Essays on Network Formation Games

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(ABSTRACT)

This dissertation focuses on studying various network formation games in Economics. We explore a different model in each chapter to capture various aspects of networks. Chapter 1 provides an overview of this dissertation.

Chapter 2 studies the possible Nash equilibrium configurations in a model of signed network formation as proposed by Hiller (2017). We specify the Nash equilibria in the case of heterogeneous agents. We find 3 possible Nash equilibrium configurations: Utopia network, positive assortative matching, and disassortative matching. We derive the specific conditions under which they arise in a Nash equilibrium.

In Chapter 3, we study a generalized model of signed network formation game where the players can choose not only positive and negative links but also neutral links. We check whether the results of the signed network formation model in the literature still hold in our generalized framework using the notion of pairwise Nash equilibrium.

Chapter 4 studies inequality in a weighted network formation model using the notion of Nash equilibrium. As a factor of inequality, there are two types of players: Rich players and poor players. We show that both rich and poor players designate other rich players as their best friends. As a result, We present that nested split graphs are drawn from survey data because researchers tend to ask respondents to list only a few friends.

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(GENERAL AUDIENCE ABSTRACT)

This dissertation focuses on studying various network formation games in Economics. We explore a different model in each chapter to capture various aspects of networks. Chapter 1 provides an overview of this dissertation.

Chapter 2 studies the possible singed network configurations in equilibrium. In the signed network, players can choose a positive (+) relationship or a negative (-) relationship toward each other player. We study the case that the players are heterogeneous. We find 3 possible categories of networks in equilibrium: Utopia network, positive assortative matching, and disassortative matching. We derive the specific conditions under which they arise in equilibrium.

In Chapter 3, we study a generalized model of signed network formation game where the players can choose not only positive and negative links but also neutral links. We check whether the results of the signed network formation model in the literature still hold in our generalized framework.

Chapter 4 studies inequality in a weighted network formation model using the notion of Nash equilibrium. In this weighted network model, each player can choose the level of relationship. As a factor of inequality, there are two types of players: rich players and poor players. We show that both rich and poor players choose other rich players as their best friends. As a result, we present that nested split graphs are drawn from survey data because these social network data are censored due to the limit of the number of responses.

Dedication

I dedicate this work to all my family:

my grandfather, Seong Youl Kim,

my grandmother, Kyeong Sun Lee,

my grandmother, Tae-Soon Kwon,

father, Jaeng Hwan Kim,

and mother Hae Ja Yoo.

They are my root and my haven.

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Chapter 1

Introduction

In economics, network formation theory studies what kind of network economic agents form under the assumption that they respectively maximize their utility. In particular, we explore what kind of network is derived when applying the Nash equilibrium or its refinement, which has been studied for the simultaneous move game. Since Jackson and Wolinsky (1996) showed the importance of the theory of network formation, many researchers have studied friendship networks. In particular, game theorists have done a lot of research on discrete decision-making models where players decide whether or not to form friendships with other players.

Recently, there have been attempts to develop network formation games in various directions, such as signed networks or weighted networks. In this dissertation, we study what kinds of configurations can be derived in equilibrium using various networks. Particularly, we mainly study the network models when different types of players exist.

1.1 Signed Network Formation with Heterogeneous Players

In Chapter 2, we study a singed network formation game where there are heterogeneous players. In signed networks, each link (edge) can have a positive or negative sign. In the context of social networks in Economics, the positive link represents friendship (cooperation), and the negative one means conflict (competition). The notion of signed networks was first discussed by Heider (1946) to study social psychology. Since he suggested the idea, the signed network model has been studied out of Economics. In Economics, for the first time, Hiller (2017) studied the signed network formation model, using the game-theoretical approach such as a notion of Nash equilibrium. In his model, players can increase their strength by finding friends or can choose conflicts so that a player with high strength can plunder another player with weaker strength.

While Hiller (2017) mainly focused on the homogeneous player model, Chapter 2 studies which network configurations emerge in equilibrium when there are heterogeneous players. We assumed that there are two types of players: strong players and weak players. Thus it is possible to define the difference between the two types to study how the network changes in equilibrium, depending on how different the players are. While each strong player is stronger than each weak player, the weak players can be stronger than the strong players if the friendship network of weak players is well organized.

We sort all Nash equilibrium network formations into three categories: Utopia network, Positive assortative matching, and Disassortative matching. In Utopia network, everyone is a friend to everyone. This network is a Nash equilibrium when conflict cost is high. Positive assortative matching means that only the players of the same type become friends. When the difference between the two types of players is large and the conflict cost is low, every Nash equilibrium network is a positive assortative matching, and all strong players bully the weak players. On the other hand, there is a positive assortative matching where the weak players cooperate in bullying all strong players. This network can be a Nash equilibrium if the weak players are much more than the strong players, the difference between the players is small, and the conflict cost is low. Lastly, if the difference between the players is low, and the conflict cost is low, disassortative matchings, where some strong players and some weak players are friends, can be a Nash equilibrium.

1.2 Signed Network Formation with Neutral Links

Chapter 3 studies a more generalized signed network formation model, where players can choose neutral links as well as positive links and negative links. In the real world, there exist the neutral links (no link) in social networks. Switzerland, for example, as a neutral country, has not formed any alliances. Hence, it has been asked to construct the signed network model with neutral links. This chapter focuses on testing whether some results in previous researches hold in this general model using the notion of pairwise Nash equilibrium.

First, we derived a result regarding the empty network similar to Jackson and Nei (2015). They analyzed another signed network formation model using a notion of war stable networks. They showed that only the empty network could be the war stable network unless there is no third factor. Similarly, we show that the empty network is a pairwise Nash equilibrium. Jackson and Nei (2015) and our research imply that the empty network is in equilibrium. Second, Hiller (2017) showed that bullying network where the other players bully one player is a Nash equilibrium. We modify the notion of this bullying network to the general model and determine when these bullying networks can be pairwise Nash equilibria. Lastly, we study positive assortative matching discussed in Chapter 2 in the general model. We show that when the difference between strong players and weak players is significantly large and the conflict cost is low, strong players bully weak players. In most of the case, this network formation is a positive assortative matching. However, there exists a case where even the same type of players are not friends, but every strong player bullies every weak player.

1.3 Weighted Network Formation with Heterogeneous Players

In Chapter 4, I study another kind of network formation, a weighted network formation. While the players in Chapters 2 and 3 can choose different kinds of links, the players in this model can choose a weight (level) of the link (relationship). Salonen (2016), Baumann (2021), and Griffith (2019a) studied these weighed network formation models. In their models, each player spends her endowment in self-investment and one to one relation-investment. The authors used the notion of Nash equilibrium, and focused on the homogeneous player model.

This chapter focuses on the weighted network formation model with inequality. There are two types of players: rich and poor players. I study how the players behave in any symmetric Nash equilibrium. Overall, in equilibrium, the players prefer rich players regardless of their types. For rich players, the result can be interpreted as homophily, while it is heterophily for poor players.

This finding induces an interesting property of networks on social network data. In surveys investigating social networks, respondents are required to designate a limited number of their friends, such as "List your closest friends [1] .". Thus, only the strongest links are reported due to the limit of the number of responses. Since every player prefers rich players, rich players form the core in the data. On the other hand, the poor players constitute a periphery set (independent set). As a result of this limit, nested split networks appear in the data.

[1]This instruction is from the National Longitudinal Study of Adolescent to Adult Health (Add Health).

Chapter 2

Weak Players, Strong Players, and Signed Network Formations

2.1 Introduction

A signed network, which is an application of graph theory, is a useful tool to analyze a society where friendly and antagonistic relationships coexist. The signed network consists of nodes, positive links, and negative links. The nodes denote people, the positive link denotes friendship, and the negative link means conflict. The literature on network economics focuses on the positive links between players, which have been well documented in the research on peer effects. Also, economists have studied hostile behavior and antagonistic relationships, which lead to conflicts. However, little work has been done on the study of the interaction between the positive relationships and the negative relationships within a group of agents. Also, economists have studied hostile behavior and antagonistic relationships, which lead to conflicts.

In history, there are many examples where conflict has affected friendship and vice-versa. World War II is perhaps the most famous instance. During the war, almost all nations on earth belonged to one of the two warring factions. Most of these countries were not related to the war at first but became friends or enemies with one another because of indirect relationships. Nazi Germany engaged the U.S.A. after the bombing of Pear Harbour by

Japan. Also, the Soviet Union joined the Allies because Nazi Germany attacked the Soviet Union. As another example, in the 18th century, many European countries forged an alliance against Napoleon's French Empire, even though they were enemies against each other before the war. In the Eastern world, Kuomintang and Communist party collaborated to fight against the Empire of Japan in World War II. However, after World War II, they terminated the collaboration and initiated the Chinese Civil War.

The signed network model was, for the first time, used in social psychology as the name of the signed graph theory to describe a balanced relationship among three entities (Heider, 1946). They had the idea that if two people were friends, it is balanced to have the same opinion to the other entity. If they were enemies, it was also balanced to have different opinions. They realized that the notion of the signed graph was useful to formalize the idea because the signed graph consists of two kinds of links. The signed graph theory has been applied not only in sociology but in other disciplines such as the study of international relations (Brown, 1979) and politics (Laumann and Pappi, 1973). A common assumption in these works is that the balanced signed graph is a kind of equilibrium, and there is a force guaranteeing a structurally balanced network.

Hiller (2017) was the first to propose a model of signed network formation model using a game theoretical approach. In his model, every player can extend either the friendly (positive) link or the antagonistic (negative) link to each of the remaining agents. Even though each agent has an intrinsic strength given exogenously, an agent's real strength in conflict is the sum of her intrinsic strength plus the intrinsic strength of her friends (or allies). This sum is called the network strength of the player. That is, each player can increase her network strength by making alliances. Moreover, if the relationship between two given players is antagonistic, then the player with higher network strength plunders payoff from the other. There are conflict costs associated with the negative links. His main finding is that every

Nash Equilibrium obeys weak structural balance even though there is no direct benefit for each player when their relationship satisfies structural balance in the network. It also means that the Nash equilibrium configurations are such that either all links are positive or players can be divided into distinct sets where the people in each set are friends with each other. He further characterizes pure strategy Nash equilibria for a general class of payoff functions which map strength of agents into extraction payoff under the antagonistic relationship.

The present paper extends the work of Hiller (2017) by specifically examining the case of heterogeneous players. We analyze Nash equilibrium configurations when players are either homogeneous or have one of two types: strong or weak. We consider a model with two types for tractability. As we shall see, this simple model with two types entails significant changes in comparison with the model with homogeneous agents.

One of our findings is a sufficient condition for all Nash equilibrium network configurations to exhibit assortative matching. Positive assortative matching is a matching between the same type of players. In matching theory, positive assortative matching has been a major research topic. For example, it is a stylized fact that marriage markets exhibit positive assortative matching in marriage markets with respect to education levels[1]. It means that highly educated males tend to get married to highly educated females. Moreover, some labor economists borrowed this terminology from the marriage market to describe the matching of firms and workers. In labor economics, positive assortative matching means that highly productive firms hire highly productive workers. Different from the marriage market, researchers have controversy as to whether positive assortative matching is a general property characterizing the labor market. In our signed network formation model consisting of heterogeneous players, positive assortative matching means that the positive relationships exist only between the players of the same type. As the antithesis, naturally, there is only the

[1] In this field, positive assortative matching is also called positive assortative mating.

negative relationship between the players of the different types. In that different types do not have any cooperative relationships, positive assortative matching can be interpreted as a segregation of the society by inherency such as gender, race, or economic background.

In this paper, we use the model with homogeneous players as a benchmark model. We characterize conditions under which complete networks or segregation into two uneven groups can be sustained in a Nash equilibrium. Then, we also describe all possible equilibrium configurations in the case of four players with two type (strong and weak). In the heterogeneous model, all Nash equilibrium configurations are classified into three categories: Utopia network, positive assortative matching, and disassortative matching. In Utopia network, each player is a friend to every other regardless of their types. We find that Utopia network arises in the Nash equilibrium if the cost of conflict is sufficiently high (with all else kept constant). In a positive assortative matching, players coalesce only with the same type of players. We categorize positive assortative matching according to who is dominant in the network configuration. If every strong player has higher network strength than every weak player, we call the network strong dominant. On the other hand, if every weak player has higher network strength than every strong player (because the number of weak players is relatively large) in the positive assortative matching, the network is called weak dominant. Lastly, there exist positive assortative matchings neither strong dominant nor weak dominant. Our major finding is that every Nash equilibrium exhibits strong dominant positive assortative matching when the gap between the two types is large enough and when the conflict cost is small enough. Also, we derive a sufficient condition for weak dominant positive assortative matching networks to be Nash equilibria. Positive assortative matching gives an intuition for social phenomena such as discrimination, segregation, and bullying. Lastly, in disassortative matching, there exists at least one friendship link between two players of different types. Bullying networks, where the rest of the players bullies one player, are examples of disassortative matching. We derive the condition for these bullying networks to be Nash equilibria, too.

The paper is structured as follows. Section 2.2 reviews the relevant literature. In Section 2.3, we introduce our model and definitions. In Section 2.4, we show the result of the research. Lastly, We conclude in Section 2.5.

2.2 Literature Review

The signed network began to be studied in sociological psychology as the name of the signed graph theory. Heider (1946) proposed the first idea regarding the interaction between positive relationships and negative relationships among people. He proposed a model where there were two people and one entity. He argued that this triangular graph is balanced if they are friends and their opinions to the entity are the same, or if their relationship is antagonistic and their opinions are different. Luce (1950) defined a clique. A clique is a subset where there are reciprocated directed links for every pair of nodes of the clique, and the clique is not a proper subset of any other set of elements satisfying this property. Following Heider (1946), Cartwright and Harary (1956) and Harary et al. (1953, 1955) formally developed this idea of the balanced graph using the notions of graph theory. They defined the structurally balanced graph with a local property that all triads in the graph are connected either with three positive links or with one positive link and two negative links. Thus, in any structurally balanced graph, my friend's friend is my friend, my enemy's friend is my enemy, and my enemy's enemy is my friend. They also showed that any structurally balanced graph satisfies a global property that the whole graph is segregated to two cliques. Between the two groups, there are only negative links. Davis (1967) defined clustering which had the same meaning to weakly structural balance. Clustering is a partition of the node set into multiple subsets where each positive link connects a node with another point in the same subset and each negative line connects a point with another point in the different subset. When a network has clustering, my friend's friend is always my friend, my enemy's friend is my enemy, but my enemy's enemy is not necessarily my friend. On the other hand, in any structurally balanced network, an enemy of my enemy is my friend.

Recently, some researchers approached the signed network model by using game-theoretical tools. They focused on the endogenous network formation, where players are free to choose their relationships with the other players for their interest. Hiller (2017) studied a signed network formation model with positive and negative links. In his model, players can choose friendship and enmities. Their respective choices uniquely determine a network, which in turn determines their payoffs. This networks formed by the players' respective strategies may a priori be structurally unbalanced. However, any network in equilibrium is structurally balanced.

Jackson and Nei (2015) analyzed another signed network formation model where players can choose the positive, negative, and neutral (no) links. They introduced a production factor such as trade. They showed that trade and high war cost decreased conflict in the network formation game and examined the theoretical result with data. Pandey (2021) is an application of Hiller (2017). Her paper is similar to our paper because she studied Nash equilibrium configurations when the players are heterogeneous. However, her model has some different aspects from our paper. We directly use Hiller (2017)'s model and generalize the equilibrium configuration results in the binary heterogeneous model. On the other hand, she modified some settings of his model. First, in her paper, when the players are friends, the most powerful player's intrinsic strength became the other friend's strength. Except for the most powerful friend, the other friends do not affect the strength[2]. Second, the conflict cost

[2] In Hiller (2017), the network strength is the sum of the intrinsic strength of the player herself and her friends in the network.

happens only if bilateral players show negative attitudes against each other[3]. . If only one player extends the negative link, then it is called coercion. She founds some results from the different setting. First, in any Nash equilibrium, only coercions happen, but the conflicts, where bilateral players fight against each other, do not occur. Second, if two players' intrinsic strengths are different and they are friends in a Nash equilibrium, then the player with the less intrinsic strength has another friend with higher intrinsic strength. Third, if the number of the strongest players is more than (or equal to) two, a network where everyone is a friend of everyone is a Nash equilibrium.

The signed network formation model is based on the network formation model. Jackson and Wolinsky (1996) defined pairwise stability in an idea that both players have to consent to form the positive link between them. Before they invented the condition, Nash equilibrium concept was not enough to derive significant network formation. Suppose two people are not friends. If it is better for both of them to have the new friendship, they will make a consensus to be friends. However, this deviation cannot be examined with Nash equilibrium concept because Nash equilibrium only considers deviations by a single player. Calvó-Armengol and İlkılıç (2009) incorporated the condition of pairwise stability and Nash equilibrium, and defined a pairwise Nash equilibrium network regarding a simultaneous move game of network formation. In pairwise stable networks, only single links are respectively examined whether it is pairwisely stable or not. On the other hand, in pairwise Nash networks, we also check multiple link deviations made by a single player in a spirit of Nash stability.

Besides the network formation model based on the game-theoretical approach, the signed network model could be applied from other approaches. König et al. (2017) obtained a Nash equilibrium fighting effort when there are alliance and enmity network given. They

[3] In Hiller (2017), there is two fold conflict cost. When a player shows the negative attitude, the first kind of conflict cost occurs. Also, if a player engages in a conflict, the second kind of conflict cost occurs, adding to the first conflict cost.

implemented the equilibrium result in the empirical analysis with the Second Congo War data. They estimated the effects of the network and predicted an expected impact of policy intervening in the conflict. Antal et al. (2006) and Cisneros-Velarde and Bullo (2019) studied network formation process, but still assumed that people prefer structural balance. Antal et al. (2006) analyzed the formation dynamically by using an updating rule which pursues a structurally balanced network. On the other hand, Cisneros-Velarde and Bullo (2019) used the Nash equilibrium concept to analyze the signed network. The utility function in this model counts the number of balanced triads.

Besides the signed network model, researchers actively have studied the interaction between cooperative behavior and uncooperative behavior in various ways. Goyal et al. (2014) analyzed a game where firms compete against each other to take consumers on a social network in an emerging market. They implemented dynamic analysis and equilibrium analysis to specify the conditions for efficiency. Grandjean et al. (2017) modeled a sequential game where players formed an alliance network in the first stage and had a contest in the second stage. They found that social designer can increase the total surplus by making the players more asymmetric when free exit is not allowed. They also showed that a barrier to entry may hurt the total surplus, and the networking can act as the barrier to entry. Bozbay and Vesperoni (2018) axiomatically analyzed Tullock contest success function for all against all contest given an alliance network.

In this paper, we derived many results using the general form of functions to describe pillage occurring on the negative relationships, but also used a normalized contest success function to describe the conflict in some parts. The origin of this normalized contest success function is from contest theory. Contest theory is a discipline studying competition dealing with conflict. Tullock (1967) initiated a discussion of Contest. He argued that tariff and monopoly were inefficient because these brought a welfare transfer from the import sector to the domestic production and from consumers to the monopolist, not generating new welfare. Moreover, this welfare transfer triggers a competition to take or keep the transferred welfare. This competition was named rent-seeking activity, later. To describe the conflict, Tullock (1980) introduced a mathematical model. This model is now known as "Tullock contest success function". Hirshleifer (1989) analyzed Tullock contest success function and suggested another contest success function[4]. Hiller (2017) defined a normalized contest success function by subtracting $\frac{1}{2}$ from the contest success function. This normalized contest success function describes a zero-sum game conflict on the negative relationship. One good thing of this function is that we can observe how a change of zero-sum game conflict by controlling the level of conflict technology. In this function, there is a parameter implying the conflict technology. If the parameter is high, small difference in network strength between the players results in a huge difference in this conflict.

Lastly, positive assortative matching has been documented in economics. Roy (1951) mentioned his idea that earning had not been determined independently but had been affected by various factors such that who meets whom as their colleague. Becker (1973) formalized a marriage market model where men tend to marry women with similar traits. Following Becker (1973), many researchers developed the matching theory and studied the condition when matching is positive assortative (Johnson, 2013, Legros and Newman, 2007, Li et al., 2013, Shimer and Smith, 2000). Other researchers were interested in whether this positive assortative matching occurs in the real world or not. Çelikaksoy et al. (2006), Chiappori et al. (2012), Greenwood et al. (2014), and Siow (2015) showed that there exists positive assortative matching by their education level in the marriage market. In labor economics, researchers have reported the existence of positive assortative matching between productive

[4] Hirshleifer (1989) named Tullock contest success function "contest success function in ratio form" and called the new contest success functoin "contest success function in difference form".

firms and productive workers (Andrews et al., 2012, Kremer, 1993, Mendes et al., 2010)[5]. Lastly, there were trials to draw positive assortative matching using joint liability in a loan market to overcome aymmetric information problem. (Ghatak, 1999, 2000, Van Tassel, 1999)

2.3 Model

Let $N = \{1, 2, ..., n\}$ denote the set of players. We assume that there are two types of players, $t \in \{s, w\}$, where s represents strong type and w represents weak type. Let N_t, $t \in \{s, w\}$, denote the set of players whose cardinality is given by n_t. Thus, $N = N_S \cup N_W$ and $n = n_s + n_w$. Based on her own type, each player has her own intrinsic power. So player i has an intrinsic strength λ_s if $i \in N_S$, and λ_w if $i \in N_w$. By definition $\lambda_s > \lambda_w > 0$.

Every player can either extend a positive (friendly) directed link or a negative (antagonistic) directed link to the remaining players. Player $i \in N$ chooses $\mathrm{g}_{i,j} \in \{1, -1\}$ for all $j \in N \setminus \{i\}$ where 1 denotes the friendly link and -1 denotes the negative link. Aggregating all choices, player i's strategy is a vector $\mathbf{g}_i = (\mathrm{g}_{i,1}, \mathrm{g}_{i,2}, ..., \mathrm{g}_{i,i-1}, \mathrm{g}_{i,i+1}, ..., \mathrm{g}_{i,n})$. Each element is represented by a directed link from one player to the other. The space of player i's strategy $\mathbf{g}_i$ is defined by G_i for all $i \in N$. Let $\mathbf{g}_{-i} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_{i-1}, \mathbf{g}_{i+1}, ..., \mathbf{g}_n)$ be a set of all the players' strategies except for player i's strategy $\mathbf{g}_i$. The players' strategy profile is a directed network $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$. Therefore, $(\mathbf{g}_i, \mathbf{g}_{-i})$ is the same to $\mathbf{g}$. The joint strategy space for $\mathbf{g}$ is given by $G = G_1 \times ... \times G_n$. To express a deviation strategy, a change in directed link $\mathrm{g}_{i,j}$ is denoted as follows: Given a network $\mathbf{g}$, $\mathbf{g} + \mathrm{g}^+_{i,j}$ changes the directed link from $\mathrm{g}_{i,j} = -1$ to $\mathrm{g}_{i,j} = 1$, maintaining the rest of the directed links. Similarly, $\mathbf{g} + \mathrm{g}^-_{i,j}$ changes the directed link from $\mathrm{g}_{i,j} = 1$ to $\mathrm{g}_{i,j} = -1$. To denote player i's deviation strategy and its strategy profile, we normally use notations $\mathbf{g}'_i$ and $\mathbf{g}' = \mathbf{g} + \sum_{j \in A} \mathrm{g}^+_{i,j} + \sum_{j \in B} \mathrm{g}^-_{i,j}$ for

[5] There are other papers reporting disassortative matching between firms and workers, ((Abowd et al., 2004, Andrews et al., 2006) but they were not published.

$A = \{g_{i,j} \mid g_{i,j} = -1,\ g'_{i,j} = 1\}$ and $B = \{g_{i,j} \mid g_{i,j} = 1,\ g'_{i,j} = -1\}$. If $g_{i,j} = 1$ in $\mathbf{g}$, $\mathbf{g} + g^{+}_{i,j} = \mathbf{g}$. (If $g_{i,j} = -1$ in $\mathbf{g}$, $\mathbf{g} + g^{-}_{i,j} = \mathbf{g}$.)

Relationships between players are formed according to their attitude towards each other. The relationship between i and j is denoted by an undirected link $\bar{g}_{i,j} = \bar{g}_{j,i} = min\{g_{i,j}, g_{j,i}\} \in \{-1, 1\}$. As this *min* function implies, the worst attitude between them determines this bilateral relationship. If both players are friendly, then they will be good friends. If one of them is antagonistic, then they will have the negative undirected link. Similar to the directed network $\mathbf{g}$, player i's relationships are represented by $\bar{\mathbf{g}}_i = (\bar{g}_{i,1}, \bar{g}_{i,2}, ..., \bar{g}_{i,i-1}, \bar{g}_{i,i+1}, ..., \bar{g}_{i,n}) \in \bar{G}_i$, and the undirected network is $\bar{\mathbf{g}} = (\bar{\mathbf{g}}_1, \bar{\mathbf{g}}_2, ..., \bar{\mathbf{g}}_n) \in \bar{G}$.

If $\bar{g}_{ij} = 1$, player i and j are friends. the players get two kinds of benefits from the positive relationship. First, these two players can strengthen each other's power. Let $N^{+}_i(\mathbf{g}) = \{j \in N \mid \bar{g}_{ij} = 1\}$ denote a set of players with whom player i has an undirected positive link, and y_i denote player i's network strength, which is a result of her intrinsic quality and her network of friends. Thus, y_i is determined as follows.

$$y_i(\mathbf{g}_i, \mathbf{g}_{-i}) = \lambda_i + \sum_{j \in N^{+}_i(\mathbf{g})} \lambda_j.$$

Therefore, apart from the type of players, each player can have different level of y_i depending on $\mathbf{g}$. An important consequence of this is that a weak player can end up with high strength simply by having strong allies. To sort the players by the network strength, let $\bar{y}_1 < \bar{y}_2 < ... < \bar{y}_m$ be the network strengths in a network $\mathbf{g}$ where $1 \leq m \leq n$. Then, let $P_i \in \{P_1, P_2, ..., P_m\}$ denote the set of the players whose network strengths are $\bar{y}_i$. That is, $P_1(P_m)$ is the set of the weakest (strongest) players. Second, there may exist a direct benefit from the positive relationship, such as benefit from trade. The direct benefit is denoted by $\beta \geq 0$, and both player i and j acquire it equally. We assume that β is identical regardless of the type of the players i and j.

If $\bar{g}_{ij} = -1$, player i and j are enemies against each other. They do not share their strength and enter a zero-sum competition. $N_i^{e-}(\mathbf{g}) = \{j \in N \mid g_{i,j} = -1\}$ denotes the set of players to whom player i extends a negative link. $N_i^{-}(\mathbf{g}) = \{j \in N \mid \bar{g}_{i,j} = -1\}$ denotes the set of players with whom i has the negative link $\bar{\mathbf{g}}$. i.e., the set of players with whom player i is engaged in conflict. In this zero-sum game, the player with the higher network strength extracts payoff from the player with lower network strength. Let $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$ denote player i's extraction from player j in a network $\mathbf{g}$. The network $\mathbf{g}$ determines y_i and y_j, which determines $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$. Thus, the extraction function $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$ is ultimately a function of $\mathbf{g}$, but it is also possible to define some property of the extractions function regarding the value of y_i and y_j. We will abbreviate the extraction function to $f(y_i, y_j)$ when there is no confusion.

Following Hiller (2017), we define the general extraction function as follows. Given n players, for any $\mathbf{g}$, if $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$ has a lower bound, $\underline{f}(\mathbf{g}) = inf(f(y_i(\mathbf{g}), y_j(\mathbf{g}))$. In the same way, given n players, for any $\mathbf{g}$, if $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$ has an upper bound, $\bar{f}(\mathbf{g}) = Sup(f(y_i(\mathbf{g}), y_j(\mathbf{g}))$. When $y_i > y_j$ the value of $f(y_i, y_j)$ is positive, but when $y_i < y_j$, it is negative. Moreover, if a pair of players with equal strength is negatively connected, then no payoff extraction takes place: $f(y_i, y_j) = 0 \; \forall y_i, y_j : y_i = y_j$. Next, we assume that $f(y_i, y_j)$ is strictly increasing in y_i ($\frac{df(y_i,y_j)}{dy_i} > 0$) and decreasing in y_j ($\frac{df(y_i,y_j)}{dy_j} < 0$). Following the definition of zero-sum game, the value a player extracts through an antagonistic link is assumed to be another player's loss, hence $f(y_i, y_j) + f(y_j, y_i) = 0, \; \forall y_i, y_j$.

We further make the following assumption: (i) $f(y_i, y_j)$ is homogeneous of degree 0. That is, $f(y_i, y_j) = f(\frac{y_i}{y_j}, 1)$. Therefore, this function can be used to describe conflicts where the relative size of strength matters. Using this property, let's denote $a = \frac{y_i}{y_j}$, a ratio of the different intrinsic strengths. Then $f(y_i, y_j) = f(a, 1)$. (ii) $\frac{d^2 f(y_i,y_j)}{dy_i^2} < 0$ when $y_i > y_j$. It means that a marginal return of extraction in network strength is decreasing.

Following Hiller (2017), we also consider a normalized contest success function which satisfies all the properties of $f(y_i, y_j)$ mentioned above. This is shown in Appendix.

Definition 2.1 (Hiller (2017))**.** The normalized contest success function (in ratio form) is

$$h(y_i, y_j, \phi) = \frac{y_i^{\phi}}{y_i^{\phi} + y_j^{\phi}} - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$$

where $\phi > 0$.

This normalized contest success function is a special case of extraction function which incorporates an interesting parameter. This parameter ϕ characterizes the technology of extraction, where $\frac{dh}{d\phi} > 0$ when $y_i > y_j$ This property implies that as ϕ increases, $h(y_i, y_j, \phi)$ increases for given y_i and y_j. This allow us to examine some interesting cases. When ϕ is high, small differences can result in the large size of the extraction. When ϕ is low, i.e., when the extraction technology is inefficient, for any value of y_i and y_j, the value of the normalized contest success function (the value of extraction) is close to zero. Formally, if ϕ goes to infinity, $h(y_i, y_j, \phi)$ is close to $\frac{1}{2}$ when $y_i > y_j$, and if ϕ goes to 0, $h(y_i, y_j, \phi)$ is close to 0. This is because $h(y_i, y_j, \phi) \in (-\frac{1}{2}, \frac{1}{2})$ for any y_i, y_j and ϕ The proof of the property is also in Appendix.

In this model, there is no cost associated with friendship[6] , but there is a cost associated with conflict. Extending a negative link results is akin to picking a fight, which imposes a cost $\varepsilon > 0$. When there is at least one directed negative link between players, both players are assumed to engage in the bilateral conflict, and the cost of war (cost of conflict) is $\omega \geq 0$.

The utility of player i under a strategy profile $\mathbf{g}$ is given by

[6] The benefit of the friendship $\beta(\geq 0)$ can be regarded as the net benefit (the benefit minus the friendship cost). Then, this model deals with the case where the friendship benefit is larger than (or equal to) the cost of friendship.

$$v_i(\mathbf{g}_i, \mathbf{g}_{-i}) = \sum_{j \in N_i^-(\mathbf{g})} f(y_i(\mathbf{g}_i, \mathbf{g}_{-i}), y_j(\mathbf{g}_j, \mathbf{g}_{-j})) + \mid N_i^+(\mathbf{g}) \mid \beta - \mid N_i^{e-}(\mathbf{g}) \mid \varepsilon - \mid N_i^-(\mathbf{g}) \mid \omega.$$

Note that $\mid N_i^+(\mathbf{g}) \mid + \mid N_i^-(\mathbf{g}) \mid = n - 1$, so $\mid N_i^+(\mathbf{g}) \mid \beta = (n - 1 - \mid N_i^-(\mathbf{g}) \mid)\beta$. To make the utility function simpler, let's transform it monotonically by i) deducting $(n-1)\beta$ and ii) integrating β and ω to κ as follows.

$$u_i(\mathbf{g}_i, \mathbf{g}_{-i}) = \sum_{j \in N_i^-(\mathbf{g})} f(y_i(\mathbf{g}_i, \mathbf{g}_{-i}), y_j(\mathbf{g}_j, \mathbf{g}_{-j})) - \mid N_i^{e-}(\mathbf{g}) \mid \varepsilon - \mid N_i^-(\mathbf{g}) \mid \kappa. \tag{2.1}$$

$\kappa = \omega - \beta$ is an opportunity cost to increase one negative undirected link. If a positive undirected link $\bar{\mathrm{g}}_{i,j} = 1$ is changed to the negative link $\bar{\mathrm{g}}_{i,j} = -1$, the direct benefit β from $\bar{\mathrm{g}}_{i,j}$ disappears and the war cost ω occurs. As a result, the change incurs κ.

In this normalized utility function $u_i(\mathbf{g})$ if there is no conflict, then player i's utility is 0 as a result of the normalization. However, the ranking of utility is still preserved. Thus, it is possible to apply every result derived from the normalized utility function $u_i(\mathbf{g})$ to the original utility function $v_i(\mathbf{g})$ with β.

Note that the same cost value may either exclude the possibility of conflict or not, depending on the parameters of networks such as the number of players. For example, for the homogeneous case, when there are five players, $\bar{f}(\mathbf{g}) = f(4,1)$, and if there are six players, $\bar{f}(\mathbf{g}) = f(5,1)$. If $f(4,1) < \varepsilon + \kappa < f(5,1)$, this $\varepsilon + \kappa$ excludes the possibility of conflict in a network with five players, but it allows for conflict in a network with six players. Our research considers this aspect by relaxing the restriction on $\varepsilon + \kappa$[7].

[7] Hiller (2017) assumed that $\varepsilon + \kappa < f(\sum_{j \in N \setminus \{k\}} \lambda_j, \lambda_k)$ where $\lambda_k \leq \lambda_j \ \forall j \in N$. It means that the cost of conflict in his model is always lower than the maximum possible extraction. He focused on conflict phenomenon by using the assumption. He did not discuss the case where the networks with no conflict are the Nash equilibrium because of the high conflict cost. Our model relaxes the assumption, so $\varepsilon + \kappa$ is only

As mentioned above, we also use the normalized contest success function $h(y_i, y_j, \phi)$ instead of $f(y_i, y_j)$. Let $u_i^h(\mathbf{g}_i, \mathbf{g}_{-i}, \phi)$ the utility of a player i under a strategy profile $\mathbf{g}$ with $h(y_i, y_j, \phi)$.

$$u_i^h(\mathbf{g}_i, \mathbf{g}_{-i}, \phi) = \sum_{j \in N_i^-(\mathbf{g})} h(y_i(\mathbf{g}_i, \mathbf{g}_{-i}), y_j(\mathbf{g}_j, \mathbf{g}_{-j}), \phi) - \mid N_i^{e-}(\mathbf{g}) \mid \varepsilon - \mid N_i^-(\mathbf{g}) \mid \kappa. \qquad (1.1)$$

In Section 3, we use $f(y_i, y_j)$ and $u_i(\mathbf{g})$ in general, but sometimes use $h(y_i, y_j, \phi)$ and $u_i^h(\mathbf{g})$ to study the results related to ϕ. If the results are about $h(y_i, y_j, \phi)$ and $u_i^h(\mathbf{g})$, we specify that h is used instead of f in each result. Otherwise, the result holds for f as well as h.

We study only pure strategy Nash equilibria which is defined in the usual way. A strategy $\mathbf{g}^*$ is a Nash equilibrium if and only if

$$u_i(\mathbf{g}_i^*, \mathbf{g}_{-i}^*) \geq u_i(\mathbf{g}_i, \mathbf{g}_{-i}^*) \ \forall \mathbf{g}_i \in G_i, \ \forall i \in N.$$

2.3.1 Structurally balanced networks

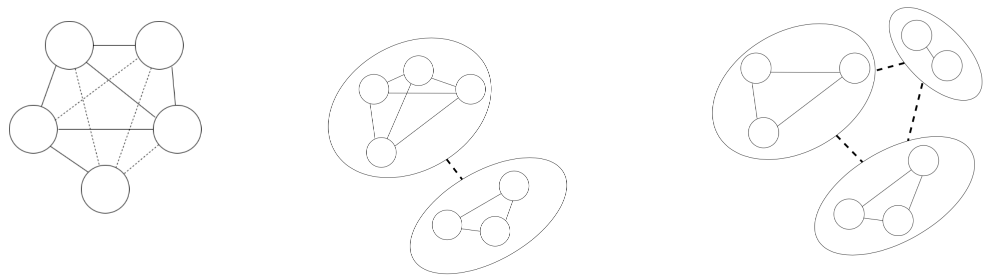
Equilibrium networks share some common properties. First, every Nash equilibrium strategy $\mathbf{g}$ shows clustering. Cartwright and Harary (1956) introduced the term "clique" from graph theory to this literature[8].

They redefined clique as a subset of nodes (players) within which all links are positive and across which they are negative. They also redefined structurally balanced graphs (networks) in the context of a signed graph as follows.

Definition 2.2. (Cartwright and Harary, 1956, p. 286, Structure theorem) A signed graph is balanced if and only if its nodes can be separated into two mutually exclusive cliques such

required to be larger than 0.

[8]In the graph theory, the clique is defined as a subset of nodes which are fully connected but not a subset of another subset satisfying this property. For more detail, see Luce (1950)



(a) Not structurally balanced (b) Structurally balanced (c) Weakly structurally balanced

Figure 2.1: Networks with different properties

that each positive undirected link joins two players within the same clique and each negative undirected link joins players between different cliques.

Davis (1967) relaxed the condition the definition in terms of cliques. He suggested calling the multiple cliques' phenomenon clustering. By borrowing terms from Cartwright and Harary (1956), a formal definition can be written as follows.

Definition 2.3. A signed graph has a clustering if and only if its nodes can be separated into multiple mutually exclusive cliques such that each positive undirected link joins two players within the same clique and each negative undirected link joins players between different cliques.

A signed network which has a clustering is also called a weakly structurally balanced signed network.

2.3.2 Types of network configurations

Before going to the next section for analysis, let us define some significant signed network configurations that arise with two types of players. Note that there are only four categories

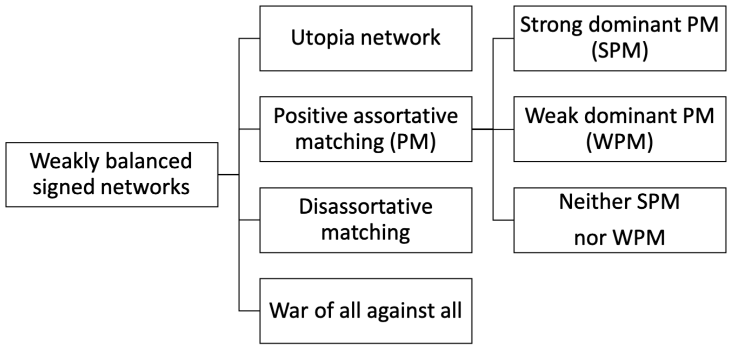


Figure 2.2: Categories in weakly balanced signed networks

of the weakly balanced signed networks with the heterogeneous players.

Definition 2.4. Utopia network is the strategy profile $\mathbf{g}$ where $\bar{g}_{ij} = 1\ \forall\{i,j\} \in N \times N \setminus \{i\}$.

Utopia network is a peaceful and ideal situation without any conflicts. However, these friendships can only exist between similar agents in some networks.

Definition 2.5. Positive assortative matching (PM) is a network configuration $\mathbf{g}$ where there exists $\bar{\mathbf{g}}_{ij} = 1$ for $\{i,j\} \in N_s \times N_s \setminus \{i\}$ or for $\{i,j\} \in N_w \times N_w \setminus \{i\}$, and $\bar{\mathbf{g}}_{ij} = -1$ $\forall\{i,j\} \in N_s \times N_w$.

We further classify positive assortative matching by order of the network strength related to the type of players. This classification indicates which type is dominant in the given positive assortative matching. There are three subcategories in this positive assortative matching. In the first category, each strong player has a higher network strength than every weak player. In the second category, each weak player has higher network strength than all of the strong players. This network configuration, which can be called the tyranny of the weak, can be formed when the number of weak players is more than that of the strong players. Lastly, the rest of the cases is in the third category. Here is a formal definition for the first and second categories.

Definition 2.6. i) Strong dominant positive assortative matching (SPM) is a positive assortative matching where $y_i > y_j$ for every $i \in N_s$ and $j \in N_w$. ii) Weak dominant positive assortative matching (WPM) is a positive assortative matching where $y_j > y_i$ for every $i \in N_s$ and $j \in N_w$.

While there is no dominant type in the third category, the strong/weak type is dominant in strong/weak dominant positive assortative matching. It implies that there is a clear direction of extraction between the types. Therefore, when a network is either strong or weak dominant positive assortative matching, people can feel that there is structural inequality by their attribute in their society.

As extreme cases of these dominant positive assortative matchings, there are network configurations where all of the same type players are fully connected with the positive links, and the negative links only exist between the players of the different types. In other words, the same type players compose one complete subnetwork with the positive links. Since there are two types of players in this model, there exist two complete friendship subnetworks. Let's call these networks, consisting of two complete friendship subnetworks, complete strong dominant positive assortative matching (CSPM) and complete weak dominant positive assortative matching (CWPM).

Definition 2.7. i) Complete strong dominant positive assortative matching (CSPM) is a strong dominant positive assortative matching such that $\bar{\mathbf{g}}_{i,j} = 1 \; \forall \; i, j \in N_s$ and $i, j \in N_w$.

ii) Complete weak dominant positive assortative matching (CWPM) is a weak dominant positive assortative matching such that $\bar{\mathbf{g}}_{i,j} = 1 \; \forall \; i, j \in N_s$ and $i, j \in N_w$.

Besides Utopia network and positive assortative matching, there are network configurations where the different types of players are friends, but there exist the negative relationships, too. Let's call this family of networks disassortative matching[9].

Definition 2.8. Disassortative matching is a network configuration such that

(i) there exists $\bar{\mathbf{g}}_{i,j} = 1$ for $i \in N_s$, $j \in N_w$, and

(ii) there exists $\bar{\mathbf{g}}_{i,j} = -1$ for any $i, j \in N$.

Among disassortative matching, there is a network where one player is bullied by the other players. Depending on the type of the bullied player, it is named *Bullying a strong network* or *Bullying a weak network* and the relevant strategy is called *Bulling a strong strategy* or *Bullying a weak strategy*.

Definition 2.9. i) Bullying a strong strategy is a strategy that $n-1$ players extend the negative directed link to one strong player and extend the positive directed link to each other, and the bullied strong player extends the positive directed links to the other players. ii) Bullying a weak strategy is a strategy that $n-1$ players extend the negative directed link to one weak player and extend the positive directed link to each other, and the bullied weak player extends the positive directed links to the other players.

In the homogeneous case, where there is only one type of player, these networks are called Bullying networks. These Bullying networks indicate networks where one player is bullied by the other players.

Lastly, there is a configuration where only negative links exist. It can be called the war of all against all. In this study, I do not consider this category of networks. When there are two types of players, if there are more than two players, there always exists two players whose network strengths are the same in this configuration. By definition of the war of all against

[9] In the literature, it is also called negative assortative matching.

all, they have the negative link between them. Then, the players whose network strengths are the same does not have an incentive to maintain the negative link. Also, it violates Remark 2.11, which will be mentioned in the first part of the section of Analysis. Due to the reasons, the war of all against all networks is always not in Nash equilibrium. Hence we do not study this network in the section of analysis.

2.4 Analysis

There is a deterministic relationship with a negative undirected link and negative directed links in equilibrium. In the model, if there is at least one negative directed link in a bilateral relationship, then the undirected link between the players has a negative sign. In equilibrium, a player with stronger network strength extends negative links to another player with weaker network strength. On the other hand, the player with weaker network strength extends positive links to the player with stronger network strength in equilibrium. Hiller (2017) mentioned the result as given below:

Remark 2.10. (Hiller, 2017, p.1073, Lemma 1 and 2)

1. In any Nash equilibrium, there does not exist a pair of agents i and j such that $\mathrm{g}_{i,j} = \mathrm{g}_{j,i} = -1$.

2. In any Nash equilibrium, if $\bar{\mathrm{g}}_{i,j} = -1$ with $y_i(\mathbf{g}) < y_j(\mathbf{g})$, then $\mathrm{g}_{i,j} = 1$.

Based on Remark 2.10, Hiller (2017) showed that every network configuration has a clustering in equilibrium.

Remark 2.11. (Hiller, 2017, p.1066, Proposition 1) In any Nash equilibrium, if $y_i(\mathbf{g}) = y_j(\mathbf{g})$, then $\bar{\mathrm{g}}_{i,j} = 1$, and if $y_i(\mathbf{g}) \neq y_j(\mathbf{g})$, then $\bar{\mathrm{g}}_{i,j} = -1$.

In Appendix, we prove Remarks 2.10 and 2.11. We shortly show the logic of the proof here. Regardless of the types, if $y_i(\mathbf{g}) = y_j(\mathbf{g})$ but $\bar{g}_{i,j} = -1$ in $\mathbf{g}$, $f(y_i(\mathbf{g}), y_j(\mathbf{g})) = 0$ but $\varepsilon + \kappa > 0$. Therefore, either i or j want to terminate this futile conflict. Regarding the second statement, it is possible to prove it indirectly by showing that every player in the same $P_a(\mathbf{g})$ has the same strategy to the other players out of $P_a(\mathbf{g})$. For example, suppose i and j in $P_m(\mathbf{g})$ has nonidentical strategies to the other players. Without loss of generality, if $u_i(\mathbf{g}) < u_j(\mathbf{g})$, then i imitates j's strategy in her deviation $\mathbf{g}'_i$. Then it increases i's utility. Roughly speaking, suppose $\bar{g}_{i,k} = 1$ and $\bar{g}_{j,k} = -1$ for $k \in P_{m-1}$. If player i imitates j's strategy so $\bar{g}'_{i,k} = -1$, it decreases k's network strength. Therefore, $u_j(\mathbf{g}') > u_j(\mathbf{g})$. $u_i(\mathbf{g}') = u_j(\mathbf{g}')$, so $u_i(\mathbf{g}') > u_i(\mathbf{g})$. This example is just for the players in $P_m(\mathbf{g})$, but it is possible to repeat this way for the other relationships between $P_a(\mathbf{g})$.

Remark 2.11 is a necessary condition for any network $\mathbf{g}$ to be a Nash equilibrium. Note that this logic is not relevant of the type of each player. Therefore, Remark 2.11 is useful to analyze the heterogeneous player model.

If a network does not consist of cliques (not weakly structurally balanced), there is a triad i, j, and k where $\bar{g}_{i,j} = \bar{g}_{j,k} = 1$ but $\bar{g}_{i,k} = -1$. Then, it violates the statement that if $y_i(\mathbf{g}) = y_j(\mathbf{g})$, then $\bar{g}_{i,j} = 1$, and if $y_i(\mathbf{g}) \neq y_j(\mathbf{g})$, then $\bar{g}_{i,j} = -1$. Hence, any network including this triad cannot be a Nash equilibrium.

With Remark 2.11 in the endogenous signed network formation game, first, we can only consider weak balanced networks in equilibrium. Note that players in the same clique have the same network strength y_i. It also simplifies the study of this endogenous network formation model. Second, if players in different cliques have the same network strength y_i, then $\mathbf{g}$ is not a Nash equilibrium. This property excludes some weakly balanced network configurations from Nash equilibria before checking each player's incentive to deviate from the networks. Third, in every Nash equilibrium, players in the same clique have the same

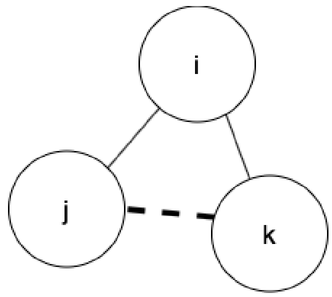


Figure 2.3: If a network contains the above triad with two positive links and one negative link, it violates Remark 2.11 and the definition of a clustered signed network in Definition 2.3.

network strength. If a pair of players have different network strength although they are in the same clique, it contradicts Remark 2.11. As mentioned above, P_i denotes a set which ranks the network strength of the players. Therefore, in equilibrium, P_i can represent each clique according to the rank of network strength which the players in the clique have.

For every bilateral relationship, a negative relationship is only profitable when an involved player has a higher network strength. Thus, the player with the higher network strength extends the negative link with the cost ε for picking a fight. However, the player who has the lower network strength does not want to pay the conflict cost. Thus, he would avoid ε by extending the positive link. Like Remark 2.11, in this logic of Remark 2.10, there is no mention of the players' type and their intrinsic strengths. Therefore, Remark 2.10 is also valid in the heterogeneous player model. Hence, in the following analysis, when we analyze networks in equilibrium, $\text{g}_{i,j} = \text{g}_{j,i} = 1$ for each $\bar{\text{g}}_{i,j} = 1$, and $\text{g}_{i,j} = -1$ and $\text{g}_{j,i} = 1$ for each $\bar{\text{g}}_{i,j} = -1$ where $y_i(\mathbf{g}) > y_j(\mathbf{g})$ by the definition of positive undirected links and Lemma 2.10.

2.4.1 The model with homogeneous players

We study the homogeneous player model as a benchmark. Let's assume that the players are homogeneous. The assumption in this benchmark model will be generalized in the latter section. In the following sections, we categorize networks according to the number of cliques. In each category, we summarize Hiller (2017)'s results and generalize some of them.

Networks consisting of one clique: Utopia networks

Hiller (2017) mentioned that Utopia network is the unique Nash equilibrium if $\varepsilon + \kappa$ is sufficiently large[10]. He also showed that if the conflict cost $\varepsilon + \kappa$ is appropriately small[11], then Utopia network is a Nash equilibrium. Therefore, Utopia network is always in equilibrium in the homogeneous model.

Remark 2.12. Let $\mathbf{g}$ be Utopia network where $\lambda_i = \lambda \ \forall i \in N$.

(a) If $\varepsilon + \kappa > f(n-1, 1)$, then $\mathbf{g}$ is the unique Nash equilibrium.

(b) If $\varepsilon + \kappa \leq f(n-1, 1)$, then $\mathbf{g}$ is a Nash equilibrium. (Hiller, 2017, p. 1066, Proposition 2)

In the homogeneous model, $\bar{f}(\mathbf{g}) = f(n-1, 1)$. If $f(n-1, 1) \geq \varepsilon + \kappa$, then Bullying network, where $n-1$ players extend the undirected negative links to the other player, is a Nash equilibrium because this extraction is profitable. If n increases, then $f(n-1, 1)$ also increases. It means that for the same level of conflict cost $\varepsilon + \kappa$, if there are few people, Utopia network can be the unique Nash equilibrium. However, if there are more people, then Utopia network is not the unique Nash equilibrium more, but also conflict can occur in equilibrium.

As mentioned in the model, the normalized contest success function $h(y_i, y_j, \phi)$ can be used for $f(y_i, y_j)$. Here, it is possible to characterize a change in the level of conflict cost with respect to the number of players and technology of extraction in equilibrium. Given n and

[10] $\varepsilon + \kappa \geq f(\sum_{j \in N \setminus \{k\}} \lambda_j, \lambda_k)$ where $\lambda_k \leq \lambda_j \ \forall j \in N$.

[11] $\varepsilon + \kappa < f(\sum_{j \in N \setminus \{k\}} \lambda_j, \lambda_k)$ where $\lambda_k \leq \lambda_j \ \forall j \in N$.

ϕ, $h(n-1, 1, \phi) = \frac{(n-1)^{\phi}}{(n-1)^{\phi}+1} - \frac{1}{2}$ is the minimum of $\varepsilon + \kappa$ for Utopia network to be the unique Nash equilibrium. If $\varepsilon + \kappa$ is larger than this threshold, peace is always the rational choice for every player. However, if $\varepsilon + \kappa$ is less than this threshold, initiating conflict can be a rational choice, too. Let $\underline{c}^u(n)$ denote the minimum of the conflict cost $\varepsilon + \kappa$ for Utopia network to be the unique Nash equilibrium given n with the function $f(y_i, y_j)$. Also, let $\underline{c}^{uh}(n, \phi)$ denote that given n and ϕ with the function $h(y_i, y_j, \phi)$.

Corollary 2.13.

(i) As n increases, $\underline{c}^u(n)$ increases.

(ii) If f is h, as ϕ increases, $\underline{c}^{uh}$ increases.

Proof. (i) $\underline{c}^u(n) = f(n-1, 1)$. $f(n-1, 1)$ increases as $n-1$ increases as defined.
(ii) $\underline{c}^{uh}(n, \phi) = \frac{(n-1)^{\phi}}{(n-1)^{\phi}+1} - \frac{1}{2}$. The first derivatives are $\frac{\partial \underline{c}^{uh}}{\partial \phi} = \frac{(n-1)^{\phi}\ln(n-1)}{\left((n-1)^{\phi}+1\right)^2} > 0$. □

Corollary 2.13 indicates that when there are small number of players and low level of extraction technology, Utopia network is unique in equilibrium. As the number of players increases and the level of technology is developed, there should be a higher level of cost for conflict for Utopia network to be the unique Nash equilibrium. Therefore, for the same level of the conflict cost, Utopia network can be the unique equilibrium or one of the Nash equilibria according to the number of players and the level of extraction technology.

Networks consisting of two cliques

This section considers networks consisting of two cliques. Hiller (2017) showed that Bullying strategy, where $n-1$ players form a clique and extend negative links to the remaining player, is a Nash equilibrium under a condition: $\varepsilon + \kappa < f(n-1, 1)$. Now, let us generalize his result to a case that there are more than one bullied players. Let C_1 denote the set of players who bully and C_2 denote the set of players who are bullied, where $n_1 = |C_1|$ and $n_2 = |C_2|$. Also, let $\mathbf{g}^{2C}$ denote the generalized bullying strategy profile. In $\mathbf{g}^{2C}$, $\mathrm{g}_{ij} = -1$ if $i \in C_1$ and $j \in C_2$, and $\mathrm{g}_{ij} = 1$ otherwise. Lemma 2.14 gives conditions under which these network configurations with two cliques can be sustained in a Nash equilibrium.

Lemma 2.14. *Suppose $\lambda_i = \lambda$ $\forall i \in N$. $\mathbf{g}^{2C}$ is a Nash equilibrium if and only if $\varepsilon + \kappa \leq f(n_1+1, n_2) - n_2(f(n_1+1, n_2) - f(n_1, n_2))$.*

The condition $\varepsilon + \kappa \leq f(n_1+1, n_2) - n_2((f(n_1+1, n_2) - f(n_1, n_2))$ is equivalent to another condition $u_i(\mathbf{g}^{2C}) \geq u_i(\mathbf{g}^{2C} + \mathrm{g}^{+}_{ij})$ where $i \in C_1$ and $j \in C_2$. When the above conditions are satisfied, then the players in C_1 do not have an incentive to deviate from $\mathbf{g}^{2C}$ by extending one positive link. Also, when this condition is satisfied, the other deviations extending multiple links are also non-profitable by the property of decreasing marginal extraction to network strength. The weak players cannot change the relationship between the cliques, so it is enough to check the strong players' incentive. The more detailed proof is in Appendix.

Based on Lemma 2.14, it is possible to argue that there is a minimum size of C_1 for $\mathbf{g}$ to be a Nash equilibrium. Let's define the minimum size $\bar{n}_1$, given n and $\varepsilon + \kappa$. If the number of players in C_1 is larger than this minimum size value, then this configuration is in equilibrium.

Proposition 2.15. *Suppose $\lambda_i = \lambda$ $\forall i \in N$. For $\varepsilon+\kappa \leq f(n-1, 1)$, there exists $\bar{n}_1(n, \varepsilon+\kappa) < n$ such that every network configurations with two cliques is a Nash equilibrium if and only if $n_1 \geq \bar{n}_1$.*

Example 2.16. Suppose there are ten players and $f = h(y_i, y_j, 1)$. When $n_1 = 6$, $\mathbf{g}^{2C}$ is not a Nash equilibrium for $\varepsilon+\kappa > 0$. When $n_1 = 7$, $\mathbf{g}^{2C}$ is a Nash equilibrium if $\varepsilon+\kappa \leq \frac{16}{110}$. When $n_1 = 8$, $\mathbf{g}^{2C}$ is a Nash equilibrium if $\varepsilon + \kappa \leq \frac{31}{110}$. When $n_1 = 9$, $\mathbf{g}^{2C}$ is a Nash equilibrium if $\varepsilon + \kappa \leq \frac{4}{10}$. Lastly, Utopia network is the unique Nash equilibrium if $\varepsilon + \kappa > \frac{4}{10}$.

Once again, this proposition is a generalized version of Hiller (2017)'s Proposition 2. In his proposition, he showed that when there are $n-1$ players in C_1, this configuration is always a Nash equilibrium. His proposition was proved without the assumptions that $f(y_i, y_j)$ is homogeneous of degree 0, and the marginal return is decreasing in y_i. With these assumptions, this Proposition 2.15 shows that other configurations with two cliques are can be Nash equilibria.

Note that all structurally balanced network configurations consist of two cliques. If there are more than two cliques in the network[12], it is not structurally balanced by definition (but weakly structurally balanced). Therefore, it is the characterization of Nash equilibrium for any structurally balanced network formation when the players are homogeneous.

When there are four homogeneous players, only Utopia network and Bullying network (three players bully one player) are all the possible Nash equilibria. We attach the detailed explanation and conditions for each Nash equilibrium in Appendix.

2.4.2 The model with two types of players

Now, let us introduce heterogeneity within the player set. As mentioned in Section 2.3, each player is either strong or weak, with the assumption that there exists at least one player of each type.

Example: The four players model with two types

In this example, we find all configurations which can be in equilibrium when there are four heterogeneous players.

[12] This configuration is called a weakly balanced network.

As mentioned in the model section, there are three categories of network configurations which can be Nash equilibria: Utopia, positive assortative matching, and disassortative matching. First, there are three Utopia networks in this model of four players. These three networks have a different number of strong (weak) players: (i) $n_s = 3$, $n_w = 1$, (ii) $n_s = 1$, $n_w = 3$, and (iii) $n_s = 2$, $n_w = 2$. Second, there are six positive assortative matchings in equilibrium. Five of them are strong dominant ones, and the other one is weak dominant. In the following list, players in a higher-numbered clique have higher network strength than players in a lower numbered clique. When $N_s = \{1, 2, 3\}$ and $N_w = \{4\}$, there are networks possible to be equilibria such that (i) $C_1 = \{1, 2, 3\}$, $C_2 = \{4\}$, and (ii) $C_1 = \{1, 2\}$, $C_2 = \{3\}$, $C_3 = \{4\}$. When $N_s = \{1\}$ and $N_w = \{2, 3, 4\}$, there are networks possible to be equilibria such that (iii) $C_1 = \{1\}$, $C_2 = \{2, 3, 4\}$, (iv) $C_1 = \{2, 3, 4\}$, $C_2 = \{1\}$ (the only weak dominant positive assortative matching), and (v) $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4\}$. When $N_s = \{1, 2, 3\}$ and $N_w = \{4\}$, there is a network possible to be an equilibrium such that (vi) $C_1 = \{1, 2\}$, $C_2 = \{3, 4\}$.

Lastly, there are four disassortative matching in equilibrium. When $N_s = \{1, 2, 3\}$ and $N_w = \{4\}$, there is a network possible to be an equilibrium such that (i) $C_1 = \{1, 2, 4\}$, $C_2 = \{3\}$. When $N_s = \{1\}$ and $N_w = \{2, 3, 4\}$, there is a network possible to be an equilibrium such that (ii) $C_1 = \{1, 2, 3\}$, $C_2 = \{4\}$. When $N_s = \{1, 2, 3\}$ and $N_w = \{4\}$, there are networks possible to be equilibria such that (iii) $C_1 = \{1, 3, 4\}$, $C_2 = \{2\}$, and (iv) $C_1 = \{1, 2, 3\}$, $C_2 = \{4\}$.

We draw all figures of possible Nash equilibrium network configurations in Table A.1 in Appendix.

Except for these configurations, there are no possible Nash equilibrium configurations in the model with the four heterogeneous players. In both of this example with four players and the general n players case, all Nash equilibrium configurations can be classified into three

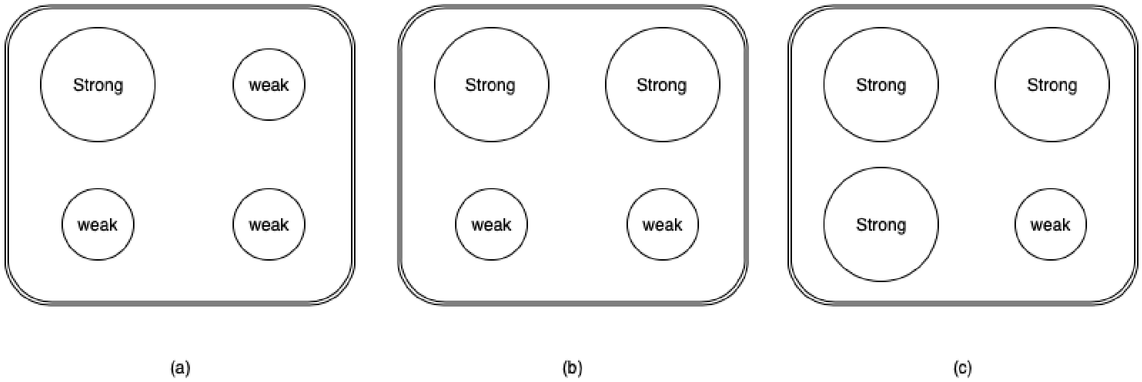


Figure 2.4: Utopia networks

categories: Utopia networks, positive assortative matching, and disassortative matching. We specify the condition for each configuration in Table A.1 to be a Nash equilibrium in Appendix. Furthermore, from the following section, we generalize these conditions to the general case with n players.

Utopia networks

Given the number of strong players and weak players and the ratio of intrinsic strength $a = \frac{\lambda_s}{\lambda_w}$, there is a condition of the conflict costs $\varepsilon+\kappa$ for Utopia network to be a Nash equilibrium. There are some specific deviations which always give higher payoff than the other deviations. Therefore, if Utopia strategy gives more payoff than these particular deviations, then the Utopia strategy profile is a Nash equilibrium. Furthermore, Utopia network is the unique equilibrium when benefit from extending any negative links is always less than the conflict cost.

Theorem 2.17. *Suppose there are n_s strong players and n_w weak players. (i) Utopia network is a Nash equilibrium if and only if $\varepsilon + \kappa \geq f^e = f(n_s\lambda_s + (n_w - 1)\lambda_w, (n_s - 1)\lambda_s + n_w\lambda_w)$. (ii) Utopia network is the unique Nash equilibrium if and only if $\varepsilon + \kappa > f^u = f(n_s\lambda_s +$*

$(n_w - 1)\lambda_w, \lambda_w)$[13].

Theorem 2.17 determines the minimum of the conflict cost for Utopia network to be a Nash equilibrium and the other minimum conflict cost for Utopia network to be the unique Nash equilibrium. Given n_s strong players and n_w weak players, f^e is the extraction value when a strong player suddenly attacks another weak player in Utopia network. Even though this strong player initiates the conflict toward the weak player, the other players ($n_s - 1$ strong players and $n_w - 1$ weak players) are still friends with both of the strong player and the weak player. On the other hand, when a weak player with no friend is bullied by one of the other players, each bullying player extracts f^u from the lonely weak player. Note that $\bar{f}(\mathbf{g}) = f^u$. Therefore, if the conflict cost $\varepsilon + \kappa$ is higher than f^u, then any possible conflict given the set of players is unprofitable. Hence, Utopia network where there are no conflicts becomes the unique Nash equilibrium.

It is possible to implement comparative statics analysis with the conditions in Theorem 2.17.

Corollary 2.18.

(i) $\frac{\partial f^e}{\partial a} > 0$, $\frac{\partial f^e}{\partial n_s} < 0$, *and* $\frac{\partial f^e}{\partial n_w} < 0$.

Also, $\frac{\partial f^u}{\partial a} > 0$, $\frac{\partial f^u}{\partial n_s} > 0$, *and* $\frac{\partial f^u}{\partial n_w} > 0$.

(ii) Let $n_w = n - n_s$. *For any given* n, $\frac{df^e}{dn_s} < 0$ *and* $\frac{df^u}{dn_s} > 0$.

Given the difference λ_s and λ_w, if either n_s or n_w increases, f^e decreases. It implies that Utopia network is stable when there are lots of players. Although a strong player suddenly attacks a weak player in Utopia network, any weak player can still have support from the rest of the players. Hence, it functions as counterchecking a new conflict. Different from f^e, f^u is increasing when either n_s or n_w increases. f^u is the possible maximum extraction value,

[13] $f^u > f^e$.

and it increases when the number of players gets larger. Therefore, the level of conflict cost to block any non-Utopia network increases as the number of players increases. Collectively, in the large society, Utopia network can be stable if it is attained once, but there are also many other possible networks with conflicts in equilibrium.

On the other hand, given the number of players, as $a = \dfrac{\lambda_s}{\lambda_w}$ increases, f^e and f^u increase in the same direction. It implies that as the disparity between the different types grows, Utopia network gets unstable and is also hard to be the unique equilibrium.

Meanwhile, as shown in Corollary 2.18 (ii), fixing the total number of players, it is possible to observe the direction of change in f^e or f^u regarding the ratio of the number of strong players to the number of weak players. If there are more strong players (and the number of weak players decreases), then f^e decreases, but f^u increases. First, regarding f^e, increasing n_s means there are more strong colleagues helping each weak player. If every other strong player continues to support the weak player, then it is hard for the strong player to bully the weak player. Thus, without the possibility of collective action, each strong player has to consider the existence of the other strong players more if there are more strong players. Second, if there are many strong players and if they bully a weak player having no friends, this attack must be effective. The effect would increase as the number of strong players rises. The cost level to make this attack unprofitable increases as the number of strong players increases. Lastly, an increase of n_w has the opposite effect because it means decreasing n_s.

Additionally, by applying the normalized contest function, it is possible to observe variations of the conditions with respect to ϕ. The following corollary is directly derived from Theorem 2.17 by using the normalized contest success function h with $\phi = 1$.

Corollary 2.19. *When $f = h$ and $\phi = 1$, (i) Utopia network is a Nash equilibrium if and* *only if*

$$\varepsilon + \kappa \geq \frac{a-1}{2((2n_s-1)a+2n_w-1)}$$

It is satisfied if

$$\varepsilon + \kappa < \frac{1}{4n-2} \text{ and } a \leq \frac{-4(\varepsilon+\kappa)n_w + 2(\varepsilon+\kappa) - 1}{4(\varepsilon+\kappa)n_s - 2(\varepsilon+\kappa) - 1}$$

or if

$$\varepsilon + \kappa \geq \frac{1}{4n-2} \; \forall a.$$

(ii) Utopia network is the unique Nash equilibrium if and only if

$$\frac{n_w+n_s-2}{2n_w+2n_s} < \varepsilon + \kappa < \frac{1}{2} \text{ and } a \leq \frac{-2(\varepsilon+\kappa)n_w + n_w - 2}{2(\varepsilon+\kappa)n_s - n_s}$$

or

$$\varepsilon + \kappa \geq \frac{1}{2} \; \forall a$$

We omit the proof because the conditions are directly derived from Theorem 2.17.

How about cases when ϕ is large or small? In simulations, the minimum boundary for the conflict costs $\varepsilon + \kappa$ increases as ϕ increases. In Appendix, there is a graphical result of the simulations. Also, for the extreme cases of ϕ, it is possible to derive conditions for Utopia network to be a Nash equilibrium or the unique Nash equilibrium.

Corollary 2.20. *When $f = h$ and $\phi \to \infty$, if and only if $\varepsilon + \kappa \geq \frac{1}{2}$, Utopia network is a Nash equilibrium and it is unique.*

Corollary 2.21. *When $f = h$ and $\phi \to 0$, for any given $\varepsilon + \kappa > 0$, Utopia network is a Nash equilibrium and it is unique.*

Corollary 2.19 confirms the implications from Proposition 2.17. In the condition for existence, the lower bound of $\varepsilon + \kappa$ increases as a rises. Also, in the condition for uniqueness, if n_w or n_s is large, $\frac{n_w+n_s-2}{2n_w+2n_s}$ gets closer to $\frac{1}{2}$. Corollary 2.20 and 2.21 discuss the extreme cases of ϕ. When the technology of extraction is highly efficient, then only high conflict cost can interfere with the existence of conflict at the equilibrium. However, if the technology is not developed, then a low conflict cost can prevent the conflicts and attain Utopia network in the equilibrium.

Utopia network is a special case consisting of only the positive links. Except for Utopia network, there is at least one negative link in every network configurations. Contrary to Utopia network, a configuration with only negative links, which can be called "the war of all against all" cannot be a Nash equilibrium except when there are only one strong player and one weak player. If there are at least three players in these two types heterogeneous player model, at least two players have the same type and have the same network strength. This is a contradiction to Hiller's Proposition 2.11.

Positive assortative matching

In these configurations, friendship (or alliance) only exists between the same type of players. However, negative relationships, which represent conflicts, can exist between the same type of players. In other words, my own type may be my friend or enemy, but the other typee is always my enemy in a positive assortative matching network. As mentioned in Remark 2.11, all Nash equilibrium network configurations consists of cliques. Therefore, discussing Nash equilibrium regarding positive assortative matching, I consider only the network configurations, which consist of cliques made of the same type of players.

Remark 2.22. Every Nash equilibrium positive assortative matching consists of cliques $C_1, C_2, ..., C_m$, and there are only one type of players in each clique.

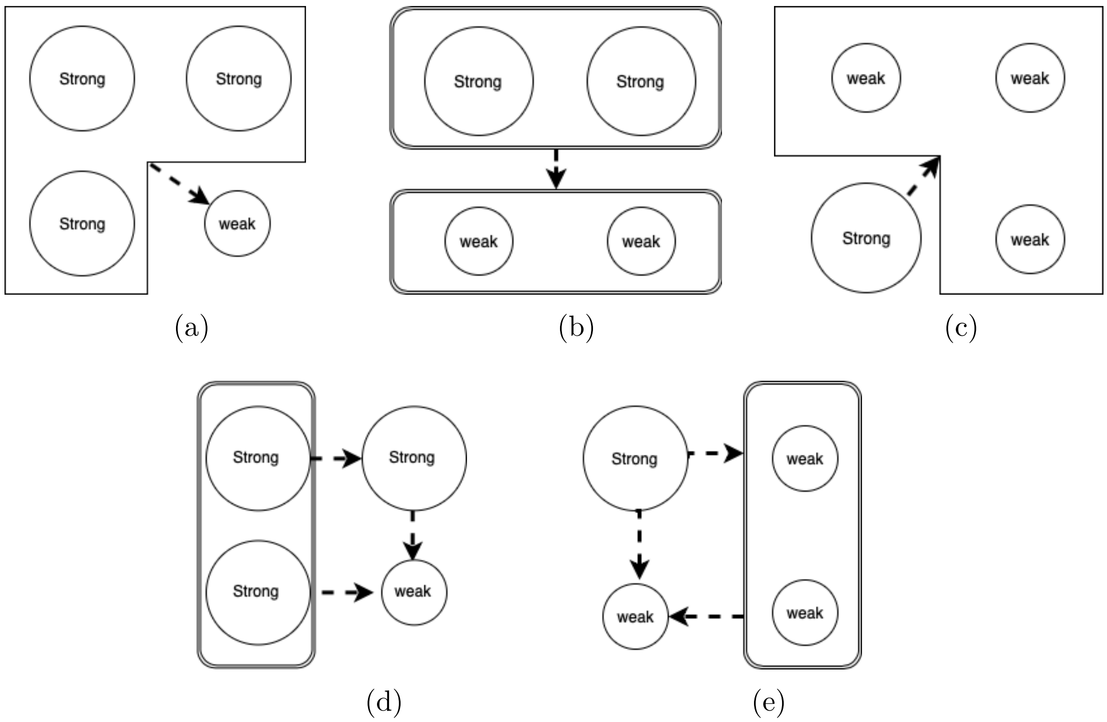


Figure 2.5: Strong dominant positive assortative matching

Once again, by definition of clique, $\bar{\mathbf{g}}_{i,j}$=1 if $i, j \in C_k$ and $\bar{\mathbf{g}}_{i,j}$=-1 if $i \in C_k$, $j \in C_l$ where $k \neq l$. As mentioned in the model, among positive assortative matching, there are special cases where a type of players dominating the other type of players. For example, suppose there are three alliances (teams). Two alliances consist of incumbents, and one alliance consists of challengers. Among the incumbents' alliances, one has the strongest marketing power, and the other has the second strongest marketing power. The challenger's alliance is the weakest one, so they always lose their customers to the other alliances. It is an example of strong dominant positive assortative matching.

Let us see strong dominant positive assortative matching first. As described in Figure 2.5, there are five cases of Nash equilibrium strong dominant positive assortative matching in the four players model. These five configurations can be Nash equilibria depending on the parameters $\varepsilon + \kappa$ and a. The conditions are specified in Appendix.

For complete strong dominant positive assortative matching (CSPM, Figure 2.5 (a)) to be a Nash equilibrium, every deviation has to give lower payoff for each payer than this strategy profile. Similar to Utopia strategy, there exist particular deviations from CSPM always more profitable than the other deviations. This particular deviations vary from case to case, so we sorted these in Lemma 2.23 below. Note that any deviation strategy by a weak player is always not profitable in CSPM. Lemma A.1 proves it in Appendix. Simply speaking, in CSPM, if the weak player extends negative links to the strong player, nothing is changed, but only the cost ε happens. If this player extends negative links to the other weak players, it is also not profitable. The payoff from the conflicts is always zero or negative because $f((n_w - x)\lambda_w, (n_w - 1)\lambda_w) \leq 0$ when $x \geq 1$. Similarly, regarding CWPM, any strong players do not have a profitable deviation.

Lemma 2.23. *Complete strong dominant possitive assortative matching* $\mathbf{g}$ *is a N.E. if and only if*

(i) when $(n_s - 1)\lambda_s \geq n_w\lambda_w$, $u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \mathrm{g}^+_{i,j})$, *or*

(ii) when $(n_s - 1)\lambda_s < n_w\lambda_w$, $u_i(\mathbf{g}) \geq u(\mathbf{g} + \mathrm{g}^+_{i,j})$ *and* $u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \sum_{k\in K} \mathrm{g}^-_{i,k} + \sum_{l\in N_w} \mathrm{g}^+_{i,l})$

for all $i \in N_s$, $j \in N_w$, $K \subset N_s$.

$u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \sum_{j\in J} \mathrm{g}_{i,j})$ for all $J \subset N_w$ is the condition for each strong player not to embrace some of the weak players. If $u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \mathrm{g}_{ij})$, then $u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \sum_{j\in J} \mathrm{g}_{ij})$ for all $J \subset N_w$. It is because of the property of decreasing marginal extraction to network strength ($\frac{\partial^2 f}{\partial y_i^2} < 0$). The more detailed proof is in the appendix. Stories about the deviations are as follows. When one group exploits another group, the attacking group seeks turncoats. The newly formed positive relationship with the turncoats is for exploiting the rest of the exploited group members. On the other hand, $\mathbf{g} - \sum_{k\in K} \mathrm{g}^-_{ik} + \sum_{j\in N_w} \mathrm{g}^+_{ij}$ means to initiate new conflicts against the same kind of players by taking advantage of the other group. Thus, player i is a casting voter here. If player i betrays the old strong type friends, then she may get a higher network strength than the old (betrayed) friends' network strength with the help of the weak players. This condition also means that this deviation is profitable only when this betrayal efficiently weakens the remaining strong players' network strength. Hence there is a tradeoff in switching alliances.

When

$$(n_s - 1)\lambda_s \geq n_w \lambda_w$$

(the first case of Lemma 2.23), any deviation switching alliance is an unattractive choice as mentioned in Lemma A.3. The new weak friend is not useful to increase the network strength. Also, the new enemy who was an old friend before the deviation is still stronger than the new friend. On the other hand, in the second case (ii), the old friends are weaker than the new friends after the deviation. However, these new friends are not as helpful as the old friends to increase the network strength.

From Lemma 2.23, it is possible to derive the necessary and sufficient condition concerning the parameters for CSPM to be a Nash equilibrium. Since CSPM is a special case of SPM, it is also a sufficient condition that there exists a Nash equilibirum SPM. Moreover, there is a condition that all Nash equilibria are SPM.

Theorem 2.24. *For $\varepsilon + \kappa < f(n_s, n_s - 1)$, there exists $\bar{a}^a(n_s, n_w, \varepsilon + \kappa)$ such that all Nash equilibria are strong dominant positive assortative matching if $a \geq \bar{a}^a$, and for $\varepsilon + \kappa < \bar{f}$ there also exists $\bar{a}^e(n_s, n_w, \varepsilon + \kappa)$ such that CSPM is a Nash equilibrium if and only if $a \geq \bar{a}^e$.*

Theorem 2.24 indicates that positive assortative matching can be a Nash equilibrium only if there is a significant difference between λ_s and λ_w. Furthermore, if the difference gets larger, every Nash equilibrium configuration is strong dominant positive assortative matching. The result implies that first, gathering and discrimination by the strong player can happen when the strong people are competitive enough compared to the discriminated people. Second, when strong players are quite stronger than weak players, if the conflict cost is small enough, only strong dominant positive assortative matchings can be Nash equilibria.

It is also available to consider extreme cases such as $\phi \to \infty$, where the technology of conflict is efficient.

Corollary 2.25. *When $f = h$ and $\phi \to \infty$, complete strong dominant positive assortative matching is a Nash equilibrium for a such that $n_s a > n_w$ and $\varepsilon + \kappa < \frac{1}{2}$.*

Remark 2.26. When $f = h$, suppose $\phi \to 0$. For any $\varepsilon + \kappa > 0$, any strong dominant positive assortative matching is not a Nash equilibrium regardless of a.

High ϕ amplifies the effectiveness of extraction from the difference between λ_s and λ_w. Therefore, sufficiently large ϕ allows any a to satisfy the condition for CSPM to be a Nash equilibrium in Lemma 2.23. Remark 2.26 is derived from Corollary 2.21. When ϕ is sufficiently low, complete strong dominant positive assortative matching cannot be a Nash equilibrium because Utopia network is the unique Nash equilibrium. The relative simulations are in Appendix.

To glance a case where ϕ is moderate, let's consider the special case that $\phi = 1$ and $\varepsilon + \kappa \to 0$.

Corollary 2.27. *When $f = h$, suppose $\phi = 1$ and $\varepsilon + \kappa$ is small enough. Complete strong dominant positive assortative matching is a N.E. if and only if*

i) When $n_s = 2$ and $n_w = 2$, $a > \bar{a}^e(2, 2, \varepsilon + \kappa \to 0) = 1.5$, or

ii) otherwise, $a > \bar{a}^e(n_s, n_w, \varepsilon + \kappa \to 0) = \dfrac{1}{2n_s}(-1 + \sqrt{12n_w^2 - 4n_w + 1})$.

Corollary 2.27 is directly derived from Lemma 2.23. The first condition, when $n_s = 2$ and $n_w = 2$, is derived from the condition $u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \mathrm{g}_{ij}^{+})$, and the other condition is derived

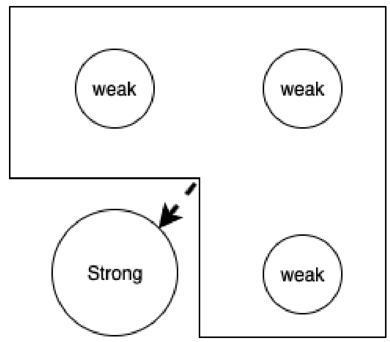


Figure 2.6: Weak dominant positive Assortative Matching

from $u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \sum_{k \in K} \mathrm{g}_{ik}^{-} + \sum_{j \in N_w} \mathrm{g}_{ij}^{+})$. If n_s increases then $\bar{a}^e$ decreases, and if n_w increases then $\bar{a}^e$ increases.

Now let's consider weak dominant positive assortative matching. In the example with four players, there is only one weak dominant positive assortative matching configuration as drawn in Figure 2.6. This undirected network is the same to Figure 2.5 (c), but the strategy profile denoted by the directed network is different. In Figure 2.5 (c), the strong player extends the negative directed links, but in Figure 2.6, the weak players extend the negative directed links. For each negative link to be profitable, the players who extend the negative links should have larger network strength. Therefore, in Figure 2.5 (c), the strong player should have his intrinsic strength larger than 3, and in Figure 2.6, the strong player should have his intrinsic strength less than 3. For the configuration in Figure 2.6 to be a Nash equilibrium, the weak players should not have any incentive to deviate from the weak dominant positive assortative matching strategy. We specify the condition in Appendix.

Like the strong dominant positive assortative matching, there is a condition for weak dominant positive assortative matching to be a Nash equilibrium for an arbitrary number of players. Weak dominant positive assortative matchings can be Nash equilibria when the difference between the types is not significant. Proposition 2.28 focuses on complete weak dominant positive assortative matching. It is about the condition for complete weak dominant positive assortative matching to be a Nash equilibrium.

Proposition 2.28. *For any n_w and n_s such that $n_w > 2n_s - 1$, and for small enough $\varepsilon + \kappa > 0$, there exists $\underline{a}(n_s, n_w)$ such that the complete weak dominant positive assortative matching is a Nash equilibrium if and only if $a \leq \underline{a}$.*

Weak dominant positive assortative matchings describe situations in which the weak players form a majority in society because of their number. Proposition 2.28 implies that this network configuration can be a Nash equilibrium only when the difference between λ_s and λ_w is not so large. If it is significantly large, then there is always a weak player who has an incentive to form new friendships with the strong type enemies. Note that not every Nash equilibrium is weak dominant positive assortative matching even when the difference between the types of players is small enough. When the difference is small, other Nash equilibrium configurations also exist, such as Utopia, disassortative matching, and even strong dominant positive assortative matching. When a is small, this heterogeneous player model brings a similar result to the homogeneous player model.

In Corollary 2.29 and Remark 2.30, we show how the extreme assumption such as $\phi \to \infty$ or $\phi \to 0$ changes the condition of the Nash equilibrium weak dominant positive assortative matching.

Corollary 2.29. *When $f = h$, suppose $\phi \to \infty$. Then complete weak dominant positive assortative matching cannot be a Nash equilibrium for any a.*

Remark 2.30. When $f = h$, suppose $\phi \to 0$. Then any weak dominant positive assortative matching cannot be a Nash equilibrium for any a.

According to Corollary 2.29, when the extraction technology is highly efficient, the strong players are never discriminated. For the weak type, it is always beneficial to embrace the talented minority whose intrinsic power is λ_s and to establish other conflicts with some of the old friends. If the extraction technology is highly efficient, the weak players always prefer to betray the old friends with the help from the strong players. Similar to Remark 2.26, Remark 2.30 is derived from Corollary 2.21 that when $\phi \to 0$, Utopia network is the unique Nash equilibrium. If the technology is extremly not efficient, each weak player give up the extraction, and choose peace in the Utopia network.

Before moving to the next category, it is worth mentioning that neither SPM nor WPM can be a Nash equilibrium, even though these configurations do not appear in the example with four players. For example, suppose there are many weak players and one strong player. Then n-2 weak players form a clique C_1, the strong player form C_2, and the remaining weak player form C_3. It can be a Nash equilibrium depending on the parameters. Figure 2.7 shows another example of positive assortative matching that is neither strong dominant nor weak dominant.

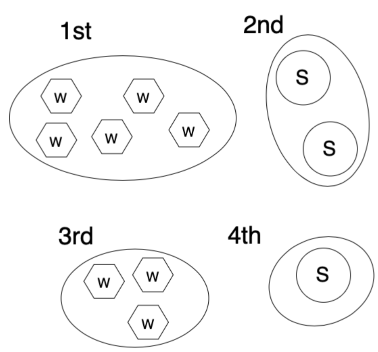


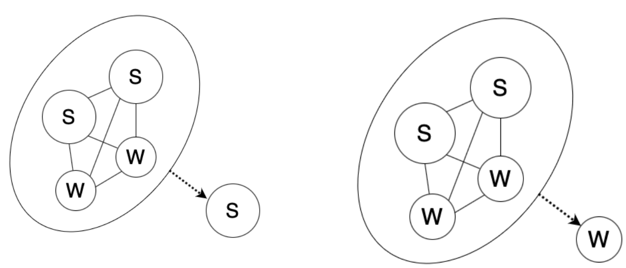
Figure 2.7: An example of positive assortative matching that is neither strong dominant nor weak dominant

Disassortative matching

Disassortative matching is any network where there exists friendship between heterogeneous players. Since the requirement in the definition is simple, there are diverse configurations in this category.s Firstly, let's mention a condition where any disassortative matching is not in equilibrium. Theorems 2.17 and 2.24 state the conditions when there are only either Utopia network or strong dominant positive assortative matchings in equilibrium. If the conflict cost $\varepsilon+\kappa$ is large enough, Utopia network is the unique Nash equilibrium. If $\varepsilon+\kappa < f(n_s, n_s-1)$ and a is large enough, every Nash equilibrium is positive assortative matching. When these conditions are satisfied, any disassortative matching cannot be a Nash equilibrium.

Remark 2.31. Any disassortative matching is not in equilibrium (i) if $\varepsilon + \kappa > f(n_s\lambda_s + (n_w - 1)\lambda_w, \lambda_w)$ (ii) or if $\varepsilon + \kappa < f(n_s, n_s - 1)$ and a is large enough.

Among many kinds of disassortative matchings, let's consider strong bullied network and weak bullied network. We study the bullying networks for two reasons. First, there are intuitive results. Second, a related network was studied in the literature. As specified in Section 2.4.1, Hiller (2017) showed that bullying strategy is a Nash equilibrium when the conflict cost is not too big ($\varepsilon + \kappa < f(n-1, 1)$). Is this bullying strategy still a Nash equilibrium when there are two types of players? If the conflict cost is moderate, Bullying a strong network and Bullying a weak network can be Nash equilibria regardless of the difference between the players.



(a) Bullying a strong network (b) Bullying a weak network

Figure 2.8: Examples of Bullying networks

Proposition 2.32. *For any $a > 1$, (i) when $n_s \geq 3$, if $f(n_s, n_s - 2) \leq \varepsilon + \kappa \leq f(n_s - 1, 1)$, Bullying a strong network is a Nash equilibrium.*

(ii) When $n_s \geq 2$, if $f(n_s, n_s - 1) \leq \varepsilon + \kappa \leq f(n - 1, 1)$, Bullying a weak network is a Nash equilibrium.

The proof is in Appendix.

Note that both of the bullying networks can be Nash equilibria depending on n_s. In particular, as n_s increases, then the range of $\varepsilon + \kappa$ for Nash equilibrium bullying network expands. For example, the lower bounds $f(n_s, n_s - 2)$ and $f(n_s, n_s - 1)$ go to 0 and the upper bounds $f(n_s - 1, 1)$ and $f(n - 1, 1) = f(n_s + n_w - 1, 1)$ go to $\bar{f}$ as n_s increases. Regardless of the difference between the players and the number of weak players, if there are enough strong players, then for $\varepsilon + \kappa < \bar{f}$, the bullying networks are Nash equilibria.

Corollary 2.33. *If $n_s \to \infty$, then Bullying a strong network and Bullying a weak network are Nash equilibria for any a and $\varepsilon + \kappa < \bar{f}$.*

The proof is straightforward, so we omit it.

2.5 Conclusion

The paper has built upon Hiller (2017) by studying different equilibrium configurations. First, we investigate all Nash equilibrium network configurations in the example with four players. While there are only two network configurations with four homogeneous players that can be sustained in equilibrium, there are much more network configurations sustained in equilibrium with two types of players. In the model with four heterogeneous players, there are thirteen network configuration categories in equilibrium. Except for the network configurations, there is no other network configuration in equilibrium in this four players model.

After we identify these thirteen Nash equilibrium configurations, we sorted them to three categories: Utopia network, positive assortative matching, and disassortative matching. In the case of an arbitrary number of players and two types, we derive the generalized condition for existence of an equilibrium exhibiting Utopia networks and positive assortative matching. In both homogeneous and heterogeneous model, Utopia network can be a Nash equilibrium or the unique Nash equilibrium if the conflict cost is significant. If the difference between intrinsic strengths gets larger, then the level of conflict cost for Utopia network to be a Nash equilibrium also increases. Generally, when the heterogeneity is introduced, strong players have an incentive to deviate from Utopia networks. However, if the number of strong players increases, then Utopia network can be sustainable in the lower level of conflict cost. It is because when there are lots of strong players, each of them holds each other in check. It is not profitable for the strong players to extend a negative link to a weak player when the weak player has lots of strong friends. Therefore, while introducing strong players into a network can impede cooperation in the beginning, but adding more strong players also promotes cooperation.

In the boundary of positive assortative matching, there are strong dominant positive assortative matching, weak dominant positive assortative matching, and the other positive assortative matching, which are neither strong dominant nor weak dominant. If the strong player's intrinsic strength is sufficiently larger than the weak player's intrinsic strength, all Nash equilibrium network configurations are these positive assortative matchings. We also show that if the difference between the intrinsic strengths is small enough, then complete weak dominant positive assortative matching can be a Nash equilibrium. Lastly, disassortative matching cannot be a Nash equilibrium when the difference is large enough. As a representative of disassortative matching, we study bullying networks. These bullying networks can be Nash equilibria if there are enough strong players.

Chapter 3

Equilibrium configurations in signed network formation model with neutral links

3.1 Introduction

In the real world, whether in the specific contexts of international relations or social relations, we witness neutrality as well as friendship and enmity between two nations or individuals. For example, during the World wars and the cold war, when the world was divided into enemies and allies, some nations, such as Switzerland, remained neutral. This paper investigates the various bilateral interactions in a single framework, the signed network model. To do so, we consider a model where two agents can form friendships through a bilateral agreement. And they will be enemies if one of them declares a conflict with the other. Each player bears a cost when extending a friendship link or an enmity link. However, remaining neutral is costless.

This study is an extension of Hiller (2017). He analyzed a signed network formation model, where each player could choose only the positive directed link or the negative directed link.

In his model, the players did not have the option of choosing the neutral directed link. In the analysis, he used Nash equilibrium and strong Nash equilibrium. His main finding is that any Nash equilibrium network is weakly structurally balanced. A signed network is weakly structurally balanced if its nodes can be separated into multiple mutually exclusive subsets. In the weakly structurally balanced network, each positive undirected link joins two players within a subset, and each negative undirected link joins players between different subsets. Chapter 2 provides another extension of Hiller (2017). We categorized all Nash equilibrium networks into four cases when the players are divided into two types. We also characterized a condition when every Nash equilibrium network is a positive assortative matching. On the other hand, Jackson and Nei (2015) studied the signed network formation model with neutral links in another way. They analyzed war-stable networks, where any subset of alliance does not have an incentive to maintain and initiate conflict. They showed that only the empty network could be war stable.

Unlike the existing literature, we analyze the model using the pairwise Nash equilibrium concept. In the network formation theory, researchers have studied the equilibrium concepts for the unsigned network formation game first. In this paper, we borrow the equilibrium concept in the unsigned network model and modify it for the signed network formation model. Pairwise Nash equilibrium network is a network that is derived from a Nash equilibrium profile and satisfies the pairwise stability condition of Jackson and Wolinsky (1996). Goyal and Joshi (2003) and Belleflamme and Bloch (2004) used the Nash equilibrium concept and the notion of pairwise stability. Goyal and Joshi (2006) formally defined pairwise Nash equilibrium networks. They defined the notion but used different terminology of pairwise equilibrium networks. Calvó-Armengol and İlkılıç (2009) firstly used the term of "Pairwise Nash equilibrium network" for this notion. For the same notion, Gilles and Sarangi (2005) used a different name, "strictly pairwise network". Here, we use the term of pairwise Nash equilibrium because it is widely used now.

While Hiller (2017) showed that the signed networks of players exhibit weakly structural balance in Nash equilibrium, we show that the relevant conflict networks are multipartite. Davis (1967) defined Weakly structural balance in a graph. In any weakly structurally balanced network, a friend of a friend is a friend, but an enemy of an enemy is not necessarily enemies. It allows multiple cliques, within which nodes are positively connected and between which nodes are negatively connected. Huremovic (2019) studied a weighted conflict network formation model. He showed that when players are limited farsighted, any Nash stable network is multipartite. Compared to the weak structural balance, multipartiteness is a weaker condition. If a signed graph is weakly structurally balanced, then the relevant conflict graph is always multipartite. However, the converse is not true.

In our model, as Hiller (2017), each player starts out with intrinsic strength. Forming allies through bilateral agreements, the players build their network strengths. When players become enemies, they enter into a zero-sum game, where the one with a higher network strength emerges as the winner. No such games are played when the relationships are neutral.

Our results can be summarized as follows. In the general model where intrinsic strengths vary across players, first, in every pairwise Nash equilibrium, the conflict network, which consists of only negative links, is multipartite. Second, when the players are homogeneous, Utopia network, where the players are all friends, is not a pairwise Nash equilibrium. Also, Dystopia network (the war of all against all) is not a pairwise Nash equilibrium. However, empty network is always a pairwise Nash equilibrium. Moreover, friendship and conflict are interdependent in equilibrium. Hence, friendship without conflict does not appear in any pairwise Nash equilibrium and vice versa. Moreover, there exists a k-regular bullying network in pairwise Nash equilibrium, where one player is an enemy of everyone else, and the others are not enemies of each other. Third, when the players are heterogeneous, Utopia network is still not a pairwise Nash equilibrium. But Dystopia network can be a pairwise Nash equilibrium. Empty network is not always a pairwise Nash equilibrium. Friendship without conflict is still not available, but conflict without friendship can appear in equilibrium. We also investigate how matching takes place in building one's network strength when there are two types of players with different intrinsic strengths.

3.2 Model

There are n players. Let $N = \{1, 2, 3, ...n\}$ denote the set of players. Each player has an intrinsic strength. Player i's intrinsic strength is λ_i, with $\lambda_i > 0$.

Every player can extend either a friendly (positive) directed link, an antagonistic (negative) directed link, or a neutral directed link. Player $i \in N$ chooses $\mathrm{g}_{i,j} \in \{1, 0, -1\}$ for all $j \in N \setminus \{i\}$. Here 1 denotes the friendly link, 0 denotes the neutral link, and -1 denotes the negative link. In the following graphical expressions, solid lines represent the positive link 1. If there is a no line, it means there is neutral link 0. Dotted lines represent the negative link -1. A player's strategy can be defined as a $1 \times (n-1)$ vector. A player i's strategy is $\mathbf{g}_i = (\mathrm{g}_{i,1}, \mathrm{g}_{i,2}, ..., \mathrm{g}_{i,i-1}, \mathrm{g}_{i,i+1}, ..., \mathrm{g}_{i,n})$. Each element can be interpreted as a directed link from one player to another. The set of a player i's strategy $\mathbf{g}_i$ is defined by G_i for all $i \in N$. $\mathbf{g}_{-i}$ is a set of all the players' strategies except for player i's strategy $\mathbf{g}_i$, i.e. $\mathbf{g}_{-i} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_{i-1}, \mathbf{g}_{i+1}, ..., \mathbf{g}_n)$. Also, $\mathbf{g}_{-i-j} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_{i-1}, \mathbf{g}_{i+1}, ..., \mathbf{g}_{j-1}, \mathbf{g}_{j+1}, ...\mathbf{g}_n)$. The players' strategy profile is a directed network $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_n)$. Therefore, $(\mathbf{g}_i, \mathbf{g}_{-i})$ and $(\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_{-i-j})$ are the same as $\mathbf{g}$. The strategy space is defined by $G = G_1 \times ... \times G_n$.

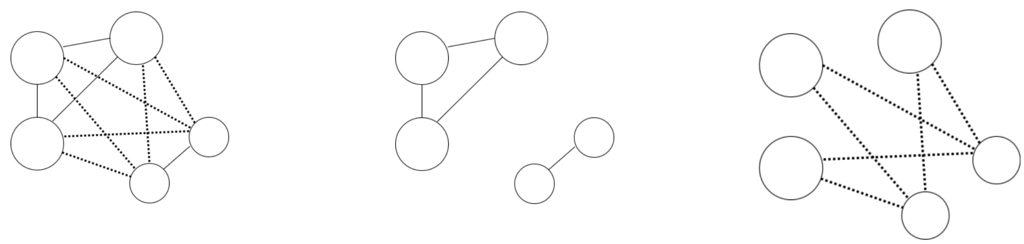
The players form their relationships according to their attitude (the directed links) towards each other. In a bilateral relationship, if both players are friendly, then they will be good friends. However, if one of them is not cooperative by extending a negative link, then they will have the undirected negative link. Otherwise, there is a undirected neutral link between the players. To represent these relationships, let us define an undirected network: $\bar{\mathbf{g}} = (\bar{\mathbf{g}}_1, \bar{\mathbf{g}}_2, ..., \bar{\mathbf{g}}_n)$, where $\bar{\mathbf{g}}_i = (\bar{\mathrm{g}}_{i,1}, \bar{\mathrm{g}}_{i,2}, ..., \bar{\mathrm{g}}_{i,i-1}, \bar{\mathrm{g}}_{i,i+1}, ..., \bar{\mathrm{g}}_{i,n})$. $\bar{\mathrm{g}}_{i,j} \in \{1, 0, -1\}$ represents player i's relationship with j. If $\mathrm{g}_{i,j} = \mathrm{g}_{j,i} = 1$, then $\bar{\mathrm{g}}_{i,j} = 1$. If $\mathrm{g}_{i,j} = -1$ or $\mathrm{g}_{j,i} = -1$, then $\bar{\mathrm{g}}_{i,j} = -1$. Otherwise, $\bar{\mathrm{g}}_{i,j} = 0$. In the same way, $\mathrm{g}_{i,j}$ and $\mathrm{g}_{j,i}$ determine $\bar{\mathrm{g}}_{j,i}$, hence $\bar{\mathrm{g}}_{i,j} = \bar{\mathrm{g}}_{j,i}$ [1]. In $\mathbf{g}$ (and in the resulting undirected network $\bar{\mathbf{g}}$), each player has a set of friends and a set of enemies. $N_i^+(\mathbf{g}) = \{j \in N \mid \bar{\mathrm{g}}_{i,j} = 1\}$, represents a set of players to which player i reciprocates a positive link. $N_i^-(\mathbf{g}) = \{j \in N \mid \bar{\mathrm{g}}_{i,j} = -1\}$, meaning a set of players such that a player i extends and/or receives a negative link. In a similar way, it is possible to specify the set of the negative direct link which player i extends as $N_i^{e-}(\mathbf{g}) = \{j \in N \mid \mathrm{g}_{i,j} = -1\}$, and $N_i^{e-}(\mathbf{g}) \subset N_i^-(\mathbf{g})$. Also, $N_i^{e+}(\mathbf{g}) = \{j \in N \mid \mathrm{g}_{i,j} = 1\}$, and $N_i^+(\mathbf{g}) \subset N_i^{e+}(\mathbf{g})$.

To express a deviation strategy, a change in directed link $\mathrm{g}_{i,j}$ is denoted as follows: Given a network $\mathbf{g}$, $\mathbf{g} + \mathrm{g}_{i,j}^+$ changes the directed link from $\mathrm{g}_{i,j} = 0$ or -1 to $\mathrm{g}_{i,j} = 1$. Similarly, $\mathbf{g} + \mathrm{g}_{i,j}^-$ changes the directed link from $\mathrm{g}_{i,j} = 0$ or 1 to $\mathrm{g}_{i,j} = -1$. On the other hand, $\mathbf{g} - \mathrm{g}_{i,j}^+$ changes the directed link from $\mathrm{g}_{i,j} = 1$ to $\mathrm{g}_{i,j} = 0$, and $\mathbf{g} - \mathrm{g}_{i,j}^-$ changes the directed link from $\mathrm{g}_{i,j} = -1$ to $\mathrm{g}_{i,j} = 0$. However, if $\mathrm{g}_{i,j} = 1$ in $\mathbf{g}$, $\mathbf{g} + \mathrm{g}_{i,j}^+ = \mathbf{g}$, i.e. there is no change. Similarly, if $\mathrm{g}_{i,j} = -1$ in $\mathbf{g}$, then $\mathbf{g} + \mathrm{g}_{i,j}^- = \mathbf{g}$. Also, if $\mathrm{g}_{i,j} = 0$, then $\mathbf{g} - \mathrm{g}_{i,j}^+ = \mathbf{g} - \mathrm{g}_{i,j}^- = \mathbf{g}$. $\mathbf{g} + \bar{\mathrm{g}}_{i,j}^+$ $(= \mathbf{g} + \mathrm{g}_{i,j}^+ + \mathrm{g}_{j,i}^+)$ is a pairwise deviation by i and j to form an undirected positive link $\bar{\mathrm{g}}_{i,j} = 1$. The summation sign Σ can be used to denote multiple link changes.

It is possible to express both directed and undirected networks as graphs consisting of solid

[1] It is possible to express this rule determining $\bar{\mathrm{g}}_{i,j}$ from $\mathrm{g}_{i,j}$ and $\mathrm{g}_{i,j}$ as a function γ: $\{\mathrm{g}_{i,j}, \mathrm{g}_{i,j}\} \rightarrow \bar{\mathrm{g}}_{i,j}$. The function $\gamma(\mathrm{g}_{i,j}, \mathrm{g}_{i,j})$ is equal to $min(\mathrm{g}_{i,j}, \mathrm{g}_{i,j})$. This minimum function implies that the player, who has a more negative attitude toward the relationship between them, determines the relationship with the other player.

and dotted lines. It is also possible to divide the whole graph into separate graphs consisting of the links of only one type. Given $\bar{\mathbf{g}}$, call a network consisting of the positive (negative) undirected links an *alliance network* $\bar{\mathbf{g}}^{all}$ (a *conflict network* $\bar{\mathbf{g}}^{con}$).



(a) Signed network (b) Alliance network (c) Conflict network

Figure 3.1: Examples of alliance network and conflict network

Each relationship has a different impact. If i and j are friends ($\bar{g}_{i,j} = 1$), then they can strengthen each other's power. Let y_i denote player i's network strength, which is a result of her intrinsic quality and her network of friends. Thus, y_i is determined as follows.

$$y_i(\mathbf{g}_i, \mathbf{g}_{-i}) = \lambda_i + \sum_{j \in N_i^+(\mathbf{g})} \lambda_j.$$

Therefore, apart from the type of players, each player can have different level of y_i according to $\mathbf{g}$. To classify the players by network strength, let $P_i(\mathbf{g}) \in \{P_1(\mathbf{g}), P_2(\mathbf{g}), ..., P_m(\mathbf{g})\}$ denote a set which ranks the network strength of the players given $\mathbf{g}$. Let us call $P_i(\mathbf{g})$ the ith lowest network strength partition in $\mathbf{g}$. $P_i(\mathbf{g})$ can be abbreviated to P_i when there is no room for confusion. $P_1(P_m)$ is the the partition containing the weakest (strongest) players. To avoid confusion, let's call a player who has higher λ_i a strong player, and a player with high (low) y_i just a player with high (low) y_i. In other words, players in P_1 have the lowest y_i, and players in P_m have the highest y_i. If they have no relationship (neutral relationship, $\bar{g}_{i,j} = 0$), then nothing happens.

If they are enemies ($\bar{g}_{i,j} = -1$), they enter a zero-sum competition. In this zero-sum game, the player with the higher network strength extracts some payoff from the weaker player with less network strength. Let $f(y_i(\mathbf{g}), y_j(\mathbf{g})) \in (\underline{f},\ \bar{f})$ denote the player i's extraction from player j at a network $\mathbf{g}$, where $\bar{f}$ ($\underline{f}$) is the least upper (greatest lower) bound for $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$. The network $\mathbf{g}$ determines y_i and y_j, which determine $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$. Thus, the extraction function $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$ is ultimately a function of $\mathbf{g}$. Also, for simplicity, let's abbreviate the extraction function to $f(y_i, y_j)$ if there is no confusion.

It is also possible to define some property of the extractions function regarding the value of y_i and y_j. First, Hiller (2017) defined this extraction function as follows to describe a zero-sum competition. When $y_i > y_j$ the value of $f(y_i, y_j)$ is positive, but when $y_i < y_j$, it is negative. If a pair of players with equal strength is negatively connected, then no payoff extraction takes place: $f(y_i, y_j) = 0\ \forall y_i, y_j : y_i = y_j$. $f(y_i, y_j)$ is strictly increasing in y_i and decreasing in y_j. Following the definition of zero-sum game, the value that a player extracts through an antagonistic link is assumed to be another player's loss, hence $f(y_i, y_j) + f(y_j, y_i) = 0,\ \forall y_i, y_j$. We further assume as follows. First, $f(y_i, y_j)$ is homogeneous of degree 0. That is, $f(y_i, y_j) = f(\frac{y_i}{y_j}, 1)$. Therefore, this function can be used to describe conflicts where the relative size of strength matters. Using this property, let's denote $a = \frac{y_i}{y_j}$, a ratio of the different intrinsic strengths. Then $f(y_i, y_j) = f(a, 1)$. Second, $\frac{d^2 f(y_i, y_j)}{dy_i^2} < 0$ when $y_i > y_j$. It means that a marginal return of extraction to network strength is decreasing.

In this model, there are two kinds of cost: friendship cost and conflict cost. But there is no cost associated with neutral links. Extending a directed positive link can be thought of as an effort to establish a friendship, which incurs a cost of $c^+ > 0$. This cost c^+ is assumed to occur even though there is no reciprocated directed positive link from the other player, where $\bar{\mathrm{g}}_{i,j} \neq 1$ [2]. Extending a negative link results in an antagonistic relationship and can be thought of as picking a fight, which incurs a cost of $\varepsilon > 0$. Even though

[2] This point leads to problems of coordination failure to form the friendship. Hence, pairwise stability should be considered.

extending the negative directed link is a unilateral decision, bilateral players get to engage in conflict, incurring another cost of $\kappa \geq 0$ to both players. Thus, this model is different from Hiller (2017) and Chapter 2 regarding the set of actions (including the neutral link) and the friendship cost. Except for these factors, the model is identical to that of Hiller (2017) and Chapter 2.

The utility of player i under a strategy profile $\mathbf{g}$ is given as

$$u_i(\mathbf{g}) = \sum_{j \in N_i^-(\mathbf{g})} f(y_i(\mathbf{g}), y_j(\mathbf{g})) - \mid N_i^{e-}(\mathbf{g}) \mid \varepsilon - \mid N_i^-(\mathbf{g}) \mid \kappa - \mid N_i^{e+}(\mathbf{g}) \mid c^+. \tag{3.1}$$

In this model, we use the notion of pairwise Nash equilibrium because neither Nash equilibrium and pairwise stability is not enough to analyze the generalized signed network model. The notion of Nash equilibrium cannot consider a pairwise deviation of two players to form their friendship. On the other hand, Jackson and Wolinsky (1996) defined the notion of pairwise stability regarding the single link formation. So, when a player only considers one link at a time, if there is no incentive to change each link, then the network is pairwise stable. However, the player may consider a deviation switching multiple links if it increases her utility. The notion of Pairwise Nash equilibrium covers the limits of both equilibrium concepts.

However, pairwise Nash equilibrium concept has not been used to analyze the signed network formation model yet. Thus, it is possible to refine pairwise Nash equilibrium for the signed network model as follows.

Definition 3.1. A strategy profile (a signed network) $\mathbf{g}$ is a pairwise Nash equilibrium if the following conditions hold:

1. Nash stability: For any $i \in N$, $u_i(\mathbf{g}_i, \mathbf{g}_{-i}) \geq u_i(\mathbf{g}'_i, \mathbf{g}_{-i})$ $\forall \mathbf{g}'_i \in G_i$, $\forall i \in N$.

2. Pairwise stability: For any $\bar{g}_{i,j} \neq 1 \in \mathbf{g}$, if $u_i(\mathbf{g}) < u_i(\mathbf{g} + \bar{g}^+_{i,j})$, then $u_j(\mathbf{g}) > u_j(\mathbf{g} + \bar{g}^+_{i,j})$.

The condition of Nash stability is related to the strategy by a single player, which take effect in changing the signed network. First, a player can degrades the relationship ($1 \to 0, 1 \to -1$, and $0 \to -1$). Similar to the normal network formation game, the positive undirected link formation requires mutual consent. Thus, if one player chooses to stop the positive attitude, then the friendship will be terminated. On the other hand, a player can initiate a conflict with another player, regardless of this player's intention. Moreover, the player can switch some positive and neutral links to the negative links towards the multiple players. Second, When only a single player extending the negative directed link between two players, this player can stop the conflict ($-1 \to 0$). In a lemma in the following section, we will show that the second deviation can stop the conflict because only a player between two players under conflict extends the negative directed link in any pairwise Nash equilibrium. The other player always extends the neutral directed link in equilibrium. Therefore, except for the positive link formation, Nash stability checks the deviation strategies such as ceasing the positive link, initiating the negative link, stopping the negative relationship, and these combination.

On the other hand, following the definition, pairwise stability only checks a single positive link formation ($0 \to 1$, and $-1 \to 1$). It means that even though player i wants to establish friendship with player j and k at the same time, and even these j and k also get benefit from the relationship with i, the pairwise stability notion does not consider it. The pairwise deviation from the negative relationship to the neutral relationship is considered by the condition of Nash stability, as explained above. Thus, pairwise stability only applies to the positive link formation.

3.3 Analysis

3.3.1 Global results

We derive a series of lemmas and propositions that apply in both models of homogeneous players and heterogeneous players. Results of the model of homogeneous players are dealt with in Section 3.3.2. Section 3.3.3 covers the model of heterogeneous players.

The following Lemmas 3.2 and 3.3 hold for every pairwise Nash equilibrium networks.

Lemma 3.2. *In any pairwise Nash equilibrium* $\mathbf{g}$*, there does not exist a pair of players* i *and* j *such that*

(i) $\mathrm{g}_{i,j} = 1, \mathrm{g}_{j,i} = -1$,

(ii) $\mathrm{g}_{i,j} = 1, \mathrm{g}_{j,i} = 0$*, and*

(iii) $\mathrm{g}_{i,j} = \mathrm{g}_{j,i} = -1$.

Lemma 3.3. *In any pairwise Nash equilibrium* $\mathbf{g}$*, if* $\bar{\mathrm{g}}_{i,j} = -1$ *with* $y_i(\mathbf{g}) > y_j(\mathbf{g})$*, then* $\mathrm{g}_{i,j} = -1, \mathrm{g}_{j,i} = 0$.

The proof for these lemmas is in Appendix.

Lemmas 3.2 and 3.3 are similar to lemmas in Hiller (2017), that every Nash equilibrium network satisfies in his model. In Lemma 3.2, (i) and (ii) imply that any unilateral friend request is not profitable so that there are only reciprocal friendships in equilibrium. (iii) means that pairwise conflicts do not exist in this model. Lemma 3.3 is a reason for (iii). Lemma 3.3 states that a player with a higher network strength extends the negative directed link to extract utility from the other player with the lower network strength. The exploited player does not want to pay the conflict cost ε because she only loses her utility from the conflicting relationship. Thus, the exploited player extends the neutral link under the conflict, at least to save her conflict cost[3] .

Lemma 3.3 shows that conflict can exist when the network strengths are not identical ($y_i(\mathbf{g}) > y_j(\mathbf{g})$). However, even though the network strengths are different, if the conflict costs are high, exploiting utility will not happen. Then, if the network strengths are identical between two players, what will happen? Lemma 3.4 shows that every player with the same network strength does not have the negative relationship with each other. it implies that any players in the same strength partition will never be enemies in a pairwise-Nash equilibrium.

It also implies that, for every conflict for an economic reason, there are always the strong and the weak, but not rivals (whose network strengths are the same).

Lemma 3.4. *In any pairwise-Nash equilibrium* $\mathbf{g}$*, if* $y_i(\mathbf{g}) = y_j(\mathbf{g})$*, then* $\bar{\mathrm{g}}_{i,j} = 1$ *or* 0*.*

The proof is in Appendix.

Note that Lemma 3.4 was a part of Hiller (2017) proposition. His proposition also include another part that if $y_i(\mathbf{g}) \neq y_j(\mathbf{g})$, then $\bar{\mathrm{g}}_{i,j} = -1$. This proposition implied that any Nash equilibrium was weakly structurally balanced in his model. A weakly structurally balanced network is a network where friend of my friend is always my friend. Also, a weakly structurally balanced network can be segregated into multiple subnetworks. In this subnetwork, every player is connected with the positive undirected link. If two players are in different subnetworks, then there exists the negative undirected link between them. This phenomenon is also called clustering.

From contraposition of Lemma 3.4, it is possible to derive Proposition 3.7, a characterization of the pairwise Nash equilibrium networks. In graph theory, an independent set means a set

[3]In Hiller (2017), the exploited player extended the positive link instead of the neutral link, because the neutral link was not allowed in his model. Hence, the positive link is the cheapest option.

of nodes that are not linked at all. In this signed network, let us apply this notion for the alliance network $\bar{\mathbf{g}}^{all}$. Then an independent set is a set of players, where any two players has the positive undirected link between them.

Definition 3.5. An independent set N_{ind} is a set in which $\bar{\mathrm{g}}_{i,j} \neq 1 \; \forall \; i, \;\; j \in N_{ind}$.

If a graph has links only between independent sets, it is a multipartite network.

Definition 3.6. a k-partite network (multipartite network) is a network of which players are partitioned into k different independent set where $k \geq 2$.

The contraposition of Lemma 3.4 is that in any conflict network $\bar{\mathbf{g}}^{con}$, the negative links only exist between independent sets in equilibrium. Hence, every pairwise Nash equilibrium conflict network is multipartite except for the empty network.

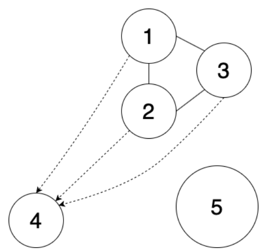
Proposition 3.7. *Suppose* $\mathbf{g}$ *is a non-empty pairwise-Nash equilibrium network and there exist* $\{P_1(\mathbf{g}), P_2(\mathbf{g}), ..., P_m(\mathbf{g})\}$. *Then the conflict network* $\bar{\mathbf{g}}^{con}$ *is a* k*-partite network where* $k \leq m$.

The proof is in Appendix.

Proposition 3.7 can give an intuition on how to detect clustering in the real data. In the network data analysis, a cluster means a set of densely connected nodes. The nodes in a cluster are sparsely connected to other clusters. Hiller (2017) showed a possibility of the weakly structurally balanced network in the signed network model. If it exists, then it is easy to detect clustering. However, in real data, it is hard to observe the weakly structural balance. Nevertheless, it is still possible to regard any network as a multipartite conflict network. Then, we can detect the independent sets based on the distribution of the negative links. The independent sets can be clusters of the alliance network as itself, or it can be a good hint to detect other plausible clusters.

Lastly, we show that the neutral links can exist in equilibrium even though the positive link cost is even zero. Note that it is possible to regard Hiller (2017)'s model as a particular case of this model where $c^+ = 0$. However, even though the friendship cost is zero, there exist situations where neutral links will be part of the equilibrium.

Example 3.8. Consider the following network $\mathbf{g}$. Player 1,2,3, and 4 has intrinsic strength of 1. Player 5's intrinsic strength λ_5 is 2.



The configuration is a pairwise Nash equilibrium when $f(3,1) - f(2,1) \geq c^+ \geq 0$ and $f(3,1) \geq \varepsilon + \kappa \geq f(2,1)$.

Example 3.8 shows that when there is a player who does not engage in any conflict, she does not have any incentive to extend any positive links regardless of the size of the friendship cost.

3.3.2 Homogeneous player model

This section investigates properties of the networks in equilibrium when the players are homogeneous. Also, we find some notable pairwise Nash equilibrium networks. All global results in the previous section are applied to this model.

First, in any pairwise-Nash equilibrium, positive links exist only when there are the negative links, and vice versa.

Proposition 3.9. *When the players are homogeneous, in any pairwise-Nash equilibrium* $\mathbf{g}$*, if and only if there exists* $i, j \in N$ *such that* $\bar{g}_{i,j} = 1$*, there exists* $k, l \in N$ *such that* $\bar{g}_{k,l} = -1$.

Proposition 3.9 shows that friendship and conflicts are prerequisites of each other in the signed network model of homogeneous players. Let us consider each statement respectively. First, conflicts are prerequisites of friendships. It is because there is no direct benefit from the positive links. However, the positive links indirectly contribute to the players' utility through the extraction under the negative relationships. It gives us two intuitions. First, even though there is no direct benefit from friendship, friendship can exist because of the conflicts. Second, conflict can facilitate a higher level of friendship. Jackson and Nei (2015) had similar results. They were interested in stable networks without conflicts and even the potential of conflicts. They called these networks war-stable networks. They showed that only the empty network could be war-stable. It implies that there is no alliance without external conflict. Even though Jackson and Nei (2015) used the different notion of the stable network, their result has a thread of connection.

On the other hand, when the players are homogeneous, having alliances is also a prerequisite for conflicts to exist. Since each player has the same intrinsic strength, their network strength is also identical without any friends. The extraction under conflicts is positive when the network strength is imbalanced. Thus, the homogeneous players need their alliances to make any conflicts profitable.

By using Proposition 3.9, it is possible to exclude some kinds of networks which cannot be in equilibrium. In this model, where there is neither complementarity nor conflicts, there is also no friendship. Thus, an ideal status, where every player is a friend, cannot be a pairwise Nash equilibrium. The network where all players are friends is named Utopia network in Chapter 2. The proposition also excludes a network consisting of only the negative links from the pairwise Nash equilibria. Hence, the worst scenario, where every player is an enemy, cannot be a pairwise Nash equilibrium, too.

Corollary 3.10. *(No Utopia and no Dystopia network) When the players are homogeneous, a network* $\mathbf{g}$ *is not a pairwise-Nash equilibrium if (i)* $\bar{\mathrm{g}}_{i,j} = 1$ *for* $\forall\ i,\ j\ \in N$*, or (ii)* $\bar{\mathrm{g}}_{i,j} = -1$ *for* $\forall\ i,\ j\ \in N$*.*

We omit the proof because it is directly derived from Proposition 3.9. Meanwhile, the network with only the negative links can be a pairwise Nash equilibrium in the heterogeneous model depending on the condition. We will discuss it in the next section of heterogeneous players.

Corollary 3.10 states that Utopia network is never a pairwise-Nash equilibrium. This result is different from Hiller (2017). He showed that when the players are homogeneous, the network where every player is a friend was always a Nash equilibrium. However, in this model, this Utopia network cannot be a pairwise Nash equilibrium with the homogeneous players. It is also not a pairwise Nash equilibrium with the heterogeneous players. We will discuss it in the next section. Instead, the empty network is a pairwise Nash equilibrium with the homogeneous players. Depending on the condition, it can be even the unique pairwise Nash equilibrium. Corollary 3.11 suggests a condition that the empty network is a pairwise Nash equilibrium or the unique pairwise Nash equilibrium.

Proposition 3.11. *When the players are homogeneous, then empty network is a pairwise-Nash equilibrium. When* $n \geq 4$*, empty network is the unique pairwise-Nash equilibrium if* $f(n-1,1) - min_{2\leq x \leq n-1} f(x,x-1)$ *is sufficiently small and* $min_{2\leq x\leq n-1} f(x,x-1) > \frac{c^+}{2}$*.*

$f(n-1,1)$ is the maximum extraction in the model of homogeneous players $(\bar{f})$, and $min_{2\leq x\leq n-1} f(x,x-1)$ is the minimum of the positive extraction. The proof is in appendix.

Then, is Proposition 3.11 different from his result? No. In fact, both results indicate the same direction. These results imply that a network without conflict can be in equilibrium regardless of the neutral links' existence. Therefore, Utopia network in Hiller (2017) has the same role as the empty network in this model.

Hiller (2017) also pointed out that specific network formations were always Nash equilibria. One of the networks is where $n-1$ players are friends to each other, and one player is an enemy against these $n-1$ players. This network can be a pairwise-Nash equilibrium, too. In this model with the neutral links, let's define a generalized version of this network using the notion of a regular network. A regular network is a network when each player has the same number of links. If every player has k links, then the network is called a k-regular network.

Definition 3.12. A network $\mathbf{g}$ is a k-regular bullying network if $n-1$ players form k regular alliance sub-network and the $n-1$ players extend the negative directed links to the remaining player.

By Lemma 3.4, there is no negative link within any regular sub-graph in any pairwise-Nash equilibrium network. Hence, between the players in the k-regular alliance network, there is no negative directed link. Proposition 3.13 states the condition when each k-regular bullying network is a pairwise Nash equilibrium.

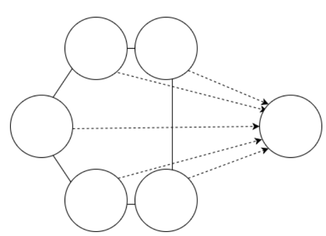


Figure 3.2: 2-regular bullying network

Proposition 3.13. *Suppose the players are homogeneous.*

i) For $2 \leq k \leq n-3$, *if* $\varepsilon + \kappa \leq f(k+1,1)$, $f(k+2,1) - f(k+1,1) \leq c^{+} \leq f(k+1,1) - f(k,1)$ *and* k*-regular networks exist, then* k*-regular bullying networks are pairwise-Nash equilibria.*

ii) If $\varepsilon + \kappa \leq f(n-1,1)$ *and* $c^{+} \leq f(n-1,1) - f(n-2,1)$, $(n-2)$*-bullying network is a pairwise-Nash equilibrium.*

The proof is in Appendix.

Note that $(n-2)$-regular bullying network is the network where the sub-network is the complete network with the positive links. Hiller (2017) mentioned that this network is always a Nash equilibrium when the conflict cost is not extremely high. Proposition 3.13 shows that for a given condition, even though this $(n-2)$-regular bullying network is not always a pairwise Nash equilibrium, the other bullying networks can be pairwise Nash equilibria.

3.3.3 Heterogeneous player model

In this section, we investigate aspects observed in the model of heterogeneous players, which are different from the result of the model of homogeneous players. Hence, while the global results in Section 3.3.1 still hold, some results in Section 3.3.2 may not hold in this section.

First, when the players are heterogeneous, conflict without alliance can exist in equilibrium. For example, if the difference among each intrinsic strength is large and the friendship cost is high, players with high intrinsic strengths enjoy the extraction under the conflict without alliances. However, conflict is still a prerequisite of friendship.

Proposition 3.14. *In any pairwise-Nash equilibrium* $\mathbf{g}$*, if there exist* $i, j \in N$ *such that* $\bar{g}_{i,j} = 1$*, there exist* $k, l \in N$ *such that* $\bar{g}_{k,l} = -1$*.*

The proof is in Appendix.

Since friendship is not a necessary prerequisite of conflict anymore, the network where all players fight against all players can be a pairwise Nash equilibrium. However, Utopia network is still not a pairwise-Nash equilibrium.

Corollary 3.15. *(No Utopia network) A network* $\mathbf{g}$ *where* $\bar{g}_{i,j} = 1$ *for* $\forall\ i,\ j\ \in N$ *is not a pairwise-Nash equilibrium.*

We omit the proof since it is trivially derived from Proposition 3.14.

While the empty network is always a pairwise-Nash equilibrium when players are homogeneous, it is not always in equilibrium if the players are heterogeneous. Here we suggest a generalized result regarding the empty network for the heterogeneous model. The empty network can be a Nash equilibrium when the conflict cost is high compared to the size of the difference between intrinsic strengths.

Proposition 3.16. *Let* $Max\{\lambda_1, ..., \lambda_n\} - min\{\lambda_1, ..., \lambda_n\} < \varepsilon + \kappa$*, then empty network is a pairwise-Nash equilibrium.*

Next, we check that positive assortative matching occurs in this model. Chapter 2 defined positive assortative matching in network formation theory as follows. Suppose there are two types of players. One is a strong type who has λ_s, and the other is a weak type with λ_w. $\lambda_s > \lambda_w$, and the difference between the types can be measured by a parameter $a = \frac{\lambda_s}{\lambda_w}$. The set of the strong players is N_s, and the set of the weak players is N_w. When the positive links exist only between the same type of players in a network, this network is a positive assortative matching.

Definition 3.17. Positive assortative matching (PM) is a network configuration $\mathbf{g}$ where there exists $\bar{\mathbf{g}}_{ij} = 1$ for $\{i, j\} \in N_s \times N_s \setminus \{i\}$ or for $\{i, j\} \in N_w \times N_w \setminus \{i\}$, and $\bar{\mathbf{g}}_{ij} \neq 1$ $\forall\{i, j\} \in N_s \times N_w$.

Definition 3.18. A strong dominant positive assortative matching (SPM) is a positive assortative matching where $y_i > y_j$ for every $i \in N_s$ and $j \in N_w$.

Chapter 2 showed that when the conflict cost is sufficiently small, there exists $\underline{a}$ such that when $a \geq \underline{a}$, all Nash equilibrium network configurations are strong dominant positive assortative matching. Here we generalize the result to the model with the neutral links.

Proposition 3.19. *For $\varepsilon + \kappa < \bar{f}$, there exists $\underline{a}(n_s, n_w, c^+, \varepsilon, \kappa)$ such that if $a \geq \underline{a}$ all pairwise-Nash equilibria exhibit either i) strong dominant positive assortative matching or that ii) there exist only the neutral undirected links between players of the same type and only the negative undirected links between players of the different types.*

The network formation in ii) is similar to positive assortative matching because there $g_{i,j} = -1$ for all $i \in N_s$ and all $j \in N_w$. However, it is not a positive assortative matching because there does not exist a positive undirected link between the same type of player. If the difference between the two types of players is extremely high so that any friendship formation is meaningless, every strong player extracts utility from all weak players respectively. The weak players also do not seek their friends because a new friend will also not be helpful to decrease the extraction.

3.4 Conclusion

We study the signed network formation model, including the neutral links. We use pairwise Nash equilibrium because pairwise stability is required for players to form the positive link from the neutral link. In our environment, each player has their own intrinsic strength. If they form the positive relationship, then they can aggregate their intrinsic strength to the bigger network strength. If they form the negative relationship, then the zero-sum conflict occurs, where the player with higher network strength pillages utility from the other player with smaller network strength. If they form the neutral relationship, nothing happens. When they form the positive or the negative links, the relevant cost is incurred, but not for the neutral link.

We show that some of the findings in the signed network formation literature still hold in this model. Here are some results we can still verify as valid in this model. First, as Jackson and Nei (2015) showed, any peaceful networks with friendship without conflicts cannot be in equilibrium. It means that conflict can be a motivation for friendship. That is, when there is no direct benefit from friendship such as complementarity, people can seek their friend to fight against their enemies. Furthermore, we show that, when the players are homogeneous, there does not exist conflict without friendship, too. Second, Hiller (2017) showed that Utopia network and bullying network were Nash equilibria when the players were homogeneous. In this model, the empty network and the bullying networks can be pairwise stable networks, too. Utopia network in his model and the empty network in this model share the property that these are the network without conflicts. Lastly, Chapter 2 showed that if the difference between the different types was large enough, any Nash equilibrium network configurations were strong dominant positive assortative matchings. In this model, in addition to it, the network, consisting of the neutral undirected links between the same type and the negative undirected links between the different types, can be a pairwise Nash equilibrium.

In future research, it will be possible to integrate the factor of complementarity accompanied with the positive relationship to the generalized model. Many researchers have pointed out complementary as a source of networking among people. However, there is no complementary effect of the positive relationship in this model. This extension will test how the results in equilibrium vary as the complementarity is introduced.

Chapter 4

A Weighted Network Formation Model of Inequality

4.1 Introduction

Inequality is a classical issue in economics. In many cases, inequality distorts the social system. At the same time, society can magnify inequality. For example, the rich tend to hang out with the rich. In the meantime, the rich often share useful information and business opportunities only with themselves. Although the rich have no intention of discriminating against the poor, this factor often cements the cartel among the rich.

This paper studies which weighted network players form when unequal wealth is given. In the existing literature, many homogeneous player models are well documented, but any heterogeneous player model was rare. I introduce simple binary heterogeneity among the players: Rich players and poor players. Rich players have higher endowments, while poor players have lower endowments. In particular, I investigate whether players prefer players of the same type or the different types when a specific preference such as homophily does not exist among players.

This paper is motivated by Salonen (2016), Baumann (2021), and Griffith (2019a). In their models, players choose self-investment levels and relation investment levels. Self-investment is independent of relation-investment in their model. When a pair of players spend non-zero relation-investment on their relationship together, they benefit from the relationship. Salonen (2016) used Cobb-Douglas function to describe the relation investment. Also, he assumed that the marginal self-investment is constant. Baumann (2021) generalized it to the general functional form and analyzed it. Griffith (2019a) also used the Cobb-Douglas function for the relation-investment but assumed that the marginal self-investment is decreasing. While all of the previous literature focused on the equilibrium analysis with the homogeneous players, I study the heterogeneous players model. My model uses Griffith (2019a), but is also based on Baumann (2021).

The paper only considers the case when the network is complete. In Griffith (2017) 's model, he focused on the complete network where every player has a relationship with every other player. The players choose only relation-investment levels without the factor of self-investment. He also performed structural estimation and analyzed a Nash equilibrium network as a part of the estimation. In my paper, I analyze the case of the complete network to study how the relationship varies by the level of inequality. While network structures can distort the effect of inequality, I start studying the weighted network formation model regarding inequality with the complete graph case as a benchmark. Even though every player is allowed to have a relationship with every other player, each player behaves differently depending on their type and on the level of inequality.

Also, I show that a specific network configuration may be investigated due to censoring in a social network survey. In the survey investigating the social network, researchers tend to set a limit of responses. For example, in the National Longitudinal Study of Adolescent to Adult Health (Add Health) conducted by Harris et al. (2009), each respondent can report her male/female friends up to five people respectively. While censoring is inevitable due to space limitations, it can be a source of distortion or momentum forcing a particular result.

Griffith (2019b) showed that this censoring could attenuate peer effects estimates. In this paper, I show that censoring can bring about nested split graphs as a result of the survey when there is inequality.

The structure of the paper is organized as follows. Section 4.2 introduces the model of weighted network formation and nested split graphs. Section 4.3 displays the result of the analysis. Section 4.3.1 presents how much the player invests in herself and in the relationship, depending on the type. Section 4.3.2 shows that the censoring in the survey data derives nested split graphs in the result. Section 4.3.3 reports the simulation result and gives intuitions regarding social welfare and inequality. Lastly, Section 4.4 concludes.

4.2 Model

Let $N = \{1, 2, ..., n\}$ denote the set of players. where n is even and $n > 3$. There are two types of players, rich player (*type* r) and poor player *type* p. The set of rich players is R and the set of poor players is P. The number of each type of player is the same, so there are $\frac{n}{2}$ rich players and $\frac{n}{2}$ poor players[1]. Each player has their endowment M_i. The endowments are exogenously given. If player i is rich (*type* r), then $M_i = M_r$. If player i is poor (*type* p), then $M_i = M_p$, where $0 < M_p < M_r \leq 1$. Player $i \in N$ chooses her self investment level and $n-1$ relation-investment level. Hence, each player's strategy is an n-dimensional vector that says how much she will invest in herself and how much she will invest in forming with others. Her self investment is $x_{i,i} \geq 0$. Her relation-investment level toward j is denoted by $x_{i,j} \geq 0$ where $j \in N \setminus \{i\}$. Hence, player i's strategy is $X_i = (x_{i,1}, x_{i,2}, ...x_{i,n}) \in \mathbb{R}^n_+$. A strategy profile X consists of every player's strategy. Thus, $X = (X_1, X_2, ..., X_n) \in \prod_{i \in N} \mathbb{R}^n_+$. Also, let X_{-i} denote a set of strategies except for player i's strategy.

[1] In this study, I assume that the first players 1 to $\frac{n}{2}$ are rich and the last players $\frac{n}{2} + 1$ to n are poor for easy understanding.

The utility of player i under a strategy profile X is given as

$$u_i(X_i \mid X_{-i}) = S(x_{i,i}) + \sum_{j \in N \setminus \{i\}} x_{i,j}^{\alpha} x_{j,i}^{\beta}.$$

$S(\cdot)$ is a self investment utility function. Following Baumann (2021) and Griffith (2019a), $S(x_{i,i})$ has the following properties. For $x_{i,i} \in [0,1]$, i) $S(x_{i,i})$ is increasing, ii) $S(x_{i,i})$ is strictly concave, iii) $S(x_{i,i})$ is continuously differentiable, and iv) $\lim_{x \to 0} S'(x) = \infty$ and $\lim_{x \to 1} S'(x) = 0$.

The Cobb-Douglas function $x_{i,j}^{\alpha} x_{j,i}^{\beta}$ is a joint production between player i and j. Only when both $x_{i,j}$ and $x_{j,i}$ are positive, the joint production is positive. This Cobb-Douglas function describes the relation investment. Here, α represents an efficiency of the relation-investment of player i (herself) on the joint product, and β means an efficiency of the relation-investment of player j (the other side) on the joint product. In this research, I assume that the joint production shows decreasing return to scale ($\alpha + \beta < 1$). The reasons are as follows. First, decreasing return to scale is an assumption that is broadly accepted for production. Second, Griffith (2019a) showed that a Nash equilibrium strategy profile always exists in this setting. Third, he also showed that there existed the unique Nash equilibrium strategy profile for any specific graph (network). For example, if a network is complete (all $x_{i,j} > 0$), then there only exists one unique Nash equilibrium strategy profile.

Player i's budget constraint is

$$\sum_{j \in N} x_{i,j} = M_i.$$

As mentioned above, the endowment is depending on the type of player r or p.

In this study, I use Nash equilibrium concept. A strategy profile X^* is a Nash equilibrium if and only if

$$u_i(X_i^* \mid X_{-i}^*) \geq u_i(X_i \mid X_{-i}^*) \ \forall X_i \in \mathbb{R}_+^{n-1}, \ \forall i \in N.$$

Social welfare is defined as the sum of individual utilities. For any X, it is given as $U(X) = \sum_{i \in N} u_i(X)$. A strategy profile $\hat{X}$ is socially efficient if and only if $U(\hat{X}) \geq U(X)$ for all $X \in \prod_{i \in N} \mathbb{R}_+^n$.

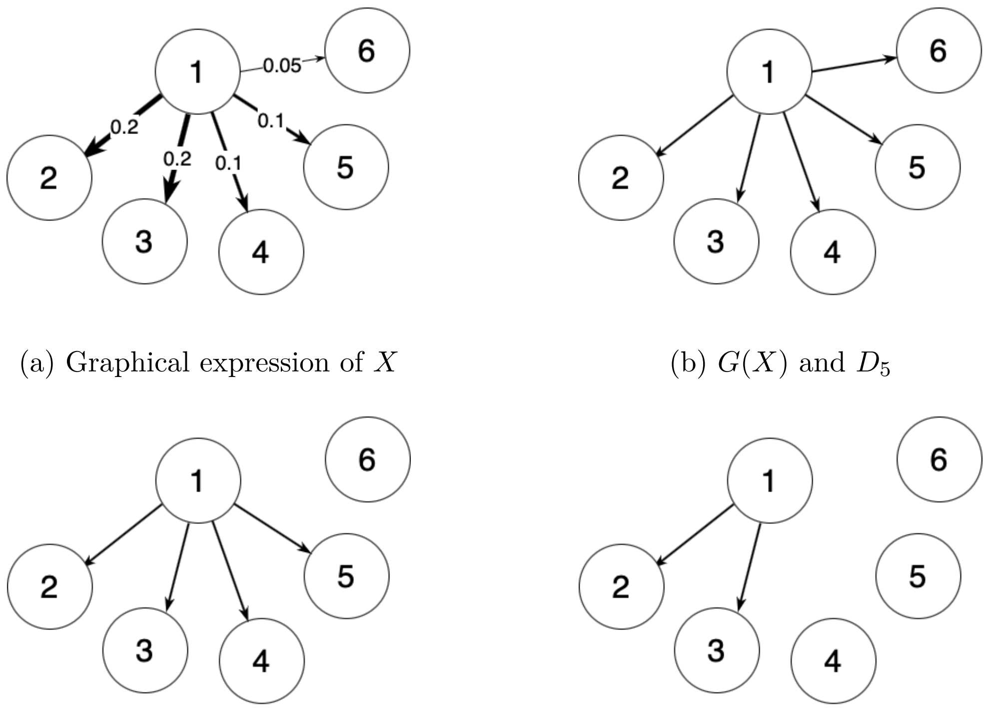
As a result of a strategy profile X, it is possible to draw a graph meaning a social network. In this paper, I consider two kinds of graphs: Normal (undirected) network $\bar{G}$ and Data (undirected) network $\bar{D}_k$. It is because researchers use undirected networks when they discuss the structure of the network. For clear understanding, I construct directed versions of both networks G and D_k and define $\bar{G}$, and $\bar{D}_k$ based on them.

First, there is a traditional concept of network, where a directed link exists ($\mathrm{g}_{i,j} = 1$) if $x_{i,j} > 0$. If $x_{i,j} = 0$, then $\mathrm{g}_{i,j} = 0$. For any $x_{i,i}$, $\mathrm{g}_{i,i}$ is defined as 0 because $x_{i,i}$ is the self-investment information. By collecting every $\mathrm{g}_{i,j}$, it is possible to construct "*Normal network*" $G(X)$ as a form of a matrix.

The second network is the data matrix. It is defined as follows. The matrix D_k is a $n \times n$ directed matrix for $k > 0$. k means the number of links for each player to report. In this model, I allow each player to report multiple links exceeding k if there exist equally close friends to remove any uncertainty [2] . The process to form D_k from X is as follows. For each i, $D_k(X)$ is defined that (i) $d_{i,j} = 1$ if the ranking of $x_{i,j}$ is the same as or higher than k among $j \in N \setminus \{i\}$. (ii) Otherwise, $d_{i,j} = 0$. (iii) Lastly, any $x_{i,i}$ gives $d_{i,i} = 0$. If there exist tied links such that $x_{i,j} = x_{i,l}$, then these have the same ranking, which is the higher one. The lower ranking is skipped for consistency. For example, if $x_{1,2} = 0.2$, $x_{1,3} = 0.2$,

[2] In the real survey, if I have six equally best friends but the number of responses is five, I should randomly exclude one of my friends. To remove the uncertainty, I assume that each respondent can report more than the limit in the situation.

$x_{1,4} = 0.1$, $x_{1,5} = 0.1$, $x_{1,6} = 0.05$, then $d_{1,2}$ and $d_{1,3}$ are the highest, $d_{1,4}$ and $d_{1,5}$ are the thirdly highest, and $d_{1,6}$ is the fifthly highest (the lowest). Figure 4.1 shows each network configuration derived from the example. In this figure, I assume that every other $x_{i,j}$ is 0 except for the ones mentioned above.



(a) Graphical expression of X

(b) $G(X)$ and D_5

(c) D_3 and D_4

(d) D_1 and D_2

Figure 4.1: Networks in the example

Now, let's define the undirected networks $\bar{G}$ and $\bar{D}_k$. I use the notations of D_k here, but $\bar{G}$ is constructed in the same way. $\bar{D}_k$ is an $n \times n$ undirected matrix. $\bar{d}_{i,j} = 1$ when $d_{i,j} = 1$ or $d_{j,i} = 1$. Naturally, $\bar{d}_{i,j} = 0$ when $d_{i,j} = 0$ and $d_{j,i} = 0$. In an undirected graph $\bar{D}_k$, $\delta_i(\bar{D}_k)$ denotes the degree of node (player) i in $\bar{D}_l$.

There are varieties of graphs in the literature. The structure of networks has been defined in unweighted graphs, in which every link has a value of 1 or 0. Following the tradition, I only consider the unweighted networks when I discuss the structure of networks. In this paper, I focus on two structures: *complete network (complete graph)* and *nested split graphs*. The complete network is the network where every link has a positive value. In this network, every player is completely connected to every other player.

Definition 4.1. A network $\bar{G}$ is the complete network where $\bar{g}_{i,j} = 1$ for every i and $j(\neq i) \in N$.

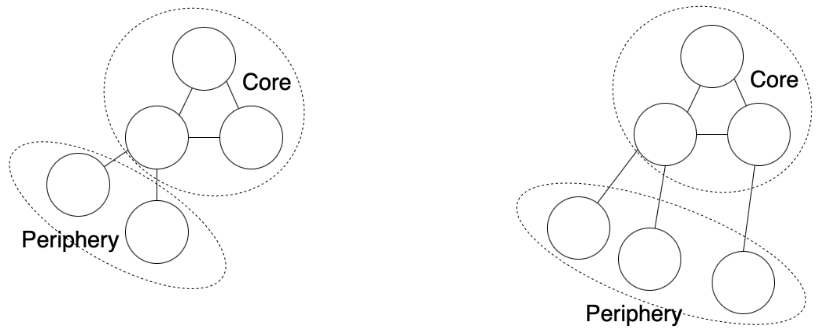
The complete network is $(n-1)$-regular network, so every player is equally connected to each other.

Contrary to the complete network, in another network, players can be connected but not equally. Some players are more connected (central), while the other players have fewer connections. If the players can be divided to two groups (the centralized group and the periphery group) in a network, the network is called a split network or a core-periphery network [3] . Furthermore, we can consider the case when the connection is hierarchical. It means that the friend set of a more connected player always includes that of a less connected player. If the condition is satisfied in a split graph, then it is called a nested split graph.

Definition 4.2. A network $\bar{G}$ is a nested split graph if and only if, when $\bar{g}_{i,j} = 1$ and $\delta_k(\bar{G}) \geq \delta_j(\bar{G})$, then $\bar{g}_{i,k} = 1$ for any $i, j, k \in N$.

Note that the complete network satisfies the definition of nested split graphs.

[3] A network $\bar{G}$ is a split network (a core-periphery network) if N is partitioned into two sets: the core $C(\bar{G})$ and the periphery set (independent set) $P(\bar{G})$, such that $\bar{g}_{i,j} = 1 \; \forall \; i, j \in C(\bar{G})$, and $\bar{g}_{k,l} = 0 \; \forall \; k, l \in P(\bar{G})$.



(a) Nested split network (b) Not nested, but split network

Figure 4.2: Examples nested (and not nested) split networks

4.3 Analysis

4.3.1 Properties of Nash equilibrium strategy profile

This section only considers a case when $\bar{G}$ is the complete network. One of the purposes of this paper is to study the configurations of networks on survey data. I presume that there is a latent network (In this paper, $\bar{G}$.). Then, only a part of the latent social network (In this paper, $\bar{D}_k$.) is investigated because of the limit of the survey form. Then, we may observe another network configurations $\bar{D}_k(X)$ different from $\bar{G}(X)$. Hence, I study how $\bar{D}_k(X)$ looks like where $\bar{G}(X)$ is complete. In future research, it will be possible to study a case when some $\mathrm{g}_{i,j}$ is not allowed to be positive ($\bar{G}$ is not complete.). However, as the first study about Data network configurations, I start the paper with a benchmark of the complete network. As mentioned above, Griffith (2019a) showed that there exists a Nash equilibrium X, and it is unique when the corresponding $\bar{G}$ is given.

Because $S(x_{i,i})$ does not have a specific functional form, it is not available to derive a closed-form of the Nash equilibrium. However, it is available to compare the size between the self-investment and the relation-investments, between the same types, or between the different types. To make the model tractable, let's consider a symmetric Nash equilibrium strategy profile. A symmetric strategy profile means that the players of the same type use an identical strategy.

Proposition 4.3. *Suppose X is a symmetric Nash equilibrium such that $\bar{G}(X)$ is the complete network. Then for any $i, j \in R$ and $k, l \in P$, the following inequalities always hold.*

(i) $x_{i,i} > x_{k,k}$

(ii) $x_{i,k} > x_{k,i}$

(iii) $x_{i,j} > x_{k,l}$

(iv) $x_{i,j} > x_{i,k}$, *and*

(v) $x_{k,i} > x_{k,l}$.

The proof is in Appendix. Even though the exact size of each investment varies depending on the self-investment function and the parameters α and β, it is possible to determine the order of the investments. Then it is available to determine which relationship is stronger (or weaker) than other relationships. Let's see each item of Proposition 4.3. First, between the same types, rich players spend more on self-investment than poor players. Second, on the relationship between the different types (rich and poor players), rich players invest more than poor players. Third, the relation investment between rich players is higher than that between poor players. Fourth, rich players invest more in the relationship with other rich players (the same type) than with poor players (the different types). Fifth, however, poor players invest more in the relationship with rich players (the different type) than that with poor players (the same type).

The first, second, and third inequalities are straightforward because rich players have more endowment that they can use. The fourth and fifth items are also similar to the others, but these can give some intuitions. Regardless of the type of player, every player prefers rich players when she builds her relationship. In the context of homophily, it can be said that rich players show homophily, but poor players show heterophily. This result can give a hint to other researchers who try to detect homophily in network data.

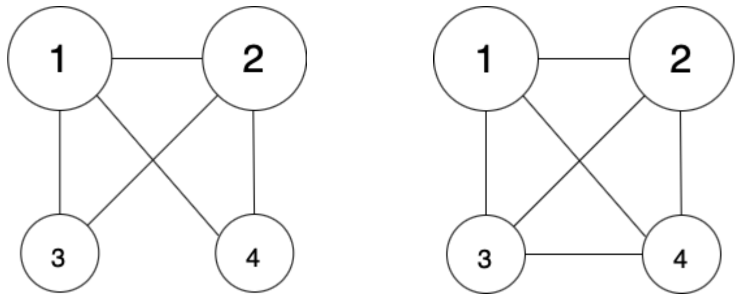
4.3.2 Nash equilibrium data network

Based on the result in Proposition 4.3, a special category of the network can be related to the Nash equilibrium strategy profile. Because I started from the complete network $\bar{G}$, let's skip the normal network part and go to the data network. Proposition 4.3-(4) and (5) indicate that every player invests more in the relationship with rich players than that with poor players. Then it is possible to find a certain data network in Nash equilibrium.

Proposition 4.4. *Suppose X is a symmetric Nash equilibrium such that $\bar{G}(X)$ is the complete network. Then $\bar{D}_k(X)$ is a nested split graph.*

The proof is in Appendix. The following example explains how the nested split graph appears in the data network.

Example 4.5. Let's consider the example of the four players' case. Player 1 and 2 are rich, and 3 and 4 are poor. When $k \leq 2$, the rich players form the core. The poor players form the periphery set, reporting only the rich players as their friends. Then $\bar{D}_k$ satisfies the definition of nested split graphs as shown in Figure 4.3 (a). When $k > 3$, the poor players report the other poor players as their friends because there are only two rich players. Then, $\bar{D}_k$ is the complete network, which is a nested split graph, too.



(a) $k \leq 2$ (b) $k > 3$

Figure 4.3: Configurations of Data network $\bar{D}_k$ in Example 4.5

4.3.3 Simulation: Numerical comparison by the level of inequality

Until the last section, I studied the model with the general self-investment utility function and the arbitrary α and β. While it has merit as a general model, it is not available to calculate utility directly. In this section, I use a specific self investment utility function $S(x) = x^{\frac{1}{2}}$ and set $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$ to perform a simulation. There are four players. Player 1 and 2 are rich, and player 3 and 4 are poor. I assume that the total endowment is fixed regardless of the level of inequality, as $M_r + M_p = 1$. Hence, if M_r increases, M_p is automatically decreased. It makes M_r represent the level of inequality in this setting. In each simulation, I report the result when $M_r = 0.5, 0.6, 0.7, 0.8,$ and 0.9 to observe the impact of inequality on the weighted network formation. Reporting the results, I also interpret the meaning regarding inequality. As I have done until the last section, I only consider the symmetric Nash equilibrium. Thus, $x_{1,1} = x_{2,2}$, $x_{1,2} = x_{2,1}$, $x_{1,3} = x_{1,4} = x_{2,3} = x_{2,4}$, $x_{3,3} = x_{4,4}$, $x_{3,1} = x_{3,2} = x_{4,1} = x_{4,2}$, and $x_{3,4} = x_{4,3}$.

Change of the investments by inequality

Above all, I check the change of each self and relation investment by the type. There are three trends in the change of investment.

M_r	0.5	0.6	0.7	0.8	0.9
$x_{1,1}$	0.274	0.327	0.382	0.442	0.510
$x_{1,2}$	0.075	0.107	0.146	0.195	0.260
$x_{1,3}$	0.075	0.082	0.085	0.081	0.064
$x_{3,3}$	0.274	0.222	0.170	0.118	0.063
$x_{3,1}$	0.075	0.064	0.049	0.033	0.016
$x_{3,4}$	0.075	0.049	0.029	0.014	0.004

Table 4.1: Change of the investments by inequality

First, as M_r increases, the rich players increases her investments except for $x_{i,k}$ where $i \in R$ and $k \in P$. The result is straightforward because each rich player will increase her investment as she has more endowment. Second, as M_r increases, M_p is decreased by the assumption. It implies that the poor players get poorer. As a result of the income reduction, the poor players decrease all of their investments. Third, as M_r increases, the rich player's relation investment toward the poor players ($x_{i,k}$ where $i \in R$ and $k \in P$) is increased at first but is decreased after a point. When M_r increases from 0.5, It is increased at first because she has more endowment. However, as M_r increases, M_p decreases, so the poor players decrease all of their investments. The outcome of the relation-investments is the multiplication (interplay) of bilateral players' relation investment. Thus, as the poor player decreases her relation investments significantly, the rich players lose an incentive to invest more in the relationship with the poor players. Relatively, as the relation-investment with the poor players gets less attractive, the other investments such as the self-investments and the relation-investments with the rich players become more profitable.

Social welfare and inequality of utility

Here, first, I compare social welfare, which is defined as the sum of utilities of every player (total utility). The notion of social welfare is one way to measure the efficiency of an economic system. On the other hand, inequality (fairness) is another important issue to evaluate economic systems. To measure the inequality of utility, I calculate the ratio of utilities of rich and poor players as $\gamma = \frac{u_i}{u_i+u_k}$, where $i \in R$ and $k \in P$. γ has a different value from M_r because of the concave self-investment utility function and the existence of the relation-investment between the players. Also, I compare γ between with the relation-investment and without it ($u_i(X) = x_{i,i}^{\frac{1}{2}}$) to check the impact of network effect on inequality. The result of simulation is as follows.

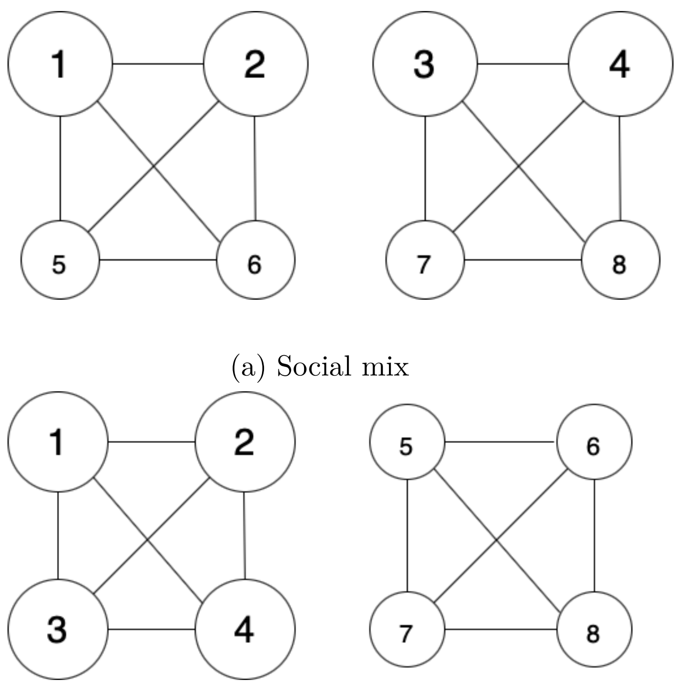
			With relationships		Without relationships	
M_r	u_1	u_3	Social welfare	γ	Social welfare	γ
0.5	0.954	0.954	3.818	0.5	2.828	0.5
0.6	1.048	0.848	3.793	0.552	3.098	0.550
0.7	1.131	0.725	3.714	0.609	3.346	0.604
0.8	1.203	0.581	3.568	0.674	3.577	0.666
0.9	1.260	0.396	3.313	0.760	3.794	0.75

Table 4.2: Social welfare and inequality of utility

First, it is possible to observe that income inequality aggravates the efficiency (social welfare). It is because every investment shows decreasing return to scale. As income inequality gets worse, the investments are concentrated on the rich players. While it increases the rich players' utility, it decreases the poor players' utility much more than the benefit of the rich players. Overall, the concentration of the investment brings about inefficiency. Second, as M_r increases, the level of inequality γ increases, but the absolute value of γ is lower than M_r. It is because every investment function is concave (decreasing marginal return). Third, the network exacerbates the inequality of utility. Compared to the case when there is no relation-investment, γ is bigger for the same M_r when there exists the factor of relation-investments.

Social mix and positive assortative matching

In this part, I use two incomplete network cases. In each case, there are two complete subnetworks. For both cases, there are eight players. Four players are rich, and the other four players are poor. In the first case, two rich players and two poor players form a complete subnetwork. Because the different types are mixed in the subnetworks, I call the case *Social mix*. In the second case, only the players of the same type form each complete subnetwork. There is no relationship between the players of the different types in the network. Thus, I call it *positive assortative matching*.



(b) Positive assortative matching

Figure 4.4: Configurations of social mix and positive assortative matching

Each case has a quality and a weak point. First, the positive assortative matching gives bigger total utility than the social mix when $M_r > 0.5$. It implies that the positive assortative matching case is better in terms of efficiency. Second, for the same level of inequality

	Social mix		Segregation policy	
	Social welfare	γ	Social welfare	γ
0.5	7.637	0.5	7.637	0.5
0.6	7.587	0.552	7.603	0.561
0.7	7.429	0.609	7.495	0.626
0.8	7.137	0.674	7.294	0.699
0.9	6.627	0.760	6.941	0.790

Table 4.3: Social mix and positive assortative matching

$M_r > 0.5$, γ, the difference in the utilities between the players of the the different types, is lower in the social mix than in the positive assortative matching. It means that in respect of fairness, the social mix shows a better result than the positive assortative matching.

4.4 Conclusion

The paper has built upon Salonen (2016), Baumann (2021), and Griffith (2019a). In this paper, I study the weighed network formation model where the players experience inequality. I investigate the symmetric Nah equilibrium profile when every player is connected. In any equilibrium strategy profile, some links are always stronger than other links. Overall, the links related to the rich players tend to be stronger than those associated with the poor players. It brings about homophily among the rich players, while the poor players show heterophily. Based on the result, I show that researchers who study social networks using survey data may observe nested split graphs because of the limit of responses on the survey.

After I derive the general properties, I perform simulations with a specific utility function. In the simulation, when the level of inequality increases, the players behave depending on their type (rich or poor). When inequality increases, the rich players increase every investment except for the relation-investment between rich players and poor players. The rich players increase the relation investment between rich and poor players at first but decrease it later.

On the other hand, the poor players decrease every investment. Based on the behavior of each type, I analyze social welfare as the sum of every player's utility and inequality of utility. The result shows that inequality decreases social welfare and aggravates the difference of utility between the types. Lastly, I simulated two special cases of social mix and positive assortative matching. While the positive assortative matching effectively increases the size of social welfare, the social mix can decrease the inequality of utility between the players of the different types.

In this chapter, I assume that the numbers of rich players and poor players are the same. Even though these numbers are different, the main result would be similar. I remain this part as future research.

Bibliography

J. M. Abowd, F. Kramarz, P. Lengermann, and S. Pérez-Duarte. Are good workers employed by good firms? a test of a simple assortative matching model for france and the united states. *Unpublished Manuscript*, 2004.

M. J. Andrews, L. Gill, T. Schank, and R. Upward. High wage workers and low wage firms: negative assortative matching or statistical artefact? Technical report, Diskussionspapiere, 2006.

M. J. Andrews, L. Gill, T. Schank, and R. Upward. High wage workers match with high wage firms: Clear evidence of the effects of limited mobility bias. *Economics Letters*, 117 (3):824–827, 2012.

T. Antal, P. L. Krapivsky, and S. Redner. Social balance on networks: The dynamics of friendship and enmity. *Physica D: Nonlinear Phenomena*, 224(1-2):130–136, 2006.

L. Baumann. A model of weighted network formation. *Theoretical Economics*, 16(1):1–23, 2021.

G. S. Becker. A theory of marriage: Part i. *Journal of Political economy*, 81(4):813–846, 1973.

P. Belleflamme and F. Bloch. Market sharing agreements and collusive networks. *International Economic Review*, 45(2):387–411, 2004.

I. Bozbay and A. Vesperoni. A contest success function for networks. *Journal of Economic Behavior & Organization*, 150:404–422, 2018.

D. J. Brown. The structuring of polopa feasting and warfare. *Man*, pages 712–733, 1979.

A. Calvó-Armengol and R. İlkılıç. Pairwise-stability and nash equilibria in network formation. *International Journal of Game Theory*, 38(1):51–79, 2009.

D. Cartwright and F. Harary. Structural balance: a generalization of heider's theory. *Psychological review*, 63(5):277, 1956.

A. Çelikaksoy, H. S. Nielsen, and M. Verner. Marriage migration: just another case of positive assortative matching? *Review of Economics of the Household*, 4(3):253–275, 2006.

P.-A. Chiappori, S. Oreffice, and C. Quintana-Domeque. Fatter attraction: anthropometric and socioeconomic matching on the marriage market. *Journal of Political Economy*, 120 (4):659–695, 2012.

P. Cisneros-Velarde and F. Bullo. Signed network formation games and clustering balance. *Dynamic Games and Applications*, pages 1–15, 2019.

J. A. Davis. Clustering and structural balance in graphs. *Human relations*, 20(2):181–187, 1967.

M. Ghatak. Group lending, local information and peer selection. *Journal of development Economics*, 60(1):27–50, 1999.

M. Ghatak. Screening by the company you keep: Joint liability lending and the peer selection effect. *The Economic Journal*, 110(465):601–631, 2000.

R. P. Gilles and S. Sarangi. Stable networks and convex payoffs. 2005.

S. Goyal and S. Joshi. Networks of collaboration in oligopoly. *Games and Economic behavior*, 43(1):57–85, 2003.

S. Goyal and S. Joshi. Unequal connections. *International Journal of Game Theory*, 34(3): 319–349, 2006.

S. Goyal, H. Heidari, and M. Kearns. Competitive contagion in networks. *Games and Economic Behavior*, 2014.

G. Grandjean, D. Tellone, and W. Vergote. Endogenous network formation in a tullock contest. *Mathematical Social Sciences*, 85:1–10, 2017.

J. Greenwood, N. Guner, G. Kocharkov, and C. Santos. Marry your like: Assortative mating and income inequality. *American Economic Review*, 104(5):348–53, 2014.

A. Griffith. Random assignment with non-random peers: A structural approach to counterfactual treatment assessment. *Unpublished working paper*, 2017.

A. Griffith. A continuous model of strong and weak ties. *Unpublished working paper*, 2019a.

A. Griffith. Name your friends, but only five? the importance of censoring in peer effects estimates using social network data. Technical report, Working paper, 2019b.

F. Harary et al. On the notion of balance of a signed graph. *The Michigan Mathematical Journal*, 2(2):143–146, 1953.

F. Harary et al. On local balance and n-balance in signed graphs. *The Michigan Mathematical Journal*, 3(1):37–41, 1955.

K. M. Harris, C. T. Halpern, E. Whitsel, J. Hussey, J. Tabor, P. Entzel, and J. R. Udry. The national longitudinal study of adolescent to adult health: Research design. 2009.

F. Heider. Attitudes and cognitive organization. *The Journal of psychology*, 21(1):107–112, 1946.

T. Hiller. Friends and enemies: A model of signed network formation. *Theoretical Economics*, 12(3):1057–1087, 2017.

J. Hirshleifer. Conflict and rent-seeking success functions: Ratio vs. difference models of relative success. *Public choice*, 63(2):101–112, 1989.

K. Huremovic. A noncooperative model of contest network formation. *arXiv preprint arXiv:1901.07605*, 2019.

M. O. Jackson and S. Nei. Networks of military alliances, wars, and international trade. *Proceedings of the National Academy of Sciences*, 112(50):15277–15284, 2015.

M. O. Jackson and A. Wolinsky. A strategic model of social and economic networks. *Journal of economic theory*, 71(1):44–74, 1996.

T. R. Johnson. Matching through position auctions. *Journal of Economic Theory*, 148(4): 1700–1713, 2013.

M. D. König, D. Rohner, M. Thoenig, and F. Zilibotti. Networks in conflict: Theory and evidence from the great war of africa. *Econometrica*, 85(4):1093–1132, 2017.

M. Kremer. The o-ring theory of economic development. *The Quarterly Journal of Economics*, 108(3):551–575, 1993.

E. O. Laumann and F. U. Pappi. New directions in the study of community elites. *American Sociological Review*, pages 212–230, 1973.

P. Legros and A. F. Newman. Beauty is a beast, frog is a prince: Assortative matching with nontransferabilities. *Econometrica*, 75(4):1073–1102, 2007.

S. Li, H. Sun, and P. Chen. Assortative matching of risk-averse agents with endogenous risk. *Journal of Economics*, 109(1):27–40, 2013.

R. D. Luce. Connectivity and generalized cliques in sociometric group structure. *Psychometrika*, 15(2):169–190, 1950.

R. Mendes, G. J. Van Den Berg, and M. Lindeboom. An empirical assessment of assortative matching in the labor market. *Labour Economics*, 17(6):919–929, 2010.

S. G. Pandey. A model of signed network formation with heterogeneous players. *Research in Economics*, 75(1):119–128, 2021.

A. D. Roy. Some thoughts on the distribution of earnings. *Oxford economic papers*, 3(2): 135–146, 1951.

H. Salonen. Equilibria and centrality in link formation games. *International Journal of Game Theory*, 45(4):1133–1151, 2016.

R. Shimer and L. Smith. Assortative matching and search. *Econometrica*, 68(2):343–369, 2000.

A. Siow. Testing becker's theory of positive assortative matching. *Journal of Labor Economics*, 33(2):409–441, 2015.

G. Tullock. The welfare costs of tariffs, monopolies, and theft. *Economic Inquiry*, 5(3): 224–232, 1967.

G. Tullock. Efficient rent seeking. jm buchanan, rd tollison, g. tullock, eds., toward a theory of the rent seeking society. *A & M University Press, College Station, TX: Texas*, 1980.

E. Van Tassel. Group lending under asymmetric information. *Journal of Development Economics*, 60(1):3–25, 1999.

Appendices

Appendix A

Appendix for Chapter 2

A.1 Proof

The properties of $h(y_i, y_j, \phi)$

The normalized contest success function $h(y_i, y_j, \phi)$ satisfies the properties of $f(y_i, y_j)$ as follows.

1. $\frac{df}{dy_i} > 0$: f increases in y_i.

$\frac{dh}{dy_i} = \frac{\phi y_i^{\phi-1} y_j^{\phi}}{(y_i^{\phi}+y_j^{\phi})^2} > 0$

2. $\frac{df}{dy_j} < 0$: f decreases in y_j.

$\frac{dh}{dy_j} = -\frac{\phi y_i^{\phi} y_j^{\phi-1}}{(y_i^{\phi}+y_j^{\phi})^2} < 0$

3. f is concave on the range of y_i.

$\frac{d^2 h(y_i,y_j)}{dy_i^2} = -\frac{\phi y_i^{\phi-2} y_j^{\phi}\left((\phi+1)y_i^{\phi}-(\phi-1)y_j^{\phi}\right)}{\left(y_i^{\phi}+y_j^{\phi}\right)^3} < 0$ when $y_i > y_j$.

4. f is homogeneous of degree 0.

$$h(\alpha y_i, \alpha y_j, \phi) = \frac{(\alpha y_i)^{\phi}}{(\alpha y_i)^{\phi} + (\alpha y_j)^{\phi}} - \tfrac{1}{2} = \frac{y_i^{\phi}}{y_i^{\phi} + y_j^{\phi}} - \tfrac{1}{2} = h(y_i, y_j, \phi)$$

In addition, in $h(y_i, y_j, \phi)$, ϕ represents the efficiency in technology of extraction. the larger ϕ is, the bigger the size of extraction is for the same y_i and y_j for $y_i > y_j$.

$\frac{dh}{d\phi} = \frac{y_i^{\phi} y_j^{\phi}(\ln(y_i)-\ln(y_j))}{\left(y_i^{\phi}+y_j^{\phi}\right)^2} > 0$ when $y_i > y_j$.

Remark 2.11

This proof includes the same part to Hiller (2017)'s Proposition 1 (Step 1 and Step 2-1). However, Step 2-2 is different from his proof.

Step 1) If $y_i(\mathbf{g}) = y_j(\mathbf{g})$, then $\bar{\mathrm{g}}_{i,j} = 1$.

Suppose $\bar{\mathrm{g}}_{i,j} = -1$ where $y_i(\mathbf{g}) = y_j(\mathbf{g})$ in a Nash equilibrium $\mathbf{g}$. To $\bar{\mathrm{g}}_{i,j} = -1$, by definition, at least one of $\mathrm{g}_{i,j}$ or $\mathrm{g}_{j,i}$ should be -1. If both $\mathrm{g}_{i,j}$ and $\mathrm{g}_{j,i}$ are negative, then it violates Remark 2.10-1. Therefore, let's consider the case where $\mathrm{g}_{i,j} = -1$ and $\mathrm{g}_{j,i} = 1$ without loss of generality. We will show that i has an incentive to change $\mathrm{g}_{i,j} = -1 \rightarrow 1$ to increase her utility. Before the deviation,

$$u_i(\mathbf{g}) = \sum_{k \in N_i^-(\mathbf{g})} f(y_i(\mathbf{g}), y_k(\mathbf{g})) - |N_i^{e-}(\mathbf{g})|\varepsilon - |N_i^-(\mathbf{g})|\kappa.$$

After the deviation,

$$u_i(\mathbf{g} + \mathrm{g}_{i,j}^+) = \sum_{k \in N_i^-(\mathbf{g}) \setminus \{j\}} f(y_i(\mathbf{g}) + \lambda_j, y_k(\mathbf{g})) - |N_i^{e-}(\mathbf{g}) - 1|\varepsilon - |N_i^-(\mathbf{g}) - 1|\kappa.$$

$$u_i(\mathbf{g}+\mathrm{g}_{i,j}^+) - u_i(\mathbf{g}) = \sum_{k \in N_i^-(\mathbf{g}) \setminus \{j\}} \big(f(y_i(\mathbf{g})+\lambda_j, y_k(\mathbf{g})) - f(y_i(\mathbf{g}), y_k(\mathbf{g}))\big) - (f(y_i(\mathbf{g}), y_j(\mathbf{g})) - \varepsilon - \kappa) > 0,$$

because $f(y_i(\mathbf{g}), y_j(\mathbf{g})) = 0$. Therefore, when $y_i(\mathbf{g}) = y_j(\mathbf{g})$ in a Nash equilibrium $\mathbf{g}$, then $\bar{\mathrm{g}}_{i,j}$ is always 1.

Step 2) If $y_i(\mathbf{g}) \neq y_j(\mathbf{g})$, then $\bar{\mathrm{g}}_{i,j} = -1$.

In the following Step 2-1 and 2-2, we will show that every player in $P_m(\mathbf{g})$ has the negative relationships $\bar{g}_{i,j}$ with every other player j out of $P_m(\mathbf{g})$. And then, it is possible to apply the same logic to the players in $P_{m-1}(\mathbf{g}), P_{m-1}(\mathbf{g}), ...,$ and $P_2(\mathbf{g})$, respectively.

Step 2-1) In any Nash equilibrium, $N_i^+(\mathbf{g}) \setminus \{j\} = N_j^+(\mathbf{g}) \setminus \{i\}$ and $N_i^-(\mathbf{g}) = N_j^-(\mathbf{g})\ \forall\ i, j \in P^m(\mathbf{g})$. (If two players are in the same group $P_m(\mathbf{g})$, then they have both the same friend set and the enemy set.)

When $|P_m(\mathbf{g})| = 1$, it is trivially holds. When $|P_m(\mathbf{g})| \geq 2$, suppose $\exists i, j \in P_m(\mathbf{g})$ such that $N_i^+(\mathbf{g}) \setminus \{j\} \neq N_j^+(\mathbf{g}) \setminus \{i\}$. Then automatically $N_i^-(\mathbf{g}) \neq N_j^-(\mathbf{g})$.

First, $\bar{g}_{i,j} = \bar{g}_{i,k} = \bar{g}_{j,k} = 1$ for $\forall\ k \in P_m(\mathbf{g})$ by Step 1. Secondly, $g_{k,i} = g_{k,j} = 1\ \forall\ k \notin P_m(\mathbf{g})$ by Remark 2.10-(2). Thus, $\forall\ k \notin N \setminus \{i, j\}$, $g_{k,i} = g_{k,j} = 1$. Therefore, $g_{i,k} = \bar{g}_{i,k}$ and $g_{j,k} = \bar{g}_{j,k}$. It means that each i or j can choose the undirected links as she wants.

Now, without loss of generality, suppose $u_i(\mathbf{g}) \leq u_j(\mathbf{g})$. Then i can choose her utility by choosing $\mathbf{g}'_i$ such that $\mathbf{g}'_i \setminus \{j\} = \mathbf{g}'_j \setminus \{i\}$ and $g_{i,j} = 1$.

$f(y_i(\mathbf{g}'), y_l(\mathbf{g}')) > f(y_j(\mathbf{g}), y_l(\mathbf{g}))$ for $l \in N_j^-(\mathbf{g}) \cap (N_i^-(\mathbf{g}))^C$, because $y_i(\mathbf{g}') = y_i(\mathbf{g}) = y_j(\mathbf{g})$ and $y_l(\mathbf{g}') = y_l(\mathbf{g}) - \lambda_i$. Also, $f(y_i(\mathbf{g}'), y_l(\mathbf{g}')) = f(y_j(\mathbf{g}), y_l(\mathbf{g}))$ for $l \in N_j^-(\mathbf{g}) \cap N_i^-(\mathbf{g})$, because $y_l(\mathbf{g}') = y_l(\mathbf{g})$. Thus,

$$u_i(\mathbf{g}') = \sum_{l \in N_j^-(\mathbf{g})} f(y_j(\mathbf{g}), y_l(\mathbf{g}')) - |N_j^{e-}(\mathbf{g})|\varepsilon - |N_j^-(\mathbf{g})|\kappa$$

$$> u_j(\mathbf{g}) = \sum_{l \in N_j^-(\mathbf{g})} f(y_j(\mathbf{g}), y_l(\mathbf{g})) - |N_j^{e-}(\mathbf{g})|\varepsilon - |N_j^-(\mathbf{g})|\kappa \geq u_i(\mathbf{g}).$$

Therefore, if their friend sets are not identical, it is not a Nash equilibrium.

Step 2-2) Suppose there exists $g_{i,j} = 1$ where $i \in P_m(\mathbf{g})$, but $j \notin P_m(\mathbf{g})$. In particular, let's pick j such that $y_j(\mathbf{g}) < y_i(\mathbf{g})$ but $y_j(\mathbf{g}) \geq y_k(\mathbf{g})$ for all $k \in N_i^+(\mathbf{g}) \cap \big(P_m(\mathbf{g})\big)^c$. It means that

j is one of the strongest players among i's friends weaker than i. Then by Step 2-1, $g_{\hat{i},j} = 1$ for all $\hat{i} \in P_m(\mathbf{g})$. Also, $N_i^+(\mathbf{g}) = N_{\hat{i}}^+(\mathbf{g}) \neq N_j^+(\mathbf{g}) =$ because $y_i(\mathbf{g}) \neq y_j(\mathbf{g})$. Hence, there exists $k' \in N_i^+(\mathbf{g}) \cap N_j^-(\mathbf{g})$ (and there may exist $l \in N_i^-(\mathbf{g}) \cap N_j^+(\mathbf{g})$). Since $y_j(\mathbf{g}) \geq y_k(\mathbf{g})$ for all $k \in N_i^+(\mathbf{g}) \cap \left(P_m(\mathbf{g})\right)^c$, $g_{k,j} = 1$ by Step 1 and Remark 2.10-2. For $i \in P_m(\mathbf{g})$, $g_{l,i} = 1$ for the same reason. Therefore, similar to Step 2-1, either i or j can imitate the strategy of each other to increase her utility and it is always profitable. Therefore, if $g_{i,j} = 1$ where $i \in P_m(\mathbf{g})$ and $j \notin P_m(\mathbf{g})$, it is not a Nash equilibrium.

Remark 2.12

(a) Suppose $\varepsilon + \kappa > f(n-1, 1)$. Let a player i have a negative link, i.e., $g_{ij} = -1$. Now the benefit from having the negative link is $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$. For $\mathbf{g}$ with n homogeneous players, $f(n-1, 1)$ is the maximum extraction value. The cost of this negative link is given by $\varepsilon + \kappa$.

Let player i deviate from the negative link to the positive link, i.e., $g_{i,j} = 1$. This deviation is profitable for player i. First, directly, $\varepsilon + \kappa > f(n-1, 1) \geq f(y_i(\mathbf{g}), y_j(\mathbf{g}))$ $\forall y_i, y_j$. Also, indirectly, this deviation increases the extractions generated on the other negative links. The positive link increases player i's network strength $y_i(\mathbf{g})$ to $y_i(\mathbf{g}') = y_i(\mathbf{g}) + 1$. Then, on another negative link with player l, the amount of extraction increases because $f(y_i(\mathbf{g}) + 1, y_l(\mathbf{g})) > f(y_i(\mathbf{g}), y_l(\mathbf{g}))$. Therefore, the deviation from $\mathbf{g}_{i,j} = -1$ to $\mathbf{g}_{i,j} = 1$ is always profitable given the condition. Thus there is no conflict when $\varepsilon + \kappa > f(n-1, 1)$.

(b) Suppose $\varepsilon + \kappa \leq f(n-1, 1)$. Let a player i's strategy $\mathbf{g}_i$ be given by $\mathbf{g}_i = \{\mathbf{g}_{i,1}, \mathbf{g}_{i,2}, ..., \mathbf{g}_{i,i-1}, \mathbf{g}_{i,i+1}, ..., \mathbf{g}_{i,n}\} = \{1, ..., 1\}$. Player i's utility $u_i(\mathbf{g})$ is 0.

Let player i deviate to $\mathbf{g}_i + \sum_{j \in J} g_{i,j}^-$ where $J \in N \setminus \{i\}$. Take $|J| = 1$, then $y_i = n - 1$ and $y_j = n - 1$ where $j \in J \subset N$. In this case, $f(y_i, y_j) = 0$. But $\varepsilon + \kappa > 0$. Hence, $u_i(\mathbf{g} + g_{i,j}^-) = -\varepsilon - \kappa < 0$. Thus, it is not profitable to deviate to the negative link.

Let's take $|J| \geq 2$, then $y_i \leq n-2$ and $y_j = n-1$, so $f(y_i, y_j) < 0$ and $\varepsilon + \kappa > 0$. Then, $u_i(\mathbf{g} + \sum_{j \in J} \mathrm{g}^-_{i,j}) = \sum_{j \in J} \big(f(y_i, y_j) - \varepsilon - \kappa\big) < 0$. Thus, it is not profitable to deviate by extending the new negative links. Hence Utopia network is a Nash equilibrium.

Lemma A.1. *In any network formation strategy* $\mathbf{g}$*, which has a clustering,* $u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \sum_{k \in K} \mathrm{g}^-_{i,k})$ *where* $K \subset N^+_i(\mathbf{g})$ *and* $\forall\ \lambda_k = \lambda_i$*.*

Proof. This lemma means that initiating new conflicts with the players of the same type in the same clique is always unprofitable. To prove it, consider the deviation strategy $\mathbf{g} + \sum_{k \in K} \mathrm{g}^-_{i,k}$. Before the deviation, in the network configuration $\mathbf{g}$, a player i has $y_i(\mathbf{g})$, and a player k in the set K also has $y_i(\mathbf{g})$. After the deviation, $y_i(\mathbf{g} + \sum_{k \in K} \mathrm{g}^-_{i,k}) = y_i - |K|\lambda_i$. However, $y_k(\mathbf{g} + \sum_{k \in K} \mathrm{g}^-_{i,k}) = y_i - \lambda_i$. $y_i - |K|\lambda_i$ is the same to or less than $y_i - \lambda_i$. Hence, $f(y_i - |K|\lambda_i, y_i - \lambda_i) \leq 0$. □

Lemma A.2. *Suppose a network configuration with a strategy* $\mathbf{g}$ *consists of two cliques*[1]*, and the players in* C_1 *have the same* λ_1*, and those in* C_2 *have* λ_2*. If* $u_i(\mathbf{g}) < u_i(\mathbf{g} + \sum_{j \in J} \mathrm{g}^+_{i,j})$ *for some set* J *such that* $J \subset N^-_i(\mathbf{g})$ *and* $|J| \geq 2$*, then* $u_i(\mathbf{g}) < u_i(\mathbf{g} + \mathrm{g}^+_{i,j})$ *where* $j \in N^-_i(\mathbf{g})$*.*

Proof. Let C_1 and C_2 denote each clique. The players in C_1 have an identical intrinsic power λ_1, and those in C_2 have λ_2. Then player's network strength is $|C_1|\lambda_1$ in C_1, and $|C_2|\lambda_2$ in C_2. Suppose $|C_1|\lambda_1 > |C_2|\lambda_2$ without loss of generality. By Remark 2.10, a corresponding strategy profile $\mathbf{g}$ to this network configuration is as follows. If player i is in C_1, $\mathrm{g}_{i,j} = 1$ for $j \in C_1$, and $\mathrm{g}_{i,j} = -1$ for $j \in C_2$. If player i is in C_2, $\mathrm{g}_{i,j} = 1$ for $j \in N \setminus \{i\}$.

Note that this lemma is only about the players in C_1. For player i in C_2, $u_i(\mathbf{g}) = u_i(\mathbf{g} + \sum_{j \in J} \mathrm{g}^+_{i,j})$, because $\mathrm{g}_{i,j} = 1$ for $j \in J$ so $\mathbf{g} + \sum_{j \in J} \mathbf{g}^+_{i,j} = \mathbf{g}$.

Let's consider a player i in $|C_1|$. Her utility after using the deviation $\mathbf{g} + \sum_{j \in J} \mathrm{g}^+_{i,j}$ is as

[1]It implies that $\mathbf{g}$ is structurally balanced.

follows.

$$u_i(\mathbf{g} + \sum_{j \in J} \mathrm{g}^{+}_{i,j}) = (|C_2| - |J|)f(|C_1|\lambda_1 + |J|\lambda_2, |C_2|\lambda_2) - (|C_2| - |J|)(\varepsilon + \kappa)$$

The first and second derivatives of $u_i(\mathbf{g} + \sum_{j \in J} \mathrm{g}^{+}_{i,j})$ with respect to $|J|$ are as follows.

$$\frac{\partial u_i}{\partial |J|} = -f(|C_1|\lambda_1 + |J|\lambda_2, |C_2|\lambda_2) + \lambda_2(|C_2| - |J|)f'(|C_1|\lambda_1 + |J|\lambda_2, |C_2|\lambda_2) + \varepsilon + \kappa$$

$$\frac{\partial^2 u_i}{\partial |J|^2} = -2\lambda_2 f'(|C_1|\lambda_1 + |J|\lambda_2, |C_2|\lambda_2) + \lambda_2^2(|C_2| + |J|)f''(|C_1|\lambda_1 + |J|\lambda_2, |C_2|\lambda_2)$$

The second derivative is less than zero when $f''(|C_1|\lambda_1 + |J|\lambda_2, |C_2|\lambda_2) < 0$. By assumption, $\frac{\partial^2 f}{\partial y_i^2} < 0$ so the second derivative $\frac{\partial^2 u_i}{\partial |J|^2}$ is always less than zero. Since the second derivative is less than zero, it is impossible that $u_i(\mathbf{g} + \mathrm{g}^{+}_{i,j}) < u_i(\mathbf{g}) < u_i(\mathbf{g} + \sum_{j \in J} \mathrm{g}^{+}_{i,j})$ for $|J| \geq 2$. Therefore, if $u_i(\mathbf{g} + \mathrm{g}^{+}_{i,j}) < u_i(\mathbf{g})$, then $u_i(\mathbf{g} + \sum_{j \in J} \mathrm{g}^{+}_{i,j}) < u_i(\mathbf{g})$ for $|J| \geq 2$.

□

Lemma A.2 implies that it is enough check only the case of $\mathbf{g} + \mathrm{g}^{+}_{i,j}$ instead of checking every case of $|J| \geq 2$ to check an incentive for player i in C_1 to deviate from the strategy $\mathbf{g}$.

Lemma A.3. *Suppose there is a network configuration consisting of two cliques C_1 and C_2. The players in C_1 have the same λ_1, and those in C_2 have λ_2, such that $\lambda_1 \geq \lambda_2$. If $(|C_1| - 1)\lambda_1 \geq |C_2|\lambda_2$, For any $\mathbf{g}'' = \mathbf{g} + \sum_{j \in J_2} \mathrm{g}^{+}_{i,j} + \sum_{k \in K_2} \mathrm{g}^{-}_{i,k}$, there exists $\mathbf{g}' = \mathbf{g} + \sum_{j \in J_1} \mathrm{g}^{+}_{i,j}$ or $\mathbf{g}' = \mathbf{g} + \sum_{k \in K_1} \mathrm{g}^{-}_{i,k}$, such that $u_i(\mathbf{g}') \geq u_i(\mathbf{g}'')$, where $J_1 \& J_2 \subset C_2 = N_i^{-}(\mathbf{g})$ and $K_1 \& K_2 \subset C_1 \setminus \{i\} = N_i^{+}(\mathbf{g})$.*

Proof. The corresponding strategy to this network configuration is as mentioned in the proof of Lemma A.2. Player i's network strength is given by $y_i(\mathbf{g}'') = (|C_1| - |K_2|)\lambda_1 + |J_2|\lambda_2$.

Now, let's consider another network:

$$\tilde{\mathbf{g}} = \mathbf{g}'' + \mathrm{g}_{i,j}^{-} + \mathrm{g}_{i,k}^{+}$$

where $j \in J_2$ and $k \in K_2$. Then

$$\begin{aligned} y_i(\tilde{\mathbf{g}}) &= (|C_1| - |K_2| + 1)\lambda_1 + (|J_2| - 1)\lambda_2 \\ &= (|C_1| - |K_2|)\lambda_1 + |J_2|\lambda_2 + \lambda_1 - \lambda_2 \\ &\geq y_i(\mathbf{g}'') \end{aligned}$$

In $\mathbf{g}''$, for $j \in J_2 \cap N_i^-(\tilde{\mathbf{g}})$ and $k \in K_2 \cap N_i^-(\tilde{\mathbf{g}})$ player j's network strength is smaller than player k's network strength as $y_j(\mathbf{g}'') = y_j(\tilde{\mathbf{g}}) = |C_2|\lambda_2 \leq y_k(\mathbf{g}'') = y_k(\tilde{\mathbf{g}}) = (|C_1| - 1)\lambda_1$. The number of the negative links extended by player i is the same in both $\mathbf{g}''$ and $\tilde{\mathbf{g}}$. Therefore,

$$\begin{aligned} u_i(\tilde{\mathbf{g}}) =& (|C_2| - |J_2|) f\Big((|C_1| - |K_2| + 1)\lambda_1 + (|J_2| - 1)\lambda_2, |C_2|\lambda_2\Big) \\ &+ |K_2| f\Big((|C_1| - |K_2| + 1)\lambda_1 + (|J_2| - 1)\lambda_2, (|C_1| - 1)\lambda_1\Big) \\ &+ f\Big((|C_1| - |K_2| + 1)\lambda_1 + (|J_2| - 1)\lambda_2, |C_2|\lambda_2\Big) \\ &- f\Big((|C_1| - |K_2| + 1)\lambda_1 + (|J_2| - 1)\lambda_2, (|C_1| - 1)\lambda_1\Big) \\ &- \Big(|C_2| - |J_2| + |K_2|\Big)(\varepsilon + \kappa) \geq \\ u_i(\mathbf{g}'') =& \Big(|C_2| - |J_2|\Big) f\Big((|C_1| - |K_2|)\lambda_1 + |J_2|\lambda_2, |C_2|\lambda_2\Big) \\ &+ |K_2| f\Big((|C_1| - |K_2|)\lambda_1 + |J_2|\lambda_2, (|C_1| - 1)\lambda_1\Big) \\ &- \Big(|C_2| - |J_2| + |K_2|\Big)(\varepsilon + \kappa) \end{aligned}$$

Therefore, $\tilde{\mathbf{g}}$ gives a utility higher than or equal to $\mathbf{g}''$. $u_i(\tilde{\mathbf{g}}) = u_i(\mathbf{g}'')$ can happen when every player is homogeneous. It is possible to iterate this procedure until $\tilde{\mathbf{g}} = \mathbf{g} + \sum_{j \in J_1} \mathrm{g}_{i,j}^{+}$ or $\tilde{\mathbf{g}} = \mathbf{g} + \sum_{k \in K_1} \mathrm{g}_{i,k}^{-}$. □

Lemma A.3 implies that it is enough to calculate whether $u_i(\mathbf{g}) \geq u_i(\tilde{\mathbf{g}})$ or not, to check whether player i has an incentive to deviate from $\mathbf{g}$.

Lemma 2.14

We will check whether each deviation gives higher utility than the suggested strategy profile for each player. As a result of this procedure, it is possible to find a condition which makes any deviations unprofitable.

Once again, in the network $\mathbf{g}$ in equilibrium, there are n_1 players in C_1 who extend negative links to n_2 players in C_2 where $n_1 > n_2 = n - n_1$. Except for these negative links, there are only positive links. The full strategy profile of $\mathbf{g}$ is as suggested in the proof of Lemma A.2.

First, there are three kinds of deviation for any player i in C_1. (i) Player i can extend positive links to players in C_2 and negative links to players in C_1 together. (ii) She can only extend negative links in C_1. Lastly, (iii) she can only extend positive links in C_2.

When the first type of deviation (i) can be profitable than $\mathbf{g}$, then the second type (ii) or the third type deviation (iii) is also profitable than $\mathbf{g}$ by Lemma A.3. In this homogeneous model with two cliques, Lemma A.3 is applicable as the assumptions are satisfied. Therefore, to check whether $\mathbf{g}$ is a Nash equilibrium or not, it is enough not to check the deviations of the first type (i). Second,by Lemma A.1, the deviations of the second type (ii) are always non-profitable. Third, regarding (iii), it is enough to check only whether $\mathbf{g} + \mathrm{g}_{i,j}^{+}$, where $j \in C_2$, gives a higher utility than $\mathbf{g}$ following Lemma A.2.

$n_2 f(n_1, n_2) - n_2(\varepsilon + \kappa) \geq (n_2 - 1)f(n_1 + 1, n_2) - (n_2 - 1)(\varepsilon + \kappa)$ is a different expression of $u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \mathrm{g}^{+}_{i,j})$ using the parameters given in the model. Therefore, if and only if this condition is satisfied, then $\mathbf{g}$ is a Nash equilibrium.

Second, the players in C_2 can only change some of their positive links to the negative links in any deviation strategy. However, by Lemma A.1, extending the negative links to other players in C_2 is not profitable. Also, extending the negative links to players in C_1 does not change the undirected network $\bar{\mathbf{g}}$, but generates the additional conflict cost (ε). Therefore, for the players in C_2, $\mathbf{g}$ describes the best response strategy.

Proposition 2.15

Let's use the condition $n_2 f(n_1, n_2) - (n_2 - 1)f(n_1 + 1, n_2) \geq \varepsilon + \kappa$ in Lemma 2.14. This condition is equivalent to

$$f(n_1, n_2) \geq (n_2 - 1)(f(n_1 + 1, n_2) - f(n_1, n_2)) + \varepsilon + \kappa.$$

Given n_2 and $\varepsilon + \kappa < f(n - 1, 1)$, if n_1 is large enough, then this equation holds.

Also, we will show that there always exists $\bar{n}_1(n_2, \varepsilon + \kappa) \leq n$. Let's consider the case where $n_1 = n - 1$. In this case, this configuration is always a Nash equilibrium[2]. A player i in C_1 does not have an incentive to deviate from this strategy because when $f(n - 1, 1) > \varepsilon + \kappa$, $\mathbf{g} + \mathrm{g}^{+}_{i,j}$ where $j \in C_2$ is unprofitable. Also, $\mathbf{g} + \sum_{j \in J} \mathbf{g}^{-}_{i,j}$ where $J \subset C_1 \setminus \{i\}$ is always not profitable by Lemma A.1. Lastly, $\mathbf{g} + \mathrm{g}^{+}_{i,j} + \sum_{j \in C_1 \setminus \{i\}} \mathbf{g}^{-}_{i,j}$ where $J \subset C_1 \setminus \{i\}$ and $k \in C_2$ is also not profitable than $\mathbf{g}$ by Lemmas A.2 and A.3.

The player in C_2 also do not have an incentive to change his strtegy by Remark 2.10.

[2]This is what Hiller (2017) proved in his Proposition 2

Theorem 2.17

(a) If and only if condition when Utopia network is a Nash equilibrium: By definition, $\mathbf{g}$ is a Nash equilibrium if and only if there is no $\mathbf{g}'_i \in G_i$ such that $u_i(\mathbf{g}'_i, \mathbf{g}_{-i}) > u_i(\mathbf{g})$ for all i. In Utopia network, a player i's strategy $\mathbf{g}_i = \{\mathbf{g}_{i,1}, \mathbf{g}_{i,2}, ..., \mathbf{g}_{i,n}\} = \{1, ..., 1\}$.

Any weak player does not have an incentive to deviate from Utopia network strategy. First, $\mathbf{g}'' = \mathbf{g} + \mathrm{g}^-_{i,j}$ where $j \in N_w$ is unprofitable. If a weak player i betrays one strong player j by extending one negative link, it is an unprofitable deviation because $y_i(\mathbf{g}') = (n_s - 1)\lambda_s + n_w\lambda_w < y_j(\mathbf{g}') = n_s\lambda_s + (n_w - 1)\lambda_w$, so $f(y_i(\mathbf{g}'), y_j(\mathbf{g}')) < 0$. If weak player i betrays one weak player j by extending one negative link, it is also unprofitable by Lemma A.1. If the weak player i betrays more than one player, then it is always unprofitable, too. When she betrays x_s strong players and x_w weak players, $y_i(\mathbf{g}') = (n_s - x_s)\lambda_s + (n_w - x_w)\lambda_w$. However, any other betrayed player j's network strength is $y_j(\mathbf{g}') = n_s\lambda_s + (n_w - 1)\lambda_w$, so $f(y_i(\mathbf{g}'), y_j(\mathbf{g}')) < 0$. Therefore, any weak players do not have an incentive to deviate from $\mathbf{g}$.

On the other hand, each strong player does not have an incentive to deviate from Utopia network strategy if and only if $u(\mathbf{g}) \geq u(\mathbf{g}'') = \mathbf{g} + \mathrm{g}^-_{i,j}$. It is possible to transform $u(\mathbf{g}) \geq u(\mathbf{g}'')$ to $\varepsilon + \kappa \geq f(n_s\lambda_s + (n_w - 1)\lambda_w, (n_s - 1)\lambda_s + n_w\lambda_w)$. First, a player i's deviation $\mathbf{g}_i + \sum_{j\in J} \mathrm{g}^-_{i,j}$ where $J \in N_s \setminus \{i\}$ is always unprofitable by Lemma A.1. Second, let's consider two other deviations: (i) $\bar{\mathbf{g}} = \mathbf{g} + \sum_{j\in J} \mathrm{g}^-_{i,j}$ for $J \subset N_s \cup N_w$ and $J \cap N_s \neq \varnothing$ and $J \cap N_w \neq \varnothing$, and (ii) $\bar{\mathbf{g}}' = \bar{\mathbf{g}} + \sum_{k\in J\cap N_s} \mathrm{g}^+_{i,k} = \bar{\mathbf{g}} + \sum_{j\in J\cap N_w} \mathrm{g}^-_{i,j}$. Note that $\bar{\mathbf{g}}'$ always gives higher utility than $\bar{\mathbf{g}}$. The reason is as follows. Once again, x_s is the number of the betrayed strong players and x_w is the number of the betrayed weak players. In $\bar{\mathbf{g}}$, for $j \in N_s$, $y_i(\bar{\mathbf{g}}) = (n_s - x_s)\lambda_s + (n_w - x_w)\lambda_w$, and $y_j(\bar{\mathbf{g}}) = (n_s - 1)\lambda_s + n_w\lambda_w$. Then, $f(y_i(\bar{\mathbf{g}}), y_j(\bar{\mathbf{g}})) < 0$. Therefore, for the strong player i, $\bar{\mathbf{g}}'$ always gives more utility than $\bar{\mathbf{g}}$ by excluding the negative value of the extraction $f(y_i(\bar{\mathbf{g}}), y_j(\bar{\mathbf{g}}))$. Lastly, we should consider the deviation $\bar{\mathbf{g}}' = \mathbf{g} + \sum_{l \in L} \mathrm{g}_{i,l}^{-}$ where $L \in N_w$. We will show that checking $u_i(\mathbf{g}) \geq u_i(\mathbf{g}')$ where $|L| = 1$ is enough. Let x_w denote $|L|$. Note that $\bar{\mathbf{g}}' = \mathbf{g} + \sum_{l \in L} \mathrm{g}_{i,l}^{-} = \bar{\mathbf{g}} + \sum_{k \in J \cap N_s}^{+} \mathrm{g}_{i,k}$. The utility from $\bar{\mathbf{g}}'$ is as follows.

$$\begin{aligned} u_i(\bar{\mathbf{g}}') &= \sum_{l \in L} \Big(f(y_i(\bar{\mathbf{g}}'), y_l(\bar{\mathbf{g}}')) - \varepsilon - \kappa \Big) \\ &= x_w \Big(f(n_s\lambda_s + (n_w - x_w)\lambda_w, (n_s - 1)\lambda_s + n_w\lambda_w) - \varepsilon - \kappa \Big) \end{aligned}$$

By using the method used in Lemma A.2,

$$\begin{aligned} \frac{\partial u(\bar{\mathbf{g}}')}{\partial x_w} &= f(n_s\lambda_s + (n_w - x_w)\lambda_w, (n_s - 1)\lambda_s + n_w\lambda_w) - \varepsilon - \kappa \\ &\quad - x_w\lambda_w f'(n_s\lambda_s + (n_w - x_w)\lambda_w, (n_s - 1)\lambda_s + n_w\lambda_w) \\ \frac{\partial^2 u(\bar{\mathbf{g}}')}{\partial x_w^2} &= -2\lambda_w f'(n_s\lambda_s + (n_w - x_w)\lambda_w, (n_s - 1)\lambda_s + n_w\lambda_w) \\ &\quad + \lambda_w^2 f''(n_s\lambda_s + (n_w - x_w)\lambda_w, (n_s - 1)\lambda_s + n_w\lambda_w). \end{aligned}$$

When $f'' < 0$, $\frac{\partial^2 u(\bar{\mathbf{g}}')}{\partial x_w^2} < 0$. Therefore, if $u_i(\mathbf{g}) \geq u_i(\bar{\mathbf{g}}')$ for $x_w = 1$, then $u_i(\mathbf{g}) \geq u_i(\bar{\mathbf{g}}')$ for $x_w \geq 2$.

(b) If and only if condition when Utopia network is the unique Nash equilibrium: In the network with heterogeneous players given the parameters n_s, n_w, λ_s and λ_w, $f(n_s\lambda_s + (n_w - 1)\lambda_w, \lambda_w)$ is the maximum of extraction. This is because $n_s\lambda_s + (n_w - 1)\lambda_w$ is the maximum network strength and λ_w is the minimum network strength given n_s, n_w, λ_s, and λ_w.

First, if $\varepsilon + \kappa > f(n_s\lambda_s + (n_w - 1)\lambda_w, \lambda_w)$, then any networks including negative links cannot be Nash equilibria. In these networks, players who are extending negative links have an incentive to change these links to the positive links. However, Utopia network is still a Nash equilibrium because $\varepsilon+\kappa > f(n_s\lambda_s+(n_w-1)\lambda_w, \lambda_w) > f(n_s\lambda_s+(n_w-1)\lambda_w, (n_s-1)\lambda_s+n_w\lambda_w)$. Thus, Utopia network is the unique Nash equilibrium.

Second, if Utopia network is the unique Nash equilibrium, then $\varepsilon + \kappa > f(n_s\lambda_s + (n_w - 1)\lambda_w, \lambda_w)$. As shown above, Utopia network is a Nash equilibrium iff $\varepsilon+\kappa \geq f(n_s\lambda_s+(n_w-1)\lambda_w, (n_s-1)\lambda_s+n_w\lambda_w)$. When $\varepsilon+\kappa < f(n_s\lambda_s+(n_w-1)\lambda_w, (n_s-1)\lambda_s+n_w\lambda_w)$, Utopia network is not a Nash equilibrium. When $\varepsilon+\kappa \leq f(n_s\lambda_s+(n_w-1)\lambda_w, \lambda_w)$, there is another Nash equilibrium network configuration where $n-1$ players are in a clique C_1 and they extend the negative links to the other player in C_2. If $\varepsilon+\kappa > f(n_s\lambda_s+(n_w-1)\lambda_w, \lambda_w)$, then any networks containing negative links cannot be a Nash equilibrium. Therefore, if the Utopia network is the unique Nash equilibrium, $\varepsilon+\kappa > f(n_s\lambda_s+(n_w-1)\lambda_w, \lambda_w)$.

Corollary 2.18

By the assumption, when $f(y_i, y_j)$ is homogeneous of degree 0, so that $f(n_s\lambda_s + (n_w - 1)\lambda_w, (n_s-1)\lambda_s + n_w\lambda_w) = f(\frac{n_s\lambda_s+(n_w-1)\lambda_w}{(n_s-1)\lambda_s+n_w\lambda_w}, 1) = f(\frac{n_s a+n_w-1}{(n_s-1)a+n_w}, 1)$, where $a = \frac{\lambda_s}{\lambda_w}$.

(i) If $\frac{n_s a+n_w-1}{(n_s-1)a+n_w}$ increases, $f(\frac{n_s a+n_w-1}{(n_s-1)a+n_w}, 1)$ increases by assumption. Let x denote $\frac{n_s a+n_w-1}{(n_s-1)a+n_w}$ here.

$$\frac{\partial x}{\partial a} = \frac{n_s+n_w-1}{(an_s-a+n_w)^2} > 0$$
$$\frac{\partial x}{\partial n_s} = -\frac{(a-1)a}{(an_s-a+n_w)^2} < 0$$
$$\frac{\partial x}{\partial n_w} = \frac{1-a}{(an_s-a+n_w)^2} < 0$$

When Utopia network is the unique equilibrium, it is trivial to determine the directions of $f^u = f(n_s\lambda_s+(n_w-1)\lambda_w, \lambda_w) = f(n_s a+(n_w-1), 1)$ regarding a, n_s, and n_w.

(ii) $n_w = n - n_s$, so $\frac{n_s a + n_w - 1}{(n_s - 1)a + n_w} = \frac{n_s a + n - n_s - 1}{(n_s - 1)a + n - n_s}$. Let x denote $\frac{n_s a + n - n_s - 1}{(n_s - 1)a + n - n_s}$. Then

$$\frac{\partial x}{\partial n_s} = -\frac{(a-1)^2}{((a-1)n_s - a + n)^2} < 0.$$

It is also trivial to prove the uniqueness condition because $\lambda_s > \lambda_w$.

Corollary 2.20 and 2.21

First, when $\phi \to \infty$, every $f(y_i, y_j) \to \frac{1}{2}$ when $y_i > y_j$. Then, $f^e = f^u \to \frac{1}{2}$, too. Thus, following Theorem 2.17, only for $\varepsilon + \kappa \geq \frac{1}{2}$, Utopia network is the unique Nash equilibrium. Second, when $\phi \to 0$, every $f(y_i, y_j) \to 0$. Thus, $f^e = f^u \to 0$, too. Hence, for any $\varepsilon + \kappa > 0$, Utopia network is the unique Nash equilibrium.

Lemma 2.23

By definition of Nash equilibrium, complete strong dominant positive assortative matching $\mathbf{g}^*$ is a Nash equilibrium if and only if $u_i(\mathbf{g}_i^*, \mathbf{g}_{-i}^*) \geq u_i(\mathbf{g}_i', \mathbf{g}_{-i}^*)$ for every deviation $\mathbf{g}_i' \in G_i$, for all $i \in N$. The strategy profile $\mathbf{g}$ for complete strong dominant positive assortative matching is as follows. (i) If player i's type is strong, $\mathrm{g}_{i,j} = 1$ for $j \in N_S \setminus \{i\}$, and $\mathrm{g}_{i,j} = -1$ for $j \in N_W$. (ii) If player i's type is weak, $\mathrm{g}_{i,j} = 1$ for all $j \in N \setminus \{i\}$.

There are four kinds of deviations from the complete strong dominant positive assortative matching. First, a strong player changes his positive links toward the other strong players to the negative links. Second, a strong player changes his negative links toward weak players to the positive links. Third, a strong player changes his positive links toward the other strong players to the negative links and changes his negative links toward weak players to the positive links. Lastly, a weak player changes his positive links toward the other players to the negative links.

First, the deviation $\mathbf{g}+\sum_{j\in J}\mathrm{g}^{-}_{i,j}$ where $i \in N_S$ and $j \subset S\setminus\{i\}$ is unprofitable by Lemma A.1. Second, the deviation $\mathbf{g}+\sum_{j\in J}\mathrm{g}^{+}_{i,j}$ where $i \in N_S$ and $J \subset N_W$ is unprofitable when $u_i(\mathbf{g}) \geq u_i(\mathbf{g}+\mathrm{g}^{+}_{i,j})$ where $j \in N_W$ by Lemma A.2. Third, the deviation $\mathbf{g}+\sum_{k\in K}\mathrm{g}^{-}_{i,k}+\sum_{j\in N_W}\mathrm{g}^{+}_{i,j}$ where $i \in N_s$ and $K \subset N_s$ is unprofitable when $(n_s-1)\lambda_s \geq n_w\lambda_w$ by Lemma A.3. If $(n_s-1)\lambda_s < n_w\lambda_w$, each deviation $\mathbf{g}+\sum_{k\in K}\mathrm{g}^{-}_{i,k}+\sum_{j\in N_W}\mathrm{g}^{+}_{i,j}$ should be compared with the suggested strategy $\mathbf{g}$ whether it is profitable or not. Lastly, each weak player does not have any incentive to deviate from the suggested strategy. When a weak player changes his positive links to strong players, it cannot change the undirected links between the weak player and the strong players, but this deviation brings about additional conflict cost ε. If the weak player changes his positive links toward other weak players, it is unprofitable by Lemma A.1.

Theorem 2.24

The condition that all Nash equilibria are SPM

Suppose a is large enough and $f(n_s, n_s-1) > \varepsilon+\kappa$. Then CSPM is a Nash equilibrium, and any other network configuration except for SPM cannot be a Nash equilibrium. We will show the proof that CSPM is a Nash equilibrium in the next part after this part that there is no other equilibrium than SPM.

The network configurations which are not SPM can be classified into three categories: (i) disassortative matching, (ii) other positive assortative matchings which are not SPMs (there exist weak players whose network strength is higher than some strong players' network strength.), and (iii) Utopia network. I do not consider the war of all against all network, where there are only negative links. It is because this network is always not a Nash equilibrium when there are more than two players.

First, if a is large enough, any disassortative matching cannot be a Nash equilibrium. To show it, suppose there are K cliques $C_1, C_2, ..., C_K$ in a disassortatvie matching $\mathbf{g}$. By definition of disassortative matching, there exists a clique C_I consisting of both strong and weak players. Suppose there are s strong players and w weak players in C_I, and players in this clique C_I has the C_Ith strongest network strength in $\mathbf{g}$. For a strong player i in this clique C_I, her utility is as follows.

$$u_i(\mathbf{g}) = \sum_{j \in \sum_{k=1}^{I-1} C_k} (f(y_i, y_j) - \kappa) + \sum_{j \in \sum_{k=I+1}^{K} C_k} (f(y_i, y_j) - (\varepsilon + \kappa))$$

where $y_i = s\lambda_s + w\lambda_w$. Note that $f(y_i, y_j) < 0$ for $j \in \sum_{k=1}^{I-1} C_k$ and $f(y_i, y_j) > 0$ for $j \in \sum_{k=I+1}^{K} C_k$. Now, let's consider a deviation strategy $\mathbf{g} + \mathrm{g}_{i,l}^{-}$ where player l is a weak player in the same clique C_I. The payoff from the deviating strategy is

$$\begin{aligned} u_i(\mathbf{g} + \mathrm{g}_{il}^{-}) = &\sum_{j \in \sum_{k=1}^{I-1} C_k} (f(y_i - \lambda_w, y_j) - \kappa) + \sum_{j \in \sum_{k=I+1}^{K} C_k} (f(y_i - \lambda_w, y_j) - (\varepsilon + \kappa)) \\ &+ f(y_i - \lambda_w, y_i - \lambda_s) - (\varepsilon + \kappa) \end{aligned}$$

To compare the sizes of $u(\mathbf{g})$ and $u_i(\mathbf{g} + \mathrm{g}_{i,l}^{-})$, let's substract $u(\mathbf{g})$ from $u_i(\mathbf{g} + \mathrm{g}_{i,l}^{-})$.

$$\begin{aligned} u_i(\mathbf{g} + \mathrm{g}_{i,l}^{-}) - u_i(\mathbf{g}) = &f(y_i - \lambda_w, y_i - \lambda_s) - (\varepsilon + \kappa) \\ &- \sum_{j \in \sum_{k=1}^{I-1} C_k} (f(y_i, y_j) - (f(y_i - \lambda_w, y_j)) \\ &- \sum_{j \in \sum_{k=I+1}^{K} C_k} (f(y_i, y_j) - f(y_i - \lambda_w, y_j)) \end{aligned}$$

Using the property of f that f is homogeneous of degree zero and $f(y_i, y_j) + f(y_j, y_i) = 0$,

$$\begin{aligned} u_i(\mathbf{g} + \mathrm{g}_{i,l}^{-}) - u_i(\mathbf{g}) =& f(y_i - \lambda_w, y_i - \lambda_s) - (\varepsilon + \kappa) \\ &+ \sum_{j \in \sum_{k=1}^{I-1} C_k} (f(\frac{y_j}{y_i}, 1) - (f(\frac{y_j}{y_i - \lambda_w}, 1)) \\ &- \sum_{j \in \sum_{k=I+1}^{K} C_k} (f(\frac{y_i}{y_j}, 1) - (f(\frac{y_i - \lambda_w}{y_j}, 1)) \end{aligned}$$

If $a = \frac{\lambda_s}{\lambda_w}$ is large enoguh, $f(y_i - \lambda_w, y_i - \lambda_s) = f(\frac{sa+w-1}{(s-1)a+w}, 1) \to f(\frac{s}{s-1}, 1)$ while $f(\frac{y_j}{y_i}, 1) - f(\frac{y_j}{y_i - \lambda_w}, 1) \to 0$ and $f(\frac{y_i}{y_j}, 1) - f(\frac{y_i - \lambda_w}{y_j}, 1) \to 0$. $f(s, s-1) \geq f(n_s, n_s - 1)$, and by the assumption, $f(n_s, n_s - 1) > \varepsilon + \kappa$. Thus, $f(\frac{s}{s-1}, 1) > \varepsilon + \kappa$, and the disassortative matching $\mathbf{g}$ is not a Nash equilibrium.

Second, if $\bar{a}^a$ is large enough, any positive assortative matching which is not SPM cannot be a Nash equilibrium. A network configuration, which is PM but not SPM, has at least one strong player i whose network strength y_i and one weak player j whose network strength y_j such that $y_i < y_j$. If $\bar{a}^a$ is large enough, y_i is always larger than y_j. So it is a contradiction.

Lastly, Utopia network is not a Nash equilibrium if a is large enough and if $f(n_s, n_s - 1) > \varepsilon + \kappa$. In Theorem 2.17, Utopia is a Nash equilibrium if $\varepsilon + \kappa \geq f^e = f(n_s\lambda_s + (n_w - 1)\lambda_w, (n_s - 1)\lambda_s + n_w\lambda_w)$. As $a \to \infty$, $f^e \to f(n_s, n_s - 1)$. Therefore, when a is large enough and $f(n_s, n_s - 1) > \varepsilon + \kappa$, Utopia network cannot be a Nash equilibrium.

If and only if condition that CSPM is a Nash equilibrium

If a is large enough, then the condition of Lemma 2.23-(i) is satisfied given $\varepsilon + \kappa < f(n-1, 1)$. Suppose a is sufficiently large. Then the condition $(n_s - 1)a \geq n_w$ is satisfied. Let us check the condition $u(\mathbf{g}) \geq u(\mathbf{g} + \mathrm{g}_{i,j}^{+})$. The condition can be transformed as follow.

$$f(n_s a, n_w) > (n_w - 1)(f(n_s a + 1, n_w) - f(n_s a, n_w)) + \varepsilon + \kappa.$$

If $a \to \infty$, $f(n_s a, n_w) \to \bar{f}$ and $f(n_s a + 1, n_w) - f(n_s a, n_w) \to 0$. Therefore, as long as $\varepsilon + \kappa < \bar{f}$, there exists a satisfying the condition in Lemma 2.23.

Corollary 2.25

Using Lemma 2.23, for complete strong dominant positive assortative matching to be a Nash equilibrium, the following two conditions should be satisfied for $0 < d < n_s$.

$$Condition(A) : \frac{n_w}{2}\frac{(n_s a)^\phi - n_w^\phi}{(n_s a)^\phi + n_w^\phi} - n_w(\varepsilon + \kappa) \geq \frac{(n_w - 1)}{2}\frac{(n_s a + 1)^\phi - n_w^\phi}{(n_s a + 1)^\phi + n_w^\phi} - (n_w - 1)(\varepsilon + \kappa)$$

$$Condition(B) : \frac{n_w}{2}\frac{(n_s a)^\phi - n_w^\phi}{(n_s a)^\phi + n_w^\phi} - n_w(\varepsilon + \kappa) \geq \frac{d}{2}\frac{((n_s - d)a + n_w)^\phi - ((n_s - 1)a)^\phi}{((n_s - d)a + n_w)^\phi + ((n_s - 1)a)^\phi} - d(\varepsilon + \kappa)$$

First, if $(n_s - 1)\lambda_s > n_w \lambda_w$, we only need Condition (A). If $(n_s - 1)a > n_w$, then $n_s\lambda_s + \lambda_w > n_s\lambda_s > n_w\lambda_w$. Next, let's consider Condition (B). If $(n_s - d)a + n_w > (n_s - 1)a$, then $\frac{1}{2}\frac{((n_s - d)a + n_w)^\phi - ((n_s - 1)a)^\phi}{((n_s - d)a + n_w)^\phi + ((n_s - 1)a)^\phi}$ converge to $\frac{1}{2}$ as ϕ goes to infinite. $d < da \leq (n_s - 1)a < n_w$, so condition (ii) is satisfied for $\varepsilon + \kappa < \frac{1}{2}$. If $(n_s - d)a + n_w \leq (n_s - 1)a$, RHS of Condition (B) is always less than zero. Hence, Condition (B) is also satisfied, too.

Thus, if ϕ goes to infinity, $\frac{1}{2}\frac{(n_s a)^\phi - n_w^\phi}{(n_s a)^\phi + n_w^\phi}$, $\frac{1}{2}\frac{(n_s a + 1)^\phi - n_w^\phi}{(n_s a + 1)^\phi + n_w^\phi}$, and $\frac{1}{2}\frac{((n_s - d)a + n_w)^\phi - ((n_s - 1)a)^\phi}{((n_s - d)a + n_w)^\phi + ((n_s - 1)a)^\phi}$ converge to $\frac{1}{2}$. $n_w > n_w - 1$ and $d < da \leq (n_s - 1)a < n_w$, so the Conditions (A) and (B) are always satisfied if $\varepsilon + \kappa < \frac{1}{2}$.

Proposition 2.28

In this proof, we only show a condition for each weak player does not have an incentive to deviate from the complete weak dominant positive assortative matching $\mathbf{g}$. It is because each strong type player decreases his/her utility when he/she extends any negative directed links.

First, we show there exists a value of a making $\mathbf{g}$ a Nash equilibrium given n_w and n_s. Suppose $\lambda_s = \lambda_w + \delta$ where $\delta > 0$ is small enough. For a weak player i, $\mathbf{g}$ and any deviation $\mathbf{g}' = \mathbf{g} + \sum_{j \in J} \mathrm{g}^+_{i,j} + \sum_{k \in K} \mathrm{g}^-_{i,k}$ where $K \subset N_w \setminus \{i\}$ and $J \subset N_s$ gives utilities are as follows.

$$
\begin{aligned}
u_i(\mathbf{g}) =& n_s f(n_w \lambda_w, n_s \lambda_s) - n_s(\varepsilon + \kappa) && \text{(A.1)} \\
u_i(\mathbf{g}') =& (n_s - |J|) f((n_w - |K|)\lambda_w + |J|\lambda_s, n_s \lambda_s) + \\
& |K| f((n_w - |K|)\lambda_w + |J|\lambda_s, (n_w - 1)\lambda_w) - (n_s - |J| + |K|)(\varepsilon + \kappa) && \text{(A.2)} \\
<& (n_s - |J| + |K|) f((n_w - |K|)\lambda_w + |J|\lambda_s, n_s \lambda_s) - (n_s - |J| + |K|)(\varepsilon + \kappa) && \text{(A.3)}
\end{aligned}
$$

(A.3) > (A.2) because $n_w - 1 > n_s$ and δ is small enough. Also, (A.1) > (A.3). Note that (A.3) is $u(\mathbf{g}'')$ where $\mathbf{g}'' = \mathbf{g} + \sum_{l \in L} \mathrm{g}^+_{i,l}$ where $|L| = |J| - |K|$ and $L \subset N_s$. By lemma A.2, if $u_i(\mathbf{g}) \geq u_i(\mathbf{g} + \mathrm{g}^+_{i,j})$ for a $j \in N_s$, $u_i(\mathbf{g}) \geq u_i(\mathbf{g}'')$. $u_i(\mathbf{g})$ is always greater than $u_i(\mathbf{g} + \mathrm{g}^+_{i,j})$ for small enough δ as follows. By the assumptions, $\frac{\partial^2 f}{\partial n_i^2} < 0$, $n_w > 2n_s - 1$, and $\varepsilon + \kappa$ is small enough. Then

$$
n_w \lambda_w > (2n_s - 1)\lambda_s
$$

$$
n_w \lambda_w - n_s \lambda_s > (n_s - 1)\lambda_s
$$

$$
f(n_w \lambda_w, n_s \lambda_s) - f(n_s \lambda_s, n_s \lambda_s) - (\varepsilon + \kappa) > (n_s - 1)(f(n_w \lambda_w + \lambda_s, n_s \lambda_s) - f(n_w \lambda_w, n_s \lambda_s)) \tag{A.4}
$$

The last line holds, because $\frac{\partial^2 f}{\partial \pi_i^2} < 0$. Rearranging the last line,

$$n_s f(n_w\lambda_w, n_s\lambda_s) - n_s(\varepsilon + \kappa) > (n_s - 1)f(n_w\lambda_w + \lambda_s, n_s\lambda_s) - (n_s - 1)(\varepsilon + \kappa)$$

Therefore, if a is close to 1 enough and $n_w > 2n_s - 1$, complete weak dominant positive assortative matching is a Nash equilibrium.

Corollary 2.29

First, let us consider the case when $\varepsilon + \kappa < \frac{1}{2}$. There is a weak player i's deviation $\mathbf{g}'_i = \mathbf{g}_i + \sum_{j\in N_s} \mathrm{g}^+_{i,j} + \sum_{k\in K} \mathrm{g}^-_{i,k}$ where $K \subset N_w \setminus \{i\}$ and $|K| = n_s + 1$, which is more profitable for i when $\phi \to \infty$. After the deviation, $y_i(\mathbf{g}') = (n_w - n_s - 1)\lambda_w + n_s\lambda_s$ and $y_k(\mathbf{g}') = n_w - 1$. Thus $y_i(\mathbf{g}') > y_k(\mathbf{g}')$. Player i's utility from $\mathbf{g}'$ is approximately

$$u(\mathbf{g}') \approx (n_s + 1)(\frac{1}{2} - (\varepsilon + \kappa)),$$

and the utility from CWPM is approximately

$$u(\mathbf{g}) \approx n_s(\frac{1}{2} - (\varepsilon + \kappa)).$$

Therefore, the deviation is always more profitable than CWPM.

If $\varepsilon + \kappa \geq \frac{1}{2}$, any extraction is not profitable because this cost is higher than the upper bound of the possible extraction. Therefore, CWPM cannot be a Nash equilibrium.

Proposition 2.32

There is one clique with n-1 players in the networks and one left player bullied by the clique members. There are three kinds of deviations in large.

i) A player in the clique extends a positive link to the bullied player.

ii) A player in the clique extends the negative link to other players in the clique.

iii) A player in the clique extends the positive link to the bullied player and extends the negative link to other players in the clique.

Any deviations by the bullied player are not profitable for her because she cannot change the undirected network $\bar{\mathbf{g}}$ by her deviation, but it incurs additional conflict costs.

1) The case of Bullying a strong network

When the number of strong players is more than two, then there always exists at least one strong player in the clique. Because it is not the homogeneous model, there exists at least one weak player in the clique. The utility of i in the clique is $u_i(\mathbf{g}) = f((n_s-1)a+n_w, a)-\varepsilon-\kappa$.

1-i) The first deviation is unprofitable if $f(n_s-1, 1) \geq \varepsilon+\kappa$.

Regardless of type of i, $u_i(\mathbf{g}') = 0$. $f((n_s-1)a+n_w, a) > f(n_s-1, 1)$. Therefore, if $f((n_s-1)a+n_w, n_s) > \varepsilon+\kappa$, then $u_i(\mathbf{g}) \geq u_i(\mathbf{g}') = 0$.

1-ii) The second deviation is unprofitable if $\varepsilon+\kappa \geq f(n_s-1, n_s-2)$. In the second and third deviation, let's b_s denote the number of betrayed strong players by i, and b_w denote the number of betrayed weak players, in the deviation strategy. That is $b_s = |B_s|$ such that $B_s = N_i^+(\mathbf{g}) \cap N_i^-(\mathbf{g}') \cap N_s$, and $b_w = |B_w|$ such that $B_w = N_i^+(\mathbf{g}) \cap N_i^-(\mathbf{g}') \cap N_w$. Suppose

i is type s. Then after the deviation,

$$
\begin{aligned}
u_i(\mathbf{g}') =& f((n_s-1-b_s)a+n_w-b_w,a)+(b_s+b_w)f((n_s-1-b_s)a+n_w-b_w,(n_s-2)a+n_w) \\
&-(b_s+b_w+1)(\varepsilon+\kappa).
\end{aligned}
$$

If $b_s > 1$, then $f((n_s-1-b_s)a+n_w-b_w,(n_s-2)a+n_w) \leq 0$ for all b_w. So $u_i(\mathbf{g}') < u_i(\mathbf{g})$. If $b_s = 0$, then $\mathbf{g}'$ is unprofitable when

$$
f((n_s-1)a+n_w,a)-\varepsilon-\kappa \geq f((n_s-1)a+n_w-b_w,a)+b_w f((n_s-1)a+n_w-b_w,(n_s-2)a+n_w)-(b_w+1)(\varepsilon+\kappa).
$$

It can be arranged as

$$
f((n_s-1)a+n_w,a)-f((n_s-1)a+n_w-b_w,a)+b_w(\varepsilon+\kappa) \geq b_w f((n_s-1)a+n_w-b_w,(n_s-2)a+n_w),
$$

$$
\varepsilon+\kappa \geq f((n_s-1)a+n_w-b_w,(n_s-2)a+n_w)-\frac{1}{b_w}\big(f((n_s-1)a+n_w,a)-f((n_s-1)a+n_w-b_w,a)\big).
$$

The RHS is smaller than $f(n_s-1,n_s-2)$. So if $\varepsilon+\kappa \geq f(n_s-1,n_s-2)$, then $\mathbf{g}'$ is not profitable for i.

If i is type w, then

$$
\begin{aligned}
u_i(\mathbf{g}') =& f((n_s-1-b_s)a+n_w-b_w,a)+(b_s+b_w)f((n_s-1-b_s)a+n_w-b_w,(n_s-1)a+n_w-1) \\
&-(b_s+b_w+1)(\varepsilon+\kappa).
\end{aligned}
$$

For all $b_s \geq 1$ or $b_w \geq 1$, $u_i(\mathbf{g}) \geq u_i(\mathbf{g}')$.

1-iii) As a result of the third deviation, i has the enemy set $N_i^-(\mathbf{g}')$ consisting of the players in the clique. The third deviation is unprofitable if $\varepsilon+\kappa \geq f(n_s,n_s-2)$. The third deviation gives

$$u_i(\mathbf{g}') = (b_s + b_w)f((n_s - b_s)a + n_w - b_w, (n_s - 2)a + n_w) - (b_s + b_w)(\varepsilon + \kappa).$$

$u_i(\mathbf{g}') \geq u_i(\mathbf{g})$ when

$$f((n_s-1)a+n_w, a)-\varepsilon-\kappa \geq (b_s+b_w)f((n_s-b_s)a+n_w-b_w, (n_s-2)a+n_w)-(b_s+b_w)(\varepsilon+\kappa).$$

Then

$$(b_s + b_w - 1)(\varepsilon + \kappa) \geq (b_s + b_w)f((n_s - b_s)a + n_w - b_w, (n_s - 2)a + n_w) - f((n_s - 1)a + n_w, a),$$

$$\varepsilon + \kappa \geq \frac{b_s + b_w}{b_s + b_w - 1} f((n_s - b_s)a + n_w - b_w, (n_s - 2)a + n_w) - \frac{1}{b_s + b_w - 1} f((n_s - 1)a + n_w, a).$$

If $b_s \geq 2$, then it always holds because $(n_s - b_s)a + n_w - b_w \leq (n_s - 2)a + n_w$.

If $b_s = 1$,

$$\begin{aligned}
\frac{1 + b_w}{b_w} f((n_s - 1)a + n_w - b_w, (n_s - 2)a + n_w) - \frac{1}{b_w} f((n_s - 1)a + n_w, a) \\
\leq & f((n_s - 1)a + n_w - b_w, (n_s - 2)a + n_w) \\
\leq & f((n_s - 1)a + n_w, (n_s - 2)a + n_w) \\
\leq & f(n_s - 1, n_s - 2).
\end{aligned}$$

Thus, if $\varepsilon + \kappa \geq f(n_s, n_s - 2)$, i does not have an incentive to choose the third deviation.

2) The case of Bullying a weak network

When the number of strong players is more than two, there always exists at least two strong players in the clique. The utility of i in the clique is $u_i(\mathbf{g}) = f(n_s a + n_w, 1) - \varepsilon - \kappa$.

2-i) The first deviation is unprofitable if $f(n-1,1) \geq \varepsilon + \kappa$. After the first deviation the utility of i is 0. Thus, if $f(n_s a + n_w, 1) - \varepsilon - \kappa \geq 0$, the deviation is unprofitable. $f(n-1,1) < f(n_s a + n_w - 1, 1)$ for any n_w and a. Hence, if $f(n-1,1) \geq 0$, the condition is automatically satisfied, and the deviation is unprofitable.

2-ii) The second deviation is not profitable if $\varepsilon + \kappa > f(n_s, n_s - 1)$. Suppose i is type s. Then after the deviation,

$$
\begin{aligned}
u_i(\mathbf{g}') =& f((n_s - b_s)a + n_w - 1 - b_w, 1) + (b_s + b_w) f((n_s - b_s)a + n_w - 1 - b_w, (n_s - 1)a + n_w - 1) \\
& - (b_s + b_w + 1)(\varepsilon + \kappa).
\end{aligned}
$$

If $u_i(\mathbf{g}) \geq u_i(\mathbf{g}')$, then $\mathbf{g}'_i$ is unprofitable for i. $u_i(\mathbf{g}) \geq u_i(\mathbf{g}')$ is equivalent to

$$
\begin{aligned}
(b_s + b_w)(\varepsilon + \kappa) \geq& (b_s + b_w) f((n_s - b_s)a + n_w - 1 - b_w, (n_s - 1)a + n_w - 1) \\
& - (f(n_s a + n_w, 1) - f((n_s - b_s)a + n_w - 1 - b_w, 1)).
\end{aligned}
$$

The RHS is smaller than $(b_s + b_w) f((n_s - b_s)a + n_w - 1 - b_w, (n_s - 1)a + n_w - 1)$. If $b_s \geq 1$, $f((n_s - b_s)a + n_w - 1 - b_w, (n_s - 1)a + n_w - 1) \leq 0$. When $b_s = 0$, it is $f(n_s a + n_w - 1 - b_w, (n_s - 1)a + n_w - 1)$. For all a and n_w, it is smaller than $f(n_s, n_s - 1)$. Therefore, if $\varepsilon + \kappa \geq f(n_s, n_s - 1)$, then the deviation is unprofitable for strong type i.

Secondly, suppose i is weak type. Then

$$
\begin{aligned}
u_i(\mathbf{g}') =& f((n_s - b_s)a + n_w - 1 - b_w, 1) + (b_s + b_w) f((n_s - b_s)a + n_w - 1 - b_w, n_s a + n_w - 2) \\
& - (b_s + b_w + 1)(\varepsilon + \kappa).
\end{aligned}
$$

This $u_i(\mathbf{g}')$ is smaller than $u_i(\mathbf{g})$ if $b_s \geq 1$ or $b_w \geq 1$, because $(n_s - b_s)a + n_w - 1 - b_w > n_s a + n_w - 2$ if $b_s \geq 1$ or $b_w \geq 1$.

2-iii) The third deviation is unprofitable if $\varepsilon + \kappa \geq f(n_s, n_s - 2)$. If i is type s,

$$u_i(\mathbf{g}') = (b_s + b_w)f((n_s - b_s)a + n_w - b_w, (n_s - 1)a + n_w - 1) - (b_s + b_w)(\varepsilon + \kappa).$$

If $b_s = 0$ and $b_w \geq 2$, $f(n_s a + n_w - b_w, (n_s - 1)a + n_w - 1) < f(n_s, n_s - 1)$. Thus, if $f(n_s, n_s - 1) \leq \varepsilon + \kappa$, then $u_i(\mathbf{g}') < 0$. If $b_s = 1$ and $b_w \geq 1$ or $b_s \geq 2$, $(n_s - b_s)a + n_w - b_w < (n_s - 1)a + n_w - 1$, so $u_i(\mathbf{g}') < 0$. If $b_s + b_w = 1$, then

$$u_i(\mathbf{g}') = f((n_s - b_s)a + n_w - b_w, (n_s - 1)a + n_w - 1) - (\varepsilon + \kappa).$$

$u_i(\mathbf{g}) > u_i(\mathbf{g}')$ because $f(n_s a + n_w, 1) > f((n_s - b_s)a + n_w - b_w, (n_s - 1)a + n_w - 1)$.

Secondly, if i is weak type,

$$u_i(\mathbf{g}') = (b_s + b_w)f((n_s - b_s)a + n_w - b_w, n_s a + n_w - 2) - (b_s + b_w)(\varepsilon + \kappa).$$

If $b_s + b_w \geq 2$, $(n_s - b_s)a + n_w - b_w \leq n_s a + n_w - 2$, so $u_i(\mathbf{g}') < 0$. If $b_s + b_w = 1$, $f(n_s a + n_w, 1) > f((n_s - b_s)a + n_w - b_w, n_s a + n_w - 2)$. Hence $u_i(\mathbf{g}) > u_i(\mathbf{g}')$.

A.2 Four players example: The condition for each configuration to be a Nash equilibrium

Homogeneous players

There are two network configuration, which can be Nash equilibria. The first case is Utopia network. According to Remark 2.12, it is always a Nash equilibrium for any $\varepsilon + \kappa$, and it is unique when $\varepsilon + \kappa > f(3, 1)$. The second case is a network where three players are a friend to each other, and they extend negative links to the other player. It is a Nash equilibrium when $\varepsilon + \kappa \leq f(3,1)$ by Lemma 2.14.

Except for these two configurations, any other network configurations cannot be Nash equilibria by Remark 2.11. In the other weakly structually balanced configurations, there always exist players whose network strengths are the same. It violates Remark 2.11.

Two types of players

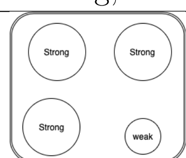
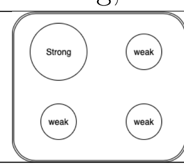
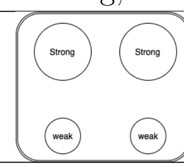
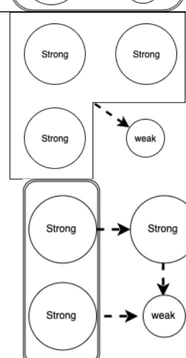
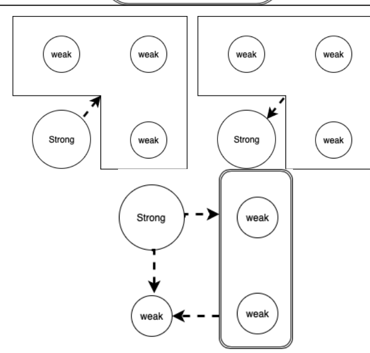
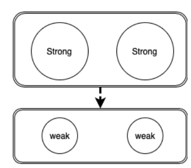
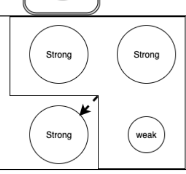
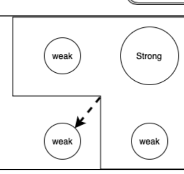
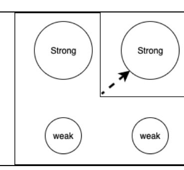
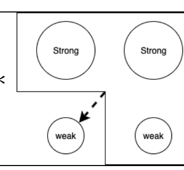
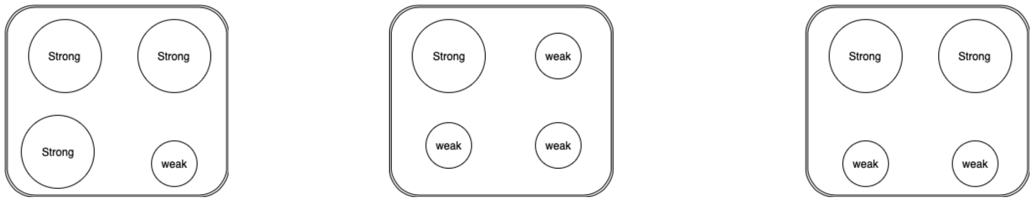
	3 strong, 1 weak	1 strong, 3 weak	2 strong, 2 weak
Utopia	 *	 *	 *
Positive assortative matching	 *	 *	
Disassortative matching	 *	 *	 *  *

Table A.1: Possible network configurations with four heterogeneous players in equilibrium

In Table A.1, we categorize all possible Nash equilibrium networks in the example of four heterogeneous players. If the network configurations are identical to the case with homogeneous players, the they are marked with an asterisk. In Figures A.1, A.2, and A.3 we specify each network configuration with the condition to be a Nash equilibrium.



(a) $\varepsilon + \kappa \geq f(3\lambda_s, 2\lambda_s + \lambda_w)$ (b) $\varepsilon + \kappa \geq f(\lambda_s + 2\lambda_w, 3\lambda_w)$ (c) $\varepsilon + \kappa \geq f(2\lambda_s + \lambda_w, \lambda_s + 2\lambda_w)$

Figure A.1: Utopia networks

A.3 Simulation

Using the normalized contest success function, we perform simulations to observe changes of conditions of each configuration to be a Nash equilibrium by parameters.

2.4.1 Utopia networks: Homogeneous players' model

In Figure A.4, the blue part is a region of the parameters $\varepsilon + \kappa$, ϕ and n for Utopia network to be the unique Nash equilibrium. The yellow part represents the boundary, and it is corresponding to $\underline{c}^{uh}$. Note that $\frac{1}{2}$ is the maximum value of $h(n_i, n_j, \phi)$. When ϕ is larger than 2 and n is larger than 6, $\underline{c}^{uh}$ is already close to $\frac{1}{2}$.

2.4.2 Utopia networks: The case of two types of players

Figure A.5 compares the conditions for Utopia networks to be a Nash equilibrium and to be the unique Nash equilibrium in the example of four players. Figure A.6 presents variations of ranges of the parameters supporting Utopia network as a Nash equilibrium with respect to ϕ when there are two strong players and two weak players. It also indicates that the conflict cost should increase and a should decrease as ϕ increases to maintain the Nash equilibrium.

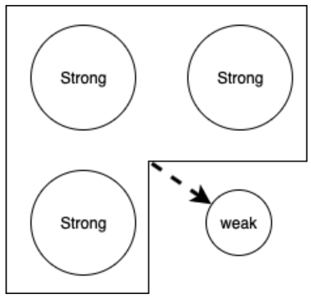
2.4.2 Positive assortative matching

Figure A.7 shows the ranges for each positive assortative matching of Figure 2.5 (a), (b), and (c) to be a Nash equilibrium. Figure A.7 (a) shows the difference with respect to the number of players when $\phi = 1$. Figure A.7 (b) describes a three-dimensional range for the configuration with two strong players and two weak players to be a Nash equilibrium. Figure A.7 (c) is a two dimensional expression of Figure A.7 (b). The figures indicate the following. First, if a is sufficiently large, then this CSPM is a Nash equilibrium. Second, If the number of strong players increases, less a can make CSPM in equilibrium. Third, if the conflict cost is sufficiently high, then CSPM can not be a Nash equilibrium. It is consistent with Proposition 2.12 mentioning the uniqueness of the Nash equilibrium Utopia network. Lastly, as the number of strong players increases or ϕ increases, CSPM can be a Nash equilibrium with the higher conflict cost.

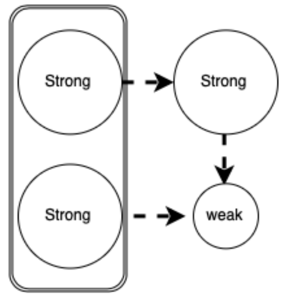
In Figure A.7, (b) and (c) also shows consistent observations with Corollary 2.25. It presents the variation in respect of ϕ on the range of parameters satisfying the condition for the Nash equilibrium complete strong dominant positive assortative matching when there are two strong and two weak players. In this diagram, as ϕ increases (decreases), the range expands (shrinks).

As mentioned above, Figure A.7 (a) demonstrates variations in respect to the number of players on the range of parameters satisfying the condition. The figure indicates that the range expands (shrinks) when there are more (less) strong players and less (more) weak players.

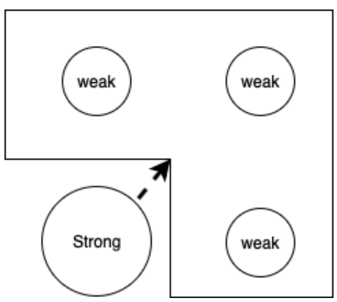
Figure A.8 demonstrates variations of the range of parameters satisfying the conditions for the Nash equilibrium complete weak dominant positive assortative matching when there are one strong and three weak players. These diagrams in Figure A.8 give implications consistent with Proposition 2.28, Corollary 2.29 and Remark 2.30. When ϕ increases, the range of $\varepsilon+\kappa$ increases and the range of a decreases. For example, in the extreme case such as $\phi > 100$, no a satisfies the condition for this Nash equilibrium configuration. Lastly, there is an upper bound of $\varepsilon + \kappa$ for the configuration in equilibrium. It is consistent with Proposition 2.17 in respect of the uniqueness of the Nash equilibrium Utopia network.



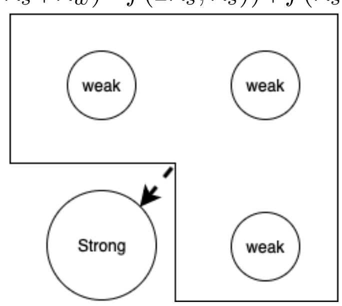
(a) $f(3\lambda_s, \lambda_w) \geq \varepsilon + \kappa$



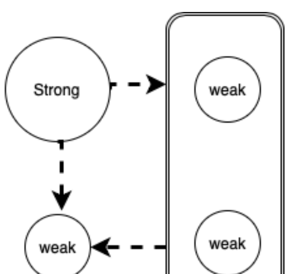
(b) $f(2\lambda_s, \lambda_s) + f(2\lambda_s, \lambda_w) \geq 2(\varepsilon + \kappa)$,
$f(2\lambda_s, \lambda_s) - (f(3\lambda_s, \lambda_w) - f(2\lambda_s, \lambda_w)) \geq \varepsilon + \kappa$,
$2(f(2\lambda_s, \lambda_s + \lambda_w) - f(2\lambda_s, \lambda_s)) + f(\lambda_s, \lambda_w) \geq 0$



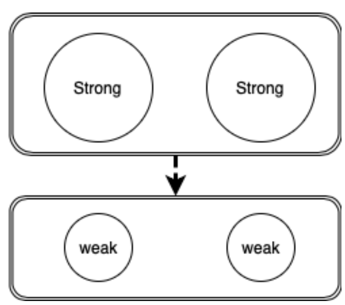
(c) $f(\lambda_s, \lambda_w) \geq \varepsilon + \kappa$,
$2(f(\lambda_s + \lambda_w, 3\lambda_w) - f(\lambda_s, 3\lambda_w)) \leq$
$f(\lambda_s, 3\lambda_w) - \varepsilon - \kappa$



(d) $f(3\lambda_w, \lambda_s) \geq \varepsilon + \kappa$,
$3f(\lambda_w, \lambda_s) \geq f(2\lambda_w + \lambda_s, 2\lambda_w)$,
$f(3\lambda_w, \lambda_s) \geq 2f(\lambda_w + \lambda_s, 2\lambda_w) - \varepsilon - \kappa$

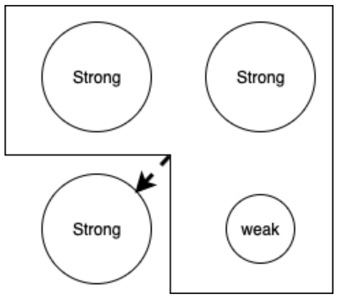


(e) $2f(\lambda_s, 2\lambda_w) + f(\lambda_s, \lambda_w) \geq \varepsilon + \kappa$,
$f(\lambda_s, 2\lambda_w) \geq (f(\lambda_s + \lambda_w, 2\lambda_w) - f(\lambda_s, 2\lambda_w))$
$+ (f(\lambda_s + \lambda_w, \lambda_w) - f(\lambda_s, \lambda_w)) + \varepsilon + \kappa$,
$2f(\lambda_s, 2\lambda_w) \geq$
$(f(\lambda_s + 2\lambda_w, \lambda_w) - f(\lambda_s, \lambda_w)) + 2(\varepsilon + \kappa)$,
$f(2\lambda_w, \lambda_w) + f(\lambda_s, 3\lambda_w) - f(\lambda_s, 2\lambda_w) \geq \varepsilon + \kappa$

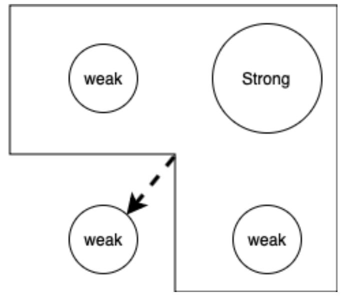


(f) $f(2\lambda_s, 2\lambda_w) \geq \varepsilon + \kappa$
$2f(2\lambda_s, 2\lambda_w) \geq f(2\lambda_s + \lambda_w, 2\lambda_w) + \varepsilon + \kappa$

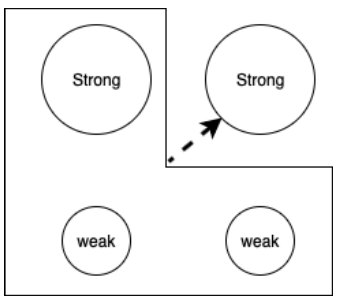
Figure A.2: Positive assortative matching



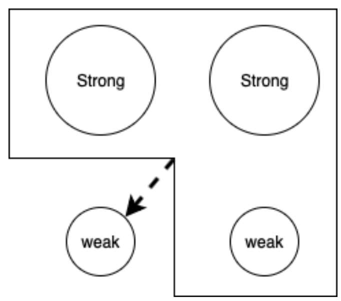
(a) $f(2\lambda_s + \lambda_w, \lambda_s) \geq \varepsilon + \kappa$
$f(3\lambda_s, \lambda_s + \lambda_w) \leq f(2\lambda_s + \lambda_w, \lambda_s)$



(b) $f(\lambda_s + 2\lambda_w, \lambda_w) \geq \varepsilon + \kappa$,
$f(\lambda_s + \lambda_w, 2\lambda_w) \leq$
$f(\lambda_s + 2\lambda_w, \lambda_w) - f(\lambda_s + \lambda_w, \lambda_w)) + \varepsilon + \kappa$



(c) $f(\lambda_s + 2\lambda_w, \lambda_s) \geq \varepsilon + \kappa$,
$f(2\lambda_s + \lambda_w, 2\lambda_w) \leq f(\lambda_s + 2\lambda_w, \lambda_s)$,
$f(\lambda_s + \lambda_w, 2\lambda_w) \leq$
$f(\lambda_s + 2\lambda_w, \lambda_s) - f(\lambda_s + \lambda_w, \lambda_s) + \varepsilon + \kappa$,
$f(\lambda_s + 2\lambda_w, \lambda_s) + \varepsilon + \kappa \geq 2f(\lambda_s, 2\lambda_w)$



(d) $f(2\lambda_s + \lambda_w, \lambda_w) \geq \varepsilon + \kappa$,
$f(2\lambda_s + \lambda_w, \lambda_w) \geq$
$f(2\lambda_s, \lambda_s + \lambda_w) + f(2\lambda_s, \lambda_w) - (\varepsilon + \kappa)$

Figure A.3: Disassortative matching

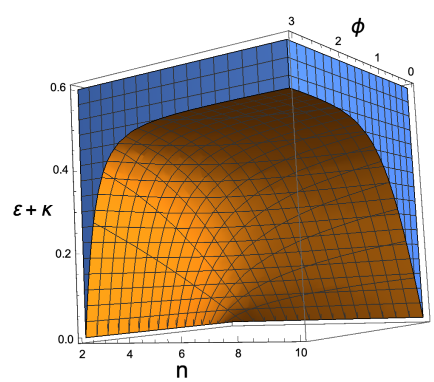
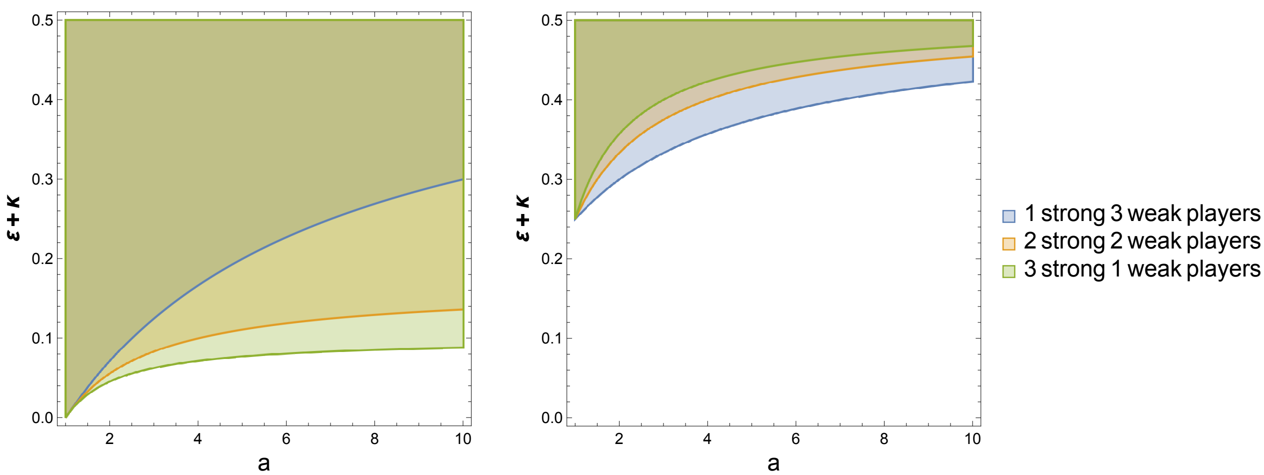


Figure A.4: Region of the parameters for Utopia networks as the unique Nash equilibrium



(a) Existence condition (b) Uniqueness condition

Figure A.5: Variation in ranges of the parameters with respect to the number of players when $\phi = 1$ for Utopia network

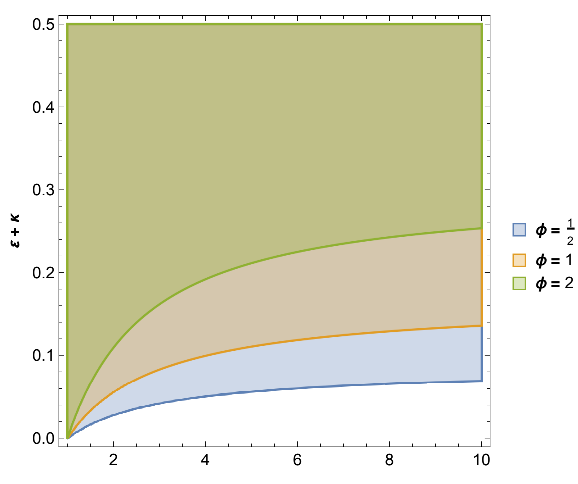
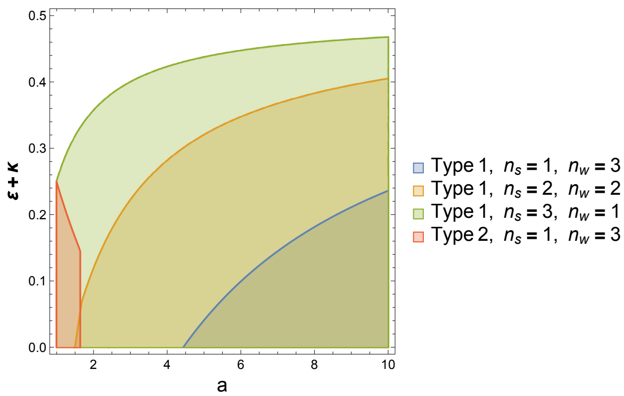
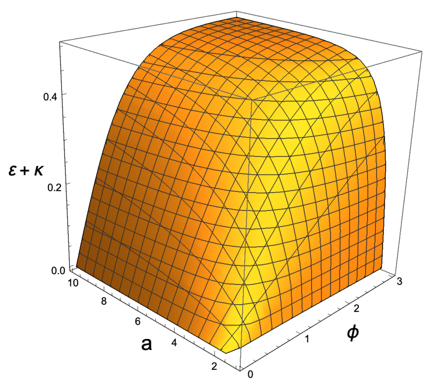


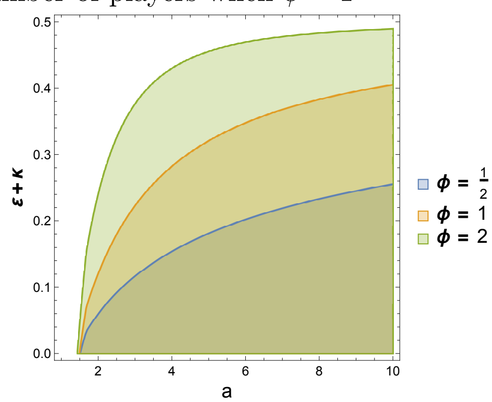
Figure A.6: Variation of uniqueness condition for Utopia network with respect to ϕ with 2 strong and 2 weak players



(a) Variation with respect to the number of players when $\phi = 1$

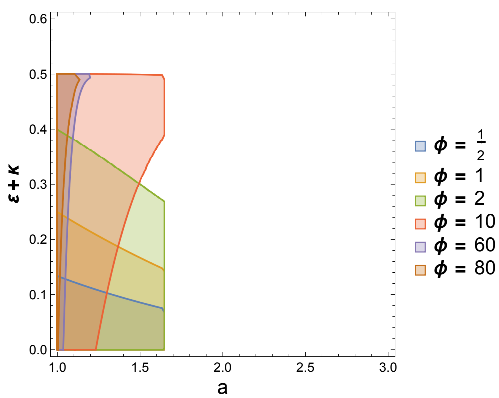


(b) CSPM, when $n_s = 2$ $n_w = 2$

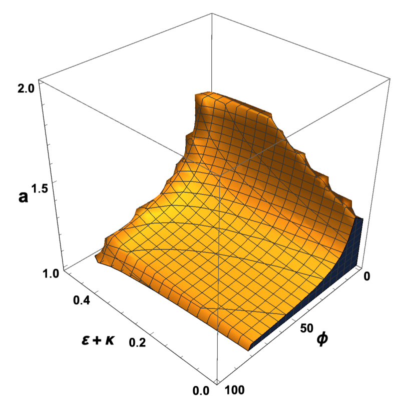


(c) Variation with respect to ϕ when $n_s = 2$ & $n_w = 2$ using CSPM

Figure A.7: Ranges of the parameters supporting positive assortative matching



(a) Variation with respect to ϕ when $n_s = 1$ & $n_w = 3$ using the weak dominant



(b) Using the weak dominant, n_s=1 n_w=3

Figure A.8: Ranges of the parameters supporting the Nash equilibrium CWPM by ϕ

Appendix B

Appendix for Chapter 3

B.1 Proof

Lemma 3.2

(i) $\mathrm{g}_{i,j} = 1, \mathrm{g}_{j,i} = -1$

Between player i and j, there is the resulting undirected link $\bar{\mathrm{g}}_{i,j} = -1$. Player i always has an incentive to change $\mathrm{g}_{i,j} = 1$ to $\mathrm{g}_{i,j} = 0$. Then player i's utility increases by c^+ but there is no change in the undirected network $\bar{\mathbf{g}}$.

(ii) $\mathrm{g}_{i,j} = 1, \mathrm{g}_{j,i} = 0$

In this case, $\bar{\mathrm{g}}_{i,j} = 0$. Given $\mathrm{g}_{j,i} = 0$, player i has an incentive to change $\mathrm{g}_{i,j} = 1$ to 0. Then she can increase her utility by c^+ with the same $\bar{\mathbf{g}}$.

(iii) $\mathrm{g}_{i,j} = \mathrm{g}_{j,i} = -1$

In this case, $\bar{\mathrm{g}}_{i,j} = -1$. One of the players i and j has an incentive to deviate from -1 to 0. Because she can increase her utility by $\varepsilon + \kappa$ with the same $\bar{\mathbf{g}}$.

Lemma 3.3

This lemma has the same implication to Hiller (2017)'s Lemma 2. First, I will show that when $y_i(\mathbf{g}) > y_j(\mathbf{g})$, no $\mathrm{g}_{i,j}$ and $\mathrm{g}_{j,i}$ except for $\mathrm{g}_{i,j} = -1$ and $\mathrm{g}_{j,i} = 0$ is possible in any PNE. By Lemma A.1, $\mathrm{g}_{i,j} = 1, \mathrm{g}_{j,i} = -1$ and $\mathrm{g}_{i,j} = \mathrm{g}_{j,i} = -1$ are not available in any pairwise Nash equilibrium. Now, for a $\bar{\mathrm{g}}_{i,j} = -1$, suppose $\mathrm{g}_{i,j} = -1$ and $\mathrm{g}_{i,j} = 0$, but $y_i(\mathbf{g}) \leq y_j(\mathbf{g})$. In this condition, $f(y_i(\mathbf{g}), y_i(\mathbf{g})) \leq 0$, but the player i spends $\varepsilon + \kappa$. Player i has an incentive to deviate from $\mathrm{g}_{i,j} = -1$ to 0. Then, her utility increases by $-f(y_i(\mathbf{g}), y_i(\mathbf{g})) + \varepsilon + \kappa$.

Lemma 3.4

Suppose $\exists\ i, j \in P_k(\mathbf{g})$ but $\bar{\mathrm{g}}_{i,j} = -1$. This undirected negative link $\bar{\mathrm{g}}_{i,j} = -1$ exists because at least one of i and j is extending the directed negative link $\mathrm{g}_{i,j} = -1$ or $\mathrm{g}_{j,i} = -1$. Then, the one who is extending this directed negative link can save $\varepsilon + \kappa > 0$ when she ceases extending this directed negative link and extends the directed neutral link because $f(y_i(\mathbf{g}), y_j(\mathbf{g})) = 0$.

Proposition 3.7

By definition, players in the same $P_i(\mathbf{g})$ have the same network strength. Therefore, by Lemma 3.4, there is no negative link between the players in the same $P_i(\mathbf{g})$. Then, the negative links exist only between the players in the different $P_i(\mathbf{g})$. Thus, it is possible to regard $P_i(\mathbf{g})$ (or an union of several $P_i(\mathbf{g})$) as an independent set. As a result, the relevant conflict network is k-partite. If there is an independent set consisting of multiple $P_i(\mathbf{g})$, $k < m$. Otherwise, $k = m$.

Propositions 3.9 and 3.14

Step 1 proves "if" condition of Proposition 3.9, and Proposition 3.14. Step 2 proves "only if" condition of Proposition 3.9.

Step 1) If there exists $i, j \in N$ such that $\bar{g}_{i,j} = 1$, there exists $k, l \in N$ such that $\bar{g}_{k,l} = -1$.

Suppose there are only the undirected positive or neutral links but are not the undirected negative links. Then for a player i, whose directed link $g_{i.j} = 1$ in $\mathbf{g}$, $\sum_{j \in N_i^-(\mathbf{g})} f(y_i, y_j) = 0$, because $N_i^-(\mathbf{g}) = \varnothing \; \forall i \in N$. Hence, $u_i(\mathbf{g}) = -|N_i^+(\mathbf{g})|c^+ \leq 0$. Instead, player i has a more profitable deviation $\mathbf{g}_i' = \mathbf{g}_i - \sum_{j \in N_i^+(\mathbf{g})} g_{i,j}^+$. It gives the utility $u_i(\mathbf{g}_i', \mathbf{g}_{-i}) = 0 \geq u_i(\mathbf{g})$.

Step 2) If there exist $i, j \in N$ such that $\bar{g}_{i,j} = -1$, there exist $k, l \in N$ such that $\bar{g}_{k,l} = 1$.

Suppose there are only the undirected negative or neutral links, but not the undirected positive links. Then every player has the same intrinsic strength, $\lambda_i = \lambda$. There is no undirected positive link by the assumption. Thus every network strength $y_i(\mathbf{g}) = \lambda$. Then from any undirected negative links, the extraction $f(n_i, n_j) = f(\lambda, \lambda) = 0$. Therefore, if any player i, who extends the directed negative links in $\mathbf{g}$, stops extending all negative links by changing them to the neutral links, it will increase player i's utility from $u_i(\mathbf{g}) = -|N_i^-(\mathbf{g})|(\varepsilon + \kappa) < 0$ to 0.

Proposition 3.11

(i) The empty network is a pairwise-Nash equilibrium.

The empty network $\mathbf{g}$ is a PNE. First, it is Nash stable. Every homogeneous player has 0 utility in the empty network. For any player $i \in N$, she cannot form the positive undirected link alone, because $\mathbf{g}_{j,i} = 0$ for all $j \in N \setminus \{i\}$. Also, any negative directed links do not increase her utility because $y_i(\mathbf{g}) = y_j(\mathbf{g}) = 1$ so $f(y_i(\mathbf{g}), y_j(\mathbf{g})) = 0$ but $\varepsilon + \kappa > 0$.

Second, it is pairwise stable for the same reason as Step 1 in the proof of Proposition 3.9. There is no existing extraction in $\mathbf{g}$, so making new friends does not increase her utility but incurs the friendship cost. (ii) The empty network is the unique pairwise-Nash equilibrium.

Suppose a network $\mathbf{g}$ is not empty. Then by Proposition 3.9, there should be positive and negative links together. First, suppose there exist i and j such that $y_i(\mathbf{g}) > y_j(\mathbf{g}) + 1$ where $i \in P_m(\mathbf{g})$ and $j \in P_{m-1}(\mathbf{g})$. By Proposition 3.9, there exist the negative undirected links between i and the other players in $P_{m-2}(\mathbf{g})$ and in $P_{m-3}(\mathbf{g})$ .... and in $P_1(\mathbf{g})$. If $\mathrm{g}_{i,j} = -1$, then i has an incentive to cut any one positive link. Let us call this deviation cutting one positive link $\mathbf{g}'$. $y_i(\mathbf{g}) > y_i(\mathbf{g}') = y_i(\mathbf{g}) - 1 > y_j(\mathbf{g})$. Then $f(y_i(\mathbf{g}), y_j(\mathbf{g})) \approx f(y_i(\mathbf{g}'), y_j(\mathbf{g}'))$. Player i can save c^+, but the difference between $f(y_i(\mathbf{g}), y_j(\mathbf{g}))$ and $f(y_i(\mathbf{g}'), y_j(\mathbf{g}'))$ is negligible. Also, if there exist $\mathrm{g}_{i,k} = -1$ for $k \in N \setminus \{P_m(\mathbf{g}) \cup P_{m-1}(\mathbf{g})\}$, the difference between $f(y_i(\mathbf{g}), y_k(\mathbf{g}))$ and $f(y_i(\mathbf{g}'), y_k(\mathbf{g}'))$ is negligible too, because $y_i(\mathbf{g}) > y_j(\mathbf{g}) + 1 > y_k(\mathbf{g})$. Therefore, i can increase her utility using the deviation.

Now, it is clear that $\mathbf{g}$ such that $|P_m(\mathbf{g})| > |P_{m-1}(\mathbf{g}) + 1|$ cannot be a PNE. By using inductive way, it is possible to repeat the procedure for $|P_m - 1(\mathbf{g})| > |P_{m-2}(\mathbf{g}) + 1|$, and $|P_m - 2(\mathbf{g})| > |P_{m-3}(\mathbf{g}) + 1|$, so on. Then, we have the last case of $\mathbf{g}$ where $|P_m(\mathbf{g})| = |P_{m-1}(\mathbf{g}) + 1| = |P_{m-2}(\mathbf{g}) + 2| = ... = |P_1(\mathbf{g}) + m - 1|$. If $|P_1(\mathbf{g})| \neq 1$, then it is not a pairwise Nash equilibrium. Because a player in $P_1(\mathbf{g})$ has an incentive to cut a positive link. Then she can save c^+ but the change in extraction $\sum f(\cdot)$ is negligible.

We assumed that there are more than or equal to four homogeneous players. Then at least, there should be more than three P_x. Let's consider a player i's deviation $\mathbf{g}'$ in $P_2(\mathbf{g})$. She can propose a new positive directed link $\mathrm{g}'_{i,j} = 1$ for $j \in P_1(\mathbf{g})$. This pairwise deviation is profitable for both players i and j. Let $\bar{f}$ denote $f(n-1, 1)$ and f_{min} denote $min_{2 \leq x \leq n-1} f(x, x-1) > \frac{c^+}{2}$. From this deviation, i loses at most $\bar{f} - (\varepsilon + \kappa) + c^+$ because $\mathrm{g}_{i,j} = -1$ is replaced by $\mathrm{g}'_{i,j} = 1$. However, her network strength $y_i(\mathbf{g}') = y_i(\mathbf{g}) + 1 =$

$y_k(\mathbf{g}')$, for $k \in P_3(\mathbf{g}')$. Then she can increase her utility at least by $3f_{min}$ from the three negative links from the players in $P_3(\mathbf{g}')$. Overall, i can increase her utility at least by $3f_{min} - \bar{f} + \varepsilon + \kappa - c^+ > 0$. On the other hand, this deviation is also profitable for j. $y_j(\mathbf{g}') = y_j(\mathbf{g}) + 1 = y_l(\mathbf{g})$ for $l \in P_2(\mathbf{g}')$. Then j can increase her utility at least by $2f_{min} - c^+ > 0$, because i ceased her negative link, and $f(y_l(\mathbf{g}'), y_j(\mathbf{g}')) = 0$.

Proposition 3.13

i) In a k-regular bullying network, there are two sets of players. In a set A, there are $n-1$ players. In the other set B, there is one player v. Every player in A extends the negative directed link to player v, so $\mathrm{g}_{i,v} = -1\ \forall i \in A$. By Lemma 3.3, $\mathrm{g}_{v,i} = 0\ \forall i \in A$. Let's check Nash stability of $\mathbf{g}$. First, any player $i \in A$ does not have an incentive to decrease the number of positive links. Suppose i ceases several $\mathrm{g}_{i,l} = -1 \rightarrow \mathrm{g}'_{i,l} = 0$ for all $l \in L \subset A \setminus \{i\}$. It decreases i's utility $u_i(\mathbf{g}) = f(k+1,1) - kc^+ - (\varepsilon+\kappa)$ to $u_i(\mathbf{g}') = f(k+1-|L|,1) - (k-|L|)c^+ - (\varepsilon+\kappa)$. $u_i(\mathbf{g}) - u_i(\mathbf{g}') = f(k+1,1) - f(k+1-|L|,1) - |L|c^+ \geq 0$, because $c^+ \leq f(k+1,1) - f(k,1) \leq f(k+2-|L|,1) - f(k+1-|L|,1)$ for any $|L| \geq 1$. Also, $\mathrm{g}_{i,v} = -1 \rightarrow \mathrm{g}'_{i,v} = 0$ decreases player i's utility $u_i(\mathbf{g}) = f(k+1,1) - kc^+ - (\varepsilon+\kappa)$ to $u_i(\mathbf{g}') = -(k+1)c^+ < 0$. $u_i(\mathbf{g}) - u_i(\mathbf{g}') > 0$ because $\varepsilon + \kappa \leq f(k+1,1)$. So the deviation $\mathbf{g}'$ is not profitable for player i.

Second, player v does not have any valid choice of single deviations. Even though $\mathrm{g}_{v,i} = 0 \rightarrow \mathrm{g}'_{v,i} = -1$ for any $i \in A$, it does not change the undirected network $\bar{\mathbf{g}}$ because $\mathrm{g}_{i,v} = -1\ \forall i \in A$, already.

Let's check pairwise stability of $\mathbf{g}$. For a pair of $i, j \in A$ where $\mathrm{g}_{i,j} = 0$, they do not have an incentive to add a positive undirected link between them, $\mathrm{g}_{i,j} = 1$, because $f(k+2,1) - f(k+1,1) \leq c^+$. On the other hand, for a pair of $i \in A$ and v, $\mathrm{g}_{v,i} = 0 \rightarrow \mathrm{g}'_{v,i} = -1$ gives

$u_i(\mathbf{g}') = -c^+ < 0$, so i does not choose the deviation.

ii) If $k = n-2$, then these $n-1$ players are fully connected, so they cannot add more positive links between them. Therefore, the condition $f(k+2,1) - f(k+1) \leq c^+$ is not required.

Proposition 3.16

If the condition $Max\{\lambda_1, ..., \lambda_n\} - min\{\lambda_1, ..., \lambda_n\} < \varepsilon + \kappa$ is satisfied, the empty network is a PNE. First, it is Nash stable. Given the empty network $\mathbf{g}$, the players cannot form the positive undirected link alone, because $\mathbf{g}_{j,i} = 0$ for all $j \in N \setminus \{i\}$, too. Regarding the negative undirected link, for any $i \in N$, $y_i = \lambda_i$. Therefore, the condition implies $Max\{y_1(\mathbf{g}), ..., y_n(\mathbf{g})\} - min\{y_1(\mathbf{g}), ..., y_n(\mathbf{g})\} < \varepsilon + \kappa$. Then any player cannot increase her utility by extending the negative links.

Second, because there is no negative link for any player, they cannot increase their utility by forming a positive undirected link, as shown in Proposition 3.14

Proposition 3.19

Before the proof, let $\lambda_s = a$ and $\lambda_w = 1$ for simplicity. It does not change the result because $f(y_i, y_j)$ is homogeneous of degree 0.

Suppose there exists a network $\mathbf{g}$, which is not positive assortative matching in equilibrium. Then, it is one of (i) the empty network, (ii) $\exists \bar{\mathrm{g}}_{i,j} = 1$ for $i \in N_s$ and $j \in N_w$, and (iii) $\nexists \bar{\mathrm{g}}_{i,j} = 1$ for $i,\ j \in N_s$ and $i,\ j \in N_w$ First, the empty network is not a pairwise-Nash equilibrium. For any $\varepsilon + \kappa < \bar{f}$, there exists a which satisfies $f(a,1) > \varepsilon + \kappa$. Then a strong player i's deviation $\mathrm{g}_{i,j} = 0 \rightarrow -1$ to $j \in N_w$ is profitable for i.

Second, any network in (ii) cannot be in equilibrium. Let's consider a player i's deviation

$g_{i,j} = 1 \to 0$. Before the deviation,

$$u_i(\mathbf{g}) = \sum_{k \in N_i^-(\mathbf{g})} f(y_i(\mathbf{g}), y_k(\mathbf{g})) - |N_i^{e+}(\mathbf{g})|c^+ - |N_i^{e-}(\mathbf{g})|(\varepsilon + \kappa).$$

After the deviation,

$$u_i(\mathbf{g}') = \sum_{k \in N_i^-(\mathbf{g})} f(y_i(\mathbf{g}) - 1, y_k(\mathbf{g})) - |N_i^{e+}(\mathbf{g}) - 1|c^+ - |N_i^{e-}(\mathbf{g})|\varepsilon + \kappa.$$

$u_i(\mathbf{g}) - u_i(\mathbf{g}') = \sum_{k \in N_i^-(\mathbf{g})} \big(f(y_i(\mathbf{g}), y_k(\mathbf{g})) - f(y_i(\mathbf{g}) - 1, y_k(\mathbf{g}))\big) - c^+$. Let $y_i(\mathbf{g}) = \lambda_s + x$. $x = y_i(\mathbf{g}) - \lambda_s$ is the strength borrowed from the other players. Then $u_i(\mathbf{g}) - u_i(\mathbf{g}') = \sum_{k \in N_i^-(\mathbf{g})} (f(\frac{\lambda_s + x}{y_k(\mathbf{g})}, 1) - f(\frac{\lambda_s + x - 1}{y_k(\mathbf{g})}, 1)) - c^+$. As a increases, $\sum_{k \in N_i^-(\mathbf{g})} (f(\frac{\lambda_s + x}{y_k(\mathbf{g})}, 1) - f(\frac{\lambda_s + x - 1}{y_k(\mathbf{g})}, 1))$ decreases and goes to 0. Thus, for any given c^+, $u_i(\mathbf{g}) - u_i(\mathbf{g}') < c^+$ if a is sufficiently large.

Lastly, suppose $\nexists \bar{g}_{i,j} = 1$ for $i, j \in N_s$ and for $i, j \in N_w$. In the paragraph just above, we showed $\nexists \bar{g}_{i,j} = 1$ for $i \in N_s$ and $j \in N_w$. Therefore, there do not exist any positive links in the network, so $y_i(\mathbf{g}) = \lambda_i$ for every $i \in N$. Then, there are three possible cases. First, if $\forall \bar{g}_{i,j} = 0$ for $i, j \in N_s$ and for $i, j \in N_w$. it is the empty network, which is proven that it cannot be in equilibrium. Second, suppose $\exists \bar{g}_{i,j} = -1$ for $i \in N_w$ and $j \in N_w$, but not all ($\exists \bar{g}_{i,j} = 0$). Then player i can deviate from $\mathbf{g}$ changing $g_{i,j} = 0 \to -1$. Then i can increase her utility by $f(a, 1) - (\varepsilon + \kappa) > 0$, because a is sufficiently large. Third, if $\forall \bar{g}_{i,j} = -1$, then There are only the neutral undirected links between the same type and only the negative undirected links between the different types. It is the network mentioned in the proposition.

Appendix C

Appendix for Chapter 4

C.1 Proof

Proposition 4.3

(i) For any rich player $i,\ j$, and any poor player k, l, the following equalities are derived from the first-order conditions.

$$S'(x_{i,i}) = \alpha x_{i,j}^{\alpha-1} x_{j,i}^{\beta} \tag{C.1}$$

$$S'(x_{i,i}) = \alpha x_{i,k}^{\alpha-1} x_{k,i}^{\beta} \tag{C.2}$$

$$S'(x_{k,k}) = \alpha x_{k,l}^{\alpha-1} x_{l,k}^{\beta} \tag{C.3}$$

$$S'(x_{k,k}) = \alpha x_{k,i}^{\alpha-1} x_{i,k}^{\beta} \tag{C.4}$$

Using symmetry, $x_{i,j} = x_{j,i}$ and $x_{k,l} = x_{l,k}$. From (C.1) and (C.3),

$$S'(x_{i,i}) = \alpha x_{i,j}^{\alpha+\beta-1} = \frac{\alpha}{x_{i,j}^{1-\alpha-\beta}}$$

$$S'(x_{k,k}) = \alpha x_{k,l}^{\alpha+\beta-1} = \frac{\alpha}{x_{k,l}^{1-\alpha-\beta}}$$

Because the players use the symmetric strategy, each rich or poor player has the budget constraint as follows.

$$x_{i,i} + (\frac{n}{2} - 1)x_{i,j} + \frac{n}{2}x_{i,k} = M_r \tag{C.5}$$

$$x_{k,k} + (\frac{n}{2} - 1)x_{k,l} + \frac{n}{2}x_{k,i} = M_p \tag{C.6}$$

Regarding $x_{i,j}$ and $x_{k,l}$, there are two cases. First, if $x_{i,j} > x_{k,l}$, then $S'(x_{i,i}) < S'(x_{k,k})$ and $x_{i,i} > x_{k,k}$. Second, if $x_{i,j} \leq x_{k,l}$ then $S'(x_{i,i}) \geq S'(x_{k,k})$ and $x_{i,i} \leq x_{k,k}$. Then to satisfy the budget constraints (C.5) and (C.6), $x_{i,k} > x_{k,i}$. However, it does not hold. From (C.2) and (C.4),

$$\begin{aligned} \frac{S'(x_{i,i})}{S'(x_{k,k})} &= \frac{\alpha x_{i,k}^{\alpha-1} x_{k,i}^{\beta}}{x_{k,i}^{\alpha-1} x_{i,k}^{\beta}} \\ &= (\frac{x_{k,i}}{x_{i,k}})^{1-\alpha+\beta} \end{aligned}$$

If $\frac{S'(x_{i,i})}{S'(x_{k,k})} \geq 1$, then $\frac{x_{k,i}}{x_{i,k}} \geq 1$. Hence, it is a contradiction.

(ii) From the inequality $x_{i,i} > x_{k,k}$, $S'(x_{i,i}) < S'(x_{k,k})$. Then from (C.2) and (C.4),

$$S'(x_{i,i}) = \frac{\alpha x_{k,i}^{\beta}}{x_{i,k}^{1-\alpha}} < S'(x_{k,k}) = \frac{\alpha x_{i,k}^{\beta}}{x_{k,i}^{1-\alpha}}$$

Thus, $x_{i,k} > x_{k,i}$.

(iii) As shown in (ii), $S'(x_{i,i}) < S'(x_{k,k})$. Then from (C.1) and (C.3),

$$S'(x_{i,i}) = \frac{\alpha}{x_{i,j}^{1-\alpha-\beta}} < S'(x_{k,k}) = \frac{\alpha}{x_{k,l}^{1-\alpha-\beta}}$$

Thus, $x_{i,j} > x_{k,l}$.

(iv) From (C.1) and (C.2),

$$\frac{\alpha x_{i,j}^{\beta}}{x_{i,j}^{1-\alpha}} = \frac{\alpha x_{k,i}^{\beta}}{x_{i,k}^{1-\alpha}}. \tag{C.7}$$

I show that only the case $x_{i,j} > x_{i,k} > x_{k,i}$ is possible. First, neither $x_{i,j} = x_{i,k}$ nor $x_{k,i} = x_{i,j}$ is possible. If $x_{i,j} = x_{i,k}$, then $x_{k,i} = x_{i,j}$. But it contradicts the inequality $x_{i,k} > x_{k,i}$ in (ii). Second, $x_{i,k} > x_{i,j} > x_{k,i}$ is impossible. If so, $\frac{\alpha x_{i,j}^{\beta}}{x_{i,j}^{1-\alpha}} > \frac{\alpha x_{k,i}^{\beta}}{x_{i,k}^{1-\alpha}}$. Third, $x_{i,k} > x_{k,i} > x_{i,j}$ is not available. Let's consider a case when $M_r = \frac{1}{2}$ and $M_p = \epsilon$ for a small enough ϵ. $x_{k,k}$, $x_{k,i}$, and $x_{k,l}$ are close to 0 because of the budget constraint. Because, $x_{k,i} > x_{i,j}$, $x_{i,j}$ should be close to 0. Then $S'(x_{i,i})$ is not close to zero, but $\frac{\alpha}{x_{i,j}^{1-\alpha-\beta}}$ is close to 0. Hence, in this case, $x_{i,j} > x_{i,k} > x_{k,i}$. Nash equilibrium solution $x_{i,j}$ is continuous on M_r and M_p because the utility function is continuous on M_r and M_p. Therefore, if there exists Nash equilibrium $x_{i,j}$ such that $x_{i,k} > x_{k,i} > x_{i,j}$, there should be a case that $x_{i,k} > x_{i,j} > x_{k,i}$, which I showed that it is impossible. Therefore, only $x_{i,j} > x_{i,k}$ can be in equilibrium.

(v) From (C.3) and (C.4),

$$\frac{\alpha x_{k,l}^{\beta}}{x_{k,l}^{1-\alpha}} = \frac{\alpha x_{i,k}^{\beta}}{x_{k,i}^{1-\alpha}} \tag{C.8}$$

I show that only the case $x_{i,k} > x_{k,i} > x_{k,l}$ is possible. First, neither $x_{i,k} = x_{k,l}$ nor $x_{k,i} = x_{k,l}$ is possible. If $x_{i,k} = x_{k,l}$, then $x_{k,i} = x_{k,l}$. But it contradicts the inequality $x_{i,k} > x_{k,i}$ in (ii). Second, $x_{i,k} > x_{k,l} > x_{k,i}$ is impossible. If so, $\frac{\alpha x_{k,l}^{\beta}}{x_{k,l}^{1-\alpha}} < \frac{\alpha x_{i,k}^{\beta}}{x_{k,i}^{1-\alpha}}$. Third, $x_{k,l} > x_{i,k} > x_{k,i}$ is not available. As shown in (ii), $x_{i,j} > x_{k,l}$. Let's multiply (C.1) and (C.3), and (C.2) and (C.4).

Then

$$\alpha^2 x_{i,j}^{\alpha-1} x_{j,i}^{\beta} x_{k,l}^{\alpha-1} x_{l,k}^{\beta} = S'(x_{i,i}) S'(x_{k,k}) = \alpha^2 x_{i,k}^{\alpha-1} x_{k,i}^{\beta} x_{k,i}^{\alpha-1} x_{i,k}^{\beta}$$

But $(x_{i,j}x_{k,l})^{1-\alpha-\beta} > (x_{i,k}x_{i,k})^{1-\alpha-\beta}$. Hence, it is a contradiction.

Then $S'(x_{i,i})$ is not close to zero, but $\frac{\alpha}{x_{i,j}^{1-\alpha-\beta}}$ is close to 0. Hence, in this case, $x_{i,j} > x_{i,k} > x_{k,i}$. Nash equilibrium solution $x_{i,j}$ is continuous on M_r and M_p because the utility function is continuous on M_r and M_p. Therefore, if there exists Nash equilibrium $x_{i,j}$ such that $x_{i,k} > x_{k,i} > x_{i,j}$, there should be a case that $x_{i,k} > x_{i,j} > x_{k,i}$, which I showed that it is impossible. Therefore, only $x_{i,j} > x_{i,k}$ can be in equilibrium.

Proposition 4.4

In this proof, I show X induces a specific Data $\bar{D}_k(X)$ depending on k. The specific configurations satisfy the definition of nested split graphs.

1) When $k < \frac{n}{2}$, the rich players form the core, and the poor players form the periphery set (independent set). Each rich players report the other $\frac{n}{2}-1$ fiends as their best friends. In the symmetric Nash equilibrium, for every $i, j \in R$, $x_{i,j}$ is identical. By Proposition 4.3-(iv), the rich players do not point out the poor players because the number of the other rich player ($\frac{n}{2}-1$) is bigger than (or equal to) k. As a result, the rich players form a core. Secondly, the poor players point out the rich players as their best friends by Proposition 4.3-(v). Because X is the symmetric Nash equilibrium, every $x_{i,k}$ is identical for $i \in R$ and $k \in P$, each poor player designates every $\frac{n}{2}$ rich player. However, they do not report the other poor players as their best friends because the number of rich players $\frac{n}{2}$ is already bigger than the limit of response k. The degree of each rich player δ_i is $n-1$, and that of each poor player δ_k is $\frac{n}{2}$. Because every rich player is connected to all of the other rich players, the configuration satisfies the definition of nested split networks.

2) When $k = \frac{n}{2}$, the rich players still form the core, and the poor players form the periphery set (independent set), too. Each rich player reports every other player as their friend because she has the space of response for $\frac{n}{2} - 1$ rich friends and one more friend. Because $x_{i,k}$ is identical for any $i \in R$ and $k \in P$, the rich players report all poor players as their friends. The poor players still report $\frac{n}{2}$ rich players as their best friends. Because the configuration and the degrees are the same, this case also satisfies the definition of nested split networks.

3) When $k = \frac{n}{2}$, $\bar{D}_k$ is the complete network. As the case when $k = \frac{n}{2}$, each rich player reports every other player as her friend because k is bigger than the number of the other rich players $\frac{n}{2} - 1$. Each poor player also reports every other player as her friend because k is more than $\frac{n}{2}$, the number of the rich players. Because all players report the other players as their friends, $\bar{D}_k$ is the complete network. The complete network is a nested split graph satisfying the definition.