

RESPONSE OF NONLINEAR STRUCTURES TO DETERMINISTIC
AND RANDOM EXCITATIONS

by

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(ABSTRACT)

Three methods are developed to study the response of nonlinear systems to combined deterministic and stochastic excitations. They are a second-order closure method, a generalized method of equivalent linearization, and a reduced non-Gaussian closure scheme. Primary resonances of single- and two-degree-of-freedom systems in the presence of different internal (autoparametric) resonances are investigated. We propose these methods to overcome some of the limitations of the existing methods of solution and explain some observed experimental results in the context of the response of randomly excited nonlinear systems. These include non-Gaussian responses, broadening effects, and shift in the resonant frequency. When available, the results of these methods are compared with those obtained by using the exact stationary solutions of the Fokker-Planck-Kolmogorov equation. It is found that the presence of the nonlinearities bend the frequency-response curves and this causes multi-valued regions for the mean-square responses. The multi-valuedness is responsible for a jump phenomenon. The results show

that for some range of parameters, noise can expand the stability region of the mean response. As applications, we study the response of a shallow arch, a string, and a hinged-clamped beam to random excitations.

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CHAPTER 1

INTRODUCTION

1.1 Scope of Work

This dissertation presents the research performed towards the analysis and development of new methods for predicting the response of nonlinear systems to deterministic and stochastic excitations. These are a second-order closure method, a generalized method of equivalent linearization, and a reduced non-Gaussian closure scheme. In the former, we take the following typical steps. First, we use the method of multiple scales to determine the equations describing the modulation of the amplitude and phase of the narrow-band response with time. Second, we separate the averaged equations into mean and fluctuating components. Third, we use these equations to obtain closed-form expressions for the stationary mean and mean-square response. The results show that the presence of the nonlinearity bends the frequency-response curves and this causes multi-valued regions for the mean-square responses. The multi-valuedness is responsible for a jump phenomenon. The results show that for some range of parameters, noise can expand the stability region of the mean response.

The method of equivalent linearization has the widest range of applicability for solving nonlinear stochastic problems. It is based on replacing the nonlinear governing equations by equivalent linear equations whose solution furnishes an approximate solution to the nonlinear response. In achieving this goal, the method of equivalent linearization creates many limitations and overlooks many interesting

phenomena caused by the nonlinearity. In Chapter 3 we propose a generalized method of equivalent linearization to overcome some of these limitations and explain some experimental results in the context of the response of randomly excited nonlinear structures. These include non-Gaussian responses, broadening effects, and shifts in the resonant frequencies. The proposed approach is illustrated by investigating the response of a class of nonlinear systems to a Gaussian white noise and a superharmonic narrow-band random excitation of order-three. The results of this method are compared with those obtained by using the method of equivalent linearization and some exact stationary solutions of the Fokker-Planck-Kolmogorov equation.

The second-order closure method is direct and can be easily extended to the case of multi-degree-of-freedom systems. As an application, the second-order closure method is used to determine the forced response of a two-degree-of-freedom system with quadratic nonlinearities in Chapter 4. The excitation is taken to be the sum of a deterministic harmonic component and a random component. The latter may be either white noise or a narrow-band random excitation. The case of primary resonance of the second mode in the presence of a two-to-one internal (autoparametric) resonance is investigated. Contrary to the results predicted by the linear analysis, the nonlinear analysis reveals that under a narrow-band random excitation of the second mode, it may saturate and spill the energy over to the indirectly excited first mode due to the autoparametric resonance.

In Chapter 5 three closure techniques via the Markov vector procedure are used to determine the response moments of a two-degree-of-freedom system with quadratic and cubic nonlinearities. They are the Gaussian, non-Gaussian, and reduced non-Gaussian closure techniques. The cases of two-to-one, one-to-one, and three-to-one internal resonances are investigated. The system is externally excited by a zero-mean Gaussian white noise. Its stochastic calculus is used to derive a hierarchy of first-order differential equations governing the modulation of the response moments with time. The Gaussian closure scheme generates 14 coupled equations for the first- and second-order response moments and the non-Gaussian closure scheme, based on setting the fifth- and sixth-order cumulants equal to zero, generates 69 coupled equations for the first-order through the fourth-order response moments. The method of multiple scales is used to describe the nature of the response and its characteristics. This information is utilized to drastically reduce the number of the response-moment equations generated by the non-Gaussian closure scheme. The results reveal major discrepancies between the Gaussian and non-Gaussian closure schemes. Close to internal resonance, the response displays an energy exchange between the two internally resonant modes. The response, however, does not display the saturation phenomenon for the case of two-to-one internal resonances. The results are applied to a shallow arch, a string, and a hinged-clamped beam.

1.2 Applications

Most of nature's excitations exhibit randomly fluctuating characteristics and contain a wide spectrum of frequencies that may result in severe vibrations. Due to the highly unpredictable trends of these natural hazard excitations in a deterministic sense, stochastic processes may be used to describe these excitations, which are completely characterized by probability-density functions or certain statistical measures. This explains the wide scope of applications of random vibrations to the solution of practical engineering problems, such as strong motion earthquakes (earthquake engineering), the motion of a ship in a confused sea (ocean engineering), road roughness (transportation engineering), the motion of an aircraft or a missile through a turbulent air (aeronautical engineering), computing machines, the design of particle accelerators, time measurements and coherent radar detection (statistical radio engineering), gusty winds (wind engineering), thermionic noise, and the effect of sea waves on offshore structures (civil engineering).

1.3 System Characterization

1.3.1 Excitations

An excitation is described as an ideal excitation if it has enough power so that there will be no feedback from the system to the source of excitation. Both ideal and nonideal excitations are described as random excitations if it is only possible to characterize them by statistical measures. However, random and deterministic excitations could be

categorized as parametric or nonparametric (external) excitations.

Parametric excitations appear as variable coefficients in the differential equations of motion and/or the boundary conditions; they may be caused by axial loads. A common problem for systems excited by parametric forces is their instability. A nonlinear model will limit the unlimited exponential amplitude of the response predicted by the linear model. A random excitation is said to be strict-sense stationary if its statistical characteristics are invariant under time shifts and wide-sense stationary if its mean is constant and its auto-correlation function depends only on the separation time.

1.3.2 Nonlinearities

Dynamical systems subjected to random excitations exhibit nonlinearities for sufficiently large motions [1,2]. The sources of the nonlinearities may be material (nonlinear elasticity, perfect plasticity, strain hardening, etc.), geometric (excessive displacements, curvature, etc.), damping (dry friction, hysteretic damping, etc.), and inertia. The output of a linear system to a Gaussian input is Gaussian. Unignorable nonlinearities in the system may deviate the output from being normal (Gaussian distribution). The deviated output yields different probability-density functions that influence the investigation of the failure criteri of such systems [3-5]. Linear problems can be solved exactly by a variety of techniques, see for example, Wang and Uhlenbeck [6] and Bharucha-Reid [7]; a comprehensive treatment is available in standard textbooks [8-10]. Many techniques

have been developed to determine the response of nonlinear systems to random excitations. Section 1.4 provides a brief review of these techniques.

1.4 Methods of Solution

The early start and development of the problem of nonlinear random vibration were completely formed and mixed up with the contributions and formulation of the solution of the Fokker-Planck-Kolmogorov (in short, FPK) equation.

The general problem of random vibration has been included in the literature for many years through the study of Brownian motion. The problem was first studied by Einstein [11] and later extended and developed by Fokker [12], Planck [13], Feller [14], Ito [15,16], and Doob [17]. In his pioneering research in the development of the FPK equation, Kolmogorov [18] gave the mathematical formulation of the equations describing the diffusion of the response probability-density function. The starting point in this parabolic differential equation is the Chapman-Kolmogorov-Smoluckowski equation, which must be satisfied by all Markov processes.

Over the years, a number of different techniques has been used for the study of nonlinear mechanical and structural systems disturbed by random excitations. So far, exact solutions of the FPK equation are only available for specific systems. However, a large class of problems is only amenable to approximate solutions where additional research is currently being conducted in this area.

Subsections 1.4.1-1.4.9 provide an update of the techniques used for the determination of the response of nonlinear systems to random excitations.

Many methods have been developed in the context of the FPK equation to approximate the response. They include

- a) Markov Methods
- b) Sequence or Iterative Solutions
- c) Stochastic Averaging
- d) Series Expansion Solutions
- e) Random Walk Analogy
- f) Moment Closures

1.4.1 Markov Methods

Discrete dynamical systems subjected to broad-band random excitations are examples of Markov processes, which could be reduced to processes without aftereffect. Markov processes are completely described by second-order transition probability-density functions obtained as solutions of the FPK equation. The Markov methods are potentially useful and have an advantage over all other methods because exact solutions may be obtained. On the other hand, because of the restrictions on the spectral density and the form of the nonlinearities, their use is somewhat restricted. Markov methods were first suggested by Pontriagin et al [19] and later used by Chuang and Kazada [20],

Ariaratnam [21], Lyon [22], Caughey [23], Iwan [24], Ludwig [25], Dimentberg [26], and Caughey and Ma [27,28].

We start by defining some quantities. Let $p_n(x_1, t_1; x_2, t_2; \dots; x_n, t_n) dx_1 dx_2 \dots dx_n$ be the probability of finding x in the range $(x_1, x_1 + dx_1)$ at time t_1 , in the range $(x_2, x_2 + dx_2)$ at time $t_2, \dots$, and in the range $(x_n, x_n + dx_n)$ at time t_n . Let $p_n(x_n, t_n | x_1, t_1; x_2, t_2; \dots; x_{n-1}, t_{n-1}) dx_1 dx_2 \dots dx_{n-1}$ be the probability of finding x in the range $(x_n, x_n + dx_n)$ given that x was in the range $(x_1, x_1 + dx_1)$ at time t_1 , in the range $(x_2, x_2 + dx_2)$ at time $t_2, \dots$, and in the range $(x_{n-1}, x_{n-1} + dx_{n-1})$ at time t_{n-1} .

A random process is completely defined by an infinite hierarchy of probability-density functions $p_1, p_2, \dots$, and p_n . A purely random process has independent successive values so that what is going to happen in the future does not depend on the present or the past values of this process. A purely random process could be defined mathematically as

$$p_n(x_1, t_1; x_2, t_2; \dots; x_n, t_n) = p_1(x_1, t_1) p_1(x_2, t_2) \dots p_1(x_n, t_n) \quad (1.1)$$

Equation (1.1) implies that full information about a purely random process can be obtained by the first-order probability-density function p_1 .

A Markov process can be defined by

$$p_n(x_n, t_n | x_1, t_1; x_2, t_2; \dots; x_{n-1}, t_{n-1}) = p_2(x_n, t_n | x_{n-1}, t_{n-1}) \quad (1.2)$$

This means that to know what will happen at time t_n , we only need to know what happened at time t_{n-1} . Every Markov process must satisfy the

Chapman-Kolmogorov-Smoluckowski equation or the Smoluchowski equation [18,29]

$$p(x_3, t_3 | x_1, t_1) = \int p(x_3, t_3 | x_2, t_2) \cdot p(x_2, t_2 | x_1, t_1) dx_2 \quad (1.3)$$

The analysis can be significantly simplified if the random process is ergodic, which is a subclass of stationary processes. In this case, $x(t)$ can be obtained experimentally from one observation, recorded over a sufficiently long time, and then divided into a number of stationary processes of duration T . For stationary processes, the Chapman-Kolmogorov-Smoluchowski equation becomes

$$p(x, t + \Delta t | x_1, t_1) = \int p(x, t + \Delta t | x_2, t) \cdot p(x_2, t | x_1, t_1) dx_2 \quad (1.4)$$

To this end, we consider the integral [6]

$$\int R(x) \cdot \left(\lim_{\Delta t \rightarrow 0} \frac{p(x, t + \Delta t | x_1, t_1) - p(x, t | x_1, t_1)}{\Delta t} \right) dx \quad (1.5)$$

where $R(x)$ is an arbitrary function that decays as $x \rightarrow \pm \infty$. Substituting equation (1.4) into (1.5), expanding $R(x)$ in a Taylor series about $(x - x_2)$, and integrating yields the general FPK equation

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (BP) - \frac{\partial}{\partial x} (AP) \quad (1.6)$$

where the moments A and B are defined as

$$A(x) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int (x_2 - x) p(x_2, t + \Delta t | x, t) dx_2 \quad (1.7)$$

and

$$B(x) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int (x_2 - x)^2 p(x_2, t + \Delta t | x, t) dx_2 \quad (1.8)$$

Conditional or transition probability-density functions of Markov processes must satisfy equation (1.6), which is sometimes referred to as the Fokker-Planck equation or forward Kolmogorov equation. In short, the FPK equation can be rewritten as

$$\frac{\partial p}{\partial t} = L_x p \quad (1.9)$$

where L is a spatial operator. The reader is referred to Gikhman and Skorokhod [30] for more details.

For an n -dimensional Markov process, the FPK equation takes the form

$$\frac{\partial p}{\partial t} = \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} [B_{ij} p] - \sum_i \frac{\partial}{\partial x_i} [A_i p] \quad (1.10)$$

The steady-state probability-density function is defined as

$$p(x) = \lim_{t \rightarrow \infty} p(x, t) \quad (1.11)$$

which is given by

$$L_x p = 0 \quad (1.12)$$

where L_x is the spatial operator on the right-hand side of equation (1.10).

As an example, we consider the following equation of motion for a single-degree-of-freedom (SDF) system:

$$\ddot{x} + g(x, \dot{x}) = F(t) \quad (1.13)$$

subject to the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \quad (1.14)$$

where $g(x, \dot{x})$ is an arbitrary function of x and $\dot{x}$, and $F(t)$ is a Gaussian white excitation with the statistics

$$\langle F(t) \rangle = 0 \quad (1.15)$$

$$\langle F(t_1) F(t_2) \rangle = 2\pi S_0 \delta(t_2 - t_1) \quad (1.16)$$

Here, δ is the Dirac delta function and the angle brackets stand for the expectation. A complete exact solution for the corresponding FPK of the oscillator described in equation (1.13) has not yet been recorded.

We let

$$y_1 = x \text{ and } y_2 = \dot{x} \quad (1.17)$$

and rewrite equation (1.13) as

$$\dot{y}_1 = y_2 \quad (1.18)$$

$$\dot{y}_2 = -g(y_1, y_2) + F(t) \quad (1.19)$$

Using the definitions of the moments equations (1.7) and (1.8), we have

$$A_1 = \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_1 \rangle}{\Delta t} = y_2 \quad (1.20)$$

$$A_2 = \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_2 \rangle}{\Delta t} = -g(y_1, y_2) \quad (1.21)$$

$$B_{11} = \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_1^2 \rangle}{\Delta t} = 0 \quad (1.22)$$

$$B_{12} = \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_1 \Delta y_2 \rangle}{\Delta t} = 0 \quad (1.23)$$

$$B_{22} = \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_2^2 \rangle}{\Delta t} = 2\pi S_0 \quad (1.24)$$

The associated FPK equation can be written as

$$\frac{\partial p}{\partial t} = \pi S_0 \frac{\partial^2 p}{\partial y_2^2} - y_2 \frac{\partial p}{\partial y_1} + \frac{\partial}{\partial y_2} [g(y_1, y_2)p] \quad (1.25)$$

or

$$\frac{\partial p}{\partial t} = \pi S_0 \frac{\partial^2 p}{\partial \dot{x}^2} - \dot{x} \frac{\partial p}{\partial x} + \frac{\partial}{\partial \dot{x}} [g(x, \dot{x})p] \quad (1.26)$$

with the initial condition

$$\lim_{t \rightarrow 0} p(x, \dot{x}; t | x_0, \dot{x}_0; 0) = \delta(x - x_0) \delta(\dot{x} - \dot{x}_0) \quad (1.27)$$

Exact stationary solutions for systems with arbitrary nonlinear restoring forces and linear damping and disturbed with stationary broadband random excitations have been recorded by Andronov, Pontryagin, and Witt [31], Kramers [32], Oliver and Wu [33], and Chuang and Kazada [20]. In this case, we have

$$g(x, \dot{x}) = 2\mu\dot{x} + h(x) \quad (1.28)$$

and equation (1.26) reduces to Kramers' equation [32] whose stationary solution has the form

$$p(x, \dot{x}) = C \exp\left[-\frac{2\mu}{\pi S_0} E\right] \quad (1.29)$$

where E is the sum of the kinetic and potential energies per unit mass of the system; that is,

$$E = \frac{1}{2} \dot{x}^2 + \int_0^x h(\xi) d\xi \quad (1.30)$$

Caughey [34] and Lin [10] investigated a more general case with a specific type of nonlinear damping given by

$$g(x, \dot{x}) = \dot{x}f(E) + h(x) \quad (1.31)$$

Substituting equation (1.31) into equation (1.26) yields

$$\frac{\partial p}{\partial t} = \pi S_0 \frac{\partial^2 p}{\partial \dot{x}^2} - \dot{x} \frac{\partial p}{\partial x} + \frac{\partial}{\partial \dot{x}} ([f(E)\dot{x} + h(x)]p) \quad (1.32)$$

To determine the steady-state probability-density function of the response, Caughey [34] replaced equation (1.32) with

$$h(x) \frac{\partial p}{\partial \dot{x}} - \dot{x} \frac{\partial p}{\partial x} = 0 \quad (1.33)$$

$$f(E)\dot{x}p + \pi S_0 \frac{\partial p}{\partial \dot{x}} = 0 \quad (1.34)$$

Integrating equations (1.33) and (1.34) gives

$$p(x, \dot{x}) = \frac{\exp[-(1/\pi S_0) \int_0^E f(\xi) d\xi]}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-(1/\pi S_0) \int_0^E f(\xi) d\xi] dx d\dot{x}} \quad (1.35)$$

Dimentberg [26] found the exact steady-state probability-density function for the following specific system under combined parametric and external excitations:

$$\ddot{x} + 2\alpha\dot{x}[1 + \eta(t)] + \beta_1\dot{x}(x^2 + \frac{\dot{x}^2}{2}) + \Omega^2 x[1 + \xi(t)] = F(t), \quad \beta_1 \geq 0 \quad (1.36)$$

where $\xi(t)$ and $\eta(t)$ are two independent normal white-noise with zero means and intensities D_ξ and D_η , respectively, such that

$$\frac{D_\xi}{D_\eta} = \frac{4\alpha^2}{\Omega^2} \quad (1.37)$$

The FPK equation associated with equation (1.36) can be written as

$$\begin{aligned} \frac{\partial p}{\partial t} = & \frac{1}{2} \frac{\partial^2}{\partial \dot{x}^2} [(4\alpha^2 D_\eta \dot{x}^2 + \Omega^2 D_\xi x^2 + 2\pi S_0)p] - \dot{x} \frac{\partial p}{\partial x} + \Omega^2 x \frac{\partial p}{\partial \dot{x}} \\ & + \frac{\partial}{\partial \dot{x}} [(2\alpha - 2\alpha^2 D_\eta + \beta_1 x^2 + \frac{\beta_1}{2} \dot{x}^2)\dot{x}p] \end{aligned} \quad (1.38)$$

whose stationary probability-density function has the form

$$p(x, \dot{x}) = \frac{C \exp[-\beta(x^2 + \dot{x}^2/\Omega^2)]}{(\nu + x^2 + \dot{x}^2/\Omega^2)^{(\delta-\nu\beta)}} \quad (1.39)$$

where

$$\nu = \frac{2\pi S_0}{D_\xi \Omega^4}, \quad \delta = \frac{2\alpha}{D_\xi \Omega^2} + \frac{1}{2}, \quad \beta = \frac{\beta_1}{D_\xi \Omega^2} \quad (1.40)$$

and

$$C = \frac{1}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, \dot{x}) dx d\dot{x}} \quad (1.41)$$

Recently during the study of the stochastic stability of a class of hypothetical elastic structures, Caughey and Ma [27] gave an exact solution for a class of nonlinear oscillators governed by equation (1.13) where

$$g(x, \dot{x}) = [H_y f(H) - \frac{H_{yy}}{H_y}] \pi S_0 \dot{x} + \frac{H_x}{H_y} \quad (1.42)$$

$$y = \frac{1}{2} \dot{x}^2 \quad (1.43)$$

$$H_x = \frac{\partial H}{\partial x}, \quad H_y = \frac{\partial H}{\partial y} \quad (1.44)$$

and $H(x, y)$ is the energy integral of the conservative oscillator

$$\ddot{x} + \frac{H_x}{H_y} = 0 \quad (1.45)$$

Here, $H(x, y)$ and $f(H)$ are functions with continuous second-order derivatives and H and $H_y \geq 0$. In addition, there exists an H_0 such that

$$f(H) \geq 0 \quad \text{if } H > H_0 \quad (1.46)$$

and

$$f^{-2} \frac{df}{dH} \rightarrow 0 \quad \text{as } H \rightarrow \infty \quad (1.47)$$

In this case, the steady-state probability-density function is

$$p(x, \dot{x}) = \frac{\exp\left[-\int_0^H f(\xi) d\xi\right] H_y}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\int_0^H f(\xi) d\xi\right] H_y dx d\dot{x}} \quad (1.48)$$

Lin and Cai [35] developed a systematic procedure for the determination of the stationary probability-density functions of the responses of nonlinear systems to broad-band random excitations. Separating the probability flow into circulatory and noncirculatory flows results in two sets of equations that must be satisfied by the probability potential.

1.4.2 Sequence or Iterative Solutions

These are general iterative schemes for studying sequence solutions that converge to the exact solutions of the FPK equation [36,37,23,38]. It is based on the parametrix method [39] for investigating the existence and uniqueness of solutions of partial-differential equations. Mayfield [38] used a modification of the parametrix method and applied it to the case of phase-locked frequency demodulation. Mayfield [38] gave a sequence solution of the form

$$p_{n+1}(x, t | \xi, \tau) = p_0(x, t | \xi, \tau) + \int_{\tau}^t \int_{\Omega} p_n(x, t | \eta, \sigma) \cdot R p_0(\eta, \sigma | \xi, \tau) d\eta d\sigma \quad (1.49)$$

where $R = L_x - \frac{\partial}{\partial t}$ and $p_0(x, t | \xi, \tau)$ is an arbitrary initial estimate of $p(x, t | \xi, \tau)$ and the system started at time τ with the initial condition $x(\tau) = \xi$. It converges to the time-varying multi-dimensional transition probability-density function from the FPK equation.

We let

$$g(x, \dot{x}) = \omega_0^2 x + 2\mu \dot{x} + \epsilon h(x, \dot{x}) \quad (1.50)$$

The corresponding FPK equation is

$$\frac{\partial p}{\partial t} = L_0 p + \epsilon \frac{\partial}{\partial \dot{x}} [h(x, \dot{x}) p] \quad (1.51)$$

where L_0 is the linear spatial operator

$$L_0 p = \pi S_0 \frac{\partial^2 p}{\partial \dot{x}^2} - \dot{x} \frac{\partial p}{\partial x} + \frac{\partial}{\partial \dot{x}} [(\omega_0^2 x + 2\mu \dot{x}) p] \quad (1.52)$$

This explains the possibility of using the transition probability-density function of the linear part ($\epsilon = 0$) of the FPK equation as an initial estimate of the probability-density function. Caughey [23] used the Chapman-Kolmogorov-Smoluckwski equation and limited the strength $|\epsilon|$ of the nonlinearity to sufficiently small values to prove the convergence of this iterative solution.

1.4.3 Stochastic Averaging

This technique yields a one-dimensional FPK equation for the amplitude envelope $a(t)$ or a two-dimensional FPK equation for the amplitude $a(t)$ and the phase $\theta(t)$. Stratonovich [40] generalized the averaging method for nonlinear deterministic problems described by Bogoliubov and Mitropolsky [41] to the case of nonlinear statistical problems. We refer the reader to Stratonovich [40], Khasminskii [42], Papanicolaou [43], Papanicolaou and Kohler [44] for a complete mathematical description of this asymptotic method.

We consider an oscillator whose response is described by the differential equation

$$\ddot{x} + \omega_0^2 x = \epsilon g^*(x, \dot{x}, t) \quad (1.53)$$

where g^* is an arbitrary function of x , $\dot{x}$, and t . The classical perturbation techniques, such as the Lindstedt-Poincaré technique and the method of renormalization, are not expected to yield an approximate solution for the transient oscillations because they do not account for variations in the amplitude. However, the method of averaging yields uniform expansions for the transient oscillations. The solution of equation (1.53) is expressed as

$$x = a(t) \cos[\omega_0 t + \theta(t)] \quad (1.54)$$

subject to the constraint

$$\dot{x} = -\omega_0 a(t) \sin[\omega_0 t + \theta(t)] \quad (1.55)$$

Differentiating equation (1.54) once with respect to t and comparing the result with equation (1.55) gives

$$\dot{a} \cos(\omega_0 t + \theta) - a \dot{\theta} \sin(\omega_0 t + \theta) = 0 \quad (1.56)$$

Differentiating equation (1.55) once with respect to t , we obtain

$$\ddot{x} = -\omega_0 \dot{a} \sin(\omega_0 t + \theta) - \omega_0 a (\omega_0 + \dot{\theta}) \cos(\omega_0 t + \theta) \quad (1.57)$$

Substituting equations (1.54), (1.55), and (1.57) into equation (1.53) yields

$$-\omega_0 \dot{a} \sin\phi - \omega_0 a \dot{\theta} \cos\phi = \epsilon g^*(a \cos\phi, -\omega_0 a \sin\phi, t) \quad (1.58)$$

Solving equations (1.56) and (1.58) for $\dot{a}$ and $\dot{\theta}$ gives

$$\dot{a} = -\frac{\epsilon}{\omega_0} g^*(a \cos\phi, -\omega_0 a \sin\phi, t) \sin\phi \quad (1.59)$$

$$\dot{\theta} = - \frac{\epsilon}{a\omega_0} g^*(a\cos\phi, -\omega_0 a\sin\phi, t)\cos\phi \quad (1.60)$$

where

$$\phi = \omega_0 t + \theta \quad (1.61)$$

The problem of investigating the response characteristics of self-excited oscillators to stochastic excitation was studied also by Blaquiere [45,46], Caughey [47], Grivet and Blaquiere [48], Zakai [49], Stratonovich [40], and Nayfeh [50]. As a special case, Stratonovich [40] considered a self-excited oscillator described by

$$g^*(x, \dot{x}, t) = \omega_0 \left(1 - \frac{e\dot{x}^2}{2}\right)\dot{x} + \omega_0^2 F(t) \quad (1.62)$$

where e is related to the system parameters. Substituting equation (1.62) into equations (1.59) and (1.60) and using some trigonometric identities, we obtain

$$\dot{a} = \frac{\epsilon\omega_0 a}{2} \left[1 - \frac{3ea^2}{4} + (ea^2 - 1)\cos 2\phi - \frac{ea^2}{4}\cos 4\phi\right] - \epsilon\omega_0 F(t)\sin\phi \quad (1.63)$$

$$\dot{\theta} = \frac{\epsilon\omega_0}{2} \left[\left(1 - \frac{ea^2}{2}\right)\sin 2\phi + \frac{ea^2}{4}\sin 4\phi\right] - \frac{\epsilon\omega_0}{a} F(t)\cos\phi \quad (1.64)$$

Averaging the deterministic parts of equations (1.63) and (1.64) (i.e., keeping only slowly varying parts) yields

$$\dot{a} = \frac{\epsilon\omega_0 a}{2} \left(1 - \frac{3ea^2}{4}\right) - \epsilon\omega_0 F(t)\sin\phi \quad (1.65)$$

$$\dot{\theta} = - \frac{\epsilon\omega_0}{a} F(t)\cos\phi \quad (1.66)$$

The terms that include $F(t)$ are replaced by the sum of their mean values and the fluctuational parts [40]; that is,

$$\dot{a} = \frac{\varepsilon \omega_0 a}{2} \left(1 - \frac{3\omega_0^2 e a^2}{4}\right) + \frac{\varepsilon^2 \omega_0^2 \pi}{2a} S(\omega_0) + \varepsilon \omega_0 [\pi S(\omega_0)]^{\frac{1}{2}} \eta(t) \quad (1.67)$$

where $S(\omega_0) = S_0$ for a Gaussian white noise and $\eta(t)$ is a zero-mean unit delta-correlated random process; that is,

$$\langle \eta(t) \eta(t + \tau) \rangle = \delta(\tau) \quad (1.68)$$

Roberts [3,4,51] applied this method to obtain the stationary response of oscillators with an arbitrary nonlinear damping force excited by a stationary random excitation. He considered the case

$$\ddot{x} + \omega_0^2 x + 2\mu V(\dot{x}) = F(t) \quad (1.69)$$

where $2\mu V(\dot{x})$ is a nonlinear damping force and μ is a small parameter. Assuming a fairly light nonlinear damping force, one finds that the response is narrow-band. Performing deterministic and stochastic averaging on the equations that describe the modulation of the response amplitude and phase with time, one obtains

$$\dot{a} = -\frac{2\mu}{\omega_0} G(a) + \frac{\pi S(\omega_0)}{2a\omega_0^2} + [\pi S(\omega_0)]^{\frac{1}{2}} \frac{\eta(t)}{\omega_0} \quad (1.70)$$

$$\dot{\theta} = -[\pi S(\omega_0)]^{\frac{1}{2}} \frac{\xi(t)}{a\omega_0} \quad (1.71)$$

where $\eta(t)$ and $\xi(t)$ are mutually independent unit white noise and

$$G(a) = -\frac{1}{2\pi} \int_0^{2\pi} V(-\omega_0 a \sin \phi) \sin \phi d\phi \quad (1.72)$$

The FPK equation associated with equation (1.70) is

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial a} \left\{ \left[\frac{2\mu}{\omega_0} G(a) - \frac{\pi S(\omega_0)}{2a\omega_0^2} \right] p \right\} + \frac{\pi S(\omega_0)}{2\omega_0^2} \frac{\partial^2 p}{\partial a^2} \quad (1.73)$$

A general stationary solution for first-order systems with the spatial operator

$$L_x = -\frac{\partial}{\partial x} R(x) + \frac{r^2}{2} \frac{\partial^2}{\partial x^2} \quad (1.74)$$

is given by Stratonovich [40] as

$$p(x) = \frac{C}{r^2} \exp\left[\frac{2}{r^2} \int_0^x R(n) dn\right] \quad (1.75)$$

where C is a normalization constant. Equation (1.73) is a special case of equation (1.74).

This approach was extended to approximate the nonstationary amplitude of lightly damped and weakly nonlinear systems [52] defined by

$$\ddot{x} + \omega_0^2 x + 2\mu\dot{x} + \epsilon f(x, \dot{x}) = F(t) \quad (1.76)$$

To this end, one defines the following linear system:

$$\ddot{x} + (\omega_0^2 + \epsilon\omega_e^2)x + 2(\mu + \epsilon\mu_e)\dot{x} = F(t) \quad (1.77)$$

where μ_e and ω_e are to be determined so that the solution of equation (1.77) furnishes an approximate solution to equation (1.76). The method of equivalent linearization (to be discussed later) was used to determine μ_e and ω_e from

$$\mu_e = -\frac{1}{2\pi\omega_n a} \int_0^{2\pi} f(a \cos\phi, -a\omega_n \sin\phi) \sin\phi d\phi \quad (1.78)$$

$$\omega_e^2 = \frac{1}{\pi a} \int_0^{2\pi} f(a \cos\phi, -a\omega_n \sin\phi) \cos\phi d\phi \quad (1.79)$$

where

$$\phi = \omega_n t + \theta \quad (1.80)$$

$$\omega_n^2 = \omega_0^2 + \epsilon \omega_e^2 \quad (1.81)$$

Eliminating the rapidly varying (i.e., oscillatory) terms $\sin\phi$, $\cos\phi$, ... that are acting with relatively large but slowly varying terms a and θ yields

$$\dot{a} = -(\mu + \epsilon \mu_e) a + \frac{\pi S(\omega_n)}{2a\omega_n^2} + \frac{[\pi S(\omega_n)]^{\frac{1}{2}}}{\omega_n} \eta(t) \quad (1.82)$$

where

$$S(\omega) = S_0 \quad (1.83)$$

for a Gaussian white noise. After expanding $(\omega_n)^{-1}$, one obtains the FPK equation associated with equation (1.82) as

$$\begin{aligned} \frac{\partial p}{\partial t} = & \frac{\partial}{\partial a} \left[\mu \left(a - \frac{\sigma^2}{a} \right) p \right] + \mu \sigma^2 \frac{\partial^2 p}{\partial a^2} + \epsilon \left\{ \frac{\partial}{\partial a} \left[\frac{\mu \sigma^2 \omega_e^2}{\omega_0^2} \left(\frac{p}{a} - \frac{\partial p}{\partial a} \right) \right. \right. \\ & \left. \left. + p \frac{\mu_e a}{\omega_0} - \frac{1}{2} \frac{\mu \sigma^2 p}{\omega_0^2} \frac{\partial}{\partial a} \omega_e^2 \right] \right\} \end{aligned} \quad (1.84)$$

where σ^2 is the stationary variance of $x(\epsilon = 0)$; that is,

$$\sigma^2 = \frac{\pi S_0}{2\mu\omega_0^2} \quad (1.85)$$

When $\epsilon = 0$, the eigenvalues and eigenfunctions of equation (1.84)

are

$$\lambda_n = 2n\mu \quad (1.86)$$

$$E_n = \frac{1}{n!} \frac{a}{\sigma^2} \exp\left[-\frac{a^2}{2\sigma^2}\right] L_n\left(\frac{a^2}{2\sigma^2}\right) \text{ for } n = 0, 1, 2, \dots \quad (1.87)$$

and L_n is the n th order Laguerre polynomial. When $\epsilon \neq 0$, the eigenvalues and eigenfunctions of equation (1.84) are defined by the asymptotic expansions

$$T_n = \lambda_n + \epsilon \gamma \quad (1.88)$$

$$Q_n = E_n + \epsilon \alpha \quad (1.89)$$

and a series expansion of the probability-density function of the nonstationary amplitude is

$$p(a,t) = \sum_{n=0}^{\infty} B_m Q_n e^{-T_n t} \quad (1.90)$$

Another approach dealing with systems with nonlinear damping and restoring forces is based on obtaining a one-dimensional FPK equation for the energy envelope E [53]. A Galerkin technique was used to obtain the solutions of the FPK equation for the response amplitude $a(t)$ of systems disturbed by modulated white excitations [54]. Roberts [55] approximated the response amplitudes $a(t)$ of systems excited by a suddenly applied stationary noise by a discrete random walk processes. Roberts [56,57] extended this approach by including hysteretic restoring forces. Other authors [58-60] used this fruitful technique for the determination of the responses of nonlinear systems to random excitations and the first passage time for the envelope crossing of linear systems.

1.4.4 Series Expansions

For a system with linear damping and a nonlinear restoring force, Stratonovich [40] expanded the response probability-density function in terms of two eigenfunctions as

$$p(x,\dot{x},t) = \sum \sum T_{ij}(t) X_i(x) Y_j(\dot{x}) \quad (1.91)$$

Atkinson [61] assumed a series expansion of the probability-density function for the arbitrary function $g(x, \dot{x})$ in the form

$$p(x, \dot{x}, t | x_0, \dot{x}_0, t_0) = \sum e^{-\lambda_i t} E_i(x, \dot{x}) E_j^*(x_0, \dot{x}_0) \quad (1.92)$$

$$(L_x + \lambda_i) E_i = 0 \quad (1.93)$$

$$(L_x^* + \lambda_i) E_i^* = 0 \quad (1.94)$$

$$\int E_i E_j^* dx d\dot{x} = \delta_{ij} \quad (1.95)$$

where δ_{ij} is the Kronecker delta, the E_i are the eigenfunctions of the FPK differential operator L_x , the E_i^* are the eigenfunctions of the adjoint operator L_x^* , and the λ_i are the eigenvalues of L_x or L_x^* . Because of the difficulty in obtaining the eigenfunctions of the parabolic differential FPK equation, Atkinson [61] approximated these eigenfunctions by a set of linearly independent functions.

An approximate solution of equation (1.13) for the case

$$g(x, \dot{x}) = f_1(x) + f_2(\dot{x}) \quad (1.96)$$

was developed by Bhandari and Sherrer [62]. The functions $f_1(x)$ and $f_2(x)$ were represented as odd polynomials to assure that the work done by the spring and the damper is continuous, positive definite, and symmetric. The normalized stationary probability-density function was represented by a double series of Hermite Polynomials [63] as

$$p(x, \dot{x}) = \frac{1}{2\pi} \exp\left[-\frac{1}{2} (x^2 + \dot{x}^2)\right] \sum \sum A_{ij} H_i(x) H_j(\dot{x}) \quad (1.97)$$

The coefficients A_{mn} ($A_{00} = 1$) were determined by the Galerkin method [64]. This implies that the resulting error of using this approximate

series expansion must be orthogonal to the weighting functions (Hermite Polynomials)

$$\int \int H_i(x) H_j(\dot{x}) L_x p dx d\dot{x} = 0 \quad \text{for } i, j = 1, 2, \dots, N \quad (1.98)$$

where the value of N represents the required accuracy of this series expansion. They developed a Fortran program for the computation of $p(x, \dot{x})$, $p(x)$, and $p(\dot{x})$ for single-degree-of-freedom and two-degree-of-freedom systems and compared the results with the exact solution of the Duffing oscillator.

In a study of random vibrations of hysteretic systems, Wen [65] extended the work of Bhandari and Sherrer [62] to obtain the nonstationary response of hysteretic systems. The nonlinear hysteretic force is

$$g(x, \dot{x}) + z(x) \quad (1.99)$$

where $z(x)$ is a hysteretic force that depends on the instantaneous displacement and its past history. The excitation is a Gaussian shot noise. In this case the dimension of the Markov vector process increases from two to three and the resulting three-dimensional FPK equation is

$$\frac{\partial p}{\partial t} = \frac{\partial^2}{\partial \dot{x}^2} [I(t) \pi S] - \frac{\partial}{\partial x} (\dot{x} p) + \frac{\partial}{\partial \dot{x}} \{ [g(x, \dot{x}) + z] p \} - \frac{\partial}{\partial z} (\dot{z} p) \quad (1.100)$$

where $I(t)$ is the intensity variation, S is the shot noise intensity, and $z(x)$ depends on many variables [66]. One of the models that describes the behavior of $z(x)$ is [67]

$$\dot{z} = -\alpha |\dot{x}| z - \beta \dot{x} |z| + A \dot{x} \quad (1.101)$$

where α , β , and A are constants that shape the hysteresis loop. The trial and weighting functions used to obtain the nonstationary response statistics were space and time dependent. A series expansion of the probability-density function was sought in the form

$$p(x, \dot{x}, z, t) = \sum T_m(t) R_m(x, \dot{x}, z, t) \quad (1.102)$$

where the trial and the weighting functions were triple series Hermite Polynomials and the time-dependent functions $T_m(t)$ were determined by the condition

$$\int \int \int H_i(x) H_j(\dot{x}) H_k(z) L_x p dx d\dot{x} dz = 0 \quad (1.103)$$

In this approach one needs to worry about using sufficiently small values of the damping force because this leads to a singular FPK equation.

1.4.5 Random-Walk Analogy

This technique is based on the analogy between discrete random walks and diffusion processes described by the continuous FPK equation [68-73]. A partial-difference equation that is used to establish a finite-difference scheme is a result of this analogy. In common with the finite-difference solution is the stability of the solution, which is highly dependent upon the time step Δt . Assumptions that are made during the solution process yield different mathematical models. Therefore, the random-walk approach lacks a unique mathematical model [72].

Toland and Yang [72] used this analogy to find the first-passage probability of systems described by equation (1.13) by the superposition

of absorbing barriers onto the phase plane. This numerical scheme starts with an initial condition and then marches out in time according to the following partial-difference equation of the FPK equation:

$$p(x_n, \dot{x}_n; t_n | x_0, \dot{x}_0, t_0) = (S-)(J-) + (S+)(J+) \quad (1.104)$$

where

$$S\pm = \left\{ \frac{1}{2} + \frac{1}{2} [\pm g(x_{n-1}, \dot{x}_{n-1})] \frac{\Delta t}{\Delta^*} \right\} \quad (1.105)$$

$$J\pm = p(x_n - \dot{x}_{n-1}\Delta t, \dot{x}_n \pm \Delta^*; t_n - \Delta t | x_0, \dot{x}_0; t_0) \quad (1.106)$$

and

$$\Delta^* = (2D\Delta t)^{\frac{1}{2}} \quad (1.107)$$

1.4.6 Moment Closures

An effective method for solving nonlinear stochastic systems is the consideration of a set of equations for the moments of the response. This yields important response statistics, such as the means and variances.

Assuming that high-order central cumulants to be zero, Bellman and Wilcox [74] considered a central-cumulant truncation scheme that gives acceptable results. However, Bover [75] presented a quasi-moment truncation scheme that is based on an approximation of higher-order central moments in terms of lower-order central moments. A series expansion of the probability-density function in terms of Hermite Polynomials was considered. The coefficients of this expansion were functions of central moments. The use of Hermite polynomials in the series-expansion solution is appropriate for the required assumption of this truncation scheme because higher-order coefficients are negligible.

1.4.7 Equivalent or Statistical Linearization

The method of equivalent linearization has the widest range of applicability because it can potentially apply to problems with nonlinear damping, nonlinear restoring forces, nonlinearity of a mixed type, and nonlinear hereditary forces.

This technique is limited to small nonlinearities and weak excitations as most of the approximate methods for solving nonlinear problems. The method of equivalent linearization is based on the method of Krylov and Bogoliubov [76] for deterministic nonlinear problems. It was adapted independently to statistical problems by Booton [77] and Caughey [47,78].

We consider the following equation describing the response of a class of single-degree-of-freedom nonlinear systems to a Gaussian white noise:

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x + \epsilon f(x, \dot{x}) = F(t) \quad (1.108)$$

An equivalent linear equation whose solution will furnish an approximate solution to equation (1.108) is assumed in the form

$$\ddot{x} + 2\mu_e\dot{x} + \omega_e^2 x = F(t) \quad (1.109)$$

where the equivalent damping coefficient μ_e and frequency ω_e are determined by minimizing the difference

$$e(x, \dot{x}) = 2(\mu - \mu_e)\dot{x} + (\omega_0^2 - \omega_e^2)x + \epsilon f(x, \dot{x}) \quad (1.110)$$

between the two equations. This minimization could be achieved by minimizing the mean-square value of $e(x, \dot{x})$ with respect to the parameters μ_e and ω_e^2 ; the result is

$$\frac{\partial \langle e^2 \rangle}{\partial \mu_e} = 0 \quad (1.111)$$

$$\frac{\partial \langle e^2 \rangle}{\partial \omega_e^2} = 0 \quad (1.112)$$

where

$$\langle e^2 \rangle = \langle \{2(\mu - \mu_e)\dot{x} + (\omega_0^2 - \omega_e^2)x + \epsilon f(x, \dot{x})\}^2 \rangle \quad (1.113)$$

We rewrite equation (1.108) as

$$f^*(\ddot{x}, \dot{x}, x) = F(t) \quad (1.114)$$

According to Atalik and Utku [79] and Iwan and Mason [80], the equivalent linear equation can be written as

$$m\ddot{x} + c\dot{x} + kx = F(t) \quad (1.115)$$

where

$$m = \left\langle \frac{\partial f^*(\ddot{x}, \dot{x}, x)}{\partial \ddot{x}} \right\rangle \quad (1.116)$$

$$c = \left\langle \frac{\partial f^*(\ddot{x}, \dot{x}, x)}{\partial \dot{x}} \right\rangle \quad (1.117)$$

$$k = \left\langle \frac{\partial f^*(\ddot{x}, \dot{x}, x)}{\partial x} \right\rangle \quad (1.118)$$

if the excitation is a stationary zero-mean Gaussian process. A

knowledge of the transition probability-density function

$p(x, \dot{x}; t | x_0, \dot{x}_0; 0)$ is needed to find the expectations that are involved in equations (1.111) and (1.112) for evaluating μ_e and ω_e^2 . As an

approximation, one uses the stationary probability-density function

$p(x, \dot{x})$ of the original nonlinear problem (if it is available) or one

uses $p(x, \dot{x})$ of the equivalent linear problem. Since the input to a linear system is Gaussian, using the second approach, one finds that the output is also Gaussian with the probability-density function

$$p(x, \dot{x}) = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{\dot{x}^2}{\sigma_{\dot{x}}^2}\right)\right] \quad (1.119)$$

where

$$\sigma_x^2 = \frac{\pi S_0}{2\mu_e \omega_e^2} \text{ and } \sigma_{\dot{x}}^2 = \frac{\pi S_0}{2\mu_e} \quad (1.120)$$

Kazakov [81] generalized the method of equivalent linearization for the case of multi-degree-of-freedom nonlinear systems. Spanos and Iwan [82] considered the existence and uniqueness of solutions generated by the method of equivalent linearization. Iwan and Mason [80] applied the method of equivalent linearization to systems subjected to nonstationary random excitations. A comparison between the results of an analog-computer simulation of a Duffing oscillator and those obtained by using the method of equivalent linearization was done by Bulsara et al [83] for a wide range of parameter values. They considered an oscillator with a linear damping force and a nonlinear restoring force such that

$$g(x, \dot{x}) = \omega_0^2 x + 2\mu\dot{x} + \alpha h(x) \quad (1.121)$$

Performing the minimization technique that is described in equations (1.111) and (1.112) yields

$$2\mu_e = 2\mu \quad (1.122)$$

$$\omega_e^2 = \omega_0^2 + \alpha \frac{\langle xh(x) \rangle}{\langle x^2 \rangle} \quad (1.123)$$

The expectation terms in equation (1.123) could be calculated by using the following steady-state probability-density function obtained from the FPK equation of the original nonlinear equation:

$$p(x) = C_1 \exp\left[-\frac{\mu}{\pi S_0} (\omega_0^2 x^2 + 2\alpha \int^x h(\xi) d\xi)\right] \quad (1.124)$$

or by using the following probability-density function of the equivalent linear equation:

$$p(x) = C_2 \exp\left[-\frac{\omega_e^2}{\pi S_0} x^2\right] \quad (1.125)$$

where C_1 and C_2 are normalization constants. For the case of the Duffing oscillator, the stiffness parameter ω_e^2 that is obtained by using equation (1.125) is

$$\omega_e^2 = \frac{\omega_0^2}{2} \left[1 + \left(1 + \frac{6\alpha\pi S_0}{\mu\omega_0^2}\right)^{\frac{1}{2}}\right] \quad (1.126)$$

and that is obtained by using equation (1.124) is

$$\omega_e^2 = \frac{\omega_0^2}{z} \frac{U(0,z)}{U(1,z)} \quad (1.127)$$

where

$$z = \omega_e^2 \left(\frac{\mu}{\alpha\pi S_0}\right)^{\frac{1}{2}} \quad (1.128)$$

and $U(n,z)$ is the parabolic cylinder function [84].

Several authors [85-89] obtained the response statistics of hysteretic degrading systems. Baber and Wen [85] generalized the linearization technique of Atalik and Utku [79] to nonzero mean problems.

A number of experimental studies tested the accuracy of the method of equivalent linearization [90-93].

1.4.8 Perturbation Methods

If the dynamical system has sufficiently weak nonlinearities, the response can be represented as an expansion in powers of the strength $|\epsilon|$ of the nonlinearity. Lyon [22,97] and Crandall [98] applied the straightforward perturbation techniques developed for treating nonlinear deterministic problems [94-96] to nonlinear statistical problems.

To explain this technique, we consider

$$\ddot{x} + 2\mu\dot{x} + \omega_0^2 x + \epsilon f(x, \dot{x}) = F(t) \quad (1.129)$$

We seek a straightforward expansion of x in powers of ϵ as

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots \quad (1.130)$$

Substituting equation (1.130) into equation (1.129) and equating coefficients of like powers of ϵ to zero, we obtain

$$\epsilon^0 \quad \ddot{x}_0 + 2\mu\dot{x}_0 + \omega_0^2 x_0 = F(t) \quad (1.131)$$

$$\epsilon^1 \quad \ddot{x}_1 + 2\mu\dot{x}_1 + \omega_0^2 x_1 = -f(x_0, \dot{x}_0) \quad (1.132)$$

The solutions of equations (1.131) and (1.132) can be expressed as

$$x_0(t) = \int_0^\infty F(t - \tau)h(\tau)d\tau \quad (1.133)$$

$$x_1(t) = \int_0^\infty f[x_0(t - \tau), \dot{x}_0(t - \tau)]h(\tau)d\tau \quad (1.134)$$

where

$$h(\tau) = \frac{1}{(\omega_0^2 - \mu^2)^{\frac{1}{2}}} e^{-\mu\tau} \sin[(\omega_0^2 - \mu^2)^{\frac{1}{2}}\tau] \quad (1.135)$$

is the response function of the linear oscillator (i.e., $\epsilon = 0$) to a unit impulse. To first order,

$$\langle x \rangle = \langle x_0 \rangle + \epsilon \langle x_1 \rangle + \dots \quad (1.136)$$

$$\langle x^2 \rangle = \langle x_0^2 \rangle + 2\epsilon \langle x_0 x_1 \rangle + \dots \quad (1.137)$$

A considerable simplification in the evaluation of $\langle x \rangle$ and $\langle x^2 \rangle$ and higher-order moments can be made if the excitation is a Gaussian white noise. This means that $x_0(t)$ is Gaussian because it is described by a linear oscillator and excited by a Gaussian noise.

This technique was used to obtain the response spectra by several investigators [99-102]. Crandall et al [103] used this method to study the response of oscillators with nonlinear damping. Soni and Surrendran [104] determined the transient response of nonlinear systems to stationary random excitations.

A tedious work is needed to extend the series expansion beyond first order. It should be noted that the resulting expansion may not be valid for long times because of the presence of secular and small-divisor terms.

1.4.9 Digital- and Analog-Computer Simulation

The simulation approach is mainly used when analytical methods are not available or to verify analytical results. A large number of samples is needed to reduce statistical uncertainty and to account for ensemble averaging. Many authors [105-114] used a random signal generator and filters to generate sample functions of the excitation. Next, they performed statistical processing of the analog-computer output to determine the response statistics. Digital-computer

simulation procedures are usually applied to nonlinear systems. For a white noise, a sequence of independent random-number generators can directly produce sample functions of the excitation [3-5,57,115,116]. For a non-white noise, a digital filter is also needed [117-120]. A sum of sinusoidal functions could be used to generate the excitation process where implementation on a digital computer can be made considerably more efficient by using an FFT algorithm [121-138]. In this case, we can write the excitation process as

$$F(t) = \left(\frac{2}{N}\right)^{\frac{1}{2}} \sigma_F \sum_{n=1}^N \cos(\omega_n t + \beta_n) \quad (1.138)$$

where σ_F is the standard deviation of F , the $\omega_n = n\Delta\omega$ are independent random frequencies distributed with a probability-density function proportional to $S(\omega)$, and the β_n are uniformly distributed in $[0, 2\pi]$.

CHAPTER 2

A SECOND-ORDER CLOSURE METHOD

A second-order closure method is presented for determining the response of nonlinear systems to random excitations. The excitation is taken to be the sum of a deterministic harmonic component and a random component. The latter may be white noise or harmonic with separable nonstationary random amplitude and phase. The method of multiple scales is used to determine the equations describing the modulation of the amplitude and phase. Neglecting the third-order central moments, we use these equations to determine the stationary mean and mean-square responses. The effect of the system parameters on the response statistics is investigated. The presence of the nonlinearity causes multi-valued regions where more than one mean-square value of the response is possible. The local stability of the stationary mean and mean-square responses is analyzed. Alternatively, assuming the random component of the response to be small compared with the mean response, we determine steady-state periodic responses to the deterministic part of the excitation. The effect of the random part of the excitation on the stable periodic responses is analyzed as a perturbation and a closed-form expression for the mean-square response is obtained. Away from the transition zone separating stable and unstable periodic responses, the results of these two approaches are in good agreement. Comparisons of the results of these methods with that obtained by using the method of equivalent linearization are presented.

2.1 Introduction

In this Chapter we describe a second-order closure method based on the method of multiple scales to determine the mean and mean-square responses of nonlinear single-degree-of-freedom systems to an excitation that is the sum of a deterministic primary resonant component and a random component. The method is described by using the Duffing-Rayleigh oscillator

$$\ddot{u} + \omega_0^2 u - \epsilon \delta_1 \dot{u} + \frac{1}{3} \epsilon \delta_2 \dot{u}^3 + \epsilon \alpha u^3 = \epsilon [h_0 \cos \omega t - k_0 \sin \omega t + \xi(t)] \quad (2.1)$$

Here, ξ is a general zero-mean random excitation. The excitation is taken to be $O(\epsilon)$ so that the primary resonance, the damping, and the nonlinearity balance each other.

In the present analysis, the random component of the excitation is taken to be either white noise or sinusoid having a random amplitude and phase. The method of multiple scales [95,96] is used to obtain two averaged first-order differential equations. They describe the modulation of the amplitude and phase of the response with the slow time scale. Three alternate approaches are used to determine the mean and mean-square values of the amplitude and phase from these modulation equations.

2.2 The Method of Multiple Scales

To use the method of multiple scales [95,96] for analyzing the case of primary resonance (i.e., $\Omega \approx \omega_0$), we describe the nearness of the resonance by introducing the detuning parameter ν defined according to

$$\Omega = \omega_0 + \epsilon \nu \quad (2.2)$$

Then, we seek a uniform approximate solution of equation (2.1) in the form

$$u = u_0(T_0, T_1) + \epsilon u_1(T_0, T_1) + \dots \quad (2.3)$$

where $T_0 = t$ is a fast scale, characterizing motions occurring with the frequencies ω_0 and Ω , and $T_1 = \epsilon t$ is a slow scale, characterizing the modulation of the amplitude and phase with the nonlinearity, damping and resonance. In terms of the T_n , where $T_n = \epsilon^n t$, the ordinary-time derivatives can be transformed into

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \dots \quad (2.4)$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \dots \quad (2.5)$$

where $D_n = \partial/\partial T_n$. Substituting equations (2.3)-(2.5) into equation (2.1) and equating coefficients of like powers of ϵ to zero gives

Order ϵ^0 :

$$D_0^2 u_0 + \omega_0^2 u_0 = 0 \quad (2.6)$$

Order ϵ :

$$\begin{aligned} D_0^2 u_1 + \omega_0^2 u_1 = & -2D_0 D_1 u_0 + \delta_1 D_0 u_0 \\ & - \frac{1}{3} \delta_2 (D_0 u_0)^3 - \alpha u_0^3 + h_0 \cos \Omega T_0 - k_0 \sin \Omega T_0 + \xi \end{aligned} \quad (2.7)$$

The general solution of equation (2.6) can be written as

$$u_0 = A(T_1) e^{i\omega_0 T_0} + cc \quad (2.8)$$

where cc stands for the complex conjugate of the preceding terms. To this order, the function A is arbitrary and it is determined by imposing

the solvability condition at the next level of approximation. Substituting equation (2.8) into equation (2.7) yields

$$\begin{aligned} D_0^2 u_1 + \omega_0^2 u_1 = & -i\omega_0(2A' - \delta_1 A + \omega_0^2 \delta_2 A^2 \bar{A})e^{i\omega_0 T_0} \\ & + \frac{1}{3} i\delta_2 \omega_0^3 A^3 e^{3i\omega_0 T_0} - \alpha(A^3 e^{3i\omega_0 T_0} + 3A^2 \bar{A} e^{i\omega_0 T_0}) \\ & + \frac{1}{2} h_0 e^{i\Omega T_0} + \frac{1}{2} i k_0 e^{i\Omega T_0} + cc + \xi \end{aligned} \quad (2.9)$$

Using equation (2.2) to eliminate the terms that lead to secular terms from equation (2.9) yields

$$\begin{aligned} i\omega_0(2A' - \delta_1 A + \omega_0^2 \delta_2 A^2 \bar{A}) + 3\alpha A^2 \bar{A} - \frac{1}{2} h_0 e^{i\nu T_1} - \frac{1}{2} i k_0 e^{i\nu T_1} \\ - \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \xi e^{-i\omega_0 T_0} dT_0 = 0 \end{aligned} \quad (2.10)$$

Starting from the averaged equation (2.10), we use three methods to determine the statistics of the response.

2.3 Linearization Method

A low-intensity noise is assumed to separate the strong mean motion from its weak fluctuations. A similar method was used by Bernstein [139] and Stratonovich [40] to determine the influence of noise on the amplitude of the limit cycle of the van der Pol oscillator and by Nayfeh [50] to investigate the response of the van der Pol oscillator with delayed amplitude limiting to a sinusoidal excitation. Nayfeh chose the amplitude of the excitation to be either constant or white noise. For the latter case, he obtained the conditional probability function for deviations of the amplitude from its limit cycle value.

It is convenient to express A in the polar form

$$A = \frac{1}{2} a e^{i(\nu T_1 - \gamma)} \quad (2.11)$$

Substituting equation (2.11) into equation (2.10), separating real and imaginary parts, and replacing T_1 with ϵt , we obtain

$$\dot{a} = \frac{1}{2} \epsilon a (\delta_1 - \frac{1}{4} \omega_0^2 \delta_2 a^2) + \frac{\epsilon h_0}{2\omega_0} \sin \gamma + \frac{\epsilon k_0}{2\omega_0} \cos \gamma + \xi_1 \quad (2.12)$$

$$a \dot{\gamma} = \epsilon \nu a - \frac{3\epsilon \alpha}{8\omega_0} a^3 + \frac{\epsilon h_0}{2\omega_0} \cos \gamma - \frac{\epsilon k_0}{2\omega_0} \sin \gamma + \xi_2 \quad (2.13)$$

where

$$\xi_1 = -\frac{\epsilon}{2\pi\omega_0} \int_0^{2\pi} \xi \sin \phi d\phi, \quad \xi_2 = \frac{\epsilon}{2\pi\omega_0} \int_0^{2\pi} \xi \cos \phi d\phi, \quad (2.14)$$

and

$$\phi = \omega t - \gamma \quad (2.15)$$

According to this approach, we assume that the ξ_n are small compared with h_0 and k_0 . Next, we determine the mean steady-state values a_0 and γ_0 of the amplitude and phase when $\xi_1 = \xi_2 = 0$. They correspond to $\dot{a} = 0$ and $\dot{\gamma} = 0$ and hence they are given by

$$-\frac{1}{2} \delta_1 a_0 + \frac{1}{8} \omega_0^2 \delta_2 a_0^3 = \frac{h_0}{2\omega_0} \sin \gamma_0 + \frac{k_0}{2\omega_0} \cos \gamma_0 \quad (2.16)$$

$$\frac{3\alpha}{8\omega_0} a_0^3 - \nu a_0 = \frac{h_0}{2\omega_0} \cos \gamma_0 - \frac{k_0}{2\omega_0} \sin \gamma_0 \quad (2.17)$$

Squaring and adding equations (2.16) and (2.17) yields the frequency-response equation

$$\frac{\omega_0^6 \delta_2^2 + 9\alpha^2}{16\omega_0^2} a_0^6 - \frac{\omega_0^3 \delta_1 \delta_2 + 6\alpha\nu}{2\omega_0} a_0^4 + (\delta_1^2 + 4\nu^2) a_0^2 - \frac{k_0^2}{\omega_0^2} - \frac{h_0^2}{\omega_0^2} = 0 \quad (2.18)$$

Hence, to the first approximation, the steady-state response is given by

$$u = a_0 \cos(\omega t - \gamma_0) + \dots \quad (2.19)$$

where a_0 and γ_0 are given by equations (2.16)-(2.18).

Next, we determine the effect of the noise (i.e., ξ_1 and $\xi_2 \neq 0$) on the mean motion. To this end, we let

$$a = a_0 + a_1, \quad \gamma = \gamma_0 + \gamma_1 \quad (2.20)$$

Substituting equations (2.20) into equations (2.12) and (2.13) and linearizing the resulting equations, we obtain

$$\dot{\underline{x}}(t) = C\underline{x}(t) + \underline{G}(t) \quad (2.21)$$

where

$$\underline{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} a_1(t) \\ \gamma_1(t) \end{Bmatrix} \quad (2.22)$$

$$\underline{G}(t) = \begin{Bmatrix} g_1(t) \\ g_2(t) \end{Bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \quad (2.23)$$

$$g_1(t) = \xi_1(t, \gamma_0), \quad g_2(t) = \frac{\xi_2(t, \gamma_0)}{a_0} \quad (2.24)$$

$$c_1 = \frac{1}{2} \varepsilon (\delta_1 - \frac{3}{4} \omega_0^2 \delta_2 a_0^2), \quad c_2 = \frac{3\varepsilon\alpha}{8\omega_0} a_0^3 - \varepsilon v a_0, \quad (2.25)$$

$$c_3 = \frac{\varepsilon v}{a_0} - \frac{9\varepsilon\alpha}{8\omega_0} a_0 \quad \text{and} \quad c_4 = \frac{1}{2} \varepsilon (\delta_1 - \frac{1}{4} \omega_0^2 \delta_2 a_0^2)$$

We note that C is a real nonsymmetric matrix.

The general solution of equation (2.21) can be obtained by modal analysis. The eigenvalues and eigenvectors of C are given by

$$C\underline{x} = \lambda \underline{x} \quad (2.26)$$

while the eigenvectors of the adjoint problem are given by

$$C^T \underline{v} = \lambda \underline{v} \quad (2.27)$$

Then, the modal matrices are

$$X = \begin{matrix} \alpha_1 & \alpha_2 \\ \frac{\alpha_1(\lambda_1 - c_1)}{c_2} & \frac{\alpha_2(\lambda_2 - c_1)}{c_2} \end{matrix}, \quad V = \begin{matrix} 1 & 1 \\ \frac{\lambda_1 - c_1}{c_3} & \frac{\lambda_2 - c_1}{c_3} \end{matrix} \quad (2.28)$$

where

$$\alpha_n = c_2 c_3 [c_2 c_3 + (\lambda_n - c_1)^2]^{-1} \text{ for } n = 1 \text{ and } 2 \quad (2.29)$$

It is convenient to introduce a transformation that decouples equations (2.21). To this end, we let

$$\underline{x}(t) = X \underline{z}(t) \quad (2.30)$$

Substituting equation (2.30) into equation (2.21) and premultiplying by V^T yields

$$\dot{z}_1(t) = \lambda_1 z_1(t) + \eta_1 \quad (2.31)$$

$$\dot{z}_2(t) = \lambda_2 z_2(t) + \eta_2 \quad (2.32)$$

where

$$\eta_n = \xi_1 + \frac{\lambda_n - c_1}{a_0 c_3} \xi_2 \quad \text{for } n = 1 \text{ and } 2 \quad (2.33)$$

Then, the statistics of the uncoupled coordinates z_1 and z_2 are

$$\langle z_m \rangle = 0 \quad (2.34)$$

$$\langle z_m(t_1) z_n(t_2) \rangle = e^{(\lambda_m t_1 + \lambda_n t_2)} \int_0^{t_1} \int_0^{t_2} e^{-(\lambda_m \tau_1 + \lambda_n \tau_2)} \cdot$$

$$\langle \eta_m(\tau_1) \eta_n(\tau_2) \rangle d\tau_1 d\tau_2 \quad \text{for } m, n = 1, 2 \quad (2.35)$$

where the angle brackets stand for the expectation. As examples, we consider the cases of sinusoids having nonstationary random amplitudes and white noise.

2.3.1 Nonstationary-Random-Amplitude Sinusoids

In this case, we let

$$\xi(t) = h_1 \cos \omega t - k_1 \sin \omega t \quad (2.36)$$

where h_1 and k_1 are separable nonstationary random functions described by

$$h_1 = f_1(T_1)W_1 \quad k_1 = f_2(T_1)W_2 \quad (2.37)$$

Here, the f_n are slowly varying deterministic functions of time, whereas the W_n are slowly varying random functions. Road roughness and seismic ground motion could be modeled by separable nonstationary random processes. Performing the integrations in equations (2.14) and using equations (2.33), we can write the η_n as

$$\eta_n(t) = d_{1n}h_1(t) - d_{2n}k_1(t) \quad \text{for } n = 1, 2 \quad (2.38)$$

where

$$d_{1n} = \frac{\epsilon}{2\omega_0} \left[\sin \gamma_0 + \frac{\lambda_n - c_1}{a_0 c_3} \cos \gamma_0 \right] \quad (2.39)$$

$$d_{2n} = \frac{\epsilon}{2\omega_0} \left[-\cos \gamma_0 + \frac{\lambda_n - c_1}{a_0 c_3} \sin \gamma_0 \right] \quad (2.40)$$

A narrow-band random excitation is a special case of the excitation given by equations (2.36) and (2.37). Here, $\xi(t)$ is chosen to be a

zero-mean Gaussian narrow-band random excitation. It could be obtained by filtering a white noise through a linear filter [40,140,141]; that is,

$$\ddot{\xi} + \Omega^2 \xi + \beta \dot{\xi} = \beta^{1/2} \Omega W \quad (2.41)$$

where Ω is the center frequency of ξ and β is the bandwidth of the filter. The autocorrelation function of the white noise W is given by

$$R_W(\tau) = 2\pi S_0 \delta(\tau) \quad (2.42)$$

where δ is the Dirac delta function. Substituting equation (2.36) into equation (2.41) and performing deterministic and stochastic averaging of the equations describing the modulations of h_1 and k_1 with time, one obtains

$$\dot{h}_1 + \frac{1}{2} \beta h_1 = \sqrt{\beta/2} W_1 \quad (2.43)$$

$$\dot{k}_1 + \frac{1}{2} \beta k_1 = \sqrt{\beta/2} W_2 \quad (2.44)$$

The white noise components W_1 and W_2 are independent and their autocorrelation functions are given by equation (2.42). The autocorrelation functions of h_1 and k_1 are

$$R_{h_1}(\tau) = R_{k_1}(\tau) = \pi S_0 e^{-\beta|\tau|/2} \quad (2.45)$$

The correlation time of h_1 and k_1 is $O(1/\beta)$. This means that for sufficiently small values of β , h_1 and k_1 are slowly varying functions of time. Using equations (2.35), (2.38)-(2.40), and (2.45), we obtain the following stationary statistics of the uncoupled coordinates

$$z_n: \quad \langle z_n z_m \rangle = \frac{\pi S_0 (d_{1n} d_{1m} + d_{2n} d_{2m})}{\lambda_n + \lambda_m} \left[\frac{1}{\lambda_n - \beta/2} + \frac{1}{\lambda_m - \beta/2} \right] \text{ for } n, m = 1, 2, \quad (2.46)$$

To the first approximation, the solution of equation (2.1) can be written as

$$u = a_0 \cos(\Omega t - \gamma_0) + a_1 \cos(\Omega t - \gamma_0) + \gamma_1 a_0 \sin(\Omega t - \gamma_0) + \dots \quad (2.47)$$

where a_0 is given by equation (2.18), γ_0 is given by equation (2.17), and the statistics of a_1 and γ_1 are

$$\langle a_1 \rangle = 0, \quad \langle \gamma_1 \rangle = 0 \quad (2.48)$$

$$\langle a_1^2 \rangle = \alpha_1^2 \langle z_1^2 \rangle + \alpha_2^2 \langle z_2^2 \rangle + 2\alpha_1 \alpha_2 \langle z_1 z_2 \rangle \quad (2.49)$$

$$\langle \gamma_1^2 \rangle = \frac{\alpha_1^2 (\lambda_1 - c_1)^2}{c_2^2} \langle z_1^2 \rangle + \frac{\alpha_2^2 (\lambda_2 - c_1)^2}{c_2^2} \langle z_2^2 \rangle + \frac{2\alpha_1 \alpha_2 (\lambda_1 - c_1)(\lambda_2 - c_1)}{c_2^2} \langle z_1 z_2 \rangle \quad (2.50)$$

$$\langle a_1 \gamma_1 \rangle = \frac{\alpha_1^2 (\lambda_1 - c_1)}{c_2} \langle z_1^2 \rangle + \frac{\alpha_2^2 (\lambda_2 - c_1)}{c_2} \langle z_2^2 \rangle + \frac{\alpha_1 \alpha_2}{c_2} (\lambda_1 + \lambda_2 - 2c_1) \langle z_1 z_2 \rangle \quad (2.51)$$

Using equations (2.20) and (2.48)-(2.51), we obtain the following statistics for the amplitude and phase of the response:

$$\langle a \rangle = a_0, \quad \sigma_a^2 = \sigma_{a1}^2 \quad (2.52)$$

$$\langle \gamma \rangle = \gamma_0, \quad \sigma_\gamma^2 = \sigma_{\gamma1}^2 \quad (2.53)$$

2.3.2 White Noise

In this case, we have

$$R_\xi(\tau) = 2\pi S_0 \delta(\tau) \quad (2.54)$$

The autocorrelation function given by equation (2.54) implies that ξ is a fast varying random function of time. Terms including $\xi(t)$ in equations (2.12) and (2.13) can be replaced by their means and fluctuating components. This stochastic averaging is restricted to random excitations with small correlation time compared with the relaxation time of the oscillator. The main advantage of stochastic averaging is to decouple the amplitude and phase equations (2.12) and (2.13). However, the presence of the deterministic excitation will keep this coupling. It is applicable to any low-intensity random excitation.

Using equations (2.14), (2.33)-(2.35), and (2.54), we can express the expected values and the mean-square values of the uncoupled coordinates z_n as

$$\langle z_n \rangle = 0 \quad (2.55)$$

$$\langle z_n z_m \rangle = - \frac{\epsilon^2 \pi S_0}{\omega_0^2 (\lambda_n + \lambda_m)} \left[1 + \frac{(\lambda_n - c_1)(\lambda_m - c_1)}{a_0^2 c_3^2} \right] \quad \text{for } n, m = 1, 2 \quad (2.56)$$

2.4 Second-Order Closure Method

The mean motion given by equations (2.16)-(2.18) has no feedback from the perturbations due to the noise. This limits the applicability of the linearization method in the transition zone separating stable and unstable periodic responses. In fact, the preceding linearization scheme predicts infinite mean-square responses at the bifurcation points separating stable from unstable mean responses. To overcome this problem, we propose a second-order closure method. To describe the

method, we find it convenient to express A in equation (2.10) in the form

$$A = \frac{1}{2} (x - iz)e^{i\nu T_1} \quad (2.57)$$

Substituting equation (2.57) into equation (2.10), separating real and imaginary parts, and replacing T_1 with ϵt yields

$$\dot{x} = -\frac{1}{2} \epsilon \delta_e x - \epsilon \nu_e z + \frac{\epsilon k_0}{2\omega_0} - \frac{\epsilon}{2\pi} \int_0^{2\pi/\omega_0} \xi \sin \Omega t \, dt \quad (2.58)$$

$$\dot{z} = -\frac{1}{2} \epsilon \delta_e z + \epsilon \nu_e x + \frac{\epsilon h_0}{2\omega_0} + \frac{\epsilon}{2\pi} \int_0^{2\pi/\omega_0} \xi \cos \Omega t \, dt \quad (2.59)$$

where

$$\delta_e = -\delta_1 + \frac{1}{4} \omega_0^2 \delta_2 (x^2 + z^2) \quad (2.60)$$

$$\nu_e = \nu - \frac{3\alpha}{8\omega_0} (x^2 + z^2) \quad (2.61)$$

Next, we separate each of x and z into mean components x_0 and z_0 and random components y_1 and y_2 according to

$$x = x_0 + y_1 \quad \text{and} \quad z = z_0 + y_2 \quad (2.62)$$

Substituting equations (2.62) into equations (2.58)-(2.61) and separating mean and fluctuating components, we have

mean motion

$$\begin{aligned}
 \dot{x}_0 &= \frac{1}{2} \varepsilon \delta_1 x_0 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0^3 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0 z_0^2 - \varepsilon v z_0 + \frac{3\varepsilon\alpha}{8\omega_0} x_0^2 z_0 + \frac{3\varepsilon\alpha}{8\omega_0} z_0^3 \\
 &- \left[\frac{3\varepsilon}{8} \omega_0^2 \delta_2 x_0 - \frac{3\varepsilon\alpha}{8\omega_0} z_0 \right] \langle y_1^2 \rangle - \left[\frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0 - \frac{9\varepsilon\alpha}{8\omega_0} z_0 \right] \langle y_2^2 \rangle + \left[\frac{3\varepsilon\alpha}{4\omega_0} x_0 \right. \\
 &- \left. \frac{1}{4} \varepsilon \omega_0^2 \delta_2 z_0 \right] \langle y_1 y_2 \rangle - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 (\langle y_1^3 \rangle + \langle y_1 y_2^2 \rangle) + \frac{3\varepsilon\alpha}{8\omega_0} (\langle y_1^2 y_2 \rangle + \langle y_2^3 \rangle) + \frac{\varepsilon k_0}{2\omega_0}
 \end{aligned} \tag{2.63}$$

$$\begin{aligned}
 \dot{z}_0 &= \frac{1}{2} \varepsilon \delta_1 z_0 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0^2 z_0 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 z_0^3 + \varepsilon v x_0 - \frac{3\varepsilon\alpha}{8\omega_0} x_0^3 - \frac{3\varepsilon\alpha}{8\omega_0} x_0 z_0^2 \\
 &- \left[\frac{9\varepsilon\alpha}{8\omega_0} x_0 + \frac{1}{8} \varepsilon \omega_0^2 \delta_2 z_0 \right] \langle y_1^2 \rangle - \left[\frac{3\varepsilon\alpha}{8\omega_0} x_0 + \frac{3}{8} \varepsilon \omega_0^2 \delta_2 z_0 \right] \langle y_2^2 \rangle \\
 &- \left[\frac{1}{4} \varepsilon \omega_0^2 \delta_2 x_0 + \frac{3\varepsilon\alpha}{4\omega_0} z_0 \right] \langle y_1 y_2 \rangle - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 (\langle y_1^2 y_2 \rangle + \langle y_2^3 \rangle) \\
 &- \frac{3\varepsilon\alpha}{8\omega_0} (\langle y_1 y_2^2 \rangle + \langle y_1^3 \rangle) + \frac{\varepsilon h_0}{2\omega_0}
 \end{aligned} \tag{2.64}$$

fluctuating motion

$$\dot{\underline{y}}(t) = \underline{C} \underline{y}(t) + \underline{G}(t) + \underline{N}(y_1, y_2), \quad \underline{N} = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} \tag{2.65}$$

$$g_1(t) = -\frac{\varepsilon}{2\pi} \int_0^{2\pi/\omega_0} \xi \sin \omega t \, dt, \quad g_2(t) = \frac{\varepsilon}{2\pi} \int_0^{2\pi/\omega_0} \xi \cos \omega t \, dt \tag{2.66}$$

$$c_1 = \frac{1}{2} \varepsilon \delta_1 - \frac{3}{8} \varepsilon \omega_0^2 \delta_2 x_0^2 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 z_0^2 + \frac{3\varepsilon\alpha}{4\omega_0} x_0 z_0 \tag{2.67}$$

$$c_2 = -\frac{1}{4} \varepsilon \omega_0^2 \delta_2 x_0 z_0 - \varepsilon v + \frac{3\varepsilon\alpha}{8\omega_0} x_0^2 + \frac{9\varepsilon\alpha}{8\omega_0} z_0^2 \quad (2.68)$$

$$c_3 = -\frac{1}{4} \varepsilon \omega_0^2 \delta_2 x_0 z_0 + \varepsilon v - \frac{9\varepsilon\alpha}{8\omega_0} x_0^2 - \frac{3\varepsilon\alpha}{8\omega_0} z_0^2 \quad (2.69)$$

$$c_4 = \frac{1}{2} \varepsilon \delta_1 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0^2 - \frac{3}{8} \varepsilon \omega_0^2 \delta_2 z_0^2 - \frac{3\varepsilon\alpha}{4\omega_0} x_0 z_0 \quad (2.70)$$

$$\begin{aligned} n_1 = & -\frac{1}{8} \varepsilon \omega_0^2 \delta_2 [3x_0(y_1^2 - \langle y_1^2 \rangle) + x_0(y_2^2 - \langle y_2^2 \rangle) \\ & + 2z_0(y_1 y_2 - \langle y_1 y_2 \rangle) + y_1^3 - \langle y_1^3 \rangle + y_1 y_2^2 - \langle y_1 y_2^2 \rangle] \\ & + \frac{3\varepsilon\alpha}{8\omega_0} [3z_0(y_2^2 - \langle y_2^2 \rangle) + z_0(y_1^2 - \langle y_1^2 \rangle) + 2x_0(y_1 y_2 \\ & - \langle y_1 y_2 \rangle) + y_2^3 - \langle y_2^3 \rangle + y_1^2 y_2 - \langle y_1^2 y_2 \rangle] \end{aligned} \quad (2.71)$$

$$\begin{aligned} n_2 = & -\frac{1}{8} \varepsilon \omega_0^2 \delta_2 [z_0(y_1^2 - \langle y_1^2 \rangle) + 2x_0(y_1 y_2 - \langle y_1 y_2 \rangle) \\ & + 3z_0(y_2^2 - \langle y_2^2 \rangle) + y_1^2 y_2 - \langle y_1^2 y_2 \rangle + y_2^3 - \langle y_2^3 \rangle] \\ & - \frac{3\varepsilon\alpha}{8\omega_0} [3x_0(y_1^2 - \langle y_1^2 \rangle) + x_0(y_2^2 - \langle y_2^2 \rangle) + 2z_0(y_1 y_2 \\ & - \langle y_1 y_2 \rangle) + y_1^3 - \langle y_1^3 \rangle + y_1 y_2^2 - \langle y_1 y_2^2 \rangle] \end{aligned} \quad (2.72)$$

For noise whose intensity is small compared with h_0 and k_0 , y_1 and y_2 are small compared with x_0 and z_0 , and hence central moments higher than the second can be neglected in equations (2.63)-(2.65). Consequently, it follows from equations (2.63) and (2.64) that the steady-state mean motion is given by

$$\begin{aligned} & \frac{1}{2} \epsilon \delta_1 x_0 - \frac{1}{8} \epsilon \omega_0^2 \delta_2 x_0^3 - \frac{1}{8} \epsilon \omega_0^2 \delta_2 x_0 z_0^2 - \epsilon \nu z_0 + \frac{3\epsilon\alpha}{8\omega_0} x_0^2 z_0 + \frac{3\epsilon\alpha}{8\omega_0} z_0^3 \\ & - \left[\frac{3\epsilon}{8} \omega_0^2 \delta_2 x_0 - \frac{3\epsilon\alpha}{8\omega_0} z_0 \right] \langle y_1^2 \rangle - \left[\frac{1}{8} \epsilon \omega_0^2 \delta_2 x_0 - \frac{9\epsilon\alpha}{8\omega_0} z_0 \right] \langle y_2^2 \rangle + \left[\frac{3\epsilon\alpha}{4\omega_0} x_0 \right. \\ & \left. - \frac{1}{4} \epsilon \omega_0^2 \delta_2 z_0 \right] \langle y_1 y_2 \rangle + \frac{\epsilon k_0}{2\omega_0} = 0 \end{aligned} \quad (2.73)$$

$$\begin{aligned} & \frac{1}{2} \epsilon \delta_1 z_0 - \frac{1}{8} \epsilon \omega_0^2 \delta_2 x_0^2 z_0 - \frac{1}{8} \epsilon \omega_0^2 \delta_2 z_0^3 + \epsilon \nu x_0 - \frac{3\epsilon\alpha}{8\omega_0} x_0^3 - \frac{3\epsilon\alpha}{8\omega_0} x_0 z_0^2 \\ & - \left[\frac{9\epsilon\alpha}{8\omega_0} x_0 + \frac{1}{8} \epsilon \omega_0^2 \delta_2 z_0 \right] \langle y_1^2 \rangle - \left[\frac{3\epsilon\alpha}{8\omega_0} x_0 + \frac{3}{8} \epsilon \omega_0^2 \delta_2 z_0 \right] \langle y_2^2 \rangle \\ & - \left[\frac{1}{4} \epsilon \omega_0^2 \delta_2 x_0 + \frac{3\epsilon\alpha}{4\omega_0} z_0 \right] \langle y_1 y_2 \rangle + \frac{\epsilon h_0}{2\omega_0} = 0 \end{aligned} \quad (2.74)$$

We neglect central moments beyond the second, and hence neglect N in equations (2.65). Using modal analysis to decouple these equations, we find that the statistics of the uncoupled coordinates z_1 and z_2 for a narrow-band random excitation are given by equations (2.34) and (2.46) where

$$d_{1n} = \frac{\epsilon(\lambda_n - c_1)}{2\omega_0 c_3}, \quad d_{2n} = -\frac{\epsilon}{2\omega_0} \quad (2.75)$$

For the case of a white noise, these statistics are given by equations (2.55) and (2.56). The mean-square values of y_1 and y_2 have the form described in equations (2.48)-(2.51) for a_1 and γ_1 .

To the first approximation, the solution of equation (2.1) can be expressed as

$$u = (x_0 + y_1)\cos\omega t + (z_0 + y_2)\sin\omega t + \dots \quad (2.76)$$

Equations (2.48)-(2.51), (2.73) and (2.74) and either equations (2.46) or equations (2.56) represent a system of five simultaneous nonlinear algebraic equations. A modified Powell's hybrid algorithm is used to solve this system.

For the case of a narrow-band random excitation, premultiplying equations (2.65), (2.43) and (2.44) by y_1 , y_2 , h_1 and k_1 and taking the expectations of the resulting equations, we have

$$\dot{\underline{P}} = \underline{B}\underline{P} + \underline{Q} \quad (2.77)$$

where

$$\underline{P} = \{ \langle y_1^2 \rangle, \langle y_2^2 \rangle, \langle y_1 y_2 \rangle, \langle y_1 h_1 \rangle, \langle y_2 h_1 \rangle, \langle y_1 k_1 \rangle, \langle y_2 k_1 \rangle \}^T \quad (2.78)$$

and $\underline{B}$ and $\underline{Q}$ are given in Appendix I. The stability of a particular fixed point of equations (2.73), (2.74) and (2.77) to a perturbation proportional to $\exp(\lambda t)$ depends on the eigenvalues of $\underline{C}$ and $\underline{B}$.

2.5 Semi-Linearization Method

Rajan and Davies [141] developed a method for the determination of the response of single-degree-of-freedom systems to zero-mean narrow-band random excitations. In this section, we generalize this method for the case of nonzero-mean random excitations and point out its limitations.

To determine the stationary mean-square value $\langle x^2 + z^2 \rangle$ of the amplitude, we follow Rajan and Davies and assume average values for δ_e and v_e . Substituting equation (2.36) into the integrands in equations (2.58) and (2.59) and performing the deterministic averaging, we find that the integral terms in equations (2.58) and (2.59) become $\epsilon k_1/2\omega_0$ and $\epsilon h_1/2\omega_0$, respectively. Then, we combine equations (2.58) and (2.59) into

$$\frac{d\langle x^2 + z^2 \rangle}{dt} = -\delta_e \langle x^2 + z^2 \rangle + \frac{1}{\omega_0} \langle zh + xk \rangle \quad (2.79)$$

where

$$h = h_0 + h_1 \quad \text{and} \quad k = k_0 + k_1 \quad (2.80)$$

Next, we add k times equation (2.58) to h times equation (2.59), use equations (2.43) and (2.44), and obtain

$$\begin{aligned} \frac{d\langle zh+xk \rangle}{dt} &= v_e \langle xh - zk \rangle - \frac{1}{2} (\delta_e + \beta) \langle zh + xk \rangle \\ &+ \frac{1}{2} \beta (h_0 \langle z \rangle + k_0 \langle x \rangle) + \frac{\langle h^2 + k^2 \rangle}{2\omega_0} + \sqrt{\frac{\beta}{2}} \langle zW_1 + xW_2 \rangle \end{aligned} \quad (2.81)$$

Subtracting k times equation (2.59) from h times equation (2.58) and using equations (2.43) and (2.44), we obtain

$$\begin{aligned} \frac{d\langle xh - zk \rangle}{dt} = & -v_e \langle zh + xk \rangle - \frac{1}{2} (\delta_e + \beta) \langle xh - zk \rangle + \frac{1}{2} \beta (h_0 \langle x \rangle - k_0 \langle z \rangle) \\ & + \sqrt{\frac{\beta}{2}} \langle xW_1 - zW_2 \rangle \end{aligned} \quad (2.82)$$

Taking the expected values of equations (2.58) and (2.59) yields

$$\frac{d\langle x \rangle}{dt} = -\frac{1}{2} \epsilon \delta_e \langle x \rangle - \epsilon v_e \langle z \rangle + \frac{\epsilon k_0}{2\omega_0} \quad (2.83)$$

$$\frac{d\langle z \rangle}{dt} = -\frac{1}{2} \epsilon \delta_e \langle z \rangle + \epsilon v_e \langle x \rangle + \frac{\epsilon h_0}{2\omega_0} \quad (2.84)$$

Following Rajan and Davies, we assume that $\langle xW_1 \rangle = 0$ and $\langle zW_1 \rangle = 0$. Then, the stationary solutions (long-time averaging) of equations (2.79) and (2.81)-(2.84) are given by

$$\langle x^2 + z^2 \rangle = \frac{2(\delta_e + \beta)}{\omega_0^2 \delta_e [4v_e^2 (\delta_e + \beta)^2]} \left[\langle F^2 \rangle + \frac{\beta F_0^2}{\delta_e^2 + 4v_e^2} \left(\delta_e - \frac{4v_e^2}{\delta_e + \beta} \right) \right] \quad (2.85)$$

where

$$F_0^2 = \frac{1}{2} (h_0^2 + k_0^2) \quad \text{and} \quad \langle F^2 \rangle = F_0^2 + \frac{1}{2} \langle h_1^2 + k_1^2 \rangle \quad (2.86)$$

A short-time averaging is implied in equation (2.86). Equation (2.85) is a nonlinear algebraic equation in $\langle x^2 + z^2 \rangle$. The response is assumed to be psuedo sinusoidal and the frequency shift and damping parameters of the system are given by

$$\delta_e = -\delta_1 + \frac{1}{4} \omega_0^2 \delta_2 \langle x^2 + z^2 \rangle \quad (2.87)$$

$$v_e = v - \frac{3\alpha}{8\omega_0} \langle x^2 + z^2 \rangle \quad (2.88)$$

Richard and Anand [142] used a generalized form of Van der Pol's method to investigate the stability of the response of the Duffing oscillator to a narrow-band random excitation. With probability one, they found the conditions under which the perturbations of the averaged equations tend to zero. Davies and Nandlall [143] used the phase-plane diagram [$\langle \dot{u}^2 \rangle$ vs. $\langle u^2 \rangle$] for their stability analysis. Mean-square values were obtained from the time dependent Fokker-Planck-Kolmogorov equation. A Gaussian closure approximation was adopted to handle higher-order moments. Fast varying terms were averaged out to obtain smooth time histories of $\langle u^2 \rangle$ and $\langle \dot{u}^2 \rangle$.

The stability of a particular fixed point of equations (2.79), (2.81)-(2.84), (2.87) and (2.88) to a perturbation proportional to $\exp(\lambda t)$ depends on the eigenvalues of the Jacobian matrix.

2.6 Equivalent Linearization for Non-Zero Mean Random Excitations

An equivalent linear equation that will furnish an approximate solution of equation (1.1) can be written as

$$\ddot{u} + \omega_e^2 u + \mu_e \dot{u} = \epsilon [h_0 \cos \omega t - k_0 \sin \omega t + \xi(t)] \quad (2.89)$$

The equivalent stiffness ω_e^2 and damping μ_e can be determined by minimizing the mean-square value of the difference between equations (2.1) and (2.89); that is

$$\omega_e^2 = \omega_0^2 + \epsilon \frac{\langle u g(u, \dot{u}) \rangle - \mu_e \langle u \dot{u} \rangle}{\langle u^2 \rangle} \quad (2.90)$$

$$\mu_e = \epsilon \frac{\langle \dot{u} g(u, \dot{u}) \rangle - \omega_e^2 \langle u \dot{u} \rangle}{\langle \dot{u}^2 \rangle} \quad (2.91)$$

where

$$g(u, \dot{u}) = -\delta_1 \dot{u} + \frac{1}{3} \delta_2 \dot{u}^3 + \alpha u^3 \quad (2.92)$$

Next, we separate u into a mean component u_0 and a random component u_1 according to

$$u = u_0 + u_1 \quad (2.93)$$

where

$$\ddot{u}_0 + \omega_e^2 u_0 + \mu_e \dot{u}_0 = \varepsilon [h_0 \cos \omega t - k_0 \sin \omega t] \quad (2.94)$$

$$\ddot{u}_1 + \omega_e^2 u_1 + \mu_e \dot{u}_1 = \varepsilon \xi(t) \quad (2.95)$$

Substituting equation (2.92) into equations (2.90) and (2.91) and using equation (2.93) yields

$$\omega_e^2 = \omega_0^2 + [u_0^2 + \langle u_1^2 \rangle]^{-1} [\varepsilon \alpha u_0^4 + 6\varepsilon \alpha u_0^2 \langle u_1^2 \rangle + \varepsilon \alpha \langle u_1^4 \rangle] \quad (2.96)$$

$$\mu_e = -\varepsilon \delta_1 + [\dot{u}_0^2 + \langle \dot{u}_1^2 \rangle]^{-1} \left[\frac{1}{3} \varepsilon \delta_2 \dot{u}_0^4 + 2\varepsilon \delta_2 \dot{u}_0^2 \langle \dot{u}_1^2 \rangle + \frac{1}{3} \varepsilon \delta_2 \langle \dot{u}_1^4 \rangle \right] \quad (2.97)$$

Using equation (2.94), we obtain

$$u_0^2 = \frac{1}{2} D^2 (h_0^2 + k_0^2), \quad \dot{u}_0^2 = \Omega^2 u_0^2 \quad (2.98)$$

$$u_0^4 = \frac{3}{8} D^4 (h_0^4 + 2h_0^2 k_0^2 + k_0^4), \quad \dot{u}_0^4 = \Omega^4 u_0^4 \quad (2.99)$$

where

$$D = [(\omega_e^2 - \Omega^2)^2 + (\mu_e \Omega)^2]^{-\frac{1}{2}} \quad (2.100)$$

A short-time averaging is implied in equations (2.98) and (2.99). For a Gaussian response $\langle u_1^4 \rangle = 3\langle u_1^2 \rangle^2$ and for a psuedo sinusoidal response $\langle u_1^4 \rangle = \frac{3}{2} \langle u_1^2 \rangle^2$.

When ξ is a narrow-band random excitation, the complex frequency response function $H(\omega)$ can be obtained by combining equations (2.95) and (2.41); that is,

$$H(\omega) = \frac{\epsilon \beta^{\frac{1}{2}} \Omega}{(\omega_e^2 - \omega^2 + i\mu_e \omega)(\Omega^2 - \omega^2 + i\beta \omega)} \quad (2.101)$$

The mean-square values of u_1 and $\dot{u}_1$ can be written as

$$\langle u_1^2 \rangle = \frac{\epsilon^2 \pi S_0 [\mu_e (\omega_e^2 + \beta^2) + \beta (\Omega^2 + \mu_e^2)]}{\mu_e \omega_e^2 [(\Omega^2 - \omega_e^2)^2 + (\mu_e + \beta)(\mu_e \Omega^2 + \beta \omega_e^2)]} \quad (2.102)$$

$$\langle \dot{u}_1^2 \rangle = \frac{\epsilon^2 \pi S_0 \Omega^2 (\mu_e + \beta)}{\mu_e [(\Omega^2 - \omega_e^2)^2 + (\mu_e + \beta)(\mu_e \Omega^2 + \beta \omega_e^2)]} \quad (2.103)$$

For the case of a white noise, we have

$$\langle u_1^2 \rangle = \frac{\epsilon^2 \pi S_0}{\mu_e \omega_e^2}, \quad \langle \dot{u}_1^2 \rangle = \frac{\epsilon^2 \pi S_0}{\mu_e} \quad (2.104)$$

2.7 Numerical Results

In the ρv -plane, where $\rho = \frac{1}{4} \omega_0^2 a_0^2$, the frequency-response equation (2.18) provides a one-parameter family of response curves. When $\alpha = 0$, the response curves are symmetric with respect to the ρ -axis. Let $f_0 = (h_0^2 + k_0^2)^{\frac{1}{2}}$. When $f_0 = 0$, the curves of the family degenerate into the line $\rho = 0$ and the point $v = 0$ and $\rho = 1$. As f_0 increases, the curves first consist of two branches - a branch running near the v -axis and a branch consisting of an oval, which can be approximated by an ellipse

having its center at $(0,1)$. As f_0 increases, the ovals expand and the branch near the v -axis moves away from this axis. When f_0^2 reaches the critical value $16/27\delta^2$ ($\delta = \delta_1 = \delta_2$), the two branches coalesce, and the resultant curve has a double point at $(0, \frac{1}{3})$ as shown in Figure 2.1. As f_0 increases further, the response curves become open curves, which continue to be multi-valued functions until f_0^2 exceeds the second critical value $32/27\delta^2$. Beyond this critical value, the response curves are single-valued functions for all values of v .

When $\alpha \neq 0$, the response curves lose their symmetry with respect to the ρ axis and bend to the right for $\alpha > 0$ (hardening nonlinearity) and bend to the left for $\alpha < 0$ (softening nonlinearity). Representative curves are shown in Figure 2.2 when $\alpha > 0$. When $f_0 = 0$, the family of curves degenerates into the line $\rho = 0$ and the point $\rho = 1$ and $v = 3\alpha/2\omega_0^3$. As f_0 increases, the response curves consist of two branches - a branch running near the ρ axis and a branch consisting of an oval. As f_0 increases further, the oval expands and the branch near the v -axis moves away from it. When f_0 reaches a critical value f_{01} , the two branches coalesce and have a double point. As f_0 increases beyond f_{01} , the corresponding response curve continues to be a multi-valued function of v until a second critical value f_{02} is exceeded. For all values of f_0 greater than f_{02} , the response curve is a single-valued function of v .

Not all of the solutions given by the frequency-response equation (2.18) are realizable because some of them are unstable. The solution given by the frequency-response equation (2.18) is stable if none of the

eigenvalues defined in equation (2.26) has a positive real part. The eigenvalues are given by the characteristic equation

$$\lambda^2 - \epsilon \delta (1 - 2\rho) \lambda + \epsilon^2 \Delta = 0 \quad (2.105)$$

where

$$\Delta = \frac{1}{4} \delta^2 (1 - 4\rho + 3\rho^2) + \nu^2 - \frac{6\nu\alpha\rho}{\omega_0^3} + \frac{27\alpha^2\rho^2}{4\omega_0^6} \text{ and } \rho = \frac{1}{4} \omega_0^2 a_0^2 \quad (2.106)$$

When $\Delta < 0$, the roots of equation (2.105) are real and have different signs. Hence, the predicted mean motions correspond to saddle points and are unstable. These correspond to the interior points of the ellipse $\Delta = 0$ in the ρ - ν plane. Since the discriminant of equation (2.105) is

$$D = \delta^2 \rho^2 - 4\nu^2 + \frac{24\nu\alpha\rho}{\omega_0^3} - \frac{27\alpha^2\rho^2}{\omega_0^6} \quad (2.107)$$

periodic motions corresponding to $D < 0$ are focal points, while those corresponding to $D > 0$ are nodal points. For $\delta > 0$, these points are stable or unstable depending on whether ρ is greater or less than $\frac{1}{2}$. The dark lines in Figures 2.1 and 2.2 separate stable from unstable solutions; all solutions corresponding to points above the dark lines are stable and those corresponding to points below the dark lines are unstable.

Figure 2.3 shows that the mean response predicted by the perturbation expansion is in good agreement with that obtained by numerically integrating equation (2.1) using a fifth-order Runge-Kutta Verner technique (numerical simulation).

The effect of the excitation frequency on the primary mean response is shown in Figure 2.4. The linearization method is used in Figure 2.4a to predict the effect of a narrow-band random excitation on the mean response. Near a saddle-node bifurcation point (the fixed point loses its stability with a real eigenvalue crossing the imaginary axis) and Hopf bifurcation point (the fixed point of the mean motion loses its stability with the real part of a complex conjugate pair of eigenvalues changing sign from negative to positive), the mean motion is sensitive to any outside perturbations and the linearization method fails. In Figure 2.4b, the second-order closure method is used to obtain the mean-square response. The second-order closure method provides the necessary feedback from the perturbations caused by the noise to the mean motion. This limits the second-order moments predicted by the linearization method at the bifurcation points separating stable from unstable mean responses. Since $\alpha > 0$, the response curves in Figure 2.4 are bent to the right, causing multi-valued regions. The multi-valuedness is responsible for a jump phenomenon at the saddle-node bifurcation points that leads to an abrupt change in the response.

The semi-linearization method is based on assuming an average value $\langle x^2 + z^2 \rangle$ of the amplitude. With this assumption, substituting equations (2.62) into equations (2.58)-(2.61) and separating mean and fluctuating components for the case of a narrow-band random excitation, we have

mean motion

$$\begin{aligned}\dot{x}_0 = & \frac{1}{2} \varepsilon \delta_1 x_0 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0^3 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0 z_0^2 - \varepsilon \nu z_0 + \frac{3\varepsilon\alpha}{8\omega_0} x_0^2 z_0 \\ & + \frac{3\varepsilon\alpha}{8\omega_0} z_0^3 - \left[\frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0 - \frac{3\varepsilon\alpha}{8\omega_0} z_0 \right] \langle y_1^2 \rangle - \left[\frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0 \right. \\ & \left. - \frac{3\varepsilon\alpha}{8\omega_0} z_0 \right] \langle y_2^2 \rangle + \frac{\varepsilon k_0}{2\omega_0}\end{aligned}\quad (2.108)$$

$$\begin{aligned}\dot{z}_0 = & \frac{1}{2} \varepsilon \delta_1 z_0 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0^2 z_0 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 z_0^3 + \varepsilon \nu x_0 - \frac{3\varepsilon\alpha}{8\omega_0} x_0^3 \\ & - \frac{3\varepsilon\alpha}{8\omega_0} x_0 z_0^2 - \left[\frac{3\varepsilon\alpha}{8\omega_0} x_0 + \frac{1}{8} \varepsilon \omega_0^2 \delta_2 z_0 \right] \langle y_1^2 \rangle \\ & - \left[\frac{3\varepsilon\alpha}{8\omega_0} x_0 + \frac{1}{8} \varepsilon \omega_0^2 \delta_2 z_0 \right] \langle y_2^2 \rangle + \frac{\varepsilon h_0}{2\omega_0}\end{aligned}\quad (2.109)$$

fluctuating motion

$$\begin{aligned}\dot{y}_1 = & \left[\frac{1}{2} \varepsilon \delta_1 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0^2 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 z_0^2 \right] y_1 + \left[-\varepsilon \nu + \frac{3\varepsilon\alpha}{8\omega_0} x_0^2 \right. \\ & \left. + \frac{3\varepsilon\alpha}{8\omega_0} z_0^2 \right] y_2 + \frac{\varepsilon k_1}{2\omega_0}\end{aligned}\quad (2.110)$$

$$\begin{aligned}\dot{y}_2 = & \left[\varepsilon \nu - \frac{3\varepsilon\alpha}{8\omega_0} x_0^2 - \frac{3\varepsilon\alpha}{8\omega_0} z_0^2 \right] y_1 + \left[\frac{1}{2} \varepsilon \delta_1 - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 x_0^2 \right. \\ & \left. - \frac{1}{8} \varepsilon \omega_0^2 \delta_2 z_0^2 \right] y_2 + \frac{\varepsilon h_1}{2\omega_0}\end{aligned}\quad (2.111)$$

The mean motion given by equations (2.108) and (2.109) provides a feedback from the noise through $\langle y_1^2 \rangle$ and $\langle y_2^2 \rangle$. However, the cross correlation $\langle y_1 y_2 \rangle$ has no contribution and the coefficients of the terms $x_0 \langle y_1^2 \rangle$ and $z_0 \langle y_2^2 \rangle$ are one-third of those given by equations (2.63) and (2.64) obtained by the second-order closure method. The linearization method and the second-order closure method have the same form with respect to the fluctuating motion. According to equations

(2.65), the fluctuating motion given by equations (2.110) and (2.111) does not include the $x_0 z_0$ terms. The coefficients of $x_0^2 y_1$ and $z_0^2 y_2$ are one-third those given by equations (2.65). As a result, the accuracy of the semi-linearization method is somewhere between the linearization method and the second-order closure method. Figure 2.5 shows a comparison among the results of these three methods and that obtained by the method of equivalent linearization. Away from the transition zone between stable and unstable responses, the results are in good agreement. The linearization method is simple and direct but it fails to give good approximations to the mean-square response at the stability boundaries. The second-order closure method is expected to be the most accurate but it requires the solution of a system of five simultaneous nonlinear algebraic equations.

The local stability of the stationary mean and mean-square response is analyzed. Figure 2.6 shows the effect of a low intensity noise, around the natural frequency ω_0 of the system, on the stability of the mean response. The strength of the noise is taken as $1/450$ of the strength of the deterministic excitation. Since $\alpha = 0$, the response curve is symmetric with respect to the $\langle u^2 \rangle$ - axis. The response curve has its peak at the perfectly-tuned frequency $\Omega = \omega_0$ (i.e., $\nu = 0$). The mean motion loses stability when $|\nu| \geq 0.093$. The presence of the noise expands the stability region $|\nu| \geq 0.1$.

As the amplitude of the mean excitation increases, the effect of the noise on the steady-state mean response decreases as shown in Figure 2.7.

2.8 Conclusions

A second-order closure method is presented for determining the response of nonlinear systems to random excitations. As an example, the method is used to determine the response of the Duffing-Rayleigh oscillator to an excitation consisting of the sum of a deterministic harmonic component and a random component. The random component is assumed to be white noise or sinusoidal whose amplitude and phase have two parts - a constant mean and a separable nonstationary random process. The method of multiple scales is used to determine the modulation of the amplitude and phase with the nonlinearity and the excitation. Three methods have been used to determine the mean and mean-square responses from the modulation equations. The local stability of the stationary mean and mean-square responses is analyzed. For some range of parameters, the presence of noise can expand the stability region of the mean response. Comparisons of the results of these methods with the results obtained by using the method of equivalent linearization are presented.

The present analysis attacks the problem in a way that the statistics of the response can be determined for general types of excitation and nonlinearity. The nonlinearity bends the frequency-response curves, causing multi-valued regions in the mean-square response. The multi-valuedness is responsible for a jump phenomenon.

CHAPTER 3

A GENERALIZED METHOD OF EQUIVALENT LINEARIZATION

The application of the method of equivalent linearization for solving nonlinear vibration problems is reviewed. The conceptual drawbacks and limitations of this method are discussed. A generalization of the method of equivalent linearization is presented. The method is based on replacing the nonlinear governing equation by two equivalent linear equations. This generalization scheme overcomes some of the drawbacks and limitations and also contributes to a better understanding of the basic phenomena introduced by the presence of nonlinearities. It provides some means to describe the non-Gaussian feature of the response of a nonlinear system. An attempt to explain some experimental phenomena in the large-amplitude response of aircraft panels to an acoustic excitation is presented. These include broadening effects and a shift in the resonant frequency. The proposed approach is illustrated by investigating the response of a class of nonlinear systems to random excitations. The effect of the system parameters on the response is analyzed for the cases of a Gaussian white noise and a superharmonic narrow-band random excitation of order-three.

3.1 Introduction

Over the years, a number of approximate methods for solving nonlinear stochastic problems has been developed and recently emphasis has been placed on their adoption for computers. These include the

method of equivalent or stochastic linearization, perturbation methods, functional series representations, and digital- and analog-computer simulations. However, the method of equivalent linearization has the widest range of applicability. This method is potentially useful for systems having nonlinear damping, nonlinear restoring forces, nonlinearity of a mixed type, and nonlinear hereditary forces, and it can be easily extended to the case of multi-degree-of-freedom systems.

The method of equivalent linearization was first proposed by Jacobsen [144]; it is based on the method of averaging of Krylov and Bogoliubov [76] for deterministic nonlinear problems. Booton [77] and Caughey [78,47] introduced independently this method to statistical problems.

To describe the method, we consider

$$\ddot{x} + \omega_0^2 x + \varepsilon h(x, \dot{x}) = F(t) \quad (3.1)$$

which describes the response of a single-degree-of-freedom (SDOF) system to a random excitation. Here, ω_0 is the linear natural frequency, $h(x, \dot{x})$ is a nonlinear force, ε is a small but finite constant, and $F(t)$ is a zero-mean random excitation. According to the method of equivalent linearization, equation (3.1) is replaced with the linear equation

$$\ddot{x} + \omega_e^2 x + 2\varepsilon\mu_e \dot{x} = F(t) \quad (3.2)$$

where the equivalent stiffness ω_e^2 and damping μ_e parameters are determined by minimizing the mean-square value of the difference between equations (3.1) and (3.2); that is,

$$\omega_e^2 = \omega_0^2 + \epsilon \frac{\langle xh(x, \dot{x}) \rangle}{\langle \dot{x}^2 \rangle} \quad (3.3a)$$

$$\mu_e = \frac{\langle \dot{x}h(x, \dot{x}) \rangle}{2\langle \dot{x}^2 \rangle} \quad (3.3b)$$

where the angle brackets stand for the expectation and the response x is assumed to be stationary. A knowledge of the probability-density function $p(x, \dot{x}; t|x_0, \dot{x}_0; t_0)$ is needed to find the expectations in equations (3.3). If available, one can use the stationary probability-density function of the original nonlinear equation (3.1). As an approximation, one can use the probability-density function of the equivalent linear equation (3.2).

Iwan and Gates [145] investigated the accuracy of the various available techniques in approximating the equivalent linear system parameters. They considered the response of a nonlinear hysteretic SDOF oscillator to an earthquake excitation. Iwan and Patula [146] concluded that the mean-square error minimization is at least as good as any other type. It can be shown that the mean-square error minimization ensures the same $\langle x \rangle$ for both the nonlinear and equivalent linear systems and the difference in $\langle \dot{x}^2 \rangle$ for both systems is a minimum.

When the input is Gaussian and $G(x, \dot{x})$ satisfies certain smoothness conditions, the equivalent stiffness and damping parameters can be written as [79,80]

$$\omega_e^2 = \left\langle \frac{\partial q(x, \dot{x})}{\partial x} \right\rangle \quad (3.4a)$$

$$2\epsilon\mu_e = \left\langle \frac{\partial q(x, \dot{x})}{\partial \dot{x}} \right\rangle \quad (3.4b)$$

where

$$g(x, \dot{x}) = \omega_0^2 x + \epsilon h(x, \dot{x}) \quad (3.5)$$

If the matrix $\begin{bmatrix} \langle \ddot{x} \rangle & \langle \dot{x}\ddot{x} \rangle \\ \langle \dot{x}\ddot{x} \rangle & \langle \dot{x}\dot{x} \rangle \end{bmatrix}$ is not singular, equations (3.4) will yield a true minimum solution [82]. According to equations (3.4), even nonlinearities contribute nothing to the equivalent linear system parameters. This means that the applicability of the method of equivalent linearization is limited to odd nonlinearities.

When the input is Gaussian, the method of equivalent linearization predicts a Gaussian response, whereas the original nonlinear equation (3.1) may predict a non-Gaussian response. To describe the non-Gaussian feature of the response, Beaman and Hedrick [147] and Crandall [148] represented the probability-density function $p(x)$ by a truncated Gram-Charlier expansion. The first term in the expansion is Gaussian and it is identical to the one obtained by using the method of equivalent linearization.

Kazakov [81] generalized the method of equivalent linearization to multi-degree-of-freedom nonlinear systems. Spanos and Iwan [82] considered the existence and uniqueness of solutions generated by the method of equivalent linearization. Iwan and Mason [80] used the same technique for systems subject to nonstationary random excitations.

For systems with hysteresis, Caughey [149] applied a modified version of the equivalent-linearization procedure. The response is

assumed to be sinusoidal with slowly varying amplitude and phase (narrow-band process). Hence,

$$x = a \cos(\omega_0 t + \beta) \quad \text{and} \quad \dot{x} = -\omega_0 a \sin(\omega_0 t + \beta) \quad (3.6)$$

Using the method of variation of parameters and eliminating fastly varying terms, we obtain

$$\omega_e^2 = \omega_0^2 + \frac{\varepsilon}{\pi} \frac{\langle h_1(a) \rangle}{\langle a^2 \rangle} \quad (3.7a)$$

$$\mu_e = - \frac{\langle h_2(a) \rangle}{2\pi\omega_0 \langle a^2 \rangle} \quad (3.7b)$$

where

$$h_1(a) = \int_0^{2\pi} a h(a \cos\phi, -\omega_0 a \sin\phi) \cos\phi d\phi \quad (3.8a)$$

$$h_2(a) = \int_0^{2\pi} a h(a \cos\phi, -\omega_0 a \sin\phi) \sin\phi d\phi \quad (3.8b)$$

$$\text{and} \quad \phi = \omega_0 t + \beta \quad (3.8c)$$

However, the loss of energy is relatively large for hysteretic systems. This broadens the frequency content of the response and eliminates its narrow-band nature. As a result, the accuracy of the method of equivalent linearization is rather poor for inelastic structures and seriously overestimates the energy dissipation in such systems [150]. Monte-Carlo-simulation results of the response of bilinear hysteretic systems to a Gaussian white noise indicate that the accuracy of the method of equivalent linearization deteriorates as the

damping increases and the response probability-density function is significantly non-Gaussian [65]. The proposed method in Section 3.3 improves the accuracy of the solution by accounting for some of the wide-band character of the response.

Many authors [85-89] used the method of equivalent linearization to determine the response statistics of hysteretic degrading systems. A generalization of the method of Atalik and Utku to the case of nonzero-mean problems is proposed by Baber and Wen [85].

For large-amplitude responses, the method of equivalent linearization fails to give a decent approximation to the response. It does not account for all interactions between the nonlinearity and the excitation. The nonlinearity creates a whole range of interesting phenomena in the response of structures to deterministic and random excitations. These include non-Gaussian responses, multi-valued responses, jumps, modal frequency shifts, broadening effects, and superharmonic, subharmonic, ultrasubharmonic, and combination resonances. The equivalent linear model equation (3.2) fails to account for most of these phenomena. This limits the applicability of the method of equivalent linearization to weak nonlinearities and soft (very small) excitations [151,51].

A number of experimental studies tested the applicability of the method of equivalent linearization. They include the studies of Nikiforuk and West [90], Smith [91], Gelb and Van der Velde [93], and Morton [92]. Bulsara et al. [83] made a comparison between the results of an analog-computer simulation and those obtained by the method of

equivalent linearization for the Duffing oscillator. The results show that the method of equivalent linearization underestimates the low-frequency part and overestimates the peak height of the spectral density. When $h(x, \dot{x}) = \alpha x^3$ or $\beta x^2 \dot{x}$ and $F(t)$ is a Gaussian white noise, the percent error in the mean-square response obtained by using the method of equivalent linearization is about 15% [79]. The method of equivalent linearization underestimates the root-mean-square response of a nearly elasto-plastic system by up to 50-60 percent [150].

Many authors [152-158] investigated the sonic fatigue design of aircraft structures based on a linear or a nonlinear analysis. The linear structural theory predicts larger mean-square responses and smaller modal frequencies than the corresponding experimental results. This leads to a poor estimate of the service life of structural elements. Mei and Prasad [155] investigated the effect of nonlinear damping on the random response of beams to acoustic excitation. The excitation was taken to be a white noise. Test results show an increase in the resonant frequency, a broadening of the frequency-response curves, and a shift in the response peak. Assuming that the beams allow large deflections and the fundamental frequency is predominant leads to a modal equation of the Duffing type. The method of equivalent linearization can only account for a nonlinear shift in the frequency of the Duffing oscillator. Mei and Prasad [155] considered the influence of the two nonlinear terms

$$\delta_1 \dot{x} x^2 + \delta_2 \dot{x}^2 x \quad (3.9)$$

on the broadening of the response; the first term is a nonlinear damping term, whereas the second term is a nonlinear inertia term. The experimental results of Smith et al. [156] suggest the presence of the nonlinear damping term in the model of a simple clamped panel. The proposed method explains some of the broadening behavior without introducing nonlinear damping models.

In Section 3.3, a generalized method of equivalent linearization is presented. It provides some means for describing the non-Gaussian feature of the response and preserving some of its wide-band character. The nonlinear governing equation of motion is replaced by two equivalent equations: one is linear and the other is nonlinear. According to the method of equivalent linearization, the latter is replaced by an approximate linear equation. The interaction and feedback between the nonlinearity and the excitation creates nonlinear resonances. These lead to a linear shift in the natural frequency and an adjustment in the strength of the primary excitation. The latter contributes to the broadening effects. The proposed method is illustrated by investigating the response of the Duffing and Van der Pol-Rayleigh oscillators to random excitations. To test the soundness of the method, we compare the mean-square responses obtained by using the proposed scheme with those obtained by using the Fokker-Planck-Kolmogorov equation.

3.2 Limitations of the Method of Equivalent Linearization

One of the major limitations of the method of equivalent linearization is that the response of a system to a deterministic excitation occurs at the driving frequencies with no contribution from other harmonics. As an example, we consider the response of the Duffing oscillator to a deterministic harmonic excitation. The problem is governed by

$$\ddot{x} + \omega_0^2 x + 2\varepsilon\mu\dot{x} + \varepsilon\alpha x^3 = G \cos\Omega t \quad (3.10)$$

The linearized version of the Duffing equation (3.10) has the form

$$\ddot{x} + \omega_e^2 x + 2\varepsilon\mu\dot{x} = G \cos\Omega t \quad (3.11)$$

The steady-state solution of equation (3.11) can be written as

$$x = A \cos(\Omega t - \zeta) \quad (3.12)$$

where

$$A = \frac{G}{[(\omega_e^2 - \Omega^2)^2 + (2\varepsilon\mu\Omega)^2]^{\frac{1}{2}}}, \quad \zeta = \tan^{-1} \left[\frac{2\varepsilon\mu\Omega}{\omega_e^2 - \Omega^2} \right] \quad (3.13)$$

and

$$\omega_e^2 = \omega_0^2 + \frac{3}{4} \varepsilon\alpha \bar{a}^2 \quad (3.14)$$

Next, we consider different resonances and compare the dynamic responses obtained by using the method of averaging and the method of equivalent linearization with those obtained by numerically integrating equation (3.10) using a fifth-order Runge-Kutta Verner technique (numerical simulation).

3.2.1 Primary Resonance

In this case, we let

$$\Omega = \omega_0 + \epsilon \nu_1 \text{ and } G = \epsilon \hat{G} \quad (3.15)$$

Using the method of averaging [95,96], one can write the steady-state solution of equation (3.10) as

$$x = a_0 \cos(\Omega t - \gamma_0) + \dots \quad (3.16)$$

where

$$-\mu a_0 + \frac{\hat{G}}{2\omega_0} \sin \gamma_0 = 0 \quad (3.17)$$

$$\nu_1 a_0 - \frac{3}{8\omega_0} \alpha a_0^3 + \frac{\hat{G}}{2\omega_0} \cos \gamma_0 = 0 \quad (3.18)$$

Comparing equations (3.16)-(3.18) with equations (3.12)-(3.14), we conclude that the solution obtained by using the the method of equivalent linearization has the same frequency content as that obtained by using the method of averaging. Figure 3.1 shows that the responses predicted by using the method of averaging and the method of equivalent linearization are in good agreement with that obtained by using numerical simulation.

3.2.2 Superharmonic Resonance of Order-Three

In this case, we let

$$3\Omega = \omega_0 + \epsilon \nu_2 \quad (3.19)$$

Using the method of averaging [95,96], one can write the steady-state solution of equation (3.10) as

$$x = a_0 \cos(3\Omega t - \gamma_0) + \Lambda \cos \Omega t + \dots \quad (3.20)$$

where

$$\mu a_0 + \frac{\alpha \Lambda^3}{8\omega_0} \sin \gamma_0 = 0 \quad (3.21)$$

$$\nu_2 a_0 - \frac{3\alpha \Lambda^2 a_0}{4\omega_0} - \frac{3\alpha a_0^3}{8\omega_0} - \frac{\alpha \Lambda^3}{8\omega_0} \cos \gamma_0 = 0 \quad (3.22)$$

and

$$\Lambda = \frac{G}{\omega_0^2 - \Omega^2} \quad (3.23)$$

Comparing equations (3.12)-(3.14) with equations (3.20)-(3.23), we conclude that the method of equivalent linearization gives only one part of the two-part solution obtained by using the method of averaging. Figure 3.2 shows a comparison of the time histories obtained by using the different methods. The method of equivalent linearization fails to predict the peak amplitude and the overall shape of the time history.

3.2.3 Subharmonic Resonance of Order One-Third

In this case, we let

$$\Omega = 3\omega_0 + \epsilon \nu_3 \quad (3.24)$$

Using the method of averaging, one can write the steady-state solution of equation (3.10) as

$$x = a_0 \cos\left(\frac{1}{3} \Omega t - \frac{1}{3} \gamma_0\right) + \Lambda \cos \Omega t + \dots \quad (3.25)$$

where

$$\mu \ddot{a}_0 + \frac{3\alpha a_0^2 \Lambda}{8\omega_0} \sin \gamma_0 = 0 \quad (3.26)$$

$$v_3 \ddot{a}_0 - \frac{9\alpha a_0^2 \Lambda^2}{4\omega_0} - \frac{9\alpha a_0^3}{8\omega_0} - \frac{9\alpha a_0^2 \Lambda}{8\omega_0} \cos \gamma_0 = 0 \quad (3.27)$$

Comparing equations (3.12)-(3.14) with equations (3.25)-(3.27), we conclude that the method of equivalent linearization fails to give the whole frequency content obtained by using the method of averaging as shown in Figure 3.3.

The above analysis shows that the method of equivalent linearization fails to account for nonlinear resonances.

3.3 Proposed Method

The present work is concerned with the response of single-degree-of-freedom systems to random excitations governed by the equation

$$\ddot{x} + \omega_0^2 x + 2\mu \dot{x} + k(x, \dot{x}) = F(t) \quad (3.28)$$

where k is a nonlinear function of x and $\dot{x}$. The excitation F represents a stationary random excitation with a zero mean and a Gaussian probability-density function. The main idea underlying the proposed method is to divide the excitation into two parts as $F = f_1 + f_2$. The first part f_1 has a spectral density around ω_0 , whereas the second part f_2 has a spectral density away from ω_0 . Moreover, the solution of equation (3.28) is expressed as

$$x = u + v \quad (3.29)$$

where

$$\ddot{u} + \omega_0^2 u + 2\mu\dot{u} = f_2(t) \quad (3.30)$$

The solution of equation (3.30) can be written in terms of Duhamel's integral; that is,

$$u(t) = \int_0^{\infty} h_1(\tau) f_2(t - \tau) d\tau \quad (3.31)$$

where

$$h_1(\tau) = (\omega_0^2 - \mu^2)^{-\frac{1}{2}} e^{-\mu\tau} \sin(\omega_0^2 - \mu^2)^{\frac{1}{2}} \tau \quad (3.32)$$

Since f_2 has no spectral component around ω_0 , the solution given by equation (3.31) has no irregularities. It follows from equation (3.30) that the output spectral density of u is

$$S_u(\omega) = \frac{S_{f_2}(\omega)}{(\omega_0^2 - \omega^2)^2 + 4\mu^2 \omega^2} \quad (3.33)$$

Substituting equation (3.29) into equation (3.28) and using equation (3.30), we obtain the following equation governing v :

$$\ddot{v} + \omega_0^2 v + 2\mu\dot{v} + k_1(u, \dot{u}, v, \dot{v}) = f_1(t) - k(u, \dot{u}) \quad (3.34)$$

where

$$k_1(u, \dot{u}, v, \dot{v}) = k(u + v, \dot{u} + \dot{v}) - k(u, \dot{u}) \quad (3.35)$$

The nonlinear function k adjusts the external excitation by $k(u, \dot{u})$ and leads to parametric excitations and shifts in the linear stiffness and

damping parameters through k_1 . Next, we replace equation (3.34) with the equivalent linear equation

$$\ddot{v} + \omega_e^2 v + 2\mu_e \dot{v} = \Gamma(t) \quad (3.36)$$

where

$$\omega_e^2 = \omega_0^2 + \frac{\langle vk_1 \rangle}{\langle v^2 \rangle}, \quad 2\mu_e = 2\mu + \frac{\langle \dot{v}k_1 \rangle}{\langle \dot{v}^2 \rangle} \quad (3.37)$$

and

$$\Gamma = f_1 - k(u, \dot{u}) \quad (3.38)$$

The term $k(u, \dot{u})$ in the excitation Γ accounts for part of the non-Gaussian feature of the nonlinear response x . Because the method of equivalent linearization cannot account for parametric excitations, we neglect the parametric terms in equations (3.36)-(3.38). We note that Brückner and Lin [161] used the method of equivalent linearization to replace the original Itô equations with linear Itô equations under certain optimal conditions for parametric excitations. The solution of equation (3.36) can be written in terms of Duhamel's integral as

$$v = \int_0^\infty h_2(\tau) f_1(t - \tau) d\tau - \int_0^\infty h_2(\tau) k[u(t - \tau), \dot{u}(t - \tau)] d\tau \quad (3.39)$$

where

$$h_2(\tau) = (\omega_e^2 - \mu_e^2)^{-\frac{1}{2}} e^{-\mu_e \tau} \sin(\omega_e^2 - \mu_e^2)^{\frac{1}{2}} \tau \quad (3.40)$$

3.4. Examples and Accuracy

3.4.1. The Duffing Oscillator

In this case, we have

$$k(x, \dot{x}) = \alpha x^3 \quad (3.41)$$

The equivalent stiffness and damping parameters and the excitation for equation (3.36) can be written as

$$\omega_e^2 = \omega_0^2 + \langle 3\alpha u^2 v^2 + 3\alpha uv^3 + \alpha v^4 \rangle / \langle v^2 \rangle \quad (3.42)$$

$$\mu_e = \mu \quad (3.43)$$

$$\Gamma = f_1 - \alpha u^3 \quad (3.44)$$

The presence of αu^3 in the excitation Γ provides some of the non-Gaussian feature of the nonlinear response x . The term $\langle u^2 v^2 \rangle$ accounts for both an increase in the linear natural frequency and ultra-subharmonic resonances, the term $\langle uv^3 \rangle$ accounts for the subharmonic resonance of order one-third, and the term u^3 in Γ accounts for the superharmonic and combination resonances. Neglecting the subharmonic and ultrasubharmonic resonances, we rewrite equation (3.42) as

$$\omega_e^2 = \omega_0^2 + 3\alpha \langle u^2 \rangle + \alpha \langle v^4 \rangle / \langle v^2 \rangle \quad (3.45)$$

If we assume that v is Gaussian, then $\langle v^4 \rangle = 3\langle v^2 \rangle^2$ and equation (3.45) becomes

$$\omega_e^2 = \omega_0^2 + 3\alpha [\langle u^2 \rangle + \langle v^2 \rangle] \quad (3.46)$$

Identical results could be obtained by using the method of averaging (details in Appendix II). A nonlinear algebraic equation in $\langle v^2 \rangle$ for the equivalent linear equation (3.36) can be written as

$$\begin{aligned}
\langle v^2 \rangle &= \int_{-\infty}^{\infty} \frac{S_{f_1}(\omega)}{(\omega_e^2 - \omega^2)^2 + 4\mu^2 \omega^2} d\omega + \alpha^2 \int_{-\infty}^{\infty} \frac{S_{u^3}(\omega)}{(\omega_e^2 - \omega^2)^2 + 4\mu^2 \omega^2} d\omega \\
&- 2\alpha \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} h_2(\tau_1) h_2(\tau_2) h_1(\tau_3) h_1(\tau_4) h_1(\tau_5) < f_1(t - \tau_1) \\
&f_2(t - \tau_2 - \tau_3) f_2(t - \tau_2 - \tau_4) f_2(t - \tau_2 - \tau_5) > d\tau_1 d\tau_2 d\tau_3 d\tau_4 d\tau_5 \quad (3.47)
\end{aligned}$$

Using equation (3.29), we can express the mean-square response as

$$\langle x^2 \rangle = \langle u^2 \rangle + \langle v^2 \rangle + 2\langle uv \rangle \quad (3.48)$$

where $\langle v^2 \rangle$ is given by equation (3.47) and

$$\langle uv \rangle = -\alpha \int_0^{\infty} h_2(\tau) \langle u(t) u^3(t - \tau) \rangle d\tau \quad (3.49)$$

Substituting equation (3.31) into equation (3.49), we obtain

$$\begin{aligned}
\langle uv \rangle &= -\alpha \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} h_2(\tau) h_1(\tau_1) h_1(\tau_2) h_1(\tau_3) h_1(\tau_4) \\
&< f_2(t - \tau_1) f_2(t - \tau - \tau_2) f_2(t - \tau - \tau_3) f_2(t - \tau - \tau_4) d\tau d\tau_1 d\tau_2 d\tau_3 d\tau_4 \quad (3.50)
\end{aligned}$$

The probability-density functions of f_1 and f_2 are needed to perform the integrations in equations (3.47) and (3.50). In the present work f_1 and f_2 are assumed to be Gaussian functions. Using the Gaussian moment relations, we find that the third integral in equation (3.47) vanishes and hence

$$S_{u^3}(\omega) = 9\langle u^2 \rangle^2 S_u(\omega) + 6 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_u(r_1) S_u(r_2) S_u(\omega - r_1 - r_2) dr_1 dr_2 \quad (3.51)$$

and

$$\langle uv \rangle = -3\alpha \langle u^2 \rangle \int_0^{\infty} h_2(\tau) R_u(\tau) d\tau \quad (3.52)$$

or

$$\langle uv \rangle = -3\alpha \langle u^2 \rangle \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} h_2(\tau) h_1(\tau_1) h_1(\tau_2) R_{f_2}(\tau + \tau_1 - \tau_2) d\tau d\tau_1 d\tau_2 \quad (3.53)$$

A Superharmonic Narrow-Band Random Excitation

Raghunandan and Anand [159,160] considered the problems of superharmonic resonance of order three and subharmonic resonance of order one-third in stretched strings. They started by assuming solutions similar to those obtained by using the method of averaging, that is;

$$x = u + v = H_C \cos \omega t + H_S \sin \omega t + Y_C \cos 3\omega t + Y_S \sin 3\omega t \quad (3.54)$$

Following Raghunandan and Anand [159,160], Rajan [140] used the method of harmonic balance to separate the random primary response u from the random secondary response v of the Duffing oscillator. He did not account for the variations in $\langle u^2 \rangle$ and assumed that $\langle uv \rangle$ is negligible. Assuming v to be Gaussian, he used the method of equivalent linearization to replace the nonlinear equations governing Y_C and Y_S by equivalent linear equations.

Here, $F(t)$ is chosen to be a narrow-band random excitation with a center frequency $\Omega \approx \frac{1}{3} \omega_0$, thereby producing a superharmonic resonance of order-three. The excitation could be obtained by filtering a white noise through a linear filter; that is,

$$\ddot{F} + \Omega^2 F + \varepsilon \dot{F} = W(t) \quad (3.55)$$

where ε is the bandwidth of the filter and the autocorrelation function of the white noise W is given by

$$R_W(\tau) = 2\pi S_0 \delta(\tau) \quad (3.56)$$

We describe the nearness of the resonance by introducing the detuning parameter ν defined according to

$$3\Omega = \omega_0 + \nu \quad (3.57)$$

In this case, we have

$$f_1 = 0, f_2 = F \quad (3.58)$$

Using equations (3.30) and (3.55), we write the mean-square value of u as

$$\langle u^2 \rangle = \frac{\pi S_0 [2\mu(\omega_0^2 + \varepsilon^2) + \varepsilon(\Omega^2 + 4\mu^2)]}{2\mu\varepsilon\omega_0^2 [(\Omega^2 - \omega_0^2)^2 + (2\mu + \varepsilon)(2\mu\Omega^2 + \varepsilon\omega_0^2)]} \quad (3.59)$$

The mean-square response $\langle x^2 \rangle$ is given by equation (3.48), where $\langle u^2 \rangle$ and $\langle uv \rangle$ are given by equations (3.59) and (3.53), respectively, and

$$\langle v^2 \rangle = \alpha^2 \int_{-\infty}^{\infty} \frac{S_{u3}(\omega)}{(\omega_e^2 - \omega^2)^2 + 4\mu^2 \omega^2} d\omega \quad (3.60)$$

$$R_{f_2}(\tau) = \frac{\pi S_0}{\beta \Omega^2} e^{-\frac{1}{2} \beta |\tau|} \cos \Omega \tau \quad (3.61)$$

According to the method of equivalent linearization, equation (3.28) can be replaced with

$$\ddot{x} + \omega_e^2 x + 2\mu \dot{x} = F(t) \quad (3.62)$$

where

$$\omega_e^2 = \omega_0^2 + 3\alpha \langle x^2 \rangle \quad (3.63)$$

Using equations (3.55) and (3.62), we write the mean-square response as

$$\langle x^2 \rangle = \frac{\pi S_0 [2\mu(\omega_e^2 + \beta^2) + \beta(\Omega^2 + 4\mu^2)]}{2\mu\beta\omega_e^2 \Omega^2 [(\Omega^2 - \omega_e^2)^2 + (2\mu + \beta)(2\mu\Omega^2 + \beta\omega_e^2)]} \quad (3.64)$$

In Figure 3.4, we show the variation of the mean-square response with the detuning parameter ν obtained by using the proposed generalized method and the method of equivalent linearization. For all values of ν , the method of equivalent linearization predicts a single-valued mean-square response. However, the proposed method displays multi-valued regions. In the multi-valued regions there are three response branches, one of them is unstable. The stable branches are the upper and lower response branches. Because $\alpha \neq 0$, the response curves lose their symmetry with respect to the $\langle x^2 \rangle$ -axis and because $\alpha > 0$, the response-peak occurs when $\nu > 0$ (hardening nonlinearity). As β

increases, the multi-valuedness disappears as shown in Figure 3.5. Similarly, increasing the linear damping parameter μ results in single-valued response curves as shown in Figure 3.6. Dependence of the mean-square response on the spectral amplitude S_0 of the excitation is presented in Figure 3.7. The method of equivalent linearization predicts a response curve, which is a single-valued function of S_0 , but the proposed method shows multi-valued regions.

A White Noise

In this case the spectral densities of F , f_1 , and f_2 are shown in Figure 3.8. The auto-correlation function of f_2 can be written as

$$R_{f_2}(\tau) = \frac{2S_0}{\tau} \left[\sin(\omega_0 - \frac{1}{2}\beta)\tau - \sin(\omega_0 + \frac{1}{2}\beta)\tau \right] + 2\pi S_0 \delta(\tau) \quad (3.65)$$

The mean-square value of u is

$$\begin{aligned} \langle u^2 \rangle = & \frac{S_0}{4\omega_0^2\omega_d} \left[\ln \frac{(\omega_1 + \omega_d)^2 + \mu^2}{(\omega_1 - \omega_d)^2 + \mu^2} - \ln \frac{(\omega_c - \omega_d)^2 + \mu^2}{(\omega_2 - \omega_d)^2 + \mu^2} + \ln \frac{(\omega_c + \omega_d)^2 + \mu^2}{(\omega_2 + \omega_d)^2 + \mu^2} \right] \\ & + \frac{S_0}{2\mu\omega_0^2} \left[\tan^{-1}\left(\frac{\omega_1 - \omega_d}{\mu}\right) + \tan^{-1}\left(\frac{\omega_1 + \omega_d}{\mu}\right) + \tan^{-1}\left(\frac{\omega_c - \omega_d}{\mu}\right) + \tan^{-1}\left(\frac{\omega_c + \omega_d}{\mu}\right) \right. \\ & \left. - \tan^{-1}\left(\frac{\omega_2 + \omega_d}{\mu}\right) \right] \end{aligned} \quad (3.66)$$

where ω_c is the cut-off frequency and

$$\omega_1 = \omega_0 - \frac{1}{2}\beta, \quad \omega_2 = \omega_0 + \frac{1}{2}\beta, \quad \text{and} \quad \omega_d = (\omega_0^2 - \mu^2)^{\frac{1}{2}} \quad (3.67)$$

The mean-square response $\langle x^2 \rangle$ is given by equation (3.48), where $\langle u^2 \rangle$, $\langle uv \rangle$, and $\langle v^2 \rangle$ are given by equations (3.66), (3.53), and (3.47), respectively.

Using equations (3.28) and (3.41), one writes the Fokker-Planck-Kolmogorov equation for the stationary probability-density function $p(x, \dot{x})$ as

$$-\dot{x} \frac{\partial p}{\partial x} + \frac{\partial}{\partial \dot{x}} [(\omega_0^2 x + 2\mu \dot{x} + \alpha x^3)p] + \pi S_0 \frac{\partial^2 p}{\partial \dot{x}^2} = 0 \quad (3.68)$$

The solution of equation (3.68) after integrating over $\dot{x}$ has the form

$$p(x) = c \exp\left[-\frac{\mu}{\pi S_0} \left(\omega_0^2 + \frac{1}{2} \alpha x^2\right) x^2\right] \quad (3.69)$$

where c is a normalizing constant. Then the exact mean-square response is [83]

$$\langle x^2 \rangle = \frac{1}{2\alpha \sqrt{\frac{\mu}{\pi \alpha S_0}}} \cdot \frac{U(1, \omega_0^2 \sqrt{\frac{\mu}{\pi \alpha S_0}})}{U(0, \omega_0^2 \sqrt{\frac{\mu}{\pi \alpha S_0}})} \quad (3.70)$$

where $U(\xi, \eta)$ is the parabolic cylinder function.

According to the method of equivalent linearization, equation (3.28) can be replaced with equation (3.62) where ω_e^2 is given by equation (3.63). The mean-square response of equation (3.62) is

$$\langle x^2 \rangle = \frac{\pi S_0}{2\mu \omega_e^2} \quad (3.71)$$

Substituting equation (3.63) into equation (3.71) and solving for $\langle x^2 \rangle$ yields

$$\langle x^2 \rangle = \frac{\omega_0^2}{6\alpha} \left[-1 + \left(1 + \frac{6\pi\alpha S_0}{\mu\omega_0} \right)^{1/2} \right] \quad (3.72)$$

Figure 3.9 shows a comparison of the variations of the mean-square response obtained by using the proposed generalized method of equivalent linearization with those obtained by using the Fokker-Planck-Kolmogorov equation and the method of equivalent linearization. It is clear that the proposed method yields a better approximation to the exact mean-square response than the method of equivalent linearization.

3.4.2. The Van der Pol-Rayleigh Oscillator

In this case, we have

$$k(x, \dot{x}) = 2\mu_Y(\dot{x}^2 + \omega_0^2 x^2)\dot{x} \quad (3.73)$$

The output spectral density of u is given by equation (3.33), the mean-square value of u is given by equation (3.66), and

$$\langle \dot{u}^2 \rangle = \int_{-\infty}^{\infty} \omega^2 S_u(\omega) d\omega \quad (3.74)$$

Substituting equation (3.73) into equation (3.28) and using equations (3.34) and (3.35), we obtain

$$\begin{aligned} k_1 = & 2\mu_Y(3\dot{u}^2\dot{v} + 3\dot{u}\dot{v}^2 + \dot{v}^3 + 2\omega_0^2 u\dot{u}\dot{v} + \omega_0^2 \dot{u}v^2 + \omega_0^2 u^2\dot{v} \\ & + 2\omega_0^2 uv\dot{v} + \omega_0^2 v^2\dot{v}) \end{aligned} \quad (3.75)$$

$$k = 2\mu\gamma(\dot{u}^3 + \omega_0^2 u^2 \dot{u}) \quad (3.76)$$

Substituting equations (3.75) and (3.76) into equations (3.37) and (3.38), neglecting subharmonic and ultrasubharmonic resonances, and assuming v to be a pseudo-sinusoidal function, we obtain

$$\begin{aligned} \omega_e^2 = \omega_0^2, \quad 2\mu_e = 2\mu[1 + \gamma(3\langle \dot{u}^2 \rangle + \omega_0^2 \langle u^2 \rangle \\ + \frac{3}{2} \langle \dot{v}^2 \rangle + \frac{1}{2} \omega_0^2 \langle v^2 \rangle)] \end{aligned} \quad (3.77)$$

$$\text{and} \quad \Gamma = f_1 - 2\mu\gamma(\dot{u}^3 + \omega_0^2 u^2 \dot{u}) \quad (3.78)$$

Here, the presence of the nonlinear damping provides some of the non-Gaussian feature of the nonlinear response and an increase in the damping parameter.

An equivalent linear equation for v is given by equation (3.36), where the linear parameters ω_e^2 and μ_e are given by equations (3.77) and the excitation Γ is given by equation (3.78). The second-order moment of v can be written as

$$\begin{aligned} \langle v^2 \rangle = & 2S_0 \int_{\omega_1}^{\omega_2} |H(\omega)|^2 d\omega + 4\mu^2 \gamma^2 \int_{-\infty}^{\infty} |H(\omega)|^2 S_{\dot{u}^3}(\omega) d\omega \\ & + 4\mu^2 \gamma^2 \omega_0^4 \int_{-\infty}^{\infty} |H(\omega)|^2 S_{u\dot{u}}(\omega) d\omega - 4\mu\gamma \int_0^{\infty} h_2(\tau_1) h_2(\tau_2) \\ & \langle f_1(t - \tau_1) \dot{u}^3(t - \tau_2) \rangle d\tau_1 d\tau_2 - 4\mu\gamma \omega_0^2 \int_0^{\infty} h_2(\tau_1) h_2(\tau_2) \\ & \langle f_1(t - \tau_1) u^2(t - \tau_2) \dot{u}(t - \tau_2) \rangle d\tau_1 d\tau_2 + 8\mu^2 \gamma \omega_0^2 \\ & \int_0^{\infty} \int_0^{\infty} h_2(\tau_1) h_2(\tau_2) \langle \dot{u}^3(t - \tau_1) u^2(t - \tau_2) \dot{u}(t - \tau_2) \rangle d\tau_1 d\tau_2 \end{aligned} \quad (3.79)$$

where

$$|H(\omega)|^2 = \frac{1}{(\omega_0^2 - \omega^2)^2 + 4\mu^2 \omega^2} \quad (3.80)$$

The mean-square response is given by equation (3.48), where $\langle u^2 \rangle$ and $\langle v^2 \rangle$ are given by equations (3.66) and (3.79), respectively, and

$$\begin{aligned} \langle uv \rangle &= -2\mu\gamma \int_0^\infty h_2(\tau) \langle u(t) \dot{u}^3(t - \tau) \rangle d\tau - 2\mu\gamma \omega_0^2 \int_0^\infty h_2(\tau) \\ &\langle u(t) u^2(t - \tau) \dot{u}(t - \tau) \rangle d\tau \end{aligned} \quad (3.81)$$

Using the Gaussian moment relations, we obtain

$$\begin{aligned} \langle v^2 \rangle &= 2S_0 \int_{\omega_1}^{\omega_2} |H(\omega)|^2 d\omega + 4\mu^2 \gamma^2 \int_{-\infty}^{\infty} |H(\omega)|^2 S_{\dot{u}^3}(\omega) d\omega \\ &+ 4\mu^2 \gamma^2 \omega_0^4 \int_{-\infty}^{\infty} |H(\omega)|^2 S_{u^2 \dot{u}}(\omega) d\omega + 2R(0) \end{aligned} \quad (3.82)$$

where

$$S_{\dot{u}^3}(\omega) = 9\langle \dot{u}^2 \rangle^2 S_{\dot{u}}(\omega) + 6 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{\dot{u}}(r_1) S_{\dot{u}}(r_2) S_{\dot{u}}(\omega - r_1 - r_2) dr_1 dr_2 \quad (3.83)$$

$$\begin{aligned} S_{u^2 \dot{u}}(\omega) &= \langle u^2 \rangle^2 S_{\dot{u}}(\omega) + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_u(r_1) S_u(r_2) S_{\dot{u}}(\omega - r_1 - r_2) dr_1 dr_2 \\ &- 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega^2 S_u(r_3) S_u(r_4) S_u(\omega - r_3 + r_4) dr_3 dr_4 \end{aligned} \quad (3.84)$$

$$\begin{aligned}
R(\tau) &= 12\mu^2 \gamma^2 \omega_0^2 \langle u^2 \rangle \langle \dot{u}^2 \rangle \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \omega^2 h_2(\tau_1) h_2(\tau_2) \cos[(\tau + \tau_1 - \tau_2)\omega] \\
&\quad S_u(\omega) d\tau_1 d\tau_2 d\omega - 24\mu^2 \gamma^2 \omega_0^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} h_2(\tau_1) h_2(\tau_2) \omega_1^2 \omega_2 \omega_3 \\
&\quad \cos[(\omega_1 - \omega_2 - \omega_3)(\tau + \tau_1 - \tau_2)] \\
&\quad S_u(\omega_1) S_u(\omega_2) S_u(\omega_3) d\tau_1 d\tau_2 d\omega_1 d\omega_2 d\omega_3
\end{aligned} \tag{3.85}$$

The second-order moment of $\dot{v}$ can be written as

$$\begin{aligned}
\langle \dot{v}^2 \rangle &= 2S_0 \int_0^{\omega_2} \omega^2 |H(\omega)|^2 d\omega + 4\mu^2 \gamma^2 \int_{-\infty}^{\infty} \omega^2 |H(\omega)|^2 S_{\dot{u}_3}(\omega) d\omega \\
&\quad + 4\mu^2 \gamma^2 \omega_0^4 \int_{-\infty}^{\infty} \omega^2 |H(\omega)|^2 S_{\dot{u}_2}(\omega) d\omega + \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega^2 R(\tau) e^{-i\omega\tau} d\tau d\omega
\end{aligned} \tag{3.86}$$

The cross-correlation of u and v has the form

$$\langle uv \rangle = (3\langle \dot{u}^2 \rangle + \omega_0^2 \langle u^2 \rangle) \left[-2\mu\gamma \int_0^{\infty} \int_{-\infty}^{\infty} \omega h_2(\tau) S_u(\omega) \sin\omega\tau d\omega d\tau \right] \tag{3.87}$$

In this case, the exact stationary probability-density function $p(x, \dot{x})$ is given by [34]

$$p(x, \dot{x}) = \frac{\exp\left[-\frac{2\mu H}{\pi S_0} (1+\gamma H)\right]}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{2\mu H}{\pi S_0} (1+\gamma H)\right] dx d\dot{x}} \tag{3.88}$$

where

$$H = \frac{1}{2} (\dot{x}^2 + \omega_0^2 x^2) \tag{3.89}$$

Using equation (3.88), we can write the mean-square response as

$$(\omega_0^2 = 1)$$

$$\langle x^2 \rangle = \frac{1}{2} \frac{\frac{2\pi S_0}{\mu\gamma} \exp(-\frac{\mu}{2\pi S_0\gamma}) - \frac{1}{\gamma} \operatorname{erfc} \frac{\mu}{2\pi S_0\gamma}}{\operatorname{erfc} \frac{\mu}{2\pi S_0\gamma}} \quad (3.90)$$

where erfc is the complementary error function.

According to the method of equivalent linearization, equation (3.28) is replaced with the linear equation

$$\ddot{x} + x + c\dot{x} = f(t) \quad (3.91)$$

where

$$c = 2\mu[1 + \gamma(3\langle \dot{x}^2 \rangle + \omega_0^2 \langle x^2 \rangle)] \quad (3.92)$$

Then, the mean-square response can be written as

$$\langle x^2 \rangle = \frac{-1 + (1 + \frac{8\pi S_0\gamma}{\mu})^{\frac{1}{2}}}{8\gamma} \quad (3.93)$$

The proposed method results in a better approximation to the exact mean-square response than the method of equivalent linearization as shown in Figure 3.10.

3.5 Conclusions

The method of equivalent linearization is based on replacing a nonlinear relation between the statistics of the input and those of the output by a linear relation. This linear relation overlooks many interesting phenomena introduced by the nonlinearity. These include

non-Gaussian responses, broadening effects, shift of the resonant frequency, and nonlinear resonances. A generalized method of equivalent linearization is presented for determining the response of nonlinear systems to random excitations. The original nonlinear governing equation is replaced by two equivalent linear equations. The method provides some means to describe the non-Gaussian feature of the nonlinear response and preserves some of its wide-band character. The interaction and feedback between the nonlinearity and excitation creates nonlinear resonances. These lead to a linear shift in the frequency and the damping coefficient and an adjustment of the strength of the primary excitation. The former is responsible for the increase in the modal frequency and the latter contributes to the experimentally observed broadening of the response curves. The method is illustrated by studying the response of a class of nonlinear systems to a Gaussian white noise and a superharmonic narrow-band random excitation of order-three. For the former, we show that the proposed method yields a better approximation to the mean-square response than the method of equivalent linearization. For the latter, the method of equivalent linearization predicts a single-valued function for the mean-square response. However, the proposed method displays multi-valued regions where more than one mean-square value of the response is possible. The multi-valuedness is responsible for a jump phenomenon at the saddle-node bifurcation points that results in an abrupt change in the response.

CHAPTER 4

NONLINEAR RANDOM COUPLED MOTIONS OF STRUCTURAL ELEMENTS WITH QUADRATIC NONLINEARITIES

The second-order closure method is used to analyze the nonlinear response of two-degree-of-freedom systems with quadratic nonlinearities. The excitation is assumed to be the sum of a deterministic harmonic component and a random component. The case of primary resonance of the second mode in the presence of a two-to-one internal (autoparametric) resonance is investigated. The method of multiple scales is used to obtain four first-order ordinary-differential equations that describe the modulation of the amplitudes and phases of the two modes. Applying the second-order closure method to the modulation equations, we determine the stationary mean and mean-square responses. For the case of a narrow-band random excitation, the results show that the presence of the nonlinearity causes multi-valued mean-square responses. The multi-valuedness is responsible for a jump phenomenon. Contrary to the results of the linear analysis, the nonlinear analysis reveals that the directly excited second mode takes a small amount of the input energy (saturates) and spills over the rest of the input energy into the first mode, which is indirectly excited through the autoparametric resonance.

4.1 Introduction

We consider the forced response of a two-degree-of-freedom system with quadratic nonlinearities to combined harmonic deterministic and random excitations. The problem is modeled by

$$\begin{aligned}
\ddot{u}_1 + \omega_1^2 u_1 + \varepsilon(2\mu_1 \dot{u}_1 + \delta_1 u_1^2 + \delta_2 u_1 u_2 + \delta_3 u_2^2 + \delta_4 \dot{u}_1^2 \\
+ \delta_5 \dot{u}_1 \dot{u}_2 + \delta_6 \dot{u}_2^2 + \delta_7 u_1 \ddot{u}_1 + \delta_8 u_2 \ddot{u}_1 + \delta_9 u_1 \ddot{u}_2 \\
+ \delta_{10} u_2 \ddot{u}_2) = 2\varepsilon\omega_1 (h_1 \cos\omega t - h_2 \sin\omega t) + \varepsilon\xi_1
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\ddot{u}_2 + \omega_2^2 u_2 + \varepsilon(2\alpha_1 \dot{u}_2 + \alpha_1 u_1^2 + \alpha_2 u_1 u_2 + \alpha_3 u_2^2 + \alpha_4 \dot{u}_1^2 \\
+ \alpha_5 \dot{u}_1 \dot{u}_2 + \alpha_6 \dot{u}_2^2 + \alpha_7 u_1 \ddot{u}_1 + \alpha_8 u_2 \ddot{u}_1 + \alpha_9 u_1 \ddot{u}_2 \\
+ \alpha_{10} u_2 \ddot{u}_2) = 2\varepsilon\omega_2 (k_1 \cos\omega t - k_2 \sin\omega t) + \varepsilon\xi_2
\end{aligned} \tag{4.2}$$

$$\langle \xi_i \rangle = 0 \quad \text{for} \quad i = 1, 2 \tag{4.3}$$

Here, the ξ_i are zero-mean random excitations, the ω_i , Ω , μ_i , δ_i , α_i , h_i , and k_i are constants, and ε is a small dimensionless parameter.

Special forms of equations (4.1)-(4.3) describe the forced response of many elastic and dynamic structures, such as beams, arches, shells, foundations, ships, and beams and plates under static loads. In the absence of random noise (i.e., $\xi_i = 0$) the deterministic solutions of equations (4.1) and (4.2) often lead to complicated long-time behavior when the linear natural frequencies ω_2 and ω_1 are in the ratio of two to one [162,163]. The system is said to possess an internal autoparametric resonance. To second order, external resonances occur when $\Omega \approx \omega_1$, $\Omega \approx \omega_2$, $\Omega \approx \omega_2 + \omega_1$, $\Omega \approx 2\omega_2$, or $\Omega \approx \frac{1}{2}\omega_1$. In this Chapter we consider the case of primary resonance of the second mode (i.e., $\Omega \approx \omega_2$) in the presence of a two-to-one internal (autoparametric)

resonance (i.e., $\omega_2 \approx 2\omega_1$). We investigate the effect of random excitations upon the deterministic mean solution.

Primary resonances of multi-degree-of-freedom (MDOF) systems with quadratic nonlinearities were analyzed by Mettler and Weidenhammer [164], Sethna [165], and Haxton and Barr [166]. They used the method of averaging to analyze the case when one linear natural frequency is twice another. For the case $\Omega \approx \omega_1$, Nayfeh et al. [167] used the method of multiple scales to investigate the coupled pitch and roll motions of ships. They reported the possibility of the nonexistence of stable periodic responses to a periodic excitation in the presence of positive damping. Instead, the response is an amplitude- and phase-modulated motion in which the energy is continuously exchanged between the two modes. Yamamoto and Yasuda [168] used the method of harmonic balance to analyze the response of two coupled oscillators and demonstrated amplitude- and phase-modulated motions. Miles [169] used the method of averaging to analyze the response of a two-degree-of-freedom system with quadratic nonlinearities to a harmonic excitation. He presented numerical results that demonstrate the presence of periodically and chaotically modulated motions. Nayfeh and Zavodney [170] performed an experimental study of a two-degree-of-freedom system with quadratic nonlinearities in the presence of a two-to-one internal resonance. They observed amplitude- and phase-modulated motions.

For the case $\Omega \approx \omega_2$, Nayfeh et al. [167] demonstrated the existence of a saturation phenomenon. When the amplitude of the excitation is small, only the second mode has a non-trivial response (linear

solution). As the amplitude of the excitation reaches a threshold value, which depends on the damping coefficient of the lower mode and the external and internal detunings, the second mode saturates (i.e., constant response) and the first mode starts to grow. This means that the excited second mode takes a small amount of the input energy and spills over the rest of the input energy into the first mode, which is indirectly excited through the internal resonance. Yamamoto et al. [171] demonstrated amplitude- and phase-modulated motions and Hatwal et al. [172] reported numerical and experimental results which demonstrate modulated and chaotic motions. Haddow et al. [173] conducted an experiment with a two-degree-of-freedom structure and observed the saturation phenomenon. Nayfeh and Raouf [174,175] analyzed the response of a circular cylindrical shell to a harmonic internal pressure when the natural frequency of the breathing mode is twice the natural frequency of a flexural mode and demonstrated the saturation phenomenon. Nayfeh [176] analyzed the response of a ship that is restrained to pitch and roll. He found conditions for the existence of amplitude- and phase-modulated responses when the two frequencies are in the ratio of 2:1. Nayfeh et al. [177] investigated experimentally the response of a simple structure consisting of two light-weight beams and two concentrated masses to a harmonic excitation. They reported experimental results which demonstrate the saturation phenomenon and periodically and chaotically modulated motions.

4.2 The Method of Multiple Scales

We use the method of multiple scales [95,96] to construct a first-order uniform expansion of the solutions of equations (4.1) and (4.2) in the form

$$u_1 = u_{10}(T_0, T_1) + \epsilon u_{11}(T_0, T_1) + \dots \quad (4.4)$$

$$u_2 = u_{20}(T_0, T_1) + \epsilon u_{21}(T_0, T_1) + \dots \quad (4.5)$$

where $T_n = \epsilon^n t$. In terms of T_n the ordinary-time derivatives can be transformed into partial derivatives according to

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \dots \quad (4.6)$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \dots \quad (4.7)$$

where $D_n = \frac{\partial}{\partial T_n}$. Substituting equations (4.4)-(4.7) into equations (4.1) and (4.2) and equating coefficients of like powers of ϵ to zero yields

Order ϵ^0

$$D_0^2 u_{10} + \omega_1^2 u_{10} = 0 \quad (4.8)$$

$$D_0^2 u_{20} + \omega_2^2 u_{20} = 0 \quad (4.9)$$

Order ε

$$\begin{aligned}
D_0^2 u_{11} + \omega_1^2 u_{11} = & -2D_0 D_1 u_{10} - 2u_1 D_0 u_{10} - \delta_1 u_{10}^2 - \delta_2 u_{10} u_{20} \\
& - \delta_3 u_{20}^2 - \delta_4 (D_0 u_{10})^2 - \delta_5 (D_0 u_{10})(D_0 u_{20}) - \delta_6 (D_0 u_{20})^2 \\
& - \delta_7 u_{10} D_0^2 u_{10} - \delta_8 u_{20} D_0^2 u_{10} - \delta_9 u_{10} D_0^2 u_{20} - \delta_{10} u_{20} D_0^2 u_{20} \\
& + 2\omega_1 h_1 \cos \Omega T_0 - 2\omega_1 h_2 \sin \Omega T_0 + \xi_1
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
D_0^2 u_{21} + \omega_2^2 u_{21} = & -2D_0 D_1 u_{20} - 2u_2 D_0 u_{20} - \alpha_1 u_{10}^2 \\
& - \alpha_2 u_{10} u_{20} - \alpha_3 u_{20}^2 - \alpha_4 (D_0 u_{10})^2 - \alpha_5 (D_0 u_{10})(D_0 u_{20}) \\
& - \alpha_6 (D_0 u_{20})^2 - \alpha_7 u_{10} D_0^2 u_{10} - \alpha_8 u_{20} D_0^2 u_{10} - \alpha_9 u_{10} D_0^2 u_{20} \\
& - \alpha_{10} u_{20} D_0^2 u_{20} + 2\omega_2 k_1 \cos \Omega T_0 - 2\omega_2 k_2 \sin \Omega T_0 + \xi_2
\end{aligned} \tag{4.11}$$

The general solutions of equations (4.8) and (4.9) can be written as

$$u_{10} = A_1(T_1) e^{i\omega_1 T_0} + cc \tag{4.12}$$

$$u_{20} = A_2(T_1) e^{i\omega_2 T_0} + cc \tag{4.13}$$

where cc stands for the complex conjugate of the preceding terms and the A_n are arbitrary functions of T_1 at this level of approximation. They are determined by imposing the solvability conditions at the next level of approximation.

Substituting equations (4.12) and (4.13) into equations (4.10) and (4.11), one obtains

$$\begin{aligned}
D_0^2 u_{11} + \omega_1^2 u_{11} = & -2i\omega_1 (\dot{A}_1 + \mu_1 A_1) e^{i\omega_1 T_0} - (\delta_2 + \delta_5 \omega_1 \omega_2 \\
& - \delta_8 \omega_1^2 - \delta_9 \omega_2^2) A_1 A_2 e^{i(\omega_2 - \omega_1) T_0} + \Lambda_3 A_1^2 e^{2i\omega_1 T_0} \\
& + \Lambda_4 A_2^2 e^{2i\omega_2 T_0} + \Lambda_5 A_1 A_2 e^{i(\omega_1 + \omega_2) T_0} + \Lambda_6 A_1 \bar{A}_1 + \Lambda_7 A_2 \bar{A}_2 \\
& + \omega_1 (h_1 + ih_2) e^{i\Omega T_0} + cc + \xi_1
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
D_0^2 u_{21} + \omega_2^2 u_{21} = & -2i\omega_2 (\dot{A}_2 + \mu_2 A_2) e^{i\omega_2 T_0} - (\alpha_1 \\
& - \alpha_4 \omega_1^2 - \alpha_7 \omega_1^2) A_1^2 e^{2i\omega_1 T_0} + \Lambda_8 A_2^2 e^{2i\omega_2 T_0} \\
& + \Lambda_9 A_1 A_2 e^{i(\omega_2 - \omega_1) T_0} + \Lambda_{10} A_1 A_2 e^{i(\omega_1 + \omega_2) T_0} + \Lambda_{11} A_1 \bar{A}_1 \\
& + \Lambda_{12} A_2 \bar{A}_2 + \omega_2 (k_1 + ik_2) e^{i\Omega T_0} + cc + \xi_2
\end{aligned} \tag{4.15}$$

where the prime indicates the derivative with respect to T_1 and the Λ_i are constants that depend on the values of the ω_i , δ_i , and α_i .

To proceed further, we need to specify the resonances being studied. In this Chapter, we consider primary resonances of the second mode (i.e., $\Omega \approx \omega_2$) in the presence of a two-to-one internal resonance ($\omega_2 \approx 2\omega_1$). To describe quantitatively the nearness of these resonances, we introduce the detuning parameters $\hat{\nu}_1$ and $\hat{\nu}_2$ defined as

$$\omega_2 = 2\omega_1 + \epsilon \hat{\nu}_1, \quad \Omega = \omega_2 + \epsilon \hat{\nu}_2 \tag{4.16}$$

Using equations (4.16) to eliminate the terms that lead to secular terms in equations (4.14) and (4.15) yields

$$2i(\hat{A}_1' + \mu_1 A_1) + 4\Lambda_1 A_1 A_2 e^{i\hat{v}_1 T_1} - \frac{1}{2\pi} \int_0^{2\pi/\omega_1} \xi_1 e^{-i\omega_1 T_0} dT_0 = 0 \quad (4.17)$$

$$2i(\hat{A}_2' + \mu_2 A_2) + 4\Lambda_2 A_1^2 e^{-i\hat{v}_1 T_1} - (k_1 + ik_2) e^{i\hat{v}_2 T_1} - \frac{1}{2\pi} \int_0^{2\pi/\omega_2} \xi_2 e^{-i\omega_2 T_0} dT_0 = 0 \quad (4.18)$$

where

$$\Lambda_1 = \frac{1}{4\omega_1} (\delta_2 + \delta_3 \omega_1 \omega_2 - \delta_3 \omega_1^2 - \delta_3 \omega_2^2) \quad (4.19)$$

$$\Lambda_2 = \frac{1}{4\omega_2} (\alpha_1 - \alpha_7 \omega_1^2 - \alpha_7 \omega_2^2) \quad (4.20)$$

It is convenient to express the Λ_n in equations (4.17) and (4.18) in the form

$$A_1 = \frac{1}{2} (x_1 + iy_1) e^{i v_1 T_1} \quad \text{and} \quad A_2 = \frac{1}{2} (x_2 + iy_2) e^{i v_2 T_1} \quad (4.21)$$

where

$$v_1 = \frac{1}{2} (\hat{v}_1 + \hat{v}_2), \quad v_2 = \hat{v}_2 \quad (4.22)$$

Substituting equations (4.21) and (4.22) into equations (4.17) and (4.18), separating real and imaginary parts, and replacing T_1 with ϵt , we have

$$x_1' = v_1 y_1 - \mu_1 x_1 + \Lambda_1 (y_1 x_2 - x_1 y_2) - g_1 \quad (4.23)$$

$$y_1' = -v_1 x_1 - \mu_1 y_1 + \Lambda_1 (x_1 x_2 + y_1 y_2) - g_2 \quad (4.24)$$

$$x_2' = v_2 y_2 - \mu_2 x_2 - 2\Lambda_2 x_1 y_1 + k_2 - g_3 \quad (4.25)$$

$$y_2' = -v_2 x_2 - \mu_2 y_2 + \Lambda_2 (x_1^2 - y_1^2) - k_1 - g_4 \quad (4.26)$$

where

$$g_1 = \frac{1}{2\pi} \int_0^{2\pi/\omega_1} \xi_1 \sin(\omega_1 + \varepsilon v_1) t \, dt, \quad (4.27a)$$

$$g_2 = \frac{1}{2\pi} \int_0^{2\pi/\omega_1} \xi_1 \cos(\omega_1 + \varepsilon v_1) t \, dt \quad (4.27b)$$

$$g_3 = \frac{1}{2\pi} \int_0^{2\pi/\omega_2} \xi_2 \sin(\omega_2 + \varepsilon v_2) t \, dt, \quad (4.27c)$$

$$g_4 = \frac{1}{2\pi} \int_0^{2\pi/\omega_2} \xi_2 \cos(\omega_2 + \varepsilon v_2) t \, dt \quad (4.27d)$$

Starting from the averaged equations (4.23)-(4.26), we use the second-order closure method to obtain some statistical measures of the coupled motion.

4.3 Analysis

We use the second-order closure method to determine the response moments. To this end, we separate the x_i and y_i into the mean components x_{i0} and y_{i0} and the random components x_{i1} and y_{i1} according to

$$x_1 = x_{10} + x_{11}, \quad y_1 = y_{10} + y_{11} \quad (4.28)$$

$$x_2 = x_{20} + x_{21}, \quad y_2 = y_{20} + y_{21} \quad (4.29)$$

Substituting equations (4.28) and (4.29) into equations (4.23)-(4.26) and separating mean and fluctuating components, we find that the mean motion is governed by

$$\dot{x}_{10} = v_1 y_{10} - \mu_1 x_{10} + \Lambda_1 (y_{10} x_{20} + \langle y_{11} x_{21} \rangle - x_{10} y_{20} - \langle x_{11} y_{21} \rangle) \quad (4.30)$$

$$\dot{y}_{10} = -v_1 x_{10} - \mu_1 y_{10} + \Lambda_1 (x_{10} x_{20} + \langle x_{11} x_{21} \rangle + y_{10} y_{20} + \langle y_{11} y_{21} \rangle) \quad (4.31)$$

$$\dot{x}_{20} = v_2 y_{20} - \mu_2 x_{20} - 2\Lambda_2 (x_{10} y_{10} + \langle x_{11} y_{11} \rangle) + k_2 \quad (4.32)$$

$$\dot{y}_{20} = -v_2 x_{20} - \mu_2 y_{20} + \Lambda_2 (x_{10}^2 - y_{10}^2 + \langle x_{11}^2 \rangle - \langle y_{11}^2 \rangle) - k_1 \quad (4.33)$$

Equations (4.30)-(4.33) show the presence of a feedback from the fluctuations caused by the noise into the mean motion. This coupling (terms in angle brackets) can be neglected if the random excitation is small in comparison with the mean excitation as long as the parameters considered are away from bifurcation values. We refer to this assumption as the linearization method. To first order, the fluctuating motion is given by

$$\dot{\underline{p}} = \underline{A} \underline{p} - \underline{g}, \quad (4.34)$$

where

$$\underline{A} = \begin{pmatrix} -\mu_1 - \Lambda_1 y_{20} & v_1 + \Lambda_1 x_{20} & \Lambda_1 y_{10} & -\Lambda_1 x_{10} \\ -v_1 + \Lambda_1 x_{20} & -\mu_1 + \Lambda_1 y_{20} & \Lambda_1 x_{10} & \Lambda_1 y_{10} \\ -2\Lambda_2 y_{10} & -2\Lambda_2 x_{10} & -\mu_2 & v_2 \\ 2\Lambda_2 x_{10} & -2\Lambda_2 y_{10} & -v_2 & -\mu_2 \end{pmatrix} \quad (4.35)$$

$$\underline{p} = \begin{pmatrix} x_{11} \\ y_{11} \\ x_{21} \\ y_{21} \end{pmatrix}, \quad \underline{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} \quad (4.36)$$

We note that A is a real nonsymmetric matrix.

The general solution of equations (4.34) can be obtained by modal analysis. The eigenvalue and the adjoint eigenvalue problems of equations (4.34) can be written as

$$A \underline{p} = \lambda \underline{p}, \quad A^T \underline{q} = \lambda \underline{q} \quad (4.37)$$

The normalized modal matrices P and Q satisfy the conditions

$$Q^T P = I \quad \text{and} \quad Q^T A P = \Lambda \quad (4.38)$$

where I is the identity matrix and Λ is a diagonal matrix whose elements are the eigenvalues of the matrix A .

It is convenient to introduce a transformation that decouples equations (4.34). To this end, we let

$$\underline{p} = P \underline{z} \quad (4.39)$$

Substituting equations (4.39) into equations (4.34), premultiplying by Q^T , and using equations (4.38), we obtain

$$\dot{z}_m(t) = \lambda_m z_m(t) + \eta_m(t) \quad \text{for} \quad m = 1, 2, 3, 4 \quad (4.40)$$

where

$$\eta_m(t) = - \sum_{n=1}^4 Q_{nm} g_n(t) \quad (4.41)$$

and the Q_{mn} are the elements of the modal matrix Q .

The statistics of the uncoupled coordinates z_m are

$$\langle z_m \rangle = 0 \quad (4.42)$$

$$\begin{aligned} \langle z_m(t_1) z_n(t_2) \rangle &= e^{(\lambda_m t_1 + \lambda_n t_2)} \int_0^{t_1} \int_0^{t_2} e^{-(\lambda_m \tau_1 + \lambda_n \tau_2)} \\ &\cdot \langle \eta_m(\tau_1) \eta_n(\tau_2) \rangle d\tau_1 d\tau_2 \quad \text{for } m, n = 1, 2, 3, 4 \end{aligned} \quad (4.43)$$

where the angle brackets stand for the expectation.

As an example, we consider the case $\xi_1 = 0$ and ξ_2 is a narrow-band random excitation; that is,

$$\xi_2 = 2\omega_2(f_1 \cos \omega t - f_2 \sin \omega t) \quad (4.44)$$

The modulations of f_1 and f_2 with time can be expressed as

$$\dot{f}_1 + \frac{1}{2} \beta f_1 = \frac{1}{2} \beta W_1, \quad \dot{f}_2 + \frac{1}{2} \beta f_2 = \frac{1}{2} \beta W_2 \quad (4.45)$$

where β is the bandwidth of the random part of the excitation and W_1 and W_2 are independent white noise. The autocorrelation functions are given by

$$R_{f_1}(\tau) = R_{f_2}(\tau) = \pi S_0 e^{-\frac{1}{2} \beta |\tau|} \quad (4.46)$$

Substituting equations (4.41) into equations (4.43) and using equations (4.46), we obtain the stationary mean-square values of the uncoupled coordinates z_m as

$$\langle z_m z_n \rangle = \frac{\pi S_0 (Q_{4m} Q_{4n} + Q_{3m} Q_{3n})}{\lambda_m + \lambda_n} \left[\frac{1}{\lambda_m - \beta/2} + \frac{1}{\lambda_n - \beta/2} \right]$$

for $m, n = 1, 2, 3$, and 4 (4.47)

Using equations (4.39) and (4.42), we obtain

$$\langle p_i \rangle = 0, \quad \langle p_i p_j \rangle = \sum_{m,n=1}^4 \sum_{j=1}^4 P_{im} P_{jn} \langle z_m z_n \rangle$$

for $i, j = 1, 2, 3, 4$ (4.48)

where the P_{mn} are the elements of the modal matrix P .

4.4 Results and Conclusions

Figure 4.1 shows the modal mean-square-response amplitudes as functions of the mean-square excitations $\langle F^2 \rangle$, where $\langle F^2 \rangle = \pi S_0$. The second mode is externally excited by a narrow-band random excitation with a center frequency Ω and a bandwidth β . The case $\Omega \approx \omega_2$ is considered. When $\langle F^2 \rangle < 0.167$, the first mode is not excited and the mean-square response of the second mode is proportional to the mean-square excitation $\langle F^2 \rangle$. As $\langle F^2 \rangle$ increases beyond 0.167, the mean-square response of the first mode increases while $\langle a_2^2 \rangle$ remains 0.64 = constant for all values of $\langle F^2 \rangle$, indicating the existence of a saturation phenomenon; that is, the directly excited mode is saturated and the extra energy spills over into the indirectly excited mode. Increasing the bandwidth of the narrow-band random excitation delays the start of the nonlinear solution to $\langle F^2 \rangle = 0.2$, as shown in Figure 4.2. In Figure 4.3, we show the dependence of $\langle a_1^2 \rangle$ and $\langle a_2^2 \rangle$ on $\langle F^2 \rangle$ for the case with a

multi-valued region. In addition to the saturation phenomenon, the response curves display the jump phenomenon. In this case, two stable solutions coexist when $0.284 \leq \langle F^2 \rangle \leq 0.385$. In contrast to the linear theory, the stationary response depends on the initial conditions when more than one stable solution is possible. As $\langle F^2 \rangle$ increases from zero, $\langle a_2^2 \rangle$ increases linearly with $\langle F^2 \rangle$ while the first mode does not respond as predicted by linear theory. When $\langle F^2 \rangle = 0.385$, $\langle a_1^2 \rangle$ jumps up from zero to point B. As $\langle F^2 \rangle$ increases further, $\langle a_1^2 \rangle$ increases nonlinearly with $\langle F^2 \rangle$ while $\langle a_2^2 \rangle$ remains 0.974=constant. As $\langle F^2 \rangle$ decreases from point C, $\langle a_1^2 \rangle$ decreases along the curve CBA and $\langle a_2^2 \rangle$ remains constant until point D. When $\langle F^2 \rangle$ decreases further, $\langle a_1^2 \rangle$ jumps down to zero while $\langle a_2^2 \rangle$ jumps down to point E and decreases linearly with $\langle F^2 \rangle$.

Next, we study the effect of a low-intensity noise with a center frequency $\Omega \approx \omega_2$ on the deterministic mean response. Representative force-response curves are presented in Figure 4.4 when $\langle F^2 \rangle = 0$. The deterministic response exhibits the saturation phenomenon. For the case $\langle F^2 \rangle = 0.1 \times 10^{-4}$, representative force-response curves are presented in Figure 4.5. When $k_1 = 0$, the first mode does not respond until k_1 reaches the threshold value 0.008×10^{-3} . As k_1 increases, the response displays an energy exchange between the resonant modes. When $k_1 \geq 0.07 \times 10^{-3}$, $\langle a_2^2 \rangle$ remains equal to 0.3 while $\langle a_1^2 \rangle$ increases with k_1 . This indicates that the presence of the noise delays the start of the saturation phenomenon from 0.05×10^{-3} to 0.07×10^{-3} . As a result, increasing the bandwidth of the random excitation or adding a narrow-

band random excitation to a deterministic harmonic excitation delays the onset of the saturation phenomenon. In other words, we are going toward the white-noise extreme where the response does not display the saturation phenomenon in the presence of a two-to-one autoparametric resonance (see Chapter 5). In Figure 4.6, we show frequency-response curves when the second mode is excited by a deterministic harmonic and a narrow-band random excitation. Nayfeh [162] indicated the possibility of the nonexistence of stable periodic responses when $-0.047 \leq \hat{v}_1 \leq -0.0127$ and $\langle F^2 \rangle = 0$, implying the existence of a Hopf bifurcation. In the presence of the narrow-band noise, the second-order closure method predicts a stable solution in this interval (i.e., all the eigenvalues have negative real parts).

In this Chapter, the second-order closure method is used to predict the lower-order response moments of a two-degree-of-freedom system with quadratic nonlinearities. According to the second-order closure method, the response amplitudes x_i and y_i are separated into the mean components x_{i0} and y_{i0} and the random components x_{i1} and y_{i1} . Next, the averaged equations are separated into mean and fluctuating motions. This creates a feedback from the fluctuations caused by the noise into the mean flow. This coupling can be neglected if the random excitation is small in comparison with the mean excitation (linearization method). However, this coupling is necessary to give good approximations for the mean-square responses at the boundary between stable and unstable periodic motions.

CHAPTER 5

CUMULANT-NEGLECT CLOSURE FOR SYSTEMS WITH MODAL INTERACTIONS

Statistical response moments of two-degree-of-freedom systems with quadratic and cubic nonlinearities to zero-mean Gaussian white noise are determined. The cases of two-to-one, one-to-one, and three-to-one internal resonances are investigated. Ito stochastic calculus is used to derive general first-order ordinary-differential equations describing the evolution of the response moments. The response-moment equations are an infinite hierarchy set due to the nonlinearities. Gaussian and non-Gaussian closure schemes are used to close this set, leading, respectively, to 14 and 69 equations for the lower-order response moments. Guided by the results of the method of multiple scales for the deterministic case of internally resonant oscillators, we reduce the number of response-moment equations for the case of non-Gaussian closure by approximately a factor of two. The results obtained by the reduced scheme are in excellent agreement with those obtained by the full 69 equations. The results reveal major discrepancies between the Gaussian and non-Gaussian closure schemes. Close to internal resonance, stationary bi-modal solutions exist and energy shifts between the two modes. However, the response does not display the saturation phenomenon for the case of a two-to-one internal resonance. The results are applied to a shallow arch, a string, and a hinged-clamped beam.

5.1 Introduction

In this Chapter we consider two-degree-of-freedom systems governed by equations of motion having the form

$$\begin{aligned} \ddot{u}_1 + \omega_1^2 u_1 + 2\mu_1 \dot{u}_1 + \beta_{11} u_1^3 + \beta_{21} u_1^2 u_2 + \beta_{31} u_1 u_2^2 + \beta_{41} u_2^3 \\ + \delta_1 u_1^2 + \delta_2 u_1 \dot{u}_2 + \delta_3 u_2^2 + \delta_4 \dot{u}_1^2 + \delta_5 \dot{u}_1 \dot{u}_2 + \delta_6 \dot{u}_2^2 = \xi_1(t) \end{aligned} \quad (5.1)$$

$$\begin{aligned} \ddot{u}_2 + \omega_2^2 u_2 + 2\mu_2 \dot{u}_2 + \beta_{12} u_1^3 + \beta_{22} u_1^2 u_2 + \beta_{32} u_1 u_2^2 + \beta_{42} u_2^3 \\ + \alpha_1 u_1^2 + \alpha_2 u_1 \dot{u}_2 + \alpha_3 u_2^2 + \alpha_4 \dot{u}_1^2 + \alpha_5 \dot{u}_1 \dot{u}_2 + \alpha_6 \dot{u}_2^2 = \xi_2(t) \end{aligned} \quad (5.2)$$

where the ω_i , μ_i , β_{i1} , β_{i2} , δ_i , and α_i are constants, the overdots denote derivatives with respect to time, and the ξ_i are zero-mean Gaussian white noise whose auto-correlation functions are given by

$$R_{\xi_1}(\tau) = 2\pi S_1 \delta(\tau), R_{\xi_2}(\tau) = 2\pi S_2 \delta(\tau), \text{ and } R_{\xi_1 \xi_2}(\tau) = 2\pi S_{12} \delta(\tau) \quad (5.3)$$

The following discussion provides a brief background of the present Chapter. For a comprehensive review, the reader is referred to the textbooks by Nayfeh and Mook [163] and Ibrahim [178].

Ariaratnam [179] used the Markov procedure to determine the random response of two-degree-of-freedom vibrating isolating suspensions with cubic displacement-dependent nonlinearities. For some special conditions, he obtained exact steady-state solutions of the Fokker-Planck-Kolmogorov equation. Bhandari and Sherrer [180] investigated the probabilistic behavior of systems with both displacement- and velocity-dependent nonlinearities. They represented the steady-state

probability-density function in terms of multiple series of Hermite polynomials. They used Galerkin's method to determine the coefficients of the series. They considered 81 terms in the series expansion to achieve a reasonable agreement with the results obtained by the exact solution of the FPK equation.

Ibrahim and Roberts [181] used the Gaussian closure scheme for investigating the response moments of a two-degree-of-freedom system with quadratic nonlinearities. In the neighborhood of a two-to-one internal resonance, the Gaussian closure scheme did not yield fixed-point response moments. Their results show a suppression effect on the higher mode and large random motions of the coupled system. Ibrahim and Roberts [182] analyzed the effect of the system parameters on the stability of the stationary uni-modal response of the higher mode. The experimental results of Roberts [183] indicate wider instability regions than those predicted by analytical methods. Ibrahim and Heo [184] compared the results obtained by the Gaussian-closure scheme with those obtained by the non-Gaussian closure scheme. Contrary to the Gaussian closure results, the non-Gaussian closure analysis yields complete fixed-point response moments. Ibrahim and Sullivan [185] conducted an experimental investigation of the response of a two-degree-of-freedom structural model possessing a two-to-one internal resonance to band-limited random excitations. Their results did not indicate the existence of saturation.

In this Chapter, three closure schemes are used to determine the response moments of two-degree-of-freedom systems possessing internal

resonances to zero-mean Gaussian white noise. They are the Gaussian, non-Gaussian, and reduced non-Gaussian closure schemes. The cases of two-to-one, three-to-one, and one-to-one internal resonances are treated. The method of multiple scales is used to establish the response characteristics and its narrow-band nature. This information is utilized to reduce the response-moment equations of the non-Gaussian closure scheme by about a factor of two. The results obtained by the Gaussian closure scheme do not agree with those obtained by the non-Gaussian closure scheme. Close to resonance, the results show energy exchanges between the two modes. As applications of the present results, we investigate the responses of a shallow arch, a string, and a hinged-clamped beam to broad-band random excitations.

5.2 Closure Schemes via the Markov Procedure

Equations (5.1) and (5.2) can be written as

$$dx_1 = x_3 dt, \quad dx_2 = x_4 dt, \quad (5.4)$$

$$dx_3 = - \left[\omega_1^2 x_1 + 2\mu_1 x_3 + \beta_{11} x_1^3 + \beta_{21} x_1^2 x_2 + \beta_{31} x_1 x_2^2 + \beta_{41} x_2^3 + \delta_1 x_1^2 + \delta_2 x_1 x_2 + \delta_3 x_2^2 + \delta_4 x_3^2 + \delta_5 x_3 x_4 + \delta_6 x_4^2 \right] dt + dN_1 \quad (5.5)$$

$$dx_4 = - \left[\omega_2^2 x_2 + 2\mu_2 x_4 + \beta_{12} x_1^3 + \beta_{22} x_1^2 x_2 + \beta_{32} x_1 x_2^2 + \beta_{42} x_2^3 + \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2 + \alpha_4 x_3^2 + \alpha_5 x_3 x_4 + \alpha_6 x_4^2 \right] dt + dN_2 \quad (5.6)$$

where

$$x_1 = u_1, \quad x_2 = u_2, \quad x_3 = \dot{u}_1 \quad \text{and} \quad x_4 = \dot{u}_2 \quad (5.7)$$

$$\xi_i = dN_i/dt \quad (5.8)$$

Here, the $N_i(t)$ are Brownian processes. Equations (5.4)-(5.6) are of Ito type and $\{x_1, x_2, x_3, x_4\}$ constitutes a four-dimensional Markov vector. Using Ito stochastic calculus [186], we use equations (5.4)-(5.6) to derive the response moments. To this end, we let

$$m_{i,j,k,l} = \langle x_1^i x_2^j x_3^k x_4^l \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1^i x_2^j x_3^k x_4^l p(x_1, x_2, x_3, x_4; t) dx_1 dx_2 dx_3 dx_4 \quad (5.9)$$

The Fokker-Planck-Kolmogorov equation associated with equations (5.4)-(5.6) can be written as

$$\begin{aligned} \frac{\partial p}{\partial t} = & -x_3 \frac{\partial p}{\partial x_1} - x_4 \frac{\partial p}{\partial x_2} + \frac{\partial}{\partial x_3} [(\omega_1^2 x_1 + 2\mu_1 x_3 + \beta_{11} x_1^3 + \beta_{21} x_1^2 x_2 \\ & + \beta_{31} x_1 x_2^2 + \beta_{41} x_2^3 + \delta_1 x_1^2 + \delta_2 x_1 x_2 + \delta_3 x_2^2 + \delta_4 x_3^2 + \delta_5 x_3 x_4 \\ & + \delta_6 x_4^2) p] + \frac{\partial}{\partial x_4} [(\omega_2^2 x_2 + 2\mu_2 x_4 + \beta_{12} x_1^3 + \beta_{22} x_1^2 x_2 + \beta_{32} x_1 x_2^2 \\ & + \beta_{42} x_2^3 + \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2 + \alpha_4 x_3^2 + \alpha_5 x_3 x_4 + \alpha_6 x_4^2) p] \\ & + \pi S_1 \frac{\partial^2 p}{\partial x_3^2} + 2\pi S_{12} \frac{\partial^2 p}{\partial x_3 \partial x_4} + \pi S_2 \frac{\partial^2 p}{\partial x_4^2} \end{aligned} \quad (5.10)$$

The response-moment equations of any order n ($n = i + j + k + l$) can be obtained by multiplying equation (5.10) by $x_1^i x_2^j x_3^k x_4^l$ and integrating by parts over the whole state space. Hence, the response moments of any order are governed by

$$\begin{aligned}
\frac{dm_{i,j,k,\ell}}{dt} = & i m_{i-1,j,k+1,\ell} + j m_{i,j-1,k,\ell+1} - k [\omega_1^2 m_{i+1,j,k-1,\ell} \\
& + 2\mu_1 m_{i,j,k,\ell} + \beta_{11} m_{i+3,j,k-1,\ell} + \beta_{21} m_{i+2,j+1,k-1,\ell} \\
& + \beta_{31} m_{i+1,j+2,k-1,\ell} + \beta_{41} m_{i,j+3,k-1,\ell} + \delta_1 m_{i+2,j,k-1,\ell} + \delta_2 m_{i+1,j+1,k-1,\ell} \\
& + \delta_3 m_{i,j+2,k-1,\ell} + \delta_4 m_{i,j,k+1,\ell} + \delta_5 m_{i,j,k,\ell+1} \\
& + \delta_6 m_{i,j,k-1,\ell+2}] - \ell [\omega_2^2 m_{i,j+1,k,\ell-1} + 2\mu_2 m_{i,j,k,\ell} \\
& + \beta_{12} m_{i+3,j,k,\ell-1} + \beta_{22} m_{i+2,j+1,k,\ell-1} + \beta_{32} m_{i+1,j+2,k,\ell-1} + \beta_{42} m_{i,j+3,k,\ell-1} \\
& + \alpha_1 m_{i+2,j,k,\ell-1} + \alpha_2 m_{i+1,j+1,k,\ell-1} + \alpha_3 m_{i,j+2,k,\ell-1} \\
& + \alpha_4 m_{i,j,k+2,\ell-1} + \alpha_5 m_{i,j,k+1,\ell} + \alpha_6 m_{i,j,k,\ell+1}] \\
& + \pi S_1 k(k-1) m_{i,j,k-2,\ell} + \pi S_2 \ell(\ell-1) m_{i,j,k,\ell-2} + 2\pi S_{12} k\ell m_{i,j,k-1,\ell-1}
\end{aligned} \tag{5.11}$$

Equation (5.11) shows that a statistical moment equation of order n contains moments of order $\geq n+2$. Hence, the statistical response moments are governed by an infinite hierarchy set. Closure schemes are usually employed to truncate this set and approximate the important lower-order moments.

5.3 Non-Gaussian Closure

Expanding the first- and second-order response moments in equation (5.11) yields a system of 14 coupled first-order differential equations, which contains third- and fourth-order moments. To close this set, we need to represent the third- and fourth-order moments in terms of the lower-order moments. According to the Gaussian closure scheme, if the nonlinearities are weak the response can be approximated by a Gaussian distribution function, which means that cumulants of order ≥ 3 vanish. Equating the third- and fourth-order cumulants to zero, one obtains

$$\begin{aligned} \langle XYZ \rangle &= \sum_S^3 \langle X \rangle \langle YZ \rangle - 2 \langle X \rangle \langle Y \rangle \langle Z \rangle \\ \langle XYZU \rangle &= \sum_S^3 [\langle XY \rangle - \langle X \rangle \langle Y \rangle] [\langle ZU \rangle - \langle Z \rangle \langle U \rangle] \end{aligned} \quad (5.12)$$

$$+ \sum_S^6 \langle X \rangle \langle Y \rangle [\langle ZU \rangle - \langle Z \rangle \langle U \rangle] + \langle X \rangle \langle Y \rangle \langle Z \rangle \langle U \rangle \quad (5.13)$$

in which $\sum_S^N$ indicates the sum of all different permutations of the terms under the summation and N is the number of terms.

The accuracy of the computed response moments can be improved by including terms that account for the non-Gaussian (non-normality) feature of the response. One approach to describe the non-normality is to express the probability-density function in terms of a truncated Gram-Charlier or an Edgeworth expansion [148,187]. The first term in the expansion is the Gaussian probability-density function. Another approach is to include cumulants of order higher than two [188].

Expanding the response moments of order ≤ 4 in equation (5.11) yields a system of 69 coupled first-order differential equations, which contains fifth- and sixth-order moments. To close this set, we express the fifth- and sixth-order moments in terms of the lower-order moments. To this end, we equate fifth- and sixth-order cumulants to zero; that is,

$$\begin{aligned} \langle XYZUV \rangle = & \sum_S^5 \langle X \rangle \langle YZUV \rangle - 2 \sum_S^{10} \langle X \rangle \langle Y \rangle \langle ZUV \rangle + 6 \sum_S^{10} \langle X \rangle \langle Y \rangle \langle Z \rangle \langle UV \rangle \\ & - 2 \sum_S^{15} \langle XY \rangle \langle ZU \rangle \langle V \rangle + \sum_S^{10} \langle XY \rangle \langle ZUV \rangle - 24 \langle X \rangle \langle Y \rangle \langle Z \rangle \langle U \rangle \langle V \rangle \end{aligned} \quad (5.14)$$

$$\begin{aligned} \langle XYZUVW \rangle = & \sum_S^6 \langle X \rangle \langle YZUVW \rangle - 2 \sum_S^{15} \langle X \rangle \langle Y \rangle \langle ZUVW \rangle \\ & + 6 \sum_S^{20} \langle X \rangle \langle Y \rangle \langle Z \rangle \langle UVW \rangle - 24 \sum_S^{15} \langle X \rangle \langle Y \rangle \langle Z \rangle \langle U \rangle \langle VW \rangle \\ & + \sum_S^{15} \langle XY \rangle \langle ZUVW \rangle + \sum_S^{10} \langle XYZ \rangle \langle UVW \rangle + 6 \sum_S^{45} \langle X \rangle \langle Y \rangle \langle ZU \rangle \langle VW \rangle \\ & - 2 \sum_S^{15} \langle XY \rangle \langle ZU \rangle \langle VW \rangle - 2 \sum_S^{60} \langle X \rangle \langle YZ \rangle \langle VUW \rangle + 120 \langle X \rangle \langle Y \rangle \langle Z \rangle \langle V \rangle \langle U \rangle \langle W \rangle \end{aligned} \quad (5.15)$$

Next, we use the method of multiple scales to describe the nature of the response and its characteristics. We utilize this information to reduce the 69 response-moment equations derived via the non-Gaussian closure scheme. The cases of two-to-one, one-to-one, and three-to-one internal resonances are investigated.

5.4 Two-To-One Internal Resonance

We describe the nearness of the two-to-one internal resonance by introducing the detuning parameter ν defined according to

$$\omega_2 = 2\omega_1 + \nu \quad (5.16)$$

To the first approximation, the general solution of equations (5.1) and (5.2) can be expressed in terms of two narrow-band random processes with the center frequencies ω_2 and $\frac{1}{2}\omega_2$; that is,

$$u_1 \approx x_1 \cos \frac{1}{2} \omega_2 t - y_1 \sin \frac{1}{2} \omega_2 t \quad (5.17)$$

$$u_2 \approx x_2 \cos \omega_2 t - y_2 \sin \omega_2 t \quad (5.18)$$

$$\dot{u}_1 \approx -\frac{1}{2} \omega_2 (x_1 \sin \frac{1}{2} \omega_2 t + y_1 \cos \frac{1}{2} \omega_2 t) \quad (5.19)$$

$$\dot{u}_2 \approx -\omega_2 (x_2 \sin \omega_2 t + y_2 \cos \omega_2 t) \quad (5.20)$$

where the x_i and y_i are slowly varying random functions governed by

$$\dot{x}_1 = -\mu_1 x_1 + \frac{1}{2} \nu y_1 + \Lambda_1 (y_1 x_2 - x_1 y_2) - \frac{1}{\omega_1} \overline{\xi_1 \sin(\frac{1}{2} \omega_2 + \frac{1}{2} \nu) t} \quad (5.21)$$

$$\dot{y}_1 = -\mu_1 y_1 - \frac{1}{2} \nu x_1 + \Lambda_1 (x_1 x_2 + y_1 y_2) - \frac{1}{\omega_1} \overline{\xi_1 \cos(\frac{1}{2} \omega_2 + \frac{1}{2} \nu) t} \quad (5.22)$$

$$\dot{x}_2 = -\mu_2 x_2 - 2\Lambda_2 x_1 y_1 - \frac{1}{\omega_2} \overline{\xi_2 \sin \omega_2 t} \quad (5.23)$$

$$\dot{y}_2 = -\mu_2 y_2 + \Lambda_2 (x_1^2 - y_1^2) - \frac{1}{\omega_2} \overline{\xi_2 \cos \omega_2 t} \quad (5.24)$$

the underbars denote short-time averaging over the period $2\pi/\omega_2$, and

$$\Lambda_1 = \frac{1}{4\omega_1} (\delta_2 + \omega_1 \omega_2 \delta_5), \quad \Lambda_2 = \frac{1}{4\omega_2} (\alpha_1 - \omega_1^2 \alpha_4) \quad (5.25)$$

The response-moment equations (5.11) involve short- and long-time averaging. The former kills fastly varying quantities. Substituting equations (5.17)-(5.20) into equations (5.9) and performing short-time averaging, we find that 38 out of the 69 moments are zero. As an example, we consider

$$\begin{aligned}
 \underline{u_2^3} &= \underline{(x_2 \cos \omega_2 t + y_2 \sin \omega_2 t)^3} \\
 &= \frac{\omega_2}{2\pi} \left[x_2^3 \int_0^{2\pi/\omega_2} \cos^3 \omega_2 t \, dt + 3x_2^2 y_2 \int_0^{2\pi/\omega_2} \cos^2 \omega_2 t \sin \omega_2 t \, dt \right. \\
 &\quad \left. + 3x_2 y_2^2 \int_0^{2\pi/\omega_2} \cos \omega_2 t \sin^2 \omega_2 t \, dt + y_2^3 \int_0^{2\pi/\omega_2} \sin^3 \omega_2 t \, dt \right] = 0 \quad (5.26)
 \end{aligned}$$

For the case $\omega_2 \approx 2\omega_1$, the zero-response moments are given in Appendix III.

Transition and steady-state solutions of the response-moment equations are obtained by numerically integrating the coupled response-moment equations by a fifth-order Runge-Kutta-Verner technique. Results are presented showing the variation of the mean-square responses with internal detuning, excitation spectral amplitude, and time.

Numerical integration of coupled differential equations or solving systems of simultaneous algebraic equations is time consuming and depends significantly on the number of these equations. Guided by the narrow-band nature of the response of lightly damped systems, we reduce the number of the coupled response-moment equations generated by the

non-Gaussian closure scheme. Consequently, computer time is drastically reduced.

The variation of the mean-square responses with the detuning parameter ν is shown in Figure 5.1 when $\xi_1 = 0$. The results of the non-Gaussian closure scheme are in good agreement with those obtained by using the reduced non-Gaussian closure scheme. In the neighborhood of a perfectly-tuned autoparametric resonance, the response exhibits an energy exchange between the two modes. Away from the autoparametric resonance, the effect of the coupling dies out and a uni-modal response is displayed. When $\nu = 0$, Figure 5.2 shows the effect of the spectral amplitude S_2 of the excitation on the mean-square responses. These results do not indicate the existence of saturation.

For the case of a perfectly tuned system (i.e., $\omega_2 = 2\omega_1$ and $\nu = 0$), Figure 5.3 shows that the Gaussian closure technique predicts a uni-modal response consisting of the directly excited second mode and that the unexcited first mode does not respond. Adding the third- and fourth-order cumulants and equating the higher-order cumulants to zero can help describe part of the non-Gaussian feature of the response due to the nonlinearities. Figure 5.4 shows the modulation of the mean-square response with time for the case $\omega_2 = 2\omega_1$. It shows that the non-Gaussian closure scheme predicts a bi-modal response, in contrast with the Gaussian closure scheme, which predicts a uni-modal response.

The directly excited second mode passes part of the energy it receives to the indirectly excited first mode due to the autoparametric resonance.

Beyond a transition period, Figure 5.5 shows that the Gaussian closure scheme does not yield long-time fixed-point response moments as reported by Ibrahim and Roberts [181].

As an application, we consider the nonlinear response of an arch to a random excitation. The equation of motion of a hinged-hinged arch with a rise $w_0(x)$ is given by [189]

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} - N \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w_0}{\partial x^2} \right) = Q(x,t) \quad (5.27)$$

$$w = 0 \text{ and } \frac{\partial w}{\partial x} = 0 \text{ at } x = 0, L \quad (5.28)$$

where

$$N = \frac{EA}{2L} \int_0^L \left[\left(\frac{\partial w}{\partial x} \right)^2 + 2 \frac{\partial w}{\partial x} \frac{\partial w_0}{\partial x} \right] dx \quad (5.29)$$

Here, $w(x,t)$ is the transverse displacement, E is Young's modulus, A and I are the area and moment of inertia of the cross section, ρ is the density, L is the projected length, c is the damping coefficient, and Q is the transverse load.

We consider the case of a sinusoidal rise and expand w in terms of the undamped free linear vibration modes; that is,

$$w(x,t) = 2r \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{L} \quad (5.30)$$

$$w_0(x) = 2r \sum_{n=1}^{\infty} \lambda_n \sin \frac{n\pi x}{L} \quad (5.31)$$

where r is the radius of gyration of the cross section. Using Galerkin's method, assuming modal damping, and truncating the result for two modes, we obtain equations (5.1) and (5.2) where

$$\omega_2 = 2\omega_1 + \nu \quad (5.32)$$

and the system parameters are given by Nayfeh [189].

As an example, we let $\lambda_1 = 5.35$ and $\lambda_2 = 0.26$ so that $\omega_1 = 1.0$, $\omega_2 = 2.0$, $\beta_{11} = 0.092$, $\beta_{21} = -0.172$, $\beta_{31} = 0.368$, $\beta_{41} = -0.184$, $\beta_{12} = -0.057$, $\beta_{22} = 0.368$, $\beta_{32} = -0.552$, $\beta_{42} = 0.953$, $\delta_1 = 1.275$, $\delta_2 = -0.986$, $\delta_3 = 1.328$, $\alpha_1 = -0.493$, $\alpha_2 = 2.656$, and $\alpha_3 = 0.250$. We are primarily concerned with low-amplitude motions. Consequently we let

$$u_n = \epsilon u_n^* \quad (5.33)$$

where ϵ is a measure of the motion amplitude.

Figure 5.6 shows the variation of the mean-square responses with the internal detuning parameter ν . The first and second modes are externally excited by two uncorrelated zero-mean white noise. Their spectral amplitude excitations are related by the following equation:

$$\frac{S_1}{\mu_1} - \frac{S_2}{\mu_2} = 0 \quad (5.34)$$

The results of the non-Gaussian closure scheme are in good agreement with those obtained by using the exact stationary solution of the FPK equation. For the case $\omega_2 \approx 2\omega_1$, Figure 5.7 shows the modulation of the mean-square responses with time when the first mode is excited by a

zero-mean Gaussian white noise. The non-Gaussian closure scheme predicts a bi-modal response. Variations of the mean-square responses with the internal detuning parameter ν are presented in Figure 5.8. In the neighborhood of $\nu = 0$, the response exhibits an energy exchange between the two modes. The minimum mean-square response of the directly excited mode and the peak mean-square response of the indirectly excited mode occur when $\nu < 0$. This indicates a softening nonlinearity. As $|\nu|$ increases, the effect of the autoparametric resonance decreases and finally a uni-modal response is displayed.

5.5 One-To-One Internal Resonance

Using the method of multiple scales, one finds that, to the first approximation, the general solution of equations (5.1) and (5.2) can be expressed in terms of two narrow-band processes with the center frequency ω_2 ; that is,

$$u_i = x_i \cos \omega_2 t - y_i \sin \omega_2 t \quad \text{for } i = 1, 2 \quad (5.35)$$

In this case, we find that 24 out of the 69 moments are zero. The zero-response moments are given in Appendix IV.

Figure 5.9 shows the dependence of the mean-square responses on the internal detuning parameter ν . Both modes are externally excited by zero-mean Gaussian white noise. The response curves are single-valued functions for all values of ν . The results of the non-Gaussian scheme, the reduced scheme, and those obtained by using the exact stationary solution of the FPK equation are in good agreement. When $\nu = \omega_2 - \omega_1 =$

0, Figure 5.10 shows the time history of the mean-square responses. The response-moment equations asymptote constant values (fixed-point solution) (i.e., $\frac{dm_{i,j,k,l}}{dt} = 0$). For some parameters, a false fixed-point solution might be reached. In this case, the time history displays a fixed-point solution for some time and then shows a jump to another solution. Hence, caution must be exercised to establish a true fixed-point solution by integrating for sufficiently long times. Variations of the mean-square responses with the detuning parameter ν are presented in Figure 5.11. Comparisons of the results of the non-Gaussian-closure and those obtained by using the reduced non-Gaussian closure scheme are in excellent agreement. The response curves are single-valued functions for all values of ν . When $\nu \approx 0$, the response displays an energy exchange between the two resonant modes. As $|\nu|$ increases, the effect of the internal coupling vanishes and the system displays a uni-modal response.

As an application, we consider the nonlinear response of a string to a random excitation. The equations of motion of a string with an initial tension N_0 are given by [163]

$$\frac{\partial^2 u}{\partial t^2} - c_1^2 \frac{\partial^2 u}{\partial x^2} = (c_1^2 - c_2^2) \frac{\partial}{\partial x} \left\{ \left[\frac{1}{2} - \frac{\partial u}{\partial x} \right] \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] \right\} \quad (5.36)$$

$$\frac{\partial^2 v}{\partial t^2} - c_2^2 \frac{\partial^2 v}{\partial x^2} = (c_1^2 - c_2^2) \frac{\partial}{\partial x} \left\{ \frac{\partial v}{\partial x} \left[\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \right\} \quad (5.37)$$

$$\frac{\partial^2 w}{\partial t^2} - c_2^2 \frac{\partial^2 w}{\partial x^2} = (c_1^2 - c_2^2) \frac{\partial}{\partial x} \left\{ \frac{\partial w}{\partial x} \left[\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \right\} \quad (5.38)$$

$$u = v = w = 0 \quad \text{at} \quad x = 0, L \quad (5.39)$$

where u is the longitudinal displacement and v and w are the transverse displacements. Here, c_1 and c_2 are the longitudinal and transverse speeds of sound given by

$$c_1 = \sqrt{E/\rho} \quad \text{and} \quad c_2 = \sqrt{\frac{NO}{\rho A}} \quad (5.40)$$

Neglecting the longitudinal inertia $\frac{\partial^2 u}{\partial t^2}$, considering the first mode in each of the transverse motions, adding modal damping forces and external broad-band random excitations, expanding v and w in terms of the linear free undamped modes, and using Galerkin's method, we obtain

$$\ddot{u}_1 + \omega_1^2 u_1 + 2\mu_1 \dot{u}_1 + \Gamma u_1^3 + \Gamma u_1 u_2^2 = \xi_1(t) \quad (5.41)$$

$$\ddot{u}_2 + \omega_2^2 u_2 + 2\mu_2 \dot{u}_2 + \Gamma u_2^3 + \Gamma u_1^2 u_2 = \xi_2(t) \quad (5.42)$$

where

$$\omega_2 = \omega_1 + \nu \quad (5.43)$$

$$\omega_1 = \frac{\pi C_2}{L} \quad \text{and} \quad \Gamma = \frac{\pi^4 C_1^2}{4L^2} \quad (5.44)$$

Figure 5.12 shows that the results of the non-Gaussian closure scheme are in excellent agreement with those obtained by using the exact stationary solution of the FPK equation. For the case of a perfectly tuned system (i.e., $\omega_2 = \omega_1$), Figure 5.13 shows that the non-Gaussian closure predicts a uni-modal response. As the linear damping parameters decrease, Figure 5.14 shows that the response exhibits an energy exchange between the two modes. Similarly, increasing the spectral

amplitude of the excitation results in a bi-modal response, as shown in Figure 5.15. Variations of the mean-square responses with the excitation spectral amplitude S_1 are presented in Figure 5.16. For $S_1 < 0.035$, the response displays a uni-modal response. For $S_1 \geq 0.035$, the response starts to display an energy exchange between the two internally-resonant modes. The response of the unexcited second mode builds up due to the presence of the nonlinear coupling. Figure 5.17 shows the dependence of the mean-square responses on the internal detuning parameter ν when the first mode is externally excited by a zero-mean Gaussian white noise. The results of the non-Gaussian closure scheme are in excellent agreement with those obtained by using the reduced scheme. The energy-exchange peak between the two resonant modes occurs when $\nu > 0$. This indicates a hardening nonlinearity. Away from the autoparametric resonance, the effect of the nonlinear coupling decays and the response displays a uni-modal response.

5.6. Three-To-One Internal Resonance

Using the method of multiple scales, one can show that, to the first approximation, the response is given by narrow-band processes with the center frequencies $\frac{1}{3}\omega_2$ and ω_2 ; that is,

$$u_1 = x_1 \cos \frac{1}{3} \omega_2 t - y_1 \sin \frac{1}{3} \omega_2 t, \quad u_2 = x_2 \cos \omega_2 t - y_2 \sin \omega_2 t \quad (5.45)$$

In this case, we find that 36 out of 69 moments are zero. The zero-response moments are given in Appendix V.

Figure 5.18 shows the variation of the mean-square responses with the internal detuning parameter ν . Both modes are externally excited by zero-mean Gaussian white noise. The response curves are single-valued functions for all values of ν . The results of the non-Gaussian closure scheme, the reduced scheme, and those obtained by using the exact stationary solution of the FPK equation are in good agreement. When $\nu = \omega_2 - 3\omega_1 = 0$, the modulation of the mean-square responses with time is shown in Figure 5.19. The response displays a nonstationary bi-modal solution for some time and hereafter, displays a stationary bi-modal solution. Integration for long times is needed to establish a true fixed-point solution. Figure 5.20 shows the variation of the mean-square response with the detuning parameter ν . Results of the non-Gaussian and the reduced non-Gaussian closure schemes are in good agreement. The response curves are single-valued functions for all values of ν . In the neighborhood of a perfectly-tuned internal resonance $\nu \approx 0$, the response exhibits an energy exchange between the two modes. As $|\nu|$ increases, the effect of internal coupling dies out and the response display a uni-modal response.

As an application, we consider the nonlinear response of a hinged-clamped beam to a random excitation. The governing equation of motion of a hinged-clamped beam is given by [163]

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} + c \frac{\partial w}{\partial t} - \frac{EA}{2L} \frac{\partial^2 w}{\partial x^2} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx = Q(x,t) \quad (5.46)$$

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0 \quad (5.47)$$

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = L \quad (5.48)$$

We expand $w(x,t)$ in terms of the undamped free linear modes; that is,

$$w(x,t) = \frac{r^2}{L} \sum_{n=1}^{\infty} \phi_n(x) u_n(t) \quad (5.49)$$

where

$$\phi_n(x) = \frac{2}{\sinh^2(\lambda_n L) - \sin^2(\lambda_n L)} [\sin(\lambda_n x) \sinh(\lambda_n L) - \sin(\lambda_n L) \sinh(\lambda_n x)] \quad (5.50)$$

$$\tan(\lambda_n L) - \tanh(\lambda_n L) = 0 \quad (5.51)$$

and the eigenvalues of the first and second mode are

$$\lambda_1 L = 3.9266, \lambda_2 L = 7.0686 \quad (5.52)$$

Considering the first and second mode, assuming modal damping, and using Galerkin's method, we obtain

$$\begin{aligned} \ddot{u}_1 + \omega_1^2 u_1 + 2\mu_1 \dot{u}_1 + \epsilon \Gamma_{1111} u_1^3 + \epsilon (\Gamma_{1112} + \Gamma_{1121} \\ + \Gamma_{1211}) u_1^2 u_2 + \epsilon (\Gamma_{1122} + \Gamma_{1212} + \Gamma_{1221}) u_1 u_2^2 \\ + \epsilon \Gamma_{1222} u_2^3 = \xi_1(t) \end{aligned} \quad (5.53)$$

$$\begin{aligned} \ddot{u}_2 + \omega_2^2 u_2 + 2\mu_2 \dot{u}_2 + \epsilon \Gamma_{2111} u_1^3 + \epsilon (\Gamma_{2112} + \Gamma_{2121} \\ + \Gamma_{2211}) u_1^2 u_2 + \epsilon (\Gamma_{2122} + \Gamma_{2212} + \Gamma_{2221}) u_1 u_2^2 \\ + \epsilon \Gamma_{2222} u_2^3 = \xi_2(t) \end{aligned} \quad (5.54)$$

where

$$\omega_2 = 3\omega_1 + \nu \quad (5.55)$$

$$r_{mnkl} = L^2 \left[\int_0^L \phi_m \frac{d^2 \phi_n}{dx^2} dx \int_0^L \frac{d\phi_l}{dx} \frac{d\phi_k}{dx} dx \right] \quad (5.56)$$

Figure 5.21 shows the variation of the mean-square responses with the excitation spectral amplitude S_1 when both modes are externally excited by two uncorrelated zero-mean Gaussian white noise. The results of the non-Gaussian closure scheme are in good agreement with those obtained by using the exact stationary solution of the FPK equation. Figure 5.22 shows the dependence of the mean-square responses on the excitation spectral amplitude S_1 when the first mode is externally excited by a zero-mean Gaussian white noise. The results obtained by using the reduced non-Gaussian scheme are in good agreement with those obtained in the non-Gaussian scheme. As S_1 increases, the response displays an energy exchange between the two resonant modes. The directly excited first mode passes part of the energy it receives to the indirectly excited second mode due to the autoparametric resonance.

5.6 Conclusions

Three cumulant-neglect closure techniques are used to determine the important lower-order response moments of two-degree-of-freedom systems with quadratic and cubic nonlinearities to external zero-mean Gaussian broad-band excitations. They are the Gaussian, non-Gaussian, and reduced non-Gaussian closure schemes. The cases of two-to-one, one-to-

one, and three-to-one internal resonances are investigated. The Markov vector procedure is used to obtain general first-order ordinary-differential equations governing the evolution of the response moments. The nonlinearities result in an infinite hierarchy set for the response-moment equations. Gaussian and non-Gaussian closure schemes yield, respectively, 14 and 69 equations for the lower-order response moments. Numerical integration of these coupled differential equations or solving the corresponding system of simultaneous algebraic equations (i.e., $\frac{dm_{i,j,k,l}}{dt} = 0$) is time consuming and depends significantly on the number of these equations. The method of multiple scales is used to describe the nature of the response and its characteristics. This information is utilized to reduce the number of the coupled response-moment equations generated via the non-Gaussian closure scheme. Consequently, computer time is drastically reduced. The results obtained by using the reduced scheme are in good agreement with those obtained by using the full 69 equations. The results are applied to determine the nonlinear responses of a shallow arch, a string, and a hinged-clamped beam to broad-band random excitations. Close to autoparametric resonance, the responses display an energy exchange between the interacting two modes. However, the response does not display the saturation phenomenon predicted for deterministic harmonic excitations in the case of two-to-one internal resonances. The results reveal major discrepancies between the Gaussian and non-Gaussian closure schemes. The Gaussian closure scheme does not yield stationary response moments.

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

In this dissertation we develop three methods to study the response of nonlinear systems to combined deterministic and stochastic excitations. They are a second-order closure method, a generalized method of equivalent linearization, and a reduced non-Gaussian closure scheme.

An effective method for determining the response of nonlinear systems to random excitations is to consider a set of equations for the moments of the response. This yields important response statistics, such as means and variances. The response-moment equations are an infinite hierarchy set due to the nonlinearities. Moment closure schemes are used to close this set. After a transformation from the fastly varying state vector $\{u, \dot{u}\}^T$ to the slowly varying vector $\{A, \theta\}^T$, we propose a second-order closure method to determine the first- and second-order response moments in Chapter 1. In this method, we take the following steps. First, we use the method of multiple scales to determine the equations describing the modulation of the amplitude and phase with the nonlinearity and excitation. Second, we separate the averaged equations into mean and fluctuating components. Third, we use these equations to obtain closed-form expressions for the stationary mean and mean-square response. It is found that the presence of the nonlinearities causes multi-valued regions where more than one mean-square value of the response is possible. The multi-valuedness is responsible for a jump phenomenon. The local stability of the

stationary mean and mean-square response is analyzed. For some range of parameters, the presence of noise can expand the stability region of the deterministic mean motion. The second-order closure method attacks the problem in a way that the statistics of the response can be obtained for general types of excitation and nonlinearity.

The method of equivalent linearization is based on replacing the nonlinear governing equations of motion by equivalent linear equations whose solution furnishes an approximate solution to the nonlinear response. The equivalent linear parameters are determined by minimizing the mean-square value of the difference between the two sets of equations. In achieving this goal, the method of equivalent linearization creates many limitations and overlooks many interesting phenomena introduced by the presence of the nonlinearities. A generalization of the method of equivalent linearization is presented in Chapter 3. The original nonlinear governing equation is replaced by two equivalent linear equations. This method provides some means to describe the non-Gaussian feature of the nonlinear response and preserves its wide-band character. The interaction and feedback between the nonlinearity and excitation create nonlinear resonances. These lead to linear shifts in the frequency and damping and an adjustment of the strength of the primary excitation. The former is responsible for the increase in the modal frequency and the latter contributes to the experimentally observed broadening of the response curves. To test the soundness of the proposed method, it is used to determine the mean-square response of the Duffing and Van der Pol-Rayleigh oscillators to a

Gaussian white noise and a superharmonic narrow-band random excitation of order three. For the former, it is shown that the proposed method yields a better approximation to the mean-square response than the method of equivalent linearization. For the latter, the method of equivalent linearization predicts a single-valued function for the mean-square response. However, the proposed method predicts multi-valued regions where more than one mean-square value of the response is possible.

In Chapter 4 the second-order closure method is used to analyze the response of two-degree-of-freedom systems with quadratic nonlinearities to narrow-band random excitations. The case of primary resonance of the second mode in the presence of a two-to-one internal (autoparametric) resonance is investigated. Contrary to the results of the linear analysis, the nonlinear analysis reveals that the directly excited second mode takes a small amount of the input energy (saturates) and spills over the rest of the input energy into the first mode, which is indirectly excited through the modal coupling.

Three closure schemes via the Markov vector procedure are used to determine the lower-order statistical response moments of two-degree-of-freedom systems with quadratic and cubic nonlinearities in Chapter 5. They are the Gaussian, non-Gaussian, and reduced non-Gaussian closure schemes. The system is externally excited by zero-mean Gaussian white noise. The case of two-to-one, one-to-one, and three-to-one autoparametric resonances are investigated. Ito stochastic calculus is used to derive a hierarchy set of first-order differential equations

governing the modulation of the response moments with time. Gaussian and non-Gaussian closure schemes are used to close this set, leading, respectively, to 14 and 69 equations for the lower-order response moments. The method of multiple scales is used to describe the nature of the response and its characteristics. This information is utilized to drastically reduce the number of the response-moment equations generated by using the non-Gaussian closure scheme. The results obtained by using the reduced scheme are in excellent agreement with those obtained by using the full 69 equations. Close to autoparametric resonance, the response displays an energy exchange between the two modes. The response, however, does not display the saturation phenomenon predicted for the case of two-to-one internal resonances. The results reveal major discrepancies between the Gaussian and non-Gaussian closure schemes. The results are applied to a shallow arch, a string, and a hinged-clamped beam.

In any extension of the work presented in this thesis, the following points are noteworthy:

1. The second-order closure method can be used to determine the response of two-degree-of-freedom systems with cubic nonlinearities to non-zero mean random excitations in the presence of one-to-one and three-to-one autoparametric resonances.

2. An attempt to introduce some correlation between f_1 and f_2 in the generalized method of equivalent linearization is expected to yield a better solution.
3. Extension and application of the methods indicated in this thesis to stationary non-Gaussian, nonstationary Gaussian, and nonstationary non-Gaussian random excitations are still needed.
4. Future studies need to be directed at investigating the stability of the response of dynamical systems to non-zero mean random excitations by examining the perturbed state equations.
5. The three methods developed in this study can be applied to determine the response of many elastic and dynamical structures, such as plates, shells, ships, and foundations. Extensions to composite structures are also recommended.
6. The fact that our theoretical results are based upon approximate methods points out the need for supporting experimental investigations.

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APPENDIX I

The Matrix B and Vector Q

$$\begin{aligned}
 B = & \begin{array}{ccccccc}
 2c_1 & 0 & 2c_2 & 0 & 0 & \frac{\epsilon}{\omega_0} & 0 \\
 0 & 2c_4 & 2c_3 & 0 & \frac{\epsilon}{\omega_0} & 0 & 0 \\
 c_3 & c_2 & c_1+c_4 & \frac{\epsilon}{2\omega_0} & 0 & 0 & \frac{\epsilon}{2\omega_0} \\
 0 & 0 & 0 & c_1-\frac{\beta}{2} & c_2 & 0 & 0 \\
 0 & 0 & 0 & c_3 & c_4-\frac{\beta}{2} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & c_1-\frac{\beta}{2} & c_2 \\
 0 & 0 & 0 & 0 & 0 & c_3 & c_4-\frac{\beta}{2}
 \end{array} \quad (I1)
 \end{aligned}$$

$$\underline{Q} = \left\{ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{\epsilon \pi S_0}{2\omega_0} \quad \frac{\epsilon \pi S_0}{2\omega_0} \quad 0 \right\}^T \quad (I2)$$

APPENDIX II

Method of Averaging

Using equation (3.36), the solution of equation (3.28) can be written as a narrow-band random process with slowly varying amplitude and phase; that is,

$$v(t) = A(T_1) \cos[\omega_0 t + \theta(T_1)] \quad (\text{II.1})$$

$$\dot{v}(t) = -\omega_0 A(T_1) \sin[\omega_0 t + \theta(T_1)] \quad (\text{II.2})$$

Here, $T_1 = \epsilon t$ is a slow time scale. Equations (II.1) and (II.2) represent a transformation from the fast varying state vector $\{v, \dot{v}\}^T$ to the slowly varying vector $\{A, \theta\}^T$. The modulation of A and θ with time can be written as

$$\begin{aligned} \dot{A} = & -\epsilon \mu A(1 - \cos 2\psi) + \frac{3\epsilon \alpha U^2 A}{2\omega_0} \sin 2\psi + \frac{3\epsilon \alpha U A^2}{4\omega_0} (\sin \psi + \sin 3\psi) \\ & + \frac{\epsilon \alpha A^3}{8\omega_0} (2\sin 2\psi + \sin 4\psi) + \frac{\epsilon \alpha U^3}{\omega_0} \sin \psi - \frac{f_1}{\omega_0} \sin \psi \end{aligned} \quad (\text{II.3})$$

$$\begin{aligned} \dot{\theta} = & -\epsilon \mu \sin 2\psi + \frac{3\epsilon \alpha U^2}{2\omega_0} (1 + \cos 2\psi) + \frac{3\epsilon \alpha U A}{4\omega_0} (3\cos \psi + \cos 3\psi) \\ & + \frac{\epsilon \alpha A^3}{8\omega_0} (3 + 4\cos 2\psi + \cos 4\psi) + \frac{\epsilon \alpha U^3}{\omega_0 A} \cos \psi - \frac{f_1}{\omega_0 A} \cos \psi \end{aligned} \quad (\text{II.4})$$

where

$$\psi = \omega_0 t + \theta \quad (\text{II.5})$$

Averaging equations (II.3) and (II.4) and keeping only slowly varying terms yields

$$\begin{aligned}\dot{A} = & -\epsilon\mu A + \frac{3\epsilon\alpha A}{2\omega_0} \langle u^2 \sin 2\psi \rangle + \frac{3\epsilon\alpha A^2}{4\omega_0} \langle u \sin 3\psi \rangle \\ & + \frac{\epsilon\alpha}{\omega_0} \langle u^3 \sin \psi \rangle - \frac{1}{\omega_0} \langle f_1 \sin \psi \rangle\end{aligned}\quad (\text{II.6})$$

$$\begin{aligned}\dot{\theta} = & \frac{3\epsilon\alpha}{2\omega_0} \langle u^2 \rangle + \frac{3\epsilon\alpha}{2\omega_0} \langle u^2 \cos 2\psi \rangle + \frac{3\epsilon\alpha A}{4\omega_0} \langle u \cos 3\psi \rangle \\ & + \frac{3\epsilon\alpha A^2}{8\omega_0} + \frac{\epsilon\alpha}{\omega_0 A} \langle u^3 \cos \psi \rangle - \frac{1}{\omega_0 A} \langle f_1 \cos \psi \rangle\end{aligned}\quad (\text{II.7})$$

Equations (II.6) and (II.7) shows a linear $3\epsilon\alpha\langle u^2 \rangle/2\omega_0$ and a nonlinear $3\epsilon\alpha A^2/8\omega_0$ shifts in the frequency. The following simultaneous primary, ultrasubharmonic and subharmonic resonances are possible:

- a) f_1 and u^3 around ω_0
- b) u^2 around $2\omega_0$
- c) u around $3\omega_0$

Neglecting the ultrasubharmonic and the subharmonic resonances, the equivalent linear equation for v can be written as

$$\ddot{v} + (\omega_0^2 + 3\epsilon\alpha \langle u^2 \rangle)v + 2\epsilon\mu\dot{v} + \frac{3}{2}\epsilon\alpha \langle v^2 \rangle v = f_1 - \epsilon\alpha u^3 \quad (\text{II.8})$$

APPENDIX III
Two-to-One Internal Resonance
Zero-Response Moments

- | | | |
|-------------------------------------|-------------------------------------|-------------------------------------|
| 1) $\langle x_1 \rangle$ | 2) $\langle x_2 \rangle$ | 3) $\langle x_3 \rangle$ |
| 4) $\langle x_4 \rangle$ | 5) $\langle x_1 x_2 \rangle$ | 6) $\langle x_1 x_4 \rangle$ |
| 7) $\langle x_2 x_3 \rangle$ | 8) $\langle x_3 x_4 \rangle$ | 9) $\langle x_1^3 \rangle$ |
| 10) $\langle x_2^3 \rangle$ | 11) $\langle x_3^3 \rangle$ | 12) $\langle x_4^3 \rangle$ |
| 13) $\langle x_1 x_2 x_4 \rangle$ | 14) $\langle x_2 x_3 x_4 \rangle$ | 15) $\langle x_1 x_2^2 \rangle$ |
| 16) $\langle x_1 x_3^2 \rangle$ | 17) $\langle x_1 x_4^2 \rangle$ | 18) $\langle x_2 x_4^2 \rangle$ |
| 19) $\langle x_3 x_4^2 \rangle$ | 20) $\langle x_1^2 x_3 \rangle$ | 21) $\langle x_2^2 x_3 \rangle$ |
| 22) $\langle x_2^2 x_4 \rangle$ | 23) $\langle x_1^3 x_2 \rangle$ | 24) $\langle x_1 x_4^3 \rangle$ |
| 25) $\langle x_2 x_3^3 \rangle$ | 26) $\langle x_3 x_4^3 \rangle$ | 27) $\langle x_1^3 x_2 \rangle$ |
| 28) $\langle x_1^3 x_4 \rangle$ | 29) $\langle x_2^3 x_3 \rangle$ | 30) $\langle x_3^3 x_4 \rangle$ |
| 31) $\langle x_1 x_2 x_3^2 \rangle$ | 32) $\langle x_1 x_2 x_4^2 \rangle$ | 33) $\langle x_1 x_2^2 x_4 \rangle$ |
| 34) $\langle x_1 x_3^2 x_4 \rangle$ | 35) $\langle x_1^2 x_3 x_4 \rangle$ | 36) $\langle x_2^2 x_3 x_4 \rangle$ |
| 37) $\langle x_1^2 x_2 x_3 \rangle$ | 38) $\langle x_2 x_3 x_4^2 \rangle$ | |

APPENDIX IV
One-to-One Internal Resonance
Zero-Response Moments

- | | | |
|-----------------------------------|-----------------------------------|-----------------------------------|
| 1) $\langle x_1 \rangle$ | 2) $\langle x_2 \rangle$ | 3) $\langle x_3 \rangle$ |
| 4) $\langle x_4 \rangle$ | 5) $\langle x_1^3 \rangle$ | 6) $\langle x_2^3 \rangle$ |
| 7) $\langle x_3^3 \rangle$ | 8) $\langle x_4^3 \rangle$ | 9) $\langle x_1 x_2 x_3 \rangle$ |
| 10) $\langle x_1 x_2 x_4 \rangle$ | 11) $\langle x_1 x_3 x_4 \rangle$ | 12) $\langle x_2 x_3 x_4 \rangle$ |
| 13) $\langle x_1 x_2^2 \rangle$ | 14) $\langle x_1 x_3^2 \rangle$ | 15) $\langle x_1 x_4^2 \rangle$ |
| 16) $\langle x_2 x_3^2 \rangle$ | 17) $\langle x_2 x_4^2 \rangle$ | 18) $\langle x_3 x_4^2 \rangle$ |
| 19) $\langle x_1^2 x_2 \rangle$ | 20) $\langle x_1^2 x_3 \rangle$ | 21) $\langle x_1^2 x_4 \rangle$ |
| 22) $\langle x_2^2 x_3 \rangle$ | 23) $\langle x_2^2 x_4 \rangle$ | 24) $\langle x_3^2 x_4 \rangle$ |

APPENDIX V
Three-to-One Internal Resonance
Zero-Response Moments

- | | | |
|-------------------------------------|-------------------------------------|-------------------------------------|
| 1) $\langle x_1 \rangle$ | 2) $\langle x_2 \rangle$ | 3) $\langle x_3 \rangle$ |
| 4) $\langle x_4 \rangle$ | 5) $\langle x_1 x_2 \rangle$ | 6) $\langle x_1 x_4 \rangle$ |
| 7) $\langle x_2 x_3 \rangle$ | 8) $\langle x_3 x_4 \rangle$ | 9) $\langle x_1^3 \rangle$ |
| 10) $\langle x_2^3 \rangle$ | 11) $\langle x_3^3 \rangle$ | 12) $\langle x_4^3 \rangle$ |
| 13) $\langle x_1 x_2 x_3 \rangle$ | 14) $\langle x_1 x_2 x_4 \rangle$ | 15) $\langle x_1 x_3 x_4 \rangle$ |
| 16) $\langle x_2 x_3 x_4 \rangle$ | 17) $\langle x_1 x_2^2 \rangle$ | 18) $\langle x_1 x_3^2 \rangle$ |
| 19) $\langle x_1 x_4^2 \rangle$ | 20) $\langle x_2 x_3^2 \rangle$ | 21) $\langle x_2 x_4^2 \rangle$ |
| 22) $\langle x_3 x_4^2 \rangle$ | 23) $\langle x_1^2 x_2 \rangle$ | 24) $\langle x_1^2 x_3 \rangle$ |
| 25) $\langle x_1^2 x_4 \rangle$ | 26) $\langle x_2^2 x_3 \rangle$ | 27) $\langle x_2^2 x_4 \rangle$ |
| 28) $\langle x_3^2 x_4 \rangle$ | 29) $\langle x_1 x_2^3 \rangle$ | 30) $\langle x_1 x_4^3 \rangle$ |
| 31) $\langle x_3 x_4^3 \rangle$ | 32) $\langle x_2^3 x_3 \rangle$ | 33) $\langle x_1 x_2 x_4^2 \rangle$ |
| 34) $\langle x_1 x_2^2 x_4 \rangle$ | 35) $\langle x_2^2 x_3 x_4 \rangle$ | 36) $\langle x_2 x_3 x_4^2 \rangle$ |

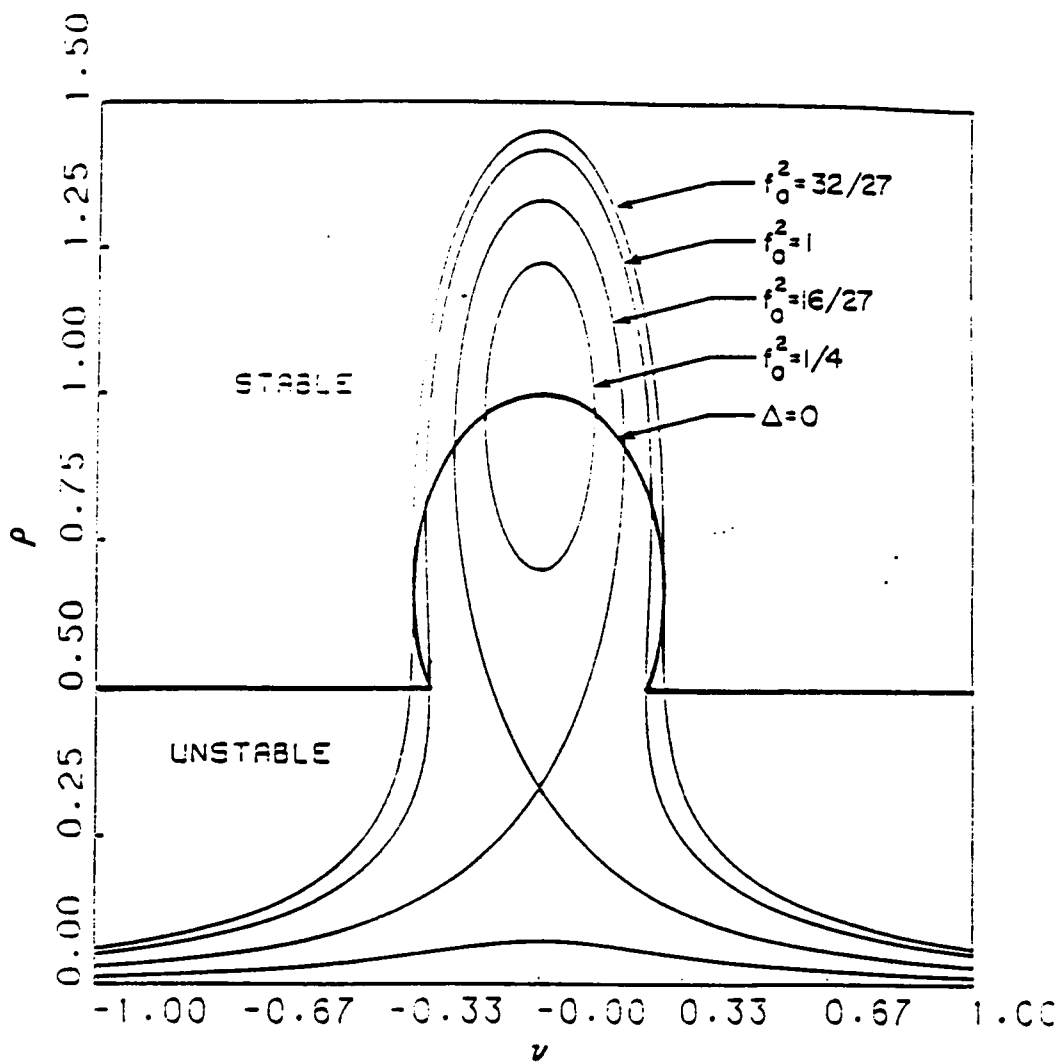


Figure 2.1. Frequency-response curves for primary resonances of the Rayleigh oscillator for $s = 1.0$, $\mu_0 = 1.0$, $k_0 = 0.0$, and $h_1 = k_1 = 0.0$.

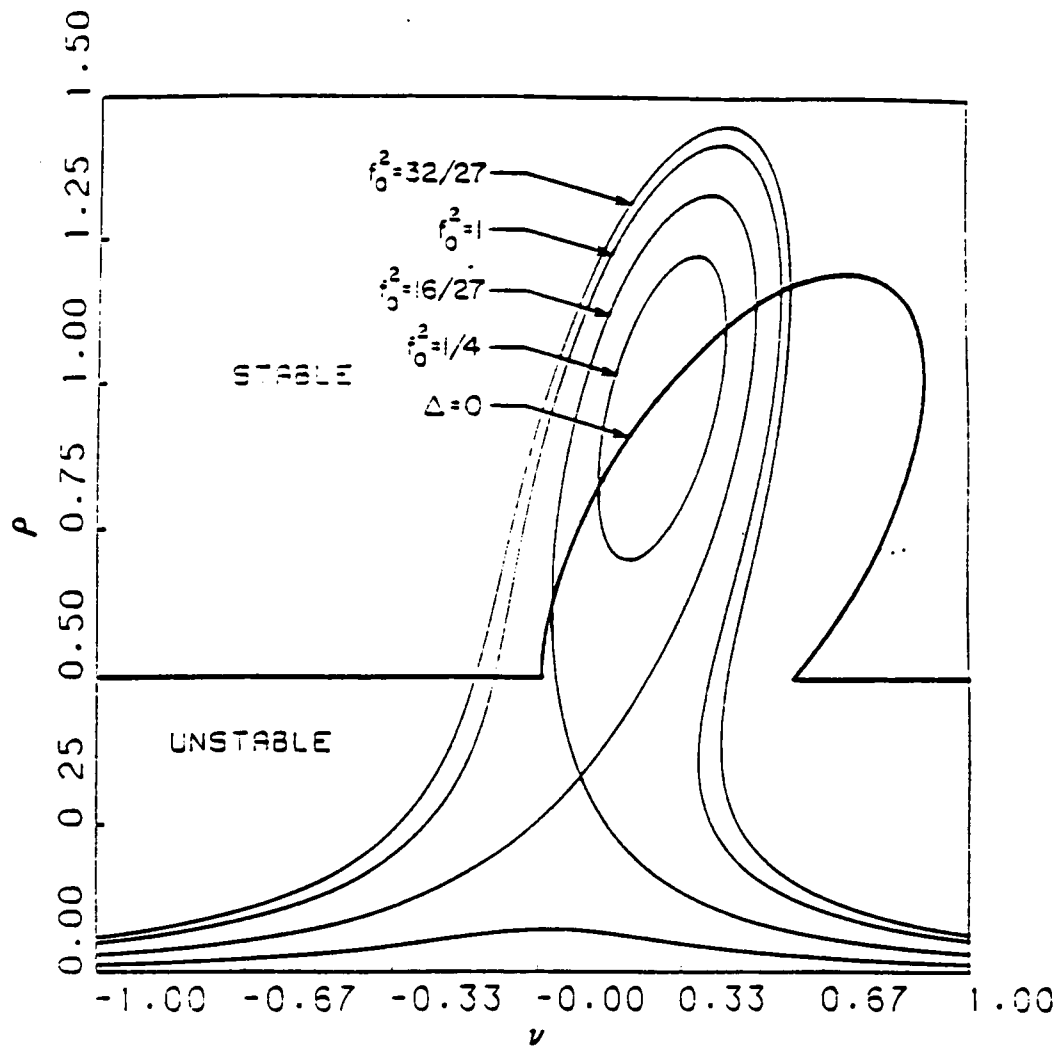


Figure 2.2. Frequency-response curves for primary resonances of the Duffing-Rayleigh oscillator for $\delta = 1.0$, $\epsilon = 0.2$, $\mu_0 = 1.0$, $k_0 = 0.0$, and $h_1 = k_1 = 0.0$.

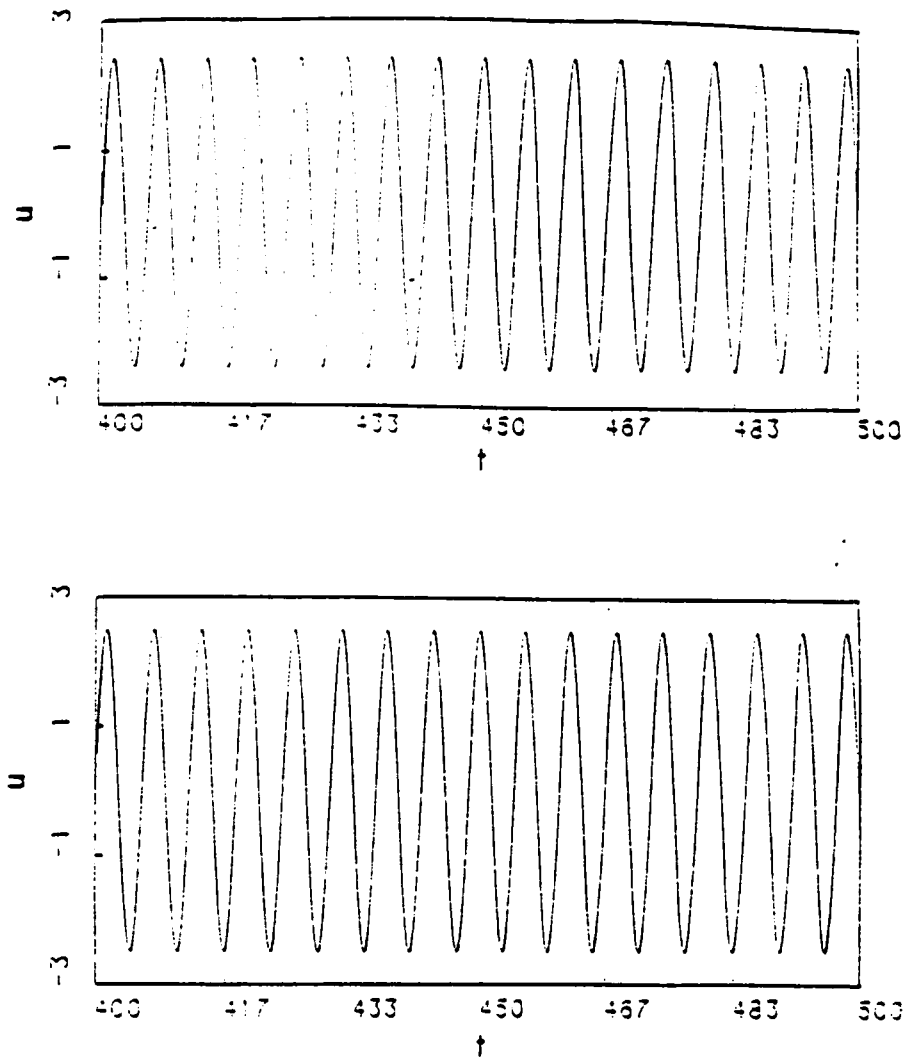


Figure 2.3. Comparison of the mean solution predicted by the perturbation analysis with that obtained by numerical simulation for $\delta = 1.0$, $\alpha = 0.2$, $u_0 = 1.0$, $n_0 = 1.414$, $\varepsilon = 0.08$, $k_0 = 0.0$, and $n_1 = k_1 = 0.0$: (a) numerical solution; (b) perturbation solution.

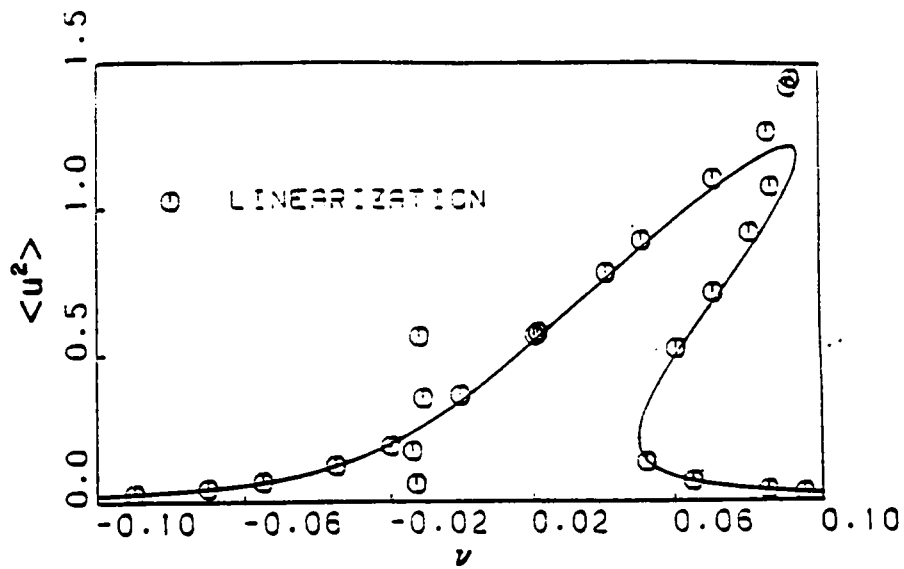


Figure 2.4. Frequency-response curves (narrow band) for $\delta_1 = 0.02$, $\delta_2 = 0.08$, $\alpha = 0.1$, $\omega_0 = 1.0$, $k_0 = 0.0316$, $h_0 = 0.0316$, $\tau S_0 = 0.00005$, and $\epsilon = 0.01$: a) linearization method; b) second-order closure method.

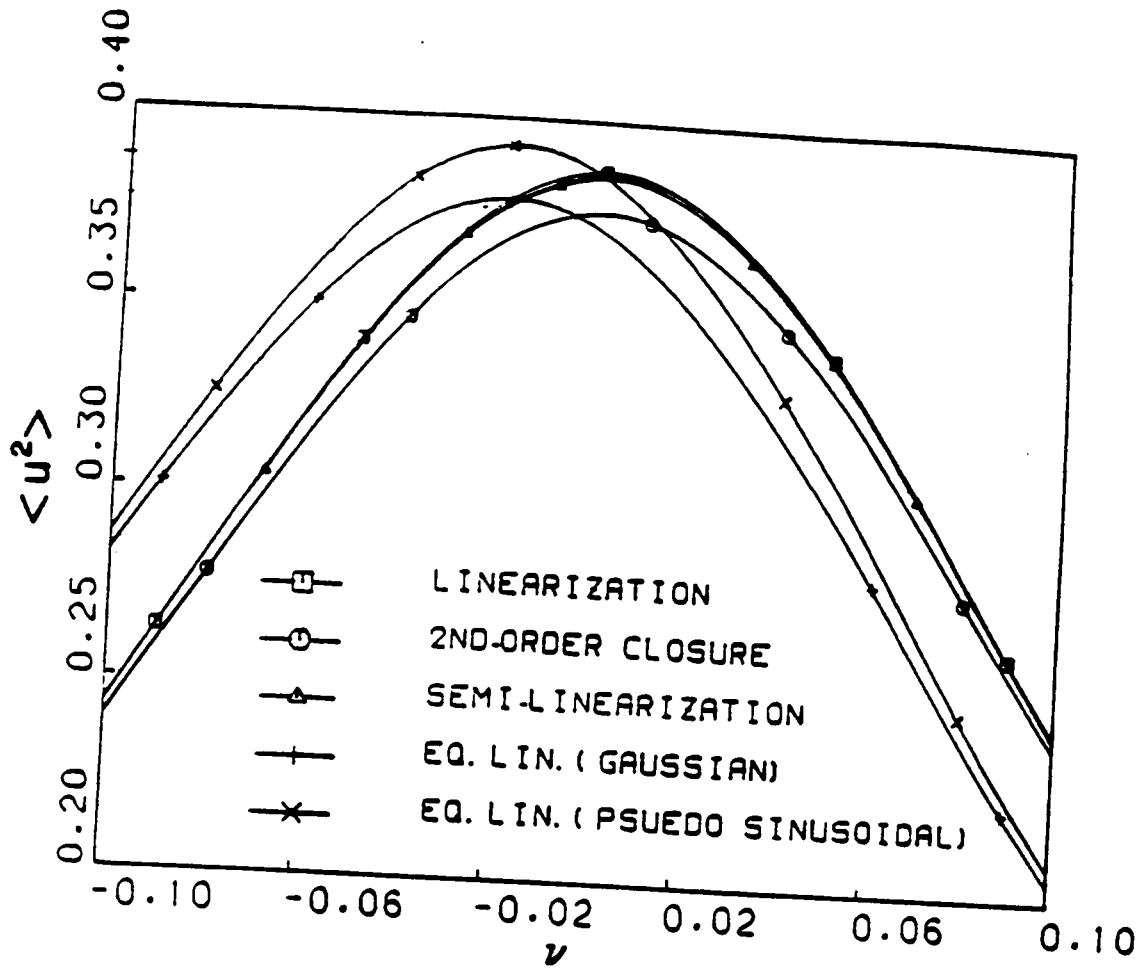
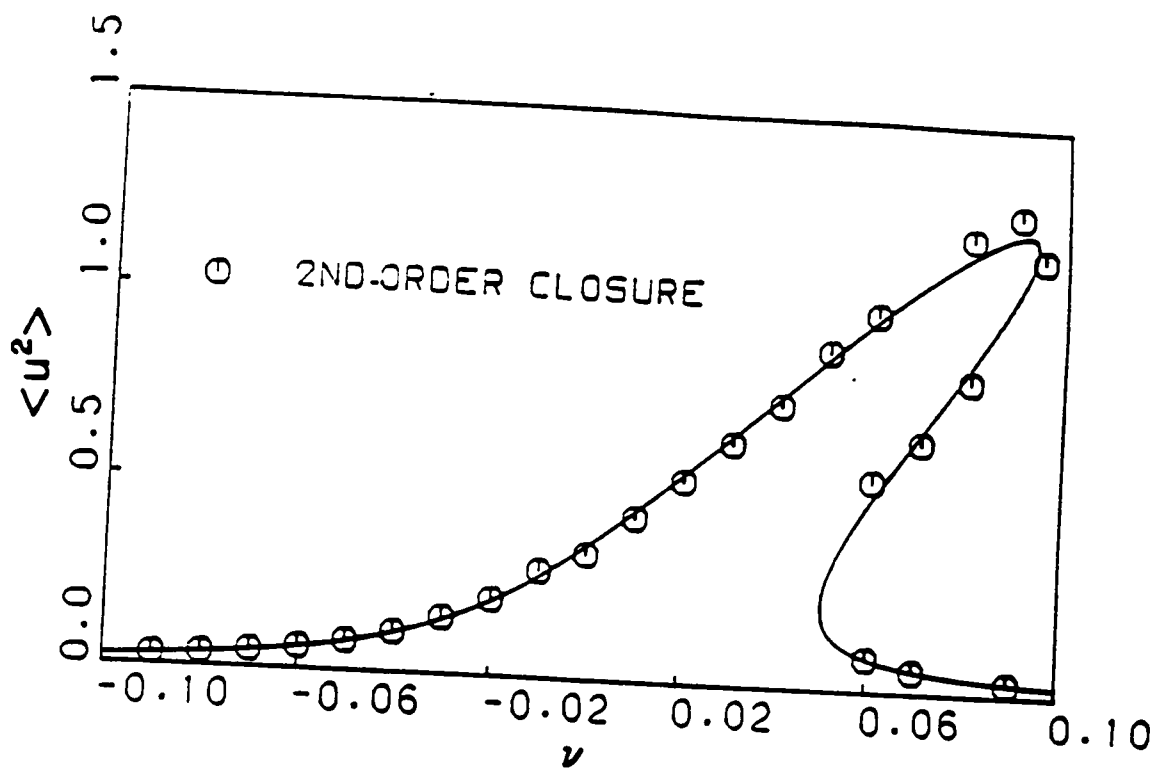


Figure 2.5. Comparison of the variations of the mean-square responses with the narrow-band excitation frequency obtained using the different methods: $\delta_1 = -0.2$, $\delta_2 = 0.2$, $\alpha = 0.0$, $\omega_0 = 1.0$, $h_0 = 0.2$, $k_0 = 0.001$, $\pi S_0 = 0.002$, and $\beta = 0.01$.



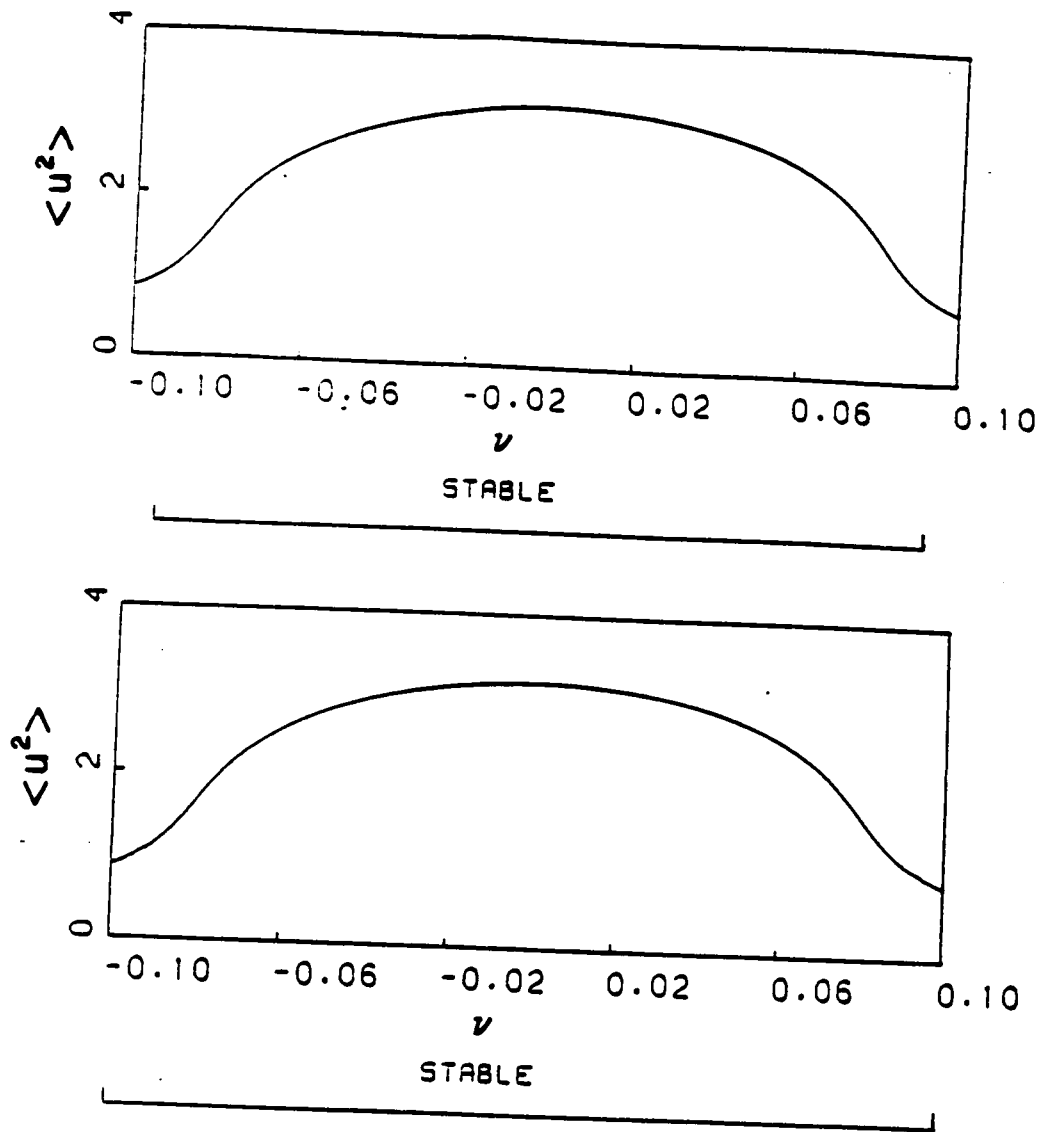


Figure 2.6. Frequency-response curves for $\delta_1 = 0.2$, $\delta_2 = 0.2$, $\alpha = 0.0$, $\omega_0 = 1.0$, $h_0 = 0.3$, $k_0 = 0.001$, $\pi S_0 = 0.0001$, and $\varepsilon = 0.01$: a) no noise; b) with noise.

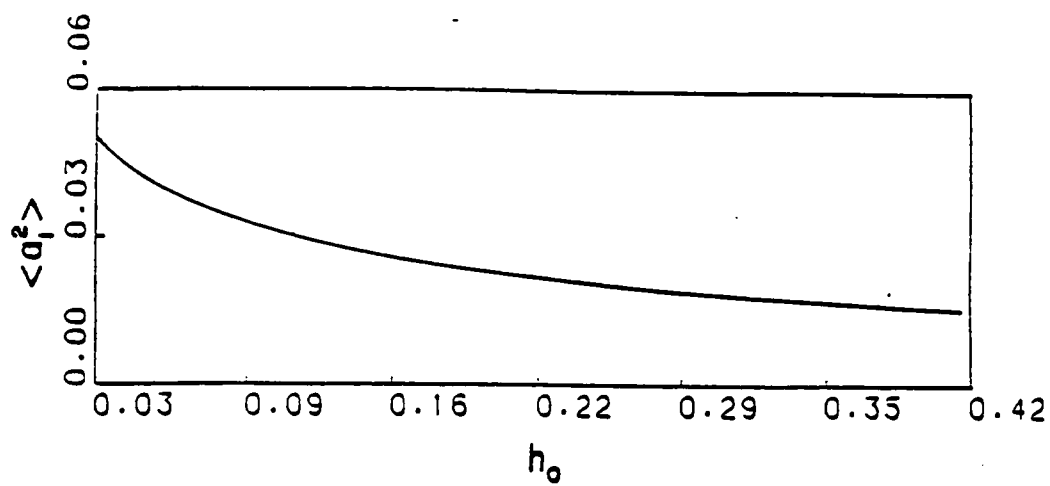


Figure 2.7. Effect of the mean amplitude of excitation upon the mean-square response to white noise for $\delta_1 = 0.02$, $\delta_2 = 0.08$, $\alpha = 0.05$, $\omega_0 = 1.0$, $k_0 = 0.0$, $\tau S_0 = 0.005$, and $\nu = 0.03$.

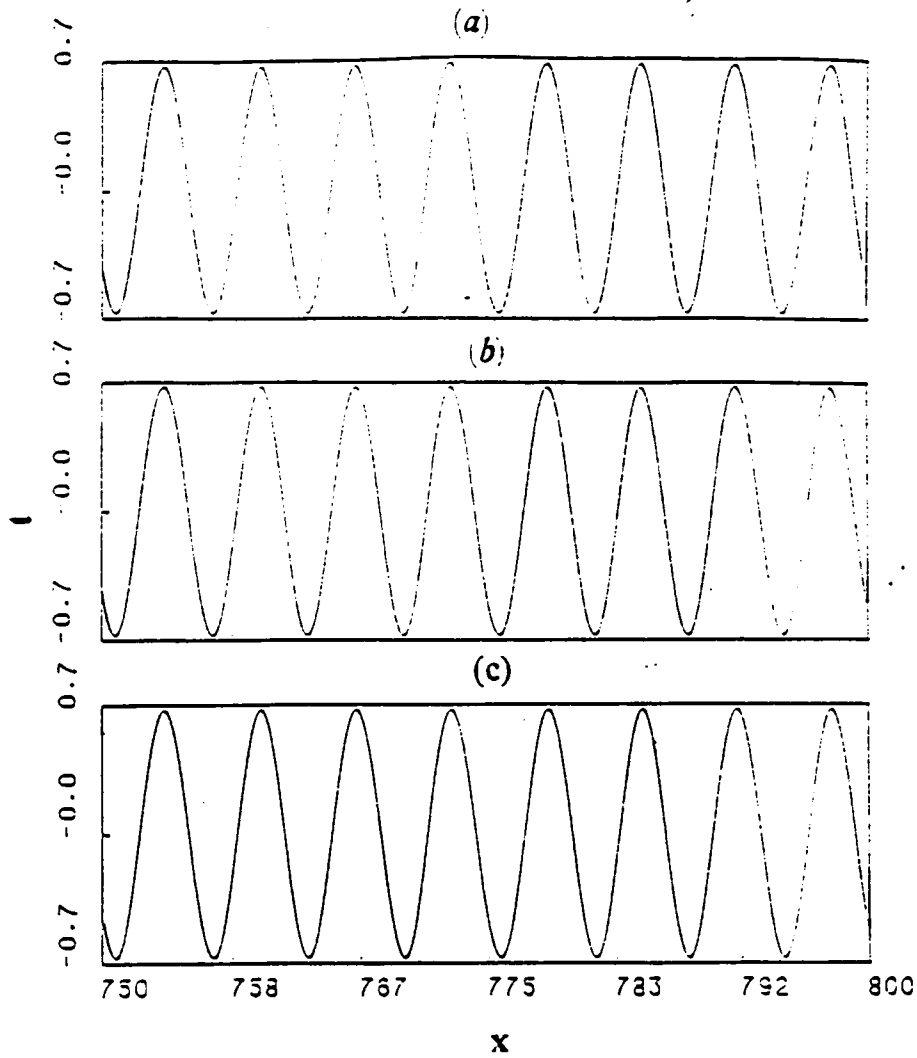


Figure 3.1. Comparison of the time-history response for primary resonance predicted by the method of equivalent linearization with those obtained by the method of averaging and numerical simulation for $\omega_0 = 1.0$, $\mu = 0.07$, $\alpha = 0.2$, $F = 0.1$, $\beta_1 = 0.01$, and $\varepsilon = 1.0$: (a) equivalent linearization; (b) averaging; and (c) numerical simulation.

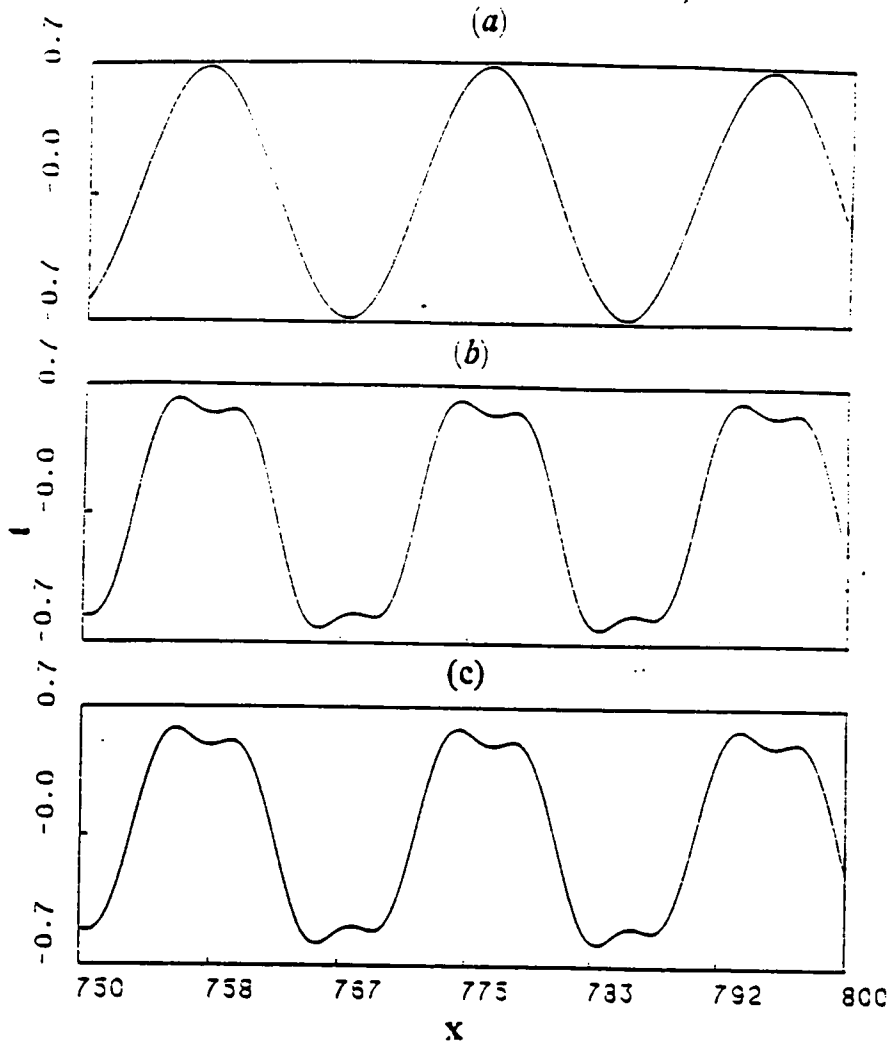


Figure 3.2. Comparison of the time-history response for superharmonic resonance of order-three predicted by the method of equivalent linearization with those obtained by the method of averaging and numerical simulation for $\mu_0 = 1.0$, $\mu = 0.02$, $\alpha = 0.3$, $\beta = 0.6$, $\gamma_1 = 0.02$, and $\varepsilon = 1.0$: (a) equivalent linearization; (b) averaging; and (c) numerical simulation.

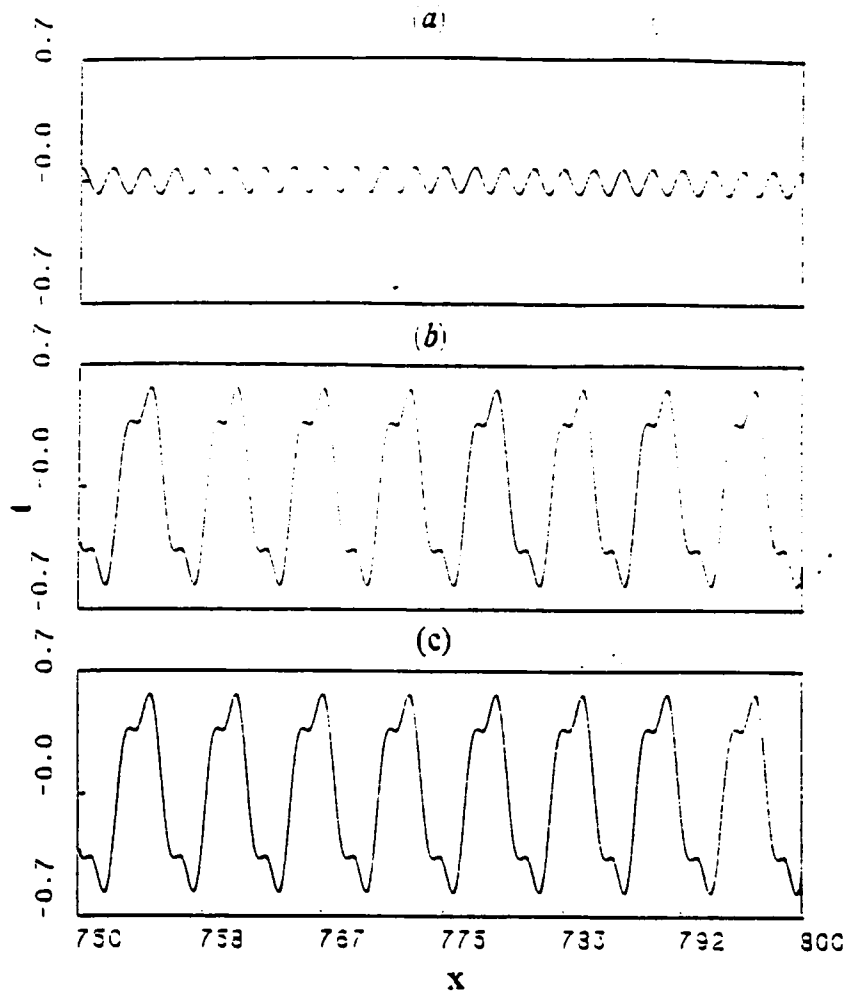


Figure 3.3. Comparison of the time-history response for subharmonic resonance of order one-third predicted by the method of equivalent linearization with those obtained by the method of averaging and numerical simulation for $\mu_0 = 1.0$, $\epsilon = 0.01$, $\alpha = 0.6$, $\beta = 1.25$, $\nu_1 = 0.15$, and $\lambda = 1.0$: (a) equivalent linearization; (b) averaging; and (c) numerical simulation.

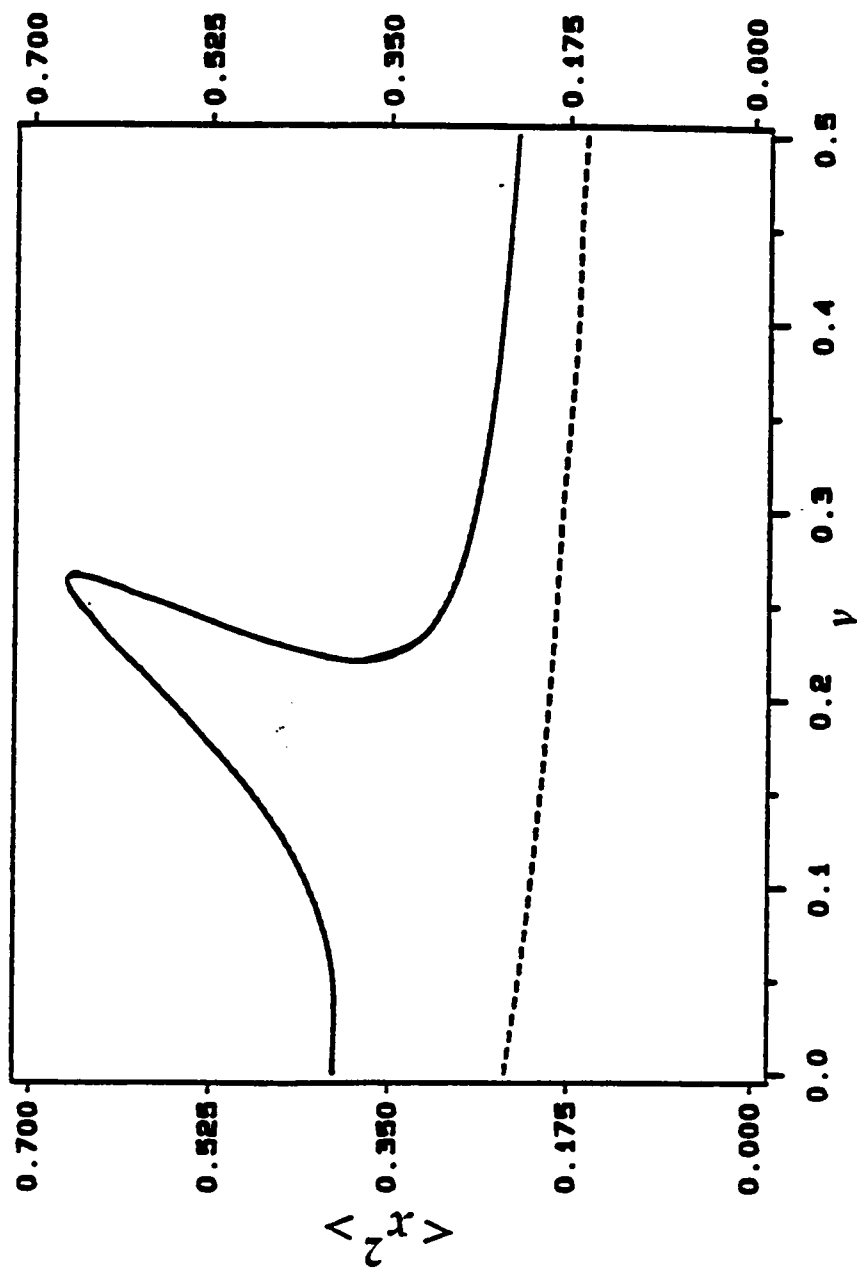


Figure 3.4. Dependence of the mean-square response of the Duffing oscillator on the detuning parameter ν for the case of a superharmonic narrow-band random excitation for $\omega_0 = 1.0$, $\mu = 0.03$, $\alpha = 0.3$, $\beta = 0.001$, and $S_0 = 1.0 \times 10^{-4}$:
 — predicted by the proposed method; --- predicted by the method of equivalent linearization.

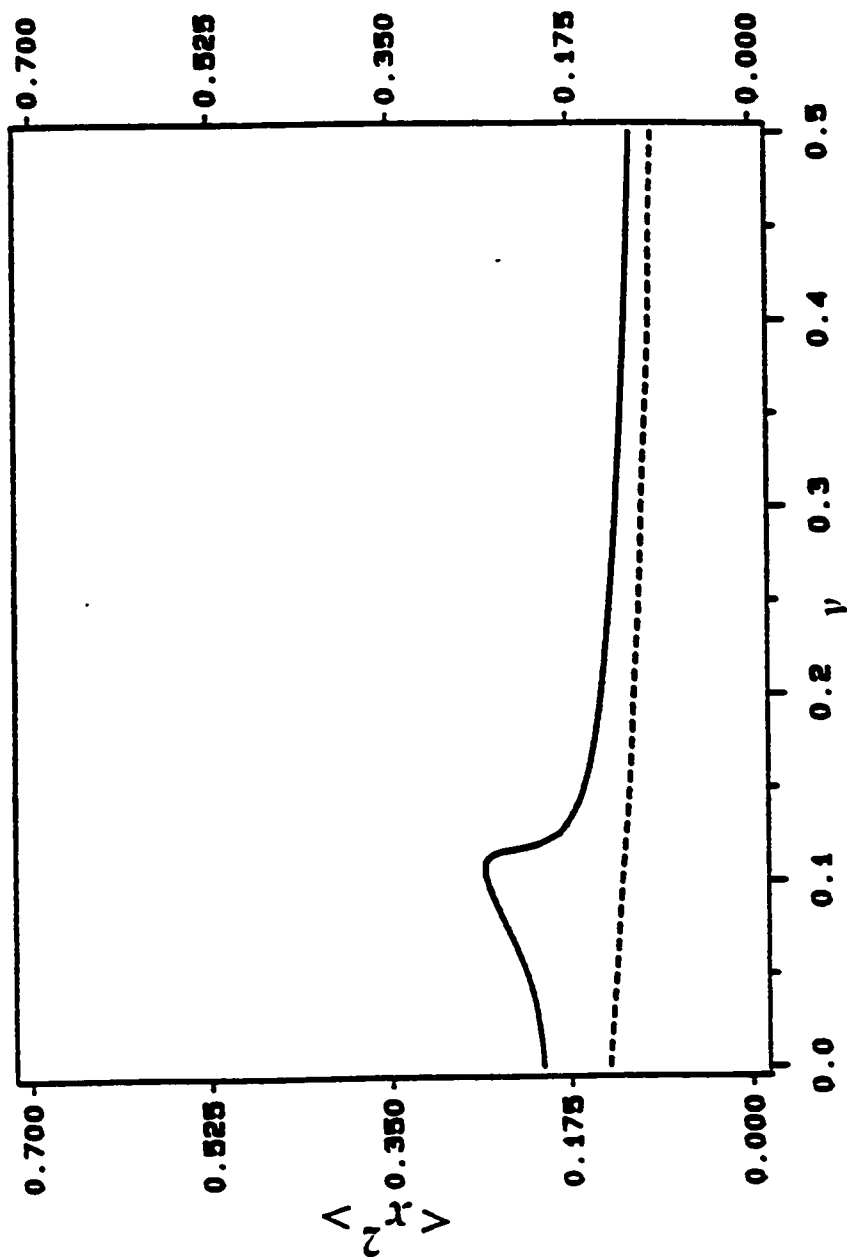


Figure 3.5. Dependence of the mean-square response of the Duffing oscillator on the detuning parameter ν for the case of a superharmonic narrow-band random excitation for $\omega_0 = 1.0$, $\mu = 0.03$, $\alpha = 0.3$, $\beta = 0.002$, and $S_0 = 1.0 \times 10^{-4}$: — predicted by the proposed method; --- predicted by the method of equivalent linearization.

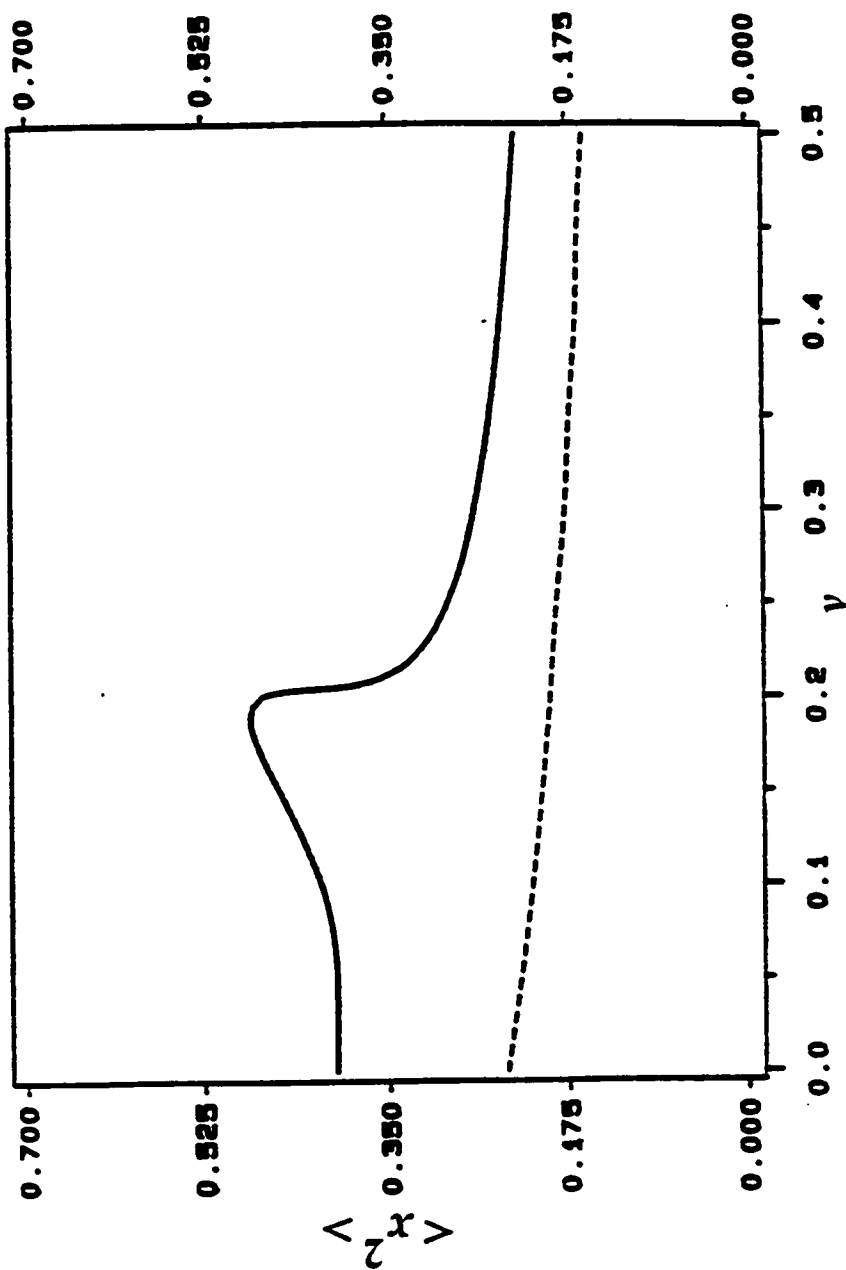


Figure 3.6. Dependence of the mean-square response of the Duffing oscillator on the detuning parameter ν for the case of a superharmonic narrow-band random excitation, for $\omega_0 = 1.0$, $\mu = 0.05$, $\alpha = 0.3$, $\beta = 0.001$, and $S_0 = 1.0 \times 10^{-4}$:
 — predicted by the proposed method, --- predicted by the method of equivalent linearization.

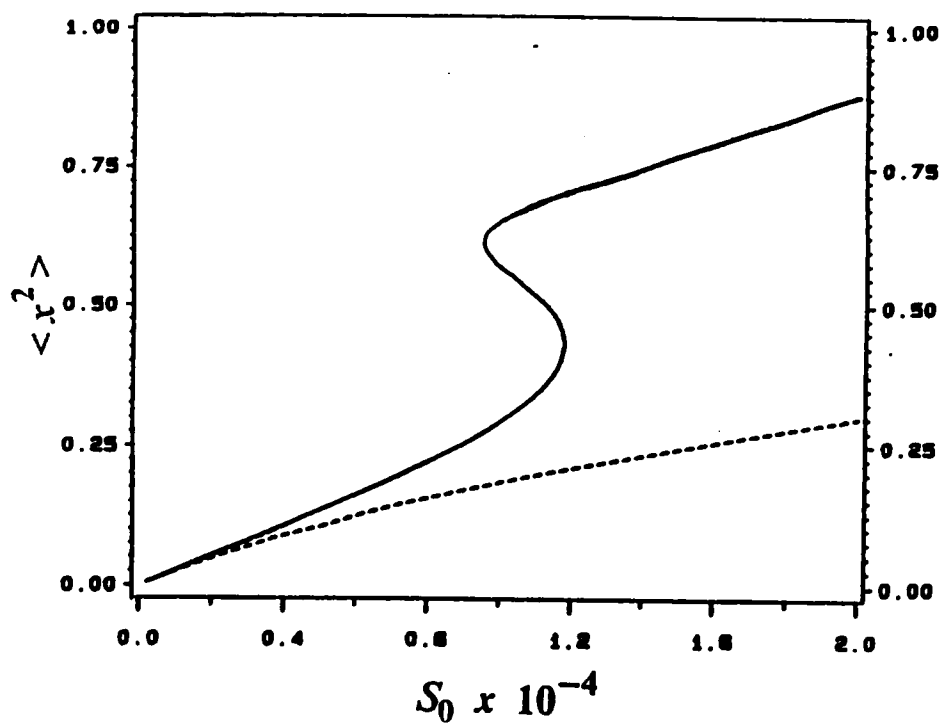


Figure 3.7. Variation of the mean-square response of the Duffing oscillator with the spectral amplitude S_0 of a superharmonic narrow-band random excitation for, $\omega_0 = 1.0$, $\mu = 0.03$, $\alpha = 0.3$, $\beta = 0.001$, and $\nu = 0.25$: — predicted by the proposed method; --- predicted by the method of equivalent linearization.

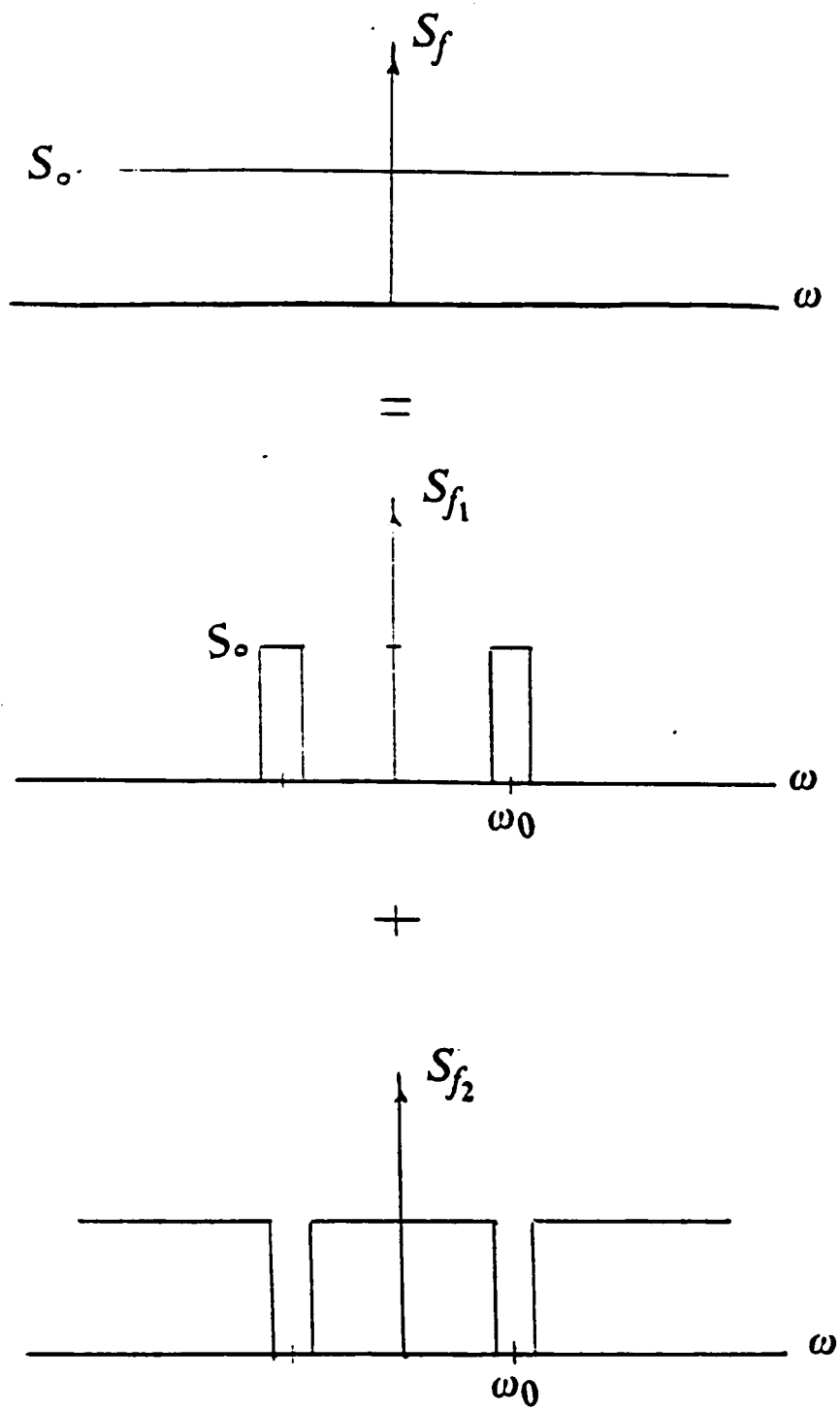


Figure 3.8. Spectral densities of F , f_1 , and f_2 .

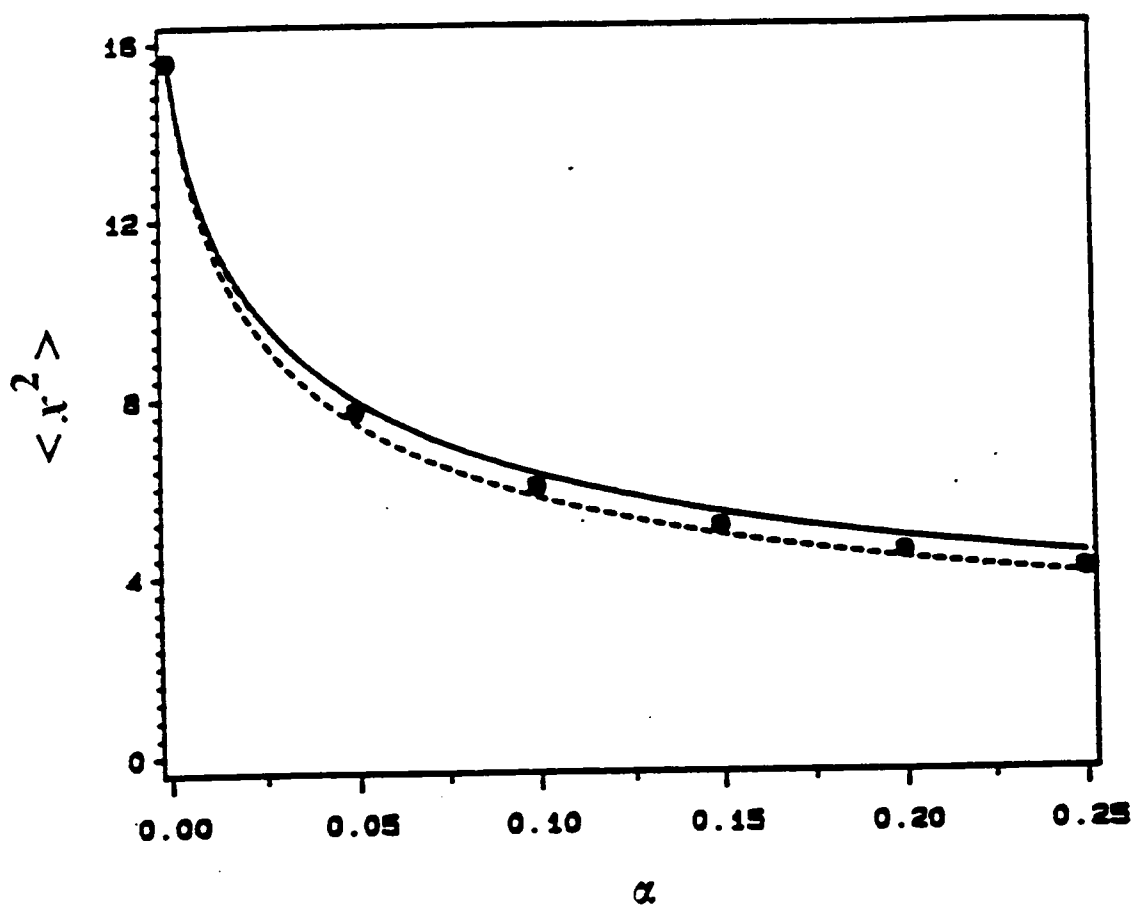


Figure 3.9. Variation of the mean-square response of the Duffing oscillator with the nonlinear parameter α for the case of a white noise, $\omega_0 = 1.0$, $\mu = 0.03$, and $S_0 = 0.3$: — exact solution; --- equivalent linearization solution; and o proposed method.

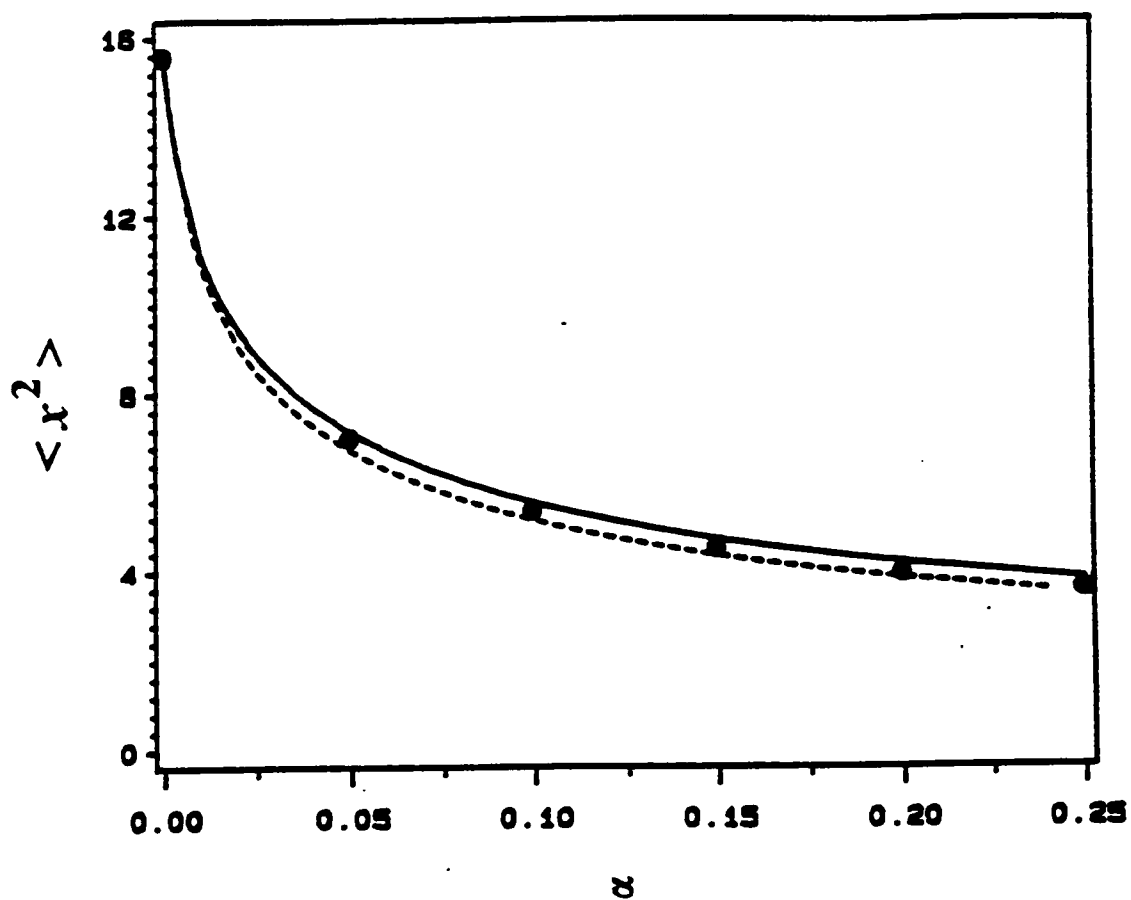


Figure 3.10. Variation of the mean-square response of the Van der Pol-Rayleigh oscillator with the nonlinear parameter γ for the case of a white noise, $\omega_0 = 1.0$, $\mu = 0.03$, and $S_0 = 0.03$: — exact solution; --- equivalent linearization solution; and o proposed method.

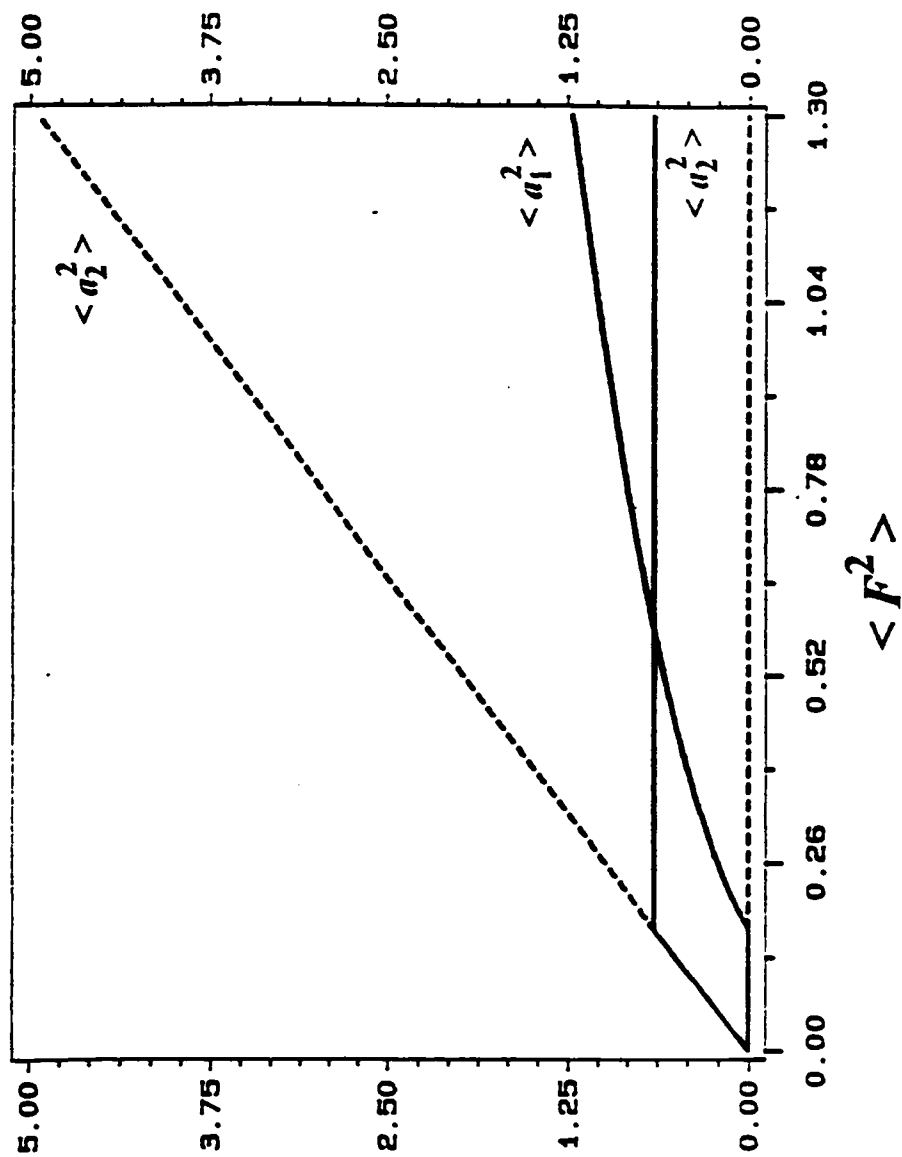


Figure 4.1. Representative force-response curves for $\mu_1 = 0.2$, $\mu_2 = 0.5$, $\lambda_1 = 0.25$, $\lambda_2 = 0.5$, $\hat{\nu}_1 = 0$, $\hat{\nu}_2 = 0$, and $\beta = 0.01$:
 — stable; --- unstable.

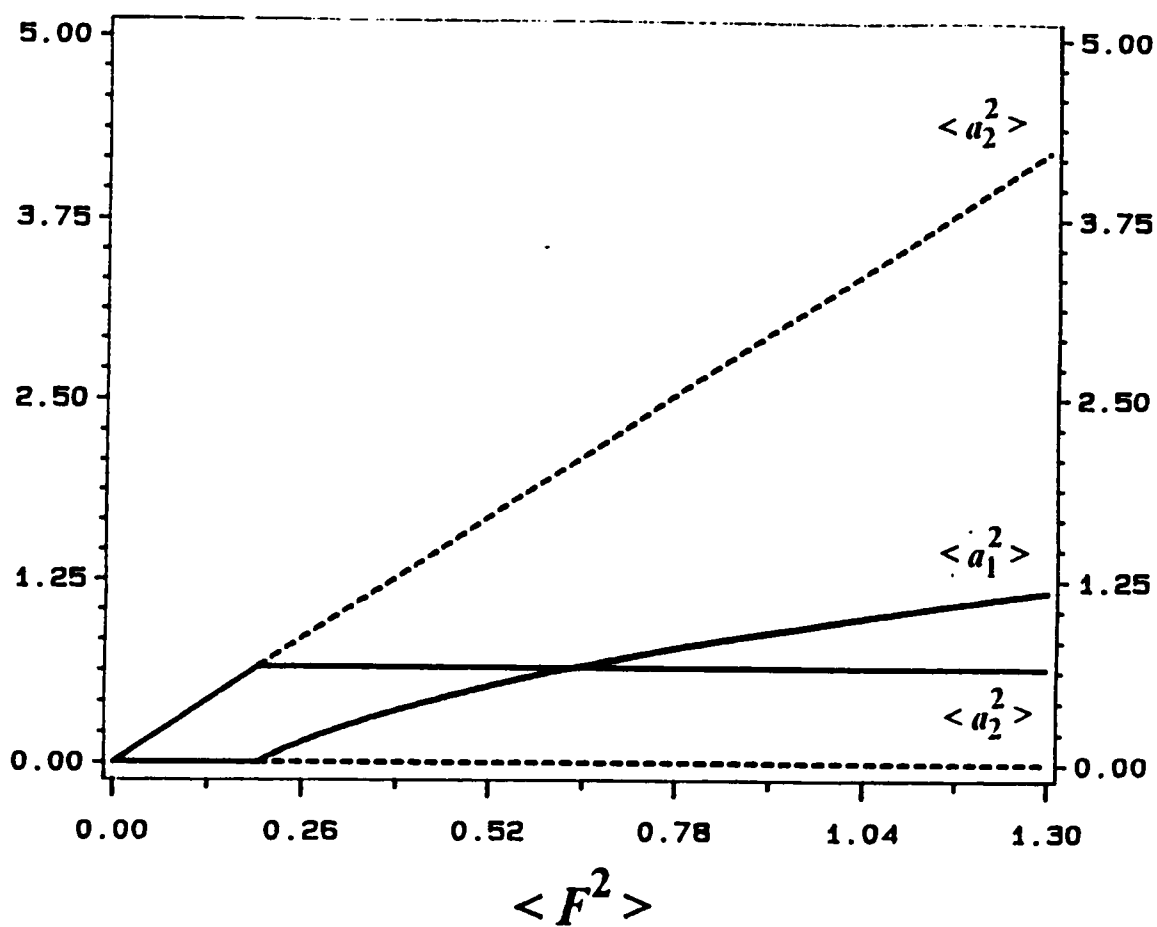


Figure 4.2. Representative force-response curves for $\mu_1 = 0.2$, $\mu_2 = 0.5$, $\Lambda_1 = 0.25$, $\Lambda_2 = 0.5$, and $\delta = 0.2$: — stable; --- unstable.

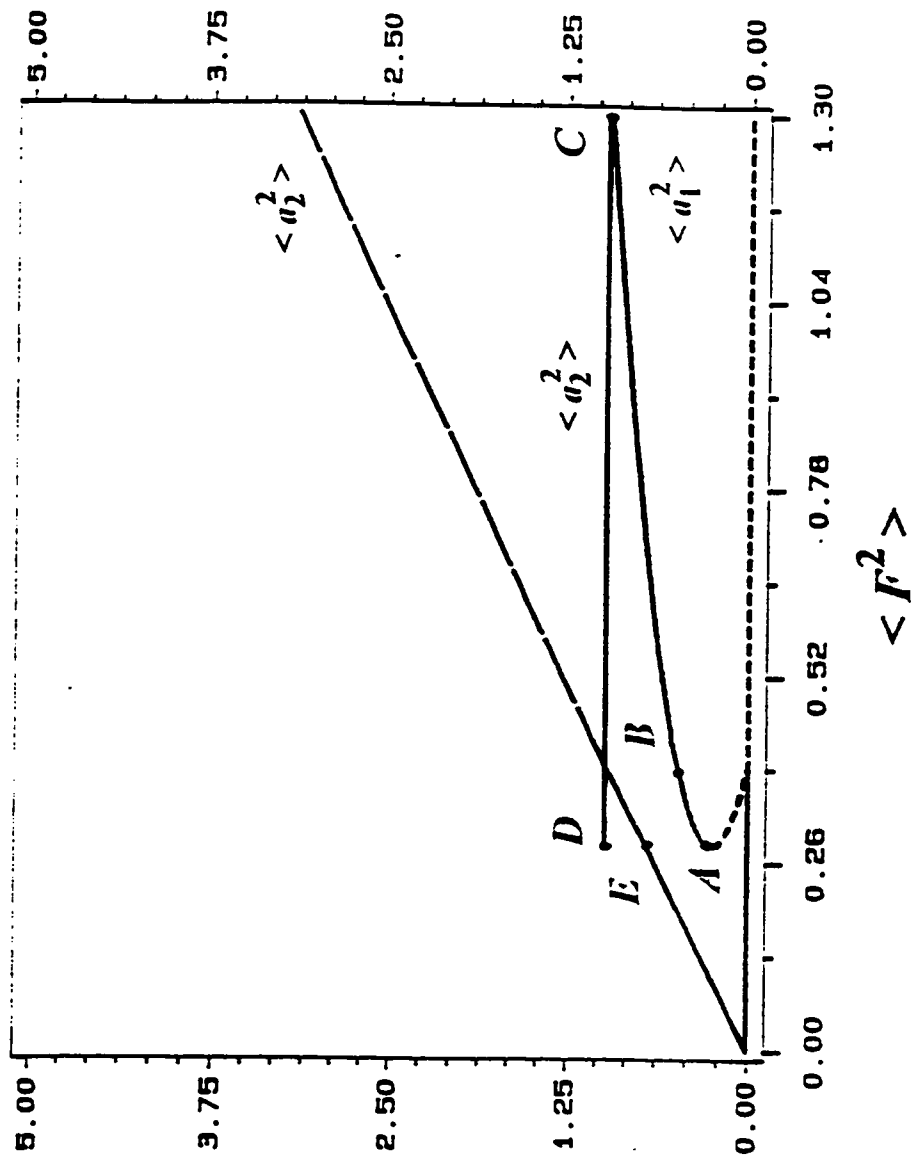


Figure 4.3. Representative force-response curves for $\mu_1 = 0.05$, $\mu_2 = 0.5$, $\lambda_1 = 0.36$, $\lambda_2 = 1.0$, $\hat{\nu}_1 = 0.6$, $\hat{\nu}_2 = 0.4$, and $\beta = 0.01$:
 — stable; --- unstable.

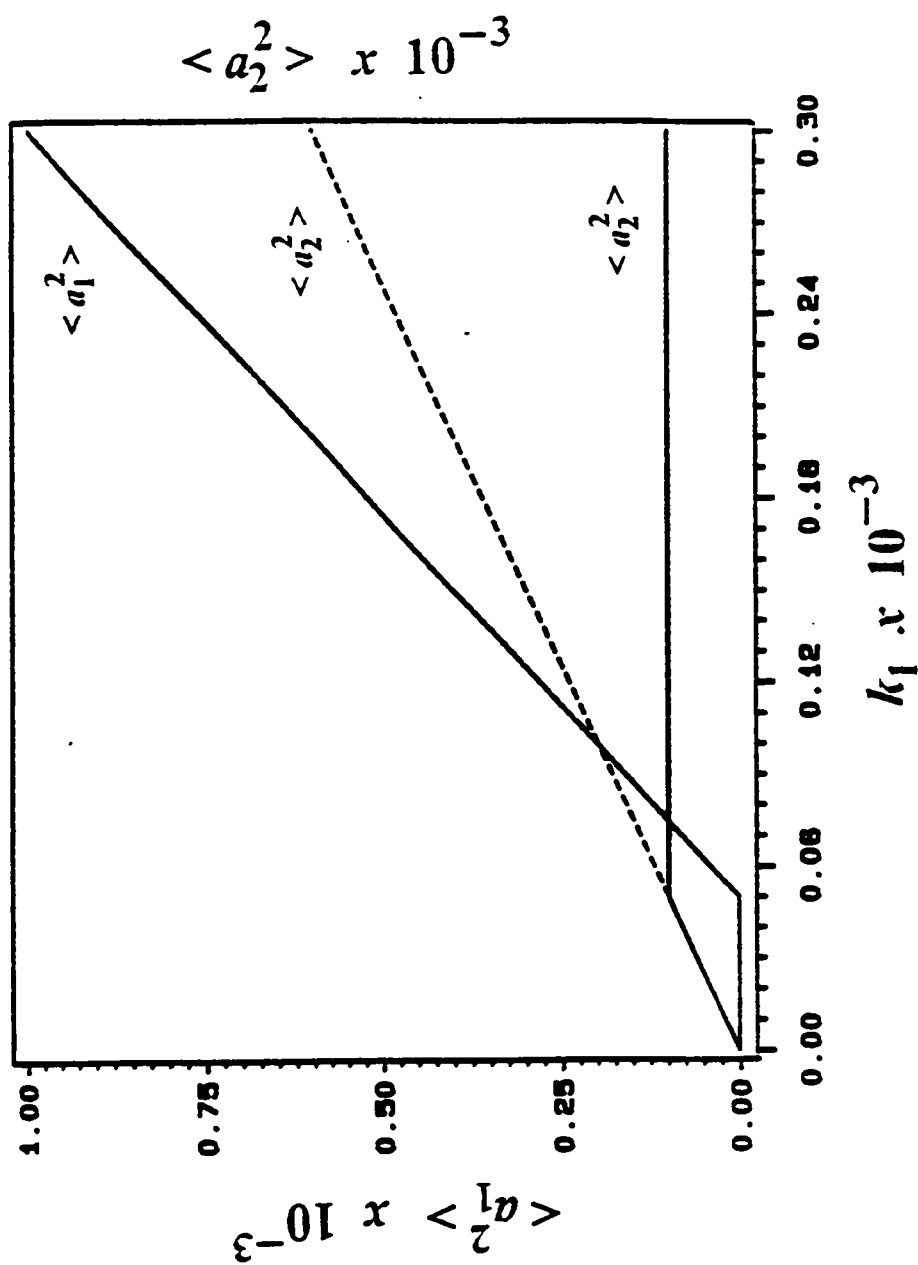


Figure 4.4. Modal mean-square-response amplitudes as functions of the amplitude of the deterministic excitation for $\mu_1 = 0.005$, $\mu_2 = 0.005$, $\Lambda_1 = 0.5$, $\Lambda_2 = 0.25$, $\nu_1 = 0.0$, $\nu_2 = 0.0$ and $\langle F^2 \rangle = 0$: — stable; --- unstable.

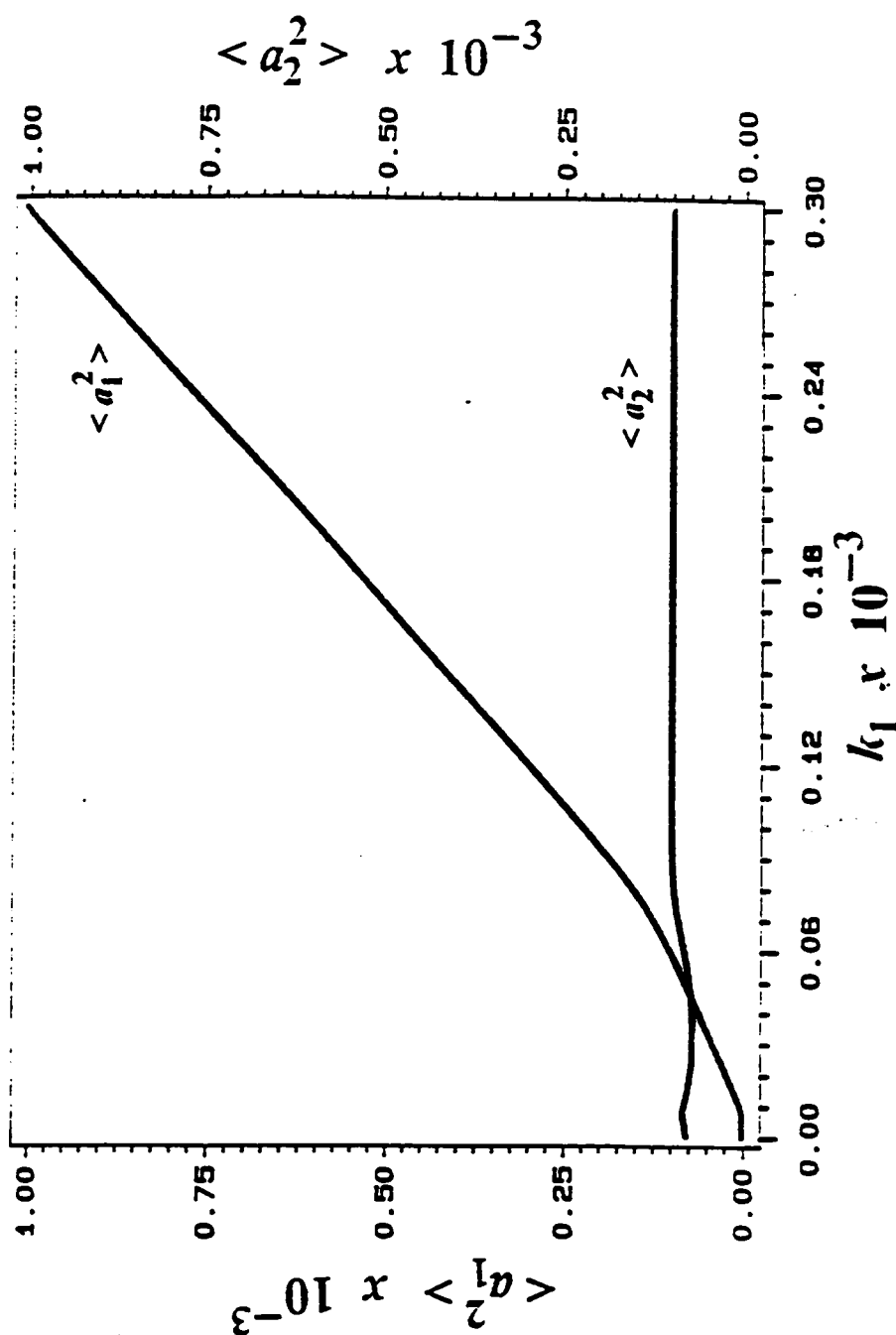


Figure 4.5. Modal mean-square-response amplitudes as functions of the amplitude of the deterministic excitation for $\mu_1 = 0.005$, $\mu_2 = 0.005$, $\Lambda_1 = 0.5$, $\Lambda_2 = 0.25$, $\nu_1 = 0.0$, $\nu_2 = 0.0$, and $\langle F^2 \rangle = 0.1 \times 10^{-5}$: — stable; --- unstable.

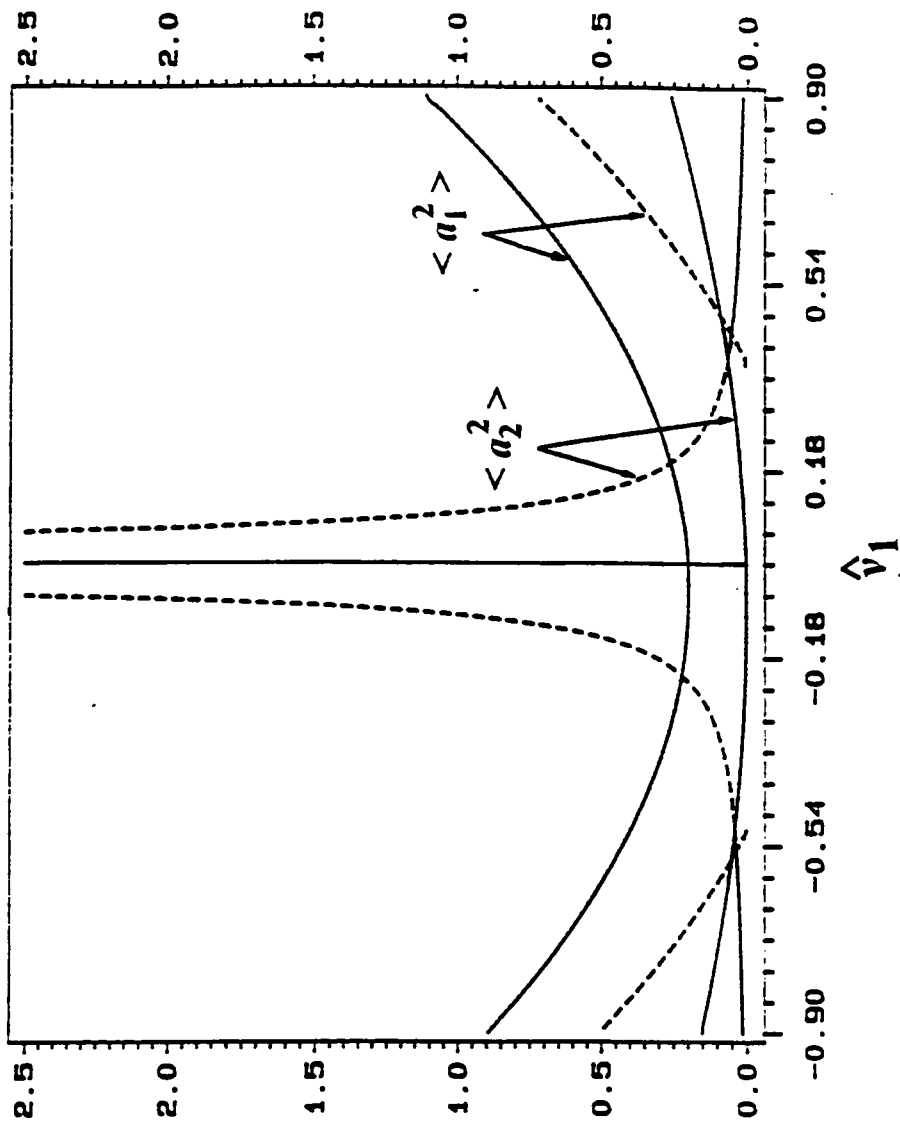


Figure 4.6. Modal mean-square-response amplitudes as functions of the autoparametric detuning parameter $\hat{\nu}_1$ for $\mu_1 = 0.02$, $\mu_2 = 0.02$, $\Lambda_1 = 1.0$, $\Lambda_2 = 0.5$, $\nu_2 = 0.12$, and $\langle F^2 \rangle = 0.1 \times 10^{-4}$: — stable; --- unstable.

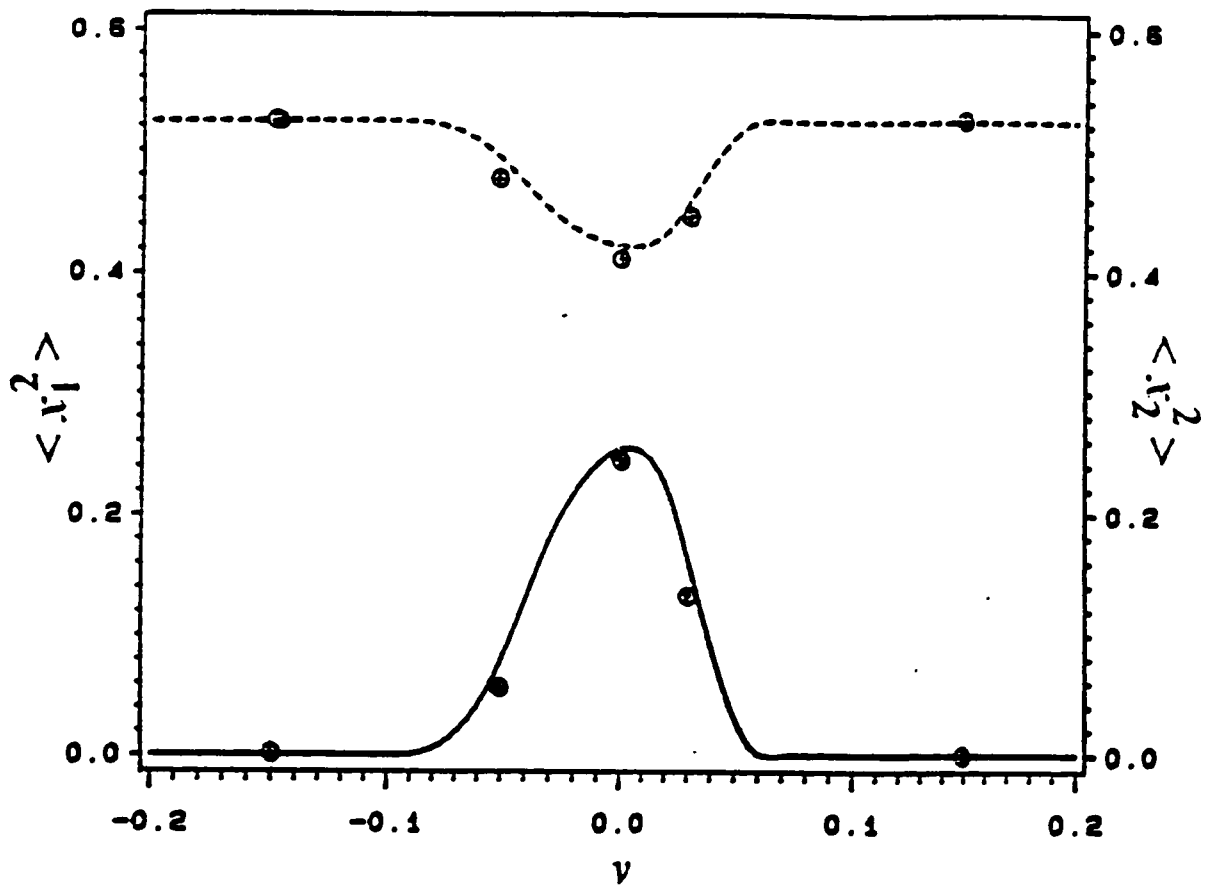


Figure 5.1. Dependence of the mean-square responses on the detuning parameter ν for the case of a two-to-one autoparametric resonance, $\omega_1 = 0.5$, $\mu_1 = 0.02$, $\mu_2 = 0.03$, $S_1 = 0.0$, $S_2 = 0.01$, $S_{12} = 0.0$, $\alpha_1 = 0.1$, and $\delta_2 = 0.08$: — $\langle x_1^2 \rangle$ and ---- $\langle x_2^2 \rangle$ predicted by the non-Gaussian closure solution; o predicted by the reduced non-Gaussian closure solution.

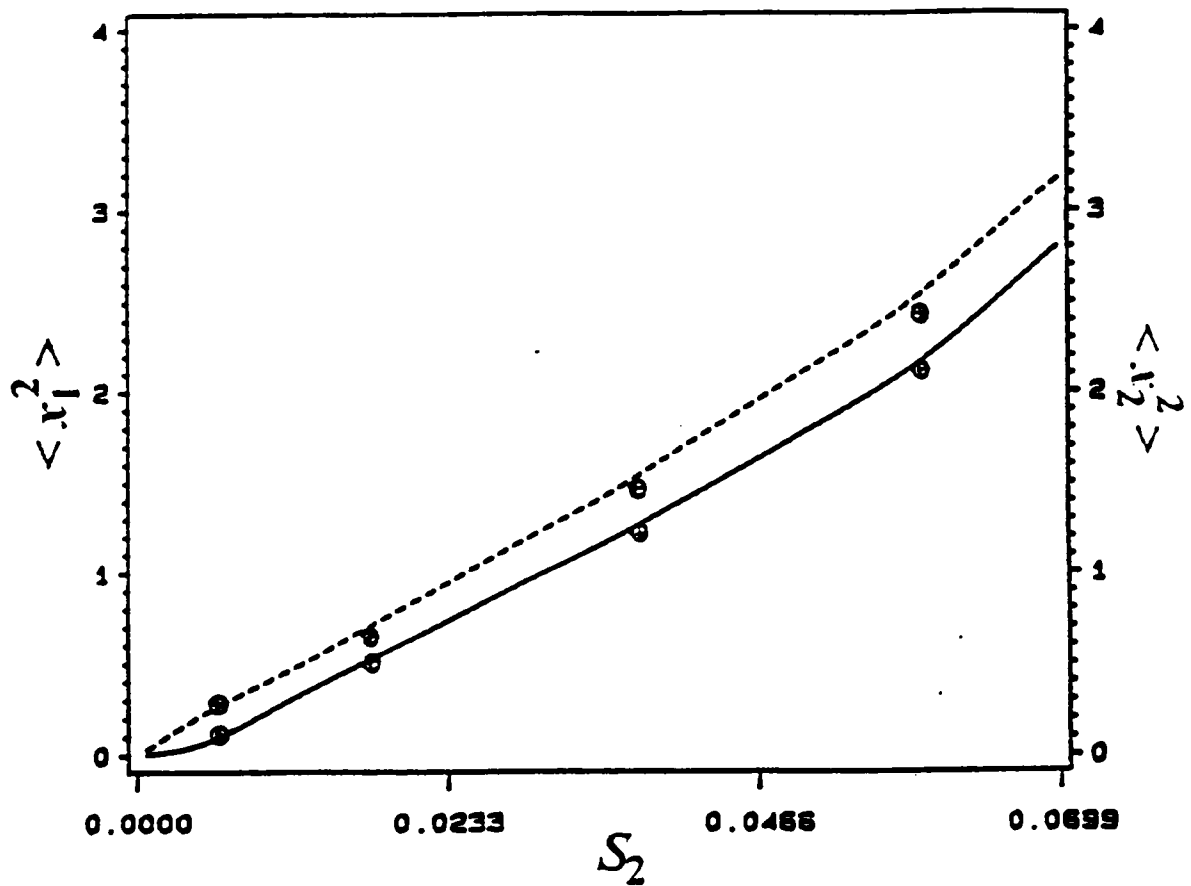


Figure 5.2. Variation of the mean-square responses with the excitation spectral amplitude S_2 for the case of a two-to-one autoparametric resonance, $\omega_1 = 0.5$, $\omega_2 = 1.0$, $\mu_1 = 0.02$, $\mu_2 = 0.03$, $S_1 = 0.0$, $S_{12} = 0.0$, $\alpha_1 = 0.1$, and $\delta_2 = 0.08$: — $\langle x_1^2 \rangle$ and ---- $\langle x_2^2 \rangle$ predicted by the non-Gaussian closure solution; o predicted by the reduced non-Gaussian closure solution.

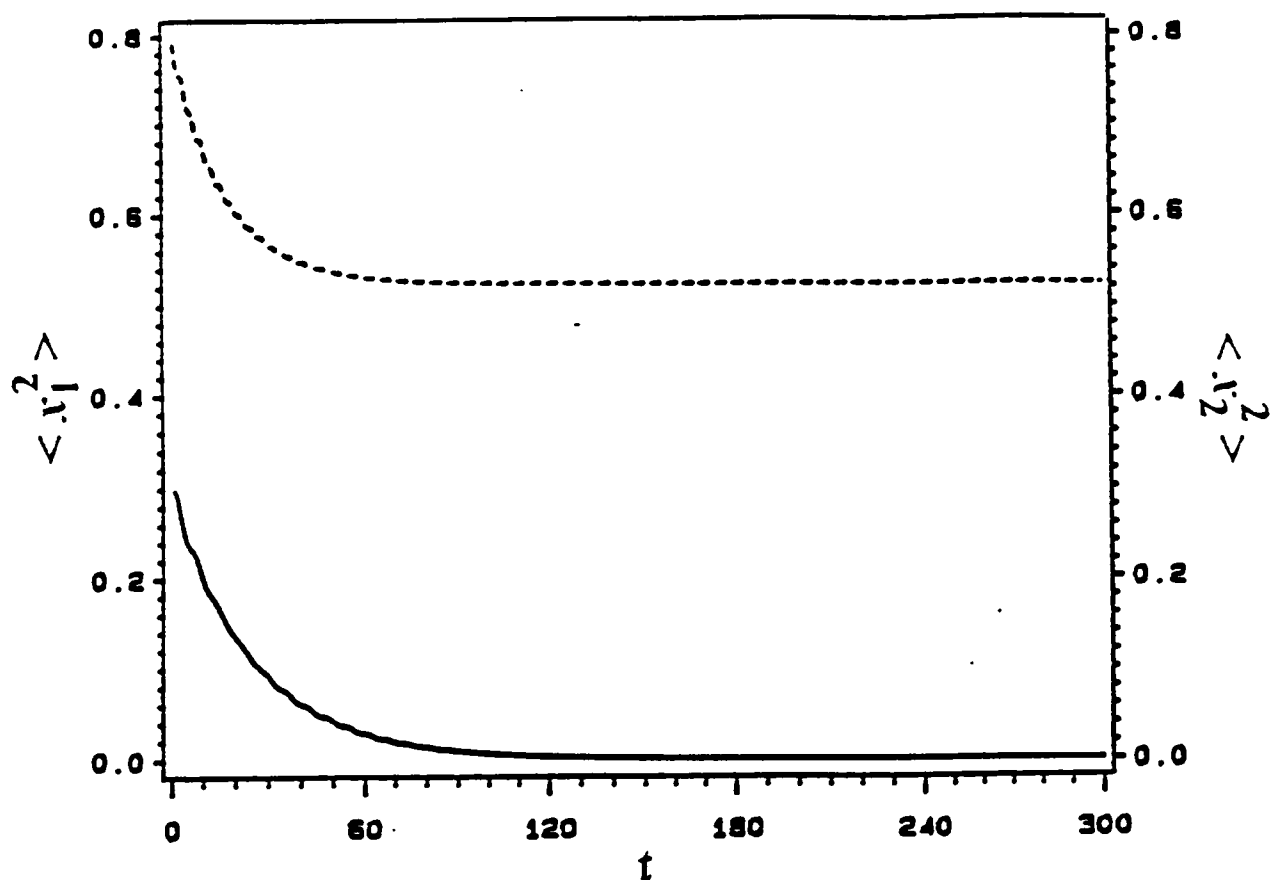


Figure 5.3. Time history of the mean-square responses predicted by using the Gaussian closure scheme for the case of a two-to-one autoparametric resonance, $\omega_1 = 0.5$, $\omega_2 = 1.0$, $\nu = 0.0$, $u_1 = 0.02$, $u_2 = 0.03$, $S_1 = 0.0$, $S_2 = 0.01$, $S_{12} = 0.0$, $\alpha_1 = 0.1$, and $\delta_2 = 0.08$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$.

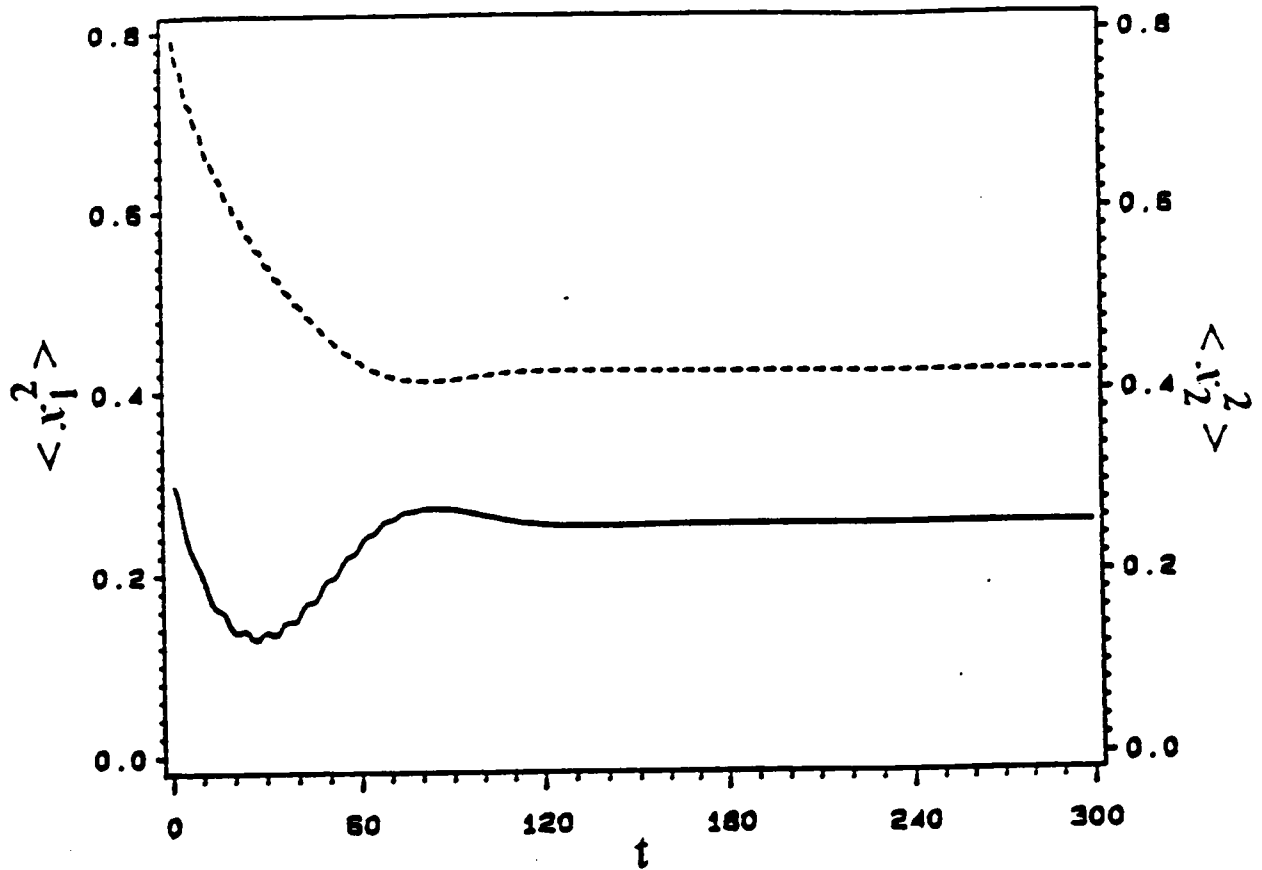


Figure 5.4. Time history of the mean-square responses predicted by using the non-Gaussian closure scheme for the case of a two-to-one autoparametric resonance, $\omega_1 = 0.5$, $\omega_2 = 1.0$, $\nu = 0.0$, $\mu_1 = 0.02$, $\mu_2 = 0.03$, $S_1 = 0.0$, $S_2 = 0.01$, $S_{12} = 0.0$, $\alpha_1 = 0.1$, and $\delta_2 = 0.08$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$.

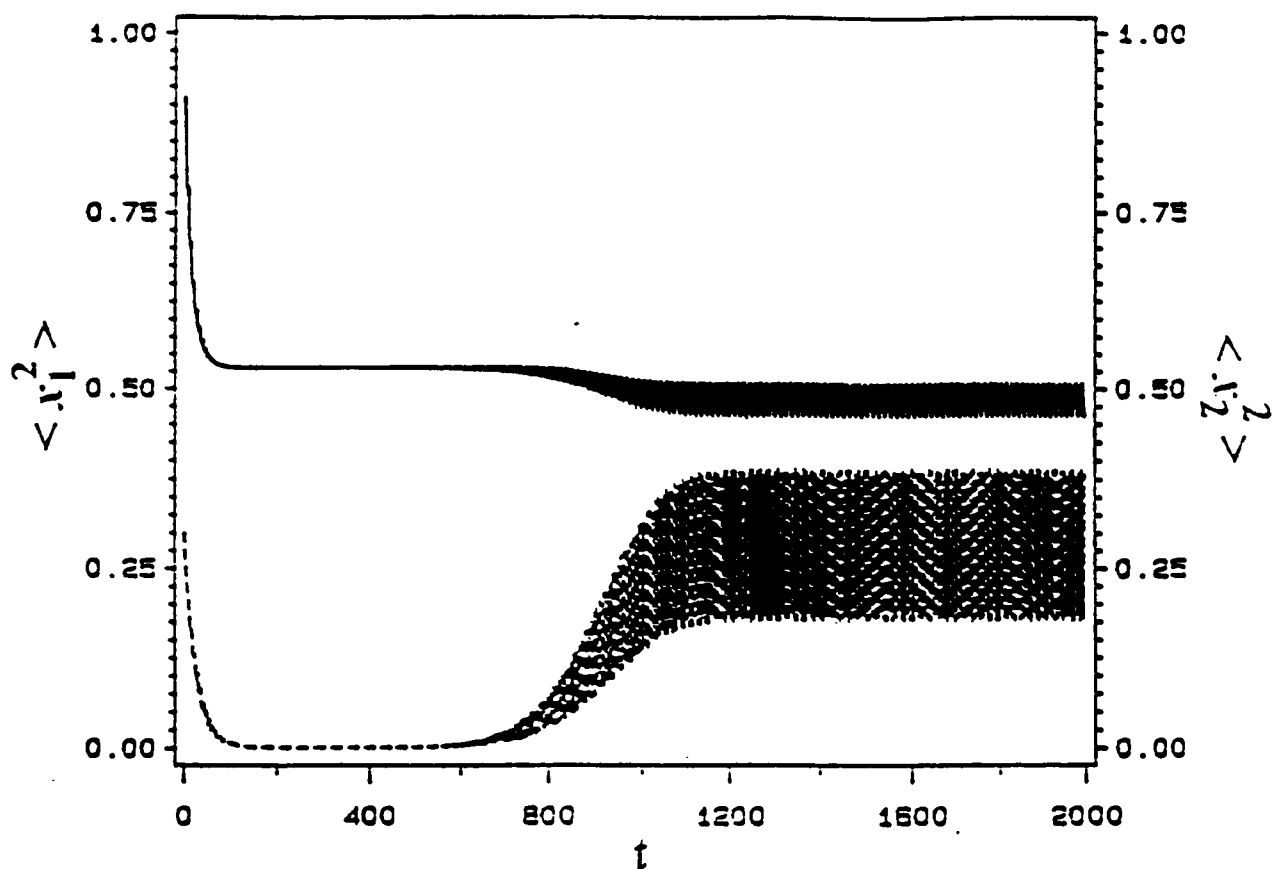


Figure 5.5. Time history of the mean-square responses predicted by using the Gaussian closure scheme for the case of a two-to-one autoparametric resonance, $\nu = 0$, $u_1 = 0.03$, $u_2 = 0.02$, $S_1 = 0.01$, $S_2 = 0$, $S_{12} = 0$, and $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \delta_6 = \alpha_2 = \alpha_3 = 0.05$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$.

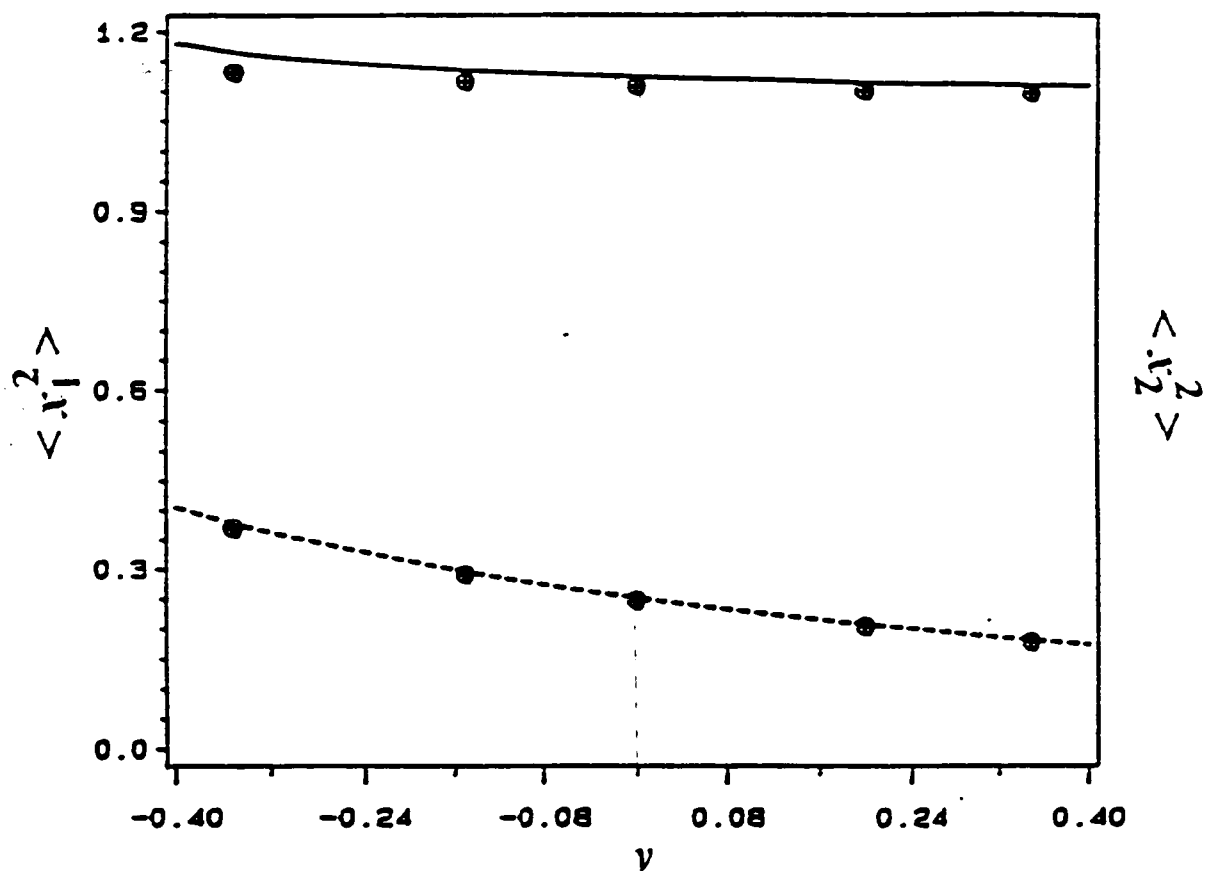


Figure 5.6. Dependence of the mean-square response of a shallow arch on the autoparametric detuning parameter ν , $\omega_1 = 1.0$, $\mu_1 = 0.08$, $\mu_2 = 0.05$, $S_1 = 0.05$, $S_2 = 0.0313$, $S_{12} = 0$, and $\varepsilon = 0.1$: — $\langle x_1^2 \rangle$ and ---- $\langle x_2^2 \rangle$ predicted by using the exact stationary solution of the FPK equation; o predicted by using the non-Gaussian closure solution.

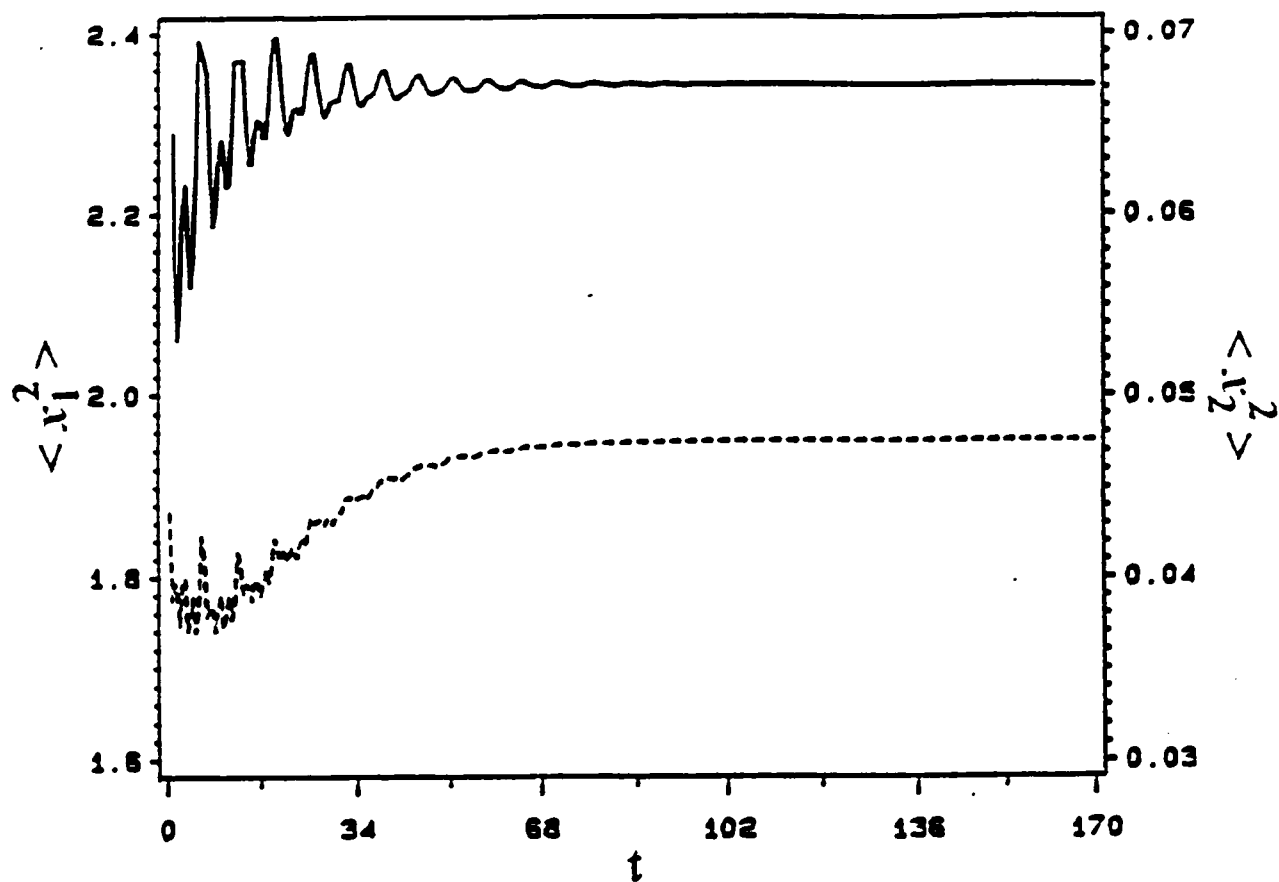


Figure 5.7. Time history of the mean-square response of a shallow arch predicted by using the non-Gaussian closure scheme $\omega_1 = 1.0$, $\nu = 0$, $u_1 = 0.04$, $u_2 = 0.03$, $S_1 = 0.06$, $S_2 = 0$, $S_{12} = 0$, and $\epsilon = 0.05$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$.

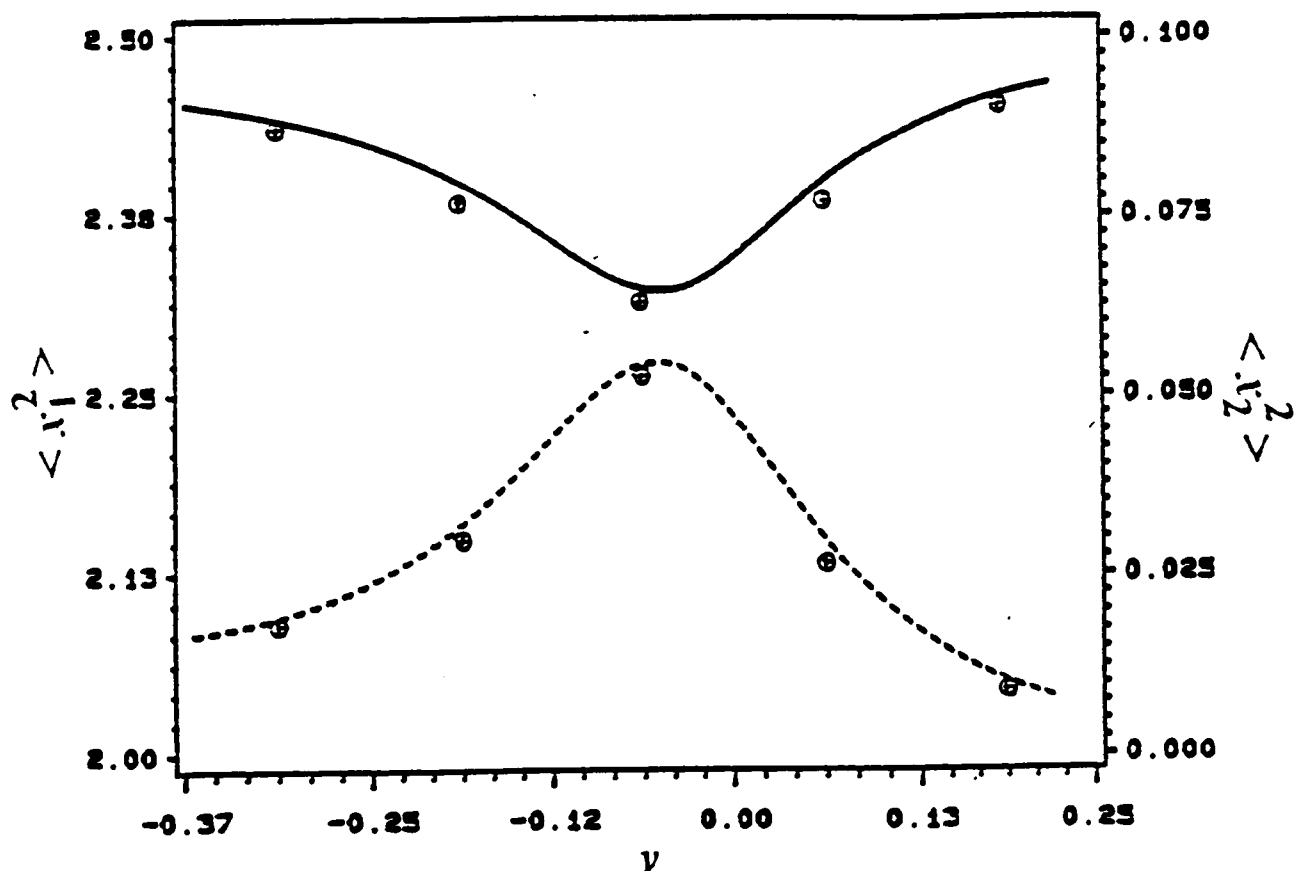


Figure 5.8. Variation of the mean-square response of a shallow arch with the autoparametric detuning parameter ν , $\omega_1 = 1.0$, $\mu_1 = 0.04$, $\mu_2 = 0.03$, $S_1 = 0.06$, $S_2 = 0$, $S_{12} = 0$, and $\epsilon = 0.05$: — $\langle x_1^2 \rangle$ and ---- $\langle x_2^2 \rangle$ predicted by using the non-Gaussian closure solution; o predicted by using the reduced non-Gaussian closure solution.

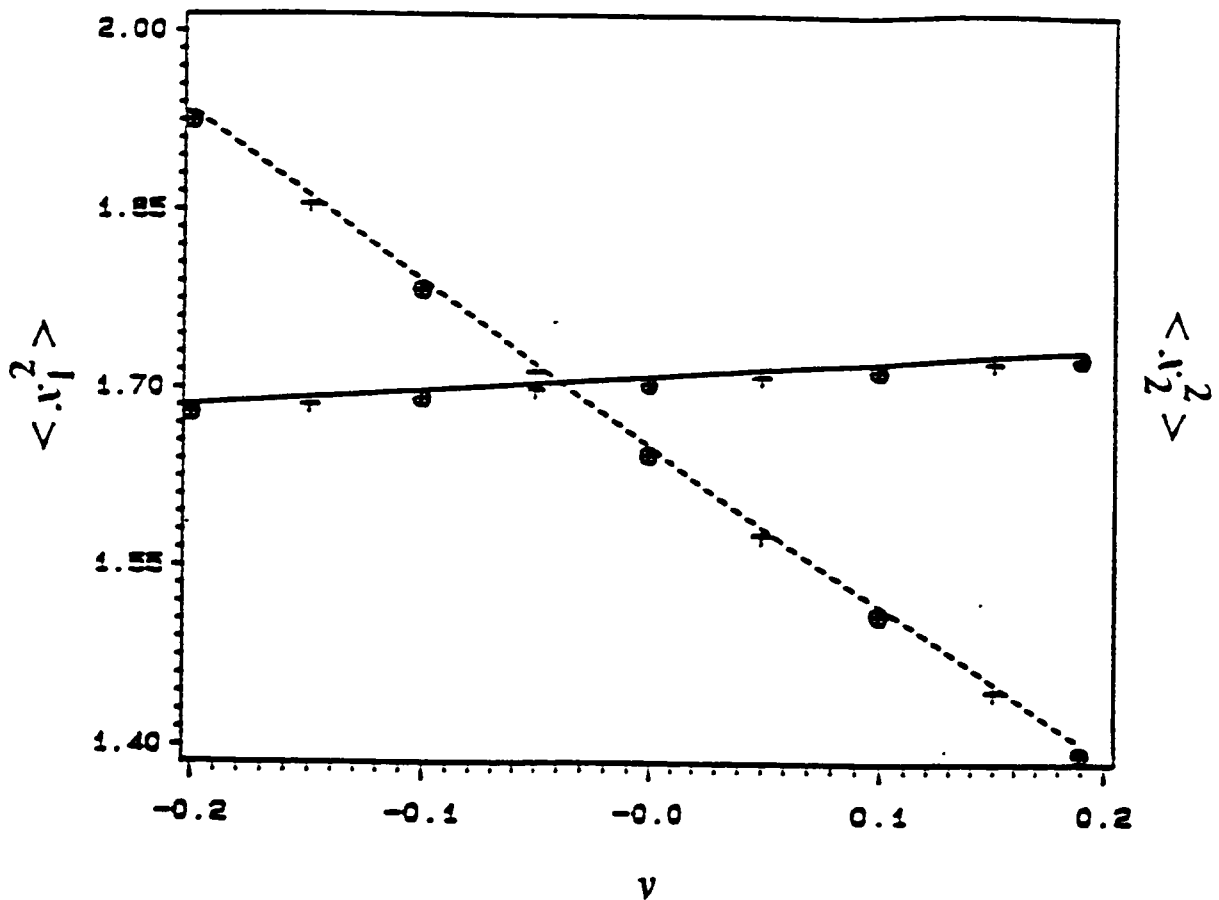


Figure 5.9. Comparison of the variation of the mean-square responses with the detuning parameter ν for the case of a one-to-one autoparametric resonance. Results are obtained by using different methods for $\omega_1 = 1.0$, $\mu_1 = 0.02$, $\mu_2 = 0.01$, $S_1 = 0.04$, $S_2 = 0.02$, $S_{12} = 0$, $\beta_{11} = 0.14$, $\beta_{12} = 0.16$, $\beta_{21} = 0.15$, and $\beta_{22} = 0.15$: — $\langle x_1^2 \rangle$ exact solution; ---- $\langle x_2^2 \rangle$ exact solution; o non-Gaussian closure solution; + reduced non-Gaussian closure solution.

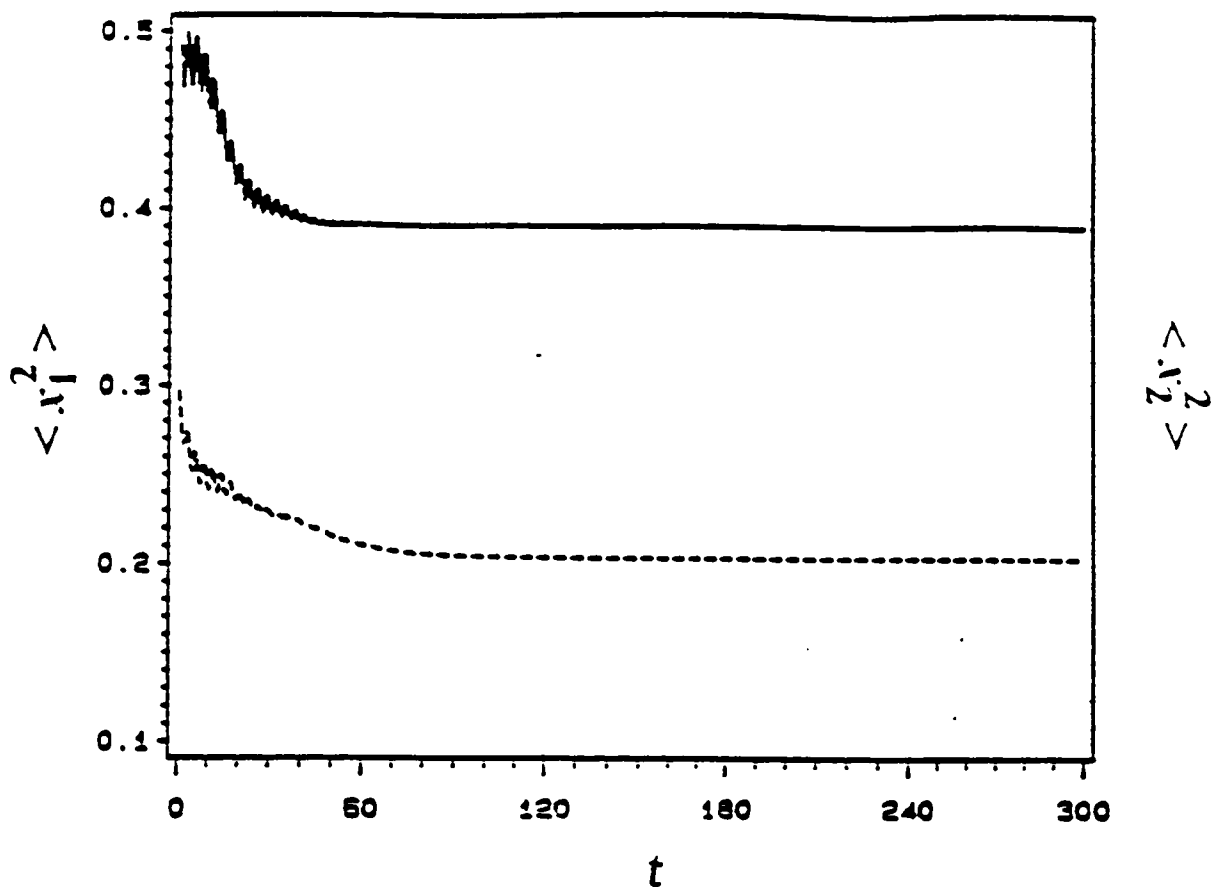


Figure 5.10. Time history of the mean-square responses for the case of a one-to-one autoparametric resonance predicted by the non-Gaussian closure scheme for $\omega_1 = 1.0$, $\nu = 0.0$, $\mu_1 = 0.03$, $\mu_2 = 0.02$, $S_1 = 0.01$, $S_2 = 0.0$, $S_{12} = 0.0$, and $\beta_{11} = \beta_{12} = 0.05$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$.

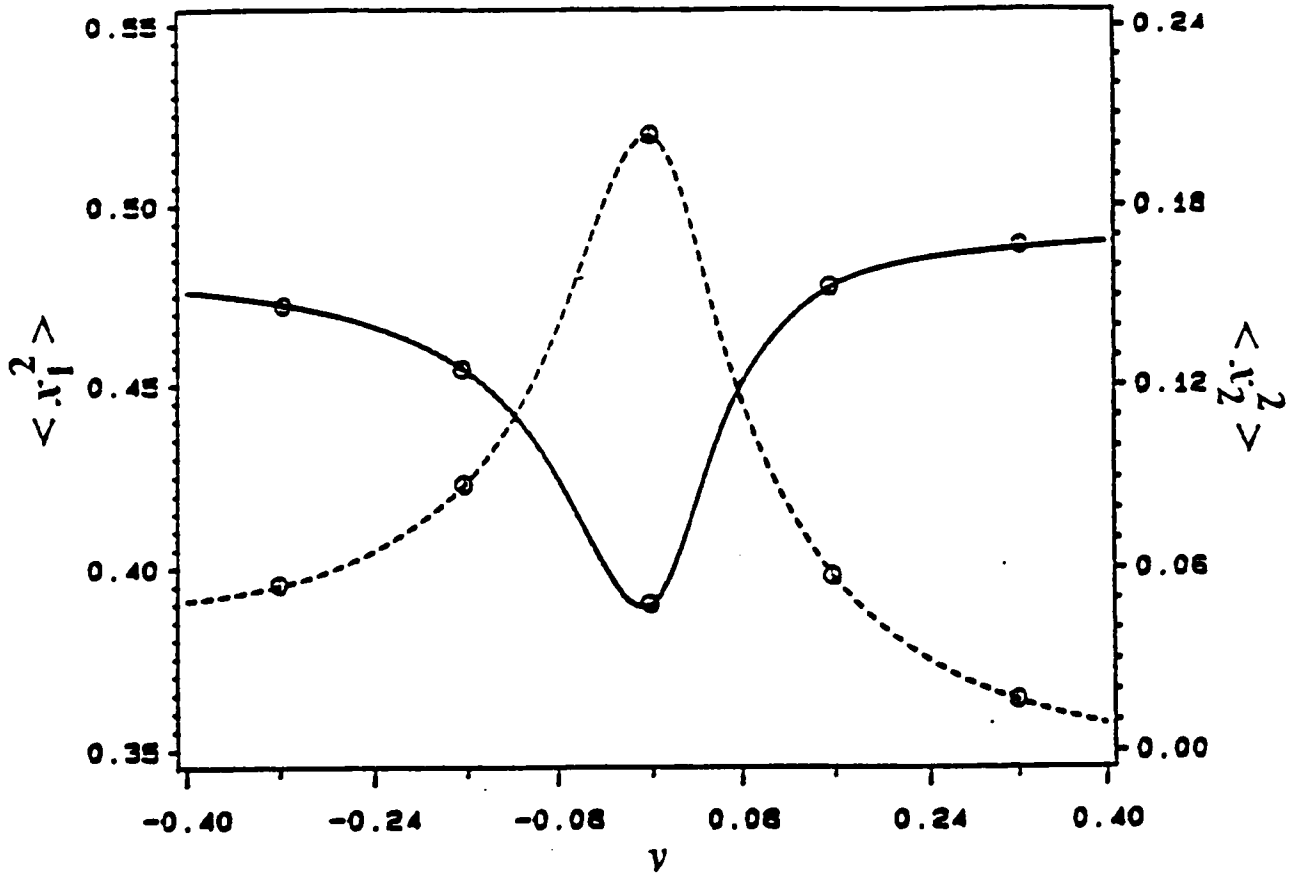


Figure 5.11. Dependence of the mean-square responses predicted by using the non-Gaussian and reduced non-Gaussian closure schemes for the case of a one-to-one autoparametric resonance on the detuning parameter ν when $\omega_1 = 1.0$, $\mu_1 = 0.03$, $\mu_2 = 0.02$, $S_1 = 0.01$, $S_2 = 0.0$, $S_{12} = 0.0$, and $\beta_{i1} = \beta_{i2} = 0.05$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$; o reduced non-Gaussian closure solution.

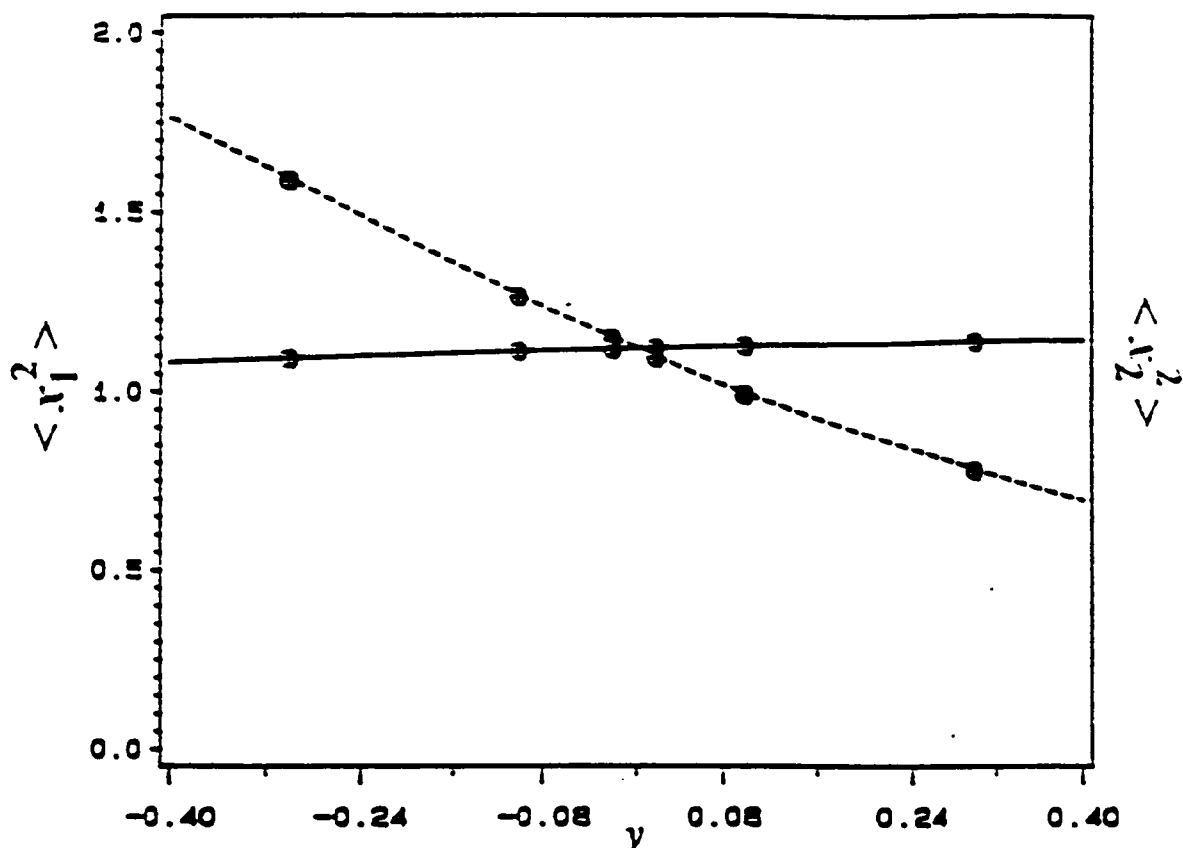


Figure 5.12. Dependence of the mean-square response of a string on the autoparametric detuning parameter ν , $\omega_1 = 1.0$, $S_1 = 0.03$, $S_2 = 0.03$, $S_{12} = 0$, $\mu_1 = 0.03$, $\mu_2 = 0.03$, and $r = 0.1$:
 — $\langle x_1^2 \rangle$ and ---- $\langle x_2^2 \rangle$ predicted by using the exact stationary solution of the FPK equation; o predicted by using the non-Gaussian closure solution.

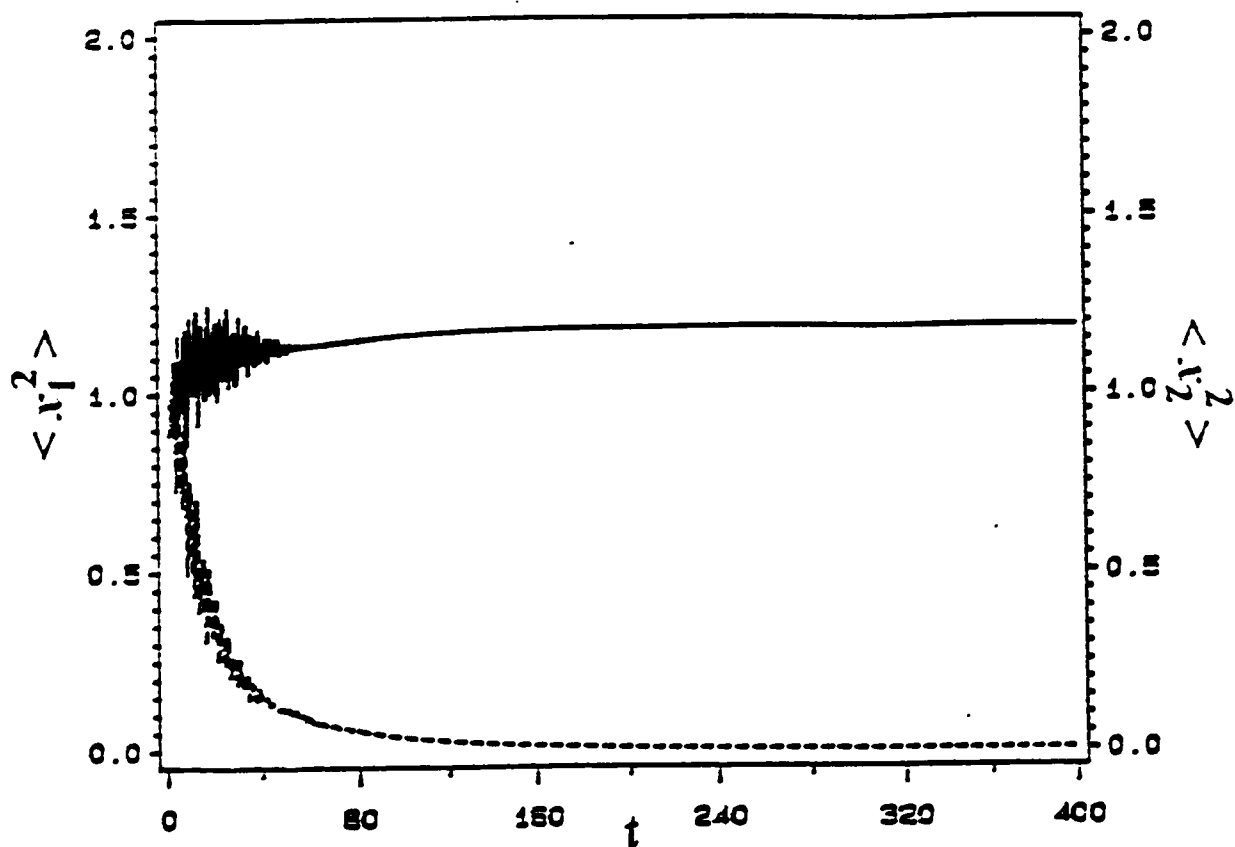


Figure 5.13. Time history of the mean-square response of a string predicted by using the non-Gaussian closure scheme, $\omega_1 = 1.0$, $\nu = 0$, $S_1 = 0.03$, $S_2 = 0$, $S_{1,2} = 0$, $u_1 = 0.03$, $u_2 = 0.03$ and $\tau = 0.1$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$.

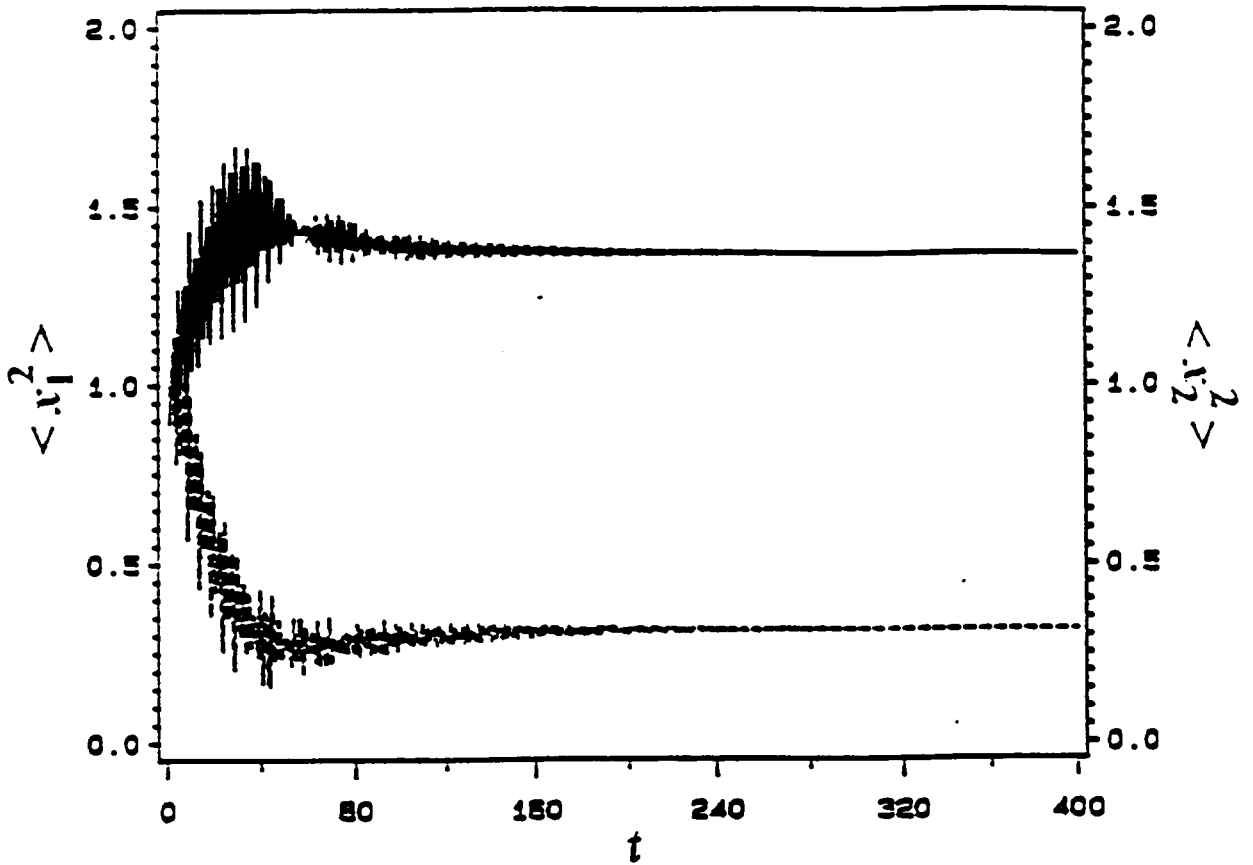


Figure 5.14. Time history of the mean-square response of a string predicted by using the non-Gaussian closure scheme, $\omega_1 = 1.0$, $\nu = 0$, $S_1 = 0.03$, $S_2 = 0$, $S_{12} = 0$, $u_1 = 0.02$, $u_2 = 0.02$, and $r = 0.1$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$.

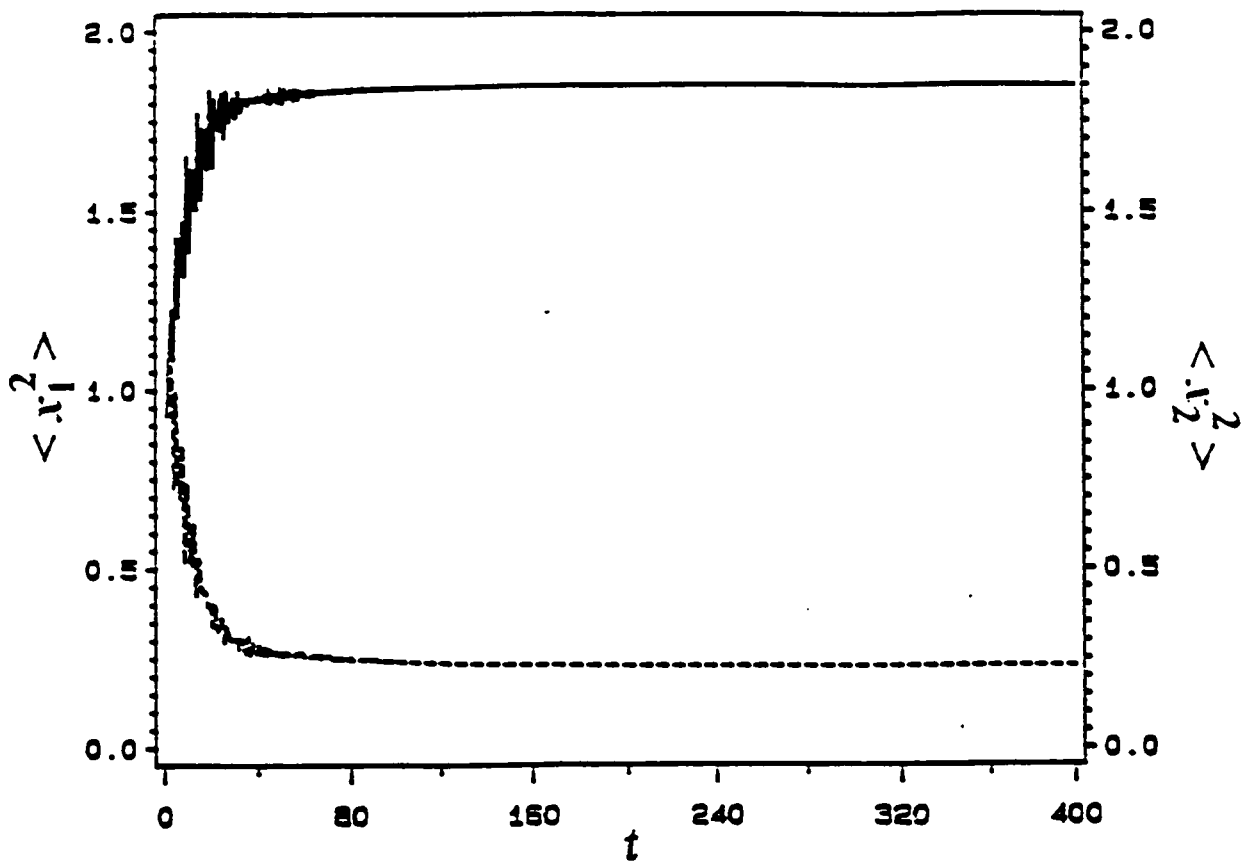


Figure 5.15. Time history of the mean-square response of a string predicted by using the non-Gaussian closure scheme, $\omega_1 = 1.0$, $\nu = 0$, $S_1 = 0.06$, $S_2 = 0$, $S_{12} = 0$, $\mu_1 = 0.03$, $\mu_2 = 0.03$, and $r = 0.1$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$.

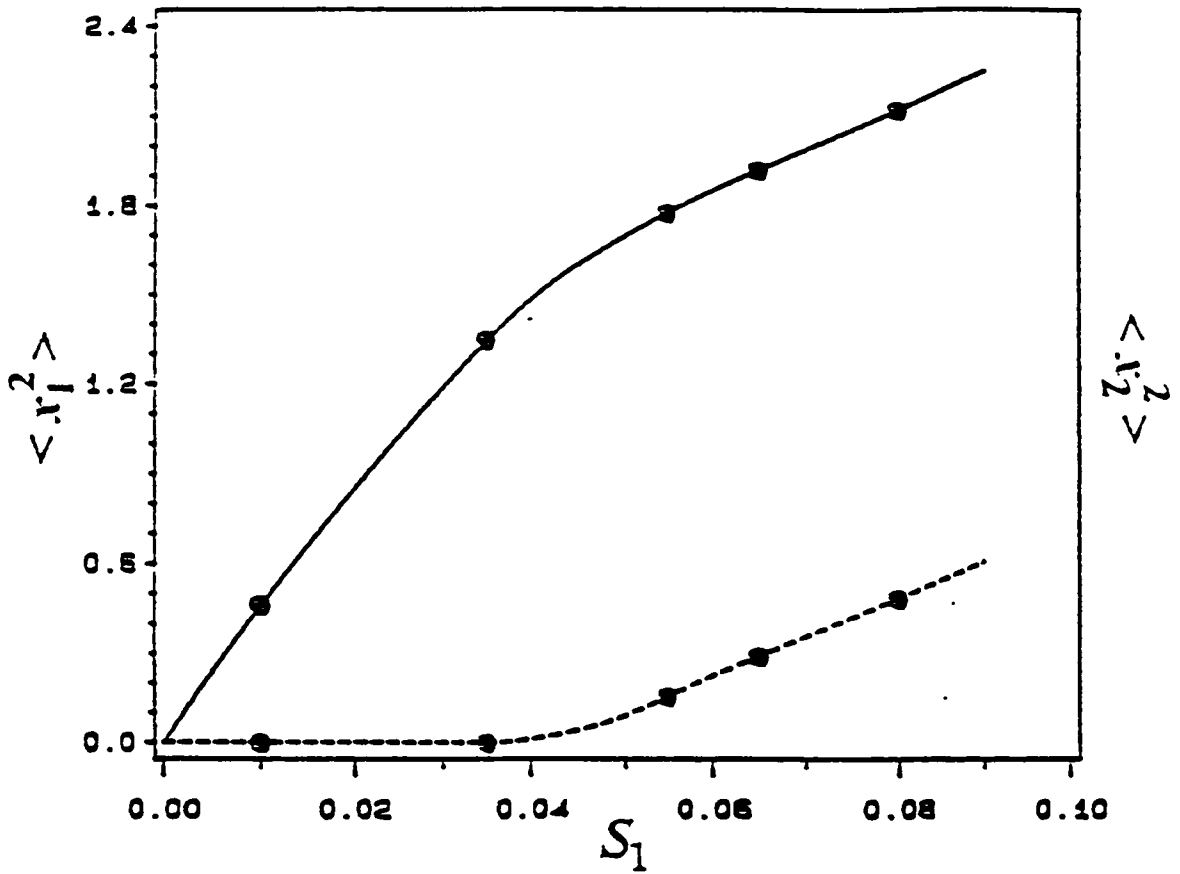


Figure 5.16. Variation of the mean-square response of a string with the excitation spectral amplitude S_1 , $\omega_1 = 1.0$, $\nu = 0$, $S_2 = 0$, $S_{1,2} = 0$, $\mu_1 = 0.03$, $\mu_2 = 0.03$, and $\tau = 0.1$:
 — $\langle x_1^2 \rangle$ and ---- $\langle x_2^2 \rangle$ predicted by using the non-Gaussian closure solution; o predicted by using the reduced non-Gaussian closure solution.

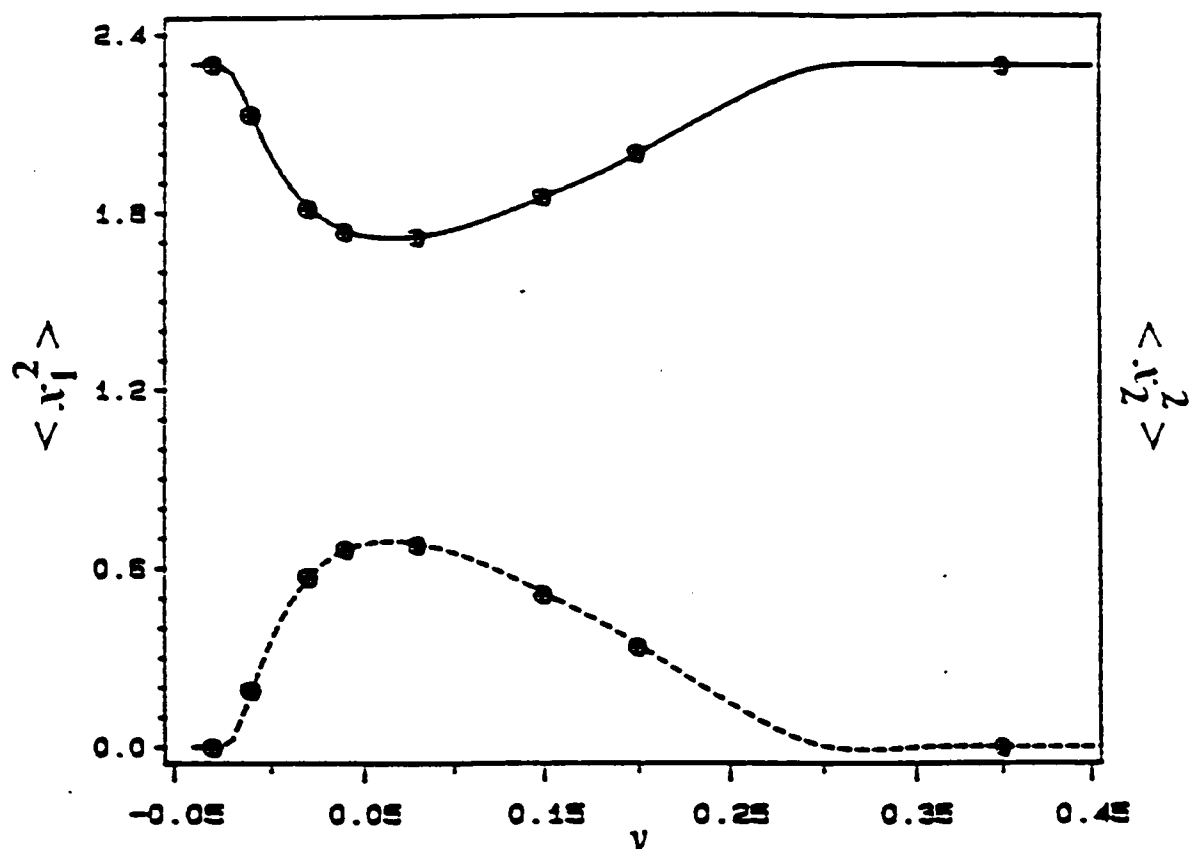


Figure 5.17. Dependence of the mean-square response of a string on the autoparametric detuning parameter ν , $\omega_1 = 1.0$, $S_1 = 0.07$, $S_2 = 0$, $S_{12} = 0$, $\mu_1 = 0.03$, $\mu_2 = 0.03$, and $\tau = 0.1$:
 — $\langle x_1^2 \rangle$ and ---- $\langle x_2^2 \rangle$ predicted by using the non-Gaussian closure solution; o predicted by using the reduced non-Gaussian closure solution.

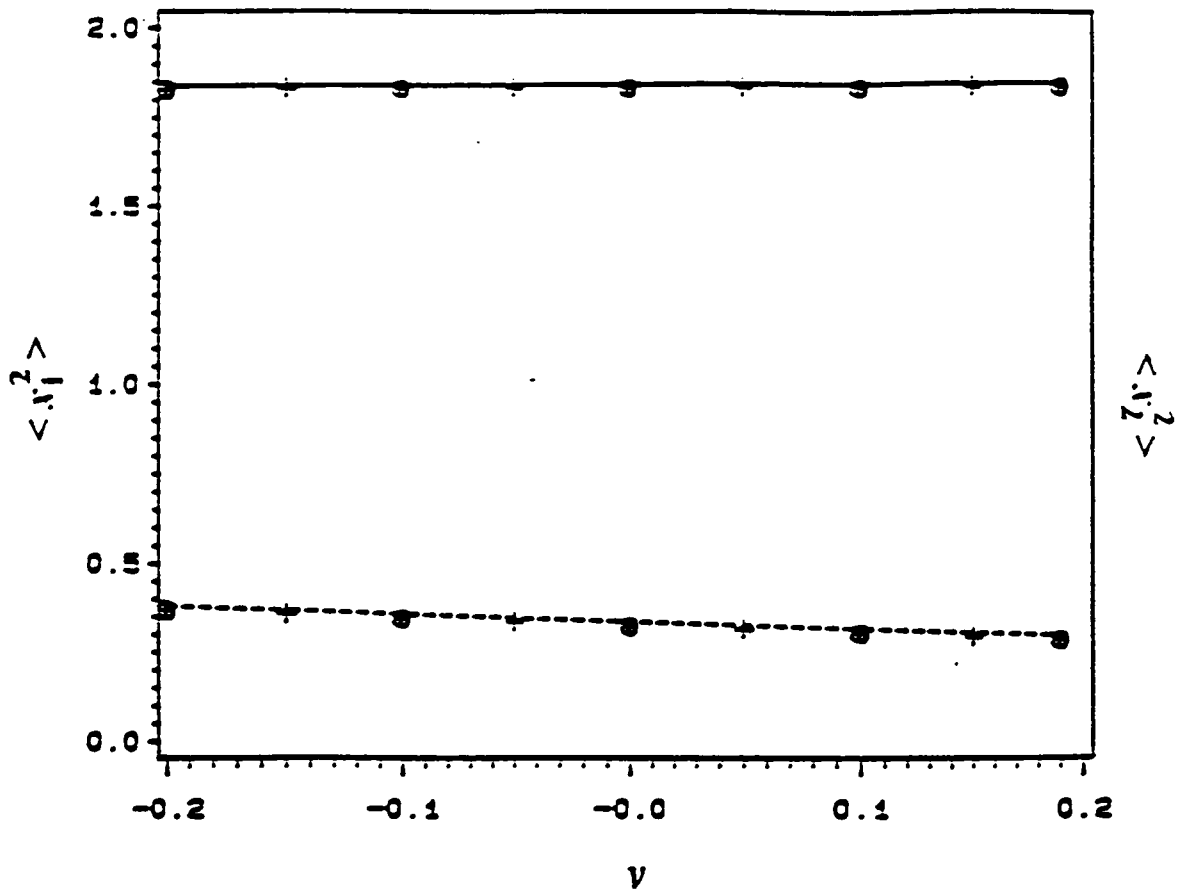


Figure 5.18. Comparison of the variation of the mean-square responses with the detuning parameter ν for the case of a three-to-one autoparametric resonance. Results are obtained by using different methods for $\omega_1 = 1.0$, $\mu_1 = 0.02$, $\mu_2 = 0.01$, $S_1 = 0.04$, $S_2 = 0.02$, $S_{12} = 0$, $a_{11} = 0.14$, $a_{12} = 0.16$, $a_{21} = 0.15$, and $a_{22} = 0.15$: — $\langle x_1^2 \rangle$ exact solution; ---- $\langle x_2^2 \rangle$ exact solution; o non-Gaussian closure solution; + reduced non-Gaussian closure solution.

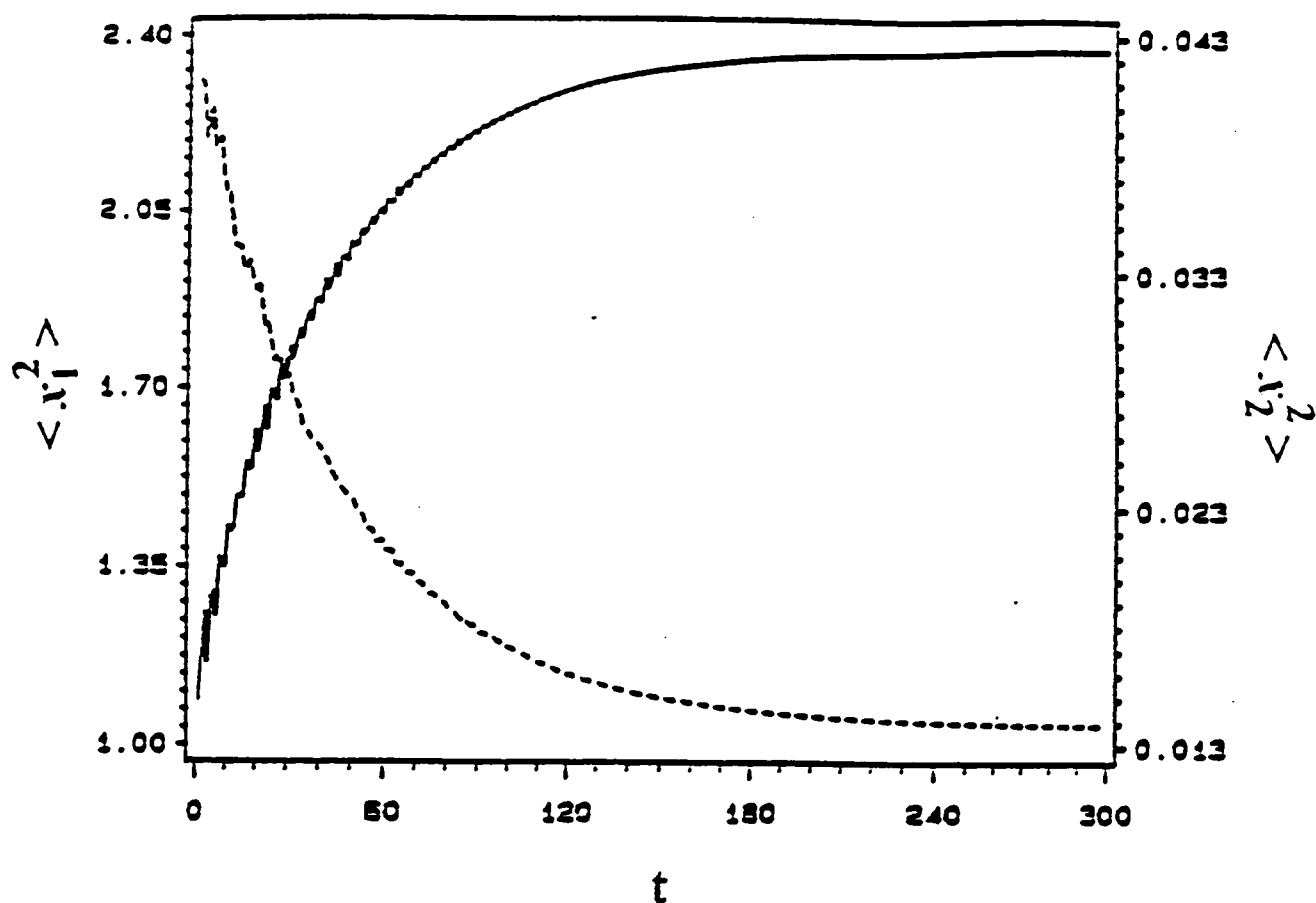


Figure 5.19. Time history of the mean-square responses predicted by using the non-Gaussian closure scheme for the case of a three-to-one autoparametric resonance, $\omega_1 = 1.0$, $\nu = 0.0$, $u_1 = 0.01$, $u_2 = 0.01$, $S_1 = 0.02$, $S_2 = 0.0$, $S_{12} = 0.0$, and $\beta_{11} = \beta_{12} = 0.05$: — $\langle x_1^2 \rangle$; ---- $\langle x_2^2 \rangle$.

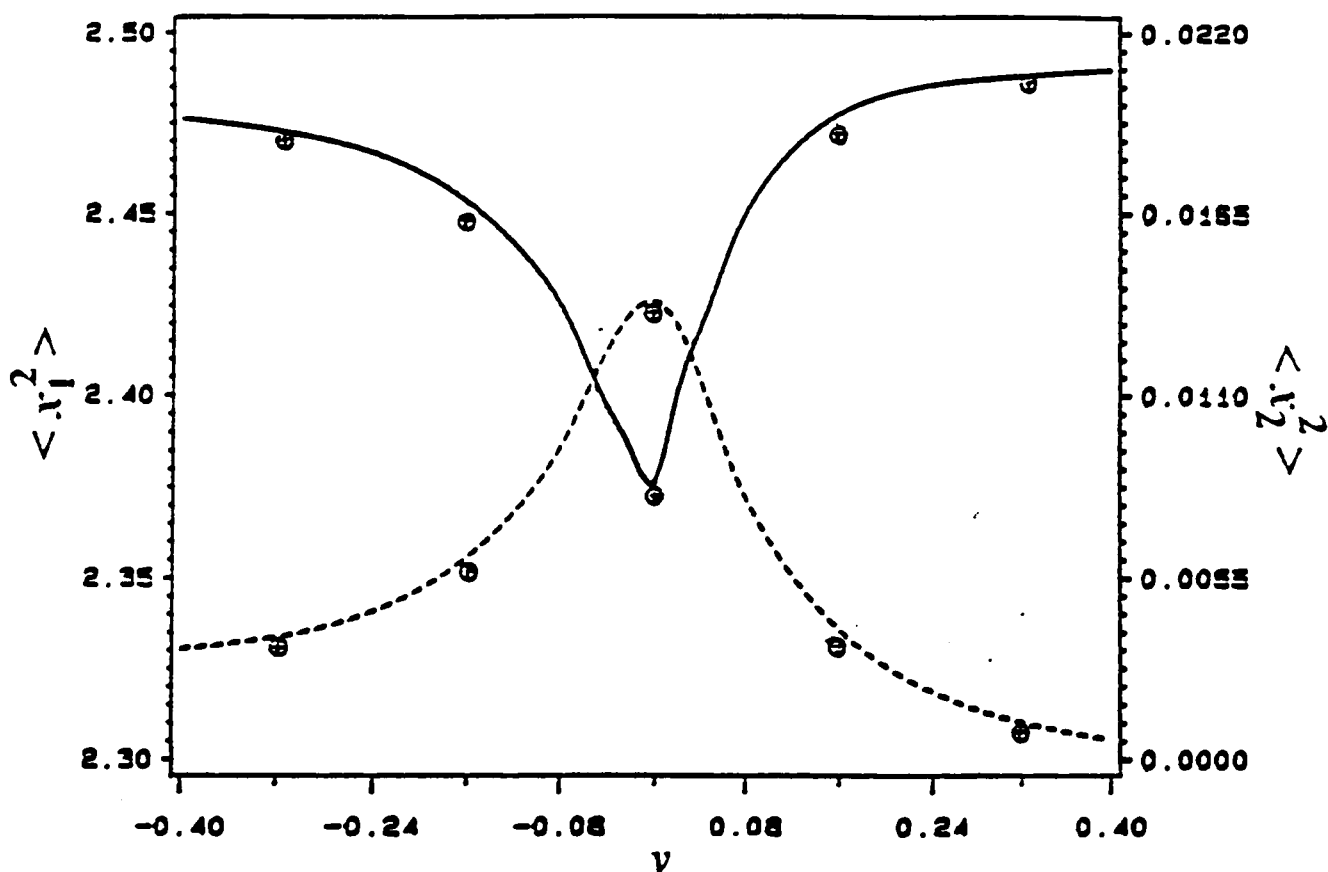


Figure 5.20. Dependence of the mean-square responses predicted by using the non-Gaussian and reduced non-Gaussian closure schemes for the case of a three-to-one autoparametric resonance on the detuning parameter ν when $\omega_1 = 1.0$, $u_1 = 0.01$, $u_2 = 0.01$, $S_1 = 0.02$, $S_2 = 0.0$, $S_{12} = 0.0$, and $s_{12} = 0.05$: — $\langle x_1^2 \rangle$; - - - $\langle x_2^2 \rangle$; o reduced non-Gaussian closure solution.

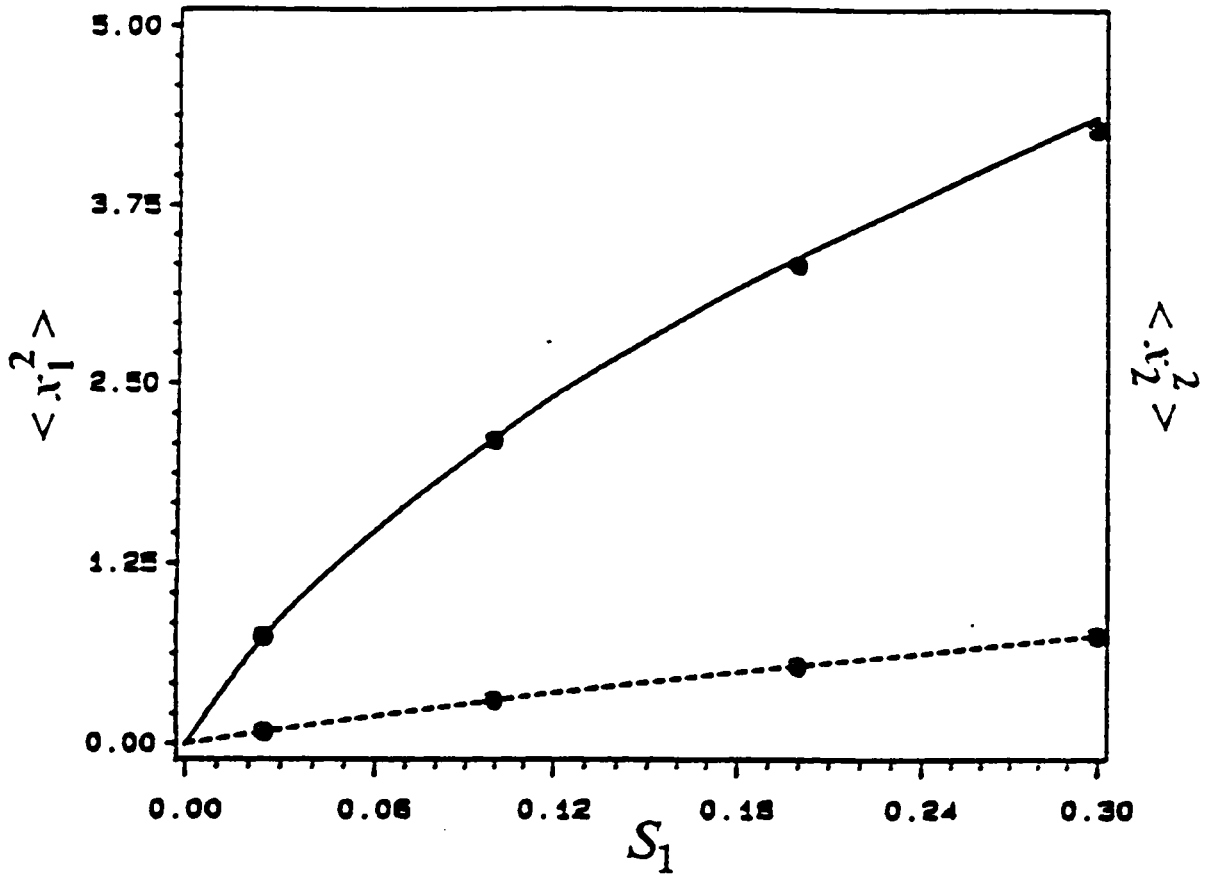


Figure 5.21. Dependence of the mean-square response of a hinged-clamped beam on the excitation spectral amplitude S_1 , $\omega_1 = 1.0$, $\mu_1 = 0.04$, $\mu_2 = 0.02$, $\nu = 0.21$, $S_{12} = 0$ and $S_2 = (\mu_2/\mu_1) S_1$; — $\langle \dot{x}_1^2 \rangle$ and ---- $\langle \dot{x}_2^2 \rangle$ predicted by using the exact stationary solution of the FPK equation; o predicted by using the non-Gaussian closure solution.

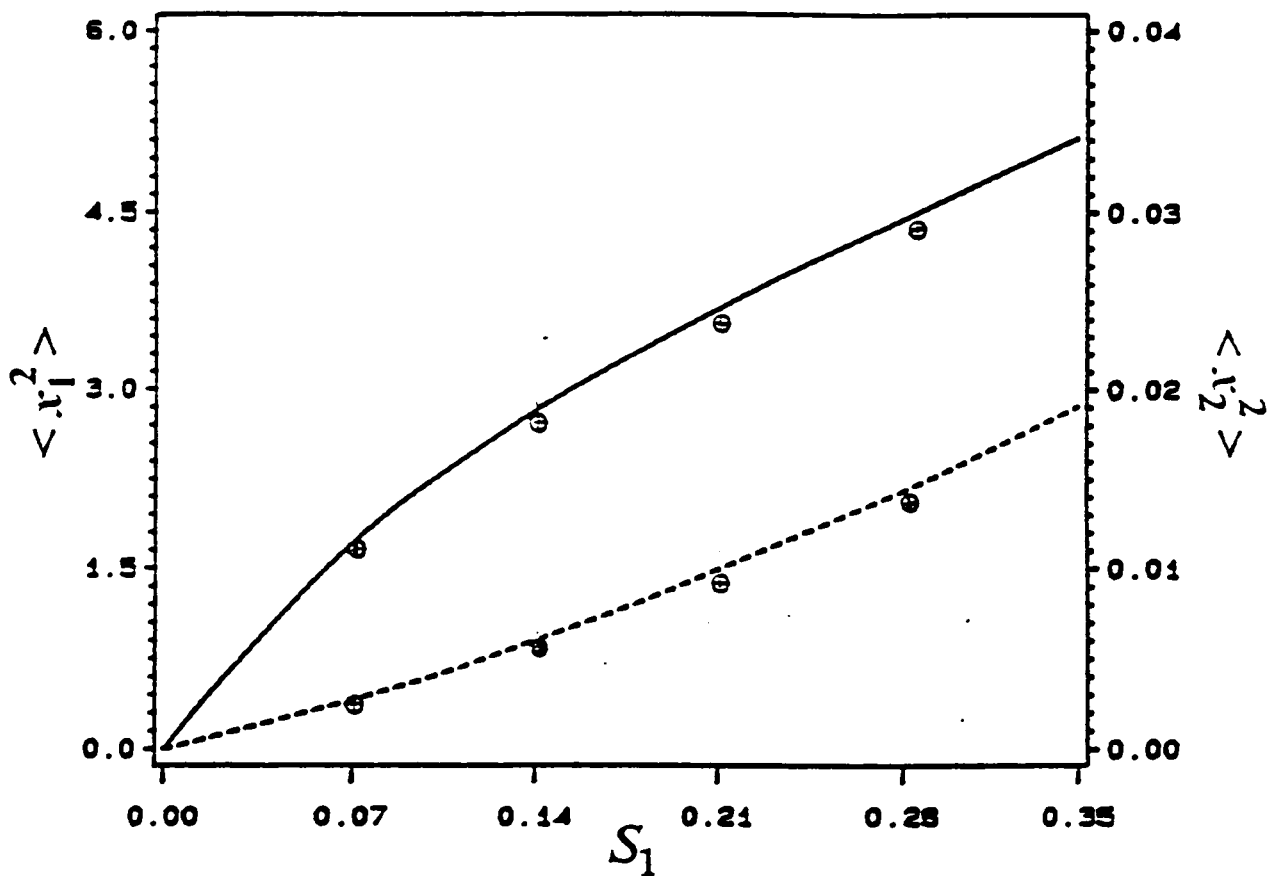


Figure 5.22. Variation of the mean-square response of a hinged-clamped beam with the excitation spectral amplitude S_1 ,
 $\nu = 0.21$, $\mu_1 = 0.04$, $\mu_2 = 0.02$, $S_2 = 0$ and $S_{12} = 0$:
 — $\langle x_1^2 \rangle$ and ---- $\langle x_2^2 \rangle$ predicted by using the non-Gaussian closure solution; o predicted by using the reduced non-Gaussian closure solution.

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