AN INTEGRAL EQUATION APPROACH TO VIBRATING PLATES

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IV. NOMENCLATURE

a, b, c	plate dimensions measured in plane of plate
A_m	the trace of the mth iterated kernel
dA	differential element of area of plate
D	plate stiffness $\left(= \frac{Eh^3}{12} \Big/ 1-\nu^2\right)$
e	eccentricity of ellipse
E	modulus of elasticity
$G(x,y;r,s)$	Green's function
$K(x,y;r,s)$	kernel of the integral equation
$K_m(x,y;r,s)$	the mth iterated kernel of the integral equation
m, i, k, j	indices of summation
N	norm of eigenfunction
p	circular frequency of natural free vibration
P	concentrated normal force
(r,s)	rectangular coordinates of normal concentrated force
r, θ	polar coordinates in the plate
$t(r,s)$	plate thickness function
$w(x,y)$	static deflection function of the plate
x, y	rectangular coordinates in the plate
α	bounding curve for the elliptical plate
β, δ	elliptical coordinates for the concentrated normal force
$\gamma(r,s)$	plate density function
λ	a parameter of the integral equation

ν	Poisson's ratio
ξ, η	elliptical coordinates in the plate
ρ, ψ	polar coordinates of normal concentrated force
$\phi(x,y)$	solution of the integral equation

V. INTRODUCTION

In the analysis of vibration problems the knowledge of the natural frequencies, and in particular the fundamental frequency, is of paramount importance. In many cases the mode shapes are required, but if any single item is needed for proper analysis it is the spectrum of frequencies. Only through a proper knowledge of these frequencies can an effective design be made which will prevent critical conditions of heavy vibration from occurring.

In the more simple cases the governing differential equation can be solved directly giving a frequency equation for the determination of the natural frequencies. However, as the problem becomes more involved one is faced with the necessity of going to approximate methods for a solution. Probably the best known is the energy method of Rayleigh-Ritz or its alternate form suggested by Galerkin. A minimization process similar to Rayleigh's method by using Lagrange multipliers has been found effective in some vibration problems. The governing differential equation has been attacked directly using finite difference, collocation and iterative procedures. All of these methods possess their own unique advantages and disadvantages. While the energy methods do give bounds to the frequency the requirement that the geometric boundary conditions be satisfied by the assumed mode shape hampers the effectiveness of the method. Finite differences require a prechosen grid size.

Collocation can approximate poorly for a small number of terms in the assumed deflection function, whereas iterative procedures can be quite lengthy. The purpose of this dissertation is to provide the engineer with still another method for the approximate determination of frequencies -- the method of integral equations. In particular, the application is made to the plate vibration problem.

Undoubtedly, the integral equation approach has its own unique disadvantages not the least of which may be the solution of the isolated force static problem. On the other hand this might be less of a real disadvantage than it would seem on the surface, inasmuch as the static solution has value in its own right and is, in general, part of the design analysis anyway. The major advantage is that the integral equation approach does not require that the approximating functions satisfy the geometric boundary conditions. Once the integral equation is formulated the satisfaction of the boundary conditions is inherent in the equation itself. No little success has been achieved using this approach in the study of the vibration of bars* although the method has not become wide-

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*A.S.Melyakovetskii,"Integral equation of free vibration of a curved bar",Dokladi Akademii Nauk SSSR, July '52.
J.L.Bogdanoff & J.E.Goldberg,"Application of Volterra Linear Integral Equations of Numerical Solution of Beam Vibration Problems",Univ. of Illinois, April 1953.
L.Kraus,"Integral Equations of Thin Walled Beams with Open Warp-Free Sections"-Ingineur-Arch.,June 1960.
W.T.White,"On the Integral Equation Approach to Problems of Vibrating Beams", Journal of the Franklin Institute, 1948.

spread in this country. The main body of mathematical literature, merely calls attention to the correctness of extending the one dimensional theory to dimensions of higher order but does nothing by way of application. It is hoped that this dissertation may partly fill a gap in the literature of vibration analysis by extending and applying a powerful tool of analysis to the thin plate vibration problem.

The understanding of integral equations, for an approach to vibration analysis, is made simpler by the fact that the nature of the vibration problem is such that the resulting governing integral equation is of a mathematically elementary kind. Integral equations attracted attention as an analytical tool about the turn of the twentieth century when Vito Volterra, Ivar Fredholm, David Hilbert and Erhard Schmidt within the space of a few short years, concentrated their attention on the investigation and development of the theory of functional equations which contained the unknown function not only explicitly but also under the integral sign. It was during the 1930's and later that N.I. Muskhelishvili, S.G. Mikhlin, and D.I. Sherman concentrated their efforts on the application of integral equation methods to elasticity problems. V.D. Kupradze applied integral equations in electromagnetic studies and N.E. Kochin used integral equation methods in his studies in fluid mechanics. During the last ten years the literature of application of integral equations to engineering problems has filled out considerably but as was mentioned previously the bulk of application has been for those problems which are mathematically governed by the one-dimensional, linear integral equation.

VI. MATHEMATICS OF SYMMETRIC INTEGRAL EQUATIONS

In this chapter that part of the mathematics of integral equations which is pertinent to the thin plate vibration problem will be discussed. As will be shown in the next chapter the integral equation which governs thin plate vibrations is of the Fredholm type of the second kind. It is homogeneous and contains a symmetric kernel, (or one that can be made symmetric). The discussions here, therefore, will be limited to linear, homogeneous, Fredholm integral equations of the second kind with symmetric kernels. Much of the material in this chapter is the two-dimensional extension of Tricomi's one dimensional development.* The nomenclature is generally his.

6.1 Eigenvalues and eigenfunctions of a symmetric kernel.

The particular type of integral equation describing free vibration may be written

$$\phi(x,y)-\lambda\int K(x,y;r,s)\,\phi(r,s)\,dA=0 \qquad (1)$$

where K (x,y;r,s) = K (r,s;x,y) is the kernel of the integral equation. Supposing that the kernel is of the form

$$K(x,y;r,s)=\sum_{i=1}^{m}P_i(x,y)\,Q_i(r,s) \qquad (2)$$

and further that K (x,y;r,s) is quadratically integrable then

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*F.G.Tricomi, "Integral Equations", Interscience Publishers, 1957 equation (1) reduces to a set of m homogeneous, linear simultaneous equations. Such kernels are called Pincherle-Goursat kernels.

By letting

$$\int Q_j(x,y)\,\phi(x,y)\,dA = \rho_j \qquad (j=1,2\ldots m) \tag{3}$$

equation (1) transforms to

$$\phi(x,y) - \lambda \sum_{j=1}^{m} \rho_j P_j(x,y) = 0. \tag{4}$$

Multiplying by $Q_h(x,y)$ and integrating over the basic area yields

$$\int Q_h(x,y)\,\phi(x,y)\,dA - \lambda \sum_{j=1}^{m} \rho_j \int P_j(x,y)\,Q_h(x,y)\,dA = 0 \tag{5}$$

and letting $a_{hj} = \int P_j(x,y)\,Q_h(x,y)\,dA$

the reduction is complete and

$$\rho_h - \lambda \sum_{j=1}^{m} a_{hj}\rho_j = 0 \qquad (h=1,2\ldots m). \tag{6}$$

Equations (6) represent a system of m homogeneous, linear equation in the m unknowns, ρ_m. To each ρ_m of (6) there will correspond a solution of (4) and therefore of (1), for distinct λ's. The set of equations, (6), will give a nontrivial solution of (1) if the determinant of the system is zero. The non-trivial solutions of (1) are the eigenfunctions and λ the eigenvalues. These are called the eigenfunctions and eigenvalues of the kernel K $(x,y;r,s)$. From the study of the determinantal equation in λ an important part of the basic Fredholm theorem can be stated: If $\lambda = \lambda_o$ is a root of multiplicity $m \geq 1$ of the determinantal equation $D(\lambda) = 0$, then the homogeneous equation

$$\phi(x,y) - \lambda \int K(x,y;r,s)\,\phi(r,s)\,dA = 0 \qquad (7)$$

has r linearly independent non-trivial solutions, called eigenfunctions, where r, the index of the eigenvalue, satisfies the condition $1 \leq r \leq m$.

It will be evident when the integral equation for the vibration of thin elastic plates is formulated that these eigenvalues are those commonly accepted as such (i.e. proportional to the natural frequency of the plate).

6.2 Hilbert Schmidt Theorem. This important theorem, usually stated in its one dimensional form, becomes the following in the two dimensional case. If f (x,y) can be written in the form

$$f(x,y) = \int K(x,y;r,s)\,g(r,s)\,dA \qquad (8)$$

where $g(r,s)$ is quadratically integrable and K $(x,y;r,s)$ is symmetric and quadratically integrable then f (x,y) can also be represented by its uniformly and absolutely convergent Fourier series with respect to the orthonormal system $\{\phi_m\}$ of eigenfunctions of K (x,y;r,s). Hence

$$f(x,y) = \sum_{h=1}^{\infty} a_h \phi_h(x,y) \tag{9}$$

where $$a_h = \iint f(x,y)\, \phi_h(x,y)\, dx\, dy.$$

6.3 Iterated kernels. Given the integral equation

$$\phi(x,y) = \lambda \int K(x,y;r,s)\, \phi(r,s)\, dA \tag{10}$$

assume a solution ϕ_{10} (x,y) as a first approximation to the first eigenfunction and call the second approximation ϕ_{11} (x,y) given by

$$\phi_{11}(x,y) = \lambda_1 \int K(x,y;r,s)\, \phi_{10}(r,s)\, dA. \tag{11}$$

Substituting ϕ_{11} (x,y) into equation (10) we get ϕ_{12} (x,y) in the form*

$$\phi_{12}(x,y) = \lambda_1^2 \iint K(x,y;t,u) \left[\iint K(t,u;r,s)\, \phi_{10}(r,s)\, dr\, ds \right] dt\, du \tag{12}$$

or

$$\phi_{12}(x,y) = \lambda_1^2 \iint K_2(x,y;r,s)\, \phi_{10}(r,s)\, dr\, ds \tag{13}$$

where $K_2(x,y;r,s) = \iint K(x,y;t,u)\, K(t,u;r,s)\, dt\,du$

Continuing in this manner an approximation to ϕ_1 (x,y) can

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*The differential variables constituting dA will be specified whenever it is not clear from the context what they are.

be arrived at in the form

$$\phi_{1m}(x,y)=\lambda_1^m\iint K_m(x,y;r,s)\,\phi_{1o}(r,s)\,dr\,ds \qquad (m=2,3\cdots) \tag{14}$$

where $K_m(x,y;r,s)=\iint K\,(x,y;t,u)K_{m-1}(t,u;r,s)dtdu.$ (15)

$K_m(x,y;r,s)$ is called the mth iterated kernel. For the jth eigenfunction (14) becomes

$$\phi_{jm}(x,y)=\lambda_j^m\iint K_m(x,y;r,s)\,\phi_{jo}(r,s)\,dr\,ds \qquad (m=2,3\ldots) \tag{16}$$

provided $\phi_{jo}(x,y)$ is not orthogonal to $\phi_j(x,y)$.

6.4 A Bound for the Eigenvalues. It can now be seen that the defining equation for the mth iterated kernel is of the same structure as equation (8) where $g(r,s)=K_{m-1}(t,u;r,s)$. (This being the case $K_m(x,y;r,s)$ can be represented by

$$K_m(x,y;r,s)=\sum_{h=1}^{\infty} a_h(r,s)\,\phi_h(x,y) \tag{17}$$

where

$$a_h(r,s)=\iint K_m(x,y;r,s)\,\phi_h(x,y)\,dx\,dy. \tag{18}$$

But

$$\phi_h(r,s)=\lambda_h^m\iint K_m(r,s;x,y)\,\phi_h(x,y)\,dx\,dy \tag{19}$$

from the discussion of section 6.3.* Hence

$$a_h(r,s) = \lambda_h^{-m} \phi_h(r,s) \tag{20}$$

and so

$$K_m(x,y;r,s) = \sum_{h=1}^{\infty} \lambda_h^{-m} \phi_h(x,y)\, \phi_h(r,s). \tag{21}$$

Now associate x,y with r,s respectively and integrate over the area where

$$\sum_{h=1}^{\infty} \lambda_h^{-m} \phi_h(x,y)\, \phi_h(r,s) \qquad (m>1)$$

is an absolute and uniformly convergent series in the basic area as a consequence of Mercer's theorem (stated in the next section). The result is

$$\iint K_m(x,y;x,y)\, dx\, dy = \sum_{h=1}^{\infty} \lambda_h^{-m} \iint \phi_h^2(x,y)\, dx\, dy. \tag{22}$$

Due to the orthonormality of the $\{\phi_h\}$ prescribed by the Hilbert-Schmidt theorem

$$\sum_{h=1}^{\infty} \lambda_h^{-m} \iint \phi_h^2(x,y)\, dx\, dy = \sum_{h=1}^{\infty} \lambda_h^{-m} \qquad (m=2,3\ldots). \tag{23}$$

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*Note that $K_n(r,s;x,y) = K_n(x,y;r,s)$ due to symmetry.

Finally define the "trace" of the Kernel as

$$A_m = \iint K_m(x,y;x,y)\,dx\,dy \tag{24}$$

yielding

$$A_m = \sum_{h=1}^{\infty} \lambda_h^{-m} \qquad (m = 2,3\cdots). \tag{25}$$

In the non-degenerate physical vibration problem the eigenvalues, which are proportional to the natural frequencies, are ordered in increasing magnitude $0<\lambda_1<\lambda_2<\cdots\lambda_h\cdots$. By neglecting the eigenvalues of index greater than 1 a <u>lower bound</u> to the true eigenvalue is obtained from the relation

$$\lambda_1 = \sqrt[m]{1/A_m} \qquad (m = 2,3\cdots). \tag{26}$$

An expression for an upper bound can be deduced from (25) by forming the quotient

$$\frac{A_{2j}}{A_{2j+2}} = \frac{\sum_{h=1}^{\infty} \lambda_h^{-2j}}{\sum_{h=1}^{\infty} \lambda_h^{-2j-2}} \qquad (j = 1,2\cdots).$$

As before, neglecting the higher indexed eigenvalues gives

$$\lambda_1^2 = \frac{A_{2j}}{A_{2j+2}}$$

or
$$\lambda_1 = \sqrt{A_{2j}/A_{2j+2}} \tag{27}$$

The validity of this can be seen by expanding the series of eq. (25).

$$\frac{A_{2j}}{A_{2j+2}} = \frac{\frac{1}{\lambda_1^{2j}} + \frac{1}{\lambda_2^{2j}} + \cdots}{\frac{1}{\lambda_1^{2j+2}} + \frac{1}{\lambda_2^{2j+2}} + \cdots}$$

$$\frac{A_{2j}}{A_{2j+2}} = \lambda_1^2 \left[\frac{1 + (\lambda_1/\lambda_2)^{2j} + (\lambda_1/\lambda_3)^{2j} + \cdots}{1 + (\lambda_1/\lambda_2)^{2j+2} + (\lambda_1/\lambda_3)^{2j+2} + \cdots} \right].$$

The terms discarded in the denominator are smaller than those discarded in the numerator so that actually the bracket is greater than one.

It can be seen then, that by neglecting the eigenvalues of index greater than 1 expression (27) gives an upper bond. Hence, the true value of the first eigenvalue satisfies the inequality.

$$\sqrt[m]{1/A_m} \leq \lambda_1 \leq \sqrt{A_{2j}/A_{2j+2}} \quad \begin{pmatrix} j = 1, 2 \cdots \\ m = 2, 3 \cdots \end{pmatrix} \tag{28}$$

Of course the magnitude of m and j for which (28) gives a good approximation depends on the degree of separation of the eigenvalues.

<u>6.5 Mercers Theorem</u>: If a continuous, symmetric and quadratically integrable kernel K (x,y;r,s) has only positive eigenvalues, then the series

$$\sum_{h=1}^{\infty} \lambda_h^{-1} \phi_h(x,y)\,\phi_h(r,s)$$

converges absolutely and uniformly to the value of the kernel K(x,y;r,s), in the basic area. As a consequence of this theorem equations (26)(27) and (28) can be used even for $m = 1$ because

$$A_1 = \int K(x,y;x,y)\,dA = \sum_{h=1}^{\infty} \lambda_h^{-1} \int \phi_h^2(x,y)\,dA = \sum_{h=1}^{\infty} \lambda_h^{-1}. \quad (29)$$

Hence

$$\lambda_1 \geqslant \frac{1}{A_1}.$$

<u>6.6. Method of Collocation.</u> Given the homogeneous, linear

integral equation where $K(x,y;r,s)$ is symmetric or non-symmetric:

$$\phi(x,y) - \lambda \int K(x,y;r,s)\, \phi(r,s)\, dA = 0 \tag{30}$$

Assume a solution of the form

$$\phi(x,y) = \sum_{i=1}^{m} a_i \phi_i(x,y). \tag{31}$$

Substituting the assumed function into the integral equation yields

$$\sum_{i=1}^{m} a_i \phi_i(x,y) - \lambda \int K(x,y;r,s) \sum_{i=1}^{m} a_i \phi_i(r,s)\, dA = 0 \tag{32}$$

or

$$\sum_{i=1}^{m} \left[\phi_i(x,y) - \lambda \int K(x,y;r,s)\, \phi_i(r,s)\, dA \right] a_i = 0 \tag{33}$$

If, now, this equation is forced to be satisfied at m specific sets of values of (x,y) there will result m simultaneous, homogeneous equations in a_i of the form

$$\sum_{i=1}^{m} \left[c_{hi} - \lambda b_{hi} \right] a_i = 0 \qquad (h = 1, 2 \cdots m) \tag{34}$$

where

$$\left. \begin{aligned} c_{hi} &= \phi_i(x_h, y_h) \\ b_{hi} &= \int K(x_h, y_h; r,s)\, dA \end{aligned} \right\}. \tag{35}$$

For a nontrivial solution the determinant of the coefficients a_i must be zero. This yields a mth order determinantal equation in λ, the solution of which gives m values of λ corresponding to the first m eigenvalues.

<u>6.7 Method of Successive Approximations.</u> Given the homogeneous, linear integral equation where K (x,y;r,s) is symmetric or non-symmetric:

$$\phi(x,y) = \lambda \int K(x,y;r,s)\,\phi(r,s)\,dA. \tag{36}$$

Assume a function ϕ_{10} (x,y) where the subscript zero means the first approximation to the first eigenfunction ϕ_1 (x,y) associated with eigenvalue λ_1 and ϕ_{10} (x,y) is not orthogonal to ϕ_1(x,y). Now obtain the second approximation by evaluating the integral

$$\phi_{11}(x,y) = \int K(x,y;r,s)\,\phi_{10}(r,s)\,dA.$$

Obtain the third approximation by evaluating the integral

$$\phi_{12}(x,y) = \int K(x,y;r,s)\,\phi_{11}(r,s)\,dA. \tag{37}$$

Continuing in this manner one obtains

$$\phi_{1m}(x,y) = \int K(x,y;r,s)\,\phi_{1m-1}(r,s)\,dA \tag{38}$$

and

$$\phi_{1m+1}(x,y) = \int K(x,y;r,s)\,\phi_{1m}(r,s)\,dA. \tag{39}$$

This convergent process will yield

$$\lim_{m\to\infty}\left\{\frac{\int K(x,y;r,s)\,\phi_{1m}(r,s)\,dA}{\phi_{1m}(x,y)}\right\}=\lambda_1^{-1} \tag{40}$$

from equation (36). Thus

$$\lambda_1=\lim_{m\to\infty}\frac{\phi_{1m}(x,y)}{\phi_{1m+1}(x,y)}. \tag{41}$$

Obviously if this process is truncated at any step the resulting ratio will give an approximate value of λ_1 provided the ratio is not a function of (x,y). If for any two successive ratios the value is the same constant then λ_1 is exact. Furthermore, this process will converge to the true eigenvalue from above.

6.8 Determination of Subsequent Eigenvalues. Each of the three methods for determining the first eigenvalue as discussed in previous sections can be used to find the higher indexed eigenvalues provided the eigenvalues are distinct.

From equation (29) a lower bound for any eigenvalue λ_h is obtained provided λ_1 through λ_{h-1} are known and all eigenvalues are distinct. As a matter of fact the discussion of Section 6.4 is available for the determination of subsequent eigenvalues as long as all lower indexed eigenvalues are known and distinct. Unfortunately, the computational difficulties for finding the upper bound, increase considerably with the index of the eigenvalue. For example, forming the ratio leading to equation (27) and neglecting eigenvalues of index higher than 2 gives

$$\frac{A_{2m}}{A_{2m+2}} = \frac{\lambda_1^{-2m} + \lambda_2^{-2m}}{\lambda_1^{-2m-2} + \lambda_2^{-2m-2}} = \lambda_1^2 \frac{(\lambda_2/\lambda_1)^{2m+2} + \lambda_2^2}{(\lambda_2/\lambda_1)^{2m+2} + 1} . \tag{42}$$

Letting $k = \lambda_2/\lambda_1 \; \therefore \lambda_2 = k\lambda_1$, the following equation, from which λ_2 can be found, results:

$$k^2 + \left(1 - \frac{A_{2m}\lambda_1^{-2}}{A_{2m+2}}\right) k^{2m+2} - \frac{A_{2m}\lambda_1^{-2}}{A_{2m+2}} = 0 . \tag{43}$$

Even the simple case of $m = 1$ this equation contains the traces A_2 and A_4 which, in general are computationally involved as can be seen from the definition of the iterated kernel in section 6.3.

When using the method of collocation to determine subsequent eigenvalues the number of terms in the assumed solution equal to the index of the desired eigenvalue are taken. This yields a determinantal equation of the proper order and has the added advantage of approximating all lower eigenvalues at the same time. It is to be noted that as more and more terms are taken in the assumed solution (eigenfunction) the lower indexed eigenvalues are increasingly more accurate. However, no statement can be made as to whether the estimate is an upper or lower bound.

Successive approximations can be used for finding the mth eigenvalue by obtaining a new kernel from which the (m-1) eigenfunctions corresponding to the (m-1) eigenvalues have been eliminated. In other words, if "successive approximations" is applied to the integral equation

$$\phi(x,y) - \lambda \int K'(x,y;r,s)\,\phi(r,s)\,dA = 0 \tag{44}$$

where $K'(x,y;r,s) = K(x,y;r,s) - \dfrac{\phi_1(x,y)\,\phi_1(r,s)}{\lambda_1}$ (45)

the second eigenvalue, λ_2 is obtained. The rationale for this comes as a consequence of Mercer's theorem. For, if it is true that

$$K(x,y;r,s) = \sum_{h=1}^{\infty} \frac{\phi_h(x,y)\,\phi_h(r,s)}{\lambda_h} \tag{46}$$

then, in forming $K'(x,y;r,s)$, the problem reduces to one of finding the first eigenvalue of the kernel $K'(x,y;r,s)$ which is the second eigenvalue of $K(x,y;r,s)$. This process can be carried out as far as desired, each time removing the previous eigenfunctions from the kernel. It must be remembered that because the eigenfunctions can be, at best, approximated the more accurate the approximation is the more exact will be the subsequent eigenvalues and eigenfunctions. Note that the corrected kernel $K'(x,y;r,s)$ is symmetric and quadratically integrable so that all theory applicable to symmetric kernels is applicable to $K'(x,y;r,s)$. Because of the expansion of the kernel in the series

$$K(x,y;r,s) = \sum_{h=1}^{\infty} \frac{\phi_h(x,y)\,\phi_h(r,s)}{\lambda_h}$$

the method, just described, of determining subsequent eigenvalues is valid only for symmetric kernels. If the kernel is not symmetric care must be taken that the successive approximations for the higher modes do not converge to the first mode. This will be assured if the first approximation to the mth mode is made orthogonal to all previous modes but not to the mth mode.

6.9 Eigenfunctions

Having found the eigenvalues of the kernel $K'(x,y;r,s)$ from a study of the "trace" the associated eigenfunctions (ie. solutions of equation (1)) are available from the system of equations (6). It is to be noted that inasmuch as the system (6) is homogeneous the eigenfunctions will be determined only to within an undetermined constant.

When the eigenvalue has been determined by the method of collocation an approximation to the associated eigenfunction can be obtained by solving the set of homogeneous equations (34) for the a_i's. Substitution into (31) will yield an approximation to within an arbitrary constant of the true eigenfunction.

Successive approximations give continually more accurate eigenfunctions as the process proceeds. As a matter of fact, we can say that

$$\Phi_1(x,y) = \lim_{m\to\infty} \Phi_{1m}(x,y).$$

6.10 Symmetrifiable Kernels

Some of the previous discussion has required symmetric kernels. In many vibration problems the kernel of the integral equation while not, in itself symmetric, is one of which can be easily made so.

Let the kernel of the integral equation

$$\phi(x,y) = \lambda \int K(x,y;r,s)\,\phi(r,s)\,dA \tag{47}$$

be of the form

$$K(x,y;r,s) = G(x,y;r,s)\,f(r,s)$$

where $G(x,y;r,s)$ is a symmetric function. Now make the substitution

$$\Phi(x,y) = \sqrt{f(x,y)}\;\phi(x,y). \tag{48}$$

This gives the new integral equation

$$\Phi(x,y) = \lambda \int K^*(x,y;r,s)\,\Phi(r,s)\,dA \tag{49}$$

where
$$K^*(x,y;r,s) = \sqrt{f(x,y)f(r,s)}\;G(x,y;r,s)$$

which is obviously symmetric.

An eigenvalue of (49) is also an eigenvalue of (47) and the eigenfunctions are related by the transformation equation.

6.11 Summary

This chapter has described the mathematical methods of integral equations pertinent to the plate vibration problem. The three basic methods available for eigenvalue determination are: (a) using the trace of the kernel which in addition to giving an approximate value of the eigenvalue also provides upper and lower bounds, (b) the method of collocation and (c) the method of successive approximations. All three methods will be used in later chapters but in general, if only the first eigenvalue (principal frequency) is desired method (a) has distinct computational advantage. Recalling that the trace of the kernel $K(x,y;r,s)$ is given by the expression $A_m = \int K_m(x,y;x,y)\,dA$

it will be seen that the association of x with r and y with s considerably simplifies the computations of A_m, especially if the eigenvalue is being determined from A_1.

For subsequent eigenvalues the method of collocation has merit, as one can simply take more terms in the assumed series of functions. Indeed, if the nth eigenvalue is desired the first n terms of the assumed function are taken.

Obtaining the eigenfunctions associated with the eigenvalues is a somewhat more difficult task. Except for the computational difficulty the method of successive approximations seems to be the best. The other two methods necessitate the solution of a system of homogeneous simultaneous equations.

No attempt has been made, in discussing the theory, to prove the theorems cited or establish the convergence of series involved as this can be found in any modern book on integral equations.

VII. FORMULATION OF THE PROBLEM

7.1 Formulation of the integral equation.

Given a thin, elastic plate restrained so that there is no general translation of the plate let it be required to obtain the integral equation governing the free vibration. Assume that the plate is vibrating in a normal mode. Under this latter assumption the deflection at any point (x,y) is given by $Z(x,y,\tau) = w(x,y)\cos p\tau$. Now define the Green's function for the plate as that static deflection at (x,y) due to a unit load at (r,s) as shown in Fig. 1.

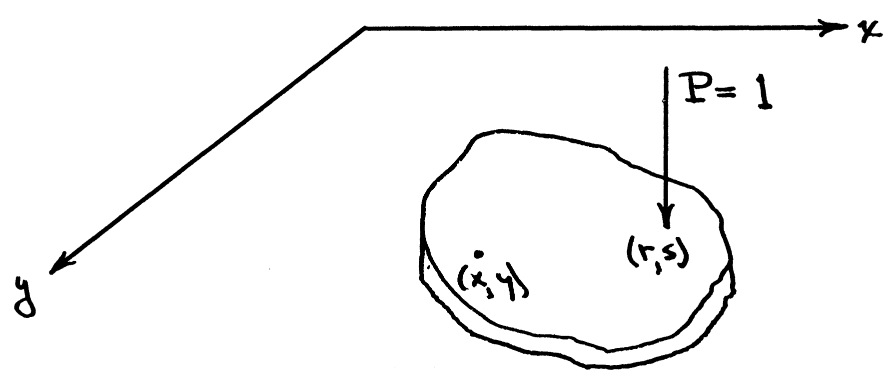

Fig. 1. A thin elastic plate subjected to a normal force.

The inertia load acting on an element of area drds is

$$-\left[t(r,s)\,\gamma(r,s)\,dr\,ds\right]\ddot{z}$$

where t(r,s) is the thickness function and γ (r,s) is the mass density. From assuming a normal mode vibration this expression becomes

$$t(r,s)\,\gamma(r,s)\,p^2\,w(r,s)\cos p\tau\,dr\,ds.$$

The deflection at (x,y) due to this infinitesimal inertia load is obtained simply by multiplying it by Green's function. Furthermore, the deflection at (x,y) due to all inertia loads is obtained by superposition. Hence

$$z(x,y,\tau) = p^2\cos p\tau\int t(r,s)\,\gamma(r,s)\,G(x,y;r,s)\,w(r,s)\,dr\,ds \qquad (50)$$

from which

$$w(x,y) = p^2\int_A t(r,s)\,\gamma(r,s)\,G(x,y;r,s)\,w(r,s)\,dA. \qquad (51)$$

Equation (51) is the governing integral equation for the free vibration of any flat, thin, elastic plate restrained against general translation.

7.2 The symmetric integral equation

If one sets $p^2 = \lambda$ and $K(x,y;r,s) = t(r,s)\,\gamma(r,s)\,G(x,y;r,s)$ in equation (51) the governing integral equation becomes

$$w(x,y) = \lambda\int K(x,y;r,s)\,w(r,s)\,dA \qquad (52)$$

which is a homogeneous Fredholm equation. By Maxwell's reciprocal theorem the Green's function $G(x,y;r,s)$ is symmetric, which is to say $G(x,y;r,s) = G(r,s;x,y)$.

However, $K(x,y;r,s)$ is not symmetric. It can be made so by following the method of section 6.10.

Eq. (52) is then transformed into a symmetric, homogeneous Fredholm equation where the λ_h that yield a non-trivial solution are the eigenvalues which are proportional to the natural frequencies, as mentioned previously at the end of section 6.1.

7.3 Boundary conditions

One of the advantages of the integral equation approach to the solution of vibrations problems is that once the governing integral equation is formulated the boundary conditions must be satisfied inasmuch as the Green's function is derived from the static deflection problem. This quality is particularly valuable when successive approximations or collocation is used for the solution of the integral equation. The assumed function need not satisfy the boundary conditions because the Green's function does. This can be a distinct advantage over the popular method of Rayleigh-Ritz in those cases where the satisfaction of the boundary conditions leads to a rather cumbersome assumed deflection function.

7.4 The physical problem

Because the integral equation was derived from the physical vibration problem in which the vibration was assumed to be non-degenerate and because then the natural frequencies of the physical vibration problem are positive, real, distinct and ordered in increasing magnitude so also must the eigenvalues of the integral equation be positive, real, distinct and ordered. Furthermore, there are eigenfunctions associated with each eigenvalue as a consequence of the basic Fredholm theorem mentioned in section 6.1. These eigenfunctions are the solutions of the governing Fredholm integral equation and hence represent the mode shape of the plate.

7.5 Degenerate eigenvalues

In the previous discussion distinct eigenvalues were postulated. Eigenvalues are said to be distinct if the eigenfunction associated with a particular eigenvalue is uniquely determined with a multiplicative factor. On the other hand, if there are r linearly independent eigenfunctions associated with a particular eigenvalue then the eigenvalue has multiplicity m and is of a degenerate case. From the basic Fredholm Theorem stated in sec. 6.1 $r \leq m$. It can be seen that multiple roots of a frequency equation give eigenvalues of a degenerate case.

If the symmetric modes are desired (these modes are non-degenerate and the Green's function is constructed for this case then the determination of eigenvalues beyond the first will result in a spectrum of distinct and ordered eigenvalues (and hence frequencies) corresponding to the symmetric modes. On the other hand when the non-symmetric modes are desired the discussion of sec. 6.8

includes a very real difficulty. When equation (29) is used to determine subsequent eigenvalues and the eigenvalues are of a degenerate case it is not known which one is being found. This difficulty is inherent in the plate problem. The frequency of a vibrating rectangular plate is given by

$$p = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\sqrt{D/\gamma t}$$

and the spectrum of degenerate frequencies can only be written in ordered form if the frequencies are determined from all combinations of m, n and then ordered. For example, if the dimensions of the plate are such that $a^2 = 3b^2$ then the three combinations $m = 1$, $n = 3$; $m = 4$, $n = 2$ and $m = 5$, $n = 1$ all give a frequency of

$$p = \frac{28}{3}\frac{\pi^2}{b^2}\sqrt{D/\gamma t}.$$

In the bulk of what follows the results will apply only to symmetric modes.

VIII. ELEMENTARY EXAMPLES

8.1 The simply supported rectangular plate.

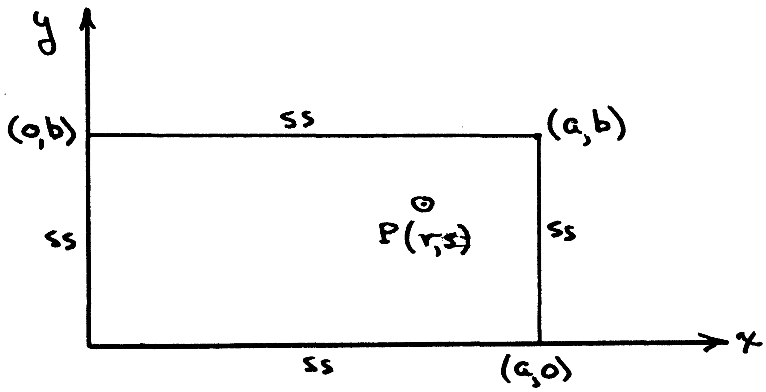

Fig. 2. A simply supported rectangular plate.

Eqtn. (51) becomes, for the case of uniform mass density and thickness,

$$w(x,y) = \gamma t p^2 \int_0^a \int_0^b G(x,y;r,s)\, w(r,s)\, dr\, ds. \tag{53}$$

The Green's function for a simply supported, rectangular plate is given in the Navier series form as

$$G(x,y;r,s) = \frac{4}{\pi^4 abD} \sum_1^\infty \sum_1^\infty \frac{\sin\frac{i\pi r}{a} \sin\frac{j\pi s}{b} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}}{\left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^2}. \tag{54}$$

Substituting in the governing equation yields

$$w(x,y) = \lambda \int_0^a \int_0^b \sum_1^\infty \sum_1^\infty \frac{\sin\frac{i\pi r}{a} \sin\frac{j\pi s}{b} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}}{\left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^2}\, w(r,s)\, dr\, ds \tag{55}$$

where $\lambda = \frac{4\gamma t P^2}{\pi^4 abD}$. Interchanging summation and integration yields

$$w(x,y) = \lambda \sum_1^\infty \sum_1^\infty \int_0^a \int_0^b \frac{\sin\frac{i\pi r}{a} \sin\frac{j\pi s}{b} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}}{\left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^2} w(r,s)\, dr\, ds. \quad (56)$$

Solving by successive approximations and choosing $w_{10}(x,y)=1$ the resulting expression for $w_{11}(x,y)$ becomes

$$w_{11}(x,y) = \sum_1^\infty \sum_1^\infty \frac{ab}{ij\pi^2}\left[1-\cos i\pi\right]\left[1-\cos j\pi\right] \frac{\sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}}{\left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^2}. \quad (57)$$

For even valued i and j this expression vanishes while for odd values i and j, $(1- \cos i\pi) = (1 - \cos j\pi) = 2$. Hence

$$w_{11}(x,y) = \sum_1^\infty \sum_1^\infty \frac{4ab}{ij\pi^2}\left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \quad (58)$$

$$i,j = 1,3,5 \cdots.$$

Likewise $w_{12}(x,y)$ is obtained as

$$w_{12}(x,y) = \sum_1^\infty \sum_1^\infty \frac{a^2b^2}{ij\pi^2}\left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^{-4} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}. \quad (59)$$

Now form the ratio

$$\frac{w_{11}}{w_{12}} = \frac{\frac{4ab}{\pi^2} \sum_1^\infty \sum_1^\infty \frac{1}{ij}\left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^{-2} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}}{\frac{a^2b^2}{\pi^2} \sum_1^\infty \sum_1^\infty \frac{1}{ij}\left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2\right]^{-4} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}}. \quad (60)$$

For the fundamental frequency we take i = j = 1 from which

$$\lambda_1 = \frac{4}{ab}\left[\frac{1}{a^2}+\frac{1}{b^2}\right]^2 = \frac{4\gamma t\, p_1^2}{\pi^4 abD}$$

$$p_1 = \pi^2\left[\frac{1}{a^2}+\frac{1}{b^2}\right]\sqrt{D/\gamma t}\ . \tag{61}$$

If further successive approximations are performed, such as determining w_{13}, w_{14}, etc., the successive ratios would be

$$\frac{w_{11}}{w_{12}} = \frac{w_{12}}{w_{13}} = \frac{w_{13}}{w_{14}} = \cdots = \frac{4}{ab}\left[\frac{1}{a^2}+\frac{1}{b^2}\right]^2,$$

hence there would be no use to carry the cycling any further. Due to the uniformity of the convergence of successive approximations it is clear that if two successive ratios are identical the exact value of λ has been found. Another point of interest is that successive approximations will always converge from above, thus yielding at any cycle an upper bound. Finally it is to be noted that w_{12} represents the approximate wave shape of eigenfunction of the plate vibrating in the first mode (where i = j = 1).

A very rapid way of determining the fundamental eigenvalue would be by using the trace of the kernel:

$$A_1 = \iint K(x,y;x,y)\,dx\,dy \tag{62}$$

where $K\ (x,y;x,y) = \sum_1^{\infty}\sum_1^{\infty}\left[\left(\frac{i}{a}\right)^2+\left(\frac{j}{b}\right)^2\right]^{-2}\sin^2\frac{i\pi x}{a}\sin^2\frac{j\pi y}{b}$

from which $A_1 = \sum_1^{\infty}\sum_1^{\infty}\frac{ab}{4}\left[\left(\frac{i}{a}\right)^2+\left(\frac{j}{b}\right)\right]^{-2}$. But $\lambda_1 = \frac{4\gamma t p_1^2}{\pi^4 abD}$

and $\lambda_1 = 1/A_1$, hence the value for p_1 (the fundamental frequency) when i = j = 1 is the same value as found by successive approximations, namely

$$p_1 = \pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\sqrt{D/\gamma t}\,. \tag{63}$$

To illustrate the use of the shortened kernel (see section 6.8) for determining higher order eigenvalues consider the final approximations to $w_1\ (x,y)$ as the eigenfunction (in this case $w_{12}(x,y)$, normalize it, and form the new kernel

$$K'(x,y;r,s) = K(x,y;r,s) - \frac{\phi_1(x,y)\,\phi_1(r,s)}{\lambda_1}. \tag{64}$$

The first eigenvalue of this new, shortened kernel will be the second eigenvalue of the original kernel.

$$N_1^2 = \left[\frac{1}{a^2}+\frac{1}{b^2}\right]^{-8}\frac{a^4b^4}{\pi^4}\int_0^a\int_0^b \sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}\,dx\,dy \tag{65}$$

$$N_1 = \left[\frac{1}{a^2}+\frac{1}{b^2}\right]^{-4}\frac{a^{5/2}b^{5/2}}{2\pi^2} \tag{66}$$

and the normalized eigenfunction becomes

$$\phi_1(x,y) = \frac{2}{\sqrt{ab}} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}. \tag{67}$$

Hence

$$\frac{\phi_1(x,y)\,\phi_1(r,s)}{\lambda_1} = \frac{\sin\frac{\pi r}{a}\sin\frac{\pi s}{b}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{\left[\frac{1}{a^2}+\frac{1}{b^2}\right]^2} \tag{68}$$

and the shortened kernel is

$$K'(x,y;r,s) = \sum_1^{\infty}{}' \sum_1^{\infty}{}' \frac{\sin\frac{i\pi r}{a}\sin\frac{j\pi s}{b}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}}{\left[\left(\frac{i}{a}\right)^2+\left(\frac{j}{b}\right)^2\right]^2}$$

$$- \frac{\sin\frac{\pi r}{a}\sin\frac{\pi s}{b}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{\left[\frac{1}{a^2}+\frac{1}{b^2}\right]^2}. \tag{69}$$

It can be seen that $K'(x,y;r,s)$ is simply the original series with the $i = j = 1$ term removed.

For the sake of brevity the second eigenvalue was determined using the trace of the kernel. Let $b < a$, then for the fundamental eigenvalue of $K'(x,y;r,s)$ the trace is, when $i=2$, $j=1$ (if $a > b$ use $i=1$, $j=2$)

$$A_1' = \frac{ab}{4}\left[\left(\frac{2}{a}\right)^2+\left(\frac{1}{b}\right)^2\right]^{-2} \tag{70}$$

from which

$$P_2 = P_1' = \pi^2\left[\frac{4}{a^2} + \frac{1}{b^2}\right]\sqrt{D/\gamma t}\ .$$

The use of successive approximations would have given us a convenient second eigenfunction associated with the second eigenvalue and the process could have been repeated by eliminating the modes for which $i=j=1$ and $i=2$, $j=1$.

8.2 The simply supported right isosceles triangular plate

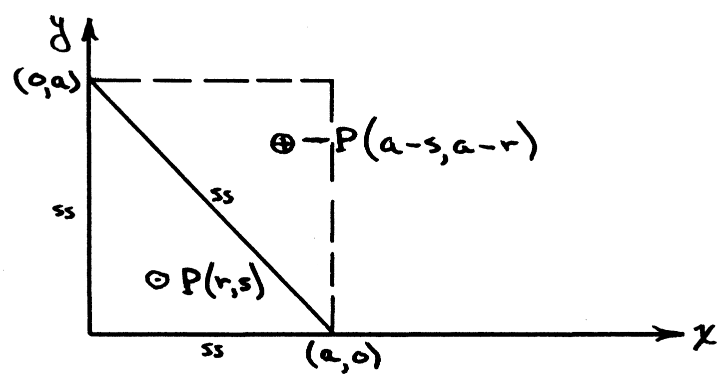

Fig. 3. Simply supported right isosceles triangular plate.

As a matter of interest consider the fundamental eigenvalue (and hence frequency) of the simply supported right isosceles triangular plate. The Green's function for this case can be deduced from that for the simply supported square plate by a method of images. If two equal and opposite concentrated loads are applied as shown in the figure the hypothenuse will have no deflection and no moment which is the requirement for a simply supported edge. By superposition the Green's function becomes

$$G(x,y;r,s)=\frac{4a^2}{\pi^4 D}\sum_1^\infty\sum_1^\infty \frac{\sin\frac{i\pi r}{a}\sin\frac{j\pi s}{a}-(-1)^{i+j}\sin\frac{i\pi s}{a}\sin\frac{j\pi r}{a}}{(i^2+j^2)^2}\sin\frac{i\pi x}{a}\sin\frac{j\pi y}{a}. \quad (71)$$

Substitution in the governing integral equation gives

$$w(x,y)=\lambda\int G(x,y;r,s)\,w(r,s)\,dA \quad (72)$$

$$\lambda=\frac{4a^2\gamma t\,p^2}{\pi^4 D}.$$

Using the trace of the kernel to determine the fundamental frequency and realizing that the fundamental frequency will occur when i=1, j=2 and i=2, j=1 first write

$$K(x,y;x,y)=\frac{1}{25}\left\{\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}+\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right\}^2 \quad (73)$$

then $$A_1=\frac{1}{25}\int_0^a\int_0^{a-y}\left[\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}+\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a}\right]^2 dx\,dy. \quad (74)$$

Integration yields $A_1=\frac{a^2}{100}$. But $\lambda_1=\frac{1}{A_1}$, so that

$$p_1=\frac{5\pi^2}{a^2}\sqrt{D/\gamma t}. \quad (75)$$

8.3 The Circular Plate Clamped at the Boundary

Given a homogeneous elastic plate clamped around the outer boundary let it be required to find the fundamental frequency of free vibration. For this case the governing integral equation (51) becomes

$$w(r,\theta) = t\gamma p^2 \iint G(r,\theta;\rho,\psi)\, w(\rho,\psi)\, d\rho\, d\psi. \tag{76}$$

The Green's function is given, in general, for a circular plate by

$$G(r,\theta;\rho,\psi) = R_o + \sum_{m=1}^{\infty} R_m \cos m\theta. \tag{77}$$

Due to the symmetry of the principal mode the θ dependent term will vanish under the integration, hence, consider only R_o for the case of the clamped boundary condition on the radius "a". For this

$$\left.\begin{aligned} R_o &= \frac{1}{8\pi D}\left[(r^2+\rho^2)\log\frac{r}{a} + \frac{1}{2a^2}(a^2+\rho^2)(a^2-r^2)\right] \quad r>\rho \\ R_o &= \frac{1}{8\pi D}\left[(r^2+\rho^2)\log\frac{\rho}{a} + \frac{1}{2a^2}(a^2+r^2)(a^2-\rho^2)\right] \quad r<\rho \end{aligned}\right\} \tag{78}$$

Note that the kernel of the governing equation is not symmetric due to the presence of the variable "ρ", but this kernel is one which is readily symmetrified (see section

6.10). Let

$$\phi(r,\theta) = \sqrt{r}\; w(r,\theta)$$

and
$$K(r,\theta;\rho,\psi) = \sqrt{r\rho}\; G(r,\theta;\rho,\psi)$$

then the integral equation becomes, by virtue of this transformation and the symmetry of the principal mode of vibration

$$\phi(r) = \gamma t p^2 \int_0^{2\pi}\int_0^a K(r,\rho)\,\phi(\rho,\psi)\,d\rho\,d\psi = 2\pi\gamma t p^2 \int_0^a K(r,\rho)\,\phi(\rho)\,d\rho \tag{79}$$

where $K(r, \rho)$ is given by

$$K(r,\rho) = \frac{r^{1/2}\rho^{1/2}}{8\pi D}\left[(r^2+\rho^2)\log\frac{r}{a} + \frac{(a^2+\rho^2)(a^2-r^2)}{2a^2}\right] \qquad r > \rho$$
$$K(r,\rho) = \frac{r^{1/2}\rho^{1/2}}{8\pi D}\left[(r^2+\rho^2)\log\frac{\rho}{a} + \frac{(a^2+r^2)(a^2-\rho^2)}{2a^2}\right] \qquad r < \rho. \tag{80}$$

Letting $\lambda = \frac{\gamma t p^2}{4D}$ the resulting integral equation is, from (79),

$$\phi(r) = \lambda\left\{\int_0^r \left[(r^2+\rho^2)\log\frac{r}{a} + \frac{(a^2+\rho^2)(a^2-r^2)}{2a^2}\right]\phi(\rho)\sqrt{r\rho}\,d\rho \right.$$
$$\left. + \int_r^a \left[(r^2+\rho^2)\log\frac{\rho}{a} + \frac{(a^2+r^2)(a^2-\rho^2)}{2a^2}\right]\phi(\rho)\sqrt{r\rho}\,d\rho\right\}. \tag{81}$$

Assuming an approximation to the deflection of the form

$$w_1(r) = C_1(a^2-r^2)^2 \qquad \text{or} \qquad \phi_1(r) = C_1 r^{1/2}(a^2-r^2)^2$$

the method of collocation can be applied by evaluating the following integral

$$\phi_1(r) = \lambda_1 r^{1/2}\left\{\int_0^r \left[(r^2+\rho^2)\log\frac{r}{a} + \frac{(a^2-r^2)(a^2+\rho^2)}{2a^2}\right]\rho C_1(a^2-\rho^2)^2\,d\rho\right.$$
$$\left.+\int_r^a \left[(r^2+\rho^2)\log\frac{\rho}{a} + \frac{(a^2-\rho^2)(a^2+r^2)}{2a^2}\right]\rho C_1(a^2-\rho^2)^2\,d\rho.\right. \qquad (82)$$

Integrating and simplifying

$$\phi_1(r) = \lambda_1 C_1 r^{1/2}\left\{\frac{r^8}{576} - \frac{r^6a^2}{72} + \frac{r^4a^4}{16} - \frac{13r^2a^6}{144} + \frac{23a^8}{576}\right\} \qquad (83)$$

where $\quad \phi_1(r) = C_1 r^{1/2}(a^2-r^2)^2.$

Equation (83) satisfies the geometric boundary conditions:

$$w_1(a) \equiv 0 \quad , \quad \left.\frac{dw_1}{dr}\right|_{r=a} \equiv \left.\frac{dw_1}{dr}\right|_{r=0} \equiv 0 .$$

Forcing the equation to be satisfied at a specific point, say at $r = a/2$, results in an approximation to the first eigenvalue :

$$\lambda_1 = \frac{26.576}{a^4} .$$

But

$$\lambda_1 = \frac{\gamma t P_1^2}{4D}$$

thus $$P_1 = \frac{10.31}{a^2}\sqrt{D/\gamma t} \qquad (84)$$

and the approximation to the modal shape is

$$w_1(r) = 26.576\, C_1\left[\frac{r^8}{576} - \frac{r^6 a^2}{72} + \frac{r^4 a^4}{16} - \frac{13 r^2 a^6}{144} + \frac{23 a^8}{576}\right]. \tag{85}$$

This problem has been solved by the Rayleigh-Ritz method*
using two terms in the assumed modal shape with the resulting frequency of

$$p_1 = \frac{10.21}{a^2}\sqrt{D/\gamma t}\ .$$

The modal shape assumed was

$$w(r) = C_1 (a^2 - r^2)^2 + C_2 (a^2 - r^2)^3.$$

A comparison of the approximate primary modal shapes between the Rayleigh-Ritz and integral equation methods is shown graphically in Fig. 4.

Consider now a solution for the fundamental frequency using the trace of the kernel. The kernel, as before, is given as

$$K(r,\rho) = \frac{r^{1/2}\rho^{1/2}}{8\pi D}\left[(r^2+\rho^2)\log\frac{r}{a} + \frac{(a^2+\rho^2)(a^2-r^2)}{2a^2}\right] \quad r > \rho$$

$$K(r,\rho) = \frac{r^{1/2}\rho^{1/2}}{8\pi D}\left[(r^2+\rho^2)\log\frac{\rho}{a} + \frac{(a^2+r^2)(a^2-\rho^2)}{2a^2}\right] \quad r < \rho \tag{86}$$

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*S. Timoshenko - Vibration Problems in Engineering, pg. 430.

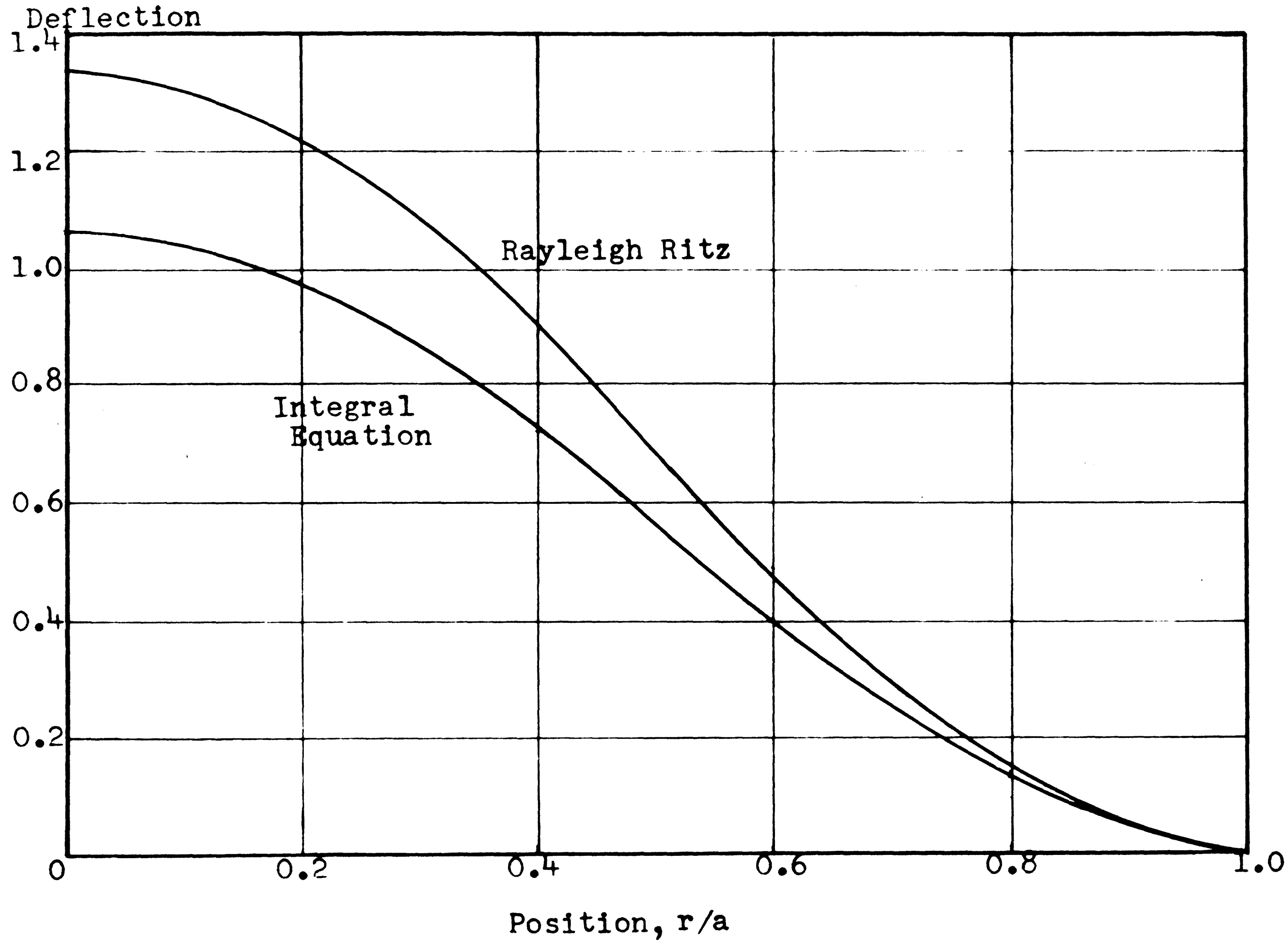

Fig. 4. Approximate first mode shape for the circular plate clamped on boundary (Deflection in arbitrary units).

$$A_1 = \int_0^a K(r,r)\,dr = \frac{1}{8\pi D}\int_0^a \left[2r^3 \log\frac{r}{a} + \frac{r(a^4-r^4)}{2a^2}\right] dr = \frac{a^4}{192\pi D}. \qquad (87)$$

Now $\lambda_1 = \frac{1}{A_1} = \frac{192\pi D}{a^4}$ but here $\lambda_1 = 2\pi\gamma t p_1^2$, which results

in $p_1 = \frac{9.8}{a^2}\sqrt{\frac{D}{\gamma t}}$. This not only has given a good approximation to the fundamental frequency but has the added benefit of being a lower bound of all possible approximate values of the fundamental frequency.

It is to be remembered that because the terms containing θ in the series solution for the static deflection were discarded the resulting solution of the integral equation for any and all eigenvalues will be only those for symmetric vibrational modes (i.e., those modes of vibration which have modal circles).

IX. THE CANTILEVER PLATE

9.1 Constant Thickness Plate

In this chapter the circular cantilever plate and in particular the stepped plate will be discussed. First, consider the uniform cantilever plate. Due to the axial symmetry of the integral equation consider, as in 8.3, only that part of the static isolated force solution which is independent of Θ. Hence

$$w(r) = A_0 + B_0 r^2 + C_0 \log r + D_0 r^2 \log r. \tag{88}$$

The boundary conditions are (see Fig. 5)

$$w_1(b) = 0 \qquad \left[\frac{dw_1}{dr}\right]_{r=b} = 0$$

$$\left[\frac{d^2 w_2}{dr^2} + \frac{\nu}{r}\frac{dw_2}{dr}\right]_{r=a} = 0 \qquad \frac{d}{dr}\left[\frac{d^2 w_2}{dr^2} + \frac{1}{r}\frac{dw_2}{dr}\right]_{r=a} = 0,$$

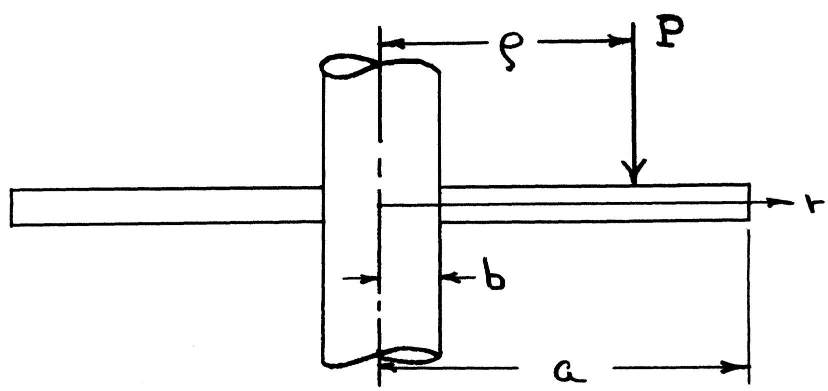

Fig. 5. Circular ring plate clamped on inner boundary subjected to concentrated normal force.

where $w_1(r)$ represents the deflection for $b \le r \le \rho$ and $w_2(r)$ represents the deflection for $\rho \le r \le a$. The continuity conditions are, at $r=\rho$,

$$w_1 = w_2 \qquad \frac{dw_1}{dr} = \frac{dw_2}{dr}$$

$$\left[\frac{d^2w_1}{dr^2} + \frac{\nu}{r}\frac{dw_1}{dr}\right] = \left[\frac{d^2w_2}{dr^2} + \frac{\nu}{r}\frac{dw_2}{dr}\right]$$

$$\frac{d}{dr}\left[\frac{d^2w_2}{dr^2} + \frac{1}{r}\frac{dw_2}{dr}\right] - \frac{d}{dr}\left[\frac{d^2w_1}{dr^2} + \frac{1}{r}\frac{dw_1}{dr}\right] = \frac{P}{2\pi D\rho}\ .$$

Substituting $w(r)$ into these conditions and evaluating the four constants gives

$$w_1 = \frac{P}{16\pi Ds}\Big\{(r^2-b^2)\Big[2a^2\left(\tfrac{1+\nu}{1-\nu}\right)+\rho^2+b^2\Big] - 2b^2\log\frac{r}{b}\Big[a^2\left(\tfrac{1+\nu}{1-\nu}\right)$$
$$+r^2+\rho^2+2a^2\left(\tfrac{1+\nu}{1-\nu}\right)\log\frac{\rho}{b}\Big] - 2a^2\left(\tfrac{1+\nu}{1-\nu}\right)\Big[r^2\log\frac{r}{\rho}+b^2\log\frac{\rho}{b}\Big]\Big\}$$
$$b \le r \le \rho$$

$$w_2 = \frac{P}{16\pi Ds}\Big\{(\rho^2-b^2)\Big[2a^2\left(\tfrac{1+\nu}{1-\nu}\right)+r^2+b^2\Big] - 2b^2\log\frac{\rho}{b}\Big[a^2\left(\tfrac{1+\nu}{1-\nu}\right) \tag{89}$$
$$+\rho^2+r^2+2a^2\left(\tfrac{1+\nu}{1-\nu}\right)\log\frac{r}{b}\Big] - 2a^2\left(\tfrac{1+\nu}{1-\nu}\right)\Big[\rho^2\log\frac{\rho}{r}+b^2\log\frac{r}{b}\Big]\Big\}$$
$$\rho \le r \le a$$

where $s = \left[a^2\left(\frac{1+\nu}{1-\nu}\right)+b^2\right]$.

Note Symmetry in r and ρ .

9.2 The Case of b = 0. Before considering, any further, the general case consider, for the moment, the case where b = 0. The Green's function then becomes

$$G(r,\rho)=\frac{\rho^2}{8\pi D}\left\{1+\frac{r^2}{2a^2}\left(\frac{1-\nu}{1+\nu}\right)-\log\frac{\rho}{r}\right\} \qquad \rho\leq r\leq a$$

$$G(r,\rho)=\frac{r^2}{8\pi D}\left\{1+\frac{\rho^2}{2a^2}\left(\frac{1-\nu}{1+\nu}\right)+\log\frac{\rho}{r}\right\} \qquad 0\leq r\leq \rho \tag{90}$$

The governing integral equation (51) becomes, as in the case of the circular plate clamped around the outer edge,

$$w(r)=\gamma t p^2\int_0^{2\pi}\int_0^a G(r,\rho)\,w(\rho)\,\rho\,d\rho\,d\alpha \tag{91}$$

$$w(r)=2\pi\gamma t p^2\int_0^a G(r,\rho)\,w(\rho)\,\rho\,d\rho\,. \tag{92}$$

To make the kernel symmetric use the transformation

$$\phi(r)=\sqrt{r}\;w(r)$$

$$K(r,\rho)=\sqrt{r\rho}\;G(r,\rho).$$

Then

$$\phi(r)=2\pi\gamma t p^2\int_0^a K(r,\rho)\,\phi(\rho)\,d\rho\,. \tag{93}$$

Substituting the Green's function, as defined, into this integral equation yields

$$\phi(r) = r^{1/2}\lambda\Big\{\int_0^r \rho^{5/2}\Big[1+\frac{r^2}{2a^2}\Big(\frac{1-\nu}{1+\nu}\Big)-\log\frac{\rho}{r}\Big]\phi(\rho)\,d\rho$$

$$+\int_r^a r^2\rho^{1/2}\Big[1+\frac{\rho^2}{2a^2}\Big(\frac{1-\nu}{1+\nu}\Big)+\log\frac{\rho}{r}\Big]\phi(\rho)\,d\rho \qquad (94)$$

where
$$\lambda = \frac{\gamma t p^2}{4D}.$$

Now, apply the method of collocation assuming a function of the form

$$w(r) = \sum_{k=1}^{m} C_k r^{k+1}. \qquad (95)$$

Evaluation of equation (94) using only the first term of series (95) yields

$$\phi(r) = C_1 r^{1/2}\lambda\Big\{\frac{3r^2a^4}{16}+\frac{r^2a^4}{12}\Big(\frac{1-\nu}{1+\nu}\Big)+\frac{r^2a^4}{4}\log\frac{a}{r}+\frac{r^6}{144}\Big\}. \qquad (96)$$

Satisfying this equation at r = a gives a fundamental frequency of

$$p_1 = \frac{4.09}{a^2}\sqrt{D/\gamma t},$$

where a Poisson ratio $\nu = 0.3$ was used. Taking the first two terms of series (95) and evaluating (94) gives

$$\phi(r) = C_1 r^{1/2}\lambda\left\{\frac{3r^2a^4}{16} + \frac{r^2a^4}{12}\left(\frac{1-\nu}{1+\nu}\right) + \frac{r^2a^4}{4}\log\frac{a}{r} + \frac{r^6}{144}\right\}$$

$$+ C_2 r^{1/2}\lambda\left\{\frac{4r^2a^5}{25} + \frac{r^2a^5}{14}\left(\frac{1-\nu}{1+\nu}\right) + \frac{r^2a^5}{5}\log\frac{a}{r} + \frac{4r^7}{1225}\right\}. \qquad (97)$$

Satisfying equation (97) at r = a and r = a/2 and evaluating the resulting determinant (2nd order) will yield the first two frequencies as

$$p_1 = \frac{3.69}{a^2}\sqrt{D/\gamma t}\ , \quad p_2 = \frac{21.9}{a^2}\sqrt{D/\gamma t}\ .$$

The first solution of this problem was suggested by Southwell* who obtained it directly from the differential equation of motion. He developed the frequency equation in terms of various Bessel functions and solved it graphically for a value of $\nu = 0.3$ for Poisson's ratio. The first two frequencies were reported as

$$p_1 = \frac{3.75}{a^2}\sqrt{D/\gamma t}\ , \quad p_2 = \frac{20.9}{a^2}\sqrt{D/\gamma t}\ .$$

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*R.V.Southwell-"Free Transverse Vibration of a Circular Disc Clamped at its Centre",Proc.Royal Soc.of London,Vol.10,1922.

By way of comparison with existing methods consider the first two frequencies using the Rayleigh-Ritz method with two terms of series (95). Note that series (95) satisfies the geometric boundary conditions.

For the application of Rayleigh-Ritz the potential and kinetic energies in polar coordinates are needed, which are given by the expressions

$$V = \pi D \int_0^a \left[\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} \right)^2 - 2(1-\nu)\frac{d^2 w}{dr^2} \cdot \frac{1}{r}\frac{dw}{dr} \right] r\,dr$$

$$T = \pi \gamma t \int_0^a \left(\frac{dw}{dt} \right)^2 r\,dr. \qquad (98)$$

Substituting these expressions in the energy balance where, for a normal mode vibration, $w(r) = w_0(r) \cos p\tau$ yields

$$\int_0^a \left[\left(\frac{d^2 w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr} \right)^2 - 2(1-\nu)\frac{d^2 w_0}{dr^2} \cdot \frac{1}{r}\frac{dw_0}{dr} - \frac{p^2 \gamma t}{D} w_0^2 \right] r\,dr = 0. \qquad (99)$$

If w_0 is chosen as an exact mode shape equation (99) will be satisfied. For an approximate mode shape such as series (95) the C_k must be adjusted to give the left side of equation (99) its minimum value. Hence

$$\frac{\partial}{\partial C_k} \int_0^a \left[\left(\frac{d^2 w_0}{dr^2} + \frac{1}{r}\frac{dw_0}{dr} \right)^2 - 2(1-\nu)\frac{d^2 w_0}{dr^2} \cdot \frac{1}{r}\frac{dw_0}{dr} - \frac{p^2 \gamma t}{D} w_0^2 \right] r\,dr = 0 \qquad (100)$$

where w_0 is given by series (95),

$$w_0(r) = \sum_{k=0}^{\infty} C_k r^{k+1} .$$

Choosing two terms and substituting in (100) yields (for $\nu = .3$) frequencies of

$$p_1 = \frac{3.92}{a^2}\sqrt{D/\gamma t} \;, \quad p_2 = \frac{40.8}{a^2}\sqrt{D/\gamma t} .$$

This comparison of the frequencies as determined by the Rayleigh-Ritz and integral equation approach indicates that with an assumed function which is very approximate the use of Green's function acting on the assumed function will yield frequencies which better approximate the true value than does the Rayleigh-Ritz method. To put it another way, if the Rayleigh-Ritz method is to approximate the true frequencies to as good a degree as the integral equation approach more terms will have to be taken in the assumed series (95) or a function which more accurately approximates the true modal shape will have to be taken. In either case the computational work will be increased considerably. The choice of series (95) was made so that a comparison could be made between the integral equation approach and Rayleigh-Ritz operating on the same function.

Consider the first and second approximate mode shapes for the circular plate clamped at b = 0. The solution of the integral equation by collocation gives two simultaneous equations in C_1 and C_2. They are

$$C_1(1-.2393\lambda a^4)+C_2 a(1-.2017\lambda a^4)=0$$

$$C_1(2-.8122\lambda a^4)+C_2 a(1-.6744\lambda a^4)=0. \qquad (101)$$

Solving these equations in terms of C_1 and substituting gives, for $k=r/a$,

$$w_1(r)=3.41\,C_1 a^2\left[.115-.132\log k+.00694k^4-.00193k^5\right]k^2$$

$$w_2(r)=120\,C_1 a^2\left[-.00457-.0132\log k+.00694k^4-.0039k^5\right]k^2. \qquad (102)$$

The modal shapes are plotted in Figs. 6 and 7.

The fundamental frequency can be obtained using the trace of the kernel, with little difficulty, in this case. From before the following equations are used:

$$\lambda_1=1/A_1 \qquad A_1=\int_0^a K(r,r)\,dr$$

$$K(r,r)=\frac{r^3}{8\pi D}\left[1+\frac{r^2}{2a^2}\left(\frac{1-\nu}{1+\nu}\right)\right]$$

$$\phi(r)=2\pi\gamma t p^2\int K(r,\rho)\,\phi(\rho)\,d\rho.$$

Letting $\lambda_1=2\pi\gamma t p_1^2$ and evaluating A_1 results in

$$A_1=\frac{1}{8\pi D}\int_0^a\left[r^3+\frac{r^5}{2a^2}\left(\frac{1-\nu}{1+\nu}\right)\right]dr=\frac{1.18a^4}{32\pi D}.$$

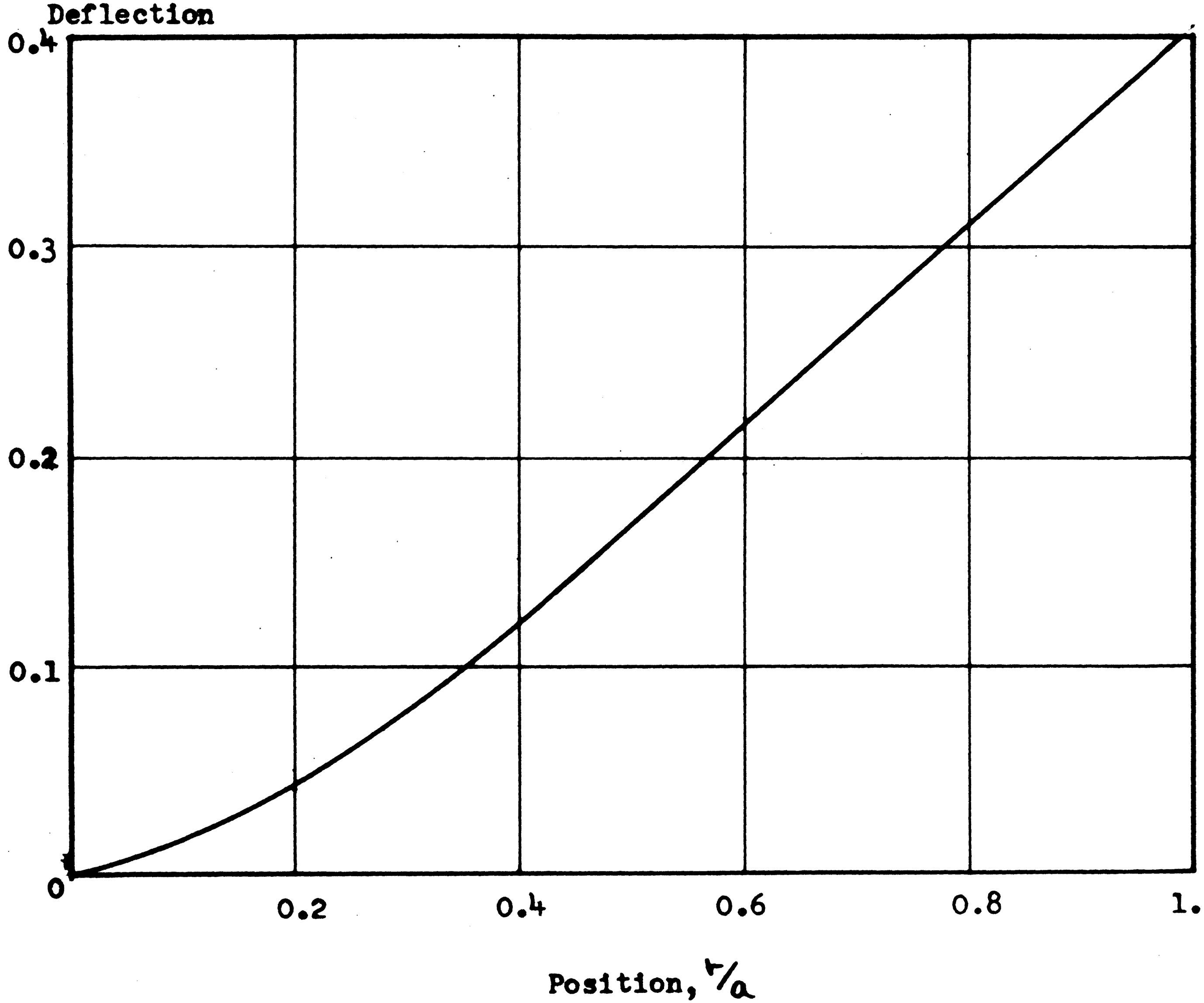

Fig. 6. Approximate first mode shape for the circular ring plate clamped on inner boundary.

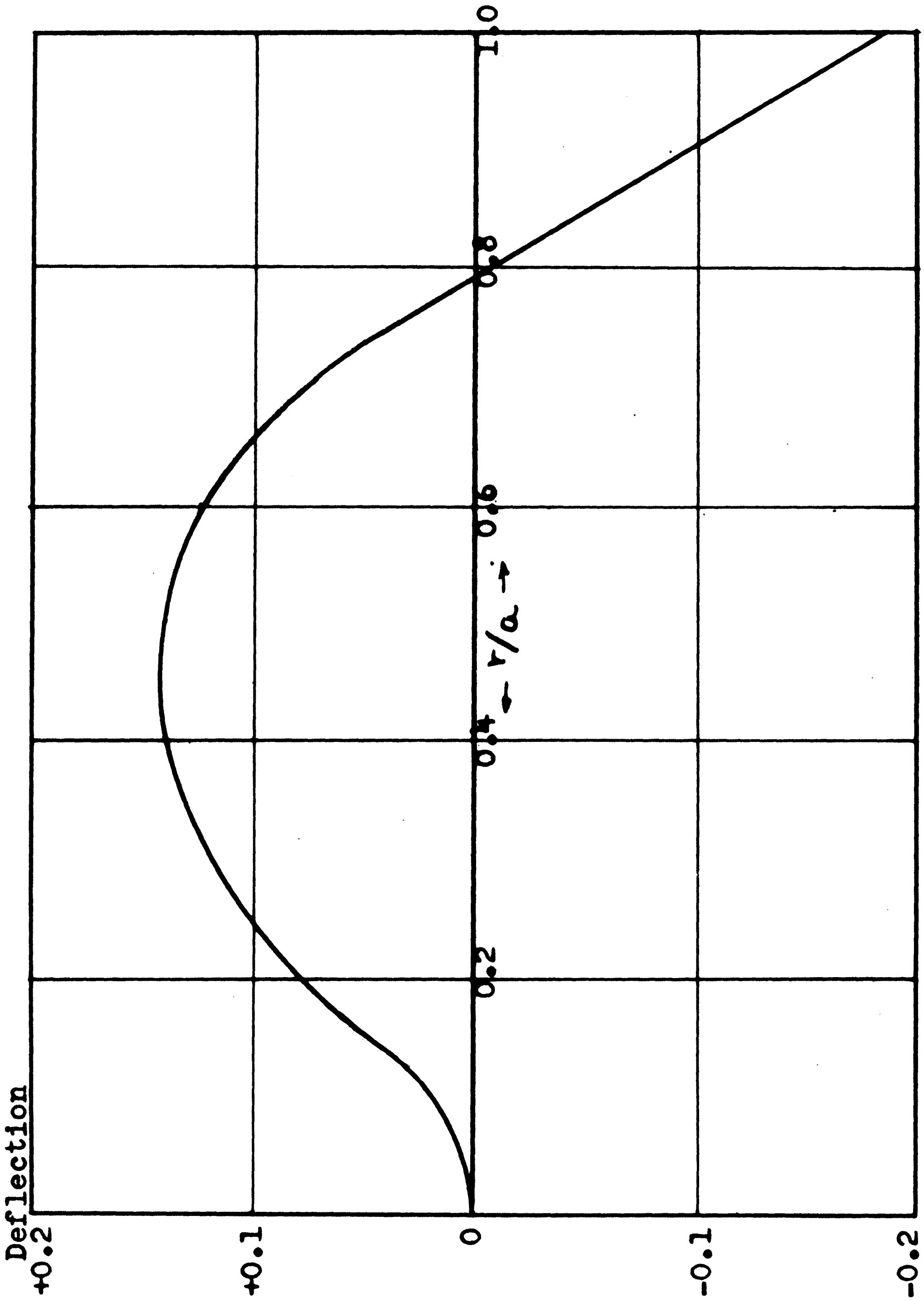

Fig. 7. Approximate second mode shape for circular ring plate clamped on inner boundary.

Hence $\lambda_1 = \frac{32\pi D}{1.18 a^4}$ which gives an approximation for

$$P_1 = \frac{3.68}{a^2}\sqrt{D/\gamma t} \;.$$

This is a lower bound and compares favorably to

$$P_1 = \frac{3.75}{a^2}\sqrt{D/\gamma t} \;;$$

Southwell's results.

A more accurate value can be determined from eqtn (28) with m = 2, repeated here,

$$\lambda_1 = \frac{1}{\sqrt{A_2}} \;.$$

By definition

$$A_2 = \int K_2(r,r)\,dr$$

and due to the symmetry of the kernel A_2 can be rewritten

$$A_2 = 2\int_0^a dr \int_0^r K^2(r,\rho)\,d\rho \qquad (103)$$

where

$$K(r,\rho) = \sqrt{r\rho}\;G(r,\rho).$$

Again due to the symmetry of $K(r,\rho)$ consider, in eqtn (103), only one region of definition of $K(r,\rho)$ say $0 \leq r \leq \rho$. This accounts for the 2 in eqtn (103) Substituting the value of the kernel found previously into eqtn (103) and evaluating the following integral

$$A_2 = 2\int_0^a dr \int_0^r \frac{r^5}{64\pi^2 D^2}\Big\{\rho + \frac{\rho^5}{4a^4}\left(\frac{1-\nu}{1+\nu}\right)^2 + \rho\log^2\frac{\rho}{r} + \frac{\rho^3}{a^2}\left(\frac{1-\nu}{1+\nu}\right) + 2\rho\log\frac{\rho}{r} + \frac{\rho^3}{a^2}\left(\frac{1-\nu}{1+\nu}\right)\log\frac{\rho}{r}\Big\}\,d\rho \qquad (104)$$

yields

$$A_2 = \frac{a^8}{32\pi^2 D^2}\left[\frac{12.19}{288}\right].$$

But $\lambda_1 = 1/\sqrt{A_2}$ and $\lambda_1 = 2\pi\gamma t p_1^2$. Hence

$$p_1 = \frac{3.70}{a^2}\sqrt{D/\gamma t}$$

which tightens the lower bound to the fundamental frequency a little.

9.3 The Case for $b \neq 0$.

Consider now the general case of $b \neq 0$. The determination of the trace of the kernel is as follows:

$$K(r,r) = \Big\{r^5 - rb^4 + 2a^2\left(\frac{1+\nu}{1-\nu}\right)(r^3 - rb^2) - 4r^3b^2\log\frac{r}{b} - 4a^2b^2\left(\frac{1+\nu}{1-\nu}\right)\Big[r\log\frac{r}{b} + r\log^2\frac{r}{b}\Big]\Big\} \qquad (105)$$

$$A_1 = \left\{ \frac{a^2}{2}\left(\frac{1+\nu}{1-\nu}\right)(a^2-b^2)^2 - 2a^4b^2\left(\frac{1+\nu}{1-\nu}\right)\log^2\frac{a}{b} - a^4b^2\log\frac{a}{b} \right.$$
$$\left. - \frac{b^4}{12}(6a^2-b^2) + \frac{a^4}{12}(2a^2+3b^2) \right\}. \qquad (106)$$

Letting $\frac{b}{a} = \rho$ and $\lambda_1 = \frac{\gamma t p_1^2}{8D}$

gives a lower bound approximation to p_1,

$$p_1^2 = \frac{\frac{8D}{\gamma t a^4}\left[\rho^2 + \left(\frac{1+\nu}{1-\nu}\right)\right]}{\frac{1}{2}\left(\frac{1+\nu}{1-\nu}\right)\left[(1-\rho^2)^2 - 4\rho^2\log^2\rho\right] + \rho^2\log\rho + \frac{1}{12}\rho^6 - \frac{1}{2}\rho^4 + \frac{1}{4}\rho^2 + \frac{1}{6}}. \qquad (107)$$

A plot of equation (107) is shown in Fig. 8. It can be seen that for $\rho = 0$ the value of the fundamental frequency is precisely the same as that for the uniform plate clamped at the center. As ρ increases (i.e. the ring plate gets narrower) the fundamental frequency increases rapidly becoming infinite at $\rho = 1$.

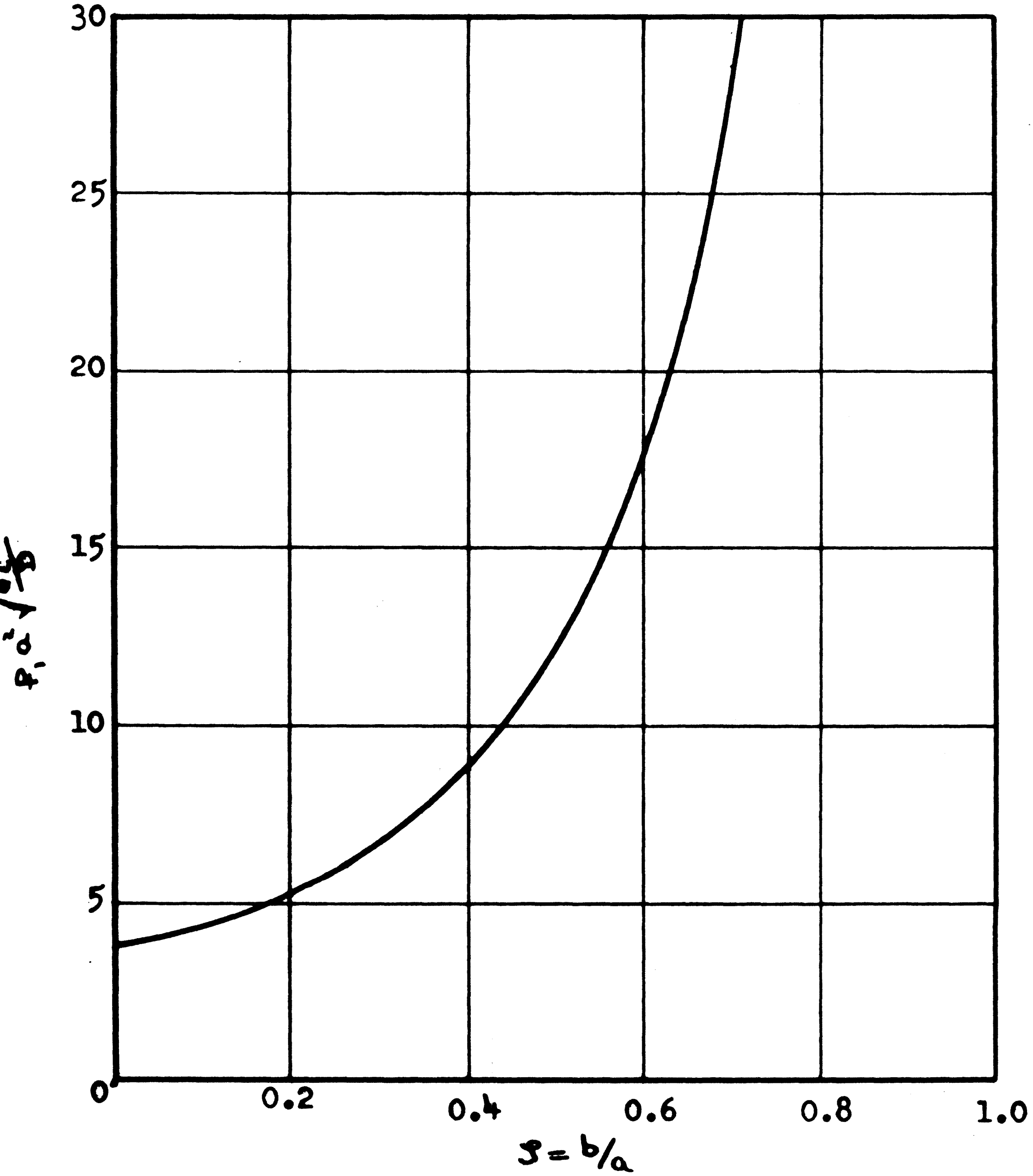

Fig. 8. Fundamental frequency curve for ring plate clamped on inner boundary.

9.4 The circular stepped-plate clamped at the centre

For the circular, cantilevered, stepped plate the methods of the preceding sections are completely applicable. When determining the fundamental frequency from the trace of the kernel the determination of Green's functions can be simplified by taking advantage of the structure of K(r,r). In general, the Green's function would be obtained from the statical deflection of a point at "r" due to a unit load at " ρ ". A look at Fig. 9 reveals that this results in six separate deflection functions satisfying the six regions.

$$0 \le \rho \le c: \qquad 0 \le r \le \rho, \; \rho \le r \le c, \; c \le r \le a$$

$$c \le \rho \le a: \qquad 0 \le r \le c, \; c \le r \le \rho, \; \rho \le r \le a.$$

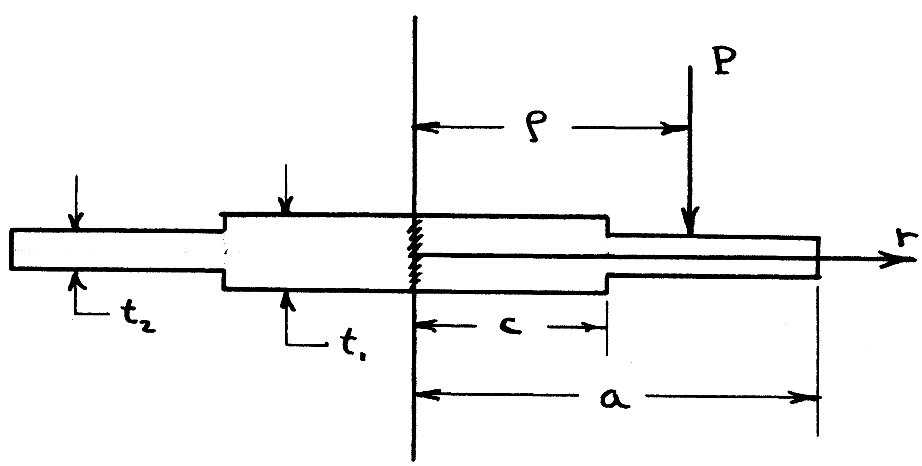

Fig. 9. A circular stepped plate clamped at the centre subjected to a concentrated normal force.

However, the meaning of $G(r,r)$ is the deflection at "r" due to a unit load at "r". This reduces the number of separate solutions to two, satisfying two regions

$$0 \leq r \leq c \qquad \text{and} \qquad c \leq r \leq a .$$

The integral equation, for a homogeneous plate whose thickness changes from t_1 to t_2, is given by

$$\phi(r) = \lambda \left\{ \int_0^c K(r,\rho_1)\,\phi(\rho_1)\,d\rho_1 + \int_c^a K(r,\rho_2)\,\phi(\rho_2)\,d\rho_2 \right. \qquad (108)$$

where $\lambda = \dfrac{\gamma t_1 p^2}{16 D_1}$, $k = t_2/t_1$, $h = D_2/D_1 = k^3$

and $K(r,\rho_1) = 32\pi D_1 \sqrt{r\rho_1}\; G(r,\rho_1)$

$K(r,\rho_2) = 32\pi D_1 k \sqrt{r\rho_2}\; G(r,\rho_2)$.

To use the trace of the kernel for the determination of the fundamental frequency evaluate the integral

$$A_1 = \int_0^c K(r_1,r_1)\,dr_1 + \int_c^a K(r_2,r_2)\,dr_2 . \qquad (109)$$

The solution of the isolated force statical problem yields

$$G(r_1,r_1)=\frac{r_1^2}{8\pi D s}\left\{c^2(1-n)+a^2\left(\frac{1+v}{1-v}+n\right)+\frac{r_1^2}{2}\left[\left(\frac{1-v}{1+v}+n\right)+\frac{a^2}{c^2}(1-n)\right]\right\}$$

(110)

$$\begin{aligned}G(r_2,r_2)=\frac{1}{32\pi n D_1 s}\Big\{&-2c^2(1-n)\left[c^2(1-n)+a^2n+2a^2\left(\tfrac{1+v}{1-v}\right)(1-2\log c+2\log^2 c)\right]\\&+4r_2^2\left[2c^2(1-n)\log c+a^2\left(\tfrac{1+v}{1-v}+n\right)\right]+2r_2^4\left[1+n\left(\tfrac{1-v}{1+v}\right)\right]\\&-8c^2(1-n)r_2^2\log r_2-8a^2c^2(1-n)\left(\tfrac{1+v}{1-v}\right)\log^2 r_2\\&-8a^2c^2(1-n)\left(\tfrac{1+v}{1-v}\right)(1-2\log c)\log r_2\Big\}\end{aligned}$$

where

$$s=c^2(1-n)+a^2\left(\frac{1+v}{1-v}+n\right).$$

Then

$$\begin{aligned}A_1=\frac{4}{s}\int_0^c r_1^3\left\{c^2(1-n)+a^2\left(\tfrac{1+v}{1-v}+n\right)+\tfrac{r_1^2}{2}\left[\left(\tfrac{1-v}{1+v}+n\right)+\tfrac{a^2}{c^2}(1-n)\right]\right\}dr_1\\+\frac{1}{k^2 s}\int_c^a\Big\{&-2c^2(1-n)\left[c^2(1-n)+a^2n+2a^2\left(\tfrac{1+v}{1-v}\right)(1-2\log c+2\log^2 c)\right]r_2\\&+4r_2^3\left[2c^2(1-n)\log c+a^2\left(\tfrac{1+v}{1-v}+n\right)\right]+2r_2^5\left[1+n\left(\tfrac{1-v}{1+v}\right)\right]\\&-8c^2(1-n)r_2^3\log r_2-8a^2c^2(1-n)\left(\tfrac{1+v}{1-v}\right)r_2\log^2 r_2\\&-8a^2c^2(1-n)\left(\tfrac{1+v}{1-v}\right)(1-2\log c)r_2\log r_2\Big\}\,dr_2.\end{aligned}$$

(111)

The evaluation of this integral, when $\rho = \frac{c}{a}$, leads to

$$A_1 = \frac{a^6}{s}\left\{\frac{2}{3}\rho^6\left[\frac{(2+\nu)-n(1+\nu)}{1+\nu}\right] + \frac{2}{3}\rho^4\left[\frac{(2+\nu)+n(1-\nu)}{1-\nu}\right]\right\}$$

$$+\frac{a^6}{k^2 s}\left\{2\rho^2(1-n)\log\rho - 4\rho^2\left(\frac{1+\nu}{1-\nu}\right)(1-n)\log^2\rho\right.$$

$$+\frac{1}{2}\rho^2(1-n)(1-2n)(1-\rho^2)^2$$

$$+\left(\frac{1+\nu}{1-\nu}+n\right)\left[(1-\rho^4)+\frac{1}{3}\left(\frac{1-\nu}{1+\nu}\right)(1-\rho^6)\right] \qquad (112)$$

$$\left.-2\rho^2\left(\frac{1+\nu}{1-\nu}\right)(1-n)(1-\rho^2)\right\},$$

where $$s = a^2\left\{\rho^2(1-n) + \left(\frac{1+\nu}{1-\nu}+n\right)\right\}.$$

Now the fundamental frequency can be approximated by the inverse of the trace of the kernel. Hence,

$$\lambda_1 = \frac{1}{A_1} = \frac{\gamma t_1 p_1^2}{16 D_1}$$

or

$$p_1 = 4A_1^{-1/2}\sqrt{D_1/\gamma t_1}\,. \qquad (113)$$

A plot of equation (113) is shown in Fig. 10. These curves have been plotted for various values of k, the ratio of outside plate thickness to the inside plate thickness. The bounds are formed by the curve for k = 1 where the frequency is given for a homogeneous plate clamped at the center and the curve for k = 0 where the inside plate thickness has become infinitely large resulting in a cantilevered ring.

It is to be remarked again that the fundamental frequency determined here is a lower bound to all possible fundamental frequencies. In working through the development of the trace of the kernel it was found that a truly significant saving in effort was realized by taking advantage of the specialized structure of the kernel K (r,r). Of course, this will limit the generality of the static deflection functions which have applicability in their own right. A word about the Green's function as the solution to a static deflection problem is in order here. The Green's function developed from section 9.1 represents the deflection of a circular cantilever plate subjected to a uniform, concentric ring loading of unit total magnitude. The deflection at any point "r" in the plate under the action of any total magnitude of load will be given by the developed Green's function multiplied by the total magnitude of load. On the other hand, the Green's function developed for section 9.4 gives only the deflections under the ring loading where the total magnitude of load is unity.

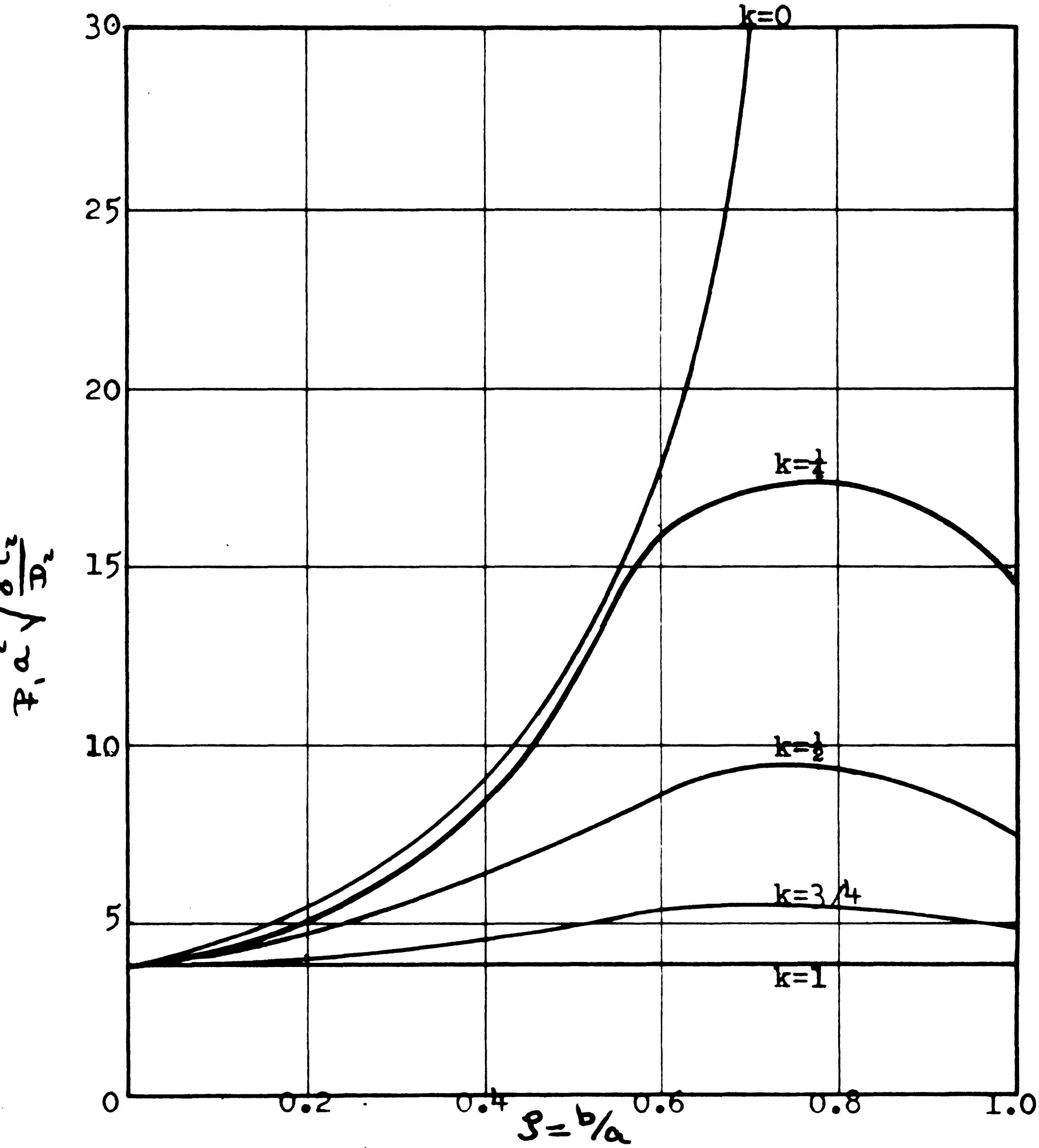

Fig.10. Fundamental frequency wave for a circular stepped plate clamped at centre.

Although the collocation method was not attempted for the stepped plate it should be noted, for completeness, that the procedure can also be significantly simplified when applied to the plate of variable thickness. If, instead of developing a general Green's function for the plate, it is recognized that the integral equation will be collocated at specific values of "r" then one can develop the Green's function (and hence the kernel) for separate unit loads at the collocation points. This technique has been applied to the determination of frequencies for the beam of variable crossection.* In this reference a tabular procedure was used and results were shown to compare favorably with those obtained by Myklestad's method.

Finally, a word should be said about the generality of the trace given by (112). For the problem considered here the only thing that changed across the step in the plate was the thickness. If instead of or in addition to the change in thickness the material changed thus changing the flexural rigidity D, of the plate by more than the thickness effect (i.e. change in modulus of elasticity) equation (112) is still applicable provided the proper values of Poisson's ratio, ν, is used as well as the

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*R.Chicural and E.Suppiger,"A Tabular Collocation Method for Beam Vibration"-ASME Paper No. 60 - WA-76, 1960.

thickness ratio, k, which would no longer be a simple cube root of $n = D_2/D_1$. The density change could also be accounted for in the same way. One change that cannot be accounted for in equation (112) is that of Poisson's ratio. Poisson's ratio was considered constant in constructing Green's function and hence ν is intermixed in equations (110).

X. THE CLAMPED ELLIPTIC PLATE

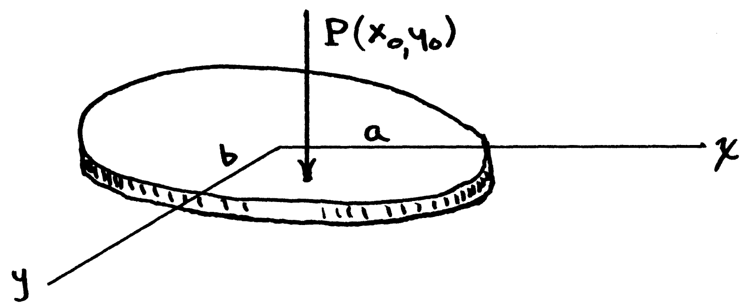

Fig. 11. Clamped elliptic plate subjected to normal force.

Consider as a final example of frequency determination using the theory of integral equations a thin, homogeneous, elliptic plate clamped around its outer boundary. For this case the application of D'Alembert's Principal, assuming normal mode vibrations leads to the integral equation in elliptic coordinates

$$w(\xi,\eta) = \gamma t_p^2 \int_0^\alpha \int_0^\pi G(\xi,\eta;\beta,\delta)\, w(\beta,\delta)\, dA(\beta,\delta). \tag{114}$$

The transformation equations necessary to carry the rectangular coordinates into elliptic coordinates are

$$x = c \cosh\xi \cos\eta \qquad x_0 = c \cosh\beta \cos\delta$$

$$y = c \sinh \sin\eta \qquad y_0 = c \sinh\beta \sin\delta$$

$$c^2 = a^2 - b^2 > 0$$

and the bounding curve is given by $\xi = \alpha$. The element of area dA in elliptic coordinates becomes

$$dA = c^2(\sinh^2\beta + \sin^2\delta)\, d\beta\, d\delta. \tag{115}$$

The solution of the isolated force problem for a clamped elliptic plate is given by H.M. Sengupta* as

$$
\begin{aligned}
w(\xi,\eta;\beta,\delta) = -\frac{Pc^2}{32\pi D}\Bigg\{ & -\sum_{i=0}^{\infty}\left[P_i(\xi)\cos i\eta + Q_i(\xi)\sin i\eta\right] \\
& + A_0' + A_0\cosh 2\xi + (A_1'\cosh\xi + A_1\cosh 3\xi)\cos\eta \\
& + (B_1'\sinh\xi + B_1\sinh 3\xi)\sin\eta \\
& + \sum_{i=2}^{\infty}\Big\{\left[A_{i-2}\cosh(i-2)\xi + A_i'\cosh i\xi + A_n\cosh(i+2)\xi\right]\cos i\eta \\
& + \left[B_{i-2}\sinh(i-2)\xi + B_i'\sinh i\xi + B_i\sinh(i+2)\xi\right]\sin i\eta\Big\}\Bigg\}
\end{aligned}
\tag{116}
$$

where a complete listing of the coefficients is given in the reference cited. The Green's function, now, is simply the deflection function quoted above when $P = 1$. As in the case of the circular plate the kernel

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*"On the bending of an elastic plate-I", H.M.Sengupta, Bull. of Calcutta Mathematical Society, Vol. 41-1949, pp 163-172.

of the integral equation (114) will be unsymmetrical. To symmetrify the kernel let

$$G'(\xi,\eta;\beta,\delta) = \frac{32\pi D}{c^4} G(\xi,\eta;\beta,\delta).$$

Then the kernel of the symmetric integral equation

$$\phi(\xi,\eta) = \lambda \int K(\xi,\eta;\beta,\delta)\,\phi(\beta,\delta)\,d\beta\,d\delta \qquad (117)$$

will be given by

$$K(\xi,\eta;\beta,\delta) = \left[(\sinh^2\beta + \sin^2\delta)(\sinh^2\xi + \sin^2\eta)\right]^{1/2} G'(\xi,\eta;\beta,\delta)$$

and

$$\lambda = \frac{c^4 \gamma t}{32\pi D} p^2.$$

As before, define the trace of the kernel by the integral

$$A_1 = \int_0^{\alpha}\int_0^{\pi} K(\xi,\eta;\xi,\eta)\,d\xi\,d\eta.$$

Choosing only those terms from the general Green's function, for which i = 0 yields

$$K(\xi,\eta;\xi,\eta) = P_0(\xi) + A_0' + A_0 \cosh 2\xi.$$

Upon substituting the values of the $P_0(\xi)$, A_0'

and A_o from those given in the reference and simplifying

$$K(\xi,\eta,\xi,\eta) = (\sinh^2\xi + \sin^2\eta)\Big[4(\xi-\alpha)\cosh 2\xi + 2(\xi-\alpha)\cos 2\eta$$

$$+(\cosh^2 2\alpha - \cosh^2 2\xi)\operatorname{csch} 2\alpha + (\cosh 2\alpha - \cosh 2\xi)\operatorname{csch} 2\alpha \cos 2\eta\Big]. \tag{118}$$

One should again note that the part of the Green's function remaining when $i = 0$ satisfies the boundary at $\xi = \alpha$ namely, that

$$G(\xi,\eta;\beta,\delta) \equiv \frac{\partial}{\partial\xi} G(\xi,\eta;\beta,\delta) \equiv 0$$

which, of course, it must in order to be useful for the present purposes.

The trace of the kernel then results from an evaluation of the integral

$$A_1 = \int_0^{\alpha}\int_0^{\pi} (\sinh^2\xi + \sin^2\eta)\Big[4(\xi-\alpha)\cosh 2\xi + 2(\xi-\alpha)\cos 2\eta$$

$$+ (\cosh^2 2\alpha - \cosh^2 2\xi)\operatorname{csch} 2\alpha \tag{119}$$

$$+ (\cosh 2\alpha - \cosh 2\xi)\operatorname{csch} 2\alpha \cos 2\eta\Big]\, d\xi\, d\eta.$$

Hence

$$A_1 = \frac{\pi}{12}\Big\{1 + \cosh^2 2\alpha - 3\alpha \tanh\alpha - 3\alpha^2\Big\}. \tag{120}$$

The approximation to the fundamental frequency now becomes

$$\lambda_1 = 1/A_1 = \frac{\gamma t c^4}{32\pi D} p_1^2$$

or

$$p_1^2 = \frac{32\pi D}{\gamma t c^4 A_1}. \qquad (121)$$

Experimental frequencies for the clamped elliptical plate seem scarce, in the literature, but a single value was found in a paper by W.H. Hoppmann II.* Hoppmann reports a value of 460 cps for the experimental fundamental frequency of an unstiffened clamped elliptical plate with the following properties:

Major axis - 11 2/16 in   Density = 0.096 lb/in^3

Minor axes = 5 19/32 in

$E = 10^7$ psi

$\nu = 0.35$

$t = 0.065$ in.

Equation (120) and (121), using Hoppmann's values, would result in a frequency of 372 cps. This is about 19% below the experimental value. Hoppmann applied the Rayleigh-Ritz method to obtain a theoretical frequency and reports a calculated fundamental frequency of 585 cps which is about 27% above the experimental value, using two terms

* -

W.H.Hoppmann II,"Flexural vibration of orthogonally stiffened circular & elliptical plates"-Proceedings of the 3rd U.S.National Congress of Applied Mechanics, 1958, ppg.181-187.

in the assumed deflection function.

As the eccentricity of the ellipse goes to zero so does the value for "c". If this limiting case is considered, using equations (120) and (121), the expression for the fundamental frequency is precisely the same as that given for the clamped circular plate as previously determined.

The deviation of the calculated value from the experimental value cannot be attributed solely to the approximate nature of the method involved. To a greater extent the deviation is explained by the fact that only n=0 terms, in Sengupta's solution, were used in constructing Green's function. It was not deemed feasible to carry out the solution for more terms of Sengupta's series. The experimental elliptic plate, considered here, had a major-minor axis ratio of 2/1 which can be considered a fairly large eccentricity. As the eccentricity decreases for various elliptic plates it can be reasonably assumed that the deviation from the actual frequency will decrease inasmuch as the solution developed here becomes that for a circle when the eccentricity becomes zero.

A plot of equation (121) is shown in Fig. 12. The plot indicates that even for moderate eccentricities the frequency differs little from that for a circular plate.

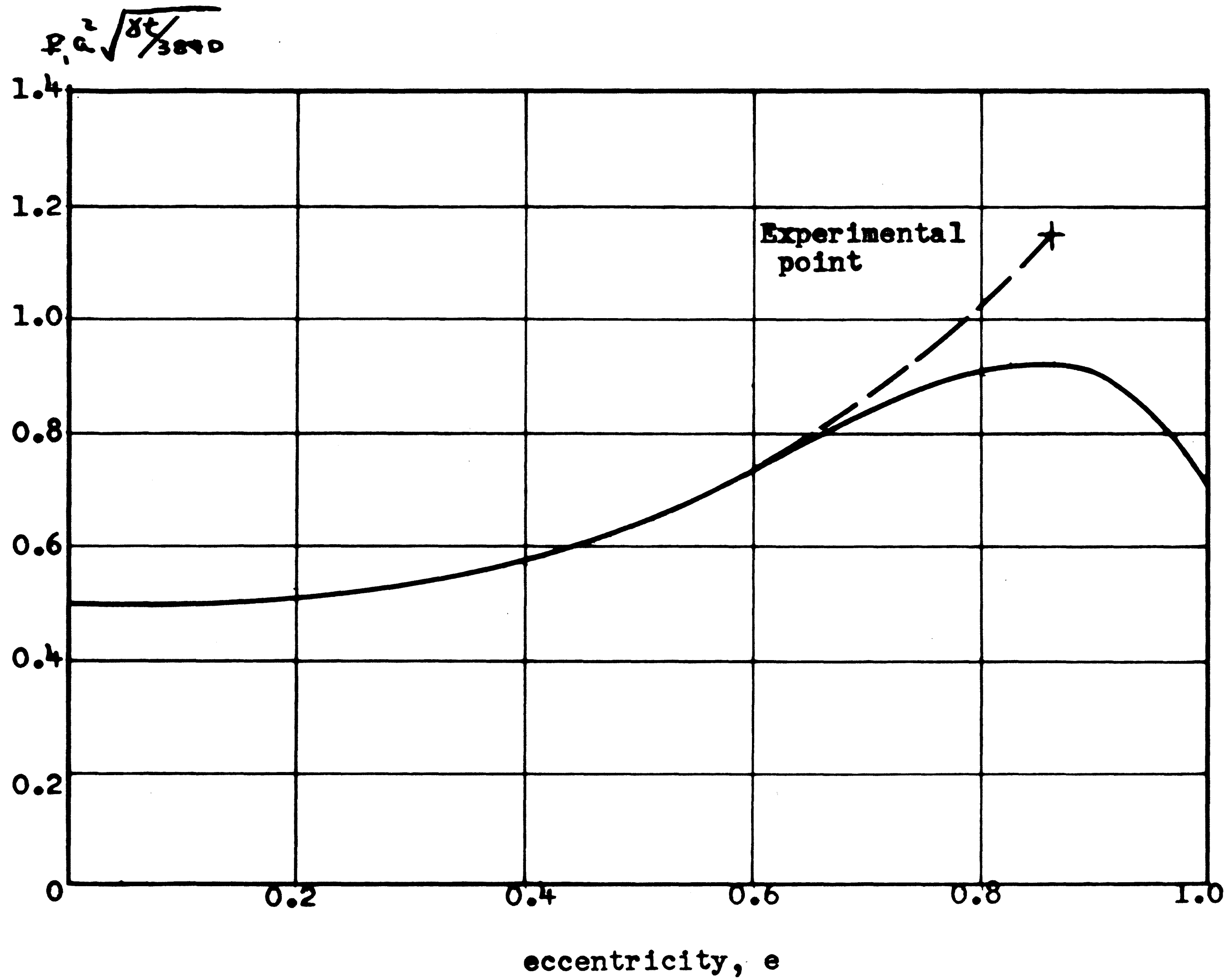

Fig. 12. Fundamental frequency curve for clamped elliptic plate.

This has been noted by H.B. Keller* in a paper deriving upper and lower bounds for the fundamental eigenvalue of clamped nearly circular plates. As has been mentioned previously the prediction equation (121) can be expected to yield values closer to the actual as the eccentricity decreases and conversely, as the eccentricity gets larger the prediction increases in error. This line of thinking would account for the reverse curvature as the eccentricity becomes larger than 0.6 and moves toward unity. As a matter of fact the equation (121) will yield a value for $p_1 a^2 \sqrt{\gamma t / 384 D}$ of 0.707 at an eccentricity of 1.0. If a curve (shown dotted, in Fig. 12) is drawn through the experimental point and joining smoothly with the curve of equation (121) (below an eccentricity of 0.6) the resulting curve would probably be closer to the shape of the true frequency variation.

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*H.B.Keller "Lowest Eigenvalue of a Nearly Circular Region", Quarterly of Applied Mathematics Vol. XII, pg. 141.

XI. SUMMARY AND CONCLUSIONS

An approach is made to the determination of the eigenvalues and eigenfunctions of a thin, elastic plate vibrating in free normal vibrations. From these eigenvalues and eigenfunctions the spectrum of resonant frequencies and the mode shapes can be determined. The approach is made through the analysis of a symmetric Fredholm, integral equation of the second kind. Three methods of obtaining the eigenvalues and eigenfunctions from the integral equation governing the free vibration of the plate are described and applied. The approach is then made to the solution of circular cantilever plates and the clamped elliptical plate. Results are obtained and compared, where possible, with existing experimental or analytical data.

The power of the method of integral equations for solving the plate vibration problem rests on the satisfaction of the boundary conditions by the very nature of the kernel of the integral equation. The disadvantage lies in the fact that, to develop the kernel, the static isolated force problem must first be solved. However, the solution of the static isolated force problem has value in its own right and as more and more of these problems are solved this disadvantage will decrease in degree.

The use of the trace of the kernel for evaluating the eigenvalue seems to be successful for fundamental eigenvalues. For subsequent eigenvalues the integrations involved may lead to a computational complexity which will damage its efficiency. A real value of this method is the convergence to the true value from below, yielding a lower bound. The method of collocation has value for the determination of both eigenvalues and eigenfunctions. As more terms are taken in the assumed deflection function convergence to the true value will result although not monotonically. "Successive approximations" is useful particularly when the kernel is given as a series of sine and cosine products. Here, as in the method of collocation the eigenfunction is a direct result of the approximation.

The results obtained here indicate that the accuracy of the integral equation approach compares favorably with that obtained using Rayleigh's method. When the trace of the kernel is used to determine the eigenvalue it was shown and frequently restated that a lower bound value is obtained. It is known that Rayleigh's method yields an upper bound although a lower bound value can be obtained as can an upper bound value from the integral equation approach. Using the two methods in conjunction a convenient bracket of the true eigenvalue can be obtained.

The case of degenerate vibration was not considered in depth as the main application was to symmetric vibration.

XII. ACKNOWLEDGEMENTS

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AN INTEGRAL EQUATION APPROACH

TO VIBRATING PLATES

by

Charles L. Best

ABSTRACT

A knowledge of the natural frequencies of a vibrating plate is of great importance if an effective design is to be made which will prevent critical conditions of heavy vibration from occurring. Those frequencies which are associated with the symmetric modes are especially important. Many approximate methods have been devised to determine these natural frequencies.

In this dissertation a method of frequency determination is suggested through an integral equation approach. The plate vibration problem is formulated as a problem in the solution of a homogeneous, linear Fredholm integral equation of the second kind in which the kernel is either symmetric or can be made so by a convenient transformation. The integral equation, as formulated, satisfies the boundary conditions in that it includes Green's function of the plate which is a solution of the isolated force problem. Three approximate methods for solving the integral equation are described mathematically and then applied to three elementary examples. The three methods

used are: 1) method of successive approximation, 2) method of collocation and 3) the trace of the kernel. It is shown that using the trace of the kernel always gives a lower bound to the frequency and is particularly useful for the determination of the fundamental frequency.

After solving the three elementary problems the integral equation approach is made to the uniform circular cantilever plate where the frequency is approximated both by collocation and by the use of the trace of the kernel. The first and second approximate mode shapes are then derived and shown graphically. The results are seen to compare favorably with results obtained from the Rayleigh-Ritz method.

Finally, the fundamental frequency is determined for the circular, stepped cantilever plate and the clamped elliptical plate. For the stepped plate fundamental frequency curves are drawn for various positions and magnitudes of the step. The fundamental frequency curve of the clamped elliptical plate is drawn as a function of the eccentricity of the ellipse. A frequency obtained from experiment is reported along with a calculated value determined from the Rayleigh-Ritz method. It is seen that the integral equation approach is about 19% below the experimental value whereas the Rayleigh-Ritz method gives a fundamental frequency about 27% above the experimental value.

The main disadvantage of the integral equation approach is the necessary solution of the isolated force problem for the construction of Green's function. However, as many of these problems have already been solved and more will be solved in the future this disadvantage will decrease in degree.