

A STUDY OF SUPER-KMS FUNCTIONALS

by

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(ABSTRACT)

We study properties of super-KMS functionals on \mathbb{Z}_2 graded von Neumann algebras. We prove that if a normal self-adjoint functional ω is weakly super-KMS, then the uniquely defined by the polar decomposition of ω positive functional $|\omega|$ is KMS.

We construct a graded representation of any von Neumann algebra with a normal self-adjoint super-KMS functional on it as an algebra of bounded operators on a Hilbert space. The grading of the algebra of operators that we obtain is induced from a natural orthogonal decomposition of the Hilbert space. In our construction we have to use the weak super-KMS property and the implications we have derived from it.

We present a generalization of the Tomita – Takesaki theorem to the case of (not necessarily positive) self-adjoint normal faithful functionals. We show that for every such functional ω there is a canonically defined *-automorphism group (the analog of the modular group) and a canonical \mathbb{Z}_2 grading of the algebra, commuting with

the automorphism group. The functional ω is weakly super-KMS with respect to them. Furthermore, the canonical automorphism group and \mathbb{Z}_2 grading are the unique pair of a σ -weakly continuous one-parameter $*$ -automorphism group and a \mathbb{Z}_2 grading, commuting with each other, with respect to which ω is super-KMS.

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CHAPTER I

INTRODUCTION

One new and exciting example of the exchange of ideas among different branches of mathematics has been the discovery in recent years of deep connections between topology and operator algebra theory which has brought some nontrivial new results, like the partial proof of the Novikov conjecture by Connes and Moscovici [1], new results in group representation theory and others. What makes this development very appealing to mathematical and theoretical physicists is that it seems that quantum statistical mechanics and quantum field theory have much to borrow and lend to the above fields in terms of ideas and methods. In particular, one of the fundamental objects in quantum statistics – the KMS state, or rather its supersymmetric generalization, the super-KMS functional – may come to play a very important role in the mathematical theory which has become known as noncommutative geometry, and which has evolved from the fusion of ideas from topology and operator algebras. Quantum field theory and its nonlinear generalizations – the nonlinear σ models may prove to be a perfect playground to test the above mentioned mathematical ideas. On one hand, a nonlinear σ model has as its configuration space (space of dynamical fields) an infinite dimensional manifold and the Hamiltonian can be viewed as a differential operator on it. On the other hand, a standard description of a quantum field theory is in terms of algebras of observables or in terms of local field operators [2, 3]. Although there is no clear understanding at present about what exactly plays the role of the local fields in a nonlinear σ model, if such understanding is achieved, one will have two descriptions of the same

object - one topological as an infinite-dimensional manifold and one in terms of a canonically defined algebra of operators. Despite that it has long been known that the theory of infinite-dimensional manifolds is very relevant to the σ model, there hasn't been yet considerable progress in using this theory to study the latter. It is possible that the methods of noncommutative geometry will help clarify some of the unresolved problems of the nonlinear σ model like the question of the relevant operator algebra that describes it.

In order to understand the origin of some of the intimate connections between topological spaces and Banach algebras one has to go back to the Gelfand representation theorem for commutative C^* -algebras [4] (see also e.g. [5]). According to it every C^* -algebra \mathcal{A} is $*$ -isometrically isomorphic to $C_0(\hat{\mathcal{A}})$ - the algebra of continuous functions, vanishing "at infinity" on the locally compact Hausdorff space $\hat{\mathcal{A}}$ of all multiplicative linear functionals on \mathcal{A} . The latter is often called the carrier space. A proper continuous map (a map is called proper if the inverse image of a compact set is compact) from one space $\hat{\mathcal{A}}$ to another space $\hat{\mathcal{B}}$ induces a $*$ -homomorphism from $C_0(\hat{\mathcal{B}})$ to $C_0(\hat{\mathcal{A}})$. In categorical language the same facts are contained in the statement that the category of commutative C^* -algebras with $*$ -homomorphisms as the morphisms of the category is contravariantly equivalent to the category of locally compact Hausdorff spaces with proper continuous maps as morphisms. The Gelfand transform is the contravariant functor providing this equivalence. This means that if one is interested in studying properties of commutative C^* -algebras, preserved by $*$ -homomorphisms, one could as well study locally compact Hausdorff spaces and properties, preserved by proper continuous maps and vice versa. One of the most fruitful ideas in present day mathematics has been to

study categories by constructing different functors on them. Usually one looks for a functor into a simpler category like commutative groups or modules over some ring, etc. Functors have the nice property that they map isomorphic objects to isomorphic ones. (Of course the converse is usually not true i.e. nonisomorphic objects can be mapped to isomorphic ones.) Thus if we know that the images are not isomorphic, like e.g. $\mathbb{Z} \oplus \mathbb{Z} \neq \mathbb{Z}$, we know that the two objects in the original category are not isomorphic. Typical examples of functors on the category of topological spaces (or some of its subcategories) are the different homology and cohomology groups, homotopy groups, K -theory. Often there exist or may be possible to construct analogous functors on the category of (not necessarily commutative) C^* -algebras or Banach algebras or even just locally convex algebras. For example, a useful functor on Banach algebras (or in fact more generally local Banach algebras [cf. e.g. 6]) is the K -theory on them. A well known theorem of Serre shows that the topological K group $K^0(X)$ for a compact Hausdorff space X is isomorphic to the operator-algebraic K group for the algebra of continuous functions on X , i.e. $K^0(X) \simeq K_0(C(X))$ [7].

The important recent works of A. Connes [8,9] on cyclic cohomology of Banach or locally convex algebras can be considered as a noncommutative extension of ordinary homology on topological spaces. In other words the idea is to construct a functor on the algebraic category which in the commutative case reduces (in a sense) to ordinary homology on topological spaces. The motivations for such a development are several. One is for the theory of operator algebras itself. Another one is that there are topological spaces, arising naturally in topology and geometry, such as the leaf space of a foliation or the orbit space of a discrete or a Lie group acting

on a manifold, which do not submit to a very useful treatment in purely topological intrinsic terms. The reason for this is that they are seldom Hausdorff in the quotient topology that they can be equipped with. In fact there are examples in which the resulting topology contains no nontrivial open sets at all [10]. However such spaces possess canonically associated with them noncommutative C^* -algebras [10] which are reminiscent of the group algebra for a discrete or a Lie group. Therefore the idea is to extract the topological information for such spaces by studying their associated C^* -algebras. The infinite-dimensional manifolds are an example of topological spaces that are not locally compact. So they fit in a category that is “bigger” than that of commutative C^* -algebras. The hope is that the theory of Connes is relevant to this case too.

In a parallel development, starting from the physical problem of constructing the two-dimensional N=2 Wess – Zumino model on a cylinder, Jaffe et al.[11,12] have provided an example of an infinite dimensional manifold \mathcal{M} (actually for now just a linear space since the target space is \mathbb{C}), a C^* -algebra of observables associated with \mathcal{M} , a superderivation on the algebra and a trace-class operator - the heat kernel e^{-H} . The authors above have shown how, using these objects, one can construct a nontrivial cocycle in Connes’ cyclic cohomology [12]. This development has been given the name Quantum K - theory. Let us describe in some detail this theory.

The N=2 Wess – Zumino model is a quantum field theory with two real bosonic fields (or one complex) and one two-component spinor and with interaction chosen so that the theory is supersymmetric. (The corresponding lagrangian is invariant under the action of the generators of certain representation of the Superpoincaré

algebra.) The important features for us are that there is a natural \mathbb{Z}_2 grading of the Hilbert space \mathcal{H} induced by the unitary operator $(-1)^{N_f}$ where N_f is the fermion number operator. The supersymmetry generator (or supercharge) is implemented as a self-adjoint Fredholm operator Q , the supersymmetry possibly being broken, i.e. Q acting on the vacuum may not be zero. Q has the property that it maps bosonic states into fermionic and vice versa. In other words it is odd with respect to the grading introduced on \mathcal{H} . Next a very important feature is that the Hamiltonian of the theory H is the square of Q . This last property makes such a theory interesting from the mathematical point of view, as it looks like an infinite-dimensional analog of the setting for the Atiyah – Singer index theory for a family of Dirac operators. (In the Wess – Zumino model both Q and H depend on the parameters of the theory – the coupling constants in the interaction Lagrangian – and in this sense are families.) The analogy was exploited by Witten [13] in his very interesting paper “Constraints on Supersymmetry breaking”. We shall describe this work in some detail in the next chapter. Jaffe et al. [14,11] treated rigorously the problem and actually computed the index of the Fredholm operator Q .

Extracting just the mathematical structure relevant to the cyclic cohomology theory from the Wess – Zumino model, one gets the following quadruple [12] $(\mathcal{A}, \mathcal{H}, \Gamma, Q)$, where $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a \mathbb{Z}_2 graded separable Hilbert space, Γ is the grading operator whose two eigenspaces are \mathcal{H}_\pm , $\mathcal{A} \subset L(\mathcal{H})$ is a C^* algebra of observables, e.g. the von Neumann algebra generated by all spectral projections of the smeared local fields of the theory. \mathcal{A} has a natural \mathbb{Z}_2 grading induced by the grading of \mathcal{H} , namely operators that preserve \mathcal{H}_\pm separately are even and those that map \mathcal{H}_\pm to \mathcal{H}_\mp are odd. The grading operator Γ takes the block-diagonal form

$\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. It induces an automorphism of \mathcal{A} by conjugation, i.e. for any $a \in \mathcal{A}$

$$a^\Gamma := \Gamma a \Gamma .$$

On homogeneous elements this gives $a^\Gamma = (-1)^{\deg a} a$. The operator Q has properties as above and its square $H := Q^2$ has a trace class heat kernel at any finite “temperature”, i.e. $\exp(-\beta H)$ is trace class for $\beta > 0$.

The supercharge Q defines a superderivation d on a dense subalgebra of $\mathcal{A}_d \in \mathcal{A}$ through taking a graded commutator with elements of \mathcal{A}_d . If $a = a_+ + a_-$ is the decomposition of the element $a \in \mathcal{A}_d$, the superderivation is defined as

$$da := [iQ, a_+] + \{iQ, a_-\} .$$

(A graded superderivation of a \mathbb{Z}_2 graded algebra \mathcal{A}_d is a linear map $d: \mathcal{A}_d \rightarrow \mathcal{A}_d$, such that a graded Leibnitz rule holds: $d(ab) = (da)b + (-1)^{\deg a} db$.)

The described structures are sufficient so that one can define a sequence of multilinear functionals on \mathcal{A}_d , which form a cocycle in the entire cyclic cohomology of \mathcal{A}_d . To avoid excessive notational complications we shall be talking from now on about cohomology of \mathcal{A} although strictly speaking the construction only defines a cocycle on the dense subalgebra \mathcal{A}_d . This is similar to the situation in ordinary cohomology theory for manifolds, where often one uses additional structure e.g. smooth structure to calculate the homology groups. Of course in that case there are powerful results, showing that the result is independent of any such choice. The entire cyclic cohomology theory of Connes will be described in more details in chapter 2. For now we just say that the entire cyclic cohomology is defined in terms of cochains, consisting of sequences of multilinear functionals, the n -th element of

each sequence being n -linear and a coboundary operator ∂ sending cochains into cochains by sending an n -linear functional to a sum of $(n-1)$ -linear and $(n+1)$ -linear functionals. There is a particular growth condition on the norms of the functionals in a sequence, which we shall describe shortly. So any entire cochain is a finite sum of sequences of the type $\tau = (\tau_0, \tau_1, \tau_2, \dots)$ where τ_i is an $(i+1)$ -linear functional. The space of cochains \mathcal{C} is a direct sum of the subspaces of even and odd cochains, denoted by \mathcal{C}^e and \mathcal{C}^o , which consist of sequences, containing only odd (resp. even)-linear functionals. The coboundary ∂ maps \mathcal{C}^e into \mathcal{C}^o and vice versa. The entire cyclic cohomology is the cohomology of the cochain complex \mathcal{A} with the coboundary operator ∂ , i.e. it consists of those cochains that are mapped to zero by ∂ (called cocycles), that are not in the range of ∂ . Recall that a coboundary operator is always defined to have the property $\partial^2 = 0$ and for that reason $\text{Ran } \partial \subset \text{Ker } \partial$.

The cocycle constructed by Jaffe et al. is defined as follows. One defines multi-linear functionals

$$\tau_{2k}(a_0, \dots, a_{2k}) := \int_{\sigma_{2k}} dt \text{Tr} (\Gamma a_0 e^{-t_1 H} \Gamma da_1 e^{(t_1-t_2)H} \Gamma da_2 e^{(t_2-t_3)H} \dots \Gamma da_{2k} e^{(t_{2k}-1)H})$$

where the region of integration is the $2k$ -simplex

$$\sigma_{2k} = \{(t_1, \dots, t_{2k}) \mid 0 \leq t_1 \leq t_2 \leq \dots \leq 1\}.$$

The integrand is of the form $\text{Tr}(\Gamma A e^{-\beta H})$ with A a bounded operator, and since $e^{-\beta H}$ is trace class, it is well defined. Remembering the explicit expression for Γ , we see that it is what has been called in the mathematical and physics literature a supertrace, i.e. the trace in \mathcal{H}_+ minus the trace in \mathcal{H}_- . (We should mention that H is an even operator or block-diagonal with respect to the decomposition of \mathcal{H} .)

The functionals τ_{2k} are constructed in such a way, as to satisfy :

(i) Growth condition: the function

$$\|\tau\|_z = \sum_{n=0}^{\infty} (n!)^{\frac{1}{2}} \|\tau\|_* z^n$$

is an entire function of z . This is the growth condition of Connes, which is in the definition of entire cochains. The norms $\|\cdot\|_*$ are defined as follows

$$\|\tau_n\|_* := \sup_{\|a_j\|_* = 1} |\tau_n(a_0, a_1, \dots, a_n)|$$

$$\|a\|_* := \|a\| + \|da\| .$$

(ii) Cocyclicity: $\partial \tau_{2k} = 0$, $\forall k$. Let us give the definition of the coboundary operator ∂ at this point[8,9,12]. First two separate coboundaries b and B are defined which act on cochains by sending n -linear functionals to $(n + 1)$ and $(n - 1)$ -linear functionals respectively.

$$(b \tau_n)(a_0, a_1, \dots, a_{n+1}) = \sum_{j=0}^n (-1)^j \tau_n(a_0, \dots, a_j a_{j+1}, \dots, a_{n+1})$$

$$+ (-1)^{n+1} \tau_n(a_{n+1}^\Gamma a_0, a_1, \dots, a_n) .$$

$$(B \tau_n)(a_0, a_1, \dots, a_{n-1}) = \sum_{j=0}^{n-1} (-1)^{(n-1)j} (\tau_n(1, a_{n-j}^\Gamma, \dots, a_{n-1}^\Gamma, a_0, \dots, a_{n-j-1}))$$

$$+ (-1)^{n-1} \tau_n(a_{n-j}^\Gamma, \dots, a_{n-j-1}, 1) .$$

A straightforward calculation shows that $B^2 = b^2 = 0$ and in addition $bB + Bb = 0$.

The coboundary operator through which the cyclic cohomology is defined is now taken to be $\partial = b + B$. It is now obvious that $\partial^2 = 0$ due to the anticommutativity of B and b .

We should point out that the coboundary we have described is the one introduced in [12] and is in fact a generalization of the one introduced by Connes to the case of \mathbb{Z}_2 graded algebras.

The linear functional that was used to define the multilinear functionals τ_n is a regularized supertrace and is analogous to the KMS (Kubo - Martin -Schwinger) functionals, describing the states of quantum statistical mechanics. One major difference is that the former is not positive.

A further abstraction of the minimal set of required structure on a C^* -algebra, so that one can define a nontrivial cocycle in the entire cyclic cohomology, has led Jaffe et al. [15] to introduce the notion of a quantum algebra and the notion of a super-KMS (sKMS) functional on a quantum algebra. A quantum algebra is defined to be a quadruple $(\mathcal{A}, \Gamma, \alpha_t, d)$ consisting of the following objects:

(i) A \mathbb{Z}_2 graded C^* -algebra $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$. The grading is given by an automorphism Γ with square 1, such that on homogeneous elements we have

$$a \rightarrow a^\Gamma = (-1)^{\deg a} a .$$

The grading Γ and the involution $*$ commute.

(ii) A continuous one-parameter group of $*$ -automorphisms α_t , preserving the grading.

$$\alpha_t(a)^* = \alpha_t(a^*) , \quad \alpha_t(a)^\Gamma = \alpha_t(a^\Gamma) .$$

It is a standard result [16] in the theory of C^* -algebras that for every continuous one-parameter $*$ -automorphism group there is a dense subalgebra \mathcal{A}_α of \mathcal{A} on which the automorphism group analytically continues to the whole complex plane, i.e. for each fixed $a \in \mathcal{A}_\alpha$ the function $\alpha_t(a)$ extends to an entire \mathcal{A} -valued function $\alpha_z(a)$.

(iii) d is a superderivation with domain a dense subalgebra \mathcal{A}_d , which contains the subalgebra \mathcal{A}_α .

iv) d^2 , which is an ordinary (even) derivation is the (hermitian) generator of the automorphism group.

$$d^2 = -i \frac{d}{dt} \alpha_t |_{t=0} .$$

Definition: A continuous linear functional ω on \mathcal{A} is called a *super-KMS functional* if it satisfies for any $a, b \in \mathcal{A}_\alpha$

$$\omega(da) = 0 , \quad \omega(ab) = \omega(b^\Gamma \alpha_i(a)) , \quad (1.1)$$

where $\alpha_i(a)$ is the value of $\alpha_z(a)$ at the imaginary unit. We shall in most cases refer to the conditions (1.1) as the strong sKMS property. In most of our work we use only the second assumption, which we shall call the weak sKMS property or some times just the sKMS property.

By comparison with (1.1), the usual KMS functionals satisfy the condition

$$\omega(ab) = \omega(b \alpha_i(a)) ,$$

The prototype of a KMS functional is a regularized trace of bounded operators on a Hilbert space, i.e. $\omega_{KMS} = Tr(\cdot e^{-H})$ with H a self adjoint operator, such that e^{-H} is trace-class and the KMS property can be understood as coming from the cyclicity of the trace ($\omega_{KMS}(ab) = Tr(abe^{-H}) = Tr(be^{-H}a) = Tr(b(e^{-H}ae^H)e^{-H}) = \omega_{KMS}(b\alpha_i(a))$ where α_t is the *-automorphism group defined by e^{iH} , that is $\alpha_t(a) := e^{iH}ae^{-iH}$). At the same time the supertrace was used as a prototype for a general sKMS functional and the second sKMS property can be seen in this case as coming again from the cyclicity of the trace. The major difference between KMS functionals and sKMS functionals is in the fact that the former are required to be positive (the regularized trace with the positive trace-class operator e^{-H} gives a positive functional) while for the latter such a requirement is

not consistent with the prototype. (Recall that the supertrace is a difference of two traces and thus in general it defines a nonpositive functional.)

The Gelfand - Naimark - Segal (GNS) construction provides a way to implement any C^* -algebra with a positive linear functional on it as an algebra of bounded linear operators on some Hilbert space. Thus any such algebra can be regarded as a concrete algebra. The functional is expressed in the GNS construction as a vacuum expectation value with the GNS vacuum Ω , which is a special case of a regularized trace, the trace-class operator being the projection onto Ω . The ability to implement the algebra has a number of very important applications both in mathematics and physics. Therefore the question arises whether one can find a way to implement an abstract Quantum algebra with a sKMS functional on it despite the fact that there is no positivity. One of the reasons to look for a suitable representation of an abstract Quantum algebra is the following: It was shown by Connes [8,9] that the entire cyclic cohomology group $H_c^e(\mathcal{A})$ of the algebra \mathcal{A} couples to the K group of \mathcal{A} , $K_0(\mathcal{A})$, to give a complex number. Let us explain this in some detail. Every element of $K_0(\mathcal{A})$ is (by definition) represented by a projection e in the matrix algebra \mathcal{A}_n over \mathcal{A} for some integer n . In other words

$$e = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

with all the entries $a_{ij} \in \mathcal{A}$ and e is self-adjoint and has the property $e^2 = e$. There is an obvious isomorphism $\mathcal{A}_n \simeq gl(n, \mathcal{C}) \otimes \mathcal{A}$ and therefore each element of \mathcal{A}_n is a finite sum of pairs of the type (m, a) with m a complex $n \times n$ matrix and $a \in \mathcal{A}$. Any multilinear functional τ_k on \mathcal{A} has a natural extension to \mathcal{A}_n

as follows

$$\tau_k^n(m_0 \otimes a_0, \dots, m_k \otimes a_k) := \text{Trace}(m_0 \dots m_k) \tau(a_0, \dots, a_k) .$$

This allows one to define a coupling as we mentioned above. A nontrivial even entire cyclic cocycle $\tau = (\tau_0, \tau_2, \dots)$ represents the cohomology class $[\tau] \in H_c^e$ and a projection $e \in \mathcal{A}_n$ represents the class $[e] \in K_0(\mathcal{A})$. The coupling is defined by

$$\langle [\tau], [e] \rangle := \sum_0^\infty \frac{(-1)^k}{k!} \tau_{2k}^n(e, e, \dots, e) .$$

The result of Connes is that the complex number above is independent of the representatives of the cohomology class and the class in K_0 , i.e. for example if one chooses an entire cocycle τ' which differs from τ by a coboundary, the result of the coupling is the same.

There are different ways to construct entire cyclic cocycles. In the cases that have been known one uses certain additional structure like the structure of the concrete Quantum algebra that was described or the existence of a θ -summable Fredholm module [17]. In these cases the number that one obtains can be related to the index of a certain Fredholm operator. For example in the framework of Jaffe et al. [12] for the simplest case one obtains

$$\langle [\tau], [1] \rangle = \text{Ind } Q .$$

The coupling between the entire cyclic cohomology and K_0 is analogous to the standard situation in differential geometry, where the index of a family of operators is an element of $K(X)$, where X is the space parametrizing the family and one obtains an integer by coupling this to a Chern character. For that reason in fact τ is called a Chern character. The integrality property of the coupling has important physical consequences, like for example the existence of integer topological charges.

When one constructs an entire cocycle using a sKMS functional on an abstract quantum algebra, it is not clear whether and under what conditions there still holds some integrality property. This has been at least part of the motivation for seeking a way to reconstruct an abstract quantum algebra.

Chapter 3 of this thesis is dedicated to the solution of the reconstruction problem. We restrict our considerations to von Neumann algebras and normal functionals on them. We do not know at present how exactly to treat the more general problem of C^* -algebras, since we do not know what condition must be imposed on ω so that it extends to a normal functional on the von Neumann algebra in which we must embed our C^* -algebra. The first section puts together mostly known results and shows how one can implement a von Neumann algebra with a nonpositive self-adjoint normal functional on it. The Hilbert space that one obtains has a natural orthogonal decomposition. However, the representation that one gets is diagonal (or reducible) with respect to this decomposition. This shows that the representation is only suitable for a nongraded algebra, because any sensible representation of a \mathbb{Z}_2 graded algebra should map odd elements to odd elements. The construction of a representation for the case of a graded algebra \mathcal{A} with a self-adjoint sKMS functional ω on it is carried out in section 3.2, which begins with a study of the properties of sKMS functionals. In fact we only assume the weak sKMS property. Two important results are the following. If ω is (weakly) sKMS, then the unique positive functional $|\omega|$, given by the polar decomposition of ω is a KMS functional. Every even element, which is in the left kernel L_{\pm} of either of the two positive functionals ω_{\pm} in the unique Jordan decomposition of ω , is mapped by the involution $*$ to an element again in the same left kernel. By contrast, an odd

element which is in the left kernel L_- is mapped to an element in the left kernel L_+ and vice versa.

These properties allow us to define a new type of representation, which unlike the representation from section 3.1, respects the grading of the algebra. Let us agree to call such a representation graded. The two representations are unitarily equivalent, but not unitarily equivalent under intertwining maps that respect the \mathbb{Z}_2 grading of the Hilbert spaces. In both cases the functional ω is expressed as a graded vacuum expectation value

$$\omega(a) = (\Omega, \Gamma\pi(a)\Omega) = (\Omega, \Gamma\pi'(a)\Omega) .$$

The graded representation π' we have constructed is unitarily equivalent to any graded cyclic representation π'' with vacuum vector Ω'' for which ω is expressed as above as a graded vacuum expectation value. The intertwining map can be chosen to respect the grading of the Hilbert spaces.

In chapter 4 we continue our study of sKMS functionals by reversing the point of view. We investigate the question whether the sKMS property can arise naturally and not be imposed, in analogy to the well known theory of Tomita and Takesaki [18], which treats the case of positive functionals. According to their results, for each faithful normal positive functional on a von Neumann algebra there is a canonical one-parameter *-automorphism group, called the modular group corresponding to this functional and the functional is KMS with respect to this group. Furthermore this is the unique group with respect to which it is KMS. In the present work we show that a completely analogous result holds for the case of a nonpositive self-adjoint functional and the sKMS property arises in the same natural way as does the KMS condition in the standard case. The main point is to realize that when the functional

is faithful the results from chapter 3 on the properties of sKMS functionals give us a way to specify the even elements as precisely those for which it is true that if they are in one of the two left kernels, their adjoints are in the same left kernel, while the odd elements are the ones for which the converse is true. So the idea now is to let the \mathbb{Z}_2 grading be also induced from the properties of the nonpositive functional. Thus we show that to each faithful normal self-adjoint functional ω there corresponds canonical \mathbb{Z}_2 grading and a *-automorphism group, commuting with the grading, such that ω is sKMS with respect to them. Finally we show that these are the unique automorphism group and \mathbb{Z}_2 grading, commuting with each other, with respect to which ω is sKMS.

The final chapter is dedicated to the problem of implementing the superderivation d which is part of the structure of a quantum algebra. We argue that the operator, which first comes to mind to construct and which by definition annihilates the GNS vacuum, is not a very interesting object from the point of view of both physics and mathematics. We find another operator, which also implements the superderivation, but does not annihilate the GNS vacuum. This is analogous to the situation when one has a generator of a spontaneously broken symmetry. Unfortunately this operator may not always exist. We show that it exists if a certain projection is in the domain of the superderivation d and in this case it turns out that the superderivation is equivalent to an inner superderivation. Obviously, far from all superderivations on a dense subalgebra are inner. In the general case we are always able to define a quadratic form on $\mathcal{H} \times \mathcal{H}$, but in order for it to come from an operator we have to assume the existence of a certain cyclic vector, different from Ω . It may turn out that such a vector exists under some reasonable assump-

tions, like d being hermitian, but we do not know the answer yet. Finally we show that the hermiticity of the superderivation as defined in [15] leads to a symmetric quadratic form.

CHAPTER II

PRELIMINARIES

The purpose of this chapter is to provide the reader with the necessary mathematical background, needed to understand the rest of this dissertation, as well as to give a brief review of some interesting new developments in mathematical physics and mathematics, to which our work has relevance.

2.1. C^* -algebras and von Neumann algebras

The theory of C^* -algebras or more generally Banach algebras historically arose as an abstract algebraic approach to the theory of bounded linear operators on Hilbert spaces. The systematic study of operator algebras was begun in the thirties mainly by Murray and von Neumann [19-22]. They dedicated their studies mostly to the type of algebras, that are now called von Neumann algebras.

The algebra $L(\mathcal{H})$ of all bounded linear operators on a Hilbert space \mathcal{H} can be equipped with different topologies. One of the most often used is the norm topology. With this topology $L(\mathcal{H})$ becomes a Banach space. There is a natural involution $*$ on $L(\mathcal{H})$ defined by taking the adjoint of the operator. It has the properties

$$\begin{aligned} (a^*)^* &= a, & (a + b)^* &= a^* + b^*, \\ (\lambda a)^* &= \bar{\lambda} a^*, & (ab)^* &= b^* a^*. \end{aligned} \tag{2.1}$$

In 1943 Gelfand and Naimark [23] gave the axioms for an abstract C^* -algebra and showed that every such algebra is isomorphic to an algebra of bounded operators on a Hilbert space.

A *Banach algebra* is a complex associative algebra, equipped with a norm $\| \cdot \|$

with respect to which it is a Banach space and the norm satisfies

$$\|ab\| \leq \|a\| \|b\| .$$

A C^* -algebra is a Banach algebra with an involution as in (2.1), satisfying

$$\|a^*a\| = \|a\|^2 .$$

Since $\|a^*a\| \leq \|a\| \|a^*\|$, it follows that $\|a\| \leq \|a^*\|$ for every a and therefore $\|a\| = \|a^*\|$.

We shall only be interested in the case when the algebra has a unit.

Let \mathcal{A} be a C^* -algebra. An element $a \in \mathcal{A}$ is called *self-adjoint* if $a^* = a$. Every element $a \in \mathcal{A}$ can be written as a linear combination of the self-adjoint elements $\frac{1}{2}(a + a^*)$ and $-\frac{1}{2}i(a - a^*)$.

The element u is called *unitary* if $uu^* = u^*u = \mathbf{1}$. From the properties of the norm it follows immediately that $\|u\| = 1$.

The *spectrum* of a in \mathcal{A} ($Sp(a)$) is defined as the set of complex numbers λ for which $\lambda\mathbf{1} - a$ is not invertible in \mathcal{A} . For normal elements (i.e. such that commute with their adjoints) it follows easily, that the spectral radius $\rho(a)$ is equal to $\|a\|$. At that point it is not hard to show that the spectrum of an unitary element is contained in the unit circle and that of a self-adjoint element is real. The first follows from the fact that if $\lambda \in Sp(u)$, then $\lambda^{-1} \in Sp(u^*)$. If one supposes that $|\lambda| < 1$ then $|\lambda^{-1}| > 1$ which is impossible because u^* is a normal element with norm one. The proof for a self-adjoint element is reduced to the one for unitary elements by exponentiating the self-adjoint element.

If \mathcal{A} is a commutative C^* -algebra (or more generally Banach algebra), the *spectrum* or *carrier space* $\hat{\mathcal{A}}$ of \mathcal{A} is defined as the set of nonzero homomorphisms

(also called multiplicative functionals) of \mathcal{A} into \mathbb{C} . It is a closed subspace of the unit ball of the dual of \mathcal{A} in the weak* topology. As such it is a compact Hausdorff space. (When \mathcal{A} has no unit $\hat{\mathcal{A}}$ is just locally compact.) The *Gelfand transform* is the homomorphism from \mathcal{A} into $C_0(\hat{\mathcal{A}})$ given by $\hat{a}(t) = t(a)$, where t is an arbitrary point in $\hat{\mathcal{A}}$, i.e. a multiplicative functional.

The *Gelfand–Naimark representation theorem* asserts that the Gelfand transform is a *-preserving isometry of \mathcal{A} onto $C_0(\hat{\mathcal{A}})$.

The following facts relate the spectrum of a normal element to the spectrum of the commutative C^* -subalgebra, generated by it. If \mathcal{B} is commutative and $a \in \mathcal{B}$, then

$$Sp(a) = \{\hat{a}(\phi) \mid \phi \in \hat{\mathcal{B}}\}.$$

Suppose that \mathcal{B} is the C^* -subalgebra generated by a normal element a . (That is, \mathcal{B} is the smallest subalgebra, containing a , a^* and the identity.) The map \hat{a} is a homeomorphism from the carrier space $\hat{\mathcal{B}}$ into the spectrum of a in \mathcal{B} .

With \mathcal{A} a unital (=having an identity) C^* -algebra, \mathcal{B} – a unital C^* -subalgebra of \mathcal{A} and $a \in \mathcal{B}$, the spectrum of a in \mathcal{B} coincides with the spectrum of a in \mathcal{A} .

All this shows that for normal operators one can use standard functional calculus techniques and define functions of normal elements as elements of the algebra.

A self-adjoint element a is called *positive* if $Sp(a) \geq 0$. Using the functional calculus one can split every self-adjoint element into a difference of two positive ones. Also if a is self-adjoint, then the element $u = a + i(1 - a^2)^{\frac{1}{2}}$ can be shown to be unitary. Since $a = \frac{1}{2}(u + u^*)$, we see that any element in a C^* -algebra is a linear combination of four unitary ones.

It is standard to denote by \mathcal{A}_{sa} the set of self-adjoint elements in \mathcal{A} and by \mathcal{A}_+ the set of positive ones.

One useful application of spectral theory is the proof of the fact, that the identity $\mathbf{1}$ is an extreme point in the unit ball. Indeed, suppose first that $\mathbf{1} = \frac{1}{2}(a + b)$ with a and b self-adjoint. It is obvious that a and b commute, since they sum up to $\mathbf{1}$. Take the commutative C^* -algebra, generated by a , b and $\mathbf{1}$ and notice that the Gelfand transforms of a and b are functions with absolute values less than or equal to one while $\hat{\mathbf{1}}$ is the constant function one and the equality above is possible only when $a = b = \mathbf{1}$. The general case $\mathbf{1} = \frac{1}{2}(a + b)$ reduces to the one above by taking the adjoint of this equality and summing up the two.

A very important and not trivial result [24] is the statement that a is positive if and only if $a = b^*b$ for some b . In one direction this is trivial – just use functional calculus to define the square root of a . In the other direction the proof goes as follows:

Since a is self-adjoint, it has the decomposition $a = a_+ - a_-$ into two positive elements.

$$(ba_-^{\frac{1}{2}})^*(ba_-^{\frac{1}{2}}) = a_-^{\frac{1}{2}}(a_+ - a_-)a_-^{\frac{1}{2}} = -a_-^2 \in -\mathcal{A}_+.$$

On the other hand, denoting $ba_-^{\frac{1}{2}} = x + iy$ with $x, y \in \mathcal{A}_{sa}$, the product in reverse order gives

$$(ba_-^{\frac{1}{2}})(ba_-^{\frac{1}{2}})^* = 2(x^2 + y^2) - (ba_-^{\frac{1}{2}})^*(ba_-^{\frac{1}{2}}) \in \mathcal{A}_+$$

where one has to use the fact that the sum of positive elements is positive. Since the spectrum of a product is known apart from zero to be independent of the order, it follows that $Sp(a_-^{\frac{1}{2}}) = 0$ and from this $a_- = 0$.

Linear functionals on C^* -algebras play an essential role in the theory. In particular the existence always of a faithful positive functional will show that any abstract C^* -algebra is $*$ -isomorphic to an algebra of bounded operators. We shall use the notation \mathcal{A}^* for the dual of \mathcal{A} . For each $\phi \in \mathcal{A}^*$ an *adjoint* ϕ^* is defined by $\phi^*(a) = \overline{\phi(a^*)}$. The functional ϕ is called *self-adjoint* if $\phi^* = \phi$. As for elements of the algebra, it is immediate that every bounded functional is the sum of a self-adjoint and a skew-adjoint functional.

A (not a priori bounded) linear functional ϕ is called *positive* if $\phi(\mathcal{A}_+) \subset \mathbb{R}_+$. Obviously positive functionals are self-adjoint. They turn out to be always bounded. Indeed, taking a sequence $\{a_i\}$, $a_i \in \mathcal{A}_+^1$, where the last notation stands for the positive elements in the unit ball, one obtains for any sequence of nonnegative real numbers $\{\lambda_i\}$, such that $\sum_{i=1}^{\infty} \lambda_i < \infty$ a convergent series $\sum_{i=1}^{\infty} \lambda_i a_i = a$. We have

$$\sum_{i=1}^n \lambda_i \phi(a_i) = \phi\left(\sum_{i=1}^n \lambda_i a_i\right) \leq \phi(a),$$

where the positivity of each term has been used. Hence $\sum_{i=1}^{\infty} \lambda_i \phi(a_i) < \infty$. Since this holds for any summable sequence of nonnegative numbers λ_i it follows that the sequence $\phi(a_i)$ is bounded. From here the boundedness of ϕ follows by representing an arbitrary element in the unit ball as a linear combination of four positive ones.

There is a *Cauchy-Schwarz* inequality for positive functionals. We shall call it just the Schwarz inequality and shall use it numerous times in the sequel. It states, that if ϕ is positive, than for any $a, b \in \mathcal{A}$

$$|\phi(a^*b)|^2 \leq \phi(a^*a)\phi(b^*b).$$

The proof is the same as for an inner product.

A bounded linear functional ϕ is positive if and only if $\|\phi\| = \phi(\mathbf{1})$. To prove this in one of the directions, assume that ϕ is positive and take a self-adjoint element a , $\|a\| \leq 1$. It follows then from spectral theory that $\mathbf{1} - a \geq 0$. Thus $\phi(\mathbf{1} - a) \geq 0$ or $\phi(a) \leq \phi(\mathbf{1})$. For an arbitrary element b the Schwarz inequality can be used to obtain

$$|\phi(b)|^2 \leq \phi(\mathbf{1})\phi(b^*b) \leq \phi(\mathbf{1})^2.$$

Since always $\phi(\mathbf{1}) \leq \|\phi\|$, it follows that $\|\phi\| = \phi(\mathbf{1})$. To prove the reverse direction, assume that $\|\phi\| = \phi(\mathbf{1}) = 1$ and let $a \in \mathcal{A}_+$ be such that $\phi(a)$ is not nonnegative. Then there is a closed disk $|z - z_0| \leq \rho$ in \mathbb{C} , which contains $Sp(a)$ but not $\phi(a)$. The spectrum of the element $a - z_0$ is contained in the disk $|z| \leq \rho$ and therefore $\|a - z_0\| \leq \rho$. We have

$$|\phi(a) - z_0| = |\phi(a) - z_0\phi(\mathbf{1})| = |\phi(a - z_0)| \leq \|\phi\| \|a - z_0\| \leq \rho,$$

which contradicts our assumption that $\phi(a)$ was outside the disk.

Two positive functionals ϕ and ψ are said to be *orthogonal* (denoted $\phi \perp \psi$) if they satisfy

$$\|\phi - \psi\| = \|\phi\| + \|\psi\|.$$

A useful alternative equivalent definition is the following:

For every $\epsilon > 0$ there is an element $b \in \mathcal{A}_+^1$ such that $\phi(\mathbf{1} - b) < \epsilon$ and $\psi(b) < \epsilon$.

The proof goes as follows:

Assume first orthogonality and notice that since $\phi - \psi$ is self-adjoint, the supremum over all elements in the unit ball in calculating the norm can be replaced by a supremum over the self-adjoint elements in the unit ball (because $\phi - \psi$ is zero on the skew-adjoint part of the elements). Hence there is $a \in \mathcal{A}_{sa}^1$ such that

$$\phi(a) - \psi(a) + \epsilon \geq \|\phi - \psi\|.$$

It follows that

$$\phi(a) - \psi(a) + \epsilon \geq \|\phi\| + \|\psi\| = \phi(\mathbf{1}) + \psi(\mathbf{1})$$

and therefore $\phi(\mathbf{1}-a) + \psi(\mathbf{1}+a) < \epsilon$. Since $0 \leq \mathbf{1}-a \leq 2 \cdot \mathbf{1}$ and $0 \leq \mathbf{1}+a \leq 2 \cdot \mathbf{1}$, one can choose $b := \frac{1}{2}(\mathbf{1}+a)$ which is in \mathcal{A}_+^1 . From the positivity of ϕ and ψ it follows that $\phi(\mathbf{1}-a) < \epsilon$ and from this $\phi(\mathbf{1}-b) = \frac{1}{2}\phi(\mathbf{1}-a) < \epsilon$. Analogously $\psi(b) = \frac{1}{2}\psi(\mathbf{1}+a) < \epsilon$. To prove the reverse implication, notice that if $\phi(\mathbf{1}-b) < \epsilon$ and $\psi(b) < \epsilon$, then

$$\begin{aligned} \|\phi\| + \|\psi\| &= \phi(\mathbf{1}) + \psi(\mathbf{1}) = \phi(2b - \mathbf{1}) + \psi(\mathbf{1} - 2b) + 2(\phi(\mathbf{1}-b) + \psi(b)) \\ &\leq (\phi - \psi)(2b - \mathbf{1}) + 4\epsilon \leq \|\phi - \psi\| + 4\epsilon, \end{aligned}$$

where $\|2b - \mathbf{1}\| \leq 1$ has been used. Since the above holds for arbitrary ϵ and since $\|\phi - \psi\| \leq \|\phi\| + \|\psi\|$ always, it follows that $\|\phi - \psi\| = \|\phi\| + \|\psi\|$.

For von Neumann algebras and normal functionals (see below) there is in fact an extremal element b as above, for which $\phi(\mathbf{1}-b) = 0$, $\psi(b) = 0$. This fact is a crucial one, and its consequences make our constructions in the next two chapters possible.

The following result of Grothendieck [25] gives what is known as the *Jordan decomposition* of a bounded self-adjoint functional. For every such functional ϕ there is a unique pair of positive functionals ϕ_{\pm} such that $\phi = \phi_+ - \phi_-$ and $\phi_+ \perp \phi_-$.

The proof of existence goes through several steps. Let S denotes the set of positive functionals of norm one, also called states (because they define states of a quantum statistical system). Since, as we have seen, the norm is additive for positive functionals, a convex combination of states is a state again and thus S is convex.

The self-adjoint part of the unit ball in \mathcal{A}^* is the same, as we have pointed out, as the unit ball of $(\mathcal{A}_{sa})^*$. (Note that it is easy to see that \mathcal{A}_{sa} is a closed subspace of \mathcal{A} and as such it is a Banach space.) The unit ball of the dual of any Banach space is weak* compact [5]. We shall denote the positive part of the unit ball of $(\mathcal{A}_{sa})^*$ by Q . It is obvious that Q is weak* closed in $(\mathcal{A}_{sa})^*$ and thus weak* compact too. Finally

$$S = \{\phi \in Q \mid \phi(\mathbf{1}) = 1\}$$

and thus S is also weak* compact.

The unit ball of $(\mathcal{A}_{sa})^*$ is the convex span of S and $-S$ [16]. Using this fact, an arbitrary self-adjoint functional ϕ can be represented as $\phi = \alpha\phi_1 - (1 - \alpha)\phi_2$ with ϕ_1 and ϕ_2 positive and with norms less than or equal to $\|\phi\|$. Putting $\phi_+ = \alpha\phi_1$ and $\phi_- = (1 - \alpha)\phi_2$ one observes that

$$\|\phi_+\| + \|\phi_-\| = \alpha\|\phi_1\| + (1 - \alpha)\|\phi_2\| \leq \|\phi\| ,$$

which gives $\|\phi_+\| + \|\phi_-\| = \|\phi_+ - \phi_-\|$.

To prove uniqueness, assume the existence of two orthogonal decompositions $\phi = \phi_+ - \phi_- = \phi'_+ - \phi'_-$. There is an element $a \in \mathcal{A}$, such that $\phi_+(1 - a) < \epsilon$ and $\phi_-(a) < \epsilon$. Then

$$\phi'_+(a) \geq \phi'_+(a) - \phi'_-(a) = \phi_+(a) - \phi_-(a) > \phi_+(\mathbf{1}) - 2\epsilon .$$

Analogously $\phi'_-(1 - a) > \phi_-(\mathbf{1}) - 2\epsilon$, so that

$$\phi'_+(a) + \phi'_-(1 - a) > \|\phi_+\| + \|\phi_-\| - 4\epsilon = \|\phi'_+\| + \|\phi'_-\| - 4\epsilon .$$

It follows that $\phi'_+(1 - a) + \phi'_-(a) < 4\epsilon$. For an arbitrary b one can write

$$\begin{aligned} \phi_+(b) - \phi'_+(b) &= \phi_+(ba) - \phi'_+(ba) + \phi_+(b(1 - a)) - \phi'_+(b(1 - a)) \\ &= \phi_-(ba) - \phi'_-(ba) + \phi_+(b(1 - a)) - \phi'_-(b(1 - a)) . \end{aligned}$$

Using the Schwarz inequality it is now easy to put a bound on each summand and to show

$$|\phi_+(b) - \phi'_+(b)| \leq \|b\| \|\phi_+ - \phi_-\|^{\frac{1}{2}} 6\epsilon^{\frac{1}{2}},$$

and since ϵ is arbitrary, $\phi_+ = \phi'_+$.

Another fundamental result in the theory of C^* -algebras is the *Gelfand–Naimark–Segal (GNS) construction* [23], to which we now turn.

A *representation* of the C^* -algebra \mathcal{A} is a pair (π, \mathcal{H}) where \mathcal{H} is a Hilbert space and π is a $*$ -homomorphism of \mathcal{A} into $L(\mathcal{H})$. (A $*$ -homomorphism is a homomorphism, commuting with the involution $*$, i.e. $\pi(a^*) = (\pi(a))^*$.)

A vector $\Omega \in \mathcal{H}$ is called a *cyclic vector* for the representation π if $\pi(a)\Omega$, $a \in \mathcal{A}$ form a dense set in \mathcal{H} . We shall often refer to a cyclic vector as a vacuum vector. A representation is called cyclic if it has a cyclic vector.

The following theorem is a statement of the results of the GNS construction.

For each positive functional ϕ on a C^ -algebra \mathcal{A} there is a cyclic representation $(\pi_\phi, \mathcal{H}_\phi)$ with a cyclic vector Ω_ϕ such that $\phi(a) = (\Omega_\phi, \pi_\phi(a)\Omega_\phi)$ for any $a \in \mathcal{A}$.*

To prove this, define the *left kernel* of ϕ as the set

$$L_\phi = \{a \in \mathcal{A} \mid \phi(a^*a) = 0\}.$$

The set L_ϕ is a left ideal due to the Schwarz inequality. Let $\eta_0: \mathcal{A} \rightarrow \mathcal{A}/L_\phi$ denote the canonical quotient map. The sesquilinear form $(\eta_0(a), \eta_0(b)) := \phi(a^*b)$ gives an inner product on \mathcal{A}/L_ϕ . (We have modded out the subspace on which the form gives zero.) The completion of \mathcal{A}/L_ϕ with respect to this inner product is denoted by \mathcal{H}_ϕ and $\eta := i \circ \eta_0$ where i is the imbedding of \mathcal{A}/L_ϕ into \mathcal{H}_ϕ . The left regular representation of \mathcal{A} onto itself induces a well defined action of \mathcal{A}

on the quotient space. This is so because L_ϕ is a left ideal and therefore elements from the subspace L_ϕ are mapped to the same subspace upon multiplication on the left. Therefore there is a well defined linear operator $\pi_\phi(a)\eta(b) := \eta(ab)$ on the dense domain $\eta(\mathcal{A}) \subset \mathcal{H}_\phi$. Furthermore

$$\begin{aligned} \|\pi_\phi(a)\eta(b)\|^2 &= \|\eta(ab)\|^2 = \phi(b^*a^*ab) \\ &\leq \|a\|^2 \phi(b^*b) = \|a\|^2 \|\eta(b)\|^2, \end{aligned}$$

where the inequality $a^*a \leq \|a\|^2 \mathbf{1}$ has been used as well as the fact that it is preserved upon multiplication by b^* and b from the left and right respectively.

The inequality above shows that $\pi_\phi(a)$ is bounded for any a and therefore extends to a bounded operator on all of \mathcal{H}_ϕ . We show now that $\pi_\phi(a^*) = (\pi_\phi(a))^*$. Indeed

$$\begin{aligned} (\eta(a), \pi_\phi(b)\eta(c)) &= \phi(a^*b.c.) = \phi((b^*a)^*c) \\ &= (\eta(b^*a), \eta(c)) = (\pi_\phi(b^*)\eta(a), \eta(c)). \end{aligned}$$

Finally it is obvious from the definition of \mathcal{H}_ϕ that the vector $\Omega_\phi := \eta(\mathbf{1})$ is cyclic for the representation π_ϕ . (In general every C^* -algebra, which has no unit, has an approximate unit u_λ and a cyclic vector can still be defined as the limit of $\eta(u_\lambda)$.)

The state space (or space of positive functionals of norm one) of a C^* -algebra was denoted by S . A subset F of S is said to be *separating* for \mathcal{A} if $\phi(a) = 0, \forall \phi \in F$ implies $a = 0$ for any positive element a of the algebra. S itself is separating for, as already stated, the unit ball of $(\mathcal{A}_{sa})^*$ is the convex span of S and $-S$ and if there was a positive element $a \in \mathcal{A}_+$ for which $\phi(a) = 0, \forall \phi \in S$, then it would follow $\phi(a) = 0$ for any $\phi \in (\mathcal{A}_{sa})^*$ which is impossible unless $a = 0$.

For each $\phi \in S$ one can define the GNS representation $(\pi_\phi, \mathcal{H}_\phi, \Omega_\phi)$ associated with ϕ . Define the direct sum of all such representations $\pi_S := \bigoplus_{\phi \in S} \pi_\phi$ acting on the Hilbert space $\mathcal{H}_S := \bigoplus_{\phi \in S} \mathcal{H}_\phi$.

The representation π_S is faithful due to the fact that S is separating. Indeed, suppose that some positive a is in the kernel of π_S . This means that π_ϕ is the zero operator for any $\phi \in S$. Then

$$\phi(a) = (\Omega_\phi, \pi_\phi \Omega_\phi) = 0$$

for each $\phi \in S$, which implies $a = 0$.

This last theorem shows that every C^* -algebra is $*$ -isomorphic to an algebra of bounded operators on a Hilbert space.

If \mathcal{A} and \mathcal{B} are two C^* -algebras, the $*$ -homomorphisms from \mathcal{A} into \mathcal{B} are often called *morphisms* in agreement with the categorical language.

Each morphism π is norm decreasing. The proof uses the connection that exists between the norm and the spectrum of self-adjoint elements. It is easy to see that the resolvent set for an element a is contained in the resolvent set for $\pi(a)$ and therefore $Sp_{\mathcal{B}}(a) \subset Sp_{\mathcal{A}}(a)$. (Again we restrict the considerations to unital algebras for simplicity, although the proof goes through for algebras without identity too.) We have

$$\begin{aligned} \|\pi(a)\|^2 &= \|(\pi(a))^* \pi(a)\| = \|\pi(a^*a)\| \\ &= \rho(\pi(a^*a)) \leq \rho(a^*a) = \|a^*a\| = \|a\|^2. \end{aligned}$$

In particular, every $*$ -automorphism of \mathcal{A} is an isometry, because the homomorphisms in both directions have to be norm decreasing.

Historically von Neumann algebras were defined as special C^* -subalgebras of $L(\mathcal{H})$. For any subset $M \subset L(\mathcal{H})$ one can define the *commutant* of M

$$M' := \{ a \in L(\mathcal{H}) \mid ab = ba \forall b \in M \} .$$

Similarly one can define the double commutant M'' as the commutant of M' and

so on. Note that $M \subset M''$ since any element of M commutes with any one from M' and thus has to be in M'' .

Let \mathcal{A} be a C^* -subalgebra of $L(\mathcal{H})$. \mathcal{A} is called *von Neumann algebra* if $\mathcal{A} = \mathcal{A}''$. Since the identity operator commutes with everything, it is obvious, that according to this definition, any von Neumann algebra contains it. The well known Double commutant (or Bicommutant) theorem [26] of von Neumann relates the algebraic condition to a topological one. We give the statement and proof of the theorem.

Let \mathcal{A} be a C^* -subalgebra of $L(\mathcal{H})$, containing the identity operator. Then the following conditions are equivalent

- (i) $\mathcal{A} = \mathcal{A}''$.
- (ii) \mathcal{A} is weakly closed.
- (iii) \mathcal{A} is strongly closed.

By the weak and strong topologies it is meant the weak and strong operator topologies on $L(\mathcal{H})$. The implication (i) \Rightarrow (ii) is shown as follows:

Let a be in the weak closure of \mathcal{A}'' , i.e. there is a net $a_\alpha \in \mathcal{A}''$, $\lim(x, a_\alpha y) = (x, ay)$ for any $x, y \in \mathcal{H}$. Then it is also true that $\lim(x, a_\alpha by) = (x, aby)$ and $\lim(x, ba_\alpha y) = (x, bay)$. In particular, choosing $b \in \mathcal{A}'$ we get

$$(x, [a, b]y) = \lim(x, [a_\alpha, b]y) = 0,$$

from which follows that $[a, b] = 0$, $\forall b \in \mathcal{A}'$ and therefore $a \in \mathcal{A}''$. The implication (ii) \Rightarrow (iii) is immediate from the fact that the strong topology is stronger than the weak. Now we prove (iii) \Rightarrow (i):

It is sufficient to show, that if $a'' \in \mathcal{A}''$, $x_1, \dots, x_n \in \mathcal{H}$ and $\epsilon > 0$ are given, there

exists an $a \in \mathcal{A}$ such that

$$\|(a'' - a)x_i\| < \epsilon, \quad \forall 1 \leq i \leq n.$$

(This last can be easily seen to mean that a'' can be strongly approximated from \mathcal{A} , and since it is strongly closed, it follows that $a'' \in \mathcal{A}$.) First let us prove the above for the case $n = 1$. Let p be the projection on the closure of the subspace $\mathcal{A}x_1$. Since $pap = ap$ for each a in \mathcal{A} , by conjugating this, one can also show $pap = pa$ and hence $p \in \mathcal{A}'$. Taking $a'' \in \mathcal{A}$ we get $a''p = pa''$ and therefore $a''x_1 = a''px_1 = pa''x_1 \in p\mathcal{H}$. Since $p\mathcal{H}$ was the closure of $\mathcal{A}x_1$, there is an element $a \in \mathcal{A}$, such that $\|(a'' - a)x_1\| < \epsilon$ for each $\epsilon > 0$. Finally, for an arbitrary n take n copies of \mathcal{H} : $\tilde{\mathcal{H}} := \bigoplus_{1 \leq i \leq n} \mathcal{H}_i$ and consider the algebra $\tilde{\mathcal{A}} := \bigoplus_{1 \leq i \leq n} \mathcal{A}_i$ acting on $\tilde{\mathcal{H}}$. Apply the first part of the proof for the vector $x_1 \oplus \cdots \oplus x_n$ to obtain the desired result. With this the proof of the double commutant theorem is complete.

The theory of von Neumann algebras is quite richer than that of just C^* -algebras due to the fact that they are closed with respect to a weaker than the norm topology. The next few propositions illustrate this richer structure.

The following conditions on a linear functional ϕ on $L(\mathcal{H})$ are equivalent:

- (i) $\phi(a) = \sum_{k=1}^n (x_k, ay_k)$ for some $x_k, y_k \in \mathcal{H}$ and all $a \in L(\mathcal{H})$.
- (ii) ϕ is weakly continuous.
- (iii) ϕ is strongly continuous.

The implications (i) \Rightarrow (ii) \Rightarrow (iii) are evident. To prove (iii) \Rightarrow (i) notice that the strong continuity of ϕ implies the existence of vectors x_1, \dots, x_n such that $\text{Max} \|ax_k\| \leq 1$ implies $|\phi(a)| \leq 1$ for all a in $L(\mathcal{H})$. Renormalizing the vectors

x_i by a coefficient $\frac{1}{n^2}$ we can write

$$|\phi(a)| \leq \left(\sum_{k=1}^n \|ax_k\|^2 \right)^{\frac{1}{2}} .$$

Again one takes the Hilbert space $\tilde{\mathcal{H}} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ of n copies of \mathcal{H} . Define $x = x_1 \oplus \cdots \oplus x_n$ and denote by $\pi(a)$ the image of a under the natural map from $L(\mathcal{H})$ into the direct sum $L(\mathcal{H}) \oplus \cdots \oplus L(\mathcal{H})$. The functional $\psi(\pi(a)x) = \phi(a)$ is a linear functional on a closed subspace of $\tilde{\mathcal{H}}$, with the property $|\psi(\pi(a)x)| \leq \|\pi(a)x\|$. It follows from the Riesz-Frechet theorem that there is a vector $y = y_1 \oplus \cdots \oplus y_n$, such that

$$\phi(a) = (y, \pi(a)x) = \sum_{i=1}^n (y_i, ax_i) .$$

The set $T(\mathcal{H})$ of trace class operators on a separable Hilbert space is well known to be a Banach space with respect to the norm $Tr(|\cdot|)$. The dual of $T(\mathcal{H})$ is $L(\mathcal{H})$, the coupling being given by the form $(a, b) \rightarrow Tr(ab)$. Thus on $L(\mathcal{H})$ is defined a topology, coming from it being the dual of a Banach space – the weak* topology. In the theory of von Neumann algebras it is accepted to call this topology the *σ -weak topology* or also the *ultra-weak topology*. In other words this is the topology, associated with the family of semi-norms $a \rightarrow |Tr(ab)|$, $a \in L(\mathcal{H})$, $b \in T(\mathcal{H})$. It is not difficult to show that this family of semi-norms is equivalent to the family of semi-norms given by

$$a \rightarrow \sum_n (x_n, ay_n) , \quad x_n , y_n \in \mathcal{H} ,$$

where $\sum_n \|x_n\|^2 < \infty$ and $\sum_n \|y_n\|^2 < \infty$.

Apparently the σ -weak topology is stronger than the weak topology. (The latter is associated to the subfamily of semi-norms as above with $x_i = y_i = 0$, $i \geq 2$.) It

is not difficult to see that the two topologies coincide on the unit ball of $L(\mathcal{H})$ or on any norm bounded subset. Indeed in this case we can truncate the infinite sum above to some n and put a uniform bound on the remainder.

Kaplanski's density theorem states that if \mathcal{B} is a C^* -subalgebra of $L(\mathcal{H})$ with a strong closure \mathcal{A} (the latter is a von Neumann algebra if $1 \in \mathcal{B}$), then the unit ball \mathcal{B}^1 of \mathcal{B} is strongly dense in the unit ball \mathcal{A}^1 of \mathcal{A} . Also the self-adjoint part of the unit ball \mathcal{B}_{sa}^1 is strongly dense in \mathcal{A}_{sa}^1 . We shall not give the proof of this theorem here.

Now take \mathcal{B} to be a unital C^* -subalgebra of $L(\mathcal{H})$ which is σ -weakly closed. It is known that the unit ball of $L(\mathcal{H})$ is weakly closed and therefore also σ -weakly closed because the two topologies coincide on bounded sets. Therefore the unit ball of \mathcal{B} is also σ -weakly closed and weakly closed. Now take \mathcal{A} to be the strong closure (=weak closure) of \mathcal{B} . Every element of \mathcal{A} is a multiple of an element from the unit ball. The latter is approximated weakly, according to Kaplanski's theorem by elements from the unit ball of \mathcal{B} . But because this is weakly closed, it follows that the element is actually in \mathcal{B} . In other words \mathcal{B} coincides with its strong closure and is thus a von Neumann algebra. Analogous arguments can be used in the reverse direction and thus it follows that a C^* -subalgebra of $L(\mathcal{H})$ is a von Neumann algebra if and only if it is σ -weakly closed.

An important corollary is that each (concrete) von Neumann algebra is isomorphic as a Banach space to the dual of a Banach space.

It is this property that gives the most elegant way to define abstract von Neumann algebras. According to a *theorem of Sakai* [27] every C^* -algebra which is the dual of a Banach space has a faithful representation as a von Neumann algebra of

operators.

Before we present the proof of this very important theorem, we need a few more definitions and results.

A positive linear map ρ between two (concrete) von Neumann algebras \mathcal{A} and \mathcal{B} is called *normal* if for each bounded, monotone increasing net $\{a_i\}$ in \mathcal{A}_{sa} with a strong limit a , the net $\{\rho(a_i)\}$ increases to $\rho(a)$ in \mathcal{B}_{sa} . (It is a simple consequence of the Double commutant theorem that bounded monotone nets always have strong limits.)

The importance of normal maps should become clear from the following *Theorem* which is stated without a proof:

A C^* -subalgebra \mathcal{A} of $L(\mathcal{H})$ is a von Neumann algebra if and only if \mathcal{A}_{sa} coincides with the set of elements that are strong limits of monotone increasing nets from \mathcal{A}_{sa} .

A *normal functional* ϕ is by definition a bounded linear functional which maps every bounded monotone increasing net $\{a_i\}$ in \mathcal{A}_{sa} to the net $\{\phi(a_i)\}$, convergent to $\phi(a)$.

It turns out that (a) ϕ being normal is equivalent to (b) ϕ being weakly continuous on the unit ball, to (c) ϕ being σ -weakly continuous and to (d) ϕ being represented as a trace with some fixed trace-class operator.

One can take ϕ to be positive. The general case then follows easily. The first implication follows from a relatively simple lemma, stating that if ϕ is a normal functional, there is a set p_i of pairwise orthogonal projections with $\sum p_i = 1$ such that each of the functionals $\phi(\cdot p_i)$ is weakly continuous. This being so, one can take a projection p close enough to 1 so that for any $\epsilon > 0$ and any bounded net

$\{a_i\}$ converging weakly to zero we have the bound

$$|\phi(a_i)| \leq |\phi(a_i p)| + \|a_i\| \|\phi\|^{\frac{1}{2}} \epsilon^{\frac{1}{2}},$$

which shows that $\{\phi(a_i)\}$ converges weakly to zero.

The σ -weak topology is stronger than the weak and thus we see that ϕ being normal implies that it is σ -weakly continuous on the unit ball and on any ball around the origin. It follows from the Krein–Smulian theorem [cf.e.g.5] that $\phi^{-1}(0)$ is σ -weakly closed and thus ϕ is σ -weakly continuous.

The fact that ϕ can be expressed as a trace with a trace-class operator now follows simply from the definition of the σ -weak topology.

The reverse implications (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a) are trivial.

We return now to the characterization of abstract von Neumann algebras and the proof of Sakai's theorem that we stated above.

The unit ball of \mathcal{A} is weak* compact. By Krein–Milman's theorem [cf.e.g.5] it has an extremal point. It is known that if a C^* -algebra has an extremal point in its unit ball, it has an identity. We shall show that \mathcal{A}_{sa} is weak* closed. By the Krein–Smulian theorem it is enough to show that \mathcal{A}_{sa}^1 is weak* closed. Let $\{a_i\}$ be a net in \mathcal{A}_{sa}^1 converging in the weak* topology to some element $a = b + ic$ with b and c self-adjoint. Then $\{a_i + n\}$ is weak* convergent to $b + i(c + n)$ for every n . Since $\|a_i + in\| \leq (1 + n^2)^{\frac{1}{2}}$ and the norm is weak* lower semicontinuous it follows that

$$(1 + n^2)^{\frac{1}{2}} \geq \|b + i(n + c)\| \geq \|n + c\|.$$

If $c \neq 0$ we may assume that there is $\lambda > 0$, $\lambda \in Sp(c)$. But then

$$\lambda + n \leq \|n + c\| \leq (n^2 + 1)^{\frac{1}{2}}$$

for any n , which is a contradiction. It follows that $c = 0$. Thus \mathcal{A}_{sa} is weak* closed. It is sufficient again to show it for \mathcal{A}_+^1 which is weak* closed because $\mathcal{A}_+^1 = \frac{1}{2}(\mathcal{A}_{sa}^1)$. The set $(\mathcal{A}_+)_*$, i.e. the predual of \mathcal{A}_+ is separating for \mathcal{A} which is the consequence of the Hahn–Banach theorem.

We show that \mathcal{A} is monotone complete, i.e. the strong limits bounded monotone nets from \mathcal{A}_{sa} are in \mathcal{A}_{sa} . Taking a bounded monotone increasing net $\{a_i\}$ in \mathcal{A}_{sa} , there is a subnet $\{a_j\}$ which is weak* convergent to some $a \in \mathcal{A}_{sa}$. Apparently $a \geq a_i$, $\forall i$ and it is easy to see that a has to coincide with the least upper bound of $\{a_i\}$ in \mathcal{A}_{sa} . Each element of $(\mathcal{A}_+)_*$ is a normal positive functional. Thus we have a separating set of normal functionals. The GNS construction for this set then gives a faithful representation of \mathcal{A} as a C^* -algebra. However the isomorphism is normal because the functionals are all normal. It follows that the image is a C^* -algebra of operators, which is monotone complete. As we already showed, this implies that it is a von Neumann algebra of operators. This completes the proof of Sakai's theorem.

In the following chapters we shall regard the von Neumann algebras in this abstract sense, i.e. as C^* -algebras with a Banach space predual.

2.2. Cohomology theories on locally convex algebras

In this section we describe some of the main definitions and results in the known cohomology theories on locally convex algebras, in particular the theory of cyclic cohomology and entire cyclic cohomology developed by Connes and others. The main references for this part are [8] and [9].

The cyclic cohomology theory contains three basic ingredients. One is a specific chain complex which is a subcomplex of a Hochschild complex (see below). The subcomplex is defined by imposing a cyclicity condition on the multilinear functionals which comprise the cochains of the Hochschild complex. The second ingredient is a construction which relates each multilinear functional above to a particular type of multilinear functionals on the universal graded algebra over the given algebra. This is the part which makes the connection to ordinary De Rham homology theory in the special case when the algebra in consideration is $C^\infty(X)$ for some smooth manifold X . The third part is the construction of nontrivial cocycles in cyclic cohomology in the case when some additional structure like p -summable or a τ -summable Fredholm module is given. (Fredholm modules were introduced first in [28-30]). The construction using sKMS functionals can be regarded as an alternative approach mainly to this third part and it also gives a graded generalization of the whole theory.

We first describe the Hochschild complex.

Let \mathcal{M} be a bimodule over the algebra \mathcal{A} . The *Hochschild complex* is the complex $(C^n(\mathcal{A}, \mathcal{M}), b)$, where $C^n(\mathcal{A}, \mathcal{M})$ is the space of n -linear maps from \mathcal{A} to \mathcal{M} and b is a coboundary (i.e. with the property $b^2 = 0$), mapping C^n to

C^{n+1} . For $T \in C^n(\mathcal{A}, \mathcal{M})$, bT is defined by

$$(bT)(a^1, \dots, a^{n+1}) = a^1 T(a^2, \dots, a^{n+1}) \\ + \sum_{i=1}^n (-1)^i T(a^1, \dots, a^i a^{i+1}, \dots, a^{n+1}) + (-1)^{n+1} T(a^1, \dots, a^n) a^{n+1} .$$

Note that $T(a^1, \dots, a^{n+1})$ is an element from the bimodule \mathcal{M} and so the multiplication from the left and right in the above definition is well defined.

The *Hochschild cohomology of \mathcal{A} with coefficients in \mathcal{M}* is the cohomology of the complex $(C^n(\mathcal{A}, \mathcal{M}), b)$. (Recall that the cohomology of a cochain complex is defined as $Ker b / Ran b$.)

Cyclic cohomology is defined by means of a subcomplex of the Hochschild complex for the special case when $\mathcal{M} = \mathcal{A}^*$ where \mathcal{A}^* is the dual of \mathcal{A} . The space \mathcal{A}^* has a natural structure of a bimodule over \mathcal{A} given by the formula $(a \phi b)(c) := \phi(bca)$, $a, b, c \in \mathcal{A}$, $\phi \in \mathcal{A}^*$. Every n -linear map from \mathcal{A} to \mathcal{A}^* can be considered as an $(n + 1)$ -linear functional on \mathcal{A} as follows

$$\tau(a^0, a^1, \dots, a^n) := T(a^1, \dots, a^n)(a^0) .$$

In this case and in this alternative picture the coboundary b has the action

$$(b\tau)(a^0, a^1, \dots, a^{n+1}) = \tau(a^0, a^1, a^2, \dots, a^{n+1}) \\ + \sum_{i=1}^n (-1)^i \tau(a^0, \dots, a^i a^{i+1}, \dots, a^{n+1}) + (-1)^{n+1} \tau(a^{n+1} a^0, \dots, a^n) .$$

The cyclic cohomology is defined by means of a complex in which the functionals τ are required to satisfy the additional cyclicity property

$$\tau(a^1, \dots, a^n, a^0) = (-1)^n \tau(a^0, \dots, a^n) .$$

It turns out that the coboundary b maps cochains with the above property to cochains of the same type. Therefore, if one denotes by C_λ^n the subspace of

$C^n(\mathcal{A}, \mathcal{A}^*)$ which has the cyclicity property, then $(C_\lambda^n(\mathcal{A}), b)$ is a subcomplex of the Hochschild complex. The n -th cyclic cohomology group of \mathcal{A} is defined as the cohomology of the complex $(C_\lambda^n(\mathcal{A}) b)$.

There is a natural map S which gives homomorphisms $S: H_\lambda^n(\mathcal{A}) \rightarrow H_\lambda^{n+2}(\mathcal{A})$ for any n . We are not going to describe here the rather complicated way in which it is defined. The map S allows the definition of the algebraic limit of the groups H_λ^n . The latter is denoted by $H^*(\mathcal{A})$ and it is this group that is actually isomorphic to the De Rham homology in the case when $\mathcal{A} = C^\infty(X)$. Because $H^*(\mathcal{A})$ is obtained as an algebraic limit with the homomorphisms S which map in dimension higher by two, it has a decomposition into $H^{ev}(\mathcal{A})$ and $H^{odd}(\mathcal{A})$, where the notation should explain the meaning. The group $H^*(\mathcal{A})$ can be expressed as the cohomology group of a certain double complex. We now describe the latter.

The cochains are the Hochschild cochains with coefficients in \mathcal{A}^* . Notice that we do not take the subset of only cyclic multilinear functionals, but rather all of them. This makes the group $H^*(\mathcal{A})$ somewhat easier to compute than the groups H_λ^n . One defines a second coboundary B mapping $(n+1)$ -cochains into n -cochains (see chapter 1). The double complex is defined by the following:

- a) $C^{n,m} = C^{n-m}(\mathcal{A}, \mathcal{A}^*)$, $\forall n, m \in \mathbb{Z}$,
- b) for $\phi \in C^{n,m}$, $d_1\phi = (n - m + 1)b\phi$,
- c) for $\phi \in C^{n,m}$, $d_2\phi = \frac{1}{n-m}B\phi$

The coboundaries d_1 and d_2 anticommute. The cohomology of the double complex C is given by

$$H^n(C) = H^{ev}(\mathcal{A}) \text{ if } n \text{ is even,}$$

$$H^n(C) = H^{odd}(\mathcal{A}) \text{ if } n \text{ is odd.}$$

The entire cyclic cohomology is a generalization of the above scheme. In the usual cohomology theories, including the ones described so far, one considers only chains with finite supports. In other words one defines cohomology as the cohomology of the direct sum of the spaces of cochains in all dimensions. It then has a filtration by dimensions. The entire cyclic cohomology of Connes is defined as the cohomology of the double complex we described with cochains that are infinite sequences of cochains in all dimensions. It turns out that such a theory is only nontrivial if one imposes certain suitable growth condition (see chapter 1).

CHAPTER III

IMPLEMENTING A \mathbb{Z}_2 GRADED VON NEUMANN ALGEBRA WITH A SUPER-KMS FUNCTIONAL ON IT

3.1. The case of a trivially graded von Neumann algebra

In this section we put together mostly known results and prepare the ground for our main construction in section 2. Let \mathcal{A} be a von Neumann algebra and ω - a continuous linear functional on it. We think of \mathcal{A} as an abstract von Neumann algebra, i.e. \mathcal{A} is a unital C^* -algebra, such that it has a Banach space predual (see e.g. sec. 3.9 of [16]). We list some definitions and facts we shall need.

A continuous functional ω is called self-adjoint if $\overline{\omega(a)} = \omega(a^*)$. Equivalently, ω is self-adjoint iff $\omega(a)$ is real for any self-adjoint element a . A (self-adjoint) functional ω is called positive if $\omega(a) \geq 0$, $\forall a \geq 0$. (Recall that a self-adjoint element a of a C^* -algebra is called positive (denoted $a \geq 0$) if its spectrum is nonnegative : $Sp(a) \in \mathbb{R}^+$)

Two positive functionals ϕ and ψ are called orthogonal (denoted $\phi \perp \psi$) if the following equality holds:

$$\|\phi - \psi\| = \|\phi\| + \|\psi\|.$$

Every self-adjoint functional ω on a C^* -algebra has a unique decomposition into two orthogonal positive functionals ω_{\pm} , called Jordan decomposition [16]:

$$\omega = \omega_+ - \omega_- , \quad \omega_+ \perp \omega_- . \tag{3.1}$$

This decomposition is a noncommutative generalization of the well known Jordan

decomposition of a signed measure and reduces to the latter in the commutative case under some assumptions.

The Jordan decomposition of a self-adjoint functional ω is preserved by any *-automorphism that leaves ω invariant, i.e.

$$\omega \circ \alpha = \omega \iff \omega_{\pm} \circ \alpha = \omega_{\pm}.$$

This fact is mentioned for example in [31] and follows from the uniqueness of the Jordan decomposition and the fact that if ω_{\pm} are positive functionals, then $\omega_{\pm} \circ \alpha$ are also positive functionals, having the same norms as ω_{\pm} and are therefore orthogonal. Since $\omega_{+} \circ \alpha - \omega_{-} \circ \alpha = \omega \circ \alpha = \omega$ is again a Jordan decomposition of ω , the statement follows.

The above statement is true in fact for any automorphism, not necessarily a *-automorphism, as long as it preserves the cone of positive elements of the algebra.

Recall that the notion of positivity makes the set of self-adjoint elements of a C^* -algebra into a partially ordered set, namely $a \geq b$ if $a - b \geq 0$. A bounded functional ω on a von Neumann algebra \mathcal{A} is called normal if for each bounded in norm, monotone increasing net $\{a_{\alpha}\}$ of self-adjoint elements of \mathcal{A} with a strong limit a , the net $\{\omega(a_{\alpha})\}$ converges to $\omega(a)$. It turns out [16] that every norm bounded monotone increasing net of self-adjoint elements in a von Neumann algebra converges strongly to a self-adjoint element. One can give an equivalent definition of a normal positive functional without any reference to the strong topology which is only defined in the case of a concrete von Neumann algebra (of operators on a Hilbert space). A positive functional ϕ is called normal [32] if for any family $\{e_i\}_{i \in I}$ of pairwise disjoint projections of \mathcal{A} we have $\phi(\sum_{i \in I} e_i) = \sum_{i \in I} \phi(e_i)$. Recall that in the abstract setting, apart from the norm topology, there is always a second

locally convex topology on any von Neumann algebra, coming from the fact that \mathcal{A} is the dual of a Banach space - the weak* topology. It is standard in von Neumann algebra theory to call it the σ -weak topology and it is also equivalent to the so-called ultraweak topology, when the latter is defined. If ϕ is a positive functional on \mathcal{A} , then ϕ being normal is equivalent to ϕ being σ -weakly continuous. For a general nonpositive functional ω one defines normal to mean that ω_{\pm} are normal or equivalently that ω is σ -weakly continuous or that ω is an element of the predual of \mathcal{A} .

In order to get a feeling for what it means for a functional ω to be normal, we look at the example $\mathcal{A} = L_{\infty}(X, \mu)$ for some measure space X with a σ -finite measure μ on it. Let us first take ω to be positive. Then the equation $\nu(E) = \omega(\chi_E)$ for any measurable set E defines a finitely additive measure on X . (χ_E denotes the characteristic function of E .) However ν is countably additive if and only if ω is normal. In that case we can write $\omega(f) = \int f g d\mu$ for some positive function $g \in L_1(X, \mu)$. The case when ω is self-adjoint but not positive is completely analogous except that ν is now a signed measure and g is an arbitrary real function from $L_1(X, \mu)$.

We point out that the regularized supertrace functional which was described in the introduction and which was taken as a prototype for a general sKMS functional is normal. Indeed, in this case $\omega(a) = Tr_{\mathcal{H}_+}(a e^{-H_+}) - Tr_{\mathcal{H}_-}(a e^{-H_-})$ is the difference of two traces with the trace-class operators $e^{-H_{\pm}}$. Now recall that the set of all trace-class operators on a separable Hilbert space with the norm $Tr(|\cdot|)$ is a Banach space and its dual is precisely $L(\mathcal{H})$. So we see that both ω_{\pm} are given by elements of the predual of $L(\mathcal{H})$ and therefore so is ω . Let ω be a self-adjoint

normal linear functional on \mathcal{A} . Then its Jordan decomposition $\omega = \omega_+ - \omega_-$ has the following additional properties [32]:

(i) ω_{\pm} are normal positive functionals with mutually singular supports, i.e. any $a \in \mathcal{A}$, $a \geq 0$ can be represented (nonuniquely in general) as a sum $a = a_+ + a_-$ so that $\omega_-(a_+) = \omega_+(a_-) = 0$.

(ii) There exist projections $\chi_{\pm} \in \mathcal{A}$ (not necessarily unique) onto the supports of ω_{\pm} with the following properties:

$$\chi_+ \chi_- = \chi_- \chi_+ = 0,$$

$$\omega_+(a) = \omega(a \chi_+), \quad \forall a \in \mathcal{A},$$

$$\omega_-(a) = -\omega(a \chi_-), \quad \forall a \in \mathcal{A}.$$

With the help of these projections one can define in a unique way a normal positive functional, which will be denoted as $|\omega|$, related to and determined by ω :

$$|\omega| := \omega_+ + \omega_- = (\chi_+ - \chi_-) \circ \omega. \quad (3.2)$$

where $(\chi_+ - \chi_-) \circ \omega$ is just a notation for the functional defined by

$((\chi_+ - \chi_-) \circ \omega)(a) := \omega(a(\chi_+ - \chi_-))$. We also have:

$$\begin{aligned} \omega &= \omega_+ - \omega_- = \chi_+ \circ \omega + \chi_- \circ \omega \\ &= (\chi_+ + \chi_-) \circ \omega = (\chi_+ - \chi_-)(\chi_+ - \chi_-) \circ \omega \\ &= (\chi_+ - \chi_-) \circ |\omega|. \end{aligned} \quad (3.3)$$

One can check immediately that $\| |\omega| \| = \|\omega\|$. Indeed, recall that by the definition of orthogonality $\|\omega\| = \|\omega_+\| + \|\omega_-\|$. Using the fact that for any positive functional ϕ , $\|\phi\| = \phi(1)$, it becomes obvious that $\| |\omega| \| = \|\omega_+\| + \|\omega_-\|$.

Formulae (3.2, 3.3) are a special case of the well known polar decomposition (due to Sakai [33]) of a normal linear functional.

We would like to dedicate some attention to the projections χ_{\pm} and the problem of their nonuniqueness. The latter is related to the fact that we do not require (for now) that $|\omega|$ be a faithful functional. This means that there can exist positive elements of \mathcal{A} on which $|\omega|$ is zero. If we take again the example $\mathcal{A} = L_{\infty}(X, \mu)$, then, as mentioned, the normal functional ω becomes an integral $\omega(f) = \int g f d\mu$ for some real integrable function g and $|\omega|$ in this case is nothing but the integral $\int |g| f d\mu$. Any two disjoint sets $X_{\pm} \subset X$ such that $g \geq 0$ on X_+ and $g \leq 0$ on X_- , provide us with projections χ_{\pm} as defined above, namely the characteristic functions of X_{\pm} . Since g can be zero on some subset of nonzero measure (which is the same as $|\omega|$ not being faithful), the subsets X_{\pm} are not unique. There are however two unique ones, which are minimal, namely $X_{\pm} = \{x \in X \mid \pm g(x) > 0\}$. This situation is reproduced in a sense in the general case [32], where the following fact is true: Among the set of all projections onto the support of ω_+ there is a unique minimal one with respect to the partial ordering of the set of self-adjoint elements. If we use the notation χ_+ for this minimal one we can now show that it is invariant under any *-automorphism α which leaves ω_+ invariant. Notice that this does not follow from the invariance of ω_+ alone. Indeed the latter implies

$$\begin{aligned} (\omega_+ \circ \alpha^{-1})(a) &= \omega_+(\alpha^{-1}(a)) = \omega(\alpha^{-1}(a)\chi_+) \\ &= \omega(a\alpha(\chi_+)) \end{aligned}$$

$$(\omega_+ \circ \alpha^{-1})(a) = \omega_+(a) = \omega(a\chi_+),$$

from which we can only conclude that $\alpha(\chi_+)$ is again a projection onto the support of ω_+ . However χ_+ being minimal means that $\chi_+ - \chi'_+ \leq 0$ for any χ'_+ . Now, any *-automorphism preserves positivity. Therefore it is true that $\alpha(\chi_+) \leq \alpha(\chi'_+)$ for any χ'_+ and from the uniqueness it follows that $\alpha(\chi_+) = \chi_+$. The same statements are of course true for the projection χ_- .

After this lengthy preparation we are now ready to start on the problem of implementing the von Neumann algebra \mathcal{A} with a normal self-adjoint functional ω on it. The Gelfand - Naimark - Segal (GNS) construction for $|\omega|$ gives a cyclic representation $(\pi, \mathcal{H}, \Omega)$ where \mathcal{H} is a Hilbert space, π is a $*$ -algebra homomorphism of \mathcal{A} into the algebra of bounded operators in \mathcal{H} and Ω is a cyclic vector (vacuum vector) in \mathcal{H} . The functional $|\omega|$ is expressed as a vacuum expectation value

$$|\omega|(a) = (\Omega, \pi(a)\Omega),$$

while ω becomes a “graded vacuum expectation value” as we shall see.

We observe several important properties of this representation. Let $\eta: \mathcal{A} \rightarrow \mathcal{H}$ denote the canonical map in the GNS construction. (Recall that \mathcal{H} is obtained by completing the linear space $\mathcal{A}/L_{|\omega|}$ with respect to the inner product $(\eta_0(a), \eta_0(b)) := |\omega|(a^*b)$. $L_{|\omega|}$ is the left kernel of $|\omega|$, i.e. $L_{|\omega|} := \{a \in \mathcal{A} \mid |\omega|(a^*a) = 0\}$ and $\eta: \mathcal{A} \rightarrow \mathcal{A}/L_{|\omega|}$ is the canonical quotient map. Finally $\eta = i \circ \eta_0$, where i is the embedding of $\mathcal{A}/L_{|\omega|}$ into its completion \mathcal{H} . In these notations $\Omega = \eta(1)$.)

The Hilbert space \mathcal{H} has a natural orthogonal decomposition as a sum of two subspaces \mathcal{H}_\pm , related to the two orthogonal functionals ω_\pm . First note that the elements $\eta(a(\chi_+ + \chi_-))$ are dense in \mathcal{H} . In fact we have $\eta(\chi_+ + \chi_-) = \eta(1) = \Omega$.

Indeed

$$\begin{aligned} |\omega|((\chi_+ + \chi_- - 1)^*(\chi_+ + \chi_- - 1)) &= |\omega|(\chi_+) - |\omega|(\chi_+) \\ &+ |\omega|(\chi_-) - |\omega|(\chi_-) - |\omega|(\chi_+) - |\omega|(\chi_-) + |\omega|(1) \\ &= |\omega|(1) - \omega_+(1) - \omega_-(1) \\ &= 0 \end{aligned}$$

The calculation shows that 1 and $\chi_+ + \chi_-$ are mapped to the same element Ω in

\mathcal{H} . Now since $\pi(a)\Omega$ are dense in \mathcal{H} and since $\eta(a(\chi_+ + \chi_-)) = \pi(a)\eta(\chi_+ + \chi_-)$, the statement is true.

We now define $\mathcal{H}_+ := (\eta(\mathcal{A}\chi_+))^-$ and $\mathcal{H}_- := (\eta(\mathcal{A}\chi_-))^-$. (Here $(...)^-$ stands for closure.) The subspaces $\eta(\mathcal{A}\chi_{\pm})$ are mutually orthogonal. Indeed taking arbitrary elements $a\chi_+$, $b\chi_-$ we have

$$\begin{aligned}
|(\eta(a\chi_+), \eta(b\chi_-))| &= ||\omega|(\chi_+ a^* b \chi_-)| \\
&\leq |\omega_+(\chi_+ a^* b \chi_-)| + |\omega_-(\chi_+ a^* b \chi_-)| \\
&\leq \omega_+(\chi_+ a^* b b^* a \chi_+)^{\frac{1}{2}} \omega_+(\chi_-)^{\frac{1}{2}} + \omega_-(\chi_+)^{\frac{1}{2}} \omega_-(\chi_- b^* a a^* b \chi_-)^{\frac{1}{2}} \\
&= 0.
\end{aligned} \tag{3.4}$$

Since $\eta(a\chi_+) + \eta(a\chi_-)$ are dense in \mathbb{H} and since $\eta(\mathcal{A}\chi_+) \perp \eta(\mathcal{A}\chi_-)$, we see that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and the decomposition is orthogonal.

The vectors $\Omega_{\pm} := \eta(\chi_{\pm})$ are then the cyclic vectors from which the subspaces \mathcal{H}_{\pm} are generated, i.e. $\pi(a)\Omega_{\pm} \equiv \eta(a\chi_{\pm})$ are dense in \mathcal{H}_{\pm} .

Next we notice that the representation π is reducible and in particular \mathcal{H}_{\pm} are invariant subspaces. Indeed, taking a vector $\eta(a\chi_+) \in \mathcal{H}_+$, we have

$$(\eta(b\chi_-), \eta(a\chi_+)) = 0, \quad \forall b \in \mathcal{A}$$

$$(\eta(b\chi_-), \pi(c)\eta(a\chi_+)) = (\eta(b\chi_-), \eta(ca\chi_+)) = 0$$

by the same reasoning as in (3.4). This shows, that if $\xi \in \mathcal{H}_+$, then $\pi(c)\xi \in \mathcal{H}_+$ for any $c \in \mathcal{A}$ and analogously for \mathcal{H}_- .

Finally we can introduce the operator $\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, where the decomposition into blocks is with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Then we get an expression for ω as a “graded vacuum expectation value”:

$$\begin{aligned}\omega(a) &= \omega_+(a) - \omega_-(a) = (\Omega, \pi(a)\Omega_+) - (\Omega, \pi(a)\Omega_-) \\ &= (\Omega, \Gamma\pi(a)\Omega_+) + (\Omega, \Gamma\pi(a)\Omega_-) = (\Omega, \Gamma\pi(a)(\Omega_+ + \Omega_-)) \\ &= (\Omega, \Gamma\pi(a)\Omega) .\end{aligned}$$

In general $\Gamma \neq \pi(\chi_+ - \chi_-)$, opposite to what intuition might suggest. However equality holds in the case when ω is a sKMS functional and one takes the representation π' that we shall construct in the next section.

Since any vacuum expectation value $(\Omega, A\Omega)$ with $A \in L(\mathcal{H})$ can be expressed as the regularized trace $Tr(A P_\Omega)$, where P_Ω is the (obviously trace-class) projection onto the one-dimensional subspace spanned by Ω , we can say that the functional ω is expressed as a graded regularized trace (or supertrace).

We look at the example $\mathcal{A} = L(\mathcal{V}_+ \oplus \mathcal{V}_-)$, \mathcal{V}_\pm -separable Hilbert spaces, in order to illustrate the construction so far. Let ρ be a positive trace-class operator on $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$, even with respect to the natural grading of $L(\mathcal{V}_+ \oplus \mathcal{V}_-)$. With the help of ρ we can define the normal self-adjoint functional $\omega(a) := Tr_{\mathcal{V}_+}(\rho a) - Tr_{\mathcal{V}_-}(\rho a)$. We can choose orthonormal bases $\{e_i\}_{i=1}^\infty$, $e_i \in \mathcal{V}_+$; $\{f_i\}_{i=1}^\infty$, $f_i \in \mathcal{V}_-$, which diagonalize ρ . We denote by $\{\rho_i^\pm\}_{i=1}^\infty$ the eigenvalues of ρ corresponding to the eigenvectors in \mathcal{V}_\pm respectively. Let us also normalize ρ so that

$$\sum_{i=1}^{\infty} \rho_i^+ + \sum_{i=1}^{\infty} \rho_i^- = 1 .$$

Obviously the functionals $\omega_\pm(a) := Tr_{\mathcal{V}_\pm}(\rho a)$ provide the unique Jordan decomposition of ω . Indeed

$$\begin{aligned}\omega_+(a^*a) &= Tr_{\mathcal{V}_+}(\rho a^*a) = \sum_{i=1}^{\infty} (e_i, \rho a^*a) \\ &= \sum_{i=1}^{\infty} \rho_i^+ (ae_i, ae_i) \geq 0 ,\end{aligned}$$

and so ω_+ is positive and so is in the same way ω_- . This by the way shows that ω is self-adjoint. We check that $\omega_+ \perp \omega_-$. Indeed we have

$$\begin{aligned} (\omega_+ - \omega_-)(\Gamma) &= \text{Tr}_{\mathcal{V}_+}(\rho) + \text{Tr}_{\mathcal{V}_-}(\rho) \\ &= \omega_+(1) + \omega_-(1) = \|\omega_+\| + \|\omega_-\| \end{aligned}$$

and since $\|\omega_+ - \omega_-\| \leq \|\omega_+\| + \|\omega_-\|$ always, we see that in fact $\|\omega_+ - \omega_-\| = \|\omega_+\| + \|\omega_-\|$. The minimal projections χ_{\pm} can be identified as the projections onto the nonzero eigenspaces of ρ in \mathcal{V}_{\pm} respectively. To avoid too cumbersome notation, let us assume that ρ has no zero eigenspaces, as is for example the case with $\rho = e^{-H}$ which we considered before. (The treatment of the general case causes no real additional difficulty.) The elements of $L(\mathcal{V}_+ \oplus \mathcal{V}_-)$ can be written in block form. In particular the projections χ_{\pm} have the representation

$$\chi_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

The support of ω_+ consists of all elements of the form

$$b = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix},$$

which is the same as all elements of the form $a\chi_+$, $a \in \mathcal{A}$. Indeed b^*b has nonzero elements in the upper left block only. Analogously all elements of the form $c = \begin{pmatrix} 0 & c_{12} \\ 0 & c_{22} \end{pmatrix}$ form the support of ω_- and can be written as $a\chi_-$. The functional $|\omega|$ gives an inner product on $L(\mathcal{H})$ and the completion of L is the GNS Hilbert space \mathcal{H} . The left regular representation of L extends by continuity to a representation π of L in \mathcal{H} . (The left regular representation is the one in which L acts on itself by multiplication from the left.) It is now evident from the block form of the elements of L that the subspaces $L\chi_{\pm}$ are invariant subspaces for π and so are their closures, which we denoted by \mathcal{H}_{\pm} . We can construct a

representation, which is unitarily equivalent to the GNS representation, using the Hilbert space \mathcal{V} we started with. Namely take $\mathcal{H}'' := \bigoplus_{i=1}^{\infty} \mathcal{V}_i$ where $\mathcal{V}_i \simeq \mathcal{V} \ \forall i$. Define a representation π'' by $\pi''(a) := \bigoplus_{i=1}^{\infty} a_i$, i.e. an infinite direct sum of the defining representation. The vacuum Ω'' can be defined as

$$\Omega'' = (e_1 \rho_1^+ \oplus e_2 \rho_2^+ \oplus \dots) \oplus (f_1 \rho_1^- \oplus f_2 \rho_2^- \oplus \dots).$$

It is quite evident that Ω'' is cyclic with respect to π'' because Ω'' has a component in each copy of \mathcal{V} . In order to check that the representation $(\pi'' \mathcal{H}'')$ is unitarily equivalent to the GNS representation (π, \mathcal{H}) , it is enough to show [16] that

$$(\Omega'', \pi''(a)\Omega'') = (\Omega, \pi(a)\Omega).$$

The GNS representation by its construction is such that the vacuum expectation value above is equal to $|\omega|(a)$. It is easy to calculate that the same is true for π'' .

In the representation (π'', \mathcal{H}'') it is also easy to identify the subspaces \mathcal{H}''_{\pm} . In fact they are again infinite sums of copies of \mathcal{V} . The vacua Ω''_{\pm} are just the first and the second summands respectively in the expression for Ω'' .

It is also instructive to look briefly at the situation when \mathcal{V} is finite dimensional. Then if $\dim \mathcal{V}_+ = n$, $\dim \mathcal{V}_- = m$, the elements of L are just $(n+m) \times (n+m)$ matrices and the GNS Hilbert space \mathcal{H} is just L itself. The subspaces \mathcal{H}_{\pm} are nothing but the subspaces of $n \times (n+m)$ and $m \times (n+m)$ matrices, respectively. In the alternative picture one has to define $\mathcal{H}' := \bigoplus_{i=1}^{n+m} \mathcal{V}_i$ and $\mathcal{H}'_+ := \bigoplus_{i=1}^n \mathcal{V}_i$ and the obvious analog for \mathcal{H}'_- .

We shall use the same example to illustrate our developments in the next section.

3.2. The case of a graded von Neumann algebra and a super-KMS functional on it

In the previous section we constructed a reducible representation π of \mathcal{A} in $\mathcal{H}_+ \oplus \mathcal{H}_-$ and expressed the σ -weakly continuous functional ω as a graded vacuum expectation value or equivalently a regularized supertrace. However the operators $\pi(x)$ are all diagonal with respect to the orthogonal decomposition of \mathcal{H} . Our initial goal was to implement a \mathbb{Z}_2 graded algebra $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$, so that it becomes a \mathbb{Z}_2 graded algebra of bounded operators on a \mathbb{Z}_2 graded Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. In such a realization the grading of the algebra of operators should be the natural one induced from the grading of the Hilbert space. In particular, the odd elements are operators, which map \mathcal{H}_+ into \mathcal{H}_- and \mathcal{H}_- into \mathcal{H}_+ . This shows that the representation π constructed in sec.1 is unsuitable for implementing the odd part \mathcal{A}_- . It turns out, however, that the super-KMS (sKMS) property of the functional ω gives a natural way to define a new representation π' , in which the odd elements of \mathcal{A} are represented as odd elements (off-diagonal) of $L(\mathcal{H}_+ \oplus \mathcal{H}_-)$. Most of the definitions which we shall need here were given in the first chapter.

We begin by studying the implications of the sKMS property. We shall assume $\omega = \omega^*$, i.e. ω is a self-adjoint functional, and let it satisfy what we shall call the weak sKMS condition;

$$\omega(ab) = \omega(b^\Gamma \alpha_i(a)).$$

PROPOSITION 3.1. *Let ω be a σ -weakly continuous functional with properties as above. Then $|\omega|$ and ω_\pm are KMS functionals when restricted to \mathcal{A}_+ .*

PROOF: First note that any element that is invariant under the action of the automorphism group α_t is in the dense subalgebra \mathcal{A}_α of analytic elements in a

trivial way, i.e. $\alpha_z(a) = a$ for any such a . Specifically we have $\alpha_z(\chi_\pm) = \chi_\pm$ if χ_\pm are the unique minimal projections. Also, if a functional ω is invariant under α_t , then it is necessarily invariant under the analytically continued group α_z . Indeed, $\omega(\alpha_z(a))$ defines an entire function for any $a \in \mathcal{A}_\alpha$. Any entire function is completely determined by its values on the real line and is therefore constant if it is constant on the real line. Similar arguments of uniqueness of the analytic continuation show that $\alpha_{z_1}(a)\alpha_{z_2}(a) = \alpha_{z_1+z_2}(a)$ for any $a \in \mathcal{A}_\alpha$.

Next we show that for $a \in \mathcal{A}_+$ the following identities hold:

$$\omega_-(a^*\chi_+a) = \omega_+(a^*\chi_-a) = 0$$

Indeed, taking a in the subspace $\mathcal{A}_\alpha \cap \mathcal{A}_+$, dense in \mathcal{A}_+ (see ch.1)

$$\begin{aligned} \omega_-((\chi_+a)^*\chi_+a) &= \omega_-(a^*\chi_+a) = -\omega(a^*\chi_+a\chi_-) \\ &= -\omega(a\chi_- \alpha_i(a^*\chi_+)) = -\omega_+(a\chi_- \chi_- \alpha_i(a^*)) \\ &= -\omega_+(a\chi_- \alpha_i((a\chi_-)^*)) = -\omega_+(\alpha_{-\frac{i}{2}}(a\chi_-) \alpha_{\frac{i}{2}}((a\chi_-)^*)) \\ &= -\omega_+(\alpha_{-\frac{i}{2}}(a\chi_-) (\alpha_{-\frac{i}{2}}(a\chi_-))^*). \end{aligned} \tag{3.5}$$

We have used the sKMS property for even elements in the first equality of the second line, the invariance of ω_+ under the (analytically continued) automorphisms in the second equality of line three and finally the fact that, if α_t is a group of *-automorphisms, then for the analytically continued group we have $\alpha_z(a^*) = (\alpha_{\bar{z}}(a))^*$ (see e.g.[15]). Since ω_+ and ω_- are both positive, the equality (3.5) shows that $\omega_-(a^*\chi_+a) = 0$. Analogously $\omega_+(a^*\chi_-a) = 0$. Now it is easy to show analogous identities for arbitrary even elements $a, b \in \mathcal{A}_\alpha$. Indeed, using the Schwarz inequality, we can write:

$$|\omega_-(a\chi_+b)| \leq \omega_-(a^*a)^{\frac{1}{2}} \omega_-((\chi_+b)^*\chi_+b)^{\frac{1}{2}} = \omega_-(a^*a)^{\frac{1}{2}} \omega_-(b^*\chi_+b)^{\frac{1}{2}} = 0$$

Next we show that $|\omega|$ is KMS when restricted to $a, b \in \mathcal{A}_\alpha \cap \mathcal{A}_+$.

$$\begin{aligned} |\omega|(ab) &= \omega(ab(\chi_+ - \chi_-)) = \omega(b(\chi_+ - \chi_-)\alpha_i(a)) \\ &= \omega_+(b\chi_+\alpha_i(a)) - \omega_+(b\chi_-\alpha_i(a)) - \omega_-(b\chi_+\alpha_i(a)) + \omega_-(b\chi_-\alpha_i(a)). \end{aligned}$$

The second and third terms are zero, so we can switch the signs in front of them.

Thus we get:

$$|\omega|(ab) = \omega_+(b(\chi_+ + \chi_-)\alpha_i(a)) + \omega_-(b(\chi_+ + \chi_-)\alpha_i(a))$$

One can replace $\chi_+ + \chi_-$ with the identity inside the functionals. Indeed, using the sKMS property of ω once again, we obtain:

$$\begin{aligned} \omega_+(b(\chi_+ + \chi_-)\alpha_i(a)) &= \omega(b(\chi_+ + \chi_-)\alpha_i(a)\chi_+) \\ &= \omega(a\chi_+b(\chi_+ + \chi_-)) = \omega(a\chi_+b) \\ &= \omega(b\alpha_i(a)\chi_+) = \omega_+(b\alpha_i(a)). \end{aligned}$$

Finally

$$|\omega|(ab) = \omega_+(b\alpha_i(a)) + \omega_-(b\alpha_i(a)) = |\omega|(b\alpha_i(a)).$$

which proves that $|\omega|$ indeed satisfies the KMS property on even elements.

It follows also that ω_\pm are KMS on \mathcal{A}_+ . Indeed both $|\omega|$ and ω have the KMS property on \mathcal{A}_+ . (Note that it is still better to avoid calling ω , when restricted to \mathcal{A}_+ , a KMS functional, because it is not positive.) Since $\omega_\pm = \pm\frac{1}{2}(\omega \pm |\omega|)$, the last statement becomes obvious.

We prove two useful lemmas.

LEMMA 3.2. *There are no positive odd elements in a C^* -algebra.*

PROOF: Suppose $a \in \mathcal{A}_-$ and $a \geq 0$. Any positive element a has the representation $a = b^*b$ for some b .

$$a = b^*b = (b_+ + b_-)^*(b_+ + b_-) = (b_+^* + b_-^*)(b_+ + b_-) = b_+^*b_+ + b_+^*b_- + b_-^*b_+ + b_-^*b_-.$$

Here b_{\pm} are the even and odd parts of b . $b_{+}^{*}b_{+} \in \mathcal{A}_{+}$ and $b_{-}^{*}b_{-} \in \mathcal{A}_{+}$. Since a is odd, it follows that $b_{+}^{*}b_{+} + b_{-}^{*}b_{-} = 0$. The sum of two positive elements can only be zero when they are zero separately, which gives

$$\|b_{\pm}\|^2 = \|b_{\pm}^{*}b_{\pm}\| = 0 \rightarrow b_{\pm} = b_{\pm}^{*} = 0.$$

It follows that $a = 0$.

Next note that a sKMS functional ω is always even, i.e. $\omega^{\Gamma} = \omega$ or in other words $\omega(a^{\Gamma}) = \omega(a)$, $\forall a \in \mathcal{A}$. To see that, apply the sKMS property to $a, \underline{1}$:

$$\omega(a) = \omega(\underline{1}a) = \omega(a^{\Gamma}\alpha_i(\underline{1})) = \omega(a^{\Gamma})$$

In particular $\omega(a) = 0$ whenever $a \in \mathcal{A}_{-}$.

LEMMA 3.3. *If ω is an even self-adjoint bounded linear functional on a C^* -algebra, then the functionals ω_{\pm} in the Jordan decomposition of ω are also even.*

PROOF: This is a special case of the general result that we stated before that any $*$ -automorphism preserving ω has to preserve ω_{\pm} separately.

We now study the implications of the sKMS property of ω for the case of odd elements.

LEMMA 3.4. *For $a \in \mathcal{A}_{-}$ the following identities hold:*

$$\omega_{+}(b\chi_{+}a) = \omega_{-}(b\chi_{-}a) = 0$$

PROOF:

$$\begin{aligned} \omega_{+}(a^{*}\chi_{+}a) &= \omega(a^{*}\chi_{+}a\chi_{+}) \\ &= \omega((a\chi_{+})^{\Gamma}\alpha_i(a^{*}\chi_{+})) = -\omega(a\chi_{+}\alpha_i(a^{*})\chi_{+}) \\ &= -\omega_{+}(a\chi_{+}\alpha_i(a^{*})) = -\omega_{+}(\alpha_{-\frac{i}{2}}(a\chi_{+})\alpha_{\frac{i}{2}}((a\chi_{+})^{*})) \\ &= -\omega_{+}(\alpha_{-\frac{i}{2}}(a\chi_{+})(\alpha_{-\frac{i}{2}}(a\chi_{+}))^{*}). \end{aligned} \tag{3.7}$$

Since ω_+ is a positive functional, both $\omega_+(a^*\chi_+a) \geq 0$ and $\omega_+(\alpha_{-\frac{i}{2}}(a\chi_+)(\alpha_{-\frac{i}{2}}(a\chi_+))^*) \geq 0$ and (3.7) shows that $\omega_+(a^*\chi_+a) = 0$. The proof for $\omega_-(a^*\chi_-a)$ is identical.

For arbitrary $b \in \mathcal{A}$, $a \in \mathcal{A}_-$ we use the Schwarz inequality:

$$|\omega_+(b\chi_+a)| \leq \omega_+(bb^*)^{\frac{1}{2}}\omega_+(a^*\chi_+^*\chi_+a)^{\frac{1}{2}} = \omega_+(bb^*)^{\frac{1}{2}}\omega_+(a^*\chi_+a)^{\frac{1}{2}} = 0$$

PROPOSITION 3.5. *The sKMS property of ω implies $\omega_{\pm}(ab) = \omega_{\mp}(b\alpha_i(a))$ for any $b \in \mathcal{A}_-, a \in \mathcal{A}_{\alpha}$.*

PROOF: First notice that if $a \in \mathcal{A}_+$, then $ab \in \mathcal{A}_-$ and $b\alpha_i(a) \in \mathcal{A}_-$ and both functionals give zero by virtue of being even. Thus we only look at the case when $a, b \in \mathcal{A}_-$

$$\begin{aligned} \omega_+(ab) &= \omega(ab\chi_+) = -\omega(b\chi_+\alpha_i(a)) = -\omega_+(b\chi_+\alpha_i(a)) + \omega_-(b\chi_+\alpha_i(a)) \\ &= \omega_-(b\chi_+\alpha_i(a)) = \omega_-(b\chi_+\alpha_i(a)) + \omega_-(b\chi_-\alpha_i(a)). \end{aligned}$$

where we have added a term that is zero according to Lemma 3.4. As in the proof of Proposition 3.1, we can replace $\chi_+ + \chi_-$ with 1 inside the functionals.

$$\omega_+(ab) = \omega_-(b(\chi_+ + \chi_-)\alpha_i(a)) = \omega_-(b\alpha_i(a))$$

COROLLARY. *If $a \in \mathcal{A}_-$ is such that $\eta(a) \in \mathcal{H}_{\pm}$ then $\eta(a^*) \in \mathcal{H}_{\mp}$.*

PROOF: $\eta(a) \in \mathcal{H}_+$ is equivalent to $\|\eta(a)\|_-^2 = 0 = \omega_-(a^*a)$. Applying Prop. 3.5 we obtain:

$$\|\eta(a^*)\|_+^2 = \omega_+(aa^*) = \omega_-(a^*\alpha_i(a)) \leq \omega_-(a^*a)^{\frac{1}{2}}\omega_-(\alpha_i(a)^*\alpha_i(a))^{\frac{1}{2}} = 0$$

COROLLARY. *If $a \in \mathcal{A}_+$ is such that $\eta(a) \in \mathcal{H}_{\pm}$, then $\eta(a^*) \in \mathcal{H}_{\pm}$.*

PROOF: The proof is analogous to the proof of the previous corollary.

Propositions 3.1 and 3.4 put together give the following result:

COROLLARY. With ω being a normal sKMS functional on \mathcal{A} and $|\omega|$ defined as before, $|\omega|$ is a KMS functional.

PROOF: The case $a, b \in \mathcal{A}_+$ was already proven. Take $a, b \in \mathcal{A}_-$.

$$|\omega|(ab) = \omega_+(ab) + \omega_-(ab) = \omega_-(b\alpha_i(a)) + \omega_+(b\alpha_i(a)) = |\omega|(b\alpha_i(a)).$$

For $a \in \mathcal{A}_+$, $b \in \mathcal{A}_-$ or vice-versa all functionals are zero.

We now decompose each of \mathcal{H}_\pm further into orthogonal direct sums. Define the subspaces $\mathcal{H}_\pm \cap \eta(\mathcal{A}_+)$ and $\mathcal{H}_\pm \cap \eta(\mathcal{A}_-)$ and their closures $\mathcal{H}_\pm^0 := (\mathcal{H}_\pm \cap \eta(\mathcal{A}_+))^-$, $\mathcal{H}_\pm^1 := (\mathcal{H}_\pm \cap \eta(\mathcal{A}_-))^-$. The subspaces $\mathcal{H}_+^{0,1} \subset \mathcal{H}_+$ are orthogonal and so are $\mathcal{H}_-^{0,1} \subset \mathcal{H}_-$, i.e. $\mathcal{H}_\pm^0 \perp \mathcal{H}_\pm^1$. We show that orthogonality holds for the dense subspaces. Take e.g. $a \in \mathcal{A}_+$ such that $\eta(a) \in \mathcal{H}_+$, $b \in \mathcal{A}_-$ such that $\eta(b) \in \mathcal{H}_+$. Then, because ω_+ is even, we get:

$$(\eta(a), \eta(b)) = \omega_+(a^*b) = 0$$

Since $\eta(a + b)$ are dense in \mathcal{H}_+ , it follows that $\eta(a)$ are dense in \mathcal{H}_+^0 and analogously $\eta(b)$ are dense in \mathcal{H}_+^1 .

Definition: Let $a \in \mathcal{A}_-$. Define an operator $J: \mathcal{H} \rightarrow \mathcal{H}$ by its action on a dense subspace:

$$J\eta(a) := \eta(\alpha_{\frac{i}{2}}(a^*))$$

(The proposition that follows will justify the statement that J is defined on all of \mathcal{H} by its action on a dense subspace, namely we will show that J is bounded.)

Those familiar with the Tomita–Takesaki theory will realize that the operator J we have defined coincides with the antiunitary operator J in the construction of Tomita–Takesaki. Recall that in that theory one defines an antilinear operator S

as the closure of the operator $S_0\pi(a)\Omega := \pi(a^*)\Omega$. Then the polar decomposition of S is given by $S = J\delta^{\frac{1}{2}}$, where J is shown to be antiunitary and δ is self-adjoint. Our starting point however is different than the one in the Tomita–Takesaki theory. In our case we have an automorphism group to begin with, while in the theory of Tomita–Takesaki the automorphism group is generated by δ .

PROPOSITION 3.6. *J is an antiunitary operator, mapping \mathcal{H}_+^1 onto \mathcal{H}_-^1 and \mathcal{H}_-^1 onto \mathcal{H}_+^1 and leaving invariant separately \mathcal{H}_\pm^0 . Furthermore it is equal to its inverse.*

PROOF: First we show the last part:

$$J^2\eta(a) = J\eta(\alpha_{\frac{i}{2}}(a^*)) = \eta(\alpha_{\frac{i}{2}}(\alpha_{\frac{i}{2}}(a^*))^*) = \eta(\alpha_{\frac{i}{2}}(\alpha_{-\frac{i}{2}}(a))) = \eta(a).$$

Next, we know from the corollary after Prop.3.5 that if $\eta(a) \in \mathcal{H}_+^1$, then $\eta(a^*) \in \mathcal{H}_-^1$, while if $\eta(b) \in \mathcal{H}_+^0$ then $\eta(b^*) \in \mathcal{H}_+^0$ also. But the automorphisms α_z leave \mathcal{H}_\pm invariant (since ω_+ and ω_- are invariant separately), so $\eta(\alpha_{\frac{i}{2}}(a^*)) \in \mathcal{H}_-^1$ and $\eta(\alpha_{\frac{i}{2}}(b^*)) \in \mathcal{H}_+^0$. J is obviously antilinear since it involves the antilinear operation $a \rightarrow a^*$. Finally J is norm preserving. Let $\eta(a) \in \mathcal{H}_+^1$. Then we calculate:

$$\begin{aligned} (J\eta(a), J\eta(a))_- &= \omega_-((\alpha_{\frac{i}{2}}(a^*))^* \alpha_{\frac{i}{2}}(a^*)) = \omega_-(\alpha_{-\frac{i}{2}}(a) \alpha_{\frac{i}{2}}(a^*)) \\ &= \omega_-(a \alpha_i(a^*)) = \omega_+(a^* a) \\ &= (\eta(a), \eta(a))_+ \end{aligned}$$

For the scalar product of two elements $\eta(a), \eta(b) \in \mathcal{H}_+^1$ we obtain:

$$\begin{aligned} (J\eta(a), J\eta(b))_- &= \omega_-(a \alpha_i(b^*)) = \omega_+(b^* a) \\ &= \omega_+((a^* b)^*) = \overline{\omega_+(a^* b)} \\ &= \overline{(\eta(a), \eta(b))_+} . \end{aligned}$$

This is exactly the defining property of an antiunitary map.

For every antiunitary map J there is a noncanonical unitary map U defined as

$$U := K J$$

where K is the operator of complex conjugation with respect to some chosen orthonormal basis in the Hilbert space. We shall investigate later on the effect of the arbitrariness of that choice on the representation we construct. Since always $K^2 = I$ we can easily see that the following hold:

$$U_0^* = JK, \quad U_0^* U = JK K J = I.$$

Indeed, choosing a basis $\{e_\alpha\}$ (not necessarily discrete), we can then expand any two elements x, y in the Hilbert space as discrete sums of basis elements. The operator K acts by complex conjugating the coefficients in this expansion. So we calculate:

$$\begin{aligned} (x, K J y) &= \sum_{i,j} \bar{x}_i y_j (e_i, K J e_j) = \sum_{i,j,l} \bar{x}_i y_j (e_i, K J_{lj} e_l) \\ &= \sum_{i,j,l} \bar{x}_i y_j (e_i, \bar{J}_{lj} e_l) = \sum_{i,j} \bar{x}_i y_j \bar{J}_{ij} \end{aligned}$$

Similarly

$$(J K x, y) = \sum_{i,j} \bar{x}_i y_j (J e_i, e_j) = \sum_{i,j} \bar{x}_i y_j \bar{J}_{i,j}.$$

We only used the fact that J is bounded and antilinear.

We observe the following properties of the different restrictions of U and $\pi(a)$:

$$\begin{aligned} U: \mathcal{H}_\pm^0 &\rightarrow \mathcal{H}_\pm^0, \quad U: \mathcal{H}_\pm^1 \rightarrow \mathcal{H}_\mp^1, \\ \pi(a): \mathcal{H}_\pm^0 &\rightarrow \mathcal{H}_\pm^0, \quad \pi(a): \mathcal{H}_\pm^1 \rightarrow \mathcal{H}_\pm^1 \quad a \in \mathcal{A}_+, \\ \pi(b): \mathcal{H}_\pm^0 &\rightarrow \mathcal{H}_\pm^1, \quad \pi(b): \mathcal{H}_\pm^1 \rightarrow \mathcal{H}_\pm^0, \quad b \in \mathcal{A}_-. \end{aligned}$$

Definition: For every $a \in \mathcal{A}$ define an operator $\pi'(a): \mathcal{H} \rightarrow \mathcal{H}$ as follows:

$$\pi'(a) := U^* \pi(a) U .$$

We now prove the main result of this chapter.

PROPOSITION 3.7. *The operators $\pi'(a)$ are bounded for any $a \in \mathcal{A}$. The map $\pi': \mathcal{A} \rightarrow L(\mathcal{H}_+ \oplus \mathcal{H}_-)$ gives a representation of the \mathbb{Z}_2 graded von Neumann algebra \mathcal{A} .*

Before proceeding with the proof we would like to make the following comments. A representation of a C^* -algebra is by definition a $*$ -algebra homomorphism (such a $*$ -homomorphism is always continuous in the norm topologies) of the given algebra into some $L(\mathcal{H})$. When we have a \mathbb{Z}_2 graded algebra, a representation is a $*$ -algebra homomorphism $\pi': \mathcal{A}_+ \oplus \mathcal{A}_- \rightarrow (L_+ \oplus L_-)(\mathcal{H}_+ \oplus \mathcal{H}_-)$, which commutes with (preserves) the grading. The grading $L_+ \oplus L_-$ is the natural one induced from the grading of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

Even though \mathcal{A} is a von Neumann algebra, it is not true in general, that $\pi'(\mathcal{A})$ is a von Neumann algebra. This is not true also for the standard GNS construction unless the positive functional used in it is a faithful normal functional.

PROOF OF PROPOSITION:

(i) Algebra homomorphism:

As π' is obviously linear, we only need to show that $\pi'(ab) = \pi'(a)\pi'(b), \forall a, b \in \mathcal{A}$.

This is obvious from the definition of π' .

(ii) π' commutes with the grading:

This is evident from the way π' was constructed. For $a \in \mathcal{A}_+$, $\pi'(a): \mathcal{H}_\pm \rightarrow \mathcal{H}_\pm$, i.e. $\pi'(a) \in L_+(\mathcal{H}_+ \oplus \mathcal{H}_-)$, and for $b \in \mathcal{A}_-$, $\pi'(b): \mathcal{H}_\pm \rightarrow \mathcal{H}_\mp$, i.e. $\pi'(b) \in$

$L_-(\mathcal{H}_+ \oplus \mathcal{H}_-)$.

(iii) A $*$ -homomorphism:

This is also immediate from the definition

$$\begin{aligned}\pi'(a)^* &= (U^*\pi(a)U)^* = U^*\pi(a)^*U \\ &= U^*\pi(a^*)U = \pi'(a^*)\end{aligned}$$

where we used the fact that the standard GNS representation π is a $*$ -homomorphism.

This completes the proof of the proposition.

In the representation π' the functional ω is again expressed as a graded vacuum expectation value.

$$\omega(a) = (\Omega, \Gamma\pi'(a)\Omega) .$$

Indeed, for $a \in \mathcal{A}$ we have

$$\begin{aligned}|\omega|(a) &= (\Omega, \pi(a)\Omega) = (U\Omega, \pi(a)U\Omega) \\ &= (\Omega, U^*\pi(a)U\Omega) ; \\ \omega(a) &= (\Omega, \Gamma\pi(a)\Omega) = (U\Omega, \Gamma\pi(a)U\Omega) \\ &= (\Omega, U^*\Gamma\pi(a)U\Omega) = (\Omega, \Gamma U^*\pi(a)U\Omega) \\ &= (\Omega, \Gamma\pi'(a)\Omega) .\end{aligned}$$

We have used the fact that $U\Omega = \Omega$ which is obvious from the definitions of U and Ω . Also both $\pi(a)$ and Γ leave \mathcal{H}^0 invariant and since $U^*|_{\mathcal{H}^0}$ is block diagonal, we can interchange U^* and Γ .

The vacuum Ω is a cyclic vector for the representation π' . Indeed, for $a \in \mathcal{A}$

$$\pi'(a)\Omega = U^*\pi(a)U\Omega = U^*\pi(a)\Omega$$

As $\pi(a)\Omega$ are dense in \mathcal{H} it follows that so are $\pi'(a)\Omega$. We investigate the effect of the arbitrariness in defining the operator K of complex conjugation with respect to some basis on the representation π' . We can in fact show a general result.

PROPOSITION 3.8. *Let $(\mathcal{H}', \pi', \Omega')$ and $(\mathcal{H}'', \pi'', \Omega'')$ be two graded representations of \mathcal{A} in \mathbb{Z}_2 graded Hilbert spaces and let Γ' and Γ'' are the operators represented both as $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ relative to the respective decompositions of \mathcal{H}' and \mathcal{H}'' . Suppose that the two representations satisfy $(\Omega', \Gamma' \pi'(a) \Omega') = (\Omega'', \Gamma'' \pi''(a) \Omega'')$ and $(\Omega', \pi'(a) \Omega') = (\Omega'', \pi''(a) \Omega'') \forall a \in \mathcal{A}$. Then π' and π'' are unitarily equivalent as graded representations with an intertwining map V that respects the gradings of the two Hilbert spaces.*

PROOF: Indeed, we can easily see that the above property assures that $\|\Omega''_{\pm}\| = \|\Omega'_{\pm}\|$. We can now define the intertwining operator to have the property $V \Omega'_{\pm} := \Omega''_{\pm}$. Now define it as usually on dense subspaces by $V \pi'(a) \Omega' := \pi''(a) \Omega'$. We show that V respects the grading of the Hilbert spaces. It is enough to prove that it intertwines Γ' and Γ'' . Because of the way we defined V , the following is true

$$V \Gamma' \Omega' = \Gamma'' \Omega'' .$$

A simple calculation shows

$$\begin{aligned} \Gamma'' V \pi'(a) \Omega' &= \Gamma'' \pi''(a) \Omega'' = \pi''(a^{\Gamma}) \Gamma'' \Omega'' \\ &= \pi''(a^{\Gamma}) V \Gamma' \Omega' = V \pi'(a^{\Gamma}) \Gamma' \Omega' \\ &= V \Gamma' \pi'(a) \Omega' . \end{aligned}$$

This proves the proposition.

We have the following

COROLLARY. In our construction of the graded representation π' , any two different choices of K lead to unitarily equivalent graded representations (with intertwining operator respecting the grading of the Hilbert space. (Note that the Hilbert space is the same in this case for the two representations.)

PROOF: We just have to recall that we showed $(\Omega, \pi'(a)\Omega) = |\omega|(a)$ and $(\Omega, \Gamma\pi'(a)\Omega) = \omega(a)$. As the right-hand sides are independent of K it is obvious that any two representations π'_1 and π'_2 satisfy the conditions of Proposition 3.8.

We would like to mention that in fact the representations π and π' are unitarily equivalent. However the first one does not respect the grading of \mathcal{A} while the second one does. The intertwining map V that realizes the unitary equivalence does not respect the grading of \mathcal{H} .

The representation π' is reducible. The two closed subspaces $\mathcal{H}_+^0 \oplus \mathcal{H}_-^1$ and $\mathcal{H}_+^1 \oplus \mathcal{H}_-^0$ are easily seen to be invariant. This situation is by no means unique, because the standard GNS construction for a functional which is the trace with some trace-class operator is also in general highly reducible.

We turn now to the question of implementing the *-automorphism group α_t as an automorphism group of $\pi'(\mathcal{A})$. We define a family of operators W_t on a dense domain:

$$W_t \pi'(a)\Omega := \pi'(\alpha_t(a))\Omega .$$

PROPOSITION 3.9. *For each fixed t the operator W_t is unitary.*

PROOF: W_t apparently has a dense range. We show that it is an isometry.

$$\begin{aligned}
(W_t \pi'(a)\Omega, W_t \pi'(a)\Omega) &= (\pi'(\alpha_t(a)\Omega, \pi'(\alpha_t(a)\Omega))\Omega) \\
&= (U^*(\pi(\alpha_t(a))\Omega, U^*\pi(\alpha_t(a))\Omega) = \pi(\alpha_t(a))\Omega, \pi(\alpha_t(a))\Omega) \\
&= |\omega|((\alpha_t(a))^*\alpha_t(a)) = |\omega|(\alpha_t(a^*a)) \\
&= |\omega|(a^*a) = (\pi(a)\Omega, \pi(a)\Omega) \\
&= (\pi'(a)\Omega, \pi'(a)\Omega) .
\end{aligned}$$

The automorphism group α_t may be defined to have different continuity properties, depending on the context in which one is working. In the context of C^* -algebras α_t is taken to be continuous in the sense of pointwise norm convergence. In other words one requires that for any fixed element $a \in \mathcal{A}$ the function $\alpha(a): t \rightarrow \alpha_t(a)$ is continuous function from \mathbb{R} into \mathcal{A} (with respect to the norm topology on \mathcal{A}). The triple $(\mathcal{A}, \mathbb{R}, \alpha)$ with α as above is an example of what is referred to as a C^* -dynamical system.

We can easily prove that if α is taken to have the continuity property as above, then W_t is a strongly continuous one-parameter unitary group. Instead we shall prove that the same result is true under a weaker continuity assumption for α_t in the situation when \mathcal{A} is a von Neumann algebra, namely we shall assume α_t to be pointwise σ -weakly continuous. We describe this situation in some detail.

Most often, when \mathcal{A} is a concrete von Neumann algebra (of bounded operators on a Hilbert space), a one-parameter $*$ -automorphism group α_t is defined to be pointwise-weakly continuous, i.e. the function $\alpha(a): \mathbb{R} \rightarrow \alpha_t(a)$ is a continuous map into \mathcal{A} with respect to the weak topology on the latter for any fixed a . It turns out [cf. e.g. 16] that this continuity coincides with pointwise strong and pointwise σ -weak continuity. This is so, because the three topologies coincide on

norm bounded subsets of \mathcal{A} , in particular they coincide on the subset of unitary elements $U(\mathcal{A})$, which is stable under $*$ -automorphisms and generates linearly all of \mathcal{A} . Therefore we can as well assume α_t to be pointwise σ -weakly continuous.

The triple $(\mathcal{A}, \mathbb{R}, \alpha)$ with α as above is an example of what is known as a W^* -dynamical system.

Assuming α_t to have the pointwise σ -weak continuity, the family of operators W_t is a strongly continuous one-parameter unitary group. The proof reduces to the standard one and we show it just for completeness of the exposition.

We have to show that $\|(W_t - W_{t_0})\pi'(a)\Omega\|$ is a continuous function of t at $t = t_0$. In fact because of the group property it is enough to take $t_0 = 0$.

$$\begin{aligned} \|(W_t - I)\pi'(a)\Omega\|^2 &= (W_t\pi'(a)\Omega, W_t\pi'(a)\Omega) + (\pi'(a)\Omega, \pi'(a)\Omega) \\ &\quad - (\pi'(a)\Omega, W_t\pi'(a)\Omega) - (W_t\pi'(a)\Omega, \pi'(a)\Omega) \\ &= 2\|\pi'(a)\Omega\|^2 - (\pi'(a)\Omega, \pi'(\alpha_t(a))\Omega) \\ &\quad - (\pi'(\alpha_t(a))\Omega, \pi'(a)\Omega) \\ &= 2|\omega|(a^*a) - |\omega|(a^*\alpha_t(a)) - |\omega|(\alpha_t(a^*)a). \end{aligned}$$

Remember that $|\omega|$ is a σ -weakly continuous positive functional and notice, that it follows that $|\omega|(a^*\cdot)$ and $|\omega|(\cdot a)$ define also σ -weakly continuous functionals. Also the involution $*$ is σ -weakly continuous. Putting these facts together we see that the functions $t \rightarrow |\omega|(a^*\alpha_t(a))$ and $t \rightarrow |\omega|(\alpha_t(a^*)a)$ are continuous functions of t . Thus we get

$$\begin{aligned} \lim_{t \rightarrow 0} \|(W_t - I)\pi'(a)\Omega\| &= 2|\omega|(a^*a) - \lim_{t \rightarrow 0} (|\omega|(a^*\alpha_t(a)) + |\omega|(\alpha_t(a^*)a)) \\ &= 2|\omega|(a^*a) - |\omega|(a^*a) - |\omega|(a^*a) \\ &= 0, \end{aligned}$$

which proves the strong continuity of W_t

At the end of this chapter we return to the example $\mathcal{A} = L(\mathcal{V}_+ \oplus \mathcal{V}_-)$. The positive trace-class operator ρ which we assumed to be strictly positive (no zero eigenvalues) can be used to define a $*$ -automorphism group α_t with respect to which ω is sKMS. We take $H := -\log \rho$. Recall that ρ was normalized so that the sum of its eigenvalues is 1 and therefore it has spectrum in the interval $(0,1]$ and it follows that H is positive. We define $\alpha_t(a) := e^{iHt} a e^{-iHt}$. The reason that we choose exactly this automorphism group to be the one in the example is because, as we shall see in the next chapter, it is in fact the unique one with respect to which ω is sKMS. We already pointed out in the introduction, that indeed if ω is a supertrace with the trace-class heat kernel e^{-H} , then it is (weakly) sKMS with respect to the unitary group e^{iHt} . It is easy to see on this example that the even elements, that are in the support of ω_+ i.e elements of the form $a = \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$ are mapped by $*$ to elements of the same type. In the same time the odd elements that are in the support of ω_+ , i.e. elements of the form $b = \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \end{pmatrix}$ are mapped by $*$ to ones, that are in the support of ω_- . So it looks “almost obvious” that the involution should provide us with a nice invertible map from the odd elements in the support of ω_+ to the odd elements in the support of ω_- . (After all the upper right and lower left blocks of a matrix “look the same”.) The amazing fact is that this idea works for any von Neumann algebra with a sKMS on it. Let us show that in the example the map J , $J\eta(a) = \eta(\alpha_{\frac{i}{2}}(a^*))$ with a odd is indeed antiunitary. It is antilinear because $*$ is such. We calculate for the norm of $\eta(a)$

$$\begin{aligned} (J\eta(a), J\eta(a)) &= (\alpha_{\frac{i}{2}}(a^*), \alpha_{\frac{i}{2}}(a^*)) = \text{Tr} \left(\left(e^{-\frac{H}{2}}(a^*)e^{\frac{H}{2}} \right)^* e^{-\frac{H}{2}}(a^*)e^{\frac{H}{2}} e^{-H} \right) \\ &= \text{Tr} \left(e^{\frac{H}{2}} a e^{-H} a^* e^{-\frac{H}{2}} \right) = \text{Tr} (a e^{-H} a^*) \\ &= \text{Tr} (a^* a e^{-H}) = (\eta(a), \eta(a)) , \end{aligned}$$

which completes the argument.

CHAPTER IV

A TOMITA - TAKESAKI THEOREM FOR NONPOSITIVE FUNCTIONALS AND THE SUPER-KMS CONDITION

Recall that the Tomita - Takesaki theory [18] gives us a canonical σ -weakly continuous one-parameter group of $*$ -automorphisms of the von Neumann algebra \mathcal{A} for every normal faithful positive linear functional ϕ . This group is called the modular group corresponding to ϕ and we shall denote it by σ_t^ϕ or simply by σ_t when the meaning is clear. It turns out that ϕ satisfies the KMS condition with respect to σ_t^ϕ and furthermore if ϕ is KMS with respect to any σ -weakly continuous one-parameter group of $*$ -automorphisms α_t , then $\alpha_t = \sigma_t^\phi$, i.e. the modular group is (up to normalization) the unique one with respect to which ϕ is KMS.

In the previous chapter we assumed the existence of a (weakly) sKMS functional on the \mathbb{Z}_2 graded von Neumann algebra under consideration. The sKMS property was used by us in an essential way in order to construct a graded (with respect to what seems to us a natural decomposition of the Hilbert space) representation of the algebra. In this section we would like to reverse the viewpoint and study the question of whether an arbitrary normal nonpositive functional ω on a von Neumann algebra \mathcal{A} induces in a canonical way a \mathbb{Z}_2 grading and a $*$ -automorphism group σ_t (the analog of the modular group) and whether ω satisfies the sKMS condition with respect to σ_t .

It becomes fairly clear from the considerations in the previous chapter that one cannot expect to start with an algebra that is already \mathbb{Z}_2 graded and an arbitrary

normal nonpositive functional and be able to find a *-automorphism group with respect to which ω is sKMS unless the grading and the functional satisfy certain consistency condition.

Throughout this section ω will always be a normal (i.e. σ -weakly continuous) nonpositive self-adjoint functional with norm 1. We shall need some notion to replace the notion of a faithful positive functional which is the assumption in the standard Tomita – Takesaki theory. Recall that a positive functional ϕ is called faithful if $a \geq 0, a \neq 0$ implies $\phi(a) > 0$. We would like ω to be such that $|\omega|$ be faithful. This justifies the following

Definition: A normal self-adjoint functional ω will be called faithful if $a \geq 0, \omega(a\chi) = 0, \forall \chi \in \mathcal{A}$ implies $a = 0$.

LEMMA 4.1. *With the above definition ω is faithful if and only if $|\omega|$ is faithful in the usual sense.*

PROOF: Suppose first that $|\omega|$ is faithful and let us take an element $a \geq 0$ such that $\omega(a\chi) = 0$ for all projections χ . Then in particular $\omega_+(a) = \omega(a\chi_+) = 0$ and $\omega_-(a) = -\omega(a\chi_-) = 0$ and it follows that $|\omega|(a) = 0$. By assumption $|\omega|$ is faithful and therefore $a = 0$. Thus it follows that ω is faithful. Conversely, suppose that ω is faithful and take $a \geq 0$ such that $|\omega|(a) = 0$. Since $|\omega| = \omega_+ + \omega_-$, it follows that $\omega_\pm(a) = 0$. Let $b := \sqrt{a}$. Then

$$\begin{aligned} |\omega_\pm(a\chi)| &= |\omega_\pm(bb\chi)| \leq \omega_\pm(bb^*)^{\frac{1}{2}} \omega_\pm(\chi^*b^*b\chi)^{\frac{1}{2}} \\ &= \omega_\pm(a)^{\frac{1}{2}} \omega_\pm(\chi a\chi)^{\frac{1}{2}} \\ &= 0. \end{aligned}$$

Therefore $\omega(a\chi) = 0$ for any (projection) χ and it follows that $a = 0$ because ω was assumed faithful.

The GNS construction with the positive faithful normal functional $|\omega|$ gives a faithful representation of \mathcal{A} as a von Neumann subalgebra of $L(\mathcal{H})$. Recall that a concrete $*$ -subalgebra of $L(\mathcal{H})$ is called a von Neumann algebra if it coincides with its second commutant. This, according to the double commutant theorem, is equivalent to it being a nondegenerate weakly closed subalgebra of $L(\mathcal{H})$. As in the previous chapter, we have a natural orthogonal decomposition of \mathcal{H} into a direct sum: $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm = (\eta(\mathcal{A}\chi_\pm))^-$.

We now take a closer look at the Tomita - Takesaki operator (see e.g. [34]) $T_0: \mathcal{H} \rightarrow \mathcal{H}$ defined on a dense domain by $T_0\eta(a) = \eta(a^*)$, or equivalently $T_0(\pi(a)\Omega) = \pi(a^*)\Omega$. The operator T_0 is a closable antilinear invertible (in the sense of unbounded operators, i.e. 1-1 and with a dense range) operator. Its closure T has the polar decomposition $T = J\Delta^{\frac{1}{2}}$, where J is an antiunitary operator with $J^2 = 1$ and Δ is a positive self-adjoint (unbounded in general) invertible operator. Δ is called the modular operator. According to the Tomita - Takesaki theorem Δ generates a σ -weakly continuous one-parameter $*$ -automorphism group. More specifically, the strongly continuous one-parameter unitary group Δ^{it} leaves $\pi(\mathcal{A})$ invariant, i.e.

$$\Delta^{it}\pi(\mathcal{A})\Delta^{-it} = \pi(\mathcal{A})$$

and the automorphism group defined by

$$\sigma_t^{|\omega|}(a) = \pi^{-1}(\Delta^{it}\pi(a)\Delta^{-it})$$

is σ -weakly continuous. (Recall that $|\omega|$ is faithful, so the GNS representation π is a $*$ -isomorphism and π^{-1} is well defined.)

As mentioned in the beginning of the chapter, $|\omega|$ satisfies the KMS condition with respect to the modular group $\sigma_t^{|\omega|}$. In particular, according to a standard

result, from the KMS property it follows that $|\omega|$ is invariant under the action of $\sigma_t^{|\omega|}$. Not every *-automorphism leaving $|\omega|$ invariant has to leave also ω_{\pm} separately invariant (unlike the situation in the previous chapter where from the invariance of ω we deduced the invariance of ω_{\pm} for any *-automorphisms). It turns out, however, that $\sigma_t^{|\omega|}$ does leave ω_{\pm} invariant as we shall now show.

PROPOSITION 4.2. *Suppose $|\omega| = \omega_+ + \omega_-$ is a decomposition of the normal positive faithful functional $|\omega|$ into two orthogonal positive functionals ω_{\pm} . Then the modular group $\sigma_t^{|\omega|}$, corresponding to $|\omega|$, leaves ω_{\pm} invariant.*

PROOF: As before we have projections $\chi_{\pm} \in \mathcal{A}$ onto the supports of ω_{\pm} , namely $\omega_{\pm}(a) = |\omega|(a\chi_{\pm})$ for any $a \in \mathcal{A}$. As $|\omega|$ is faithful, in this case χ_{\pm} are unique. Indeed, suppose that χ'_+ is a second projection onto the support of ω_+ . Then for any $a \in \mathcal{A}$ we have:

$$\begin{aligned} |\omega|(a(\chi_+ - \chi'_+)) &= |\omega|(a\chi_+) - |\omega|(a\chi'_+) \\ &= \omega_+(a) - \omega_+(a) = 0 \end{aligned}$$

This, together with $|\omega|$ being faithful, implies $\chi_+ = \chi'_+$.

In a similar fashion we show that $\chi_+ + \chi_- = 1$. Indeed, for any $a \in \mathcal{A}$

$$|\omega|(a(1 - \chi_+ - \chi_-)) = |\omega|(a) - \omega_+(a) - \omega_-(a) = 0,$$

so the statement again follows from $|\omega|$ being faithful.

Now we use the fact, that $|\omega|$ is KMS with respect to the modular group $\sigma_t^{|\omega|}$, which we shall denote by σ_t from now on. Take an element $a \in \mathcal{A}_{\sigma}$, where, as in chapter I, \mathcal{A}_{σ} is the dense subalgebra of \mathcal{A} for which $\sigma_t(a)$ continues to an

entire function $\sigma_z(a)$.

$$\begin{aligned}
\omega_+(a) &= |\omega|(a\chi_+) = |\omega|(\chi_+\sigma_i(a)) \\
&= \omega_+(\chi_+\sigma_i(a)) + \omega_-(\chi_+\sigma_i(a)) = \omega_+(\chi_+\sigma_i(a)) \\
&= \omega_+((\chi_+ + \chi_-)\sigma_i(a)) = \omega_+(\sigma_i(a))
\end{aligned} \tag{4.1}$$

where we have used the fact that $\omega_-(\chi_+a) = 0 = \omega_+(\chi_-a)$ for any $a \in \mathcal{A}$, which is an immediate consequence of the Schwarz inequality. From this point on the argument is standard and repeats the one used in the proof of the fact that a KMS functional with respect to an automorphism group is invariant with respect to it [cf e.g. 34]. We show briefly how the proof goes. Applying (4.1) to $\alpha_z(a)$ instead of a one obtains an entire function $F(z) := \omega_+(\alpha_z(a))$, which is bounded on the real line and has the property $F(z+i) = F(z)$. By a version of the Phragmen - Lindelöf theorem it follows $F(z) = \text{const}$ or $\omega_+(\alpha_z(a)) = \omega_+(\alpha_0(a)) = \omega_+(a)$. Of course an equivalent proof shows $\omega_- \circ \alpha = \omega_-$.

It is known, that the GNS representation π related to a normal faithful functional $|\omega|$ is σ -weakly continuous and the inverse isomorphism π^{-1} is also σ -weakly continuous. One consequence of this is that $\pi(\mathcal{A})$ is a von Neumann algebra whenever \mathcal{A} is and not just a C^* -algebra. We now define the following σ -weakly closed subspaces of \mathcal{A} :

$$\mathcal{A}_+^{(+)} := \{ a \in \mathcal{A} \mid \omega_-(a^*a) = 0, \omega_-(aa^*) = 0 \} \equiv \chi_+ \mathcal{A} \chi_+$$

$$\mathcal{A}_+^{(-)} := \{ a \in \mathcal{A} \mid \omega_+(a^*a) = 0, \omega_+(aa^*) = 0 \} \equiv \chi_- \mathcal{A} \chi_-$$

$$\mathcal{A}_-^{(+)} := \{ a \in \mathcal{A} \mid \omega_-(a^*a) = 0, \omega_+(aa^*) = 0 \} \equiv \chi_- \mathcal{A} \chi_+$$

$$\mathcal{A}_-^{(-)} := \{ a \in \mathcal{A} \mid \omega_+(a^*a) = 0, \omega_-(aa^*) = 0 \} \equiv \chi_+ \mathcal{A} \chi_-$$

$$\mathcal{A}_+ := \mathcal{A}_+^{(+)} \oplus \mathcal{A}_+^{(-)}; \quad \mathcal{A}_- := \mathcal{A}_-^{(+)} \oplus \mathcal{A}_-^{(-)} .$$

First we note that $\omega_{\pm}(a^*a) = 0$ if and only if $\omega_{\pm}(ba) = 0, \forall b$, which in the nontrivial direction follows from the Schwarz inequality. Next we observe that the spaces $\{a \in \mathcal{A} \mid \omega_{\pm}(a^*a) = 0\}$ are σ -weakly closed (because ω_{pm} are σ -weakly continuous) linear subspaces. The spaces $\{a \in \mathcal{A} \mid \omega_{\pm}(aa^*) = 0\}$ are obviously the images of the ones above under the involution $*$, which is σ -weakly continuous [cf.e.g.34] and so is its inverse. (The inverse of $*$ is of course $*$ itself.) Thus the subspaces $\{a \in \mathcal{A} \mid \omega_{\pm}(aa^*) = 0\}$ are also σ -weakly closed. Finally, we have

$$\mathcal{A}_+ = (\{a \mid \omega_-(a^*a) = 0\} \cap \{a \mid \omega_-(aa^*) = 0\}) \oplus (\{a \mid \omega_+(a^*a) = 0\} \cap \{a \mid \omega_+(aa^*) = 0\})$$

$$\mathcal{A}_+ = (\{a \mid \omega_-(a^*a) = 0\} \cap \{a \mid \omega_+(aa^*) = 0\}) \oplus (\{a \mid \omega_+(a^*a) = 0\} \cap \{a \mid \omega_-(aa^*) = 0\})$$

Therefore \mathcal{A}_{\pm} are σ -weakly closed.

Notice that any $a \in \mathcal{A}$ can be decomposed uniquely (since $|\omega|$ is faithful) into four parts, each in one of the four different subspaces above. Indeed, using the fact $\chi_+ + \chi_- = 1$ we can write:

$$a = (\chi_+ + \chi_-)a(\chi_+ + \chi_-) = \chi_+a\chi_+ + \chi_+a\chi_- + \chi_-a\chi_+ + \chi_-a\chi_- . \quad (4.2)$$

Now it is easy to convince ourselves that each of the four terms belongs to exactly one of the four subspaces above. (If it belongs to more than one it is zero.) Thus we see that we have a direct sum decomposition (as a linear space) of the algebra \mathcal{A} :

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- .$$

PROPOSITION 4.3. *The decomposition above makes \mathcal{A} into a \mathbb{Z}_2 graded von Neumann algebra. In particular \mathcal{A}_+ is a von Neumann subalgebra of \mathcal{A} .*

PROOF: We already showed that \mathcal{A}_\pm are σ -weakly closed subspaces. We need to show that the usual algebraic identities for a \mathbb{Z}_2 grading hold, namely that

$$\begin{aligned}\mathcal{A}_+ \cdot \mathcal{A}_+ &\subset \mathcal{A}_+, & \mathcal{A}_+ \cdot \mathcal{A}_- &\subset \mathcal{A}_-, \\ \mathcal{A}_- \cdot \mathcal{A}_+ &\subset \mathcal{A}_-, & \mathcal{A}_- \cdot \mathcal{A}_- &\subset \mathcal{A}_+.\end{aligned}$$

The proof becomes quite trivial once we use the decomposition (4.2). Indeed $a \in \mathcal{A}_+$ means that $a = \chi_+ a \chi_+ + \chi_- a \chi_-$ and $b \in \mathcal{A}_-$ means that $b = \chi_+ b \chi_- + \chi_- b \chi_+$. We check e.g. that $ab \in \mathcal{A}_-$:

$$\begin{aligned}ab &= (\chi_+ a \chi_+ + \chi_- a \chi_-)(\chi_+ b \chi_- + \chi_- b \chi_+) \\ &= \chi_+ a \chi_+ b \chi_- + \chi_- a \chi_- b \chi_+ \subset \mathcal{A}_-.\end{aligned}$$

Finally it is obvious that if $a \in \mathcal{A}_\pm$, then $a^* \in \mathcal{A}_\pm$. Indeed, taking for example $a \in \mathcal{A}_-$, it can be represented as $a = \chi_+ b \chi_- + \chi_- c \chi_+$ for some $b, c \in \mathcal{A}$. Then

$$a^* = \chi_- b^* \chi_+ + \chi_+ c^* \chi_- \subset \mathcal{A}_-.$$

This shows that the grading we have defined commutes with the involution $*$ as is the usual assumption for graded $*$ -algebras. This completes the proof.

We now state the main result of this section.

PROPOSITION 4.4. *Given a faithful normal nonpositive self-adjoint linear functional ω on a von Neumann algebra \mathcal{A} , there exist a canonically defined \mathbb{Z}_2 grading on \mathcal{A} and a σ -weakly continuous one-parameter $*$ -automorphism group σ_t of \mathcal{A} . The grading commutes with the action of the automorphism group. The functional ω has the (weak) sKMS property with respect to σ_t . Furthermore, the canonical \mathbb{Z}_2 grading and the group σ_t are the unique ones for which the above holds.*

Notice that as a consequence of the (weak) sKMS property ω is invariant with respect to σ_t .

PROOF: We take σ_t to be the modular group for $|\omega|$. Proposition 4.2 shows that ω is invariant under σ_t . Taking the \mathbb{Z}_2 grading defined above, we can check that $\sigma_t(\mathcal{A}_+) = \mathcal{A}_+$, $\sigma_t(\mathcal{A}_-) = \mathcal{A}_-$. This follows immediately from the representation $\mathcal{A}_+ = \chi_+ \mathcal{A} \chi_+ \oplus \chi_- \mathcal{A} \chi_-$, $\mathcal{A}_- = \chi_+ \mathcal{A} \chi_- \oplus \chi_- \mathcal{A} \chi_+$ and the fact that since ω_{\pm} are invariant, the projections χ_{\pm} have to be invariant too.

We now show that ω has the (weak) sKMS property. Let $a \in \mathcal{A}_+^{(+)} \equiv \chi_+ \mathcal{A} \chi_+$. This is the same as the condition $\omega_-(a^*a) = \omega_-(aa^*) = 0$. Then for any $b \in \mathcal{A}$, we can write

$$\begin{aligned} \omega(ba) &= \omega_+(ba) - \omega_-(ba) = \omega_+(ba) + \omega_-(ba) \\ &= |\omega|(ba) = |\omega|(a\sigma_i(b)) = \omega_+(a\sigma_i(b)) \\ &= \omega(a\sigma_i(b)). \end{aligned}$$

We have used again the trick of adding and subtracting terms like $\omega_-(ba)$, which is zero because $\omega_-(a^*a) = 0$, as well as the KMS property of $|\omega|$.

The same equality holds for $a \in \mathcal{A}_+^{(-)} \equiv \chi_- \mathcal{A} \chi_-$ and therefore for any $a \in \mathcal{A}_+$.

Take now $a \in \mathcal{A}_-^{(+)} \equiv \chi_- \mathcal{A} \chi_+$. This implies $\omega_-(a^*a) = \omega_+(aa^*) = 0$. Then we calculate

$$\begin{aligned} \omega(ab) &= \omega_+(ba) = |\omega|(ba) \\ &= |\omega|(a\sigma_i(b)) = \omega_-(a\sigma_i(b)) \\ &= -\omega(a\sigma_i(b)). \end{aligned}$$

The same holds for $a \in \mathcal{A}_-^{(-)} \equiv \chi_+ \mathcal{A} \chi_-$ and therefore for any $a \in \mathcal{A}_-$.

Putting the two cases together, we get

$$\omega(ba) = \omega(a^\Gamma \sigma_i(b)), \quad \forall b \in \mathcal{A}_\alpha, a \in \mathcal{A}.$$

Now suppose we are given an arbitrary \mathbb{Z}_2 grading Γ' of \mathcal{A} and an arbitrary σ -weakly continuous one-parameter group of *-automorphisms α_t , with respect to which ω is (weakly) sKMS. Then using the corollary after Prop. 3.5, we see that

$|\omega|$ is KMS with respect to α_t . (This result was independent of the particular \mathbb{Z}_2 grading chosen, as long as it commuted with the involution $*$ and the action of the automorphism group.) As mentioned in the beginning of this chapter, the modular group is the unique one with respect to which $|\omega|$ satisfies the KMS condition [cf., e.g., 34]. Thus $\alpha_t \equiv \sigma_t$.

Now we recall that the elements odd with respect to Γ' satisfy:

$$\omega_-(a^*a) = 0 \iff \omega_+(aa^*) = 0 ,$$

which was a consequence of the sKMS property of ω . At the same time the even elements satisfy:

$$\omega_{\pm}(a^*a) = 0 \iff \omega_{\pm}(aa^*) = 0 .$$

But these were exactly the conditions according to which we decomposed \mathcal{A} into the direct sum $\mathcal{A}_+ \oplus \mathcal{A}_-$ in our canonical construction. Therefore we see that the grading Γ' has to coincide with the canonical one.

Proposition 4.4 shows that while the KMS condition follows naturally through the Tomita - Takesaki theory for positive normal faithful functionals, the natural objects for the case of self-adjoint nonpositive normal faithful functionals appear to be \mathbb{Z}_2 graded algebras and the sKMS property.

CHAPTER V

THE SUPERDERIVATION

In this chapter we take the normal self-adjoint sKMS functional ω to be faithful. In the scheme of Jaffe et al. [15,35] one has to assume, as was described in the introduction, additional structure, necessary for the construction of a nontrivial cocycle in the entire cyclic cohomology of the algebra. It is required that a superderivation d is defined on a dense subalgebra $\mathcal{A}_d \in \mathcal{A}$ and that $D := d^2$ is the (hermitian) generator of the *-automorphism group α_t . (The operator D is defined as $D := -i \frac{d}{dt} \alpha_t |_{t=0}$.) We would like to find a natural operator in the Hilbert space which implements the superderivation.

Let us introduce the notation $\gamma := \chi_+ - \chi_-$. As we showed in chapter 4, it follows from ω being faithful that the minimal projections χ_{\pm} are actually the unique projections onto the supports of ω_{\pm} and that $\chi_+ + \chi_- = 1$. We present a useful lemma.

LEMMA 5.1. $\pi'(\chi_{\pm})$ are the identity operators on \mathcal{H}_{\pm} , respectively, and zero on \mathcal{H}_{\mp} , respectively. $\pi'(\gamma) = \Gamma$ with $\Gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as in sec 3.1.

PROOF: The elements $\pi'(a)\Omega$ with $a \in \chi_+ \mathcal{A} \chi_+ \oplus \chi_+ \mathcal{A} \chi_-$ form a dense subset in \mathcal{H}_+ . Indeed $\pi'(a)\Omega = U^* \pi(a)\Omega$. Let $a = b + c$ with $b = \chi_+ b \chi_+$ and $c = \chi_+ c \chi_-$ and recall that the above implies $\pi(b)\Omega \in \mathcal{H}_+^1$ and $\pi(c)\Omega \in \mathcal{H}_-^2$. U^* maps \mathcal{H}_-^2 onto \mathcal{H}_+^2 and is the identity on \mathcal{H}_+^1 , and so $\pi'(a)\Omega$ is in \mathcal{H}_+ . The fact that the elements are dense follows from the unitarity of U^* , which maps a dense set to a

dense set. But now we have

$$\begin{aligned}\pi'(\chi_+)\pi'(a)\Omega &= \pi'(\chi_+(\chi_+b\chi_- + \chi_+c\chi_-))\Omega \\ &= \pi'(\chi_+b\chi_+ + \chi_+c\chi_-)\Omega = \pi'(a)\Omega .\end{aligned}$$

Similar argument shows that a dense set in \mathcal{H}_- is given by the set of elements $\pi'(\chi_-b\chi_- + \chi_-c\chi_+)\Omega$ and it is trivial to check that $\pi'(\chi_+)$ is zero on this subspace. (Remember that $\chi_+\chi_- = \chi_-\chi_+ = 0$.) An analogous proof works for $\pi'(\chi_-)$ and the proof of the lemma is complete.

Note that a similar identity is not true for the representation π .

Let us consider possible representations of the superderivation. The first construction that comes to mind is to define an operator R on a dense domain as follows:

$$R\pi'(a)\Omega := -i\pi'(da)\Omega, \quad a \in \mathcal{A}_d .$$

The (possibly unbounded) operator R represents the superderivation, namely

$$\pi'(da) = [iR, \pi'(a_+)] + \{iR, \pi'(a_-)\}, \quad a \in \mathcal{A}_d, \quad (5.1)$$

where a_{\pm} are the even and odd parts of a . $\pi'(a_{\pm})$ is the same as $(\pi'(a))_{\pm}$ due to the fact that π' is a grading preserving representation. To prove (5.1) we can apply both sides to a vector $\pi'(b)$ and use the superderivation rule $d(ab) = (da)b + a^{\Gamma}db$. By its construction the operator R annihilates the vacuum Ω .

There are several reasons why this representation of the superderivation is not very useful. First, it is neither even nor odd with respect to the natural grading of $L(\mathcal{H}_+ \oplus \mathcal{H}_-)$. We can see this on the example $\mathcal{A} = L(\mathcal{V}_+ \oplus \mathcal{V}_-)$ with d being given by the graded commutator (by which we mean commutator with even and

anticommutator with odd elements as in the definition (5.1)) with an odd operator $Q = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix}$. In the GNS construction the elements of \mathcal{H}_+^2 are generated as $\pi'(a)\Omega$ with a being of the type $a = \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix}$. ($\pi(a)\Omega \in \mathcal{H}_-^2$, but then the operator U^* maps it to \mathcal{H}_+^2 .) Since

$$da = \{iQ, a\} = \begin{pmatrix} aQ_+ & 0 \\ 0 & Q_+a \end{pmatrix},$$

the vector $\pi'(da)\Omega = R\pi'(a)\Omega$ has components both in \mathcal{H}_+ and \mathcal{H}_- . Next recall that when one starts with a trace (or supertrace in our case) functional on $L(\mathcal{V})$ for some infinite-dimensional separable Hilbert space \mathcal{V} , the GNS Hilbert space \mathcal{H} is isomorphic to the direct sum of infinitely many copies of \mathcal{V} , and the GNS vacuum is given by the vector $\rho_1^{\frac{1}{2}}e_1 \oplus \rho_2^{\frac{1}{2}}e_2 \oplus \rho_3^{\frac{1}{2}}e_3 \oplus \dots$ where e_i is the i -th basis vector of \mathcal{V} taken as an element of the i -th space in the infinite direct sum above and ρ_i are the eigenvalues of the operator ρ defining ω (see the end of chapter 3). In this picture π is just the direct sum of copies of the defining representation. One might expect that the operator Q is represented by a similar direct sum. However, the operator R that was defined above does not act in this way, in particular $R\Omega = 0$, while an operator like $Q \oplus Q \oplus Q \dots$ does not annihilate the GNS vacuum. Since we would like our construction to recover as closely as possible the original objects in the case when our von Neumann algebra is an algebra of operators to begin with, we seek another representation of the superderivation. Let us assume just for now that $\chi_{\pm} \in \mathcal{A}_d$. As we shall see shortly, such an assumption is equivalent to d being an inner superderivation, which is too strong a restriction. Therefore we shall drop it later on. Define an element $q \in \mathcal{A}$,

$$q := d\chi_+\gamma. \tag{5.2}$$

The element q is obviously odd. Every odd element defines a superderivation on a \mathbb{Z}_2 graded algebra by its (super)adjoint action on it, i.e. by taking a graded commutator with the elements of \mathcal{A} .

PROPOSITION 5.2. *The superderivation defined by q coincides with the superderivation d in the case when ω is a sKMS functional in the strong sense. (That is, we assume in addition $\omega(da) = 0$.)*

PROOF: Let us take first $a \in \mathcal{A}_+$. We want to show that $[d\chi_+\gamma, a] - da = 0$. For this it is enough to prove

$$|\omega|(b([d\chi_+\gamma, a] - da)), \quad \forall b \in \mathcal{A}$$

because $|\omega|$ is a faithful positive functional. Since we chose a to be even, the only nontrivial case is b odd. Thus we can calculate

$$\begin{aligned} & |\omega|(b d\chi_+ \gamma a) - |\omega|(b a d\chi_+ \gamma) - |\omega|(b da) \\ &= \omega(b d\chi_+ \gamma a \gamma) - \omega(b a d\chi_+ \gamma^2) - \omega(b da \gamma) \\ &= \omega(b d\chi_+ a) - \omega(b a d\chi_+) - \omega(b da \gamma) \\ &= \omega(db \chi_+ a) - \omega(b \chi_+ da) - \omega(db a \chi_+) + \omega(b da \chi_+) - \omega(b da \gamma) \\ &= \omega_+(db a) + \omega_-(b da) - \omega_+(db a) + \omega_+(b da) - \omega_+(b da) - \omega_-(b da) \\ &= 0. \end{aligned}$$

(Strictly speaking, one has to approximate a and b by elements from \mathcal{A}_d , perform the calculation and then pass to the limit.)

We use the following simple identities in this calculation: $\gamma^2 = 1$, which follows from the fact that χ_{\pm} are disjoint projections which sum up to one; $\gamma a \gamma = a^{\Gamma} = a$ because a is even; as we showed in Proposition 3.1 terms like $\omega_+(b \chi_- a)$ are zero when a is even while if a is odd $\omega_+(b \chi_+ a)$ is zero as proven in Lemma 3.4,

and we used the strong sKMS condition in the fourth line in order to move the superderivation d . Take now a to be odd. We have to show that $\{\chi_+, \gamma, a\} - da = 0$. We take an even element b and perform a similar calculation

$$\begin{aligned}
& |\omega|(bd\chi_+ \gamma a) + |\omega|(ba d\chi_+ \gamma) - |\omega|(b da) \\
&= \omega(b d\chi_+ \gamma a \gamma) + \omega(b a d\chi_+) - \omega(b da \gamma) \\
&= -\omega(b d\chi_+ a) + \omega(b a d\chi_+) - \omega(b da \gamma) \\
&= \omega(db \chi_+ a) + \omega(b \chi_+ da) + \omega(b da \chi_+) + \omega(dba \chi_+) - \omega(b da \gamma) \\
&= -\omega_-(dba) + \omega_+(b da) + \omega_+(b da) \\
&+ \omega_+(dba) - \omega_+(b da) - \omega_-(b da) \\
&= \omega(dba) + \omega(b da) \\
&= 0 .
\end{aligned}$$

Having found an inner superderivation q , we define an operator Θ acting in the GNS Hilbert space \mathcal{H} in the obvious way

$$\Theta := \pi'(q) .$$

Note that because $q \in \mathcal{A}_-$, Θ is an odd operator in $L(\mathcal{H}_+ \oplus \mathcal{H}_-)$.

We would like now to address the case when $\chi_\pm \notin \mathcal{A}_d$. To gain some intuition, let us still use the formal expression $d\chi_+ \gamma$ (we may think that we have approximated χ_+ by elements from \mathcal{A}_d) and look at the quadratic form that Θ defines on $\mathcal{H} \times \mathcal{H}$ with domain $\eta(\mathcal{A}_d) \times \eta(\mathcal{A}_d)$:

$$\begin{aligned}
(\pi'(a)\Omega, \Theta\pi'(b)\Omega) &= |\omega|(a^* d\chi_+ \gamma b) = \omega(a^* d\chi_+ \gamma b \gamma) \\
&= \omega(a^* d\chi_+ b^\Gamma) = -\omega(a^{*\Gamma} d\chi_+ b) \quad . \\
&= \omega(da^* \chi_+ b) + \omega(a^{*\Gamma} \chi_+ db)
\end{aligned}$$

The last line is well defined even when $d\chi_+$ does not exist, as long as $a, b \in \mathcal{A}_d$. So we can use it to define the quadratic form we want. Namely, let $\varphi := \pi'(a)\Omega$, $\psi := \pi'(b)\Omega$ with $a, b \in \mathcal{A}_d$. We define the quadratic form

$$t(\varphi, \psi) := \omega(da^* \chi_+ b) + \omega(a^{*\Gamma} \chi_+ db). \quad (5.3)$$

In general in the unbounded case not every quadratic form comes from a uniquely defined, if any at all, linear operator [36]. We have not been able to find some natural conditions under which the bilinear form t does define a unique operator Θ . In the case at hand there are some additional properties which simplify the matter of recovering the operator Θ from the quadratic form t . We know what to expect for the commutation relations of Θ with any $\pi'(a)$, $a \in \mathcal{A}_d$. We expect them to be $\Theta \pi'(a) - \pi'(a^\Gamma) \Theta = \pi'(da)$. All of the domain $\eta(\mathcal{A}_d)$ is generated by the action of the operators $\pi'(a)$ on the vacuum Ω . Unfortunately, it turns out that the vacuum may not be in the domain of the operator Θ we are trying to construct. To see this we use again our formal expression $\Theta = \pi'(d\chi_+ \gamma)$. For the norm of $\Theta\Omega$ we obtain formally $\|\Theta\Omega\|^2 = |\omega|(\gamma(d\chi_+)^* d\chi_+ \gamma) = \omega((d\chi_+)^* d\chi_+ \gamma)$, which cannot be expected to be finite. Suppose there exists a vector Ω_d which is in the domain of Θ , that is, the antilinear form on $\eta(\mathcal{A}_d)$ given by $t(\varphi, \Omega_d)$ is bounded and therefore extends to an antilinear form on all of \mathcal{H} and suppose that $\pi'(a)\Omega_d$, $a \in \mathcal{A}_d$ are dense in \mathcal{H} . Then the superderivation can be implemented as a densely defined operator Θ on \mathcal{H} . We believe that the existence of such a vector Ω_d can be shown under some reasonable assumptions, for example, the assumption that the quadratic form t is symmetric, but we have not been able to show this.

Let us address briefly the question of when t is symmetric. (A quadratic form

t is called symmetric if $t(\varphi, \psi) = \overline{t(\psi, \varphi)}$.) With φ and ψ as before and using (5.3) we have

$$\begin{aligned} \overline{t(\psi, \varphi)} &= \overline{\omega(db^* \chi_+ a)} + \overline{\omega(b^{*\Gamma} \chi_+ da)} \\ &= \omega(a^* \chi_+ (db^*)^*) + \omega((da)^* \chi_+ b^\Gamma) \quad . \\ &= \omega(a^{*\Gamma} \chi_+ (db^*)^{*\Gamma}) + \omega((da)^{*\Gamma} \chi_+ b) \end{aligned}$$

Comparing this to (5.3) we see that a sufficient condition for t to be symmetric is that the superderivation d has the property $da = (da^*)^{*\Gamma}$ for every a in its domain. As was pointed out in [15], to every superderivation d there corresponds a natural adjoint superderivation d^+ defined by $d^+(a) := (da^*)^{*\Gamma}$ with domain \mathcal{A}_d^* . Thus we see that if d is symmetric, i.e. $d^+ = d$, this leads to a symmetric bilinear form t .

We point out that this condition on d does not assure that the operator R described in the beginning of the chapter will be symmetric, and, in fact, there seems to be no natural condition on the superderivation that does this.

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