Nonlinear Stochastic Vibration in Geometrically Varying Beams

by

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1.0 INTRODUCTION

Here an investigation is made into the stochastic vibration behavior of nonlinear beams with geometrical irregularities. The nonlinearities arise from midplane stretching and are cubic in nature. These nonlinearities are considered small and thus the system is said to be weakly nonlinear. The properties of the beam are allowed to vary along its length, and thus an example can be given in which an internal resonance condition exists. An analysis of such a beam is made by Markov methods, and a solution for the moment characteristics is demonstrated.

A number of methods have evolved in the study of nonlinear vibrations. In the literature, a description of which follows, a number of these methods have been applied to single degree of freedom systems. There has been, however, limited effort in applying these methods to systems with more than one degree of freedom. In this investigation a multiple degree of freedom system is considered, and the solution is examined for the effects of modal coupling and internal resonances, phenomena which occur in nonlinear multiple degree of freedom systems.

As it is the purpose of this investigation to examine the stochastic response behavior of such a beam, the external forcing function must be a random function of time. In many engineering applications it is acceptable to assume that the random forcing function is a Gaussian white noise. This assumption makes it possible to write the equations of motion as Ito type stochastic differential equations. These equations have the advantage that their solutions are Markov processes and the probability density functions of the solutions satisfy the

Fokker-Planck equation. Using Ito stochastic calculus, one can find a set of ordinary differential equations for the moment characteristics and, after applying a closure technique, solve for the transitional and steady state moments of the solution process.

2.0 LITERATURE REVIEW

2.1 Deterministic Nonlinear Systems

There has been a large number of studies of the response of nonlinear beams to deterministic excitations. In this section a few of those papers are cited which have similarities to the type of nonlinear beam analysis considered here. Bennett and Eisley [1] analyzed a beam with cubic nonlinearities due to midplane stretching by first converting the partial differential equation to a set of ordinary differential equations, and then applying the method of harmonic balance to the first three mode equations. They compared their results with experimental results for symmetric and asymmetric forcing. Nayfeh, Mook and Lobitz [2] considered a nonlinear beam whose properties varied along its length. The partial differential equation of motion was converted to a set of ordinary differential equations by the normal mode approach, where the normal modes of the associated linear system were found by the finite element method. The response to a periodic excitation near one of the natural frequencies was found by a perturbation method, the method of multiple scales. A Lagrange type of finite element analysis was used by Sarma and Varadan [3] to find the nonlinear mode shapes and corresponding frequencies, also of a beam with nonlinearities due to midplane stretching.

2.2 Stochastic Nonlinear Systems

A number of books on stochastic differential equations such as Soong [4], Stratonovich [5], Srinivasan and Vasudevan [6], and Nigam [7] can be found in the field. Two surveys of the recent developments in the field are currently available, one by Crandall and Zhu [8] and a two-part one by Roberts [9,10].

One of the methods of analysis which is currently popular is equivalent linearization. In this method the nonlinear system is approximated by a linear system whose parameters are chosen to minimize the equation difference in some sense, usually by minimizing the mean square error term. Busby and Weingarten [11] used finite element discretization and equivalent linearization to get a one mode and a two mode approximate solution to the problem of a nonlinear beam subjected to certain types of random loading. They considered simply supported and clamped beams, and in the two mode approximation used the first and third modes. Richard and Anand [12] used a single mode expansion and equivalent linearization to find the response of a string to narrow band random excitation centered on the mode considered. Harrison and Hammond [13], also using equivalent linearization, found the transient statistics of a nonlinear system excited by a non-stationary random process. The input process was of a type known as frequency modulated, a type associated with the motion of vehicles over rough ground.

Hierarchy methods are another way of analyzing nonlinear stochastic systems. In these methods a set of ordinary differential equations for the statistical moments of the response process is derived either by direct averaging of the differential equations, or, if the input process is white noise, from the Fokker-Planck equation or Ito stochastic calculus. For nonlinear systems these moment equations form an infinite hierarchy which must be closed in some way. The processes for doing this are known as closure methods. One type of closure method involves assuming the response probability density function is of a certain form with adjustable coefficients. The probability density function is used to find the moments in terms of the coefficients of the density function, and those moments are substituted into the moment equations to solve for those coefficients. The values of the response moments can then be calculated. The other closure method assumes that the moments above a certain order are related to the lower moments in some way. For example, in a Gaussian distribution the cumulants of order three and higher, which can be expressed in terms of the moments, are equal to zero. Hierarchy methods are usually applied to systems with white noise excitations because of difficulties with correlation between the response and non-white noise excitations.

Crandall, in two papers [14,15], demonstrated a closure solution in which the probability distribution is represented by a Gram-Charlier expansion. As an example he considered a nonlinear oscillator whose exact solution is known and compared the results for various orders of closure with the exact solution. Expressing the probability density function as an Edgeworth expansion, Assaf and Zirkle [16] found an approximate solution for three different nonlinear systems, and compared these solutions to the exact solutions and the solutions obtained by equivalent linearization. Bover [17] gave a truncation scheme in which the probability distribution is approximated by an expansion in terms of the quasi-moments of the solution process. He also gave a comparison between the quasi-moments, central cumulants and the central moments.

Bellman and Richardson [18] proposed a closure scheme, based on the Poincare-Lyapunov stability theorem, that guarantees that the moment properties will be preserved. In this closure scheme the third and higher order moments are related to the lower moments in a matrix form, and the coefficients relating the moments are determined by minimizing a matrix norm. Sancho [19,20] demonstrated a technique for finding the moments of a nonlinear stochastic system by Ito's stochastic calculus. He then related the third and fourth order moments to the lower order moments by the closure scheme proposed by Bellman and Richardson in the article cited above, and arrived at a closed set of ordinary differential equations for the first and second order moments.

A cumulant neglect closure, in which the higher order moments are expressed in terms of the lower moments by setting the higher order cumulants equal to zero, was given by Wu and Lin [21] for several nonlinear oscillators. The solutions for various orders of closure were compared with the exact solutions and, in one case, the results of Crandall [14]. Wilcox and Bellman [22] compared the moment closure due to Bellman and Richardson [18], mean square averaging, central moment neglect and cumulant neglect closures, getting closed equations for the first and second order moments. A two degree of freedom nonlinear system with a quadratic nonlinearity was solved for the response moments by Ibrahim and Heo [23], by cumulant neglect closure. The transient and stationary response moments were found by truncating cumulants of order three and higher and cumulants of order five and higher.

Iyengar [24] proposed a method for studying nonlinear random vibration where the excitation is not necessarily Gaussian white noise. In this method the output is assumed to have a Gaussian autocorrelation and a Gaussian input-output cross-correlation. The differential equation is averaged directly with these assumptions, and differential equations for the autocorrelation and cross-correlation are found. These equations are solved by an iterative process on a digital computer. In another paper, Iyengar and Dash [25] applied the above procedure to a hysteretic system under white noise excitation, and compared their results with results arrived at by analog simulation and equivalent linearization.

A different type of solution from those above was proposed by Langley [26]. He proposed a solution for the response probability density function of a system excited by white noise by directly solving numerically the stationary Fokker-Planck equation by the finite element method. He applied the procedure to the Duffing oscillator and the ship rolling problem, and compared his results with the exact solution in the case of the Duffing oscillator.

3.0 NONLINEAR EQUATIONS OF MOTION

This chapter contains a description of the development of the nonlinear differential equations governing the motion of beams with ends constrained against longitudinal motion. First the differential equations for the transverse and longitudinal motion of the beam are derived, then a single partial differential equation for the transverse motion is found, and finally that equation is converted to an infinite set of ordinary differential equations.

The partial differential equations of motion of a beam constrained from longitudinal motion are :

$$\rho A\ddot{u} + \frac{\partial}{\partial s}\left(EA(s)\left(u' + \frac{1}{2}w'^2\right)\right) = 0 \tag{3.1}$$

$$\rho A\ddot{w} + \frac{\partial^2}{\partial s^2}\left(EI(s)w''\right) - \frac{\partial}{\partial s}\left(EA(s)\left(u' + \frac{1}{2}w'^2\right)w'\right) = G(s,t) \tag{3.2}$$

where, as shown on the beam element in Fig. 1, w and u are, respectively, the transverse and longitudinal components of displacement of the point located at s in the undeformed beam, ρ, E, A and I are the mass density, modulus of elasticity, cross sectional area, and moment of inertia, respectively, of the beam, and $G(s,t)$ is the forcing function. Primes represent partial differentiation with respect to s, the coordinate along the length of the undeformed beam, while dots represent partial differentiation with respect to t, the time coordinate. It is convenient to introduce the following dimensionless variables.

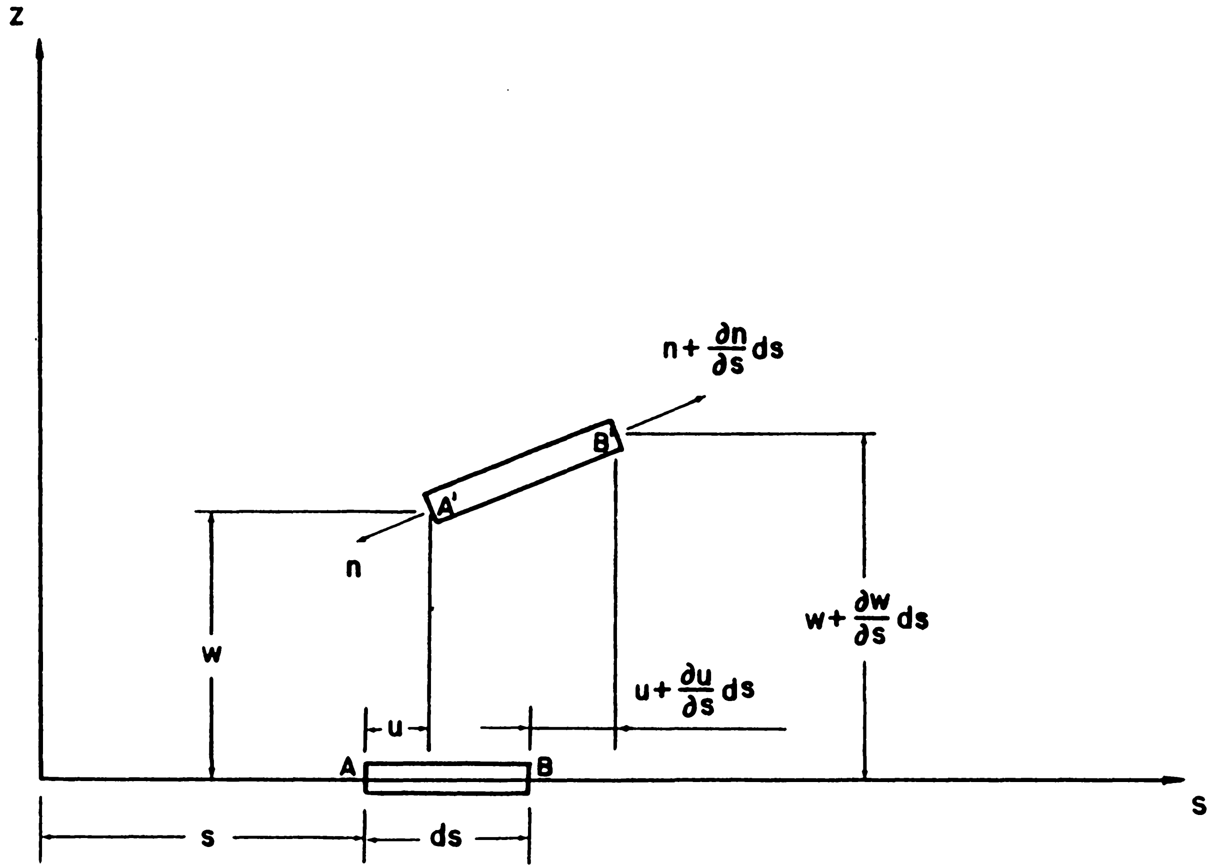

Figure 1. Beam Element

$$s^* = \frac{s}{L}, \quad t^* = \frac{R}{L^2}\sqrt{\frac{E}{\rho}}\,t, \quad G^* = \frac{L}{EA}G(s,t),$$
$$w^* = \frac{w}{L}, \quad u^* = \frac{u}{L}, \quad l^* = \frac{l}{L}, \quad r^* = \frac{R}{L},$$

where R is a characteristic value of the the radius of gyration of the cross section, l is the actual length of the beam, and L is a characteristic length, which can be the actual length of the beam or the wavelength of one of the transverse oscillation modes being considered. It is also convenient to introduce the notation

$$r(s) = R f_r(s), \quad EA(s) = EA\, f_{EA}(s),$$

in order to bring the dimensional terms out of the partial differentiation. After dropping dropping the asterisks and adding damping terms, one can write the dimensionless equations of motion as follows:

$$r^2 f_r(s)^2 \ddot{u} + r^2 \nu \dot{u} + \frac{\partial}{\partial s}(f_{EA}(s)(u' + \frac{1}{2}w'^2)) = 0 \tag{3.3}$$

$$r^2 f_r^2(s)\ddot{w} + r^2 \frac{\partial^2}{\partial s^2}(f_{EA}(s) f_r^2(s) w'') + r^2 \nu \dot{w} - \frac{\partial}{\partial s}(f_{EA}(s)(u' + \frac{1}{2}w'^2)w') = G(s,t). \tag{3.4}$$

The ratio of characteristic radius of curvature and characteristic length, r, is assumed to be small, and thus the inertia and damping terms can be dropped from Eq. (3.3). Integrating the remaining term twice with respect to s gives

$$\int_0^s u' ds + \frac{1}{2}\int_0^s w'^2 ds = f_1(t) \int_0^s \frac{1}{f_{EA}}(s) ds + f_2(t)$$

where f_1 and f_2 are arbitrary functions of time. These functions can be evaluated by imposing the boundary conditions:

$$u(0) = 0 \quad \text{and} \quad u(l) = 0$$

giving

$$f_2(t) = 0\,, \qquad f_1(t) = B\frac{1}{2}\int_0^L w'^2 ds \quad \text{where} \quad B^{-1} = \int_0^L \frac{1}{f_{EA}(s)} ds$$

It is noted that

$$f_{EA}(s)(u' + \frac{1}{2}w'^2) = f_1(t) = B\frac{1}{2}\int_0^L w'^2 ds$$

When this is substituted into Eq. (3.4) the following differential equation of motion is obtained:

$$r^2(f_r^2(s)\ddot{w} + \nu\dot{w} + \frac{\partial^2}{\partial s^2}(f_{EA}(s)f_r^2(s)w'')) - \frac{1}{2}B\int_0^L w'^2 ds\, w'' = G(s,t) \quad (3.5)$$

The solution w is assumed to be of order r^k, which makes it possible to directly study the effects of the size of the nonlinearity on the solution. Then, using that assumption, the solution is expanded in terms of the linear oscillation modes. That is,

$$w(s,t) = r^k \sum_{m=1}^{\infty} \psi_m(t)\varphi_m(s) \quad (3.6)$$

where the $\varphi_m(s)$ are the modeshapes of the associated linear beam and $\psi_m(t)$ is the generalized coordinate of the m-th mode. The $\varphi_m(s)$ are the solutions of the eigenvalue problem :

$$\frac{\partial^2}{\partial s^2}(f_{EA}(s)f_r^2(s)\varphi''_m) - \omega_m^2 f_r^2(s)\varphi_m = 0 \tag{3.7}$$

The eigenvalues, ω_m, are the natural frequencies and the eigenfunctions, φ_m, are the orthogonal free vibration modes.

Substituting the expansion, Eq. (3.6), into Eq. (3.5), multiplying by φ_n, integrating over the length, and assuming modal damping yield

$$\ddot{\psi}_n + \omega_n^2\psi_n = -2\mu_n\dot{\psi}_n + \varepsilon \sum_{m,p,q=1}^{\infty} \Gamma_{mnpq}\psi_m\psi_p\psi_q + g_n(t) \qquad n = 1, \ldots, \infty \tag{3.8}$$

where

$$\varepsilon = r^{2(k-1)}, \quad r^{2+k}g_n(t) = \int_0^L G(s,t)\varphi_n(s)ds \quad \text{and}$$

$$\Gamma_{mnpq} = \frac{1}{2}B\left(\int_0^L \varphi_n\varphi''_m\, ds\right)\left(\int_0^L \varphi'_p\, \varphi'_q\, ds\right)$$

By integrating by parts and imposing the boundary conditions $\psi_i(0) = 0$, $\psi_i(L) = 0$, $i = 1, \ldots, \infty$, one can obtain the nonlinear constants:

$$\Gamma_{mnpq} = -\frac{1}{2}B\left(\int_0^L \varphi'_n\, \varphi'_m\, ds\right)\left(\int_0^L \varphi'_p\, \varphi'_q\, ds\right) \tag{3.9}$$

The nonlinear constants have the symmetry relations

$$\Gamma_{mnpq} = \Gamma_{nmpq} = \Gamma_{nmqp} = \Gamma_{qpnm} \qquad (3.10)$$

Equation (3.8) is an infinite set of coupled differential equations for $\psi_n(t)$, the response of the beam in the nth mode. The assumption that the response in modes higher than some number is small enough to be negligible is usually made in order to convert the infinite set of equations to a finite one. In the following chapters the equations for the response of the beam to random excitation are developed for one, two and three mode approximations of the modal equations.

4.0 THE STOCHASTIC EQUATIONS

In this chapter the differential equations for the response moments will be derived using Ito's stochastic calculus. Following that there is a discussion of the techniques for closing the infinite hierarchy of moment equations.

4.1 Description of the Stochastic System

It is convenient in deriving the moment equations to use the state vector approach. Thus the system of ordinary differential equations is written in the form [17]

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x},t) + H(\mathbf{x},t)\mathbf{Y}(t) \tag{4.1}$$

Here $\mathbf{x}(t)$ is an n by 1 vector of state variables, $\mathbf{f}(\mathbf{x},t)$ is an n by 1 vector process representing the deterministic influences on the system, and $\mathbf{Y}(t)$ is an m by 1 vector of random processes which influence the system through the n by m matrix $H(\mathbf{x},t)$. This is a general representation where the system can have random coefficients as well as random inhomogeneous terms. Because the mean parts of the random processes are included in $\mathbf{f}(\mathbf{x},t)$, all the elements of $\mathbf{Y}(t)$ have zero mean.

When the random processes $\mathbf{Y}(t)$ is approximated by the mathematical idealization of Gaussian white noise, the system may be written in the Ito form [17]

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x},t)dt + H(\mathbf{x},t)d\mathbf{W}(t), \tag{4.2}$$

where $\mathbf{W}(t) = \{W_i(t)\}$, $i = 1, \ldots, m$ is an m by 1 Wiener process with the following incremental properties [16]:

$$E\left[dW_i(t)\right] = 0 \quad i = 1, \ldots, n$$
$$E\left[dW_i(t)dW_j(t)\right] = Q_{ij}\,dt \quad i,j = 1, \ldots, n$$

where Q is the covariance matrix of the incremental vector. The elements W_i are assumed to be independent of the initial conditions of the system. Equation (4.1) can be written in the more convenient form

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x},t)dt + F(\mathbf{x},t)d\mathbf{w}(t), \tag{4.3}$$

where

$$F(\mathbf{x},t) = H(\mathbf{x},t)Q^{1/2}(t)$$

$$d\mathbf{w}(t) = Q^{-1/2}(t)d\mathbf{W}(t),$$

$\mathbf{w}(t)$ is a normalized Wiener process with the incremental properties

$$E\left[dw_i(t)\right] = 0 \quad i = 1, \ldots, n$$
$$E\left[dw_i(t)dw_j(t)\right] = I dt \quad i,j = 1, \ldots, n$$

where I is the identity matrix. The process $\mathbf{x}(t)$ that satisfies the Ito differential equation is termed an Ito process and the study of the properties of Ito processes is called Ito stochastic calculus. The transient probability density function satisfies the well known Fokker-Planck differential equation

$$\frac{\partial p}{\partial t} = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[f_i p] + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[(FF^T)_{ij} p\right] \tag{4.4}$$

where the notation $(\cdot)_{ij}$ denotes the ij-th component of the matrix argument.

The integration of Eq. (4.4) and even of the steady state form of Eq. (4.4) is in general not possible. This makes it necessary to resort to approximate methods to solve the system. The method described here is one such method. However, the joint probability density function $p(\mathbf{x}, t)$ is not evaluated directly; instead a system of differential equations for the moments; mean, variance, correlations, etc., is found.

4.2 Derivation of the Moment Equations

When $h(\mathbf{x}, t)$, is considered an arbitrary function of $\mathbf{x}(t)$ whose partial derivatives $\frac{\partial h}{\partial x_i}$, $\frac{\partial^2 h}{\partial x_i \partial x_j}$ and $\frac{\partial h}{\partial t}$ are continuous and bounded on any interval of $\mathbf{x}(t)$, a differential equation for the expected value of $h(\mathbf{x}, t)$ can be derived as follows. Using a Taylor's series and Δ as a forward finite increment operator over the time increment Δt, yields [4]

$$\Delta h = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} \Delta x_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 h}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + \frac{\partial h}{\partial t} \Delta t + \mathrm{o}(\Delta \mathbf{x} \Delta \mathbf{x}^T) + \mathrm{o}(\Delta t) \tag{4.5}$$

From the properties of an Ito process, it can be shown that:

$$E\left[\Delta \mathbf{x} \mid \mathbf{x}\right] = \mathbf{f} \Delta t + \mathrm{o}(\Delta t)$$

and

$$E\left[\Delta \mathbf{x}\Delta \mathbf{x}^T \mid \mathbf{x}\right] = 2FF^T\Delta t + o(\Delta t).$$

Taking the expectation of Eq. (4.5) yields

$$\begin{aligned} E[\Delta h \mid \mathbf{x}] = & \sum_{i=1}^{n} f_i(\mathbf{x},t)\frac{\partial h}{\partial x_i}\Delta t + \sum_{i,j=1}^{n} (FF^T)_{ij}\frac{\partial^2 h}{\partial x_i \partial x_j}\Delta t \\ & + \frac{\partial h}{\partial t}\Delta t + o(\Delta t) \end{aligned} \tag{4.6}$$

The expectation of $E[\Delta h \mid \mathbf{x}]$ is observed to be

$$E[E[\Delta h \mid \mathbf{x}]] = E[\Delta h]$$

Thus taking the expectation of Eq. (4.6) gives

$$\begin{aligned} E[\Delta h] = & \sum_{i=1}^{n} E\left[f_i(\mathbf{x},t)\frac{\partial h}{\partial x_i}\right]\Delta t + \sum_{i,j=1}^{n} E\left[(FF^T)_{ij}\frac{\partial^2 h}{\partial x_i \partial x_j}\right]\Delta t \\ & + E\left[\frac{\partial}{\partial t}h\right]\Delta t + o(\Delta t) \end{aligned} \tag{4.7}$$

provided that the indicated expectations exist. Finally, this is divided by Δt and the limit $\Delta t \to 0$ is taken. This yields, interchanging expectation and differentiation, the ordinary differential equation

$$\begin{aligned} \frac{dE[h]}{dt} = & \sum_{i=1}^{n} E\left[f_i(\mathbf{x},t)\frac{\partial h}{\partial x_i}\right] + \sum_{i,j=1}^{n} E\left[(FF^T)_{ij}\frac{\partial^2 h}{\partial x_i \partial x_j}\right] \\ & + E\left[\frac{\partial h}{\partial t}\right] . \end{aligned} \tag{4.8}$$

This equation is used to obtain the moment equation for $E\left[x_1^{k_1}x_2^{k_2}\dots x_n^{k_n}\right]$ by setting

$$h(\mathbf{x}, t) = x_1^{k_1}x_2^{k_2}\dots x_n^{k_n} \tag{4.9}$$

Equation (4.8) results, except in the case of linear systems, in an infinite hierarchy of equations. This is due to the fact that the differential equation for a moment of a certain order of a nonlinear equation contains moments of higher orders as well as moments of the same and lower order. This property of the moment equations leads to a need for assumptions which will close the equations. The following section is a discussion of some of those assumptions and the resulting closure methods.

4.3 Closure Methods

Most closure methods are based on an assumption concerning the form of the response probability density function. One simple assumption is that the response is Gaussian. While it is true that a linear system with a Gaussian input process will have a Gaussian output process, this is not true, in general, for nonlinear systems. The assumption that the output process is close to Gaussian in some way, which seems reasonable for Gaussian inputs to systems with small nonlinearities, is the basis for most non-Gaussian closure schemes. One such scheme of this type is cumulant-neglect closure. In this scheme the output distribution is is assumed to resemble a Gaussian distribution in that the cumulants above a certain order are zero. A number of other near Gaussian closure schemes are based on the assumption that the response probability density function can be approximated by an expansion of a Gaussian distribution. The expansions used include a Gram-Charlier expansion, an Edgeworth expansion in terms of the central cumulants and an expansion in terms of the quasi-moments. Computing the moments of the assumed distribution results in the relations between the coefficients of the expansion and the moments of the process. Thus, in each case the truncation of the expansion results in relating the higher order moments to the lower order moments in some way. The actual computation of the moment statistics can be done by substituting the derived higher moment relations into the moment equations. It can also be done, in the stationary case, by using as $h(\mathbf{x}, t)$ the Hermite polynomials used in the expansion, and deriving a set of equations for the coefficients of the expansion.

There is another type of closure, proposed by Bellman and Richardson [18], which seeks to relate the higher order moments to the lower order moments in such a way as to guarantee that the moment inequalities will be preserved. In this closure scheme the differential moment equations are written in the form of square matrices, with the higher order moment terms contained in the matrix on the right hand side. This right hand side matrix is approximated by a matrix of the lower order moments multiplied by a coefficient matrix in such a way as to guarantee that the moment relations will be preserved. The coefficient matrix is determined by seeking to minimize some matrix norm of the difference between the original right hand side and the approximation.

Comparisons of the different closure schemes can be found in the literature. Bover [17] and Crandall [14] give the relations between the different expansion coefficients. Comparing the relations between the expansion coefficients and the central moments, one can see that the coefficients proposed by Crandall, the central cumulants and the quasi-moments are all equal for order ≤ 5, the coefficients proposed by Crandall and the quasi-moments are the same for order ≤ 7, and all are different for order ≥ 8. Wilcox and Bellman [22] compared second order solutions for a particle undergoing Brownian motion subjected to a nonlinear restoring force. The solutions were obtained by the moment preserving closure, central moment neglect, cumulant neglect and a mean square averaging procedure, and were compared with the exact steady state solution. They found that cumulant neglect closure gave the best results. Higher order closures were not considered. Bover compared quasi-moment truncation and cumulant neglect closures with an analytical solution and Monte Carlo simulation for a stochastic cubic oscillator, and found that for fourth and fifth order closures the solutions using the two closure methods were the same. For the sixth order closure both closures gave solutions closer to the analytical and Monte Carlo simulation solutions; however, the quasi-moment closure gave a closer solution than the cumulant neglect closure. Crandall [14] gives second, fourth and sixth order steady state solutions for a Duffing oscillator, and compares those with the exact solution. He shows that the solutions approach the exact solution as the order of the approximation increases, but does not compare his closure method with other methods.

In the chapter that follows, closures up to fourth order will be performed. The closure that has given the best results in the literature is the quasi-moment closure, and that closure will be used in what follows. It is noted that in the closures relations for moments up to the sixth order will be required, and that for terms up to that order the closure of Crandall is the same as the quasi-moment closure.

5.0 RESULTS AND DISCUSSION

In this chapter the method of chapter four is applied to the nonlinear beam equations that were derived in chapter three. The resulting moment equations are closed by the quasi-moment closure described by Bover [17]. One, two and three mode solutions are performed; second and fourth order closures are used for the one and two mode cases, and second order closure is used for the three mode case.

5.1 Moment Equations for the Nonlinear Beam

As an example the moment equations for the one mode case will be derived here. From Eq. (3.5) with a one mode approximation the differential equation of the beam is

$$\ddot{\psi}_1 + \omega_1^2 \psi_1 = -2\mu_1 \dot{\psi}_1 + \varepsilon \Gamma_{1111} \psi_1^3 + g_1(t). \tag{5.1}$$

In order to write this as a set of first order equations, the state variables $x_1 = \psi_1$ and $x_2 = \dot{\psi}_1$ are introduced, giving

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega_1^2 x_1 - 2\mu_1 x_2 + \varepsilon \Gamma_{1111} x_1^3 + f_1(t). \end{aligned} \tag{5.2}$$

As in Chapter 4, $g_1(t)$ is assumed to be $dW(t)$ an incremental process of the zero mean white noise $W(t)$ which has the incremental property $E\left[(dW(t))^2\right] = \sigma_1^2 dt$. Rewriting the equations in the form of Eq. (4.3), yields

$$
\begin{aligned}
f_1(\mathbf{x},t) &= x_2 \\
f_2(\mathbf{x},t) &= -\omega_1^2 x_1 - 2\mu_1 x_2 + \varepsilon\Gamma_{1111} x_1^3 \\
F(\mathbf{x},t) &= \begin{bmatrix} 0 & 0 \\ 0 & \sigma_1 \end{bmatrix}
\end{aligned}
$$

Using these in Eq. (4.8) gives

$$
\begin{aligned}
\frac{dE\,[h]}{dt} = E\left[x_2\frac{\partial h}{\partial x_1}\right] + E\left[(-\omega_1^2 x_1 - 2\mu_1 x_2 + \varepsilon\Gamma_{1111}x_1^3)\frac{\partial h}{\partial x_2}\right] \\
+ E\left[\sigma_1^2\frac{\partial^2 h}{\partial x_2^2}\right] + E\left[\frac{\partial h}{\partial t}\right].
\end{aligned}
\tag{5.3}
$$

Using $h = x_1^p x_2^q$ in Eq. (5.3) and introducing the notation $m_{p,q} = E\,[x_1^p x_2^q]$, yields the following relation for the moments of the nonlinear system described by Eq. (5.2):

$$
\begin{aligned}
\frac{d}{dt}m_{p,q} = pm_{p-1,q+1} + q(-\omega_1^2 m_{p+1,q-1} - \mu_1 m_{p,q} \\
+ \varepsilon\Gamma_{1111} m_{p+3,q-1}) + \sigma_1^2 q(q-1)m_{p,q-2}
\end{aligned}
\tag{5.4}
$$

The differential equation for any required moment can be found from this equation. The moment equations for the two mode and three mode approximations were also found by this method, and the nonlinear equations along with the associated moment equations are shown in Appendix B.

5.2 A Numerical Example

In the following section a beam is analyzed as a numerical example. The beam, shown in Fig. 2, is clamped at one end and hinged at the other, and has a varying cross sectional height as shown. The beam is analyzed for the natural frequencies and nonlinear coefficients using the finite element method described in Appendix A. The non-dimensional length of the beam is chosen to be equal to 1.0, and the coefficients found for the beam are

$$\begin{array}{lll} \omega_1 = 18.91 & \alpha_{11} = 12.84 & \alpha_{22} = 116.68 \\ \omega_2 = 58.14 & \alpha_{12} = 18.16 & \alpha_{23} = 20.48 \\ \omega_3 = 142.49 & \alpha_{13} = 21.39 & \alpha_{33} = 303.53 \end{array} \tag{5.5}$$

where $\alpha_{mn} = \int_0^L \varphi'_m \varphi'_n ds$ is used in Eq. (3.9) to calculate the nonlinear coefficient.

The geometry of the beam is chosen so that ω_2 and ω_1 are nearly in a ratio of three to one. This is done so that an internal resonance exists. In order to study the effects of this internal resonance, a detuning parameter γ is introduced, defined by

$$\omega_2 = 3\omega_1(1 + \gamma). \tag{5.6}$$

giving, for the beam chosen, $\gamma = 0.0249$. It is noted that there is also an internal resonance condition between the second and third modes when $\gamma = -0.183$

The transient and steady state solutions were closed by the quasi-moment closure. It is noted that, because the nonlinearities are cubic, the second order closure will require relations for the fourth order moments and the fourth order

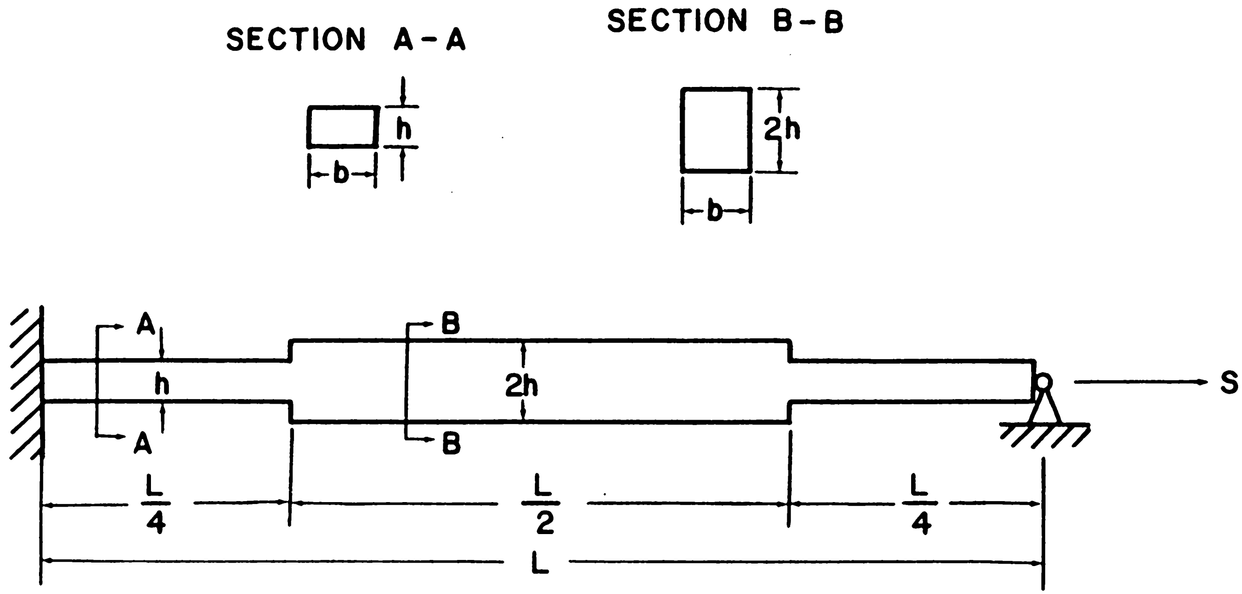

Figure 2. Beam configuration

closure will require relations for the sixth order moments. These relations are found by setting the associated quasi-moment equal to zero. The relationships between the first six quasi-moments and the central moments are given in Appendix C. The second order solutions could be called Gaussian solutions since the fourth order quasi-moments are related to the central moments in the same way as the fourth order cumulants, which would be set equal to zero in a Gaussian solution.

It is also noted that, because the nonlinearities are odd functions of the state variables, the equations for the even order moments are not coupled with the odd order moments. Also, because the excitation is not a function of the state variables, there are no inhomogeneous terms in the ordinary differential equations for the odd moments; thus the odd moments, if initially zero, will remain zero and, because of damping terms, will tend to zero as the response approaches a steady state. It is concluded that the odd moments can be set to zero and only the equations for the even moments need to be solved. The transient solutions are obtained by using the IMSL DVERK Routine (fifth and sixth order Runge-Kutta-Verner method) [27]. The steady state solutions were obtained by the secant method, described in Appendix D.

The transient problem of the system suddenly subjected to the random excitation is considered first. Attention is focused on the mean squared variance of the response in each mode. First for the second order closure, the one mode approximation first modal response variance $E\left[x_1^2\right]$ plotted against the non-dimensional time coefficient is shown in Fig. 3. The two mode approximation first modal transient response variance is shown in Fig. 4, while the second modal transient response variance is shown in Fig. 5. For the second order solution the transient variances rise smoothly to a steady value. The steady value of the first modal variance is higher for the one mode approximation than for the two mode approximation. This is expected due to the additional stiffening from the second mode nonlinear terms present in the first mode equation.

The fourth order solution of the first mode approximation for the first modal response variance is shown in Fig. 6; it is noted that the fourth order solution resembles closely the second order solution. From the transient solutions, it is observed that the system approaches a steady state for both the second and the fourth order closures. Because much less computational effort is required to solve the steady state equations, they are used to study the system in further detail.

Fig. 7 shows the ratio of nonlinear to linear first modal variances, $\frac{E[x_1^2]_{nonl}}{E[x_1^2]_{lin}}$, plotted against the size of the nonlinearity ε for the second and fourth order closures, where the equations were those due to the one mode approximation. The graph shows the expected nonlinear stiffening which increases with size of the nonlinearity. Comparing the different orders of closure it is seen that they give close to the same results. For the two mode approximation, Fig. 8 shows the second and fourth order first modal variance ratios versus ε, while Fig. 9 shows the second and fourth order second mode variance ratios versus ε. The two mode approximation first modal response shows a greater difference between the different orders of closure than does the one mode approximation first modal response. The two mode approximation second mode response, however, shows a

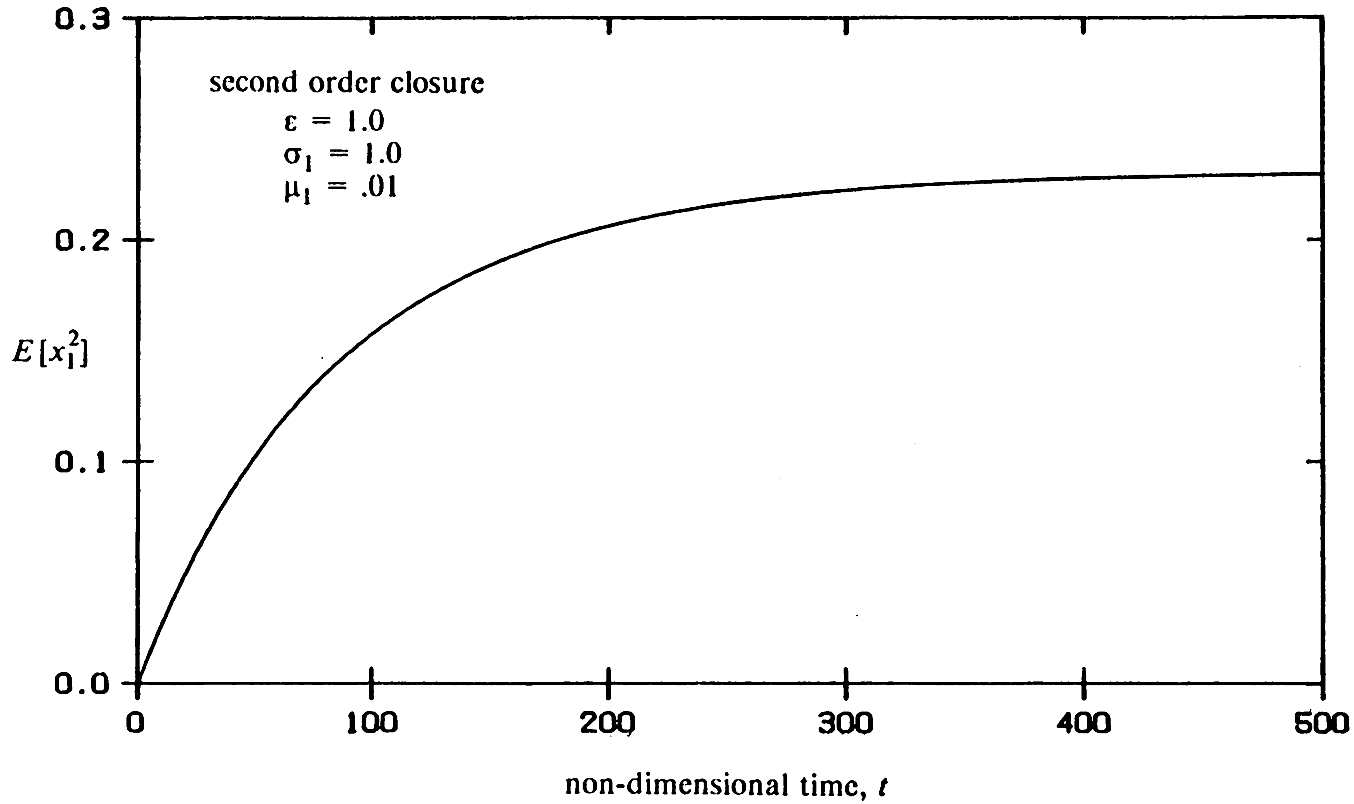

Figure 3. First modal transient variance - one mode approximation

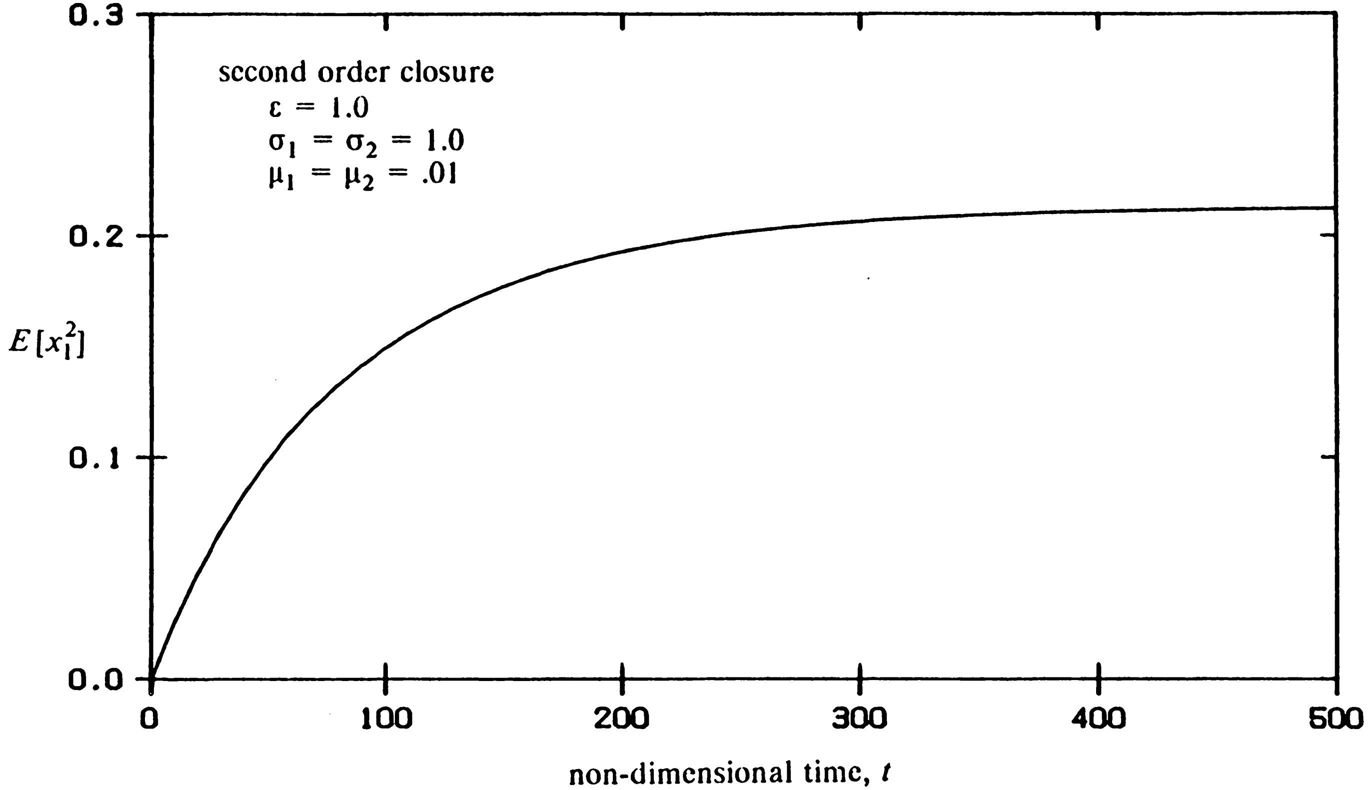

Figure 4. First modal transient variance - two mode approximation

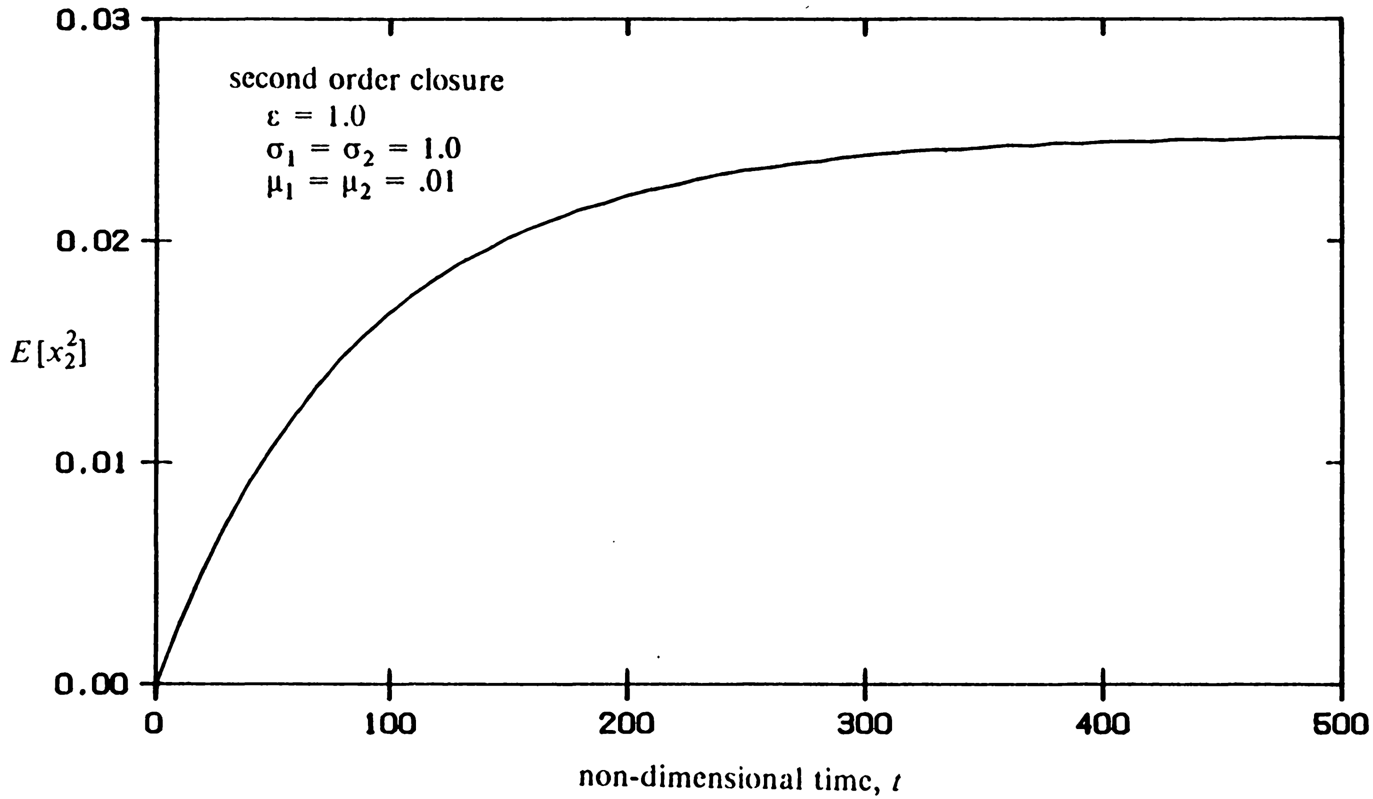

Figure 5. Second modal transient variance - two mode approximation

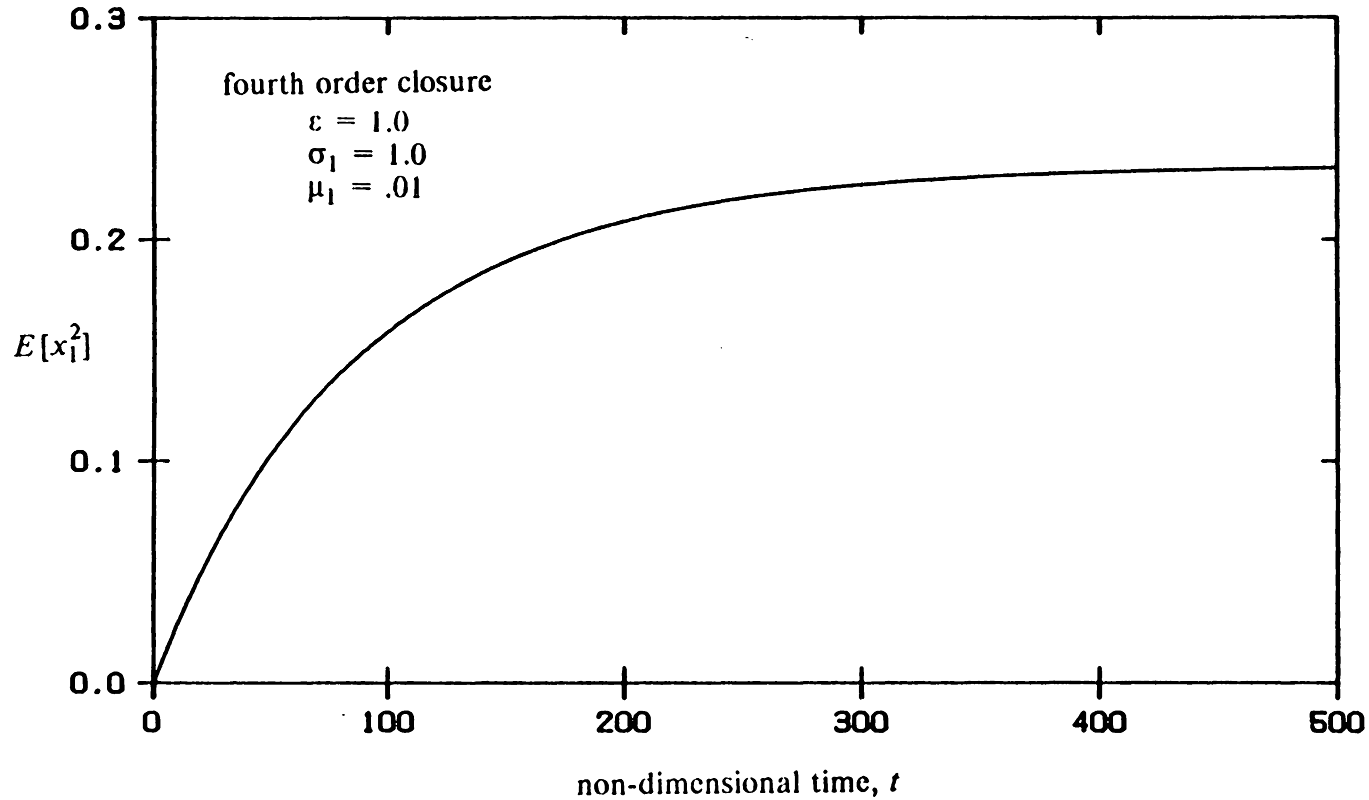

Figure 6. First modal transient variance - fourth order solution

smaller difference between the different orders of closure than does the one mode approximation. It will be shown later that this effect is due to the presence of the internal resonance condition.

Fig.s 10 and 11 show the effects of considering additional modes in the approximation of the beam equations. In Fig. 10 it is seen that the addition of the second mode causes significant drop in the first modal response, especially for relatively large ε, while the change with the addition of the third mode is much less. In Fig. 11 it is seen that the change in the second mode with the addition of the third mode in the approximation is small. The steady state magnitudes of the first three modal response variances, for a second order solution, when $\varepsilon = 1.0, \mu_1 = \mu_2 = \mu_3 = .01, \sigma_1 = \sigma_2 = \sigma_3 = 1.0$, are

$$E\left[x_1^2\right] = 0.207, \qquad E\left[x_2^2\right] = 0.0241, \qquad E\left[x_3^2\right] = 0.0046.$$

It is seen that each additional mode has a response variance that is close to an order of magnitude less than the mode before it.

In investigating the effect of the internal resonance condition, The change in the steady state variances with the change in the detuning parameter γ is examined. For the two mode approximation, the second and fourth order solution variances of the response in the first mode are shown in Fig. 12, and the variances of the response in the second mode are shown in Fig. 13. The second order solution shows no effect from the internal resonance in either of the modes, but the fourth order solution does display a change due to internal resonance. The second modal variance ratio displays a drop in the presence of the internal reso-

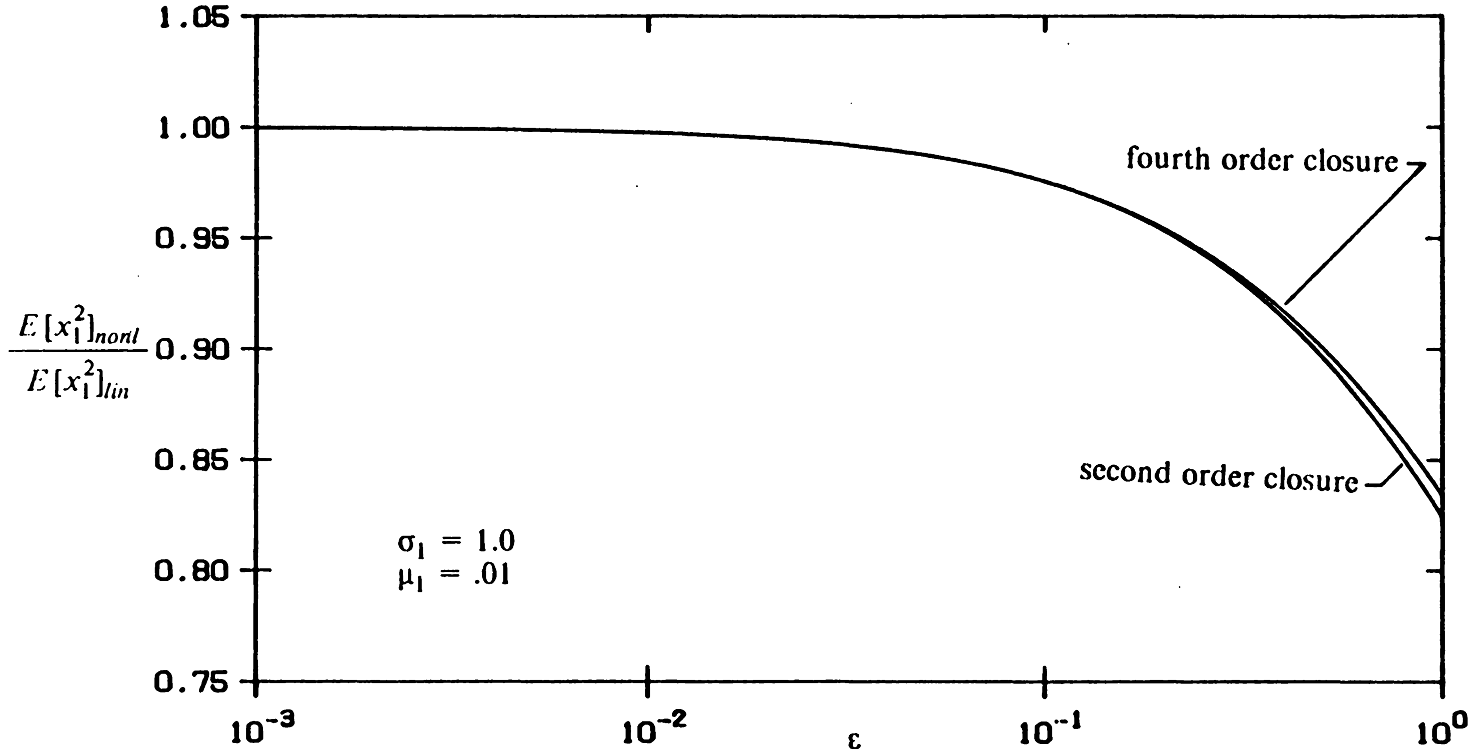

Figure 7. First modal variance vs. epsilon - one mode approximation

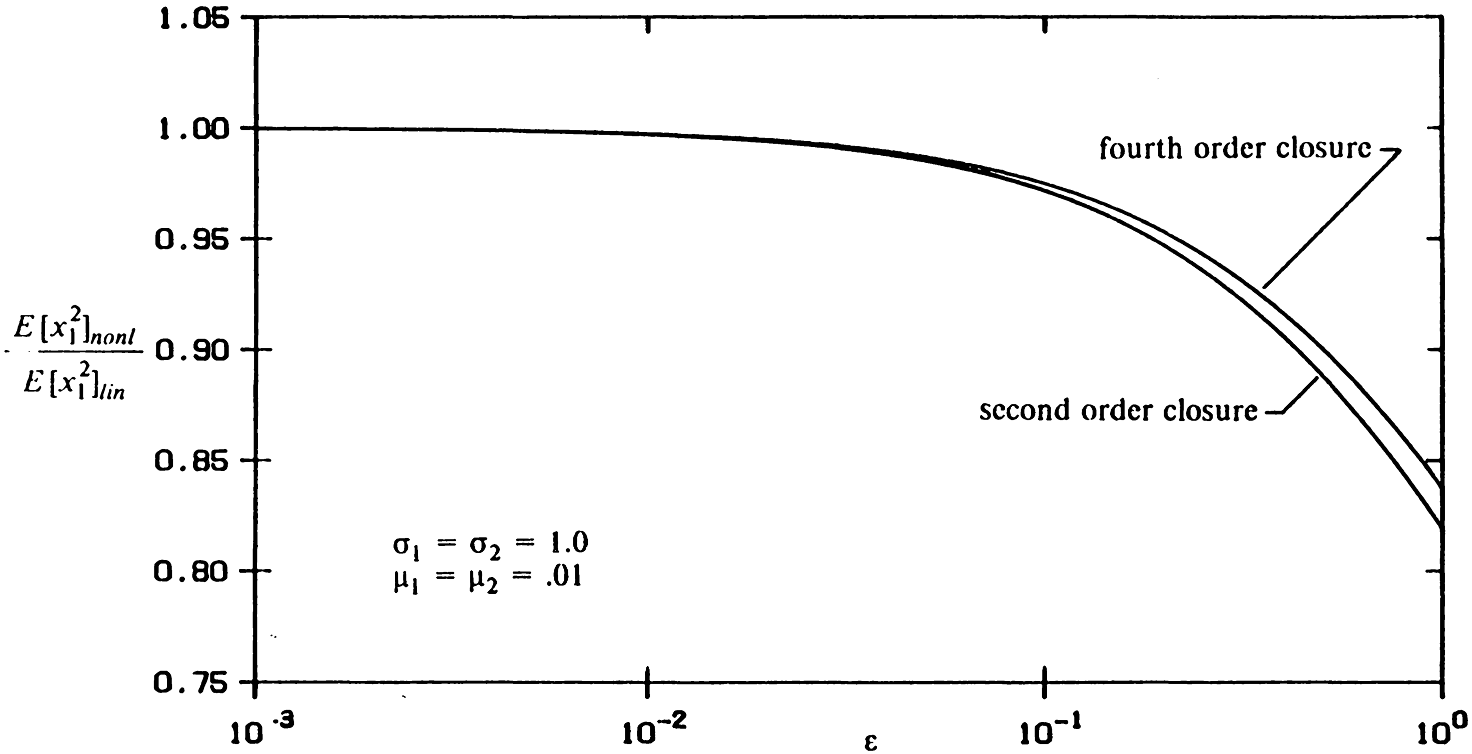

Figure 8. First modal variance vs. epsilon - two mode approximation

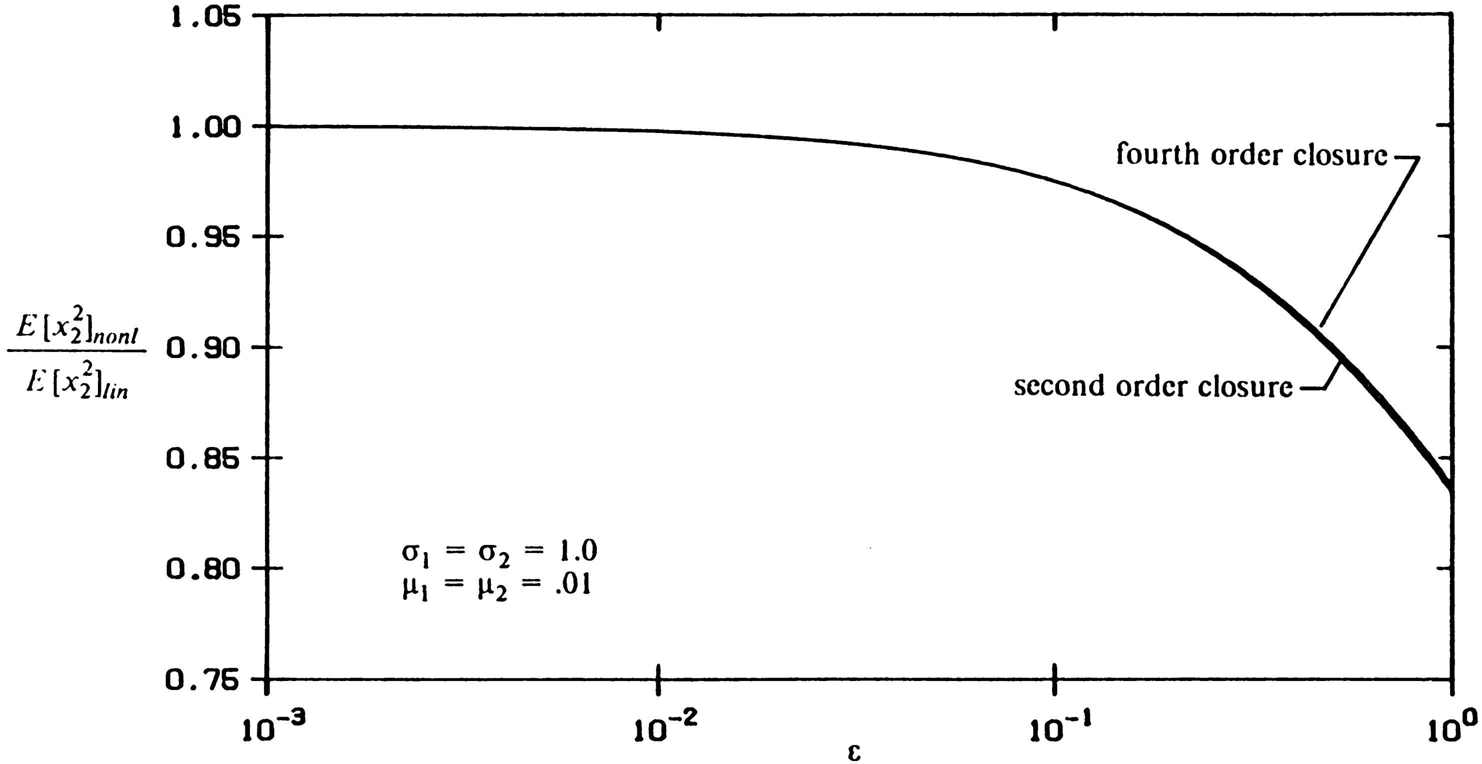

Figure 9. Second modal variance vs. epsilon - two mode approximation

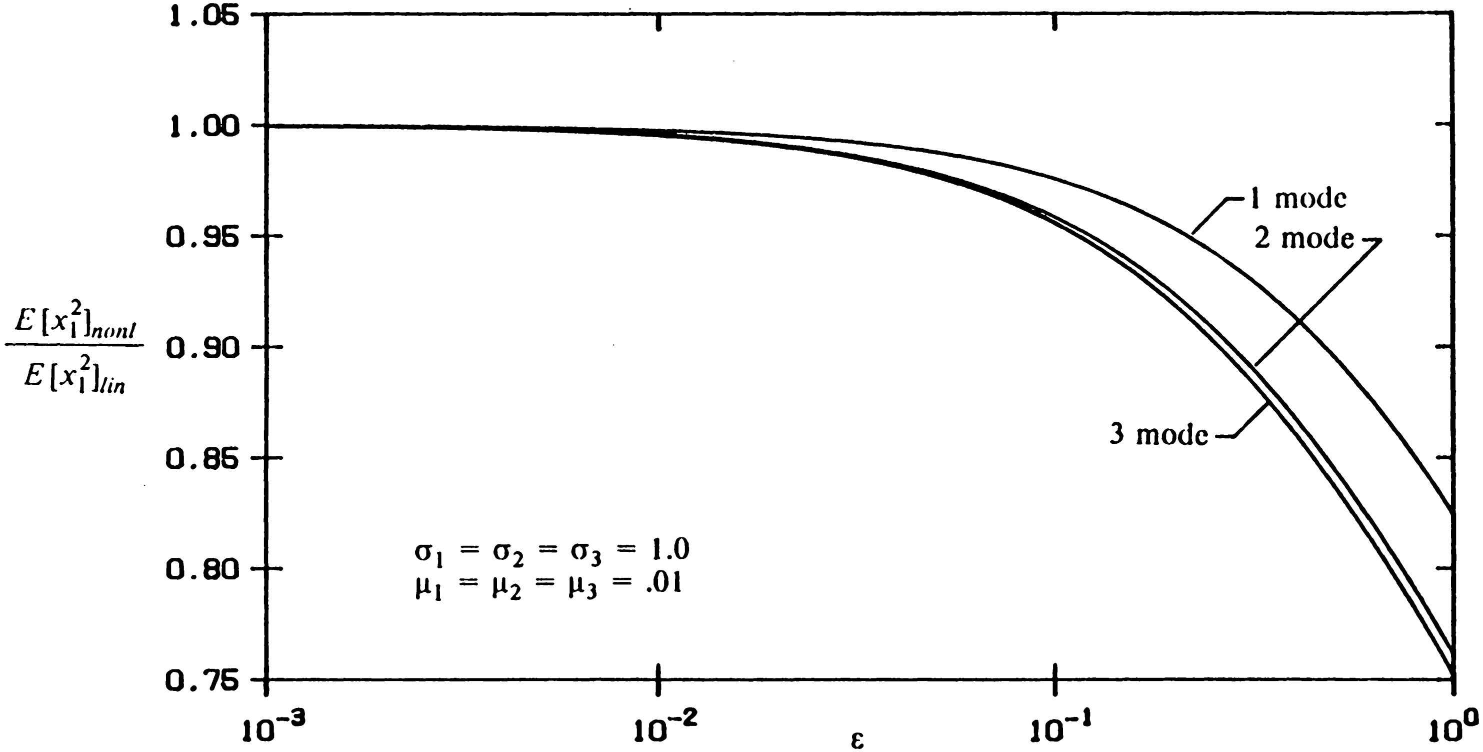

Figure 10. First modal variance vs. epsilon - one, two and three mode approximations

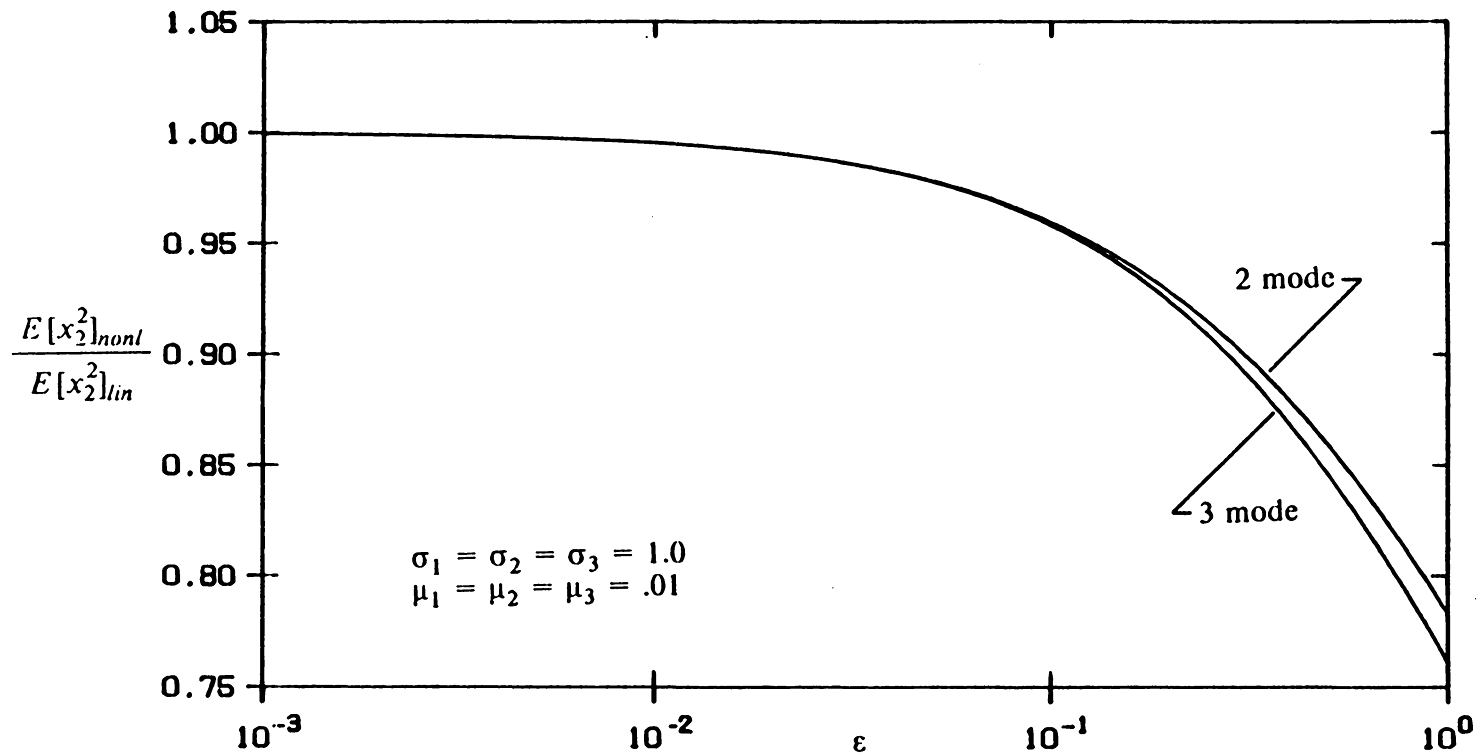

Figure 11. Second modal variance vs. epsilon - two and three mode approximations

nance, and the first mode displays a corresponding rise in the variance ratio. This indicates a transfer of energy from the second mode to the first mode in the presence of the internal resonance condition.

Next the three mode approximation is examined. Figure 14 shows the change with detuning of the ratio of first modal variances, and Fig.s 15 and 16 show the change with detuning of the ratios of second and third modal variances, respectively. As in the two mode analysis the second order solution does not display any effect from the internal resonances, but the fourth order solution does. Near the resonance between the first and second modes there is a transfer of energy from the second mode to the first mode. The third mode, as would be expected, displays no influence from the resonance in this region. The third mode does display an decrease in the region of the internal resonance between it and the second mode. The second mode shows a corresponding increase which is smaller than the drop corresponding to its resonance with the first mode. The first mode curve is unchanged in shape from the two mode analysis, and thus there is no effect on the first modal response from the resonance between the second and third modes.

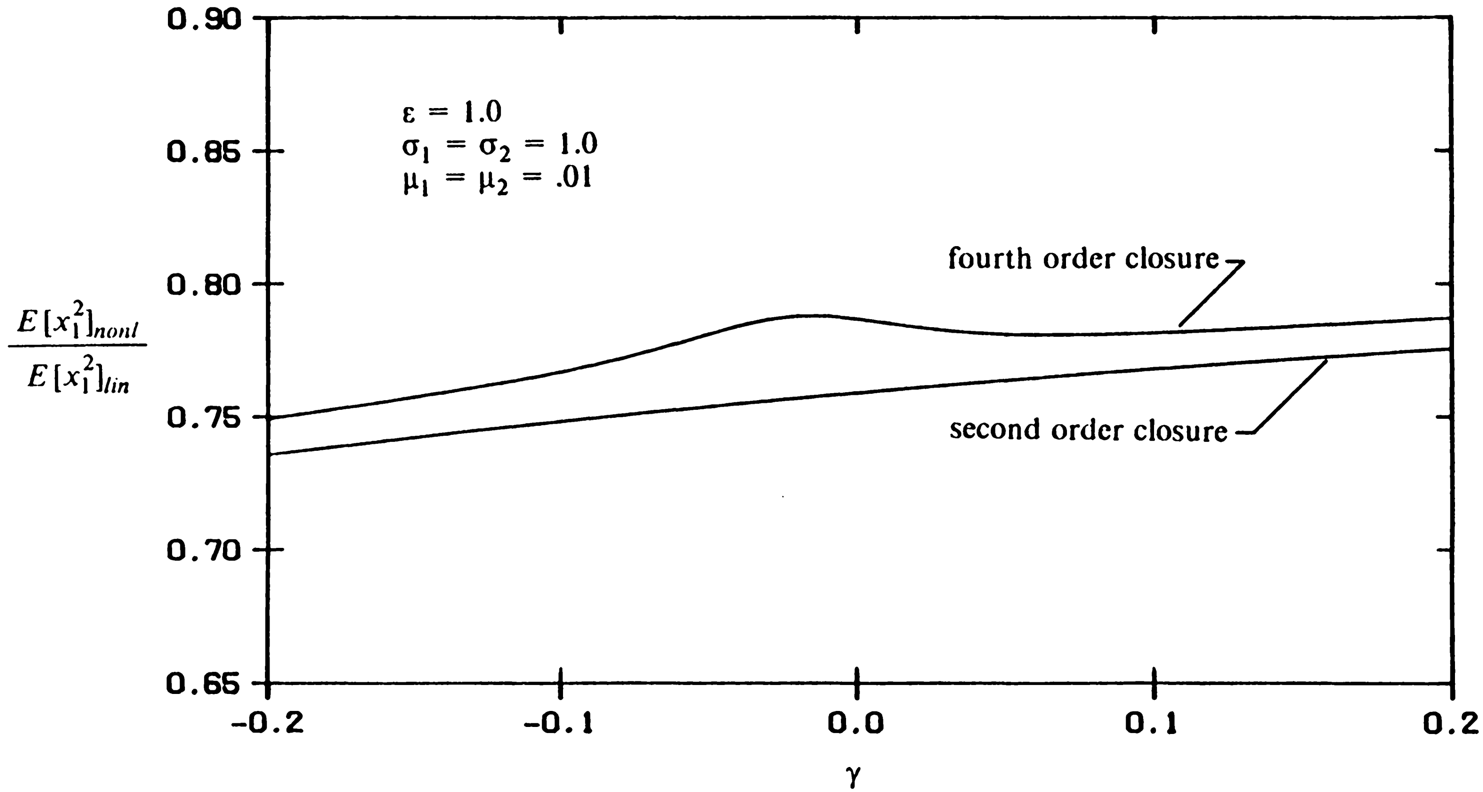

Figure 12. First modal variance vs. detuning - two mode approximation

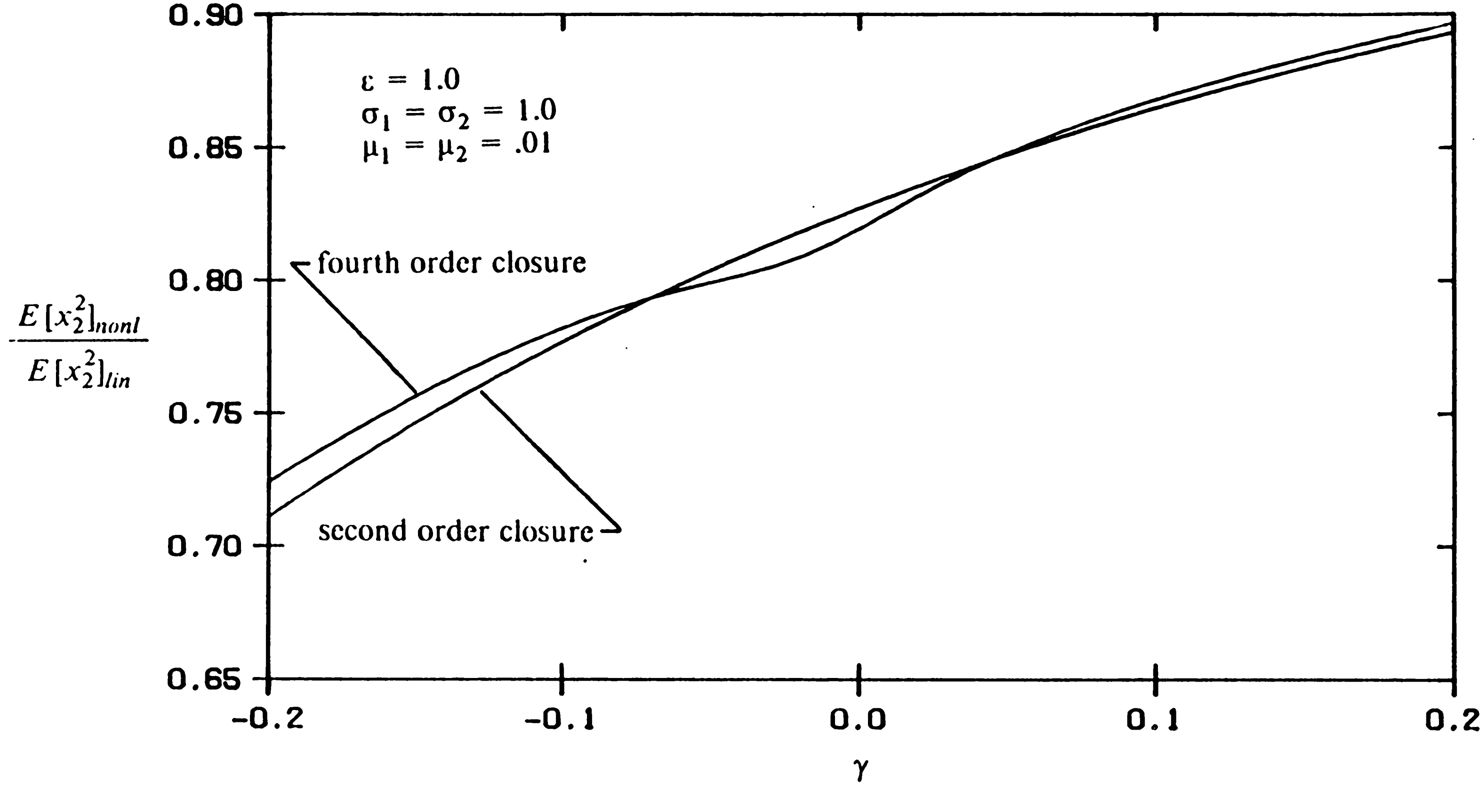

Figure 13. Second modal variance vs. detuning - two mode approximation

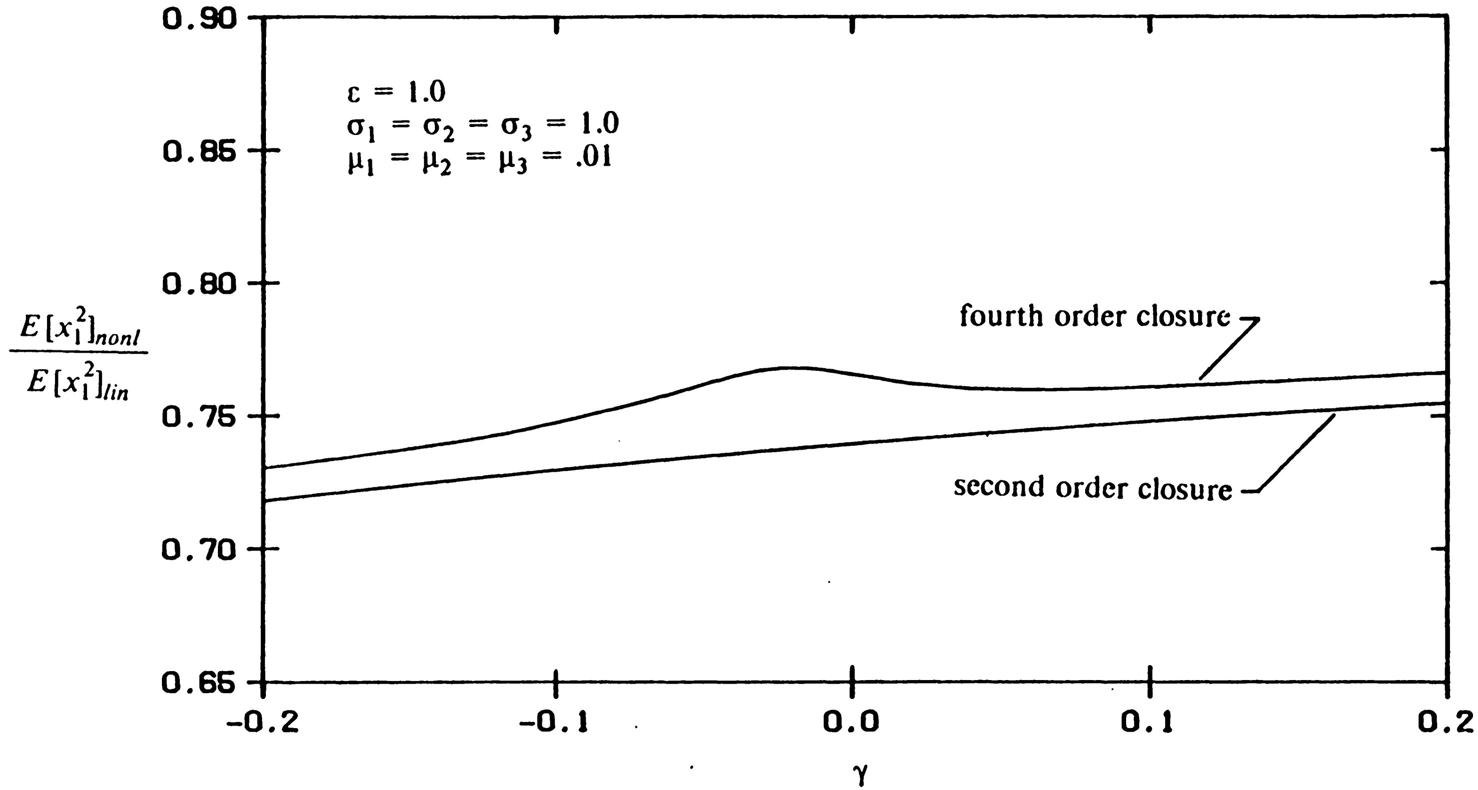

Figure 14. First modal variance vs. detuning - three mode approximation

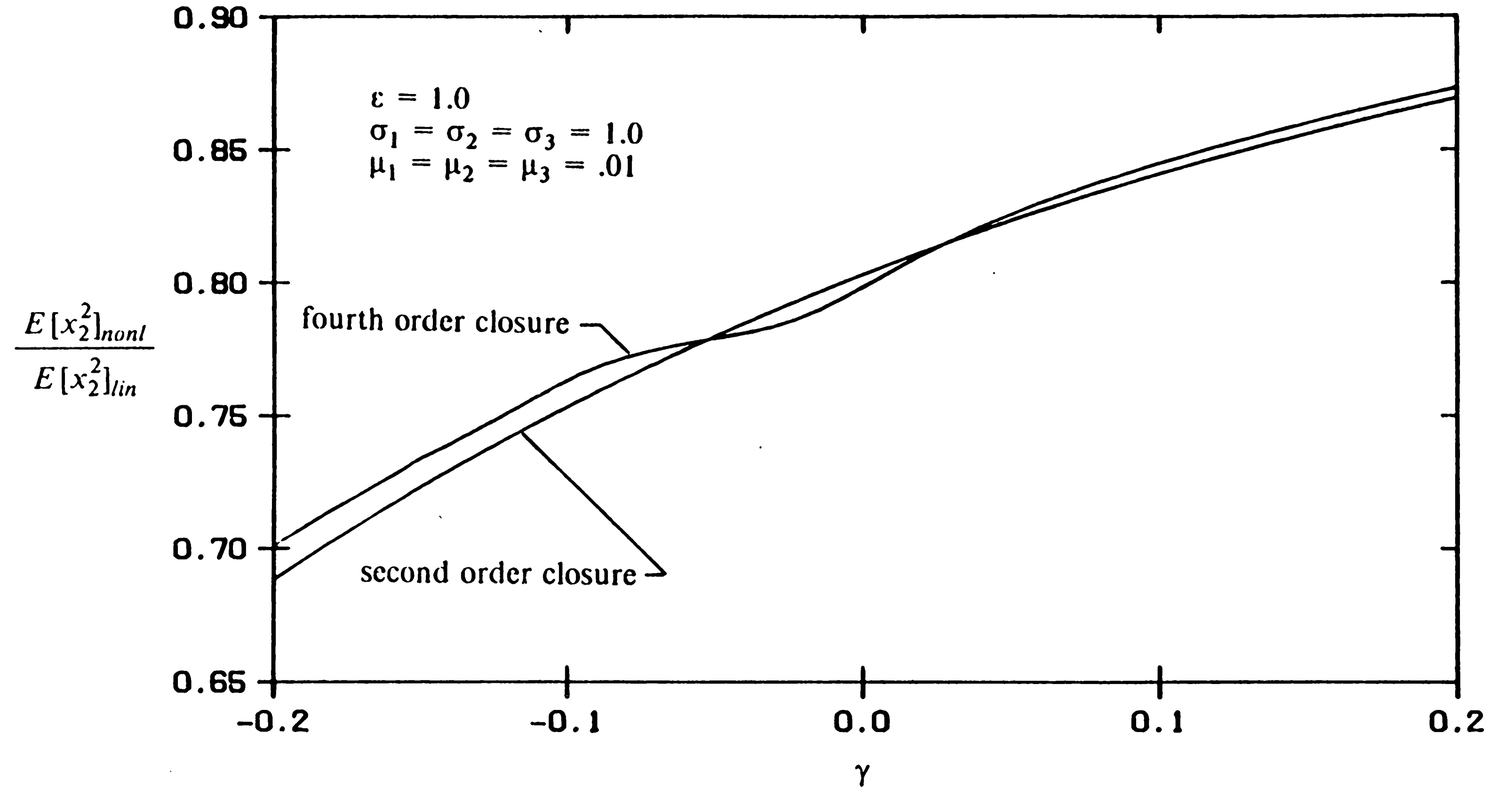

Figure 15. Second modal variance vs. detuning - three mode approximation

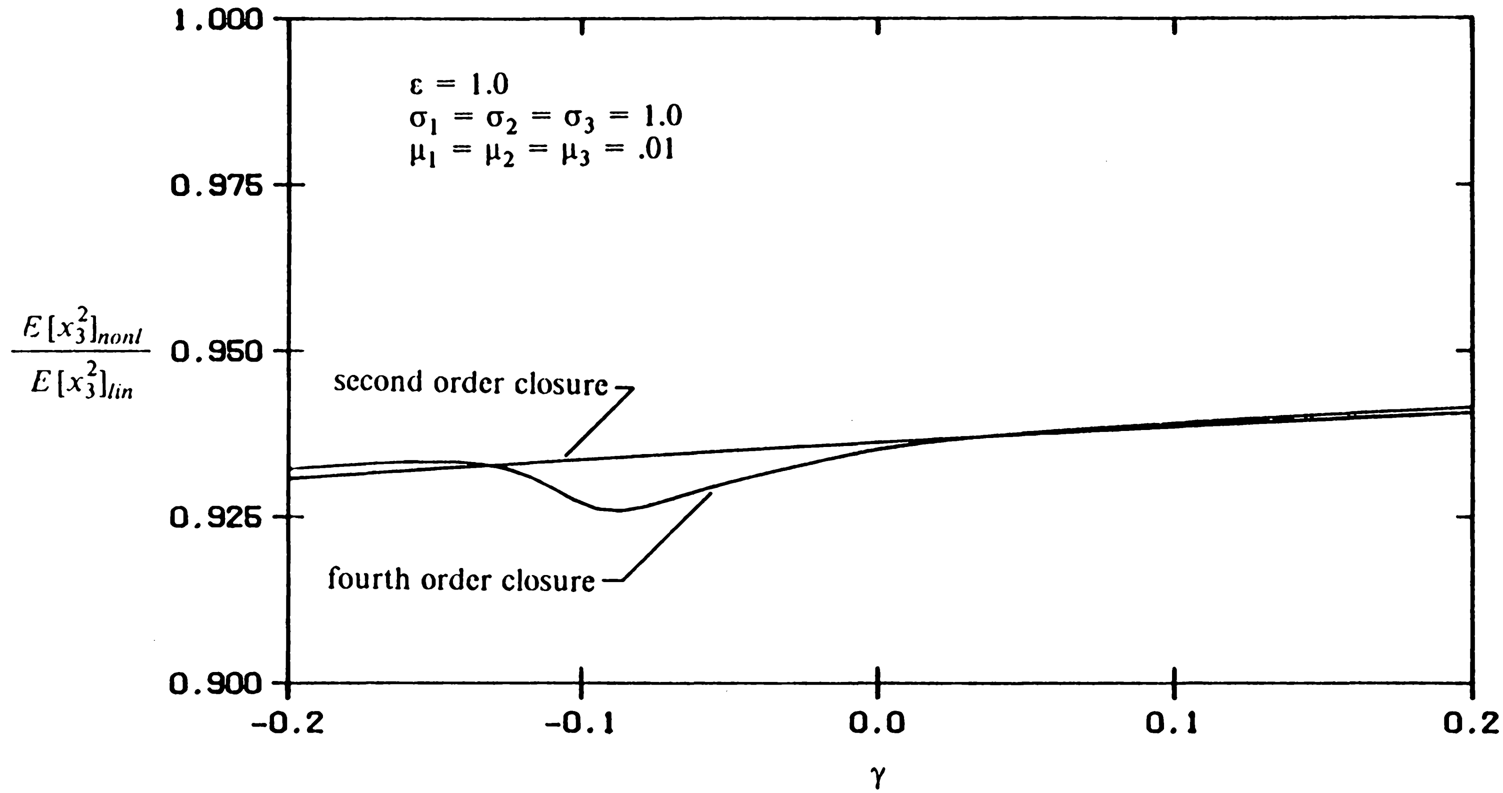

Figure 16. Third modal variance vs. detuning - three mode approximation

6.0 CONCLUSIONS

In the present study a stochastic multiple degree of freedom system is analyzed for the transient and steady state response moments. The solution is examined for different mode approximations and for different orders of closure. The results of the analysis can be summarized as follows:

The one mode approximation underestimates the nonlinear reduction in the beam response, since addition of the second mode in the solution causes a significant drop in the variance of the response in the first mode. Addition of the third mode causes a smaller drop, and this suggests that considering additional modes would not cause a significant change in the solution.

The single mode analysis is obviously not able to predict internal resonance effects, while multiple mode analyses are able to predict these effects, if internal resonance conditions exist between the modes considered. There is no effect from an internal resonance on modes which are not involved in that internal resonance. The second and fourth order closures give results which are close to each other, but second order solutions are unable to predict internal resonance phenomena. In each resonance case studied, the effect of the resonance is a transfer of energy from the higher mode to the lower mode involved in the resonance.

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APPENDIX A. THE FINITE ELEMENT METHOD

The finite element method is used to find the eigenvalues and eigenvectors of the following differential equation

$$\frac{d^2}{ds^2}\left[f(s)\,\frac{d^2}{ds^2}\,w(s)\right] - \omega_m^2 g(s)w(s)$$

For this differential equation the energy inner product is given by

$$[w, w] = \int_0^L f(s)(\frac{d^2}{ds^2}w(s))^2 ds$$

and the weighted inner product is given by

$$(mw, w) = \int_0^L g(s)w^2(s)ds$$

The domain of the equation is divided into a set of elements as shown in Fig. 17, each element having two end nodes. The solution function $f(s)$ is then approximated by a set of interpolation functions. Because the function and its derivative must be continuous at the nodes, four constants are necessary to define the interpolation functions. The interpolation functions are chosen to be cubic polynomials. The function is represented over the element by the interpolation functions times the nodal coordinates. The nodal displacements and rotations as shown in Fig. 2 are used as the nodal coordinates. Thus the function is written in the form

$$w = L_1 w_j + L_2\theta_j + L_3 w_{j+1} + L_4\theta_{j+1}$$

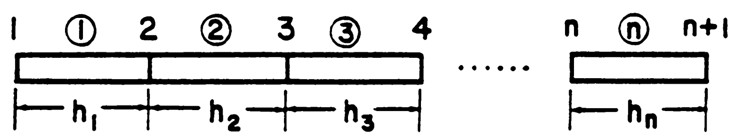

Figure 17. Finite element discretization

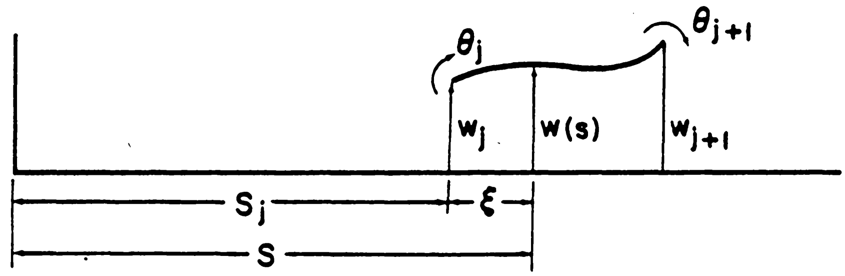

Figure 18. Nodal coordinates

where $L_k; k = 1,2,3,4$ are the interpolation functions. For convenience the following natural coordinate is introduced:

$$\xi = \frac{s - s_j}{h_j}$$

The interpolation functions are then the cubic polynomial

$$L_i(\xi) = C_{i1} + C_{i2}\xi + C_{i3}\xi^2 + C_{i4}\xi^3 ; \qquad i = 1,2,3,4$$

The coefficients C_{ik} $(i,k = 1,2,3,4)$ are evaluated by insisting that $w(s)$ and $dw(s)/ds$ take on the values w_j and θ_j at the node j or $\xi = 0$ and w_{j+1} and θ_{j+1} at the node $(j + 1)$ or $\xi = 1$. Imposing this leads to the interpolation functions

$$L_1(\xi) = 1 - 3\xi^2 + 2\xi^3$$

$$L_2(\xi) = h(- \xi + 2\xi^2 - \xi^3)$$

$$L_3(\xi) = 3\xi^2 - 2\xi^3$$

$$L_4(\xi) = h(\xi^2 - \xi^3)$$

We introduce the notation

$$w = \mathbf{L}^T\mathbf{a}_j \qquad s_j \leq s \leq s_{j+1}$$

where $\mathbf{L}^T = [L_1 \;\; L_2 \;\; L_3 \;\; L_4]$ and $\mathbf{a}_j = [w_j \;\; \theta_j \;\; w_{j+1} \;\; \theta_{j+1}]^T$.

Because the L_i are functions of ξ, one can write

$$\frac{dw}{ds} = \frac{d}{ds}(\mathbf{L}^T\mathbf{a}_j) = \frac{d\mathbf{L}^T}{ds}\mathbf{a}_j = \frac{d\xi}{ds}\frac{d\mathbf{L}^T}{d\xi}\mathbf{a}_j = \frac{1}{h_j}\frac{d\mathbf{L}^T}{d\xi}\mathbf{a}_j$$

Similarly

$$\frac{d^2w}{ds^2} = \frac{1}{h_j^2}\frac{d^2\mathbf{L}^T}{d\xi^2}\mathbf{a}_j$$

The energy inner product and the weighted inner product are therefore given by

$$[w, w] = \sum_{j=1}^{n} \mathbf{a}_j^T K_j \mathbf{a}_j$$

$$(mw, w) = \sum_{j=1}^{n} \mathbf{g}_j^T M_j \mathbf{a}_j$$

where

$$K_j = h_j \int_0^1 \frac{1}{h_j^4} f_j \frac{d^2\mathbf{L}}{d\xi^2}\frac{d^2\mathbf{L}^T}{d\xi^2}\, d\xi\,; \qquad j = 1,2,3,...,n$$

$$M_j = h_j \int_0^1 g_j \mathbf{L}\mathbf{L}^T d\xi\,; \qquad j = 1,2,3,...,n$$

Considering the functions f and g to be constant across each element and carrying out the integrations using the interpolation functions found earlier, yields the element stiffness matrices K_j

$$K_j = \frac{f_j}{h_j^3}\begin{bmatrix} 12 & -6h_j & -12 & -6h_j \\ -6h_j & -4h_j^2 & 6h_j & h_j^2 \\ -12 & 6h_j & 12 & 6h_j \\ -6h_j & h_j^2 & 6h_j & 4h_j^2 \end{bmatrix}; \qquad j = 1,2,3,\ldots,n$$

and mass matrices M_j

$$M_j = \frac{g_j h_j}{420}\begin{bmatrix} 156 & -22h_j & 54 & 13h_j \\ 22h_j & 4h_j^2 & -13h_j & -3h_j^2 \\ 54 & -13h_j & 156 & 22h_j \\ 13h_j & -3h_j^2 & 22h_j & 4h_j^2 \end{bmatrix}; \qquad j = 1,2,3,\ldots,n$$

The differential eigenvalue problem is equivalent to solving for the stationary states of the Rayleigh's quotient, R,

$$R = \frac{[w, w]}{(mw, w)}$$

Requiring that Rayleigh's quotient be stationary yields the algebraic eigenvalue problem

$$K\mathbf{a} = \Lambda^n M\mathbf{a}$$

where $\mathbf{a} = [w_1 \ \theta_1 \ w_2 \ \theta_2 \ \ldots \ w_{n+1} \ \theta_{n+1}]^T$ is the nodal displacement vector, K and M are global stiffness and mass matrices, respectively, and $\Lambda^n = \omega_m^2$. The global matrices come from assembling the respective element matrices.

The eigenvalue problem is solved by first finding the Cholesky decomposition of M then solving for the eigenvalues and eigenvectors of the system

$$L^{-T}KL^{-1}\mathbf{b} = \Lambda^{n}\mathbf{b}$$

$$\text{where} \quad M = L^{T}L \quad \text{and} \quad \mathbf{b} = L\mathbf{a}$$

The resulting eigenvectors are normalized such that

$$\mathbf{a}^{T}M\mathbf{a} = I$$

Calculating the nonlinear coefficients requires the calculation of the integral

$$\alpha_{pq} = \int_{0}^{L} w'_{p}(s) w_{q'}(s)\, ds$$

Again using the approximation

$$w = \mathbf{L}^{T}\mathbf{a}_{j} \qquad s_{j} \leq s \leq s_{j+1}$$

the integral is found to be

$$\int_{0}^{L} w'_{p}(s) w_{q'}(s)\, ds = \mathbf{a}_{p}^{T}\, D\, \mathbf{a}_{q}$$

where D is the global matrix assembled from the element matrices

$$D_{j} = \frac{1}{h_{j}} \int_{0}^{1} \mathbf{L}'\mathbf{L}'^{T}\, d\xi$$

and $\mathbf{a}_p$ and $\mathbf{a}_q$ are the p-th and q-th eigenvectors respectively. The element matrices D_j are

$$D_j = \frac{1}{30h_j}\begin{bmatrix} 36 & -3h_j & -36 & -3h_j \\ -3h_j & 4h_j^2 & 3h_j & -h_j^2 \\ -36 & 3h_j & 36 & 3h_j \\ -3h_j & -h_j^2 & 3h_j & 4h_j^2 \end{bmatrix}; \qquad j = 1,2,3,\ldots,n$$

APPENDIX B. TWO AND THREE MODE MOMENT EQUATIONS

The state variables and differential equations for the two mode approximation of the nonlinear beam equation are

$$x_1 = \psi_1, \quad x_2 = \psi_2, \quad x_3 = \dot{\psi}_1, \quad x_4 = \dot{\psi}_2$$

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\omega_1^2 x_1 - 2\mu_1 x_3 + \beta_{11} x_1^3 + \beta_{12} x_1^2 x_2^1 + \beta_{13} x_1^1 x_2^2 + \beta_{14} x_2^3 + f_1 \\
\dot{x}_4 &= -\omega_2^2 x_2 - 2\mu_2 x_4 + \beta_{21} x_1^3 + \beta_{22} x_1^2 x_2^1 + \beta_{23} x_1^1 x_2^2 + \beta_{24} x_2^3 + f_2
\end{aligned}$$

Using the notation $m_{p,q,r,s} = E\left[x_1^p x_2^q x_3^r x_4^s\right]$, the moment equations for the above system are

$$\begin{aligned}
\frac{d}{dt} m_{p,q,r,s} = {} & p m_{p-1,q,r+1,s} + q m_{p,q-1,r,s+1} \\
& + r(-\omega_1^2 m_{p+1,q,r-1,s} - \mu_1 m_{p,q,r,s} \\
& + \beta_{11} m_{p+3,q,r-1,s} + \beta_{12} m_{p+2,q+1,r-1,s} + \beta_{13} m_{p+1,q+2,r-1,s} + \beta_{14} m_{p,q+3,r-1,s}) \\
& + s(\omega_2^2 m_{p,q+1,r,s-1} - \mu_2 m_{p,q,r,s} \\
& + \beta_{21} m_{p+3,q,r,s-1} + \beta_{22} m_{p+2,q+1,r,s-1} + \beta_{23} m_{p+1,q+2,r,s-1} + \beta_{24} m_{p,q+3,r,s-1}) \\
& + \sigma_1^2 r(r-1) m_{p,q,r-2,s} + \sigma_1 \sigma_2 r s m_{p,q,r-1,s-1} + \sigma_2^2 r(r-1) m_{p,q,r,s-2}
\end{aligned}$$

where the β s are defined as in the table following. The state variables and differential equations for the three mode approximation of the nonlinear beam equation are

$$x_1 = \psi_1, \quad x_2 = \psi_2, \quad x_3 = \psi_3, \quad x_4 = \dot{\psi}_1, \quad x_5 = \dot{\psi}_2, \quad x_6 = \dot{\psi}_3$$

$$
\begin{aligned}
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_5 \\
\dot{x}_3 &= x_6 \\
\dot{x}_4 &= -\omega_1^2 x_1 - 2\mu_1 x_4 + \beta_{11} x_1^3 + \beta_{12} x_1^2 x_2 + \beta_{13} x_1 x_2^2 + \beta_{14} x_2^3 + \beta_{15} x_1^2 x_3 + \beta_{16} x_1 x_2 x_3 + \beta_{17} x_1 x_3^2 \\
&\qquad + \beta_{18} x_2^2 x_3 + \beta_{19} x_2 x_3^2 + \beta_{1,10} x_3^3 + f_1 \\
\dot{x}_5 &= -\omega_2^2 x_2 - 2\mu_2 x_5 + \beta_{21} x_1^3 + \beta_{22} x_1^2 x_2 + \beta_{23} x_1 x_2^2 + \beta_{24} x_2^3 + \beta_{25} x_1^2 x_3 + \beta_{26} x_1 x_2 x_3 + \beta_{27} x_1 x_3^2 \\
&\qquad + \beta_{28} x_2^2 x_3 + \beta_{29} x_2 x_3^2 + \beta_{2,10} x_3^3 + f_2 \\
\dot{x}_6 &= -\omega_3^2 x_3 - 2\mu_3 x_6 + \beta_{31} x_1^3 + \beta_{32} x_1^2 x_2 + \beta_{33} x_1 x_2^2 + \beta_{34} x_2^3 + \beta_{35} x_1^2 x_3 + \beta_{36} x_1 x_2 x_3 + \beta_{37} x_1 x_3^2 \\
&\qquad + \beta_{38} x_2^2 x_3 + \beta_{39} x_2 x_3^2 + \beta_{3,10} x_3^3 + f_3
\end{aligned}
$$

Using the notation $m_{p,q,r,s,t,u} = E\left[x_1^p x_2^q x_3^r x_4^s x_5^t x_6^u\right]$, the moment equations for the above system are

$$
\begin{aligned}
\frac{d}{dt} m_{p,q,r,s,t,u} = {} & p m_{p-1,q,r,s+1,t,u} + q m_{p,q-1,r,s,t+1,u} + r m_{p,q,r-1,s,t,u+1} \\
& + s\,(-\omega_1^2 m_{p+1,q,r,s-1,t,u} - \mu_1 m_{p,q,r,s,t,u} + \beta_{11} m_{p+3,q,r,s-1,t,u} \\
& + \beta_{12} m_{p+2,q+1,r,s-1,t,u} + \beta_{13} m_{p+1,q+2,r,s-1,t,u} + \beta_{14} m_{p,q+3,r,s-1,t,u} \\
& + \beta_{15} m_{p+2,q,r+1,s-1,t,u} + \beta_{16} m_{p+1,q+1,r+1,s-1,t,u} + \beta_{17} m_{p+1,q,r+2,s-1,t,u} \\
& + \beta_{18} m_{p,q+2,r+1,s-1,t,u} + \beta_{19} m_{p,q+1,r+2,s-1,t,u} + \beta_{1,10} m_{p,q,r+3,s-1,t,u}) \\
& + t\,(-\omega_2^2 m_{p,q+1,r,s,t-1,u} - \mu_2 m_{p,q,r,s,t,u} + \beta_{21} m_{p+3,q,r,s,t-1,u} \\
& + \beta_{22} m_{p+2,q+1,r,s,t-1,u} + \beta_{23} m_{p+1,q+2,r,s,t-1,u} + \beta_{24} m_{p,q+3,r,s,t-1,u} \\
& + \beta_{25} m_{p+2,q,r+1,s,t-1,u} + \beta_{26} m_{p+1,q+1,r+1,s,t-1,u} + \beta_{27} m_{p+1,q,r+2,s,t-1,u} \\
& + \beta_{28} m_{p,q+2,r+1,s,t-1,u} + \beta_{29} m_{p,q+1,r+2,s,t-1,u} + \beta_{2,10} m_{p,q,r+3,s,t-1,u}) \\
& + u\,(-\omega_3^2 m_{p,q,r+1,s,t,u-1} - \mu_3 m_{p,q,r,s,t,u} + \beta_{31} m_{p+3,q,r,s,t,u-1} \\
& + \beta_{32} m_{p+2,q+1,r,s,t,u-1} + \beta_{33} m_{p+1,q+2,r,s,t,u-1} + \beta_{34} m_{p,q+3,r,s,t,u-1} \\
& + \beta_{35} m_{p+2,q,r+1,s,t,u-1} + \beta_{36} m_{p+1,q+1,r+1,s,t,u-1} + \beta_{37} m_{p+1,q,r+2,s,t,u-1} \\
& + \beta_{38} m_{p,q+2,r+1,s,t,u-1} + \beta_{39} m_{p,q+1,r+2,s,t,u-1} + \beta_{3,10} m_{p,q,r+3,s,t,u-1}) \\
& + \sigma_1^2 s(s-1) m_{p,q,r,s-2,t,u} + \sigma_2^2 t(t-1) m_{p,q,r,s,t-2,u} + \sigma_3^2 u(u-1) m_{p,q,r,s,t,u-2} \\
& + 2\sigma_1\sigma_2 st m_{p,q,r,s-1,}t - 1,u + 2\sigma_1\sigma_3 su m_{p,q,r,s-1,t,u-1} + 2\sigma_2\sigma_3 tu m_{p,q,r,s,t-1,u-1}
\end{aligned}
$$

The β s used in the previous equations are defined as follows

$\beta_{11} = \varepsilon\Gamma_{1111}$	$\beta_{21} = \varepsilon\Gamma_{2111}$	$\beta_{31} = \varepsilon\Gamma_{3111}$
$\beta_{12} = 3\varepsilon\Gamma_{1112}$	$\beta_{22} = \varepsilon\Gamma_{2211} + 2\varepsilon\Gamma_{2112}$	$\beta_{32} = \varepsilon\Gamma_{3211} + 2\varepsilon\Gamma_{3112}$
$\beta_{13} = \varepsilon\Gamma_{1122} + 2\varepsilon\Gamma_{1212}$	$\beta_{23} = 3\varepsilon\Gamma_{2221}$	$\beta_{33} = \varepsilon\Gamma_{3122} + 2\varepsilon\Gamma_{3212}$
$\beta_{14} = \varepsilon\Gamma_{1222}$	$\beta_{24} = \varepsilon\Gamma_{2222}$	$\beta_{34} = \varepsilon\Gamma_{3222}$
$\beta_{15} = 3\varepsilon\Gamma_{1113}$	$\beta_{25} = \varepsilon\Gamma_{2311} + 2\varepsilon\Gamma_{2113}$	$\beta_{35} = \varepsilon\Gamma_{3311} + 2\varepsilon\Gamma_{3113}$
$\beta_{16} = 2\varepsilon\Gamma_{1123} + 4\varepsilon\Gamma_{1213}$	$\beta_{26} = 2\varepsilon\Gamma_{2213} + 4\varepsilon\Gamma_{2123}$	$\beta_{36} = 2\varepsilon\Gamma_{3312} + 4\varepsilon\Gamma_{3123}$
$\beta_{17} = \varepsilon\Gamma_{1133} + \varepsilon\Gamma_{1313}$	$\beta_{27} = \varepsilon\Gamma_{2133} + 2\varepsilon\Gamma_{2313}$	$\beta_{37} = 3\varepsilon\Gamma_{3133}$
$\beta_{18} = \varepsilon\Gamma_{1322} + 2\varepsilon\Gamma_{1223}$	$\beta_{28} = 3\varepsilon\Gamma_{2223}$	$\beta_{38} = \varepsilon\Gamma_{3322} + 2\varepsilon\Gamma_{3223}$
$\beta_{19} = \varepsilon\Gamma_{1233} + 2\varepsilon\Gamma_{1323}$	$\beta_{29} = \varepsilon\Gamma_{2233} + \varepsilon\Gamma_{2323}$	$\beta_{39} = 3\varepsilon\Gamma_{3233}$
$\beta_{1,10} = \varepsilon\Gamma_{1333}$	$\beta_{2,10} = \varepsilon\Gamma_{2333}$	$\beta_{3,10} = \varepsilon\Gamma_{3333}$

APPENDIX C. THE QUASI-MOMENT EQUATIONS

The following is the relations between the first six quasi-moments and the central moments.

$$
\begin{aligned}
b_a &= 0 \\
b_{ab} &= \mu_{ab} \\
b_{abc} &= \mu_{abc} \\
b_{abcd} &= \mu_{abcd} - 3\{\mu_{ab}\mu_{cd}\} \\
b_{abcde} &= \mu_{abcde} - 10\{\mu_{ab}\mu_{cde}\} \\
b_{abcdef} &= \mu_{abcdef} - 15\{\mu_{ab}\mu_{cdef}\} + 30\{\mu_{ab}\mu_{cd}\mu_{ef}\}
\end{aligned}
$$

where μ denotes a central moment, and the braces denote a symmetrizing operation of the terms enclosed. For example, the first symmetrizing operation is defined by

$$\{\mu_{ab}\mu_{cd}\} = \frac{1}{3}(\mu_{ab}\mu_{cd} + \mu_{ac}\mu_{bd} + \mu_{ad}\mu_{bc})$$

APPENDIX D. SECANT METHOD FOR A SYSTEM OF NONLINEAR EQUATIONS

In computing the steady state solutions of the moment equations it is necessary to solve the system of equations

$$\mathbf{g}(\mathbf{z}) = 0$$

where $\mathbf{z}$ is the 1 by n vector of response moments and $\mathbf{g}(\mathbf{z})$ is the nonlinear set of functions for the response moment derivatives. The procedure, due to Wolfe &wol., consists of an iteration of the following step, where it is initially assumed that $n + 1$ trial vectors $\mathbf{z}^1, \ldots, \mathbf{z}^{n+1}$exist.

We find $\pi_1, \ldots, \pi_{n+1}$ such that

$$\sum_{j=1}^{n+1} \pi_j \mathbf{g}(\mathbf{z}^j) = 0$$

$$\sum_{j=1}^{n+1} \pi_j = 1$$

we then calculate

$$\bar{\mathbf{z}} = \sum_{j=1}^{n+1} \pi_j \mathbf{z}^j$$

and obtain a new set of trial vectors by replacing the vector $\mathbf{z}^j$ for which $||\mathbf{z}^j||$ is a maximum with the vector $\bar{\mathbf{z}}$. The vector norm used is defined by

$$||\mathbf{z}|| = \sum_{i=1}^{n} |g_i(\mathbf{z})|^2.$$

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Nonlinear Stochastic Vibration in Geometrically Varying Beams

by

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(ABSTRACT)

The nonlinear stochastic vibrations of a beam with a varying cross section are investigated. The nonlinearity is caused by midplane stretching and cubic in nature, and the forcing function is wide band white noise. The analysis is carried out by expanding the deflection curve in terms of the undamped linear modes. Substituting this expansion into the partial differential equation yields a set of ordinary differential equations in terms of the modal response functions, which are coupled through the nonlinear terms. The normal modes are found by the finite element method.

The differential equations are then converted to a set of Ito's equations, from which a set of first order differential equations for the response joint moments is found using the Fokker-Planck equation. These equations form an infinite hierarchy which is closed by the quasi-moment method. The solution is investigated near an internal resonance condition and the effects of higher order cumulants in the closure scheme and of additional modes to the expansion are considered.

It is shown that the second order solution is inadequate in the presence of internal resonances, but the fourth order solution proves to be adequate. The one

mode approximation underestimates the nonlinear stiffening, and a multiple mode approach is necessary. It is also shown that the effect of an internal resonance of the stochastic vibration is to transfer of energy from the higher modes involved to the lower modes involved.