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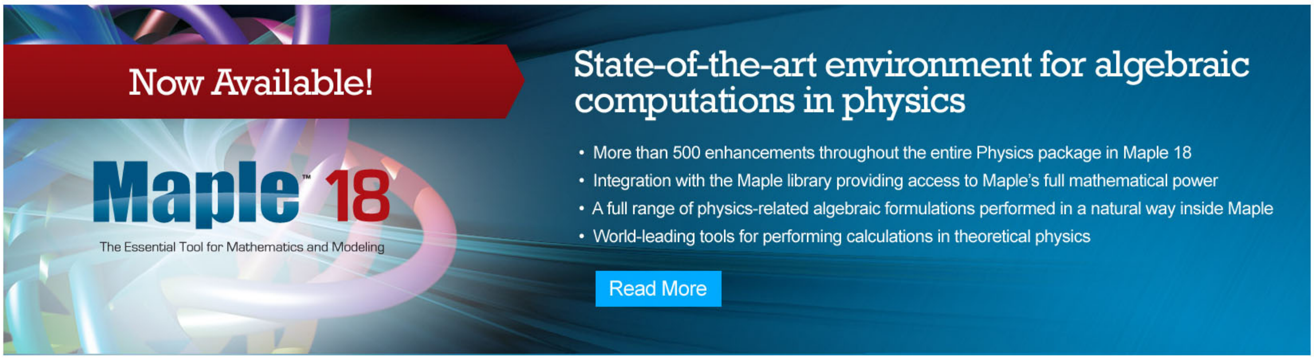

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A note on the iteration of the Chandrasekhar nonlinear H-equation

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An iteration scheme to solve the Chandrasekhar H equation in the form
$$H(\mu)=\{1-\mu\int_0^1[\Psi(s)\ H(s)]/(s+\mu)ds\}^{-1}$$
is shown to converge monotonically and uniformly.

I. INTRODUCTION

It is well known that the nonlinear H equation of radiative and neutron transport theory,

$$H(\mu)=1+\mu H(\mu)\int_0^1\frac{\Psi(s)H(s)}{s+\mu}ds, \tag{1a}$$

does not have a unique solution. However, the "physical" solution of Eq. (1a) subject to the constraints

$$\nu_j\int_0^1\frac{\Psi(s)H(s)}{\nu_j-s}ds=1,\quad j=0,\ldots,\alpha, \tag{2}$$

where $\nu_j, j=0,\ldots,\alpha$, are zeros of the dispersion function

$$\Lambda(z)=1+z\int_{-1}^{+1}\frac{\Psi(s)}{s-z}ds, \tag{3}$$

is unique, and an explicit solution can be written down. Nevertheless, a traditional approach to obtaining values for the H function is to attempt to solve numerically Eq. (1a) by iteration. However, it has been only relatively recently that the iteration of Eq. (1a) has been shown to converge. For example, Bittoni, Casadei, and Lorenzutta[1] showed that when the right-hand side of Eq. (1a) is regarded as a bilinear operator from $L_1(0,1)\times L_1(0,1)$ to $L_1(0,1)$ that the norm of its Fréchet derivative is less than unity and that the bilinear operator is contractive in the ball

$$S=\{\Psi H\in L_1(0,1)|\|\Psi(1-H)\|<\tfrac{1}{2}\} \tag{4}$$

if $\|\Psi\|<\frac{1}{2}$, where $\|\,\|$ denotes the usual L_1 norm. Thus the unique solution of Eq. (1) in S can be obtained by iteration using the scheme

$$H_{n+1}(\mu)=1+\mu H_n(\mu)\int_0^1\frac{\Psi(s)H_n(s)}{s+\mu}d\mu,\quad H_0\in S. \tag{5}$$

Subsequently, Bowden and Zweifel[2] showed that the solution so obtained was indeed the "physical" solution. Furthermore, if $\Psi(s)$ is nonnegative and even for $s\in(-1,1)$, the transformation

$$H(z)=\frac{\nu_0(1+z)}{\nu_0+z}L(z) \tag{6}$$

leads to an equation for L which is identical in form to Eq. (1), but with the characteristic function Ψ replaced by the function

$$\Psi'(s)=\frac{\nu_0^2(1-s^2)}{\nu_0^2-s^2}\Psi(s).$$

(Under the condition stated for Ψ, the dispersion function has only two zeros $\pm\nu_0$.) Bowden and Zweifel pointed out that Ψ' is also a nonnegative even function and that $\|\Psi'\|<\frac{1}{2}$. Thus in all instance in which $\Psi(s)$ is nonnegative and even, numerical values of the H function can be obtained from the iteration scheme given by Eq. (5).

On the other hand, Eq. (1a) can also be rewritten as

$$H(\mu)=\left(1-\mu\int_0^1\frac{\Psi(s)H(s)}{s+\mu}ds\right)^{-1}, \tag{1b}$$

and numerical values for the H function can be obtained by iterating on

$$H_{n+1}(\mu)=\left(1-\mu\int_0^1\frac{\Psi(s)H_n(s)}{s+\mu}ds\right)^{-1},\quad H_0(\mu)=0, \tag{7a}$$

or equivalently,

$$H_{n+1}(\mu)=1+\mu H_{n+1}(\mu)\int_0^1\frac{\Psi(s)H_n(s)}{s+\mu}ds. \tag{7b}$$

Unfortunately, rewriting Eq. (1a) removes the bilinear structure of the equation, and the proof of Bittoni, Casadei, and Lorenzutta need no longer apply. Thus, although this iteration scheme has often been used, dating perhaps from Chandrasekhar and Breen,[3] there does not appear to have been a proof that the sequence $\{H_n\}$ obtained from the iteration scheme given by Eq. (7) converges to the solution of Eq. (1), and that if a solution is obtained, that it represents the physical solution given by the constraining condition (2). The purpose of this note is to given a brief proof that the iteration scheme given by Eq. (7) does indeed converge to the "physical" solution of Eq. (1). The proof is based only on the positivity of the integral operator

$$(\mathscr{L}f)(\mu)=\mu\int_0^1\frac{\Psi(s)f(s)}{s+\mu}ds,\quad \mu\in(0,1), \tag{8}$$

where the assumptions

$$\Psi(s)=\Psi(-s),\quad \Psi(s)\geqslant 0,\quad s\in(-1,+1), \tag{9a}$$

and

$$\int_0^1\Psi(s)ds<\tfrac{1}{2} \tag{9b}$$

are used throughout this note. However, as pointed out above, if condition (9b) is not satisfied, then transformation (6) can be used to obtain an equation in which the resulting characteristic function does satisfy the inequality (9b). The assumption of evenness of $\Psi(s)$ on $(-1,+1)$ is not used in

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the proof of convergence of the iteration scheme *per se*. It is a condition that the solution obey the constraint given by Eq. (2).[2]

2. PROOF OF CONVERGENCE

In this section we present the proof of convergence of the iteration scheme given by Eq. (7). The first step is to show that

$$H_{n+1}(\mu)\geqslant H_n(\mu), \quad \mu\in(0,1), \quad n\geqslant 1, \tag{10}$$

and

$$\|\Psi H_n\| < 1-\sqrt{1-2\|\Psi\|}. \tag{11}$$

The proof of inequalities (10) and (11) proceeds by induction. First note that

$$H_1(\mu)=1, \quad \mu\in(0,1). \tag{12}$$

Further note that

$$\int_0^1 |\Psi(s)H_1(s)|ds=\|\Psi\|\leqslant 1-\sqrt{1-2\|\Psi\|}. \tag{13}$$

If the inequalities (10) and (11) are assumed true for $n=N$, then subtraction of

$$H_N=1+H_N\mathscr{L}(H_{N-1}) \tag{14}$$

from

$$H_{N+1}=1+H_{N+1}\mathscr{L}(H_N) \tag{15}$$

yields, after a little rearrangement,

$$(H_{N+1}-H_N)[1-\mathscr{L}(H_N)]=H_N\mathscr{L}(H_N-H_{N-1}). \tag{16}$$

By simple manipulation it is apparent that

$$1-(\mathscr{L}H_N)(\mu)=1+\int_0^1\frac{s\Psi(s)H_N(s)}{s+\mu}ds$$
$$-\int_0^1\Psi(s)H_N(s)ds>0, \quad \mu\in(0,1), \tag{17}$$

since by assumption,

$$\int_0^1\Psi(s)H_N(s)ds\leqslant 1-\sqrt{1-2\|\Psi\|}<1.$$

Therefore, it follows that

$$H_{N+1}(\mu)-H_N(\mu)\geqslant 0, \quad \mu\in(0,1). \tag{18}$$

This proves inequality (10). To prove inequality (11) it may be noted from Eq. (7b) and inequality (10) that

$$\begin{aligned}
&\|\Psi H_{N+1}\|\\
&=\|\Psi\|+\int_0^1\left(\mu\Psi(\mu)H_{N+1}(\mu)\int_0^1\frac{\Psi(s)H_N(s)}{s+\mu}ds\right)d\mu\\
&\leqslant\|\Psi\|+\tfrac{1}{2}\left(\int_0^1\Psi(\mu)H_{N+1}(\mu)d\mu\right)^2\\
&\quad+\tfrac{1}{2}\int_0^1\left(\Psi(\mu)H_{N+1}(\mu)\int_0^1\frac{\Psi(s)H_{N+1}(s)(\mu-s)}{s+\mu}ds\right)d\mu.
\end{aligned} \tag{19}$$

However, the last term on the right-hand side of the last inequality vanishes so that

$$2\|\Psi H_{N+1}\|-2\|\Psi\|\leqslant\|\Psi H_{n+1}\|^2. \tag{20}$$

It is then obvious that either

$$\|\Psi H_{N+1}\|\geqslant 1+\sqrt{1-2\|\Psi\|}, \tag{21a}$$

or

$$\|\Psi H_{N+1}\|\leqslant 1-\sqrt{1-2\|\Psi\|}. \tag{21b}$$

On the other hand, Eq. (7b) and inequality (10) can be used to write

$$\begin{aligned}
&\|\Psi H_{N+1}\|\\
&=\|\Psi\|+\int_0^1\left[\Psi(\mu)H_{N+1}(\mu)\int_0^1\Psi(s)H_N(s)ds\right]d\mu\\
&\quad-\int_0^1\left(\Psi(\mu)H_{N+1}(\mu)\int_0^1\frac{s\Psi(s)H_N(s)}{s+\mu}ds\right)d\mu\\
&\leqslant\|\Psi\|+\|\Psi H_{N+1}\|\cdot\|\Psi H_N\|\\
&\quad-\int_0^1\left(\Psi(\mu)H_N(\mu)\int_0^1\frac{s\Psi(s)H_N(s)}{s+\mu}ds\right)d\mu\\
&=\|\Psi\|+\|\Psi H_{N+1}\|\cdot\|\Psi H_N\|-\tfrac{1}{2}\|\Psi H_N\|^2.
\end{aligned} \tag{22}$$

Thus it follows that

$$\|\Psi H_{N+1}\|\leqslant(1-\|\Psi H_N\|)^{-1}(\|\Psi\|-\tfrac{1}{2}\|\Psi H_N\|^2). \tag{23}$$

This last inequality yields

$$\|\Psi H_{N+1}\|\leqslant 1, \tag{24}$$

since

$$(1-\|\Psi H_N\|)^{-1}(\|\Psi\|-\tfrac{1}{2}\|\Psi H_N\|^2)>1 \tag{25}$$

would imply that

$$\|\Psi\|>\tfrac{1}{2}[1+(1-\|\Psi H_N\|)^2], \tag{26}$$

which contradicts the assumption $\|\Psi\|<\frac{1}{2}$. Combining this result with the earlier estimate of $\|\Psi H_{N+1}\|$ gives inequality (11).

Therefore, the sequence $\{H_n(\mu)\}$ given by Eq. (7) is positive and monotone with $\|\Psi H_n\|$ bounded by the inequality (11). Thus from the monotone convergence theorem there exists an L_1 (0,1) function, say H^*, such that

$$\lim_{n\to\infty}\Psi H_n\to\Psi H^*, \tag{27}$$

and

$$\begin{aligned}
\|\Psi H^*\|&=\int_0^1|\Psi(s)H^*(s)|ds\\
&=\int_0^1\Psi(s)H^*(s)ds\leqslant 1-\sqrt{1-2\|\Psi\|}.
\end{aligned} \tag{28}$$

A straightforward calculation gives

$$\|\Psi H^*-\Psi-\Psi H^*\mathscr{L}(H^*)\|=0, \tag{29}$$

that is, H^* is an L_1 solution to

$$\Psi H^*=\Psi+\Psi H^*\mathscr{L}(H^*). \tag{30}$$

Now define the function H by

$$H(z)=[1-(\mathscr{L}H^*)(z)]^{-1}. \tag{31}$$

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It is obvious that

$$\Psi(\mu)H(\mu) = \Psi(\mu)[1-(\mathscr{L}H^*)(\mu)]^{-1} = \Psi(\mu)H^*(\mu) \tag{32}$$

almost everywhere for $\mu \in (0,1)$. Thus $H(z)$ satisfies the nonlinear H equation (1) and is analytic in the complex plane of z cut along $(-1,0)$ except at that value of z such that $1-(\mathscr{L}H)(z)$ vanishes. However, since $\|\Psi H\| < 1$ it follows from the results of Bowden and Zweifel[2] that $1-(\mathscr{L}H)\times(-\nu_0) = 0$, i.e., $H(z)$ represents the "physical" solution according to Eq. (2). In particular $1-(\mathscr{L}H)(z)$ is bounded in the right half complex z plane.

Finally an estimate of $|H_{n+1}(\mu) - H(\mu)|$ can be written from Eqs. (1) and (7) as

$$\begin{aligned} |H_{n+1}(\mu) - H(\mu)| &= |H_{n+1}(\mu)(\mathscr{L}H_n)(\mu) - H(\mu)(\mathscr{L}H)(\mu)| \\ &\leqslant |H_{n+1}(\mu)[\mathscr{L}(H_n - H)](\mu) \\ &\quad + [H_{n+1}(\mu) - H(\mu)](\mathscr{L}H)(\mu)|. \end{aligned} \tag{33}$$

This inequality in turn yields

$$\begin{aligned} |H_{n+1}(\mu) - H(\mu)| &\leqslant [1-(\mathscr{L}H)(\mu)]^{-1} \\ &\quad \times |H_{n+1}(\mu)[\mathscr{L}(H_n - H)](\mu)|. \end{aligned} \tag{34}$$

Since the right-hand side of this last approaches zero as $n \to \infty$, the sequence $\{H_n\}$ approaches H at least pointwise. However, since H is continuous on $(0,1)$ and the $[H_n(\mu)]$ is a positive monotone sequence, it follows from Dini's theorem that the convergence is also uniform.

In summary, it can then be stated that the sequence of functions (H_n) obtained by the iteration scheme given by Eq. (7) converges monotonically and uniformly to the physical solution of Eq. (1).

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