5. FINITE ELEMENT ANALYSIS OF THE PROPELLER

In this chapter we describe the solid model and the finite element analysis of the propeller. In order to reduce the computational cost, we have done modal condensation on the finite element model of the propeller. We present the theory of modal condensation and the calculation of the steady state response. Finally, the method developed for the calculation of the second order statistics of the response of a linear system subjected to CS excitations is extended to the case of modal condensation.

5.1 Propeller Model and Finite Element Analysis Using I-DEAS

Propeller geometry has undergone considerable changes during the last two decades. The use of wider blades with increasing skewback made the older beam and shell theories inadequate for static and dynamic strength calculation. On the other hand, finite element method (FEM) has become a powerful tool for such static and dynamic analysis because of its successful applications. (Politis, 1984). We use I-DEAS™ for the modeling and the finite element analysis (FEA) of the propeller. Data of the propeller is provided by David Taylor Model Basin and is shown in Appendix A. Figure 5.1 shows the points and the lines joining those points to construct the hydrofoils. These hydrofoils were joined to make a blade. Constructing three blades of identical shape and size, we joined them with a cylindrical hub to construct the model of the propeller. Figure 5.2 shows the wire-frame geometry of the propeller blade and hub. The solid model was meshed using TET-10 elements (Beek, 1978). It has the shape of tetrahedron and implies 10 nodes of which 4 are located at the vertices and 6 in the center of the edges (Fig. 5.3). Each node has three degrees of freedom and consequently the element stiffness matrix contains 30 $\times$ 30 components. We select TET-10 element because 1) the smooth curvature of a propeller blade enables a fair approximation by means of flat-sided tetrahedrons 2) the root section of the blade and hub are relatively thick and 3-D elements are suited for that, and 3) the TET-10 element solution contains stresses, that vary linearly in all directions, so the predominant blade bending is represented easily. The numerical results of FEA of propeller are presented in chapter 6.

5.2 Modal Condensation

The governing system of equation for the propeller response can be given by

$$[\,\boldsymbol{M}\,]_{n\times n}\left\{\ddot{\boldsymbol{X}}\right\}_{n\times 1} + [\,\boldsymbol{C}\,]_{n\times n}\left\{\dot{\boldsymbol{X}}\right\}_{n\times 1} + [\,\boldsymbol{K}\,]_{n\times n}\left\{\boldsymbol{X}\right\}_{n\times 1} = \left\{\boldsymbol{F}\right\}_{n\times 1} \tag{5-1}$$

where, $[\boldsymbol{M}]_{n\times n}$, $[\boldsymbol{C}]_{n\times n}$, and $[\boldsymbol{K}]_{n\times n}$ are mass, damping, and stiffness matrices, respectively. These matrices are obtained by the FEA of the propeller. $[\boldsymbol{X}]_{n\times 1}$ and $[\boldsymbol{F}]_{n\times 1}$ are the displacement and force vectors. $\boldsymbol{F}$ is obtained using the expression for lift and drag developed in chapter 4. To calculate the mode shapes, we construct the undamped free vibration problem as

$$[\,\boldsymbol{M}\,]\left\{\ddot{\boldsymbol{X}}\right\} + [\,\boldsymbol{K}\,]\left\{\boldsymbol{X}\right\} = \left\{\boldsymbol{0}\right\} \tag{5-2}$$

Putting $\{\boldsymbol{X}\} = \{\boldsymbol{P}\}e^{j\omega t}$ in Eq. 5-2, we get

$$[-\omega^2 M + K]\left\{P\right\} = \left\{0\right\} \tag{5-3}$$

Eq. 5-3 can be rewritten as

$$[-I\lambda + A]\left\{P\right\} = \{0\} \tag{5-4}$$

where λ = $-\omega^2$ and λ = M^{-1} K. Nontrivial solution of Eq. 5-4 implies

$$-I\lambda + A = 0 \tag{5-5}$$

Equation 5-5 is the eigenvalue problem resulting to n eigenvalues λ_1, λ_2, … λ_n and n eigenvectors $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, … $\boldsymbol{\Phi}_n$. We are not considering the case of the repeated eigenvalues. In general, for an accurate estimate of the response $\boldsymbol{X(t)}$, we need large number of elements in the model and hence the large number of modes. The disadvantage of such complicated model is that it makes the calculation of second order statistics of the response computationally very expensive. While a large number of modes are computationally expensive, it may not be also needed in some cases. Frequency domain analysis of forcing function sometimes shows that the magnitude of the forces corresponding to frequency above a certain level is not significant and hence modes of the structure, whose frequencies are much higher than this, will not be significant in the analysis. For these reasons, it is beneficial to reduce the dimensions of the matrices in Eq. 5-1. A full modal analysis would include all the eigenvectors, but for the above-mentioned reasons, we will consider only from 1 to m (< n) eigenvectors. Constructing the modal matrix $\boldsymbol{\Phi}$ consisting of eigenvectors $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, … $\boldsymbol{\Phi}_m$, we get

$$[\boldsymbol{\Phi}]_{n\times m} = [\boldsymbol{\Phi_1}, \boldsymbol{\Phi_2}, \ldots \boldsymbol{\Phi_m}] \tag{5-6}$$

Let

$$[\boldsymbol{X}(t)]_{n\times n} = [\boldsymbol{\Phi}]_{n\times m}\{\boldsymbol{N}(t)\}_{m\times 1} \tag{5-7}$$

where $\boldsymbol{N(t)}$ is response vector in principal coordinate system.

Putting ***X(t)*** from Eq. 5-7 into 5-1, we get

$$[\boldsymbol{M}][\boldsymbol{\Phi}]\left\{\ddot{\boldsymbol{N}}\right\}_{m\times 1}+[\boldsymbol{C}][\boldsymbol{\Phi}]\left\{\dot{\boldsymbol{N}}\right\}_{m\times 1}+[\boldsymbol{K}][\boldsymbol{\Phi}]\left\{\boldsymbol{N}\right\}_{n\times 1}=\left\{\boldsymbol{F}\right\}_{n\times 1} \quad (5\text{-}8)$$

Pre-multiplying Eq. 5-8 by the $\boldsymbol{\Phi}^T$ i.e., transpose of $\boldsymbol{\Phi}$, we obtain

$$[\boldsymbol{\Phi}]^T[\boldsymbol{M}][\boldsymbol{\Phi}]\left\{\ddot{\boldsymbol{N}}\right\}+[\boldsymbol{\Phi}]^T[\boldsymbol{C}][\boldsymbol{\Phi}]\left\{\dot{\boldsymbol{N}}\right\}+[\boldsymbol{\Phi}]^T[\boldsymbol{K}][\boldsymbol{\Phi}]\left\{\boldsymbol{N}\right\}=[\boldsymbol{\Phi}]^T\left\{\boldsymbol{F}\right\} \quad (5\text{-}9)$$

We assume here a special case of viscous damping such that $[\boldsymbol{\Phi}]^T[\boldsymbol{C}][\boldsymbol{\Phi}]$ is diagonal, called modal damping matrix. This assumption is adequate in representing the damping of the structure if the damping is small which is the case for a propeller. To obtain it we set r^{th} diagonal coefficient C_r of the modal damping matrix equal to $2\xi_r\omega_r$, where ξ_r is the damping ratio and ω_r is natural frequency corresponding to mode r. At this point, we replace $[\boldsymbol{\Phi}]^T[\boldsymbol{M}][\boldsymbol{\Phi}]$ by $[\boldsymbol{M'}]$, called modal mass matrix, $[\boldsymbol{\Phi}]^T[\boldsymbol{K}][\boldsymbol{\Phi}]$ by $[\boldsymbol{K'}]$, called modal stiffness matrix, $[\boldsymbol{\Phi}]^T[\boldsymbol{C}][\boldsymbol{\Phi}]$ by $[\boldsymbol{C'}]$, the modal damping matrix., and $[\boldsymbol{\Phi}]^T[\boldsymbol{F}]$ by $[\boldsymbol{F'}]$, called modal force vector. Thus all the modal matrices are $m \times m$ diagonal matrices, modal force vector has a dimension of $m \times 1$, and Eq. 5-9 a system of m decoupled linear equations given by

$$[\boldsymbol{M'}]\left\{\ddot{\boldsymbol{N}}\right\}_{m\times 1}+[\boldsymbol{C'}]\left\{\dot{\boldsymbol{N}}\right\}_{m\times 1}+[\boldsymbol{K'}]\left\{\boldsymbol{N}\right\}_{n\times 1}=\left\{\boldsymbol{F'}\right\}_{n\times 1} \quad (5\text{-}10)$$

Equation 5-10 can be solved to obtain the displacements, ***N(t)***, in principal coordinate system and the displacement in the physical coordinate system can be obtained using Eq. 5-7.

5.2.1 Steady State Response

As mentioned earlier, Eq. 5-10 is system of decoupled linear equation. To obtain the steady state response we set ***{F}*** = ***{Fo}*** *cos Ω t*, where elements of ***{Fo}*** are fr_o's, in Eq. 5-10, we get

$$\ddot{\eta}_r(t) + 2\xi_r\omega_r\dot{\eta}_r(t) + \omega_r^2\eta_r(t) = \frac{fr_o}{M_r}\cos\Omega t \tag{5-11}$$

and the natural frequencies ω_r is given by

$$\omega_r = \sqrt{\frac{K_r}{M_r}} \tag{5-12}$$

and modal damping factor ξ_r is

$$\xi_r = \frac{C_r}{2M_r\omega_r} \tag{5-13}$$

where C_r, K_r , and M_r is the element from r^{th} row and r^{th} column of the diagonal damping, stiffness and mass matrices, respectively.

Eq. 5-11 gives the solution

$$\eta_r(t) = \frac{fr_o / K_r}{\sqrt{(1-r_r^{\,2})^2 + (2\xi_r r_r)^2}}\cos(\Omega t - \alpha_r) \tag{5-14}$$

where

$$\tan\alpha_r = \frac{2\xi_r r_r}{1 - r_r^{\,2}} \tag{5-15}$$

and

$$r_r = \frac{\Omega}{\omega_r}. \tag{5-16}$$

5.2.2 Modal Condensation and Input-Output Problem

As mentioned earlier, to reduce the dimension of the matrices involved in Eq. 5-1 and hence to reduce the computational cost, the modal condensation method can be adopted and the number of modes considered in the final calculation will depend upon the frequency content of the excitations. In this section we develop a method to calculate the response of a linear system subjected to CS excitation if the dimension of the system has been reduced using modal condensation. Without loss of any generality, we assume that the means of the excitations are zero. This implies that the means of the responses are also zero. Correlation matrix of the response $\boldsymbol{X(t)}$ in terms of $\boldsymbol{N(t)}$ can be written as

$$\boldsymbol{R_{XX}}(t_1,t_2) = E[\boldsymbol{X}(t_1)\boldsymbol{X}(t_2)^T] = E[\boldsymbol{\Phi N}(t_1)\boldsymbol{N}(t_2)^T\boldsymbol{\Phi}^T] \tag{5-17}$$

Taking the constants $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^T$ out of the expectation sign, we get

$$\boldsymbol{R_{XX}}(t_1,t_2) = \boldsymbol{\Phi}E[\boldsymbol{N}(t_1)\boldsymbol{N}(t_2)^T]\boldsymbol{\Phi}^T = \boldsymbol{\Phi R_{NN}}(t_1,t_2)\boldsymbol{\Phi}^T \tag{5-18}$$

Where $\boldsymbol{R_{NN}}(t_1, t_2)$ can be obtained using Eq. 5-10 and the method to calculate the correlation matrix of the response developed in chapter 2. However, we need a relation, which will relate the correlation matrix of the forces in the physical coordinate system to the correlation matrix of the forces in the principal coordinate system. Taking the steps similar to the above, we write the correlation matrix of $\boldsymbol{F'}$ as

$$\boldsymbol{R_{F'F'}}(t_1,t_2) = E[\boldsymbol{F'}(t_1)\boldsymbol{F'}(t_2)^T] = E[\boldsymbol{\Phi}^T\boldsymbol{F}(t_1)\boldsymbol{F}(t_2)^T\boldsymbol{\Phi}] \tag{5-19}$$

Taking the constants $\boldsymbol{\Phi}^T$ and $\boldsymbol{\Phi}$ out of the expectation sign, we get

$$\boldsymbol{R}_{\boldsymbol{F}'\boldsymbol{F}'}(t_1, t_2) = \boldsymbol{\Phi}^T E[\boldsymbol{F}(t_1)\boldsymbol{F}(t_2)^T]\boldsymbol{\Phi} = \boldsymbol{\Phi}^T \boldsymbol{R}_{\boldsymbol{FF}}(t_1, t_2)\boldsymbol{\Phi} \tag{5-20}$$

For a given problem, we know the correlation matrix of the forces $\boldsymbol{R}_{\boldsymbol{FF}}(t_1, t_2)$ and then using Eq. 5-20, we calculate the correlation matrix of the forces in principal coordinate system $\boldsymbol{R}_{\boldsymbol{F}'\boldsymbol{F}'}(t_1, t_2)$, which is then used in the calculation of correlation matrix of the response using the matrices involved in Eq. 5-10 and the method to calculate the second order statistics of the response developed in chapter 4.