}.$$

Since $X_{p,q}$ is hyperbolic whenever $q > p$, the degrees of the polynomial growth rates of hyperbolic expansion complexes with respect to the fat path norm are dense in $[2, \infty)$, and the degrees of the polynomial growth rates of hyperbolic expansion complexes with respect to the skinny path norm are dense in $[2, \infty)$.

In his Ph.D. thesis [Woo06], Wood notes that hyperbolic complexes can have spherical growth rates of degree $1 + \epsilon$ for ϵ arbitrarily small.

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