COMPARISON OF APPROXIMATE AND EXACT METHODS FOR DETERMINING THE FREQUENCIES OF VIBRATING BEAMS

by

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LIST OF SYMBOLS

Symbol	Name	Units
a_i	amplitude	in.
a_{ij}	influence coefficient	in.
E	Young's modulus	lb/in.^{2}
g	acceleration of gravity	in/sec.^{2}
h_i	amplitude	in.
I	moment of inertia	in.^{4}
k	spring constant (unless otherwise defined)	lb/in
L	length	in.
m	mass	lb-sec^2/in.
M	dimensionless constant	-
p	circular frequency	rads/sec.
t	time	sec.
x_i	deflection	in.
$\ddot{x}_i$	acceleration d^2x/dt^2	in/sec.^{2}
$\ddot{y}$	acceleration d^2y/dt^2	in/sec.^{2}
W	weight of concentrated load	lb.
w	weight per unit length	lb/in.

INTRODUCTION

The purpose of this paper is to compare the exact method of computation with the various approximate methods of computation of the natural frequencies of simply supported uniform beams carrying concentrated loads.

For purposes of this investigation, the exact method of computation for a single span beam with one concentrated load will be considered as the application of the appropriate boundary conditions to the solution of the fourth order partial differential equation for uniform beams. The effects of shear, damping, and rotatory inertia are neglected. The approximate methods to be used for comparison are the Rayleigh Method[1,8], the Ritz Method[1], the Dunkerley Equation Method[2], and formulas developed by Timoshenko[1], Freberg and Kemler[3], and Den Hartog[5].

In considering the single span carrying several concentrated loads, the criterion is established by the Method of Influence Coefficients[2], in which the deflections are expressed in terms of the influence coefficients and the inertia forces. Methods used for comparison are, the Rayleigh Method, the Iteration Method[2], and the Dunkerley Equation Method[2].

For the two span beam it was assumed that the method in which D'Alembert's Principle[1] was applied presented the most accurate results. Methods used for comparison were the Iteration Method, the Rayleigh Method and the Dunkerley Equation Method.

The literature available on the subject can be divided into two categories: Standard text books which treat of the fundamentals, and papers and articles published in the scientific journals, which for the most part deal with one facet of the problem to a high degree of concentration. It is from the former that most of the material used in this paper is taken.

The material considered was divided into four parts. They are the consideration of

A. A single span uniform beam with a concentrated load at the center,

B. A single span uniform beam with a concentrated load not at the center,

C. A single span uniform beam with discrete loads,

D. A two span uniform beam with concentrated loads at mid span.

In each of these methods the fundamental frequency is computed. Where methods permit, the second and third modes were also computed. In case A, load ratios were varied so that results were obtained for the load conditions $W = 2wL$ and $W = 4wL$ as well as for $W = wL$. Where W is the concentrated load, w is the weight per unit length of the beam, and L is the length of the beam.

Approximate methods used were chosen on the basis of flexibility, accuracy, speed and ease of use.

Numerical comparisons are included in tabular form and a summary is included recapitulating the results.

IV THE REVIEW OF LITERATURE

The papers covered in the Review of Literature are those which are closely related to the subject of this paper, since coverage of the exact topic was only found in text books which are listed in the Bibliography.

R. A. Anderson in the 1953 Journal of Applied Mechanics in an article on "Flexural Vibrations in Uniform Beams According to the Timoshenko Theory", presented the general series solution for the flexural vibrations in a uniform beam. The series solution for the pin ended beam was also given. Bending moment and shear force solutions, for the case of a concentrated transient force at the midpoint of a pin-ended beam, according to the elementary and the Timoshenko equations, are compared. It was found that the form of the general solution was similar to that of the elementary equation except that two series and two sets of frequencies are produced. For a pin ended beam, the first set of frequencies approached the elementary equation frequencies for the lower modes in slender bars, and the pure shear vibration frequencies for the higher modes in thick beams.

W. H. Hoppmann published a paper in the 1952 Journal of Applied Mechanics under the title, "Forced Lateral Vibrations of Beams Carrying A Concentrated Mass". A simply supported beam with a concentrated weight at the midpoint was chosen. A force was assumed to act normal to the beam length. The homogeneous form of the Bernoulli - Euler beam equation was solved considering the problem as one with time dependent boundary conditions. The solution was transformed to give the deflection and bending strain caused by a pulse type load. Computational as well as experimental results were compared. Oscillograms showed transition to a one degree of freedom system as the concentrated mass to beam ratio increased.

W. T. Thomson in the 1950 Journal of Applied Mechanics under the title, "Matrix Solution for the Vibration of Non Uniform Beams", introduced a matrix solution for a tabular method previously introduced by N. O. Myklestad in 1944. This method assumed the beam to be divided into small sections each of which was regarded as a cantilever with a concentrated mass at the end. Shear, moment, slope and deflection for each point were expressed in terms of the free end of each cantilever. The equations thus found were put in matrix form. By applying boundary conditions to the consolidated matrix formed by multiplying all the square matrices, the frequency equation was found.

R. S. Ayre and L. S. Jacobsen published a paper in the 1950 Journal of Applied Mechanics under the title, "Natural Frequencies of Continuous Beams of Uniform Span Length". The authors have devised a graphical network which can be used to determine the flexural vibration of continuous beams having any number of spans of uniform length. It is shown that, because there exists for continuous beams of a uniform span a definite pattern of frequency, once the pattern has been established, the frequency of any mode may be determined by an application of the problem to the network. Three cases were treated; both ends simply supported, both ends clamped, and one end simply supported and the other end clamped.

Dana Young, in the 1948 Journal of Applied Mechanics under the title, "Vibration of a Beam With Concentrated Mass, Spring and Dash-pot", presented a method for finding the natural frequencies of a uniform beam which carries certain combinations of concentrated masses, dash-pots and springs. Examples are given and solutions are worked out for a cantilever beam. However, the method is general and may be applied to beams having any type of end support.

N. O. Myklestad in the Journal of Aeronautical Science of June 1944 published a paper with the title, "A New Method of Calculating Natural Modes of Uncoupled Bending Vibration of Airplane Wings and Other Types of Beams." This method is of the tabular type and is analogous to that of Holzer. It is based on the fact that for a forced vibration with a finite amplitude the shaking force becomes zero at any of the natural frequencies. The beam, considered weightless, is assumed to carry a series of concentrated masses. Equations for shear, moment, slope and deflection are written for the nth section. Assuming an arbitrary value of slope and a unit value of deflection at one end of the beam, by a tabular method, the slope and deflection are computed at the other end. By assuming that one end of the beam is shaken by an arbitrary shaking force, new values of slope and deflection are calculated. These in turn enable the calculation of the products of masses and deflections. Since at resonance, the shaking force is zero, so also, the sum of the inertia forces must be zero. A plot of the sum of the products of masses and deflections versus applied frequency will give the natural frequencies of the system.

THE INVESTIGATION

A. Single Span with Load at Center.

1. Solution by the Exact Method.

A simply supported uniform beam of length L, weight per unit length w, and having a concentrated load at mid-span W, is given. (Figure 1.)

A solution of the simplified beam equation will be obtained by the method of separation of variables. Appropriate boundary conditions will be applied thereto to obtain four homogeneous linear equations. The expansion of the determinant of these equations will produce the frequency equation. First mode frequencies will be obtained for the load conditions W = wL, W = 2wL, and W = 4wL. The first three mode frequencies will be obtained for the load condition W = wL.

The differential equation for the transverse vibration of a beam is

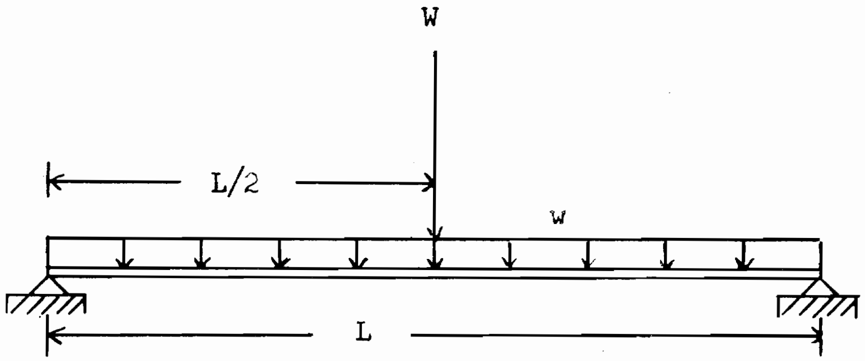
$$\frac{a^2 \partial^4 y}{\partial x^4} = -\frac{\partial^2 y}{\partial t^2}$$

where $a^2 = EIg/A\rho$ and ρ is the weight per unit volume.

A product solution $y(x,t) = X(x)\,T(t)$ is assumed and when substituted in the beam equation above produces

$$a^2 X^{IV} T = -X\ddot{T}$$

Dividing by XT we obtain



Single Span Beam with Load at Center

Fig. 1

$$a^2\frac{X^{IV}}{X} = -\frac{\ddot{T}}{T} = p^2$$

$$\text{or } X^{IV} = \frac{p^2}{a^2}X \quad \text{and} \quad \ddot{T} = -p^2 T$$

Letting $p = \frac{M^2 a}{L^2} = M^2\sqrt{EIg/\rho AL^4}$ and solving we obtain

$X = C \sin \frac{Mx}{L} + D \cos \frac{Mx}{L} + E \sinh \frac{Mx}{L} + F \cosh \frac{Mx}{L}$, and

$T = A \sin pt + B \cos pt$.

Using the following boundary conditions:

(1) $X(0,t) = 0$ (3) $X'(L/2,t) = 0$

(2) $X''(0,t) = 0$ (4) $EIX'''(L/2,t)T = \frac{W\ddot{y}}{2g}$

where $\ddot{y} = -p^2XT$

two homogeneous equations containing the constants C and E are obtained. These are

$$C \cos \frac{M}{2} + E \cosh \frac{M}{2} = 0$$

$$C\left[\left(2\rho AL\right)\cos \frac{M}{2} - \sin \frac{M}{2}\right] - E\left[\left(\frac{2\rho AL}{MW}\right)\cosh \frac{M}{2} + \sinh \frac{M}{2}\right] = 0$$

These constants have values other than zero only if the determinant of their coefficients is equal to zero. Expanding the determinant, the frequency equation in the parameter M is obtained.

$$\frac{W}{\rho AL} \cdot \frac{M}{4} = \frac{1}{\tan \frac{M}{2} - \tanh \frac{M}{2}}$$

This equation is solved graphically for M (Figure 2), first second and third mode frequencies being obtained.

Frequencies are obtained by use of the equation, $p = M^2\sqrt{\frac{EIg}{wL^4}}$

Letting $\sqrt{\frac{EIg}{wL^4}} = Q$, the results obtained are as follows:

$$W = wL \qquad p_1 = 5.66\ Q$$

$$W = 2wL \qquad p_1 = 4.37\ Q$$

$$W = 4wL \qquad p_1 = 3.27\ Q$$

$$W = wL \qquad p_2 = 67.8\ Q$$

$$W = wL \qquad p_3 = 206.5\ Q$$

2. Solution by Rayleigh Method

There are two variations of this method in the literature. Both of them will be discussed.

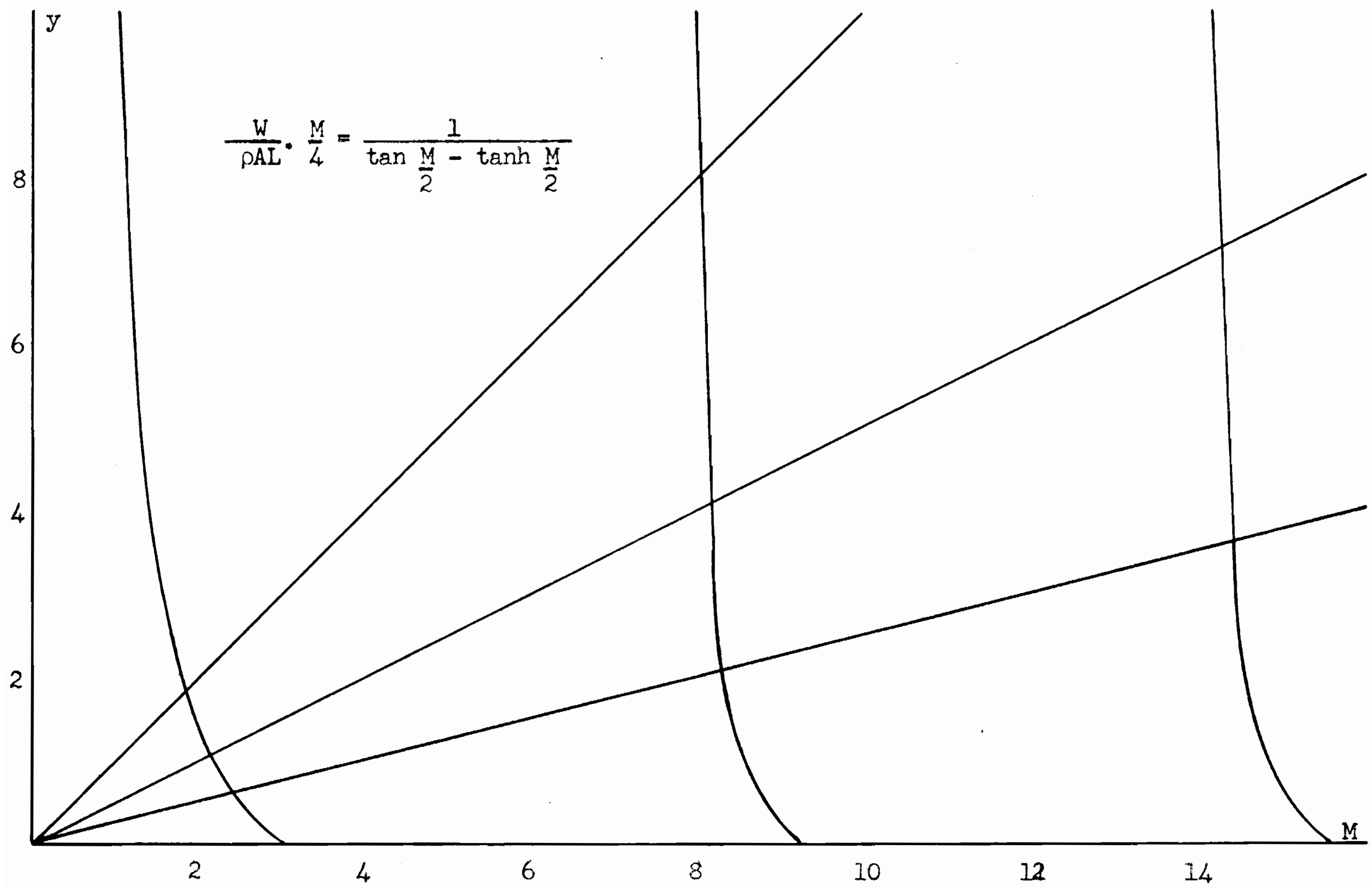
Timoshenko[1] indicates a procedure whereby the kinetic energy of the beam is computed and added to the maximum kinetic energy of the concentrated load. The weight thus involved is used as the concentrated load of a beam where the weight of the beam is neglected. The frequency is computed by the formula $p = \sqrt{kg/W}$

To illustrate:

Let $y = -\frac{Px}{48EI}(3L^2 - 4x^2)$ be the assumed deflection curve of the the beam. It may also be assumed that displacements are proportional to the static deflection of a beam carrying a concentrated load at mid-span. The displacement y_c of an element 'wdc' distant 'c' from the support will be

$$y_c = y_{L/2}\frac{(3cL^2 - 4c^3)}{L^3}.$$

The kinetic energy of the beam is found to be



$$\frac{W}{\rho AL} \cdot \frac{M}{4} = \frac{1}{\tan \frac{M}{2} - \tanh \frac{M}{2}}$$

Frequency Equation for Single Span Beam with Load at Center

Fig. 2

$$K.E. = 2\int_0^{\frac{L}{2}} \frac{w}{2}\left[\dot{y}_{L/2}\frac{(3cL^2 - 4c^3)}{L}\right]^2 dc = \frac{17}{35}\, wL\, \frac{\dot{y}^2}{2g}$$

The kinetic energy of the concentrated weight is $\frac{W\dot{y}^2}{2g}$, where y is the maximum displacement of W. Adding the two kinetic energies we obtain

$$K.E. = \frac{1}{2}\ \frac{(W + 17/35wL)}{g}\ \dot{y}^2.$$

Using W + 17/35wL as a concentrated load at midspan, and substituting in $p = \sqrt{kg/W}$ for the three load conditions the first mode frequency is obtained.

For W = wL	p = 5.69 Q
W = 2wL	p = 4.39 Q
W = 4wL	p = 3.28 Q

where $Q = \sqrt{EIg/wL^4}$

The other variation involves the computation of the maximum potential and kinetic energies of the system, equating them to form a frequency equation.

To illustrate:

A deflection curve is assumed which satisfies the boundary conditions.

$$y = a \sin \frac{\pi x}{L}$$

The potential energy is $P.E. = \frac{1}{2}\int_0^L \frac{M^2dx}{EI} = \frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx$

The kinetic energy is $\quad K.E. = \frac{1}{2}\int_0^L \dot{y}^2 dm + \frac{1}{2}\frac{W}{g}\dot{y}^2 \Big|_{x = L/2}$

Since $\dot{y}(max) = py$, by equating and solving for p^2 we obtain

$$p^2 = \frac{EI\int_0^L \left(\frac{d^2y}{dx}\right)^2 dx}{\frac{w}{g}\int_0^L y^2 dx + \frac{W}{g}y^2\Big|_{X = L/2}}$$

Substituting the assumed deflection curve into the above formula, the fundamental frequency is obtained. The results are

For $W = wL$	$p = 5.69\ Q$
$W = 2wL$	$p = 4.42\ Q$
$W = 4wL$	$p = 3.29\ Q$

Only the fundamental or first mode frequency is available by this method.

Equally accurate results were obtained by using the deflection curve $y = \frac{Px}{48EI}(3L^2 - 4x^2)$ in the second method. However, the work was more time consuming. Care should be taken, when using this curve, not to integrate across the discontinuity, since the curve represents the deflection for only half the beam.

3. <u>Solution by Dunkerley's Equation</u>

For a characteristic equation of degree n the second term of the frequency equation will be

$$-(a_{11}m_1 - a_{22}m_2 - a_{33}m_3 \ldots)\left(\frac{1}{p}\right)^{n-1}$$

If the roots of the frequency equation are assumed to be $\frac{1}{p_1^2}$,

$\frac{1}{p_2^2}, \frac{1}{p_3^2}$ . . . the second term of the equation will be

$$-\left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} \cdots\right)\left(\frac{1}{p^2}\right)^{n-1}$$

Equating corresponding coefficients and dropping the terms having the squares of the higher frequencies the Dunkerley Equation is obtained.

$$\frac{1}{p_1^2} = a_{11}m_1 + a_{22}m_2 + a_{33}m_3 \cdots$$

Since $a_{11}m_1 = \frac{W_1}{k_1 g} = \frac{1}{p_{11}^2}$ the equation may be written

$$\frac{1}{p_1^2} = \frac{1}{p_{11}^2} + \frac{1}{p^2_{22}} + \frac{1}{p_{33}^2} \cdots$$

This equation can be adapted to the solution of this problem by redefining the terms of the equation as follows:

p_1 = fundamental frequency of the system,

p_{11} = fundamental frequency of the beam alone,

p_{22} = fundamental frequency of the beam with the load at the mid-point but neglecting the weight of the beam.

Since $p_{11} = \pi^2 \sqrt{EIg/wL^4}$ and $p_{22} = \sqrt{kg/W} = \sqrt{48EIg/wL^4}$

and by letting $\sqrt{EIg/wL^4} = Q$ the frequencies were found to be

For $W = wL$ $\quad p = 5.68\ Q$

$W = 2wL$ $\quad p = 4.39\ Q$

$W = 4wL$ $\quad p = 3.26\ Q$

This method offers a rapid, accurate method for the solution of this problem, as the beam frequency can be found in beam frequency tabulations[5], and the k for the concentrated weight is the reciprocal of the maximum static deflection for the beam with the concentrated load at mid-span.

4. Solution by Ritz Method

The Ritz method is a refinement of the Rayleigh method which enables the calculation of higher modes as well as the fundamental.

In solutions by this method a deflection curve is assumed which satisfies the boundary conditions and is symmetric with the concentrated load. The number of terms employed will govern the number of modes obtained.

For examples, $y = a_1 \sin \pi x/L + a_2 \sin 3\pi x/L + a_3 \sin 5\pi x/L$ was assumed as the deflection curve. The frequencies obtained were for the first three modes. If

$$y = a_1 \sin \pi x/L + a_2 \sin 2\pi x/L + a_3 \sin 3\pi x/L$$

had been used, the frequencies for the first and second modes would have been produced by the first and third terms, and the second term would have produced the second mode of the uniform beam without the concentrated load. The nodal point for the second mode passes through the center of the beam and hence imparts zero velocity to the concentrated load. Since the presence of the concentrated load does not affect the potential energy, the conditions introduced by the term $a_2 \sin 2\pi x/L$ are analogous to the conditions for a uniform beam with no concentrated load.

The mechanics of the Ritz method are the same as the Rayleigh up to the determination of the first frequency equation. At this point, since minimum frequencies are required, the derivatives of p^2 are taken with respect to each of the coefficients of the deflection curve, a_1, a_2, . . . a_n, and are equated to zero.

The determinant of the coefficients a_1, a_2, . . . a_n is equated to zero forming another frequency equation. The degree of this equation will depend upon the number of terms in the deflection equation, and its solution will produce a mode for each term of the deflection equation.

This method is accurate to within one per cent, when compared with the exact solution. The accuracy of the method depends upon how close the assumed curve is to the actual curve. Higher modes can be improved by using more terms in the assumed deflection curve. However, by so doing the work is made considerably longer.

To illustrate:

The deflection curve assumed is

$$y = a_1 \sin \pi x/L + a_2 \sin 3\pi x/L + a_3 \sin 5\pi x/L$$

Substitution of this equation was made in the Rayleigh frequency equation,

$$p^2 = \frac{EI \int_0^L \left(\frac{d^2 y}{dx^2}\right)^2 dx}{\frac{w}{g}\int_0^L y^2 dx + \frac{W}{g} y^2 \Big|_{x = L/2}}$$

and the derivative of the resulting equation taken with respect to the coefficients a_1, a_2, a_3 of the deflection curve. These differentiations are set equal to zero. It will be helpful when performing this operation, if the differentiation of the above expression as a quotient, is taken with respect to a_n.

This will produce $$\frac{\partial}{\partial a_n}\int_0^L\left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{p^2}{EIg}\frac{\partial}{\partial a_n}\left[w\int_0^L y^2dx + Wy^2\Big|_{x=L/2}\right] = 0$$

Now substituting the deflection equation into the above, differentiating as indicated, and by letting $k = p^2/EIg$ and $q = \pi^4/L^3$ we obtain

$$(q - kwL - 2kW)a_1 + 2kWa_2 - 2kWa_3 = 0$$
$$2kWa_1 + (81q - kwL - 2kW)a_2 + 2kWa_3 = 0$$
$$-2kWa_1 + 2kWa_2 + (625q - kwL - 2kW)a_3 = 0$$

Setting the determinant of the coefficients a_1, a_2, and a_3 equal to zero and expanding the final frequency equation is obtained,

$$k^3w^3L^3 + 6k^3w^2L^2W - 707k^2w^2L^2q - 2828k^2wLWq + 51331\,kwLq^2$$
$$+ 102662kWq^2 - 50625q^3 = 0$$

Introducing the weight relationship, $W = wL = R$, the above reduces to

$$7k^3R^3 - 3535k^2R^2q + 153993kRq^2 - 50625q^3 = 0.$$

Solving we obtain $kR = 0.335q$, $47.78q$, $456.67q$.

Letting $Q = \sqrt{EIg/wL^4}$ and solving for p_i we obtain

$p_1 = 5.69\ Q, \qquad p_2 = 68.1\ Q \qquad \text{and} \qquad p_3 = 210.5\ Q$

By using a two term deflection curve, the frequency equation was found to be

$$5k^2R^2 - 246kRq - 81q^2 = 0.$$

Solving, the two frequencies are

$$p_1 = 5.69\ Q, \text{ and } \qquad p_2 = 69.0\ Q.$$

5. Solutions Based on Beam Weight Concentration

Freberg and Kemler[3] employ a solution to this problem based on the energy method. The total weight to be concentrated is expressed as a sum of the given concentrated load and the weight of the beam times a factor. The factor is governed by the ratio of beam weight to concentrated weight and may be taken from a table in the above reference. Results using $p = \sqrt{kg/W'}$, where $W' = W + UwL$ and U the constant from the weight ratio table

were for $W = wL$ $\qquad p = 5.71\ Q$

$W = 2\ wL$ $\qquad p = 4.43\ Q$

$W = 4\ wL$ $\qquad p = 3.28\ Q$ $\qquad$ with $Q = \sqrt{EIg/wL^4}$

Timoshenko[1] in his Rayleigh method, has indicated that 17/35 of the weight of the beam should be concentrated at the mid span, but makes no distinction for the various W to wL ratios. Results have been listed under section A-2.

Den Hartog[5] has incorporated in his text a formula which concentrates one half the weight of the beam at midspan.

Results were for

$W = wL \qquad p = 5.66\ Q$

$W = 2wL \qquad p = 4.38\ Q$

$W = 4wL \qquad p = 3.28\ Q$

These methods while accurate and rapid, produce only the first mode frequency.

B. Single Span Beam with Load not at the Center

1. Solution by Exact Method

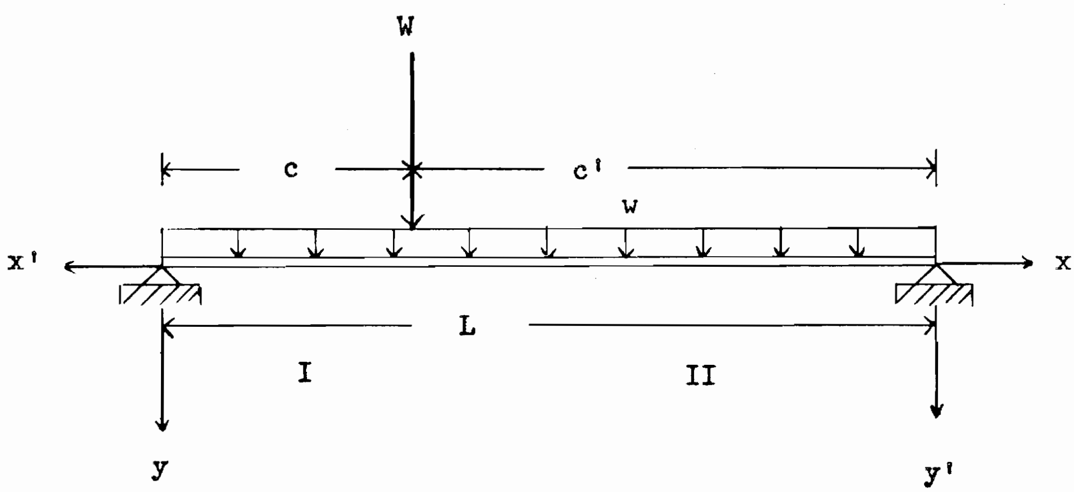
The uniformly loaded beam (figure 3) of length L, weight per unit length w, has a concentrated load W ,'c' distant from the left support. The first three modes of vibration for the load condition W = wL will be found by applying the appropriate boundary conditions to the solutions of the beam equation.

For convenience divide the beam into two parts; the part to the left of the load to be part I and the part to the right to be part II. Construct a y axis downward from the left support, and a y' axis downward from the right support, an x axis to the right from the left support, and an x' axis to the left from the right support. (Figure 3).

The differential equation of the vibrating beam is

$$a^2 \frac{\partial^4 y}{\partial x^4} = -\frac{\partial^2 y}{\partial t^2}$$

If a solution $y = X(x)T(t)$ is assumed and substituted in the differential equation above we will obtain $a^2 X^{IV} T = -X\ddot{T}$ which after



Single Span Beam with Load Not at Center

Fig. 3

division by XT becomes

$$a^2 \frac{X^{IV}}{X} = - \frac{\ddot{T}}{T} = p^2, \quad \text{where } p = \frac{M^2 a}{L^2}$$

Solving parts I and II separately the solutions are

$X_I = A \sin Mx/L + B \cos Mx/L + C \sinh Mx/l + D \cosh Mx/L$

$X_{II} = E \sin Mx'/L + F \cos Mx'/L + G \sinh Mx'/L + H \cosh Mx'/L$

The eight boundary conditions are:

(1) $X_I(0,t) = 0$ (3) $X_{II}(0,t) = 0$

(2) $X_I''(0,t) = 0$ (4) $X_{II}''(0,t) = 0$

(5) $X_I(c,t) = X_{II}(c't)$ (6) $X_I'(c,t) = - X_{II}'(c',t)$

(7) $X_I''(c,t) = X_{II}''(c',t)$ (8) $EIX_I'''(c,t) + EIX_{II}'''(c',t) = \frac{W\ddot{y}}{g}$

since $\ddot{y} = -p^2XT$ (8) becomes $EIX_I'''(c\ ,t) + \frac{W}{g} p^2 X(c,t) + EIX_{II}'''(c't) = 0$

Application of these boundary conditions to the equations for X_I and X_{II} produce four simultaneous equations in the constants A, C, E, and G. Letting $k = \frac{L^3}{M^3} \frac{W}{g} p^2 \frac{1}{EI}$ we have

$A \sin Mc/L + C \sinh Mc/L - E \sin Mc'/L - G \sinh Mc'/L = 0$

$A \cos Mc/L + C \cosh Mc/L + E \cos Mc'/L + G \cosh Mc'/L = 0$

$A \sin Mc/L - C \sinh Mc/L - E \sin Mc'/L + G \sinh Mc'/L = 0$

$A (\sin Mc/L - \cos Mc/L) + C (k \sinh Mc/L + \cosh Mc/L) - E \cos Mc'/L + G \cosh Mc'/L = 0$

Since A, C, E, and G will have values other than zero only if the determinant of their coefficients is equal to zero, such a determinant is formed. Expanding, we obtain the frequency equation as a transcendental equation in the parameter M. Letting Mc/L = B and Mc'/L = B'

$$\frac{W}{\rho AL}\,\frac{M}{2} = \frac{\sin B \cosh B \cos B' \sinh B' + \sin B \sinh B \cos B' \cosh B' - \cos B \sinh B \sin B' \cosh B' + \cos B \cosh B \sin B' \sinh B'}{\sin B \sinh B \sin B' \cosh B' + \sin B \cosh B \sin BI \sinh B' - \sin B \sinh B \cos B' \sinh B' - \cos B \sinh B \sin B' \sinh B'}$$

Solution is achieved by interpolation and graphing for the load condition W = wL with $c = \frac{L}{4}$ and $c' = \frac{3L}{4}$ (Figure 4) for the first, second and third modes. The results letting $Q = \sqrt{\frac{EIg}{wL^4}}$ were,

$p_1 = 6.84\ Q \qquad p_2 = 28.8\ Q \qquad p_3 = 80.1\ Q$

2. Solution by Rayleigh Method

Timoshenko[1] develops a formula for the unsymmetrical case by taking the frequency equation for a massless beam with a concentrated load,

$$\frac{W}{2g}\,\mathring{y}^2(\max) = \frac{k\,y_o^2}{2}$$

where y_o = maximum deflection and $\mathring{y} = y\,p$, and adding to the left side the kinetic energies acquired by both sides of the beam because of the maximum velocity $\mathring{y}$ of the concentrated load. (Figure 3)

The frequency $p = 6.93\sqrt{EIg/\rho AL^4}$ for c = L/4 and c' = 3L/4 is accurate to 1-1/2% of error over the exact method. However, unless a graph were available, of the transcendental equation for the particular load condition, the Rayleigh method would be much more convenient to use.

3. Solution by Dunkerley's Equation

The Dunkerley equation lends itself to a rapid solution which for this case is accurate to within 2%. The procedure is the same as outlined in part A, where

$$\frac{1}{p_1^2} = \frac{1}{p_{11}^2} + \frac{1}{p_{22}^2}$$

p_1 = frequency of the system.

p_{22} = frequency of the weightless beam with a concentrated load.

p_{11} = frequency of the uniform beam alone = $\pi^2 \sqrt{EIg/wL^4}$.

Since $p_{22} = \frac{m_2}{k_2^2} = a_{22}m_2 = a_{22}\frac{W}{g} = \frac{3L^3}{256EI}\ \frac{W}{g}$ where a_{22} is the influence coefficient.

If we let W = wL we have

$$\frac{1}{p_1^2} = \frac{wL^4}{\pi^4 EIg} + \frac{3wL^4}{256EIg} = \frac{wL^4}{EIg}\left(\frac{1}{\pi^4} + \frac{3}{256}\right) = .02193 \quad wL^4/EIg$$

$$p_1 = 6.70\sqrt{EIg/wL^4}$$

4. Solution by Ritz method.

In this method a deflection curve is assumed which satisfies the boundary conditions of the problem. In this case the deflection and moment at each end of the beam must be zero. For this case the symmetry restriction imposed in Case A is unnecessary. Substitution may be made directly into the differentiated form of the Rayleigh frequency equation

$$\frac{\partial}{\partial a_i}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx - \frac{p^2}{EIg}\frac{\partial}{\partial a_i}\left[w\int_0^L y^2 dx + Wy^2\ (L/i,t)\right] = 0$$

$i = 1, 2, 3, \ldots n$

This operation will produce simultaneous linear equations in a_i, the expansion of the determinant formed from them producing the frequency equation.

The method is accurate and reasonably rapid for two terms in the deflection equation, but begins to be cumbersome as the number of terms in the deflection equation is increased. The number of terms in the deflection equation governs the degree of the frequency equation, and hence the number of modes obtained.

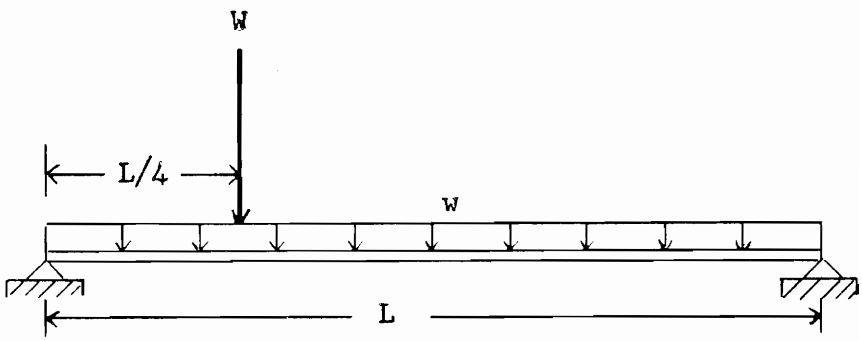
To illustrate:

Given a simply supported uniform beam of length L, and weight per unit length w. A concentrated weight W is placed L/4 distant from the left support, (Figure 4).

Assume the deflection curve

$$y = a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{2\pi x}{L} + a_3 \sin\frac{3\pi x}{L}$$

Substitute this equation directly into the differentiated form of the Rayleigh frequency equation and indicate its differentiation with respect to the coefficients a_1, a_2, and a_3. Letting $k = \frac{p^2}{EIg}$ and $q = \frac{\pi^4}{L^3}$ we obtain



Single Span Beam with Load at Quarter Point

Fig. 4

$$(q - kwL - kW)\, a_1 - 2\, kWa_2 - kWa_3 = 0$$

$$- 2\, kWa_1 + (16q - kwL - 2kW)\, a_2 - 2\, kWa_3 = 0$$

$$-kWa_1 - 2\, kWa_2 + (81q - kwL - kW)\, a_3 = 0$$

Setting the determinant of the coefficients equal to zero, inserting the weight ratio W= wL = R, and expanding the final frequency equation, a cubic in kR is obtained.

$$5\, k^3R^3 - 376\, k^2R^2q + 2876\, R\, q^2 - 1296\, q^3 = 0$$

$$kR = 0.48q,\ 8.066q,\ 66.656q$$

By letting $\sqrt{EIg/wL^4} = Q$ and solving for p_i the frequencies are obtained,

$$p_1 = 6.84\, Q, \quad p_2 = 28.05\, Q, \quad p_3 = 80.60\, Q.$$

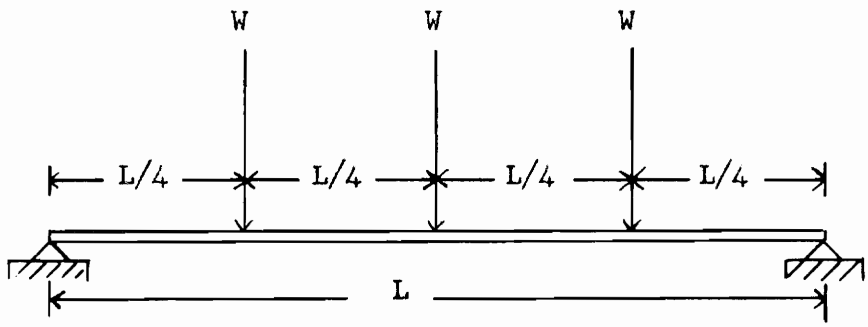
For the first three modes this method is accurate, differences of less than one percent from the exact solution being experienced.

C. Single Span Beam with Discrete Loads.

In illustrating this case a beam 80 inches long will be used, carrying equal concentrated loads W at the quarter and half points. The weight of the beam will be neglected. (Figure 5).

1. Solution by Method of Influence Coefficients. [2]

In this method the deflections under the loads are expressed in terms of the influence coefficients and the inertia forces. The equations so formed are then converted from differential equation form to algebraic form by using the relation $x_i = h_i \sin pt$.



Single Span Beam with Discrete Loads

Fig. 5

By dividing each equation by p^2 and transposing, homogeneous linear equations in h_i are formed. The determinant of these equations is then set to zero. It is expanded and produces the frequency equation, a cubic in $\frac{1}{p^2}$.

To illustrate:

The deflection at each load is

$$x_1 = - a_{11} \frac{W}{g} \ddot{x}_1 - a_{12} \frac{W}{g} \ddot{x}_2 - a_{13} \frac{W}{g} \ddot{x}_3$$

$$x_2 = - a_{21} \frac{W}{g} \ddot{x}_1 - a_{22} \frac{W}{g} \ddot{x}_2 - a_{23} \frac{W}{g} \ddot{x}_3$$

$$x_3 = - a_{31} \frac{W}{g} \ddot{x}_1 - a_{32} \frac{W}{g} \ddot{x}_2 - a_{33} \frac{W}{g} \ddot{x}_3$$

by substituting $x_i = h_i$ sin pt into each equation the deflections are obtained in algebraic form

$$h_1 = a_{11} \frac{W}{g} p^2 h_1 + a_{12} \frac{Wp^2}{g} h_2 + a_{13} \frac{W}{g} p^2 h_3$$

$$h_2 = a_{21} \frac{W}{g} p^2 h_1 + a_{22} \frac{Wp^2}{g} h_2 + a_{23} \frac{W}{g} p^2 h_3$$

$$h_3 = a_{31} \frac{W}{g} p^2 h_1 + a_{32} \frac{Wp^2}{g} h_2 + a_{33} \frac{W}{g} p^2 h_3$$

Dividing by p^2 and rearranging in determinant form as coefficients of h_i, the determinant is set equal to zero.

$$\frac{W}{g}\begin{vmatrix} (a_{11} - \frac{g}{Wp^2}) & a_{12} & a_{13} \\ a_{21} & (a_{22} - \frac{g}{Wp^2}) & a_{23} \\ a_{31} & a_{32} & (a_{33} - \frac{g}{Wp^2}) \end{vmatrix} = 0$$

Evaluating the influence coefficients, and letting $2 \cdot 10^3/3EI = S$

$$a_{11} = a_{33} = \frac{bx}{6EIL}(L^2 - b^2 - x^2) = \frac{18000}{3EI} = 9\ \frac{2 \cdot 10^3}{3EI} = 9S$$

$$a_{22} = \frac{L^3}{48EI} = \frac{32000}{3EI} = 16S$$

$$a_{12} = a_{21} = a_{32} = a_{23} = \frac{22000}{3EI} = 11S$$

$$a_{13} = a_{31} = \frac{14000}{3EI} = 7S$$

and substituting $k = 3EIg/2Wp^2 10^3$, the determinant becomes

$$\frac{2W}{3EIg} \cdot 10^3 \begin{vmatrix} (9 - k) & 11 & 7 \\ 11 & (16 - k) & 11 \\ 7 & 11 & (9 - k) \end{vmatrix} = 0$$

Expanding we obtain

$$\frac{2W\ 10^3}{3EIg}\ (k^3 - 34\ k^2 + 78\ k - 28) = 0$$

where $k = 0.445$, 2, and 31.56

Since $p^2 = \frac{3EIg\ 10^{-3}}{2kW}$ $\qquad p = \sqrt{\frac{3000}{2k}} \cdot 10^{-3} \cdot \sqrt{\frac{EIg}{W}}$

$$p_1 = 6.90\ (10^{-3})\ \sqrt{EIg/W}$$

$$p_2 = 27.4\ \text{"}\ \sqrt{EIg/W}$$

$$p_3 = 58.0 \quad (10^{-3}) \quad \sqrt{EIg/W}$$

2. Solution by Iteration Method.

In this method the deflections at the load points are expressed in terms of the influence coefficients and the inertia forces. These simultaneous equations in x_i are placed in matrix form. Assuming arbitrary values of x_i, matrix multiplication of the coefficient matrix and the column matrix containing the assumed values of x_i is carried out. The resulting column matrix is normalized and the process repeated with the normalized values substituted for x_i. This process is continued until the product matrix stabilizes when normalized.

To obtain the second mode the orthogonality relationship between the first and second mode is used to eliminate the first mode content from the second mode. The resulting matrix is substituted in the first mode matrix obtaining a second mode matrix equation. The iteration procedure as carried out for the first mode is repeated, the stabilized column matrix producing the second mode frequency, when normalized.

The third mode procedure is much the same, except that the orthogonality relationship must be expressed for the first and third modes as well as for the second and third modes, to insure the contents of these modes (first and second) being excluded from the third mode. The resulting matrix equation is substituted in the matrix equation for the second mode to obtain the matrix equation for the third mode.

To illustrate:

The deflection equations are written in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = p^2 \begin{bmatrix} a_{11}m_1 & a_{12}m_2 & a_{13}m_3 \\ a_{21}m_1 & a_{22}m_2 & a_{23}m_3 \\ a_{31}m_1 & a_{32}m_2 & a_{33}m_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Substituting the previously determined values of the influence coefficients the first mode matrix is obtained

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{2Wp^2 10^3}{3EIg} \begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Using an iteration procedure outlined by Thomson [2] we obtain

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{2Wp^2 10^3}{3EIg} \quad (31.52) \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix} \qquad p_1 = 6.90(10^{-3})\sqrt{EIg/W}$$

In obtaining the second mode the orthogonality relationship is used, where the subscripts refer to position and the superscripts refer to the mode.

$$\sum_{1}^{3} m_i\, x_i^{(1)} x_i^{(2)} = \frac{W}{g}(0.707)x_1^{(2)} + \frac{W}{g}(1)x_2^{(2)} + \frac{W}{g}(0.707)x_3^{(2)} = 0$$

$$x_1^{(2)} = -\,1.414x_2^{(2)} - x_3^{(2)}$$

$$x_2^{(2)} = x_2^{(2)}$$

$$x_3^{(2)} = x_3^{(2)}$$

Writing in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -1.414 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Substituting in the matrix equation for the first mode.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{2Wp^2 10^3}{3EIg} \begin{bmatrix} 9 & 11 & 7 \\ 11 & 16 & 11 \\ 7 & 11 & 9 \end{bmatrix} \begin{bmatrix} 0 & -1.414 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and performing the multiplication indicated we obtain the second mode matrix equation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{2Wp^2 10^3}{3EIg} \begin{bmatrix} 0 & -1.73 & -2 \\ 0 & 0.45 & 0 \\ 0 & 1.10 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Using the iteration procedure in the same manner as for the first mode

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{2Wp^2 10^3}{3EIg} \begin{bmatrix} -1.99 \\ 0.003 \\ 1.99 \end{bmatrix} = \frac{2Wp^2 10^3}{3EIg} (1.99) \begin{bmatrix} -1 \\ 0.0015 \\ 1 \end{bmatrix}$$

$$1 = \frac{2Wp^2 10^3}{3EIg} (1.99)\ (1) \qquad p_2 = 27.4\ (10^{-3})\sqrt{EIg/W}$$

To obtain the third mode the orthogonality relationships are

$$\frac{W}{g}\ x_1^{(2)} x_1^{(3)} + x_2^{(2)} x_2^{(3)} + x_3^{(2)} x_3^{(3)} = 0$$

$$\frac{W}{g}\ x_1^{(1)} x_1^{(3)} + x_2^{(1)} x_2^{(3)} + x_3^{(1)} x_3^{(3)} = 0$$

These relationships produce the equations

$$x_1^{(3)} = x_3^{(3)}$$

$$x_2^{(3)} = -1.414\ x_3^{(3)}$$

$$x_3^{(3)} = x_3^{(3)}$$

which when expressed in matrix form give

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1.414 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Substitution is made in the matrix equation for the second mode resulting in the matrix equation for the third mode

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{2Wp^2\ 10^3}{3EIg} \begin{bmatrix} 0 & 0 & 0.44 \\ 0 & 0 & -0.635 \\ 0 & 0 & 0.45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{2Wp^2\ 10^3}{3EIg} \begin{bmatrix} 0.44 & x_1 \\ -0.635 & x_2 \\ 0.45 & x_3 \end{bmatrix}$$

then normalizing we obtain

$$1 = 2W\ p^2\ 10^3\ (0.45)\ (1)$$

$$p_3 = 57.6\ (10^{-3}) \sqrt{EIg/W}$$

3. Solution by Rayleigh Method

In this method a frequency equation for the fundamental only is obtained by the equating of the expressions for potential and kinetic energy.

$$\sum \frac{1}{2} \frac{W_i}{g} x_i^2 p^2 = \frac{1}{2} W_i x_i \ , \qquad p = \sqrt{\frac{g \Sigma W_i x_i}{\sum W_i x_i^2}}$$

where W_i are the loads at the respective positions and x_i the corresponding deflections.

Writing the deflection equations in terms of influence coefficients and loads

$$x_1 = W_1a_{11} + W_2a_{12} + W_3a_{13} ,$$

$$x_2 = W_1a_{21} + W_2a_{22} + W_3a_{23} ,$$

$$x_3 = W_1a_{31} + W_2a_{32} + W_3a_{33} .$$

Computing these from influence coefficient data previously worked out and substituting in the frequency formula above, we obtain

$$p = 6.90 \; (10^{-3}) \sqrt{EIg/W}$$

4. <u>Solution by Dunkerley's Equation</u>

The Dunkerley Equation for this part of the problem takes the form

$$\frac{1}{p_1^2} = a_{11}m_1 + a_{22}m_2 + a_{33}m_3$$

Substituting influence coefficients previously computed in part C-1 we have

$$\frac{1}{p_1^2} = \frac{W}{3EIg}(68 \times 10^3)$$

$$p_1 = 6.65(10^{-3})\sqrt{\frac{EIg}{W}}$$

This method for this type of problem is rapid and accurate to within three or four per cent. The frequency obtained will be lower than those found by the other methods because of the dropping of the $\frac{1}{p_2^2}$ terms from the left side of the equation.

D. <u>Two Span Beam with Concentrated Loads</u>

A two span beam with hinged supports is assumed, with concentrated loads at mid span. The spans are of equal length. (Fig. 6)

1. Solution by the Application of D'Alembert's principle

In this method the deflections are written in terms of the influence coefficients and the inertia forces. Reduction to algebraic form is accomplished by the relationship $x_i = h_i \sin pt$. The expansion of the determinant formed from the simultaneous linear equations in h_1 and h_2 provide the frequency equation.

To illustrate:

Given the beam indicated in Figure six and neglecting the weight of the beam. Find the frequencies of the two modes of vibration.

The deflections in terms of the influence coefficients and the inertia forces are

$$x_1 = -a_{11}\frac{W}{g}\ddot{x}_1 - a_{12}\frac{W}{g}\ddot{x}_2$$

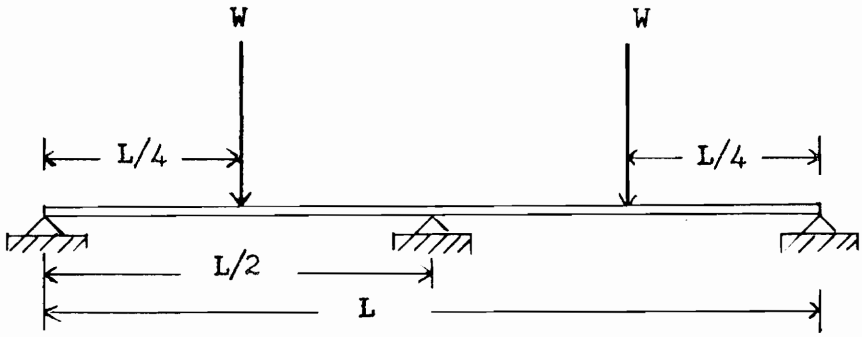
$$x_2 = -a_{21}\frac{W}{g}\ddot{x}_1 - a_{22}\frac{W}{g}\ddot{x}_2$$

Converting to algebraic form using $x_i = h_i \sin pt$ we obtain

$$h_1 = a_{11}\frac{W}{g}p^2 h_1 + a_{12}\frac{W}{g}p^2 h_2$$

$$h_2 = a_{21}\frac{W}{g}p^2 h_1 + a_{22}\frac{W}{g}p^2 h_2$$

Transposing and forming a determinant of the coefficients of h



Two Span Beam with Concentrated Loads.

Fig. 6

$$\frac{W}{g}\begin{vmatrix} (a_{11}p^2 - \frac{g}{W}) & a_{12}p^2 \\ a_{21}p^2 & (a_{22}p^2 - \frac{g}{W}) \end{vmatrix} = 0$$

Expanding $(a_{11}a_{22} - a_{12}a_{21})\, p^4 - (a_{11} - a_{22})\, \frac{g}{W}\, p^2 - \frac{g^2}{W^2} = 0$

The influence coefficients are found to be

$$a_{11} = a_{22} = 23 \quad \frac{L}{12288EI} = 23k \quad \text{where } k = \frac{L^3}{12288EI}$$

$$a_{12} = a_{21} -9 \quad \frac{L^3}{12288EI} = -9k$$

Substituting these values into the determinant and expanding the frequency equation is found to be

$$448k^2p^4 - 46kp^2\frac{g}{W} - \frac{g^2}{W^2} = 0$$

Solving for kp^2 we obtain

$$kp^2 = \frac{g}{32W} \quad \text{and} \quad \frac{g}{14W}$$

Solving for p we obtain

$$p_1 = 19.6\sqrt{\frac{EIg}{WL^3}} \qquad p_2 = 29.7\sqrt{\frac{EIg}{WL^3}}$$

2. Solution by Iteration Method

In this method deflections are expressed in terms of the influence coefficients and the inertia forces. A matrix equation is formed in terms of the deflections. Arbitrary values are assumed for the deflections and the iteration procedure is carried out as previsouly described. In this process the higher mode will be found first. Eliminating the higher mode content by using the orthogonality relationship a matrix equation is constructed. Substituting in the original matrix equation and simplifying, the matrix equation for the lower mode is found.

To illustrate:

The deflection equations are

$$x_1 = a_{11}\frac{Wp^2x_1}{g} + a_{12}\frac{W}{g}p^2x_2$$

$$x_2 = a_{21}\frac{Wp^2x_1}{g} + a_{22}\frac{W}{g}p^2x_2$$

The matrix is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = p^2\frac{W}{g}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Inserting the influence coefficients and letting $k = L^3/12288EI$ and $x_1 = x_2 = 1$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = p^2\frac{W}{g}k\begin{bmatrix} 23 & -9 \\ -9 & 23 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = p^2\frac{W}{g}k\begin{bmatrix} 14 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = p^2\frac{W}{g}k\,(14)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

therefore $p^2 = \frac{12288}{14}\frac{EIg}{WL^3}$ $\qquad p_2 = 29.6\sqrt{\frac{EIg}{WL^3}}$

Using the orthogonality relationship the equations

$$m_1x_1^{(1)} + m_2x_2^{(2)} = 0$$

$$\text{or } x_1^{(1)} = -x_2^{(2)}$$

$$x_2^{(1)} = x_2^{(1)}$$

Expressing in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Substituting in the matrix equation for the higher mode

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = p^2 \frac{W}{g} k \begin{bmatrix} 23 & -9 \\ -9 & 23 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = p^2 \frac{W}{g} k \begin{bmatrix} -32 \\ 32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 32\, p^2 \frac{W}{g} k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$1 = p^2 \frac{W}{g} \frac{L^3}{12288\ EI} (32)\ (1) \qquad p = 19.6\sqrt{\frac{EIg}{WL^3}}$$

Since the assumed deflections in the first part were both positive, the solution indicated was that of the symmetric or higher mode. In the second part the application of the orthogonality relationship excluded the first mode content and introduced deflections which were different in sign thus producing the lower mode.

3. Solution by Rayleigh Method

The frequency formula found by equating potential and kinetic energy is

$$p = \sqrt{\frac{g \sum W_i x_i}{\sum W_i x_i^2}}$$

The deflection equations are

$$x_1 = a_{11}W_1 + a_{12}W_2$$

$$x_2 = a_{21}W_1 + a_{22}W_2$$

Using the influence coefficients as previously determined and substituting in the formula above, the frequency

$p = 29.6 \sqrt{EIg/WL^3}$ is obtained.

The influence coefficients are deflections, which reflect the static load condition. And since the symmetric or higher mode closely approximates the static curve, when the influence coefficients, obtained from the static conditions, are substituted into the frequency equation, the higher mode results. If on the other hand, the loads are assumed to be of the inertia type, and that one acts downward when the other acts upward, the influence coefficients would all be positive. Substitution in the frequency formula on this basis would produce the anti-symmetric or lower mode.

4. <u>Solution by Dunkerley's Equation</u>

The formula is $\frac{1}{p_1^2} = a_{11}\frac{W_1}{g} + a_{22}\frac{W_2}{g}$.

Substituting values of influence coefficients previously determined, we have

$$\frac{1}{p_1^2} = \frac{46}{12288}\frac{L^3}{EI}\frac{W}{g}$$

$$p = 16.35 \sqrt{EIg/WL^3}$$

The error here is about 17% as compared with the $19.6\sqrt{EIg/WL^3}$ previously determined. The large percentage of error is due to the other frequency being relatively so close.

TABLE 1

Frequencies for Single Span Beam with Load at Center

Frequency Equation $p = M^2 \sqrt{EIg/wL^4}$, Weight ratio $k = \frac{W}{wL}$

M^2

k	Sep. Va.	Rayleigh 1	Rayleigh 2	Dunkerley	DenHartog	Freberg Kemler	Ritz
1	5.66	5.69	5.69	5.68	5.66	5.71	5.70
2	4.37	4.39	4.42	4.39	4.38	4.43	4.42
4	3.27	3.28	3.29	3.26	3.28	3.29	3.29

TABLE 2

Frequencies for Single Span Beam with Load Not at Center

Frequency Equation $p = M^2 \sqrt{EIg/wL^4}$ Load at 1/4 point.

M^2

Mode	Sep. Var.	Rayleigh	Ritz	Dunkerley
1	6.84	6.93	6.84	6.70
2	27.9		28.05	
3	80.0		80.70	

TABLE 3

Frequencies for Single Span Beam with Discrete Loads

Frequency Equation $p = C10^{-3}\sqrt{EIg/W}$

C

Mode	Influence Coefficient	Iteration	Rayleigh	Dunkerley
p_1	6.90	6.90	6.90	6.65
p_2	27.4	27.4		
p_3	58.0	57.6		

TABLE 4

Frequencies for Two Span Beam with Concentrated Loads

Frequency Equation $p = C \sqrt{EIg/WL^3}$

C

Mode	D'Alembert's Principle	Iteration	Rayleigh	Dunkerley
p_1	19.6	19.6		16.35
p_2	29.7	29.6	29.6	

VI DISCUSSION OF RESULTS

A. Single Span Beams with Load at Center

The Solution by the Exact Method resulted in a transcendental equation in the parameter M which was solved by graphing, for the first three modes and for three load conditions for the first mode.

The Rayleigh Method, producing only the first mode, was used in two different ways, first by a procedure outlined by Timoshenko[1] and second by a procedure suggested by Thomson[2]. Both procedures were equally effective and results were within one per cent of the exact solution.

The Dunkerley Equation gave the fundamental frequency within 2/3 per cent. It was found rapid to use and required few calculations.

The Ritz Method produced first, second and third mode frequencies which were never more than one per cent higher than the exact values. The degree of difficulty of the solution depended upon the number of terms in the assumed deflection curve, and hence the number of modes desired.

Freberg and Kemler[3], Timoshenko[1], and Den Hartog[5] have presented methods and formulas for lumping the beam weight at the center. These methods produced the first mode only. They were found to be rapid and accurate to within one per cent.

B. Single Span Beam with Load Not at Center

The Solution by the Exact Method produced a transcendental equation which was very tedious to solve. Solution was accomplished graphically, for the special case, with the concentrated load at a quarter point.

The Rayleigh Solution which has been reduced to formula form by Timoshenko[1] is accurate to within 1-1/2 per cent.

The Dunkerley Equation is rapid and easy to use. It produces the fundamental frequency only, which is about two per cent lower than the exact value.

The Ritz Method produced results which more closely coincided with the Exact Method than did any of the others. The fundamental was found to be the same and the higher modes did not differ by more than one per cent.

C. Single Span Beam with Discrete Loads.

The Influence Coefficient Method and the Iteration Method were used to produce the first three modes. Numerical results were identical for the first two modes and were within one per cent of each other for the third mode. For this problem, the former required the solution of an algebraic cubic equation, where as, the latter required the repeated application of the iteration procedure to a third order matrix.

The Rayleigh Method proved to be as accurate as the first two producing the same fundamental, and since influence coefficients were required, the same preliminaries were necessary. Solution was by substitution into an energy equation.

Dunkerley's Equation proved rapid to use but produced only the first mode which was found to be four per cent below the exact value.

D. Two Span Beam with Concentrated Loads.

The Iteration Method and the method in which D'Alembert's Principle is applied produced identical results. The former required the repeated application of the iteration procedure to a matrix equation, and the latter the solution of an algebraic equation.

The Dunkerley Equation, although rapid to use produced a result which was 17 per cent lower than the assumed exact value.

The Rayleigh Method, involving the use of influence coefficients and the energy relationship to frequency, produced the higher mode exactly. This was caused by the fact that the deflections used in the energy equation reflected the static conditions. And since the static curve closely approximated the symmetric or higher mode, that mode was obtained. Had the loading been assumed to be of the inertia type, such that when one load acted upward the other acted downward, the influence coefficients would have been positive and the anti-symmetric or lower mode would have been obtained.

VII CONCLUSIONS

The conclusions reached from the investigation will be considered under three categories; (1) Single Span Beam with One Concentrated Load, (2) Single Span Beam with Several Concentrated Loads, and (3) Two Span Beam with Concentrated Loads.

(1) <u>Single Span Beam with One Concentrated Load</u>

The method of solution requiring the application of boundary conditions to the solution of the beam equation was considered practical for the symmetrical case. However, for the unsymmetrical case, with the concentrated load not at the center, the resulting transcendental equation was considered too cumbersome to permit of a practical solution. The accuracy of this method was presumed to be good, as frequencies obtained by the Ritz Method compared very favorably.

The Ritz Method served as a check on the fundamental as obtained by the other methods as well as a practical way of finding two of the higher modes. It was found to be within one per cent of the exact method. The work involved and the time required were found to be dependent upon the number of modes desired. The number of terms in the assumed deflection equation corresponded with the degree of the final frequency equation and hence with the number of modes determined.

The Rayleigh Method for the determination of the first mode only, was found to be accurate to within one per cent, for both the symmetrical and the unsymmetrical cases. It presented no complications in manipulation and was reasonably rapid.

The Dunkerley Equation produced the fundamental frequency to within one per cent for the symmetrical case and to within two per cent for the unsymmetrical case. It presented no complication in manipulation requiring only the computation of a deflection and the frequency of a uniform beam.

(2) Single Span Beam with Several Concentrated Loads

All methods considered required the calculation of influence coefficients. Of the systems producing the fundamental only, the Rayleigh Method was the more accurate, assuming the exact solution was that produced by both the Iteration and the Influence Coefficient Methods.

The Dunkerley equation although much more rapid than the Rayleigh Method produced a frequency which was four per cent below the assumed exact value.

Results obtained from the Influence Coefficient Method and the Iteration Method coincided exactly for the first two modes and were within 3/4 per cent of each other on the third mode. The Iteration Method is probably the more practical of the two as the Influence Coefficient Method required the solution of an algebraic equation, the degree of which depended upon the number of concentrated loads.

(3) Two Span Beam with Concentrated Loads

All methods considered required the calculation of influence coefficients.

Results obtained by the application of D'Alembert's Principle and the Iteration Method were identical. The former required the solution of an algebraic equation, the degree of which depended on the number of concentrated loads. The latter method involved the application of the iteration procedure to a matrix equation, the order of the coefficient matrix being the same as the number of concentrated loads. For this particular problem the application of D'Alembert's Principle was more desirable.

The Rayleigh Method produced accurate results in this case, producing the higher mode. This was due to the influence coefficients having been computed from the static load conditions, and since the static curve closely approximated the symmetric mode, which was the higher one, that one was produced. Introduction of inertia instead of static loads would change the signs of the negative influence coefficients and would produce the fundamental frequency.

The Dunkerley Equation, although rapid to apply, produced results which were 17 per cent below the assumed exact value for the first mode frequency, due to the fact that the first and second mode frequencies were so close together.

VIII SUMMARY

The classical method, required for its solution, the application of boundary conditions to the solution of the beam equation. Except for the case of the beam with one concentrated load at the center, it was not considered a practical solution. The transcendental equation obtained in the solution of the unsymmetrical case, considered in part B, was found too cumbersome to handle. It was not attempted in parts C and D.

The Rayleigh Method proved to be a simple, accurate and reasonably rapid method for all cases considered.

The Dunkerley Equation gave very satisfactory results for parts A, B, and C. It was rapid to use, accurate and in most cases the data could be found in prepared tabulations. Results were inaccurate for the two span beam, indicating the necessity for caution in its application to multi-span beams.

The Ritz Method, which is a refinement of the Rayleigh Method, proved to be exceedingly accurate when applied to the beam with the single concentrated load. However, it was found, that as the number of terms in the assumed deflection equation increased, the work became more time consuming. It was used only in parts A and B.

The Influence Coefficient Method and the application of D'Alembert's Principle, which methods are quite similar, proved to be simple, accurate, and rapid. However, as the number of degrees of freedom increased, the degree of the algebraic equation increased, which complicated the solution.

The Iteration Method is probably the method to be used if the number of degrees of freedom exceeds three. As the number of modes increases the number of iterations would increase, but the individual operations in themselves would remain simple. This method proved simple and accurate to use. For the cases considered, it was more time consuming to use than either the Influence Coefficient Method or the application of D'Alembert's Principle. However, for higher degree situations, it should prove to be a more practical method.

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XI VITA

Yates Stirling 3rd was born in Manila, Phillipine Islands, on September 25, 1904. He attended various elementary, grammar and high schools in the United States graduating from Penn Charter in Philadelphia in 1921. He entered the United States Naval Academy in 1922 and was graduated in 1926. A Bachelor of Science Degree was awarded by act of Congress in 1938. On graduation he was commissioned an Ensign in the Navy. Service was continuous therein until 1946 when he was retired with the rank of Captain. From 1946 to 1947 he was actively affiliated with the Equitable Life Assurance Society of New York. In 1947 he began his civilian teaching career, teaching elementary mathematics at the St. Helena Extension of the College of William and Mary. In 1948 to 1951 he was a member of the faculty of the College of William and Mary - VPI at Norfolk. In 1951 he joined the faculty of the Norfolk Academy in Norfolk. He has accepted a position on the faculty of the College of William and Mary - VPI at Norfolk for the academic year 1954-1955.

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