Combined Correlation Induction Strategies

for Designed Simulation Experiments

by

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(ABSTRACT)

This dissertation deals with variance reduction techniques (VRTs) for improving the reliability of the estimators of interest through a controlled laboratory-like simulation experiment. This research concentrates on correlation methods of VRTs which include common random numbers, antithetic variates and control variates. The basic idea of these methods is to utilize the linear correlation either between the responses or between the response and control variates in order to reduce the variance of estimators of certain system parameters. Combining these methods, we develop procedures for estimating a system parameter of interest.

First, we develop three combined methods utilizing antithetic variates and control variates for improving the estimation of the mean response in a single population model. We explore how these methods may reduce the variance of the estimator of interest. A combined method (Combined Method I) using antithetic variates for the non-control variate stochastic components and independent streams for the control variates yields better results than by applying methods of either antithetic variates or control variates individually for several selected models.

Second, we develop variance reduction techniques for improving the estimation of the model parameters in a multipopulation simulation model. We extend Com-

bined Method I showing good performance in estimating the mean response of a single population model to the multipopulation context with independent simulation runs across design points. We also develop another extension of Combined Method I that incorporates the Schruben-Margolin method to estimate the parameters of a multipopulation model. Under certain conditions, this method is superior to the Schruben-Margolin method. Finally, we propose a new approach (Extended Schruben-Margolin Method) utilizing the control variates under the Schruben-Margolin strategy for improving the estimation in a first-order linear model. Extended Schruben-Margolin Method yields better results than the Schruben-Margolin method in estimating the model parameters of interest.

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CHAPTER 1. INTRODUCTION

Computer simulation has become a widely used technique to study systems too complex to be evaluated analytically. Kleijnen (1974) defined simulation as building a mathematical-logical model of a stochastic system, experimenting with it over time and collecting data to evaluate the system parameters of interest. We will consider discrete-event simulation throughout this research. That is, experimentation of a simulation model involves the changing of the state of the system model only at a finite number of points in time (discrete), simulation evolves with time (dynamic), and the simulation model contains more than one stochastic variable (a more detailed presentation of this framework for simulation is given in Section 1.2 of Law and Kelton (1982)).

Through a controlled laboratory-like simulation experiment, an experimenter is concerned with estimating the system parameters of interest from the outputs of the simulation model. Estimation of the system parameters and reliability of these estimators involve many statistical principles of experimental design and regression analysis. This dissertation focuses on a simulation technique for reducing the variability of the estimators for the parameters of interest in a designed simulation experimental model.

The precision of the estimators of the parameters is associated with collecting the necessary amount of simulation output data. Frequently large-scale systems analysis through simulation requires extensive experimentation with a simulation model to obtain acceptable precision in the estimators of the system parameters of interest. Although we expect the cost of experimenting with the simulation model to become less as the computer technology continues to increase, there remain many situations where estimating model parameters is difficult with adequate precision at an acceptable cost. Also, if we can reduce the variance of the estimator of interest at little cost, we can obtain greater precision of the estimator with the same amount of simulation. Thus, in this work we are proposing ways of obtaining significantly better information from a set of simulation runs with little additional effort.

To this end, variance reduction techniques (VRTs) have attracted considerable interest. The variance reduction problem consists of obtaining unbiased estimators of the system parameters of interest with smaller variances than the simple estimators obtained by independent replications (without applying any transformations on the input domain or the output space of the simulation experiment (see Wilson (1984))). VRTs offer a broad selection of methods useful in reducing the variance of the estimator for the system parameter. Wilson (1984) classified all VRTs into two major categories -- importance methods and correlation methods. The importance methods include the techniques of importance sampling, conditional Monte Carlo, stratified sampling, and systematic sampling. The basic idea of these techniques is that most of the contribution to the value of the parameter of interest comes from more important subregions of the input domain. The accuracy of the estimators can be improved by concentrating the sampling effort in those subregions of the input domain. Correlation methods include the techniques of common random numbers, antithetic variates, and control variates. The techniques of common random numbers and antithetic variates require the induction of positive and negative correlations, respectively, between pairs of responses from different simulation runs. The method of control variates attempts to take advantage of the correlation between the simulation response and the control variates. Under the normality assumption of the response and control variates, this method is a special case of covariance analysis in classical regression theory (see Chapter 8 in Searle (1972) and Lavenberg, Moeller and Welch (1982)).

In this research, of particular interest are correlation methods that utilize the correlations between simulation output either within a single run or across different replications. For a single population model, usually antithetic variates and control variates are applied to reduce the error of the estimator for the mean response of interest. In contrast to the approach of antithetic variates, the method of control variates attempts to exploit any inherent correlations between the response and selected concomitant variables (control variates) within a single run. We hypothesize that through correlated replications of simulation runs, we get a reduced variance of the estimator for the mean response and yet maintain the same correlation between the response and control variates as those obtained under independent replications. Then it is conjectured that we may take advantage of both antithetic variates and control variates together in one simulation run, and reduce the variance of the estimator further by applying either antithetic variates or control variates separately. Although the control variates can be external (that is, similar variates in a much simplified version of the original model which is driven by the same random number streams as the original model), this research deals only with internal (concomitant) control variates.

For a designed simulation model having multiple design points (we refer to it as a multipopulation simulation model throughout this research), several authors have developed methods combining correlation methods for estimating the model parameters of interest. To improve the precision of the estimator of the model parameters, Schruben and Margolin (1978) developed a correlation induction strategy for combining the use of common random numbers and antithetic variates in the same designed experiment. Also Tew and Wilson (1989) proposed a combined approach of the Schruben and Margolin (1978) correlation induction strategy and control variates. Under certain condition, this method yields better results than the Schruben-Margolin method in estimating the model parameters of interest.

The above two studies exploit the correlation between the responses at different design points. The combined method proposed earlier, which is fundamentally different from these two approaches, focuses on reduction in variance of the mean response of interest in a single population model. We consider that the responses with reduced variances at the design points of the experimental model may ensure improvement in the estimation of the parameters of the multipopulation model. Based on this conjecture, we explore a way of extending the combined method proposed in a single population model to the multipopulation context. Also, in extending the combined method, we consider a strategy incorporating the correlations between the responses at different design points for further improving the precision of the parameter estimation. In the same spirit of the methods mentioned above for improving the precision of the multipopulation estimation, we propose a new approach of utilizing the control variates under the Schruben-Margolin method.

1.1 Research Objectives

The objectives of this research are threefold. First, we develop methods combing antithetic variates and control variates for a single population model. Specifically, we develop three methods utilizing induced correlations between: (a) the responses of interest, (b) the response and a set of control variates, and (c) the control variates, obtained by an appropriate assignment of random numbers streams through replications, and try to improve upon the simulation efficiency of the control variates method. Second, we explore the extension of a combined approach for a single population model to a simulation experiment designed to estimate the parameters of interest of a multipopulation simulation model. Third, we develop a new method which deals with the application of the control variates under the Schruben-Margolin method for the multipopulation described as a first-order linear model.

1.2 Organization of the Dissertation

Chapter 2 presents a review of notation and the relevant literature. This review includes analysis of covariance in general and the method of control variates specifically. Chapter 3 develops the three combined methods utilizing control variates and antithetic variates in a single population model. Chapter 4 applies the three combined methodologies developed in Chapter 3 as well as methods of control variates and antithetic variates to various simulation models, and presents a summary of the simulation results as well as inferences for these results. Chapter 5 provides a development for extending the first combined method proposed in Chapter 3 to the multipopulation model, and also presents a new approach for improving the Schruben-Margolin method with the additional application of control variates. Chapter 6 describes the simulation results from the application of the two methods developed in Chapter 5 as well as the Schruben-Margolin method for the multipopulation model, and presents a summary and analysis of these simulation results. Chapter 7 summarizes the research findings and outlines directions for future research.

CHAPTER 2. LITERATURE REVIEW

This chapter presents a brief overview and review of the literature for analysis of covariance for the general linear statistical model and the variance reduction technique of control variates for simulation experimentation.

2.1. Analysis of Covariance

In this section, we briefly review analysis of covariance in terms of the structure of a linear statistical model. This review includes methods of estimation of model parameters and distributions of these estimators.

In statistical experimentation, the response of interest may be related to two types of independent variables: (a) factor variables and (b) concomitant variables (covariate or covariable). Factor variables are under the control of an experimenter, in that we assume that the experimenter can select and set levels of a factor variable without error. In contrast, the levels of the concomitant variables are not set by the experimenter but merely observed in the course of conducting the experiment. If the experimenter is interested in the effect of the levels of the factor variables to the response, he carefully chooses the levels of the factor variables in a region of interest. A concomitant variable is observed independently, typically, at each of the levels of the factor variables during the experiment and assumed to be correlated with the corresponding response. For instance, the experimenter may wish to compare the effects of different drugs on a patient by measuring a response variable of interest. It is assumed that the body weight of the patient is correlated with the response variable, but is independent of drug type. In this example, drug type is the factor variable and the body weight of the patient is the concomitant variable (see Seber (1977), pp. 279-281). If the concomitant variables are strongly correlated with the response variable, then by subtracting an appropriate linear function of the concomitant variables from the response variable, the unknown error term of the response variable can be counteracted. Thus a statistical model including the concomitant variables *may* describe the response more accurately than that with the factor variables only.

Consider the linear statistical model with a combination of factor variables and concomitant variables. While the factor variables are typically nonrandom variables, the concomitant variables are usually assumed to be random variables which have a multivariate jointly normal distribution with the response variable of interest. Given the concomitant variables, we analyze the conditional distribution of the response variable appropriately represented by the linear function of the factor variables (see Graybill (1976), Chapter 5). Analysis of a combined model with factor variables and concomitant variables is generally referred to as analysis of covariance. Analysis of covariance is based on the analysis of the factor part of the model suitably amended by the presence of the concomitant variables.

Throughout this section we present, in summary, the major results found in the literature regarding the use of concomitant variables in the context of the general linear model. In Section 2.1.1, we identify the structure of the linear statistical model with concomitant variables and contribution concomitant variables make to the overall variance of the response. Section 2.1.2 discusses the estimation of model parameters and their interpretation. Section 2.1.3 identifies the distributions of these estimators and some related quadratic functions of the response variable of interest.

2.1.1. Linear statistical model with concomitant variables

We consider the case where the experimenter is concerned with estimating the mean of a univariate response of interest and the relationship of this mean to the levels of the factor variables. We also assume the presence of at least one concomitant variable that is correlated with the response variable of interest. As indicated earlier, the factor variables must be variables whose values can be established without error by the experimenter. A variable is classified as a concomitant variable if the experimenter thinks that deliberate manipulation of its value is an impossibility (for example, natural physical phenomena), or because the experimenter is not attempting to learn about its effect (see Pratt and Schlaifer (1984)), yet it is assumed to be correlated with the response variable of interest. Thus the experimenter cannot control the concomitant variable.

Since the experimenter is interested in the effect of the factor variables, he may want to explore the response surface over a factor region of interest. The considerations connected with the exploration of the response surface over a factor region of interest include: (a) the choice of a proper model to approximate the response surface over a factor region, (b) the suitable design of a factor region, and (c) the testing of the adequacy of the model to represent the response surface (see Cornell (1981), Section 1.2). To this end, we assume that there exists some linear functional relationship between the response and the experimental variables (factor and concomitant).

First, we consider the relationship of the response y to the settings of just the factor variables. We let

$$y = \mu(\xi) + \varepsilon, \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, ..., \xi_d)'$ is a specific setting of the d factor variables, $\mu(\xi)$ is a linear function in some unknown parameters that relate the response y to the levels of the factor variables, and ε represents the inability of the postulated function $\mu(\xi)$ to determine y. Although the relationship $\mu(\xi)$ between the response and the factor variables is generally unknown, a polynomial function often is used to represent $\mu(\xi)$. The justification is due to the ability to expand $\mu(\xi)$ by using a Taylor Series expansion (see Cornell (1981), p. 9). Normally low-degree polynomials such as first or second-degree polynomials are sufficient to adequately represent the response surface (see Myers (1971), p. 26). Suppose we have p functions of the factor variables of interest and we select m ($m > p$) experimental points (settings of the factor variables); $\xi_1, \xi_2, ..., \xi_m$. The linear statistical model in (2.1) can be written as

$$y_i = \beta_0 + \sum_{k=1}^{p} \beta_k x_k(\xi_i) + \varepsilon_i, \qquad i = 1, 2, ..., m, \tag{2.2}$$

where β_k $(k = 0, 1, ..., p)$ are model parameters, $x_k(\xi_i)$ $(k = 1, 2, ..., p)$ represent p known functions of the settings of the factor variables, ε_i $(i = 1, 2, ..., m)$ is the error term at the ith design point, and y_i $(i = 1, 2, ..., m)$ is the response of interest at the ith design point. We assume that the ε_i's are IID $\sim N(0, \sigma_\varepsilon^2)$ for all experimental points. Under this assumption, the linear statistical model given by (2.2) implies that the distribution of y_i is $N(\beta_0 + \sum_{k=1}^{p} \beta_k x_x(\xi_i), \sigma_\varepsilon^2)$ if we experiment repeatedly at a given point ξ_i $(i = 1, 2, ..., m)$. The model (2.2) can be written in matrix form as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} , \tag{2.3}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_m)'$ is the $(m \times 1)$ vector of responses, $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)'$ is the $((p+1) \times 1)$ vector of unknown model coefficients, $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)'$ is the $(m \times 1)$ vector of error terms, and $\mathbf{X}$ is a $(m \times (p+1))$ matrix whose first column is the $(m \times 1)$ vector of 1's (1_m) and whose ith column consists of $(x_i(\xi_1), x_i(\xi_2), \ldots, x_i(\xi_m))'$ $(i = 2, 3, \ldots, p+1)$.

Second, we consider the linear statistical model with both factor variables and concomitant variables:

$$y_i = \beta_0 + \sum_{k=1}^{p} \beta_k x_k(\xi_i) + \sum_{l=1}^{s} c_{il}\alpha_l + \varepsilon_i^* , \qquad i = 1, 2, \ldots, m; \tag{2.4}$$

where y_i, β_0, β_k and $x_k(\xi_i)$ are defined in (2.2); c_{il} $(l = 1, 2, \ldots, s)$ is the lth concomitant variable at the ith design point; α_l $(l = 1, 2, \ldots, s)$ is the coefficient of c_{il}; and ε_i^* $(i = 1, 2, \ldots, m)$ represents the inability of the postulated model to determine y_i.

Analogous to the assumption on the distribution of ε_i in (2.2), we assume that the ε_i^*'s are IID $\sim N(0, \sigma_{\varepsilon^*}^2)$ across all experimental points. We also assume that the $\mathbf{c}'_i = (c_{i1}, c_{i2}, \ldots, c_{is})$ $(i = 1, 2, \ldots, m)$ are IID $\sim N_s(\mathbf{0}, \boldsymbol{\Sigma}_c)$. Under these assumptions, the linear statistical model given by (2.4) implies that the distribution of y_i is $N(\beta_0 + \sum_{k=1}^{p} \beta_k X_k(\xi_i),\ \boldsymbol{\alpha}'\boldsymbol{\Sigma}_c\boldsymbol{\alpha} + \sigma_{\varepsilon^*}^2)$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_s)'$, if we experiment repeatedly at a given point ξ_i $(i = 1, 2, \ldots, m)$.

It is often convenient to represent (2.4) in matrix form as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{C}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}^*, \tag{2.5}$$

where $\mathbf{y}$, $\boldsymbol{\beta}$, and $\mathbf{X}$ are defined in (2.3), $\boldsymbol{\varepsilon}^* = (\varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_m^*)'$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_s)'$ and $\mathbf{C}$ is a $(m \times s)$ matrix whose ith row consists of $\mathbf{c}'_i = (c_{i1}, c_{i2}, \ldots, c_{is})$. In this model, $\boldsymbol{\beta}$ describes the effect of the factor variables on the response y_i, *given* $\mathbf{c}'_i$. However, we can not interpret the parameter $\boldsymbol{\alpha}$ as the effect of $\mathbf{c}'_i$ on y_i. We will discuss how the parameter $\boldsymbol{\alpha}$ may be interpreted and how the inclusion of concomitant variables may improve the estimate of $\boldsymbol{\beta}$ in the next section.

Obviously the model in (2.2) is a special case of the model in (2.4). A difference in the observed responses is attributed to the factor variables only if the responses are obtained under the same external conditions which influence them. By the definition of the concomitant variables, they are correlated with the response, hence if the relationship between the concomitant variables and the response variable is known, we can reduce the residual variation of the statistical linear model significantly. For specifying the difference between the variance of the response variable in (2.2) and that in (2.4), we compare the variance of ε_i^* with that of ε_i. If we subtract the effect contributed by the concomitant variables from both sides of (2.4), we have

$$y_i - \sum_{k=1}^{s} \alpha_k c_{ik} = \beta_0 + \sum_{k=1}^{p} \beta_k x_k(\xi_i) + \varepsilon_i^*, \qquad i = 1, 2, \ldots, m. \tag{2.6}$$

This equation represents the relationship of the adjusted response to the settings of the factor variables. When the concomitant variables $\mathbf{c}'_i = (c_{i1}, c_{i2}, \ldots, c_{is})$ are observed at a specific design point, the variance of ε_i^* will be

$$Var(\varepsilon^*_i) = Var(y_i - \sum_{k=1}^{s} \alpha_k c_{ik}) = Var(y_i - \mathbf{c}'_i\boldsymbol{\alpha}), \qquad i = 1,\ 2,\ \ldots,\ m. \tag{2.7}$$

The minimum variance of ε^*_i $(i = 1, 2, \ldots, m)$ is achieved when the vector of coefficients for concomitant variables, $\boldsymbol{\alpha}$, is established as

$$\boldsymbol{\alpha}' = \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}, \tag{2.8}$$

where $\boldsymbol{\sigma}'_{yc}$ is a $(1 \times s)$ row vector of covariances between y_i and $\mathbf{c}'_i$ $(i = 1,\ 2,\ \ldots, m)$, and $\boldsymbol{\Sigma}_c$ is the $(s \times s)$ covariance matrix of the $\mathbf{c}_i$'s (see Anderson (1958), p. 32). The vector of coefficients $\boldsymbol{\alpha}$ in (2.8) induces the maximum correlation between y_i and the linear combination of the set of concomitant variables $\mathbf{c}'_i$ $(i = 1, 2, \ldots, m)$ (see Morrison (1976), p. 95). If $\boldsymbol{\alpha}$ is known, the variance of ε^*_i $(\sigma^2_{\varepsilon^*})$ is

$$Var(\varepsilon^*_i) = \sigma^2_{\varepsilon^*} = \sigma_y^{\,2} - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} = \sigma_y^{\,2}(1 - R_{yc}^{\,2}), \qquad i = 1,\ 2,\ \ldots, m, \tag{2.9}$$

where $R^2_{yc} = \sigma_y^{-2}\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}$ is the square of the multiple correlation coefficient between a single response variable y_i and a set of s concomitant variables $\mathbf{c}_i$ $(i = 1, 2, \ldots, m)$. The correlation coefficient between y_i and $\mathbf{c}'_i\boldsymbol{\alpha}$ is given by $R_{yc} = \boldsymbol{\alpha}'\boldsymbol{\sigma}_{yc}/(\sigma_y\sqrt{\boldsymbol{\alpha}'\boldsymbol{\Sigma}_c\boldsymbol{\alpha}}\ )$ (see the covariance operators in Seber (1977). pp. 10-11). When $\boldsymbol{\alpha}'$ is equal to $\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}$, we have the maximum $R_{yc} = (\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc})^{1/2}/\sigma_y$ (see Morrison (1976), p. 95), which is equivalent to the greatest correlation coefficient between y_i and the linear combination of $\mathbf{c}_i$'s $(i = 1, 2, \ldots, m)$ (see the definition of the multiple correlation coefficient in Muirhead (1982), p. 164). Therefore, in the case where the coefficient vector $\boldsymbol{\alpha}$ is known, we can reduce the effective variation of the response variable of interest by the amount $\sigma^2_y R^2_{yc}$.

2.1.2. Estimation of model parameters

In this section, we review the results on the estimation of α and β in (2.5) using the least squares method and interpret these estimators.

The least squares method is designed to provide estimators for the parameters in the linear statistical model so that the residual error sum of squares, $\varepsilon^{*\prime}\varepsilon^{*}$, is minimized (see Searle (1971), p. 87). To obtain the least squares estimators of α and β in (2.5) from one normal vector equation, we define $\mathbf{G} = (\mathbf{X}, \mathbf{C})$ and $\gamma = (\beta', \alpha')'$, where $\mathbf{X}$ and $\mathbf{C}$ are defined in the previous section. Then the linear model given in (2.5) can be written as

$$\mathbf{y} = \mathbf{G}\gamma + \varepsilon^{*}. \tag{2.10}$$

It is assumed that $\mathbf{C}$ has full column rank s and is independent of $\mathbf{X}$, and that $\mathbf{X}$ has full column rank $(p + 1)$. Then the matrix $\mathbf{G}$ has full column rank $(p + s + 1)(< m)$ and also $\mathbf{G}'\mathbf{G}$ has full column rank $p + s + 1$. Thus $(\mathbf{G}'\mathbf{G})$ is positive-definite and there exists an inverse of $(\mathbf{G}'\mathbf{G})$ (see Seber (1977), Theorems A2.4, A4.1 and A4.2). The least squares estimator for γ in (2.10) is given by

$$\hat{\gamma}_{\mathbf{G}} = (\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\mathbf{y}. \tag{2.11}$$

By using the technique for the inverse of a partitioned matrix, we recognize that the $((p + s + 1) \times (p + s + 1))$ matrix $(\mathbf{G}'\mathbf{G})^{-1}$ may be partitioned as

$$(\mathbf{G}'\mathbf{G})^{-1} = \begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{C} \\ \mathbf{C}'\mathbf{X} & \mathbf{C}'\mathbf{C} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (\mathbf{X}'\mathbf{X})^{-1} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\mathbf{W}\mathbf{C}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} & -(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\mathbf{W} \\ -\mathbf{W}\mathbf{C}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} & \mathbf{W} \end{bmatrix}, \tag{2.12}$$

where $(\mathbf{X}'\mathbf{X})^{-1} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\mathbf{W}\mathbf{C}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$ is $((p+1)\times(p+1))$, $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\mathbf{W}$ is $((p+1)\times s)$, and the $(s\times s)$ matrix $\mathbf{W}$ is given by

$$\mathbf{W} = (\mathbf{C}'\mathbf{C} - \mathbf{C}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C})^{-1} = [\mathbf{C}'(\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{C}]^{-1} = (\mathbf{C}'\mathbf{P}\mathbf{C})^{-1} \tag{2.13}$$

(see Searle (1971), p. 27). Thus the least squares estimator $\hat{\gamma}_G$ can be written as

$$\hat{\gamma}_G = \begin{bmatrix} \hat{\beta}_G \\ \hat{\alpha}_G \end{bmatrix} = \begin{bmatrix} (\mathbf{X}'\mathbf{X})^{-1} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\mathbf{W}\mathbf{C}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} & -(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\mathbf{W} \\ -\mathbf{W}\mathbf{C}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} & \mathbf{W} \end{bmatrix} \begin{bmatrix} \mathbf{X}'\mathbf{y} \\ \mathbf{C}'\mathbf{y} \end{bmatrix}$$

$$= \begin{bmatrix} (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\mathbf{W}\mathbf{C}'(\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y} \\ \mathbf{W}\mathbf{C}'[\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{y} \end{bmatrix}$$

$$= \begin{bmatrix} (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\mathbf{W}\mathbf{C}'(\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y} \\ [\mathbf{C}'(\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{C}]^{-1}\mathbf{C}'[\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{y} \end{bmatrix}. \tag{2.14}$$

Instead of using the inverse of $(\mathbf{G}'\mathbf{G})$, we can obtain the least squares estimator $\hat{\gamma}_G$ by directly solving two normal equations. Consider the linear statistical model (2.5). If we differentiate, with respect to α and β, the residual error sum of squares

$$\varepsilon^{*'}\varepsilon^{*} = (\mathbf{y} - \mathbf{X}\beta - \mathbf{C}\alpha)'(\mathbf{y} - \mathbf{X}\beta - \mathbf{C}\alpha) \tag{2.15}$$

and set it equal to $\mathbf{0}$, then the normal equations for β and α of the model (2.5) are

$$\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{C} \\ \mathbf{C}'\mathbf{X} & \mathbf{C}'\mathbf{C} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{y} \\ \mathbf{C}'\mathbf{y} \end{bmatrix}. \tag{2.16}$$

(see Searle (1971), p. 341). As assumed earlier, $\mathbf{X}$ and $\mathbf{C}$ have full column ranks $(p+1)$ and s, respectively, and the inverses of both $(\mathbf{X}'\mathbf{X})$ and $(\mathbf{C}'\mathbf{C})$ exist. Suppose that $\hat{\alpha}_G$ and $\hat{\beta}_G$ are the solutions to (2.16). Then the first set of equations in (2.16) gives

$$\hat{\beta}_{\mathbf{G}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{y} - \mathbf{C}\hat{\alpha}_{\mathbf{G}}) = \hat{\beta}_{\mathbf{X}} - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\hat{\alpha}_{\mathbf{G}}, \tag{2.17}$$

where $\hat{\beta}_X = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ is the solution of the normal equations for the linear model without concomitant variables. The second equation from (2.16) is

$$\hat{\alpha}_{\mathbf{G}} = (\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}'(\mathbf{y} - \mathbf{X}\hat{\beta}_{\mathbf{G}}). \tag{2.18}$$

Substituting for $\hat{\beta}_G$ into (2.18) gives

$$\hat{\alpha}_{\mathbf{G}} = (\mathbf{C}'\mathbf{C})^{-1}\mathbf{C}'[(\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y} + \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\hat{\alpha}_{\mathbf{G}}]. \tag{2.19}$$

Multiplying both sides of (2.19) by $(\mathbf{C}'\mathbf{C})$ and arranging (2.19) with respect to $\hat{\alpha}_G$ gives

$$[\mathbf{C}'\mathbf{C} - \mathbf{C}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}]\hat{\alpha}_{\mathbf{G}} = \mathbf{C}'[\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{y}. \tag{2.20}$$

Define $\mathbf{P}$ as

$$\mathbf{P} = \mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'. \tag{2.21}$$

In (2.3), the least squares estimated response $\hat{\mathbf{y}}$ is the m-dimensional vector in the space of $\mathbf{X}$ by the projection of $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Also $\mathbf{P}\mathbf{y}$ is equivalent to $\mathbf{y} - \hat{\mathbf{y}}$ which is orthogonal to $\hat{\mathbf{y}}$. Thus $\mathbf{P}$ can be interpreted as the linear transformation representing an orthogonal projection of $\mathbf{y}$ (in m-dimensional Euclidian space) onto the range of space orthogonal to the space of $\mathbf{X}$ (see Seber (1977), p. 46). Since $\mathbf{P}' = \mathbf{P}$ and $\mathbf{P}'\mathbf{P} = \mathbf{P}$, $\mathbf{P}$ is symmetric and idempotent. Symmetry and idempotency of $\mathbf{P}$ ensure that $\mathbf{C}'\mathbf{P}\mathbf{C}$ and $\mathbf{P}\mathbf{C}$ have the same rank since $\text{rank}(\mathbf{P}\mathbf{C}) = \text{rank}(\mathbf{C}'\mathbf{P}\mathbf{P}'\mathbf{C}) = \text{rank}(\mathbf{C}'\mathbf{P}\mathbf{C})$

(see Seber (1977) p. 385). Furthermore, **PC** has full column rank and hence **C′PC** is nonsingular (see Searle (1971), p. 342). Then $\hat{\alpha}_G$ can be written as

$$\hat{\alpha}_G = [\mathbf{C}'(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{C}]^{-1}\mathbf{C}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{y} = (\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{y}, \qquad (2.22)$$

which is the same as in equation (2.14). Substitution of (2.22) into (2.17) then gives $\hat{\beta}_G$ as

$$\hat{\beta}_G = \hat{\beta}_X - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{y} \qquad (2.23)$$

(see Searle (1971), p. 342). These results are the same as the estimator $\hat{\gamma}_G$ in (2.14) which is known to be the best linear unbiased estimator (b.l.u.e.) of γ (see Rao (1973), p. 229 and Graybill (1976) Section 6.10). (The procedures given above are discussed in Chapter 8 of Searle (1971) and Chapter 3 of Seber (1977)).

The least squares estimator $\hat{\beta}_G$ can be interpreted as the effect of the factor variables on the response y adjusted by the concomitant variables. The effect of the least squares estimator $\hat{\alpha}_G$ is discussed in the remainder of this section. The $(m \times 1)$ vector of residuals, $\mathbf{e}_y$, obtained by fitting the linear statistical model $\mathbf{y} = \mathbf{X}\beta + \varepsilon$ is defined to be

$$\mathbf{e}_y = \mathbf{y} - \mathbf{X}\hat{\beta}_X = (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y} = \mathbf{P}\mathbf{y}. \qquad (2.24)$$

Similarly, we consider the linear statistical model which fits the design matrix **X** to the $(m \times 1)$ vector of the ith concomitant variables $\mathbf{c}_i$ $(i = 1, 2, \ldots, s)$:

$$\mathbf{c}_i = \mathbf{X}\beta + \varepsilon^{**}, \qquad (2.25)$$

where $\varepsilon^{**} = (\varepsilon_1^{**}, \varepsilon_2^{**}, \ldots, \varepsilon_m^{**})'$ are IID $\sim N_m(\mathbf{0}, \sigma_\varepsilon^{**2}\mathbf{I}_m)$. Then the $(m \times 1)$ vector of residuals, $\mathbf{e}_{c_i}$, obtained by fitting the linear statistical model $\mathbf{c}_i = \mathbf{X}\boldsymbol{\beta} + \varepsilon^{**}$ is

$$\mathbf{e}_{c_i} = \mathbf{P}\mathbf{c}_i, \qquad i = 1, 2, \ldots, s. \tag{2.26}$$

Hence we can consider the matrix **PC** as the residual matrix resulting from fitting the linear statistical model $\mathbf{c}_i = \mathbf{X}\boldsymbol{\beta} + \varepsilon^{**}$ $(i = 1, 2, \ldots, s)$. The least squares estimator of $\boldsymbol{\alpha}$ in the linear statistical model

$$\mathbf{P}\mathbf{y} = \mathbf{P}\mathbf{C}\boldsymbol{\alpha} + \varepsilon^{*} \tag{2.27}$$

is $(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{y}$, which is equivalent to the least squares estimator of $\hat{\boldsymbol{\alpha}}_G$ in (2.22). Therefore, $\hat{\boldsymbol{\alpha}}_G$ can be viewed as the effect of the matrix of residuals, due to fitting the model to the concomitant variables, on the residuals due to fitting the model to the response of interest (see Searle (1971), p. 343).

2.1.3. Distributions of model estimates, regression sum of squares and residual error sum of squares

In this section, we review the results on the distributions of the parameter estimators for the models in (2.3) and (2.5) identified in the previous section under the assumptions of Section 2.1.1. We also identify the residual error sum of squares, the regression sum of squares and the variance estimators for the models in (2.3) and (2.5) and their distributions under the assumptions of Section 2.1.1.

First, we consider the distributions of $\hat{\boldsymbol{\beta}}_X$, the regression sum of squares (SSR), and residual error sum of squares (SSE) for the model in (2.3) under the assumption that $\varepsilon \sim N_m(\mathbf{0}, \sigma_\varepsilon^2\mathbf{I}_m)$. It is known that $\hat{\boldsymbol{\beta}}_X$ has the following distribution:

$$\hat{\boldsymbol{\beta}}_{\mathbf{X}} \sim N_{p+1}(\boldsymbol{\beta},\ \sigma_y^2(\mathbf{X}'\mathbf{X})^{-1}) \tag{2.28}$$

(see Graybill (1976), p. 176).

To identify SSR and SSE, we partition the total sum of squares (SST) of the response variables into two components. That is, SST is composed of the sum of squares due to fitting the linear model (SSR($\hat{\boldsymbol{\beta}}_{\mathbf{X}}$)) and the residual error sum of squares (SSE($\hat{\boldsymbol{\beta}}_{\mathbf{X}}$)):

$$\mathrm{SST}(\hat{\boldsymbol{\beta}}_{\mathbf{X}}) = \mathrm{SSR}(\hat{\boldsymbol{\beta}}_{\mathbf{X}}) + \mathrm{SSE}(\hat{\boldsymbol{\beta}}_{\mathbf{X}}) \tag{2.29}$$

(see Myers (1986), p. 16).

Let $\hat{\mathbf{y}}_{\mathbf{X}}$ be the vector of the predicted responses corresponding to the vector of the observed response variables $\mathbf{y}$ obtained by fitting the model in (2.3). Then the vector of deviations of the observed $\mathbf{y}$ from their corresponding predicted vector is

$$\mathbf{y} - \hat{\mathbf{y}}_{\mathbf{X}} = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}_{\mathbf{X}} = \mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y}, \tag{2.30}$$

and SSE($\hat{\boldsymbol{\beta}}_{\mathbf{X}}$) is given by

$$\mathrm{SSE}(\hat{\boldsymbol{\beta}}_{\mathbf{X}}) = (\mathbf{y} - \hat{\mathbf{y}}_{\mathbf{X}})'(\mathbf{y} - \hat{\mathbf{y}}_{\mathbf{X}}) = \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \tag{2.31}$$

(see Seber (1977), p. 45). In this equation, the first part of the right-hand side in SSE($\hat{\boldsymbol{\beta}}_{\mathbf{X}}$) is SST and the second part is SSR($\hat{\boldsymbol{\beta}}_{\mathbf{X}}$) (see Searle (1971), p. 94). If we rewrite SSR($\hat{\boldsymbol{\beta}}_{\mathbf{X}}$) as the inner product form of two vectors, then we get

$$\mathrm{SSR}(\hat{\boldsymbol{\beta}}_{\mathbf{X}}) = \mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \mathbf{y}'\mathbf{X}\hat{\boldsymbol{\beta}}_{\mathbf{X}} = \mathbf{y}'\hat{\mathbf{y}}_{\mathbf{X}} \tag{2.32}$$

since $\hat{\boldsymbol{\beta}}_{\mathbf{X}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ and $\mathbf{X}\hat{\boldsymbol{\beta}}_{\mathbf{X}} = \hat{\mathbf{y}}_{\mathbf{X}}$. To determine the distributions of SSE($\hat{\boldsymbol{\beta}}_{\mathbf{X}}$) and SSR($\hat{\boldsymbol{\beta}}_{\mathbf{X}}$) which are quadratic functions of the response variables, we note two theorems associated with the distribution of quadratic forms of the response variables:

Theorem 2.1: When $\mathbf{y} \sim N_m(\boldsymbol{\mu}, \sigma_y^2\mathbf{I}_m)$, then $\mathbf{y}'\mathbf{A}\mathbf{y}/\sigma_y^2 \sim \chi^2(r(\mathbf{A}), \lambda)$, where λ (noncentrality parameter) $= \frac{1}{2\sigma_y^2}\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$, if and only if $\mathbf{A}$ is an $(m \times m)$ idempotent matrix of rank $r(\mathbf{A})$ (see Graybill (1976), Theorem 4.4.2).

Theorem 2.2: When $\mathbf{y} \sim N_m(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $\mathbf{y}'\mathbf{A}\mathbf{y}$ and $\mathbf{y}'\mathbf{B}\mathbf{y}$ are distributed independently if $\mathbf{B}\boldsymbol{\Sigma}\mathbf{A} = \mathbf{0}$, where $\mathbf{A}$, $\mathbf{B}$ and $\boldsymbol{\Sigma}$ are $(m \times m)$ matrices respectively (see Graybill (1976), Theorem 4.5.3).

Since $\mathbf{y} \sim N_m(\mathbf{X}\boldsymbol{\beta}, \sigma_y^2\mathbf{I}_m)$ and $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is symmetric and idempotent with rank $(p+1)$, we have, from Theorem 2.1, that

$$\frac{\text{SSR}(\hat{\boldsymbol{\beta}}_{\mathbf{X}})}{\sigma_y^2} \sim \chi^2(p+1, \lambda), \tag{2.33}$$

where the noncentrality parameter, λ, is given by

$$\lambda = \frac{1}{2\sigma_y^2}\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} = \frac{1}{2\sigma_y^2}\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} \tag{2.34}$$

(see Searle (1971), p. 175). Similarly, SSE($\hat{\boldsymbol{\beta}}_{\mathbf{X}}$) $= \mathbf{y}'\mathbf{P}\mathbf{y}$, where $\mathbf{P}$ is defined in (2.21), is distributed as

$$\frac{\text{SSE}(\hat{\boldsymbol{\beta}}_{\mathbf{X}})}{\sigma_y^2} \sim \chi^2(m-p-1) \tag{2.35}$$

since $\mathbf{P}$ is symmetric and idempotent with rank $(m-p-1)$ (see Seber (1977), p. 174). From Theorem 2.2, equation (2.31) and (2.32), $\mathrm{SSR}(\hat{\boldsymbol{\beta}}_{\mathbf{X}})$ and $\mathrm{SSE}(\hat{\boldsymbol{\beta}}_{\mathbf{X}})$ are distributed independently because

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') = \mathbf{0} \tag{2.36}$$

by the definition of $\mathbf{P}$ in (2.21). From (2.35), the unbiased estimator of σ_y^2 is

$$\hat{\sigma}_y^2 = \frac{\mathrm{SSE}(\hat{\boldsymbol{\beta}}_{\mathbf{X}})}{m-p-1}. \tag{2.37}$$

(see Graybill (1976), Theorem 6.2.1).

Next we review the results on the distributions of $\hat{\boldsymbol{\beta}}_G$ and $\hat{\boldsymbol{\alpha}}_G$, respectively, under the assumptions of Section 2.1.1. We also identify $\mathrm{SSR}(\hat{\mathbf{y}}_G)$, $\mathrm{SSE}(\hat{\mathbf{y}}_G)$ and the unbiased estimator of σ_ϵ^2 for the model in (2.5) and review these distributions under the same assumptions.

Since $\hat{\boldsymbol{\alpha}}_G$ and $\hat{\boldsymbol{\beta}}_G$ in (2.14) are represented, respectively, as the product form of $\mathbf{Py}$, $\mathbf{C}$ and $(\mathbf{C}'\mathbf{PC})^{-1}$ which are dependent, their unconditional distributions are not simply identified. Thus, we review the distributions of $\mathbf{Py}$, $\mathbf{C}$ and $(\mathbf{C}'\mathbf{PC})$ respectively instead of identifying the unconditional distribution of $\hat{\mathbf{y}}_G$. It is known that $\mathbf{Py} \sim N_m(\mathbf{0},\ \sigma_y^2\mathbf{P})$ since $\mathbf{PX}\boldsymbol{\beta} = \mathbf{0}$ for the mean and $\mathbf{P}\sigma_y^2\mathbf{I}_m\mathbf{P}' = \sigma_y^2\mathbf{P}$ for the variance, by the definition of $\mathbf{P}$. As assumed before, since $\mathbf{c}'_i$ $(i = 1,\ 2,\ \ldots,\ m)$ are IID $\sim N_s(\mathbf{0},\ \boldsymbol{\Sigma}_c)$, the $(m \times s)$ random matrix $\mathbf{C}$ has the matrix normal distribution:

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}'_1 \\ \mathbf{c}'_2 \\ \vdots \\ \mathbf{c}'_m \end{bmatrix} \sim N_{m,\ s}(\mathbf{O},\ \mathbf{I}_s,\ \Sigma_{\mathbf{c}}) \tag{2.38}$$

where $\mathbf{O}$ is a $(m \times s)$ matrix of zeroes, $\mathbf{I}_s$ represents the independence of the rows of $\mathbf{C}$, and Σ_c denotes the covariance matrix of the columns of $\mathbf{C}$ (see Arnold (1981), pp. 310-311). Also, the $(s \times s)$ random matrix $(\mathbf{C'PC})$ has the Wishart distribution with $\mathbf{C'PC} \sim W_s(m - p - 1,\ \Sigma_c)$ (see Theorem 17.7a in Arnold (1981)).

Given $\mathbf{C}$, it is known that the conditional distribution of $\hat{\gamma}_G$ is

$$\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C} \sim N_{p+s+1}(\gamma_{\mathbf{G}},\ \sigma^2_{\varepsilon}(\mathbf{G'G})^{-1}) \tag{2.39}$$

(see (2.28) and (2.11)). Since $\hat{\gamma}_G = (\hat{\beta}'_G,\ \hat{\alpha}'_G)'$, the conditional distributions of $\hat{\alpha}_G$ and $\hat{\beta}_G$ are s-variate and $(p + 1)$-variate normal and their means and covariances can be obtained by taking the proper components of γ_G and $\sigma^2_{\varepsilon}(\mathbf{G'G})^{-1}$ in (2.38) respectively (see Anderson (1958), Theorem 2.4.3). The mean and covariance of $\hat{\alpha}_G$ correspond to α and the lower-right $(s \times s)$ submatrix of $\sigma^2_{\varepsilon}(\mathbf{G'G})^{-1}$ in (2.12), respectively. Thus, the conditional distribution of $\hat{\alpha}_G$ is

$$\hat{\alpha}_{\mathbf{G}} \mid \mathbf{C} \sim N_s(\alpha,\ \sigma^2_{\varepsilon}(\mathbf{C'PC})^{-1}). \tag{2.40}$$

Similarly, since the mean and covariance of $\hat{\beta}_G$ are given by β and the upper-left $((p + 1) \times (p + 1))$ submatrix of $\sigma^2_{\varepsilon}(\mathbf{G'G})^{-1}$, respectively, the conditional distribution of $\hat{\beta}_G$ has the following $(p + 1)$-variate normal distribution:

$$\hat{\beta}_{\mathbf{G}} \mid \mathbf{C} \sim N_{p+1}(\beta,\ \sigma^2_{\varepsilon}[(\mathbf{X'X})^{-1} + (\mathbf{X'X})^{-1}\mathbf{X'C}(\mathbf{C'PC})^{-1}\mathbf{C'X}(\mathbf{X'X})^{-1}]). \tag{2.41}$$

In order to identify $\text{SSR}(\hat{\gamma}_G \mid \mathbf{C})$ and $\text{SSE}(\hat{\gamma}_G \mid \mathbf{C})$ in (2.5), we define $\hat{\mathbf{y}}_G$ as the vector of the predicted responses corresponding to the vector of the observed responses $\mathbf{y}$ obtained by fitting the model in (2.5). As before, $\text{SSE}(\hat{\gamma}_G \mid \mathbf{C})$ is given by

$$\text{SSE}(\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C}) = (\mathbf{y} - \hat{\mathbf{y}}_{\mathbf{G}})'(\mathbf{y} - \hat{\mathbf{y}}_{\mathbf{G}}) = \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\mathbf{y}, \tag{2.42}$$

in which the first part of the right hand side is $\text{SST}(\hat{\gamma}_G \mid \mathbf{C})$ and the second part is $\text{SSR}(\hat{\gamma}_G \mid \mathbf{C})$ (see Seber (1977), p. 45 and Searle (1971), p. 94). If we rewrite $\text{SSR}(\hat{\gamma}_G \mid \mathbf{C})$ as the inner product form of two vectors, then we get

$$\text{SSR}(\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C}) = \mathbf{y}'\mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\mathbf{y} = \mathbf{y}'\mathbf{G}\hat{\gamma}_{\mathbf{G}} = \mathbf{y}'\hat{\mathbf{y}}_{\mathbf{G}} \tag{2.43}$$

since $\hat{\gamma}_G = (\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\mathbf{y}$ and $\hat{\mathbf{y}}_G = \mathbf{G}\hat{\gamma}_G$. Since $\mathbf{y} \sim N_m(\mathbf{G}\gamma, \sigma_\varepsilon^2\mathbf{I}_m)$ and $\mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'$ is symmetric and idempotent with rank $(p + s + 1)$, we obtain, from Theorem 2.1, that

$$\frac{\text{SSR}(\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C})}{\sigma_\varepsilon^2} \sim \chi^2(p + s + 1,\ \lambda), \tag{2.44}$$

where the noncentrality parameter, λ, is given by

$$\lambda = \frac{1}{2\sigma_\varepsilon^2}\gamma'\mathbf{G}'\mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'\mathbf{G}\gamma = \frac{1}{2\sigma_\varepsilon^2}\gamma'\mathbf{G}'\mathbf{G}\gamma \tag{2.45}$$

(see Searle (1971), p. 175). Also, $\text{SSE}(\hat{\gamma}_G \mid \mathbf{C}) = \mathbf{y}'(\mathbf{I}_m - \mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}')\mathbf{y}$ is distributed as, from Theorem 2.1,

$$\frac{\text{SSE}(\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C})}{\sigma_\varepsilon^2} \sim \chi^2(m - p - s - 1) \tag{2.46}$$

since $(\mathbf{I}_m - \mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}')$ is symmetric and idempotent with rank $(m - p - s - 1)$, and the noncentrality parameter, λ, is

$$\lambda = \frac{1}{2\sigma_{\varepsilon^*}^2} \gamma' \mathbf{G}'(\mathbf{I} - \mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}')\mathbf{G}\gamma = \mathbf{0} \tag{2.47}$$

(see Searle (1971), p. 344). From (2.46), the unbiased estimator of $\sigma_{\varepsilon^*}^2$ is given by

$$\hat{\sigma}_{\varepsilon^*}^2 = \frac{\mathrm{SSE}(\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C})}{m - p - s - 1}. \tag{2.48}$$

To partition $\mathrm{SSR}(\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C})$ into two parts (SSR due to the models in (2.3) and (2.27)), we rewrite equation (2.43) as

$$\mathrm{SSR}(\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C}) = \mathbf{y}'\mathbf{G}\hat{\gamma}_{\mathbf{G}} = \mathbf{y}'\mathbf{X}\hat{\boldsymbol{\beta}}_{\mathbf{G}} + \mathbf{y}'\mathbf{C}\hat{\boldsymbol{\alpha}}_{\mathbf{G}}. \tag{2.49}$$

If we substitute $\hat{\boldsymbol{\beta}}_{\mathbf{G}}$ and $\hat{\boldsymbol{\alpha}}_{\mathbf{G}}$ in (2.49), we get

$$\mathrm{SSR}(\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C}) = \mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} + \mathbf{y}'\mathbf{P}\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{y} \tag{2.50}$$

(see Searle (1971), p. 343). Clearly, the first part of the right hand side in (2.50) is the sum of squares for regression due to fitting the model in (2.3). The second part in (2.50) is the sum of squares for regression due to fitting the linear statistical model $\mathbf{Py} = \mathbf{PC}\boldsymbol{\alpha} + \varepsilon^*$, which is the sum of squares for regression attributable to fitting the concomitant variables, having already fitted the factor variables' part of the model in (2.5) (see Searle (1971), p. 344). We use the notation $\mathrm{SSR}(\hat{\boldsymbol{\alpha}}_{\mathbf{G}} \mid \hat{\boldsymbol{\beta}}_{\mathbf{X}}, \mathbf{C})$ for SSR in this case:

$$\mathrm{SSR}(\hat{\boldsymbol{\alpha}}_{\mathbf{G}} \mid \hat{\boldsymbol{\beta}}_{\mathbf{X}}, \mathbf{C}) = \mathbf{y}'\mathbf{P}\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{y}. \tag{2.51}$$

Then we can write (2.49) as

$$\mathrm{SSR}(\hat{\gamma}_{\mathbf{G}} \mid \mathbf{C}) = \mathrm{SSR}(\hat{\boldsymbol{\beta}}_{\mathbf{X}} \mid \mathbf{C}) + \mathrm{SSR}(\hat{\boldsymbol{\alpha}}_{\mathbf{G}} \mid \hat{\boldsymbol{\beta}}_{\mathbf{X}}, \mathbf{C}). \tag{2.52}$$

Since $\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ is symmetric and idempotent, from Theorem 2.1, we have

$$\frac{SSR(\hat{\boldsymbol{\beta}}_{\mathbf{X}} \mid \mathbf{C})}{\sigma^2_{\varepsilon^*}} \sim \chi^2(p+1,\ \lambda), \tag{2.53}$$

where the noncentrality parameter, λ, is given by

$$\lambda = \frac{1}{2\sigma^2_{\varepsilon^*}} (\mathbf{X}\boldsymbol{\beta} + \mathbf{C}\boldsymbol{\alpha})'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\boldsymbol{\beta} + \mathbf{C}\boldsymbol{\alpha})$$

$$= \frac{1}{2\sigma^2_{\varepsilon^*}} (\boldsymbol{\beta}'\mathbf{X}'\mathbf{X}\boldsymbol{\beta} + 2\boldsymbol{\beta}'\mathbf{X}'\mathbf{C}\boldsymbol{\alpha} + \boldsymbol{\alpha}'\mathbf{C}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{C}\boldsymbol{\alpha}) \tag{2.54}$$

(see Searle (1971), p. 344). Also, the conditional distribution of $SSR(\hat{\boldsymbol{\alpha}}_{\mathbf{G}} \mid \hat{\boldsymbol{\beta}}_{\mathbf{X}})$ is obtained, from Theorem 2.1, as

$$\frac{SSR(\hat{\boldsymbol{\alpha}}_{\mathbf{G}} \mid \hat{\boldsymbol{\beta}}_{\mathbf{X}},\ \mathbf{C})}{\sigma^2_{\varepsilon^*}} \sim \chi^2(s,\ \lambda), \tag{2.55}$$

where the parameter of noncentrality, λ, is given by

$$\lambda = \frac{1}{2\sigma^2_{\varepsilon^*}} (\mathbf{X}\boldsymbol{\beta} + \mathbf{C}\boldsymbol{\alpha})'\mathbf{P}\mathbf{C}'(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}(\mathbf{X}\boldsymbol{\beta} + \mathbf{C}\boldsymbol{\alpha})$$

$$= \frac{1}{2\sigma^2_{\varepsilon^*}} \boldsymbol{\alpha}'\mathbf{C}'\mathbf{P}\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{C}\boldsymbol{\alpha} = \frac{1}{2\sigma^2_{\varepsilon^*}} \boldsymbol{\alpha}'\mathbf{C}'\mathbf{P}\mathbf{C}\boldsymbol{\alpha}. \tag{2.56}$$

From Theorem 2.2, $SSR(\hat{\boldsymbol{\beta}}_{\mathbf{X}} \mid \mathbf{C})$ and $SSR(\hat{\boldsymbol{\alpha}}_{\mathbf{G}} \mid \hat{\boldsymbol{\beta}}_{\mathbf{X}},\ \mathbf{C})$ are distributed independently because

$$\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{P}\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P} = \mathbf{0} \tag{2.57}$$

by the definition of $\mathbf{P}$ in (2.21). Also, from Theorem 2.2, both SSR($\hat{\boldsymbol{\beta}}_{\mathbf{X}} \mid \mathbf{C}$) and the SSR($\hat{\boldsymbol{\alpha}}_{G} \mid \hat{\boldsymbol{\beta}}_{\mathbf{X}}, \mathbf{C}$) are independent of SSE($\hat{\boldsymbol{\gamma}}_{G} \mid \mathbf{C}$) since

$$[\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' - \mathbf{PC}(\mathbf{C}'\mathbf{PC})^{-1}\mathbf{C}'\mathbf{P}][\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'] = \mathbf{0} \tag{2.58}$$

and

$$[\mathbf{I}_m - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' - \mathbf{PC}(\mathbf{C}'\mathbf{PC})^{-1}\mathbf{C}'\mathbf{P}][\mathbf{PC}(\mathbf{C}'\mathbf{PC})^{-1}\mathbf{C}'\mathbf{P}] = \mathbf{0} \tag{2.59}$$

(see Searle (1971), p. 344).

2.2. Method of Control Variates

This section presents a summary of the method of control variates for variance reduction for estimating the model parameters of interest. This summary includes the concept, effects and applications of control variates to the output from a simulation experiment.

Consider a simulation run which yields a single response variable of interest as well as a set of concomitant variables. In simulation, these concomitant variables are referred to as *control variates*. Analysis of covariance in Section 2.1 tries to improve the estimation on the response(s). As a special case of analysis of covariance, in a simulation experiment, the method of control variates is applied to reduce the variability of the estimator of the mean system response of interest by taking advantage of the correlation between the simulation response and the control variates. In this section, as in Section 2.1, it is assumed that the response and control variates have a multivariate normal distribution, and that the control variates are observed independently on each setting of factors and that they are highly correlated with the response variable of interest.

Throughout this section, we summarize the major findings of the method of control variates applied to the following models: (a) single population with single response and single control variate (Section 2.2.1), (b) single population with single response and multiple control variates (Section 2.2.2), (c) multipopulation with single response and multiple control variates (Section 2.2.3), (d) single population with multiple responses and multiple control variates (Section 2.2.4), and (e) multipopulation with multiple responses and multiple control variates (Section 2.2.5).

2.2.1. Single population with single response and single control variate

This section summarizes the method of control variates for estimating the mean response of interest and the statistical problems related to this method.

Suppose that an experimenter is concerned with estimating the mean value of the response of interest through simulation experimentation. Let y be a response and c be a control variate corresponding to y. It is assumed that an appropriate function f of the control variate c has a significant relationship with response y and that the expected value of f is known.

The method of control variates tries to counteract an unknown deviation of $(y - \mu_y)$ by subtracting a known deviation $[f - E(f)]$ from y. Lewis, Ressler and Wood (1987) set the generalized form of an controlled estimator for μ_y as

$$\hat{\mu}_y = y - [f(c; \alpha) - E(f(c; \alpha))], \tag{2.60}$$

where α is a constant parameter of the function f. For any fixed constant α, $\hat{\mu}_y$ is an unbiased estimator of μ_y, and

$$Var(\hat{\mu}_y) = Var(y) - 2Cov(y, f(c; \alpha)) + Var(f(c; \alpha)). \tag{2.61}$$

Hence, if $2Cov(y, f(c; \alpha)) > Var(f(c; \alpha))$, $\hat{\mu}_y$ has a smaller variance than y. It is desirable that the function f should be chosen so that the variance of $\hat{\mu}_y$ is as small as possible.

Moy (1965) and Radema (1969) used a quadratic representation of the function f which was linear in the unknown parameters. Their experimental results indicated that the correlation between the response and the control variates is not strengthened by including higher degree terms of the control variate into the function f (see Section III 4.2 in Kleijnen (1974)). Lewis, Ressler and Wood (1987) investigated the potential effectiveness of nonlinear (piecewise linear or power transformation) adjusted controls in estimating the mean of the Anderson-Darling goodness-of-fit statistic (Anderson and Darling (1952)) and suggested the results that nonlinear types of control variates may be effective in reducing the variance of the estimator when the statistic of the estimator is a nonlinear function of the random variables.

However, in most applications of control variates, the linear parametric function, $f(c; \alpha) = \alpha c$, is assumed when considering one control variate. Thus, the controlled response is

$$\hat{\mu}_y = y - (c - \mu_c)\alpha. \tag{2.62}$$

This parameterization of f is considered in this section. From (2.62), we see that the variance of $\hat{\mu}_y$ is

$$\mathrm{Var}(\hat{\mu}_y) = \mathrm{Var}(y) - 2\alpha\mathrm{Cov}(y, c) + \alpha^2\mathrm{Var}(c), \tag{2.63}$$

and the value of α which minimizes $\mathrm{Var}(\hat{\mu}_y)$ is

$$\alpha = \sigma_{yc}\sigma_c^{-2}, \tag{2.64}$$

where σ_{yc} is the covariance between y and c and σ_c^2 is the variance of the control variate c. The resulting minimum variance of $\hat{\mu}_y$ is

$$Var(\hat{\mu}_y) = \sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2} = (1 - \rho_{yc}^2)\sigma_y^2, \tag{2.65}$$

where σ_y^2 is the variance of y and ρ_{yc} is the correlation coefficient between the random variables y and c. As shown in the above equation, the variance of $\hat{\mu}_y$ decreases as the correlation between y and c increases. Therefore, the selection of a control variate that is highly correlated with the response is an important factor to the effectiveness of this methodology (see Lavenberg, Moeller and Welch (1982), Wilson and Pritsker (1984a, b), Bauer (1987), and Lewis, Ressler and Wood (1987)).

When the length of simulation run l is sufficiently large, we consider that y and c are samples from the $N(\mu_y, \sigma_y^2)$ and $N(\mu_c, \sigma_c^2)$ distributions, respectively by the central limit theorem. Hence, it seems reasonable to assume that $(y, c) \sim N_2((\mu_y, \mu_c), \boldsymbol{\Sigma})$, where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_y^2 & \sigma_{yc} \\ \sigma_{yc} & \sigma_c^2 \end{bmatrix}. \tag{2.66}$$

(see Lavenberg, Moeller and Welch (1982), Cheng and Feast (1980), and Cheng (1978)). Under this assumption, the conditional distribution of y, given c, is the univariate normal distribution with expectation

$$E[y \mid c] = \mu_y + (c - \mu_c)\alpha, \tag{2.67}$$

where $\alpha = \sigma_{yc}\sigma_c^{-2}$ and variance

$$Var(y \mid c) = \sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2} = \sigma_{\varepsilon^*}^2, \tag{2.68}$$

where $\sigma_{\varepsilon^*}^2$ is the same as in (2.9) when $s = 1$. Therefore, conditional on c, if we perform m independent replications of the simulation run, we can represent the simulation model as a classical regression model

$$y_i = \mu_y + (c_i - \mu_c)\alpha + \varepsilon_i^*, \qquad i = 1, 2, \ldots, r, \tag{2.69}$$

where y_i and c_i are the ith observations of the response and the control variate respectively, $\mu_c = E(c_i)$, and $\varepsilon_i \sim$ IID $N(0, \sigma_{\varepsilon^*}^2)$ $(i = 1, 2, \ldots, r)$. In matrix form, equation (2.69) is written as

$$\mathbf{y} = \mu_y \mathbf{1} + \mathbf{c}\alpha + \boldsymbol{\varepsilon}^*; \tag{2.70}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_r)'$; $\mathbf{c} = (c_1 - \mu_c, c_2 - \mu_c, \ldots, c_r - \mu_c)'$; and $\boldsymbol{\varepsilon}^{*} = (\varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_r^*)'$.

First, we consider the case where $\boldsymbol{\Sigma}$ is known. When $\boldsymbol{\Sigma}$ in (2.66) is known, the coefficient of the control variate is $\alpha = \sigma_{yc}\sigma_c^{2}$. Thus, the controlled estimator in (2.69) is given by

$$\hat{\mu}_y = \bar{y} - \alpha(\bar{c} - \mu_c), \tag{2.71}$$

where $\bar{y}$ and $\bar{c}$ are the sample means of the response and the control variate, respectively. From (2.65), the variance of $\hat{\mu}_y$ is

$$Var(\hat{\mu}_y) = \frac{1}{r}(\sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2}) = \frac{1}{r}(1 - \rho_{yc}^2)\sigma_y^2. \tag{2.72}$$

Lavenberg, Moeller and Welch (1982) defined the minimum variance ratio, which is the ratio of the variance of the adjusted estimator to the variance of the unadjusted estimator when the optimal value of α is known:

$$\frac{Var(\hat{\mu}_y)}{Var(\bar{y})} = \frac{(\sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2})}{\sigma_y^2} = 1 - \rho_{yc}^2, \tag{2.73}$$

which represents the theoretical potential for variance reduction by adopting a single control variate c. From regression theory, $\hat{\mu}_y \sim N(\mu_y, \sigma_{\varepsilon^*}^2/r)$. Thus, the $(1-a)$ confidence interval of μ_y is given by

$$\hat{\mu}_y \pm Z_{a/2}\frac{\sigma_{\varepsilon^*}}{\sqrt{r}}, \tag{2.74}$$

where $Z_{a/2}$ is the upper $a/2$ percentile of the standard normal distribution.

Next we consider the case where $\mathbf{\Sigma}$ given in (2.66) is unknown. Since α is unknown in this case, we must develop an estimator. From (2.22) and (2.23), the least squares estimators of $\hat{\alpha}$ and $\hat{\mu}_y$ are, respectively,

$$\hat{\alpha} = (\mathbf{c}'\mathbf{P}\mathbf{c})^{-1}\mathbf{c}'\mathbf{P}\mathbf{y} = [\mathbf{c}'\mathbf{c} - \mathbf{c}'\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{c}]^{-1}[\mathbf{c}'\mathbf{y} - \mathbf{c}'\mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{y}]$$

$$= \left[\sum_{i=1}^{r} c_i^2 - r\bar{c}^2\right]^{-1}\left[\sum_{i=1}^{r} c_i y_i - r\bar{c}\bar{y}\right] = S_c^{-1}S_{yc}, \tag{2.75}$$

where $\mathbf{P} = (\mathbf{I}_r - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}')$ and

$$\hat{\mu}_y = (\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'(\mathbf{y} - \mathbf{c}\hat{\alpha}) = \bar{y} - \hat{\alpha}(\bar{c} - \mu_c). \tag{2.76}$$

From (2.40) and (2.41), $\hat{\alpha}$ and $\hat{\mu}_y$ are conditionally unbiased estimators for α and μ_y. The variance of $\hat{\mu}_y$, if it is finite, is given by

$$Var(\hat{\mu}_y) = E[Var(\hat{\mu}_y|c)] + Var(E[\hat{\mu}_y|c]) \tag{2.77}$$

(see Bickel and Doksum (1977), p. 76). Since the conditional expectation of $\hat{\mu}_y$ is μ_y, the second term in (2.77) reduces to 0. Hence,

$$Var(\hat{\mu}_y) = E[Var(\hat{\mu}_y \mid \mathbf{c})] = (\frac{r-2}{r-3})(1 - \rho_{yc}^2)\frac{\sigma_y^2}{r} \tag{2.78}$$

(see Lavenberg and Welch (1981)).

Lavenberg, Moeller and Welch (1982) defined the *loss factor* as the amount by which the variance is increased due to estimating α, that is:

$$loss\ factor = \frac{r-2}{r-3}, \tag{2.79}$$

which is the ratio of the variance of the estimator $\hat{\mu}_y$ when the α is unknown to that when the α is known. The *efficiency* of the control variate is measured by the product of the minimum variance ratio and loss factor, that is:

$$efficiency = \frac{r-2}{r-3}(1 - \rho_{yc}^2). \tag{2.80}$$

Therefore, the control variate is effective if $1/(r-2) < \rho_{yc}^2$ (note that the efficiency should be less than 1). From (2.39), $\hat{\mu}_y \sim N(\mu_y, \sigma_\varepsilon^2 s_{11})$, where s_{11} is the first-row first-column element of $(\mathbf{G}'\mathbf{G})^{-1}$ with $\mathbf{G} = (\mathbf{1}_r,\ \mathbf{c})$. Then,

$$t = (\hat{\mu}_y - \mu_y)/\sqrt{\hat{\sigma}_\varepsilon^2 s_{11}} \sim t(r-2), \tag{2.81}$$

where $\hat{\sigma}_\varepsilon^2$ is the sample estimator of σ_ε^2: $\hat{\sigma}_\varepsilon^2 = \mathbf{y}'[\mathbf{I}_r - \mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}']\mathbf{y}/(r-2)$ (see (2.42) and (2.48)). Thus, the $(1-a)$ confidence interval of μ_y is given by

$$\hat{\mu}_y \pm t_{r-2}^{a/2}\hat{\sigma}_\varepsilon\sqrt{s_{11}}, \tag{2.82}$$

where $t_{r-2}^{a/2}$ is the upper $a/2$ percentile of the student-t distribution with $(r-2)$ degrees of freedom.

Finally, we consider the case where the normality assumption placed on the response and control variate is untenable. From (2.76), the expectation of $\hat{\mu}_y$ is represented in general as

$$\mathrm{E}[\bar{y} - \hat{\alpha}(\bar{c} - \mu_c)] = \mu_y - \mathrm{E}[\hat{\alpha}(\bar{c} - \mu_c)] \neq \mu_y - \mathrm{E}[\hat{\alpha}]\mathrm{E}[\bar{c} - \mu_c] = \mu_y \tag{2.83}$$

since $\hat{\alpha}$ and $\bar{c}$ are not independent. Thus, the point estimator $\hat{\mu}_y$ is biased. In this situation, the Jackknife Method is applied to reduce the bias of estimator $\hat{\mu}_y$ in (2.76) (see Kleijnen (1975), pp. 158-159, and Lavenberg, Moeller and Welch (1982)).

2.2.2. Single population with single response and multiple control variates

The discussion of Section 2.2.1 can be extended to the case of more than one control variate. Lavenberg and Welch (1981) studied the use of multiple control variates for a single population with single response. Lavenberg, Moeller and Welch (1982) developed types of some control variates and applied them to a closed queuing network problem and Wilson and Pritsker (1984a, 1984b) developed procedures for using standardized control variates in conjunction with replication analysis as well as regenerative analysis. We summarize the development presented by these authors.

Suppose we have s control variates associated with the response variable of interest from a single simulation run. By extending a simple linear function of a single control variate in (2.62) to a linear combination of s control variates, the adjusted response $\hat{\mu}_y$, which is an estimator of μ_y, is given as follows:

$$\hat{\mu}_y = y - (\mathbf{c} - \boldsymbol{\mu}_c)'\boldsymbol{\alpha}, \tag{2.84}$$

where $\mathbf{c}$ is a $(s \times 1)$ vector of control variates, $\boldsymbol{\mu}_c = \mathrm{E}(\mathbf{c})$ and $\boldsymbol{\alpha}$ is a $(s \times 1)$ coefficient vector. For a fixed $\boldsymbol{\alpha}$, $\hat{\mu}_y$ is an unbiased estimator of μ_y, and

$$Var(\hat{\mu}_y) = \sigma_y^2 - 2\boldsymbol{\sigma}'_{yc}\boldsymbol{\alpha} + \boldsymbol{\alpha}'\boldsymbol{\Sigma}_c\boldsymbol{\alpha}, \tag{2.85}$$

where $\boldsymbol{\Sigma}_c$ is the $(s \times s)$ covariance matrix of the random vector $\mathbf{c}$ and $\boldsymbol{\sigma}_{yc}$ is a $(s \times 1)$ covariance vector between the random response, y, and $\mathbf{c}$. The value of $\boldsymbol{\alpha}$ which minimizes (2.85) is given by

$$\boldsymbol{\alpha}' = \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_{\mathbf{c}}^{-1} \tag{2.86}$$

and the resulting minimum variance of $\hat{\mu}_y$ is

$$Var(\hat{\mu}_y) = \sigma_y{}^2 - \boldsymbol{\sigma}_{yc}'\boldsymbol{\Sigma}_{\mathbf{c}}{}^{-1}\boldsymbol{\sigma}_{yc} = (1 - R_{yc}{}^2)\sigma_y{}^2 = \sigma_{\varepsilon}^2, \tag{2.87}$$

where $R_{yc}^2 = \sigma_y^{-2}\sigma_{yc}'\Sigma_c^{-1}\sigma_{yc}$ is the square of the multiple correlation coefficient between y and **c** (see Lavenberg, Moeller and Welch (1982)). When $s = 1$, the multiple correlation coefficient in (2.87) is equal to the correlation coefficient between y and c in (2.65). Thus, the minimum variance ration in (2.87) is the generalized form of that in (2.65). As in the case of a single control variate, the stronger the correlation between a set of control variates **c** and y, the greater the variance reduction of the estimator. Thus, choosing a set of control variates which is highly correlated with the response is an important factor in determining the efficiency of a simulation study (see Lavenberg and Welch (1981)).

By the extending the assumption of a single control variate in the previous section to the multiple control variates case, we have that

$$(y, \mathbf{c}')' \sim N_{s+1}((\mu_y, \boldsymbol{\mu}'_c)', \boldsymbol{\Sigma}), \tag{2.88}$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_y^2 \, \boldsymbol{\sigma}'_{yc} \\ \boldsymbol{\sigma}_{yc} \, \boldsymbol{\Sigma}_c \end{bmatrix}. \tag{2.89}$$

Under this assumption, the response is represented as

$$y_i = \mu_y + (\mathbf{c}_i - \boldsymbol{\mu}_c)'\boldsymbol{\alpha} + \varepsilon_i^*, \qquad i = 1, 2, \ldots, r, \tag{2.90}$$

where y_i and $\mathbf{c}_i$ are the ith observations of the response and the control variates respectively, and $\varepsilon_i^* \sim$ IID $N(0, \sigma_{\varepsilon^*}^2)$. In matrix form, equation (2.90) can be written as

$$\mathbf{y} = \mu_y \mathbf{1} + \mathbf{C}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}^*, \tag{2.91}$$

where **y** and $\boldsymbol{\varepsilon}^*$ are defined in (2.70), **C** is a $(r \times s)$ matrix of control variates.

First, we consider the case where Σ is known. In this case, we have an unbiased minimum variance estimator $\hat{\mu}_y$ in (2.84). Lavenberg, Moeller and Welch (1982) defined the quantity $(1 - R^2_{yc})$ in (2.87) as the minimum variance ratio. The adjusted response $\hat{\mu}_y$ in (2.84) is given by

$$\hat{\mu}_y = \bar{y} - (\bar{\mathbf{c}} - \mu_{\mathbf{c}})'\boldsymbol{\alpha}, \tag{2.92}$$

where $\bar{y}$ is a sample mean of the response and $\bar{\mathbf{c}}$ is a mean vector of the control variates. The $(1 - a)$ level confidence interval of μ_y is the same as in (2.74) if σ_{ε^*} in (2.74) is replaced by σ_{ε^*} in (2.87).

Next, we consider the case where Σ is unknown. The least squares estimators of Σ and μ_y in (2.91) are, respectively,

$$\hat{\boldsymbol{\alpha}}' = (\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{y} = S'_{yc}S_{\mathbf{c}}^{-1}, \tag{2.93}$$

where S_{yc} and S_c are the sample estimators of σ_{yc} and Σ_c, respectively, and

$$\hat{\mu}_y = (\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{y} - (\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{y}$$

$$= \bar{y} - (\bar{\mathbf{c}} - \mu_{\mathbf{c}})'\hat{\boldsymbol{\alpha}} = \bar{y} - (\bar{\mathbf{c}} - \mu_{\mathbf{c}})'S_{\mathbf{c}}^{-1}S'_{yc}. \tag{2.94}$$

From (2.39), $\hat{\mu}_y \sim N(\mu_y, \sigma^2_{\varepsilon^*}s_{11})$, where s_{11} is the upper left-hand corner element of $(\mathbf{G}'\mathbf{G})^{-1}$ with $\mathbf{G} = (\mathbf{1}_r, \mathbf{C})$. Let $\hat{\sigma}^2_{\varepsilon^*}$ be the sample estimator of $\sigma^2_{\varepsilon^*}$. Then

$$t = (\hat{\mu}_y - \mu_y)/\sqrt{\hat{\sigma}_{\varepsilon^*}^2 s_{11}} \sim t(r - s - 1). \tag{2.95}$$

Therefore, the $(1 - a)$ level confidence interval of μ_y is given by

$$\hat{\mu}_y \pm t^{a/2}_{r-s-1}\hat{\sigma}_{\varepsilon^*}\sqrt{s_{11}} \tag{2.96}$$

(see Lavenberg and Welch (1981) and Lavenberg, Moeller and Welch (1982)). Lavenberg, Moeller and Welch (1982) showed that the variance of $\hat{\mu}_y$ is

$$Var(\hat{\mu}_y) = (\frac{r-2}{r-s-2})(1 - R_{yc}^2)\frac{\sigma_y^2}{r}. \tag{2.97}$$

and defined the loss factor as the amount by which the variance is increased due to the use of $\hat{\alpha}$ instead of α, that is:

$$loss\ factor = \frac{r-2}{r-s-2}. \tag{2.98}$$

The effect of control variates is measured by the product form of the loss factor and the minimum variance ratio which yields:

$$(\frac{r-2}{r-s-2})(1 - R_{yc}^2). \tag{2.99}$$

Consequently, the use of control variates is effective if $s/(r-2) < R_{yc}^2$. By the trade-off relationship between the loss factor and the multiple correlation coefficient, it is important to keep the number of control variates not too large (see Lavenberg and Welch (1981), Lavenberg, Moeller and Welch (1982), Wilson and Pritsker (1984a, 1984b), and Venkatraman and Wilson (1986)).

For a general class of closed queuing networks (s service stations, d different types of customers and N customers of all types) which allow priorities and blocking, Lavenberg, Moeller and Welch (1982) developed three types of control variates (service time variable, flow variable and work variable) to estimate the response of interest. For each type of customer at station k, (a) the service time variable is defined as

$$\frac{1}{a(k,t)} \sum_{j=1}^{a(k,t)} [s_j(k) - \mu_{c_k}], \tag{2.100}$$

where $a(k, t)$ is the number of service times at station k during the simulation time period $[0, t]$, $s_j(k)$ is the random observation of the kth service station $(j = 1, 2, ..., a(k, t))$, and $\mu_{c_k} = \mathrm{E}[s_j(k)]$, (b) the flow variable (the relative frequency of each type of customer visiting station k) is defined as $a(k,t)/a(t)$ with $a(t) = \sum_{k=1}^{s} a(k, t)$, and (c) the work variable is defined as

$$\frac{1}{a(k,t)} \sum_{j=1}^{a(k,t)} [s_j(k) - \mu_{c_k}] \frac{a(k,t)}{a(t)} = \frac{1}{a(t)} \sum_{j=1}^{a(k,t)} [s_j(k) - \mu_{c_k}], \tag{2.101}$$

which incorporates information about the flow effect and service time effect. Important results obtained from an extensive experiment across many different networks by the authors included: (a) Confidence intervals of the responses of interest using the control variates method were substantially reduced in length compared to those obtained without the control variates method; (b) A loss factor appeared to inflate the minimum variance of the estimator correctly; (c) Work control variates yielded the smallest variance of the estimator provided the loss factor is not too large; and (d) A regression based method of applying the control variates method produced confidence intervals having proper coverage.

Beja (1969), Kleijnen (1974), Lavenberg and Welch (1981) and Lavenberg, Moeller and Welch (1982) used the type of control variates in (2.99). Wilson (1982) showed that the control variates in (2.99) have asymptotic mean and variance zero as the length of the statistics accumulation period increases and the covariance matrix of control variates is asymtotically singular. To improve this problem, Wilson and Pritsker (1984a) developed the standardized control variates defined for the k stations of a queuing system over the simulation time interval $[0, t]$ as

$$c_k(t) = \frac{1}{\sqrt{a(k,t)}} \sum_{j=1}^{a(k,t)} [s_j(k) - \mu_{c_k}]/\sigma_{c_k}, \tag{2.102}$$

where $a(k, t)$, $s_j(k)$, and μ_{c_k} are in (2.100) and $\sigma^2_{c_k}$ is the variance of $s_j(k)$, and showed that

$$\mathbf{c}(t) \xrightarrow{L} N_s(\mathbf{0}_s, \mathbf{I}_s) \quad \text{as } t \to \infty, \tag{2.103}$$

where $\xrightarrow{L}$ denotes converge in distribution, and $\mathbf{c}(t) = [c_1(t), c_2, \ldots, c_s(t)]'$ is a $(s \times 1)$ vector of standardized control variables.

Using these standardized control variates, Wilson and Pritsker (1984a, 1984b) conducted a set of simulation experiments on a variety of closed and mixed queuing networks (machine repair systems). With the replication estimation scheme, the standard control variates yielded variance reductions ranging from 20% to 90% and confidence interval reductions between 10% and 70%.

2.2.3. Multipopulation with single response and multiple control variates

Nozari, Arnold and Pegden (1984) extended the results of a single population case considered by Lavenberg, Moeller and Welch (1982) to a multipopulation case in a direction different from the work by Rubinstein and Marcus (1986). In a multipopulation model, we allow the population of the response variable to vary over m design points. We summarize the results derived by Nozari, Arnold and Pegden (1984): the statistical procedures of the simultaneous inference for the linear combinations of parameters of $\boldsymbol{\beta}$ in (2.5), and the efficiency of control variates for the cases of known and unknown covariance matrix $\boldsymbol{\Sigma}$.

Nozari, Arnold and Pegden (1984) considered the effect of the control variates method by comparing the variance of the least squares estimator $\boldsymbol{\beta}_{\mathbf{X}}$ in (2.3) with that of $\boldsymbol{\beta}_{G}$ in (2.5) under the assumption that

$$(y_i, \mathbf{c}'_i)' \sim \text{IID } N_{s+1}((\mu_{y_i}, \mathbf{0}')', \Sigma), \qquad i = 1, 2, \ldots, m, \tag{2.104}$$

where y_i and $\mathbf{c}_i$ are the response and the $(s \times 1)$ vector of control variates, respectively, at the ith design point, μ_{y_i} is the mean response of the ith design point, and $\boldsymbol{\Sigma}$ is the covariance matrix defined in (2.89). Under this assumption, the responses of interest at the m design points can be represented as the linear statistical model in (2.5).

First, we consider the case where $\boldsymbol{\Sigma}$ is known. Nozari, Arnold and Pegden (1984) showed that the effect of control variates is

$$Var(\hat{\boldsymbol{\beta}}_{\mathbf{X}}) - Var(\hat{\boldsymbol{\beta}}_{\mathbf{G}}) = (\sigma_y^2 - \sigma_{\varepsilon^*}^2)(\mathbf{X}'\mathbf{X})^{-1}, \tag{2.105}$$

where $\sigma_{\varepsilon^*}^2$ is defined in (2.9). (see Nozari, Arnold and Pegden (1984), p.162). Since the amount of difference in (2.105) is always positive, it is better to use the control variates than not to use them.

For some known $(q \times (p+1))$ constant matrix $\mathbf{H}$, $\mathbf{H}\hat{\boldsymbol{\beta}}_G \sim N_q(\mathbf{H}\boldsymbol{\beta}, \sigma_{\varepsilon^*}^2\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}')$ and under H_0: $\mathbf{H}\boldsymbol{\beta} = 0$,

$$\chi^2 = \frac{\hat{\boldsymbol{\beta}}'_{\mathbf{G}}\mathbf{H}'[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}']^{-1}\mathbf{H}\hat{\boldsymbol{\beta}}_{\mathbf{G}}}{\sigma^2_{\varepsilon^*}} \sim \chi^2(q). \tag{2.106}$$

(see Anderson (1958), p. 54). Hence, the statistical procedure for testing the general linear hypothesis H_0: $\mathbf{H}\boldsymbol{\beta} = 0$ vs H_1: $\mathbf{H}\boldsymbol{\beta} \neq 0$ is based on the χ^2 test: Reject H_0 if $\chi^2 > \chi^2_{q,\,a}$, where $\chi^2_{q,\,a}$ is the upper $(1 - a)$ percentile of the χ^2 distribution with q degrees of freedom. The ratio of the expected value of the square of the half-length of Scheffe's simultaneous confidence interval with control variates to that without control variates is given by

$$\frac{\sigma^2_{\varepsilon^*}\chi^2_{r,\,a}\mathbf{h}'\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\mathbf{h}}{\sigma^2_{\mathrm{y}}\chi^2_{r,\,a}\mathbf{h}'\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\mathbf{h}} = \frac{\sigma^2_{\varepsilon^*}}{\sigma^2_{\mathrm{y}}} \quad \text{for } \mathbf{h} \in R^q, \tag{2.107}$$

which is equivalent to the minimum variance ratio originally defined by Lavenberg, Moeller and Welch (1982). (see Nozari, Arnold and Pegden (1984), p. 162).

Next we consider the case where $\boldsymbol{\Sigma}$ is unknown. When $\boldsymbol{\Sigma}$ is unknown, the least squares estimators $\hat{\boldsymbol{\beta}}_{G}$ and $\hat{\boldsymbol{\alpha}}_{G}$ for parameters in (2.5) are given in (2.23) and (2.22) respectively. Their corresponding conditional distributions are given in (2.41) and (2.40) respectively. Nozari, Arnold and Pegden (1984) showed that the unconditional variance of $\hat{\boldsymbol{\beta}}_{G}$ is

$$Var(\hat{\boldsymbol{\beta}}_{\mathbf{G}}) = \left(\frac{m-p-1}{m-p-s-1}\right)\sigma^2_{\varepsilon^*}(\mathbf{X}'\mathbf{X})^{-1} \qquad \text{if} \quad m > p+s+1 \tag{2.108}$$

and measured the effect of the control variates as

$$Var(\hat{\boldsymbol{\beta}}_{\mathbf{X}}) - Var(\hat{\boldsymbol{\beta}}_{\mathbf{G}}) = \left(\sigma^2_y - \frac{m-p-1}{m-p-s-1}\sigma^2_{\varepsilon^*}\right)(\mathbf{X}'\mathbf{X})^{-1}. \tag{2.109}$$

When $p = 1$, this efficiency measure is equivalent to that obtained by Lavenberg, Moeller and Welch (1982). Inspection of (2.109) suggests use of control variates is effective if $\sigma^2_{\varepsilon}/\sigma^2_y > (m-p-s-1)/(m-p-1)$.

Similarly, as with the case of $\boldsymbol{\Sigma}$ known, for some known $(q \times p)$ constant matrix $\mathbf{H}$, $\mathbf{H}\hat{\boldsymbol{\beta}}_G \sim N_q(\mathbf{H}\boldsymbol{\beta}, \ \sigma^2_{\varepsilon}\mathbf{H}(\mathbf{G}'\mathbf{G})^{-1}_{(p+1)(p+1)}\mathbf{H}')$, where $(\mathbf{G}'\mathbf{G})^{-1}_{(p+1)(p+1)}$ is a $(p+1) \times (p+1)$ submatrix associated with $\hat{\boldsymbol{\beta}}_G$ in (2.12). Under H_0: $H\boldsymbol{\beta} = \mathbf{0}$,

$$f = \frac{\hat{\boldsymbol{\beta}}'_G\mathbf{H}'[\mathbf{H}(\mathbf{G}'\mathbf{G})^{-1}_{(p+1)(p+1)}\mathbf{H}']^{-1}\mathbf{H}\hat{\boldsymbol{\beta}}_G}{q\hat{\sigma}^2_{\varepsilon}} \sim F_{q,\ m-p-s-1}, \tag{2.110}$$

where $F_{q,m-p-s-1}$ is the F distribution with q and $(m-p-s-1)$ degrees of freedom (see Nozari, Arnold and Pegden (1984), p. 164). Thus, the statistical procedure for testing the general linear hypothesis H_0: $\mathbf{H}\boldsymbol{\beta} = \mathbf{0}$ vs H_1: $\mathbf{H}\boldsymbol{\beta} \neq \mathbf{0}$ is based on the F test: Reject H_0 if $f > F^a_{q,m-p-s-1}$, where $F^a_{q,m-p-s-1}$ is the upper $(1-a)$ percentile of the F distribution with q and $(m-p-s-1)$ degrees of freedom. Nozari, Arnold and Pegden (1984) showed that the ratio of the expected value of the square of the half-length of the simultaneous confidence interval when the control variates are neglected to that when the control variates are used, is

$$\frac{\sigma^2_{\varepsilon}F^a_{q,m-p-s-1}}{\sigma^2_y F^a_{q,m-p-1}}\left(\frac{m-p-1}{m-p-s-1}\right). \tag{2.111}$$

Note that, by the definition of σ^2_{ε}, we have $\sigma^2_{\varepsilon} < \sigma^2_y$. However, this condition does not guarantee that using the control variates will always improve the results since $F^a_{r,m-p-s-1} > F^a_{r,m-p-1}$ and $(m-p-1) > (m-p-s-1)$.

Nozari, Arnold and Pegden (1984) suggested two methods of selection of the set of control variates, which are variations of the All Regression Procedure and the Forward Selection Method, respectively, in the classical regression theory. The former method searches all set of control variates which yields the best results. The latter method requires less computational effort, however, it may not always produce the best results.

2.2.4. Single population with multiple responses and multiple control variates

The control variates method in Section 2.2.2 can be extended to the case of more than a single response. Rubinstein and Marcus (1985) developed a procedure for applying control variates to the situation of multiple response variables of interest. They defined the minimum generalized variance ratio and derived the loss of variance caused by the estimation of the coefficients of control variates to measure the efficiency of the control variates method. Venkatraman and Wilson (1986) developed an alternative procedure to quantify the loss of variance, which was substantially simpler. These two studies are extensions of the results obtained by Lavenberg, Moeller and Welch (1982).

Consider a multipopulation case in which we seek to estimate a q-dimensional response vector of interest. As before, we observe the s control variates associated with the response vector of interest from a simulation run. Then, for a $(q \times s)$ coefficient matrix $\mathbf{A}$ of control variates, the adjusted estimator can be defined as

$$\hat{\mu}_{\mathbf{y}} = \mathbf{y} - \mathbf{A}(\mathbf{c} - \mu_{\mathbf{c}}), \tag{2.112}$$

where $\mathbf{y} = (y_1, y_2, ..., y_q)'$, $\mathbf{c} = (c_1, c_2, ..., c_s)'$ and $\mu_c = E(\mathbf{c})$. For a fixed matrix $\mathbf{A}$, the covariance matrix of $\hat{\mu}_y$ is given by

$$Cov(\hat{\mu}_y) = \mathbf{\Sigma_y} + \mathbf{A\Sigma_c A'} - \mathbf{\Sigma_{yc} A'} - \mathbf{A\Sigma'_{yc}}, \tag{2.113}$$

where $\mathbf{\Sigma_y}$ and $\mathbf{\Sigma_c}$ are the covariance matrices of the random vectors, $\mathbf{y}$ and $\mathbf{c}$, respectively, and $\mathbf{\Sigma_{yc}}$ is the covariance matrix between $\mathbf{y}$ and $\mathbf{c}$. Rubinstein and Marcus (1985) showed that the coefficient matrix $\mathbf{A}$ which minimizes the generalized variance of $\hat{\mu}_y$ is equal to

$$\mathbf{A} = \mathbf{\Sigma_{yc}\Sigma_c^{-1}} \tag{2.114}$$

and the resulting minimum generalized variance is

$$|Cov(\hat{\mu}_y)| = |\mathbf{\Sigma_y} - \mathbf{\Sigma_{yc}\Sigma^{-1}\Sigma'_{yc}}| = |\mathbf{I} - \mathbf{\Sigma_{yc}\Sigma_c^{-1}\Sigma'_{yc}}|\,|\mathbf{\Sigma_y}| = \prod_{i=1}^{k}(1 - \lambda_i^2)\,|\mathbf{\Sigma_y}|, \tag{2.115}$$

where $k = \min(q, s)$ and $\lambda_i\,(i = 1, 2, ..., k)$ are the canonical correlation coefficients between $\mathbf{y}$ and $\mathbf{c}$ that satisfy $\lambda_1^2 \geq \lambda_2^2 \geq ... \geq \lambda_k^2$ (see Anderson (1958), p. 293).

Rubinstein and Marcus (1985) defined the efficiency of control variates by the ratio of the generalized variance (volume of confidence ellipsoid) of the controlled estimator to that of the uncontrolled estimator under the assumption that

$$(\mathbf{y}', \mathbf{c}')' \sim N_{q+s}((\boldsymbol{\mu}'_y, \boldsymbol{\mu}'_c)', \mathbf{\Sigma}), \tag{2.116}$$

where

$$\Sigma = \begin{bmatrix} \Sigma_y & \Sigma_{yc} \\ \Sigma'_{yc} & \Sigma_c \end{bmatrix}. \quad (2.117)$$

When the covariance matrix Σ is known, from (2.115), the ratio of the generalized variance of $\hat{\mu}_y$ to that of $\mathbf{y}$ is given by

$$\delta_1^2 = \frac{|Cov(\hat{\mu}_y)|}{|\Sigma_y|} = \prod_{i=1}^{k}(1 - \lambda_i^2). \quad (2.118)$$

When Σ is unknown, Rubinstein and Marcus (1985) estimated μ_y by the controlled vector estimator given as

$$\hat{\mu}_y = \bar{\mathbf{y}} - \hat{\mathbf{A}}(\bar{\mathbf{c}} - \mu_c) = \bar{\mathbf{y}} - \mathbf{S}_{yc}\mathbf{S}_c^{-1}(\bar{\mathbf{c}} - \mu_c), \quad (2.119)$$

where $\bar{\mathbf{y}}$ and $\bar{\mathbf{c}}$ are the sample mean vectors of $\mathbf{y}_i$ and $\mathbf{c}_i$ ($i = 1, 2, \ldots, r$), and $\mathbf{S}_c$ and $\mathbf{S}_{yc}$ are the sample estimators of Σ_c and Σ_{yc} respectively. Given $\mathbf{c}_i$'s, the controlled response vector $\hat{\mu}_y$ is distributes as

$$\hat{\mu}_y \sim N_q[\mu_y, \mathbf{d}'\mathbf{d}(\Sigma_y - \Sigma_{yc}\Sigma_c^{-1}\Sigma_{yc})], \quad (2.120)$$

where $\mathbf{d}'\mathbf{d} = r^{-1} + (r-1)^{-1}(\bar{\mathbf{c}} - \mu_c)\mathbf{S}_c^{-1}(\bar{\mathbf{c}} - \mu_c)'$ (Rao (1967)). Rubinstein and Marcus (1985) showed that the ratio of the expectation of the generalized sample variance of $\hat{\mu}_y$ to that of $\bar{\mathbf{y}}$ is

$$\delta_2^2 = \frac{E_c E_y[|\hat{\Sigma}_{y|c}(\mathbf{d}'\mathbf{d})| \mid \mathbf{c}]}{E_y[|r^{-1}\mathbf{S}_y|]} = C(r, s, q)\prod_{i=1}^{k}(1 - \lambda_i^2), \quad (2.121)$$

where $\mathbf{S}_y$ is the sample estimator of Σ_y,

$$\hat{\Sigma}_{\mathbf{y}\,|\,\mathbf{c}} = \frac{r-1}{r-s-1}\,(\mathbf{S_y} - \mathbf{S_{yc}}\mathbf{S_c^{-1}}\mathbf{S'_{yc}}), \tag{2.122}$$

and

$$C(r,s,q) = \prod_{i=1}^{q} \frac{(r-s-i)(r-1)}{(r-s-1)(r-i)}\,[1 + \sum_{i=1}^{q} \binom{q}{i} \frac{s(s+2)...(s+2(i-1))}{(r-s-2)...(r-s-2i)}]. \tag{2.123}$$

Venkatraman and Wilson (1986) formulated a different efficiency of the loss of the generalized variance due to the estimation of the optimal coefficient matrix from that of Rubinstein and Marcus (1985). Since the covariance of the controlled estimator $\hat{\mu}_{\mathbf{y}}$ is

$$Cov(\hat{\mu}_{\mathbf{y}}) \;=\; E[Cov(\hat{\mu}_{\mathbf{y}}\,|\,\mathbf{c})] \;=\; \frac{r-2}{r(r-s-2)}\,(\Sigma_{\mathbf{y}} - \Sigma_{\mathbf{yc}}\Sigma_{\mathbf{c}}^{-1}\Sigma'_{\mathbf{yc}}) \tag{2.124}$$

(Rao (1967)), the ratio of generalized variance of $Cov(\hat{\mu}_{\mathbf{y}})$ to the generalized variance of $Cov(\bar{\mathbf{y}})$ is given by

$$\omega_1^2 \;=\; \frac{|Cov(\hat{\mu}_{\mathbf{y}})|}{|Cov(\bar{\mathbf{y}})|} \;=\; (\frac{r-2}{r-s-2})^q \prod_{i=1}^{k}(1-\lambda_i^2), \tag{2.125}$$

which is the generalized form of the efficiency of simulation presented by Lavenberg, Moeller and Welch (1982). The difference between the loss factors ω_1^2 and δ_2^2 is that the operators for taking expectation and determinant do not communicate if the dimension of $\mathbf{y}$ is greater than 1:

$$E[\,|Cov(\hat{\mu}_{\mathbf{y}}\,|\,\mathbf{c})|\,] \;\neq\; |E[Cov(\hat{\mu}_{\mathbf{y}}\,|\,\mathbf{c})]| \tag{2.126}$$

(Venkatraman and Wilson (1986)).

For $q = 1$, the efficiency measures of the control variates method developed by Rubinstein and Marcus (1985) and Venkatraman and Wilson (1986) are equivalent to the efficiency measure of the control variates method originally formulated by Lavenberg, Moeller and Welch (1982). The loss factor $C(r, s, q)$ developed by Rubinstein and Marcus (1985) is too complicated to apply it to choosing the appropriate number of of the control variates. The loss factor suggested by Venkatraman and Wilson (1986) is a simpler and more tractable measure for estimating the efficiency of the control variates method than the loss factor of Rubinstein and Marcus (1985).

As with the single response case, because of the trade-off relationship between the loss factor and the minimum generalized variance ratio, the empirical results showed that we should keep the number of control variates small. Venkatraman and Wilson (1986) recommended a guideline for limiting the number of control variates when the user supplied the number of the response variables of interest, the number of the independent replications taken, and specified the upper limit of the loss factor Λ due to the estimation of the optimal coefficient matrix of the control variates. At most,

$$s^* = (r-2)(1-\Lambda^{-1/q}) \qquad (2.127)$$

control variates should be used to be effective in the simulation experiment using the control variates method.

2.2.5. Multipopulation with multiple responses and multiple control variates

Porta Nova (1985) extended the results of Nozari, Arnold and Pegden (1984) to the case of more than one response in a multipopulation model with multiple control variates. He generalized the minimum variance ratio in (2.118) of Rubinstein and Marcus (1985), and the loss factor in (2.98) defined by Lavenberg, Moeller and Welch (1982).

Consider the simulation output at the ith design point from a single simulation run: a q-dimensional vector of the response $\mathbf{y}_i$, and s control variates $\mathbf{c}_i$. Under the assumption that

$$\begin{bmatrix} \mathbf{y}_i \\ \mathbf{c}_i \end{bmatrix} \sim N_{q+s}\left[\begin{bmatrix} \mu_{y_i} \\ \mu_c \end{bmatrix}, \Sigma\right], \qquad i = 1, 2, \ldots, m, \tag{2.128}$$

where Σ is defined in (2.117), Porta Nova (1985) represented $\mathbf{y}_i$ as the linear statistical model:

$$\mathbf{y}'_i = \mathbf{x}'_i\mathbf{B} + \mathbf{c}'_i\mathbf{A} + \varepsilon'^{*}_i, \qquad i = 1, 2, \ldots, m; \tag{2.129}$$

where $\mathbf{x}'_i$ is the ith row of $\mathbf{X}$ defined in (2.5), $\mathbf{y}_i$ and $\mathbf{c}_i$ are the $(q \times 1)$ vector of the responses and the $(s \times 1)$ vector of control variates, respectively, at $\mathbf{x}'_i$; $\mathbf{B}$ is a $(p \times q)$ matrix of unknown parameters; $\mathbf{A}$ is a $(s \times q)$ matrix of control variates coefficients; and $\varepsilon'^{*}_i = (\varepsilon^{*}_{i1}, \varepsilon^{*}_{i2}, \ldots, \varepsilon^{*}_{iq})'$ is a vector of residuals. In matrix form, this model can be written as

$$\mathbf{Y} = \mathbf{XB} + \mathbf{CA} + \mathbf{E}^{*}, \tag{2.130}$$

where $\mathbf{Y}$ is a $(m \times q)$ matrix of the responses (each row of $\mathbf{Y}$, $\mathbf{y}'_i$, constitutes $\mathbf{Y}$); $\mathbf{X}$ is a $(m \times p)$ design matrix; $\mathbf{C}$ is a $(m \times s)$ matrix of control variates; and $\mathbf{E}^*$ is a $(m \times q)$ matrix of residuals.

First, we consider the case where $\mathbf{\Sigma}$ is known. When $\mathbf{\Sigma}$ is known, Porta Nova (1985) showed that the least squares estimator for $\mathbf{B}$ in (2.130) is

$$\hat{\mathbf{B}}(\mathbf{A}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{Y} - \mathbf{CA}), \tag{2.131}$$

where $\mathbf{A}' = \mathbf{\Sigma}_{\mathbf{yc}}\mathbf{\Sigma}'^{-1}_{\mathbf{c}}$, and defined the minimum variance ratio as

$$\frac{|\operatorname{Var}(vec\hat{\mathbf{B}}(\mathbf{A}))|}{|\operatorname{Var}(vec\hat{\mathbf{B}})|} = \left[\prod_{i=1}^{r}(1-\lambda_i^2)\right]^p, \tag{2.132}$$

where $vec\,\hat{\mathbf{B}}$ denotes the operation that the columns of $\hat{\mathbf{B}}$ are stacked into a single mp-dimensional vector, $r = \min(q, s)$ and λ_i $(i = 1, 2, ..., r)$ are the canonical correlations between $\mathbf{y}_i$ and $\mathbf{c}_i$. When $m = 1$ (a single design point), equation (2.132) is equivalent to (2.118). Thus, the minimum variance ratio in (2.132) is the generalized form of that of Rubinstein and Marcus (1985). The level $(1 - a)$ confidence region for $vec\,\mathbf{B}$ is given by

$$Pr\{vec(\hat{\mathbf{B}} - \mathbf{B})'[\mathbf{\Sigma}_{\mathbf{yc}}\otimes(\mathbf{X}'\mathbf{X})^{-1}]^{-1}vec(\hat{\mathbf{B}} - \mathbf{B}) \le \chi^2_{a,\,pq}\} = 1 - a, \tag{2.133}$$

where $\chi^2_{a,pq}$ is the upper a percentile of chi-square distribution with d.f. pq and $\otimes$ is the Kronecker product of two matrices (see (3.47) in Porta Nova (1985)).

Next we consider the case where $\mathbf{\Sigma}$ is unknown. When $\mathbf{\Sigma}$ is unknown, Porta Nova (1985) derived the least squares estimators of $\mathbf{B}$ and $\mathbf{A}$ in (2.130), which are given respectively by

$$\hat{\mathbf{B}}(\hat{\mathbf{A}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I} - \mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}]\mathbf{Y} \tag{2.134}$$

and

$$\hat{\mathbf{A}} = (\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{Y}, \tag{2.135}$$

where $\mathbf{P} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Also, he showed that the efficiency of control variates is

$$\frac{|\operatorname{Var}(vec\hat{\mathbf{B}}(\mathbf{A}))|}{|\operatorname{Var}(vec\hat{\mathbf{B}})|} = \left[\frac{m-p-1}{m-p-s-1}\right]^{q}\left[\prod_{i=1}^{r}(1-\lambda_i^2)\right]^{p}, \tag{2.136}$$

where $[(m-p-1)/(m-p-s-1)]^q$ is the generalized form of the loss factor defined by Lavenberg, Moeller and Welch (1982). The level $(1-a)$ confidence region for $vec\ \mathbf{B}$ is given by

$$Pr\{\frac{m-p-s-pq+1}{pq(m-p-s)}\mathrm{T}^2_{m-p-q} \le F^a_{pq,\,m-p-s-pq-1} \mid \mathbf{C}\} = 1-a, \tag{2.137}$$

where T^2_{m-p-q} is the Hotelling's T^2 statistic with d.f. $(m-p-q)$ and $F^a_{pq,m-p-s-pq-1}$ is the upper a percentile of the F-distribution with d.f. pq and $(m-p-s-pq+1)$ (see (3.71) in Porta Nova (1985)).

CHAPTER 3. EFFICIENCY OF CONTROL VARIATES WITH CORRELATED REPLICATIONS

This chapter develops three combined approaches utilizing control variates and antithetic variates to improve the estimation of the mean response of interest in a single population model.

One of the important characteristics of computer simulation is the system analyst's control over the random number streams that drive a simulation model. In computer simulation, streams of random numbers completely determine the simulation response output. Usually, antithetic variates is applied to reduce the error of the estimator of the mean response in a single population model (discussions of this method are given in Section 11.3 in Law and Kelton (1982), Section 2.2 in Bratley, Fox and Schrage (1983), and Section III.6 in Kleijnen (1974)). This method assigns complementary random numbers to pairs of simulation runs taken at a single design point to induce a negative correlation between the responses. Let y_1 and y_2 denote two responses obtained by antithetic replicates at a single design point. Suppose that we estimate the mean response of interest by the sample mean response. Then we observe that in general,

$$\mathrm{Var}[\frac{1}{2}(y_1 + y_2)] = \frac{1}{4}\mathrm{Var}(y_1) + \frac{1}{4}\mathrm{Var}(y_2) + \frac{1}{2}\mathrm{Cov}(y_1,\ y_2).$$

In this equation, if the covariance between y_1 and y_2 obtained by antithetic replicates is negative, then the variance of the estimator for the mean response is less than that obtained by two independent replicates.

Typically, in applying the method of control variates, we perform the simulation independently through replications (see Bauer (1987), Lavenberg, Moeller and Welch (1984), Nova (1985), Rubinstein and Marcus (1985), Nozari, Arnold and Pegden (1984), and Wilson and Priksker (1984a, 1894b)). However, suppose, through correlated replications, we get a variance reduction in the response of interest, but maintain the same correlations between the response and control variates as those obtained under independent replications. Then, it is conjectured that we may take advantage of both antithetic variates and control variates together in one simulation run, and reduce the variance of the estimator further than by applying either antithetic variates or control variates separately.

We now consider the random number assignment strategy of utilizing antithetic variates and control variates for a simulation model with a single response and multiple control variates. Tew (1989) suggested correlation induction techniques, across replications, for fitting a second-order metamodel in simulation experiments with $2h$ replications at each design point. We apply his correlation induction strategy (see Table 2, Tew (1989)) to induce correlations across $2h$ replications for a single population model that also includes control variates. Let the random number stream $\mathbf{r}_{ij}$ denote the sequence of the random numbers used for the jth stochastic component of the simulation model for the ith replication. Assume that the simulation model requires g such random number streams to drive all of its stochastic components for a single replication. Also let $\mathbf{R}_i$ be the set of g random number streams for the ith replication: $\mathbf{R}_i = (\mathbf{r}_{i1}, \mathbf{r}_{i2}, \ldots, \mathbf{r}_{ig})$ $(i = 1, 2, \ldots, 2h)$. We separate $\mathbf{R}_i$ into two mutually exclusive and exhaustive subsets of random number streams, $(\mathbf{R}_{i1}, \mathbf{R}_{i2})$ $(i = 1, 2, \ldots, 2h)$, and assign $\mathbf{R}_{i1}$ to non-control variate stochastic components and $\mathbf{R}_{i2}$ to control variate stochastic components. Through an appropriate assignment of a set of random number streams to the stochastic components in the simulation model, we may induce correlations between responses, between control variates, and between responses and control variates, across replications.

For instance, if we employ antithetic variates using $\mathbf{R}_{i1}$ across h pairs of replications while leaving $\mathbf{R}_{i2}$ randomly chosen through the $2h$ replications, then we have negatively correlated h pairs of the responses. However, the control variates are independently observed through the $2h$ replications. Of particular interest is the correlation between the response and the control variates when we apply antithetic replications to either the non-control variables or the control variates in the model. If this correlation is consistent with that resulting from independent replicates, we will have the additive effect of both methods in reducing the variance of the estimator for the mean response.

In this research, we develop three combined methods of antithetic variates and control variates, and investigate their simulation efficiency in estimating the mean response of interest in a single population model with a single response and s control variates. Specifically, we consider the following correlated replication strategies: (a) use antithetic variates for all stochastic components except the control variates through $2h$ replications, (b) use antithetic variates on only the control variates through $2h$ replications, and (c) use antithetic variates for all stochastic components through $2h$ replications. Through statistical analysis and simulation experimentation, we will explore: (a) how these methods may improve the simulation efficiency, (b) what conditions are necessary for each method to ensure an improvement in variance reduction, and (c) a way of extending these methods to the estimation of the parameters of a multipopulation simulation metamodel.

The remainder of this chapter is organized as follows: Section 3.1 develops Combined Method I based on the first correlated replications described above (antithetic variates on all stochastic components except the control variates through $2h$ replications) and identifies the variance of the estimator in a single population model with a single response and a single (multiple) control variate(s). We also evaluate the simulation efficiency of this method. Section 3.2 presents the procedure of applying Combined Method II, that is, antithetic replications on control variates through $2h$ replications and investigates the efficiency of this method. Section 3.3 considers Combined Method III which uses antithetic variates on all stochastic components in the model. Finally, Section 3.4 compares the simulation efficiency of these three combined methods with respect to the unconditional variance of the estimator for the mean response obtained for each method. We also compare the three combined methods with control variates, respectively, and examine necessary conditions for each combined method to yield a better result than the method of control variates.

3.1. Combined Method I

In this section, we present the method for combining antithetic variates and control variates based on correlated replications through the non-control variate stochastic components in the model for two cases: (a) a single population model with a single response and a single control variate and (b) a single population model with a single response and multiple control variates.

For the single control variate case, the problem is given in Section 2.2.1, where the response of interest y_i and the control variate c_i ($i = 1, 2, ..., r$) are assumed to be obtained by independent random number streams at each replication. In contrast with the independent case, we now consider the simulation output driven by antithetic variates through the h pairs of replications. We assign $\mathbf{R}_{i1}$ and $\mathbf{R}_{i2}$ to the non-control stochastic variables and to the control variable, respectively, for the ith replication as follows: we use antithetic variates on $\mathbf{R}_{i1}$ through the h pairs of replications while leaving $\mathbf{R}_{i2}$ randomly chosen through the $2h$ replications. The complete assignment of random number streams across $2h$ replications is given in Table 1, where $\mathbf{R}_{2i-1,1}$, $\mathbf{R}_{2i-1,2}$ and $\mathbf{R}_{2i,2}$ ($i = 1, 2, ..., h$) are sets of randomly selected random number streams, and $\overline{\mathbf{R}}_{2i-1,1}$ ($i = 1, 2, ..., h$) are their antithetic sets of random number streams.

With this replication rule, within h pairs of replications corresponding to $(\mathbf{R}_{2i-1,1}, \mathbf{R}_{2i-1,2})$ and $(\overline{\mathbf{R}}_{2i-1,1}, \mathbf{R}_{2i,2})$ ($i = 1, 2, ..., h$), we try to induce a negative correlation, but across pairs we leave the observations independent. Thus, we have negatively correlated responses across h pairs of the response: $\mathbf{y} = (y_1, y_2, ..., y_{2h-1}, y_{2h})'$, where the response y_{2i-1} is obtained by random number streams $(\mathbf{R}_{2i-1,1}, \mathbf{R}_{2i-1,2})$ and y_{2i} is obtained by random number streams $(\overline{\mathbf{R}}_{2i-1,1}, \mathbf{R}_{2i,2})$. However, through the $2h$ replications, the control variates $\mathbf{c} = (c_1, c_2, ..., c_{2h-1}, c_{2h})'$ are independently generated by random number streams $\mathbf{R}_{2i-1,2}$ and $\mathbf{R}_{2i,2}$ ($i = 1, 2, ..., h$).

In order to completely determine the joint distribution of the response and the control variate obtained through the $2h$ antithetic replications described above, we assume the following:

Table 1. Random Number Assignment of Correlated Replication I

Replication	Control Variates	Response
1	$c_1(\mathbf{R}_{12})$	$y_1(\mathbf{R}_{11}, \mathbf{R}_{12})$
2	$c_2(\mathbf{R}_{22})$	$y_2(\overline{\mathbf{R}}_{11}, \mathbf{R}_{22})$
3	$c_3(\mathbf{R}_{32})$	$y_3(\mathbf{R}_{21}, \mathbf{R}_{32})$
4	$c_4(\mathbf{R}_{42})$	$y_4(\overline{\mathbf{R}}_{21}, \mathbf{R}_{42})$
.	.	.
.	.	.
.	.	.
2h-1	$c_{2h-1}(\mathbf{R}_{2h-1,2})$	$y_{2h-1}(\mathbf{R}_{h-1,1}\mathbf{R}_{2h-1,2})$
2h	$c_{2h}(\mathbf{R}_{2h,2})$	$y_{2h}(\overline{\mathbf{R}}_{h,1}, \mathbf{R}_{2h,2})$

1. $\mathrm{Var}(y_i) = \sigma_y^2$, for $i = 1, 2, \ldots, 2h$ (homogeneity of response variances across replicates),

2. $\mathrm{Cov}(y_i, y_j) = -\rho_1\sigma_y^2$ $(\rho_1 > 0)$, if $j = i + 1$ $(i = 1, 3, \ldots, 2h - 1)$ (homogeneity of induced negative correlations across replicates pairs). Otherwise, $\mathrm{Cov}(y_i, y_j) = 0$,

3. $\mathrm{Cov}(y_i, c_i) = \sigma_{yc} > 0$, for $i = 1, 2, \ldots, 2h$ (homogeneity of control variates-response covariances across replicates), and $\mathrm{Cov}(y_i, c_j) = 0$, for $i \neq j$,

4. $\mathrm{Var}(c_i) = \sigma_c^2$, for $i = 1, 2, \ldots, 2h$ (homogeneity of control variate variances across replicates), and

5. $\mathrm{Cov}(c_i, c_j) = 0$, for $i \neq j$ (independence of control variates between replicates).

Under these assumptions, we can identify the joint distribution of the h mean responses and mean control variates within a pair of replications: $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_h)'$ and $\bar{\mathbf{c}} = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_h)'$, where $\bar{y}_i = (y_{2i-1} + y_{2i})/2$ and $\bar{c}_i = (c_{2i-1} + c_{2i})/2$ $(i = 1, 2, \ldots, h)$. The variances of $\bar{y}_i$ and $\bar{c}_i$ are given by, respectively:

$$\mathrm{Var}(\bar{y}_i) = \frac{1}{4}[\mathrm{Var}(y_{2i-1}) + \mathrm{Var}(y_{2i}) + 2\mathrm{Cov}(y_{2i-1}, y_{2i})]$$

$$= \frac{1}{4}[\sigma_y^2 + \sigma_y^2 - 2\rho_1\sigma_y^2] = \frac{1}{2}(1 - \rho_1)\sigma_y^2, \tag{3.1}$$

and

$$\mathrm{Var}(\bar{c}_i) = \frac{1}{4}[\mathrm{Var}(c_{2i-1}) + \mathrm{Var}(c_{2i}) + 2\mathrm{Cov}(c_{2i-1}, c_{2i})] = \frac{1}{4}[\sigma_c^2 + \sigma_c^2] = \frac{1}{2}\sigma_c^2. \tag{3.2}$$

Also the covariance between $\bar{y}_i$ and $\bar{c}_i$ is given by

$$\mathrm{Cov}(\bar{y}_i, \bar{c}_i) = \frac{1}{4}\mathrm{Cov}(y_{2i-1} + y_{2i}, c_{2i-1} + c_{2i})$$

$$= \frac{1}{4}[\mathrm{Cov}(y_{2i-1}, c_{2i-1}) + \mathrm{Cov}(y_{2i-1}, c_{2i}) + \mathrm{Cov}(y_{2i}, c_{2i-1}) + \mathrm{Cov}(y_{2i}, c_{2i})]$$

$$= \frac{1}{4}[\sigma_{yc} + \sigma_{yc}] = \frac{1}{2}\sigma_{yc}. \tag{3.3}$$

Under the joint normality assumption of the response and control variates (see Lavenberg, Moeller and Welch (1982) and Cheng (1978)), the joint distribution of $\bar{y}_i$ and $\bar{c}_i$ is given by

$$\begin{bmatrix} \bar{y}_i \\ \bar{c}_i \end{bmatrix} \sim N_2\left[\begin{bmatrix} \mu_y \\ 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} (1-\rho_1)\sigma_y^2 & \sigma_{yc} \\ \sigma_{yc} & \sigma_c^2 \end{bmatrix}\right], \tag{3.4}$$

where the mean of the control variate is assumed to be 0 (this can be always achieved by subtracting its known mean μ_c from $\bar{c}_i$ without loss of generality). Theorem 2.5.1 in Anderson (1984) gives the conditional variance of $\bar{y}_i$, given $\bar{c}_i$, as follows

$$\mathrm{Var}(\bar{y}_i \mid \bar{c}_i) = \frac{1}{2}[(1-\rho_1)\sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2}]. \tag{3.5}$$

From equations (3.4) and (3.5), and the independence of the simulation output across the h pairs of replications, the mean vector of the response pairs, $\bar{\mathbf{y}}$, can be represented as the linear model in (2.70) if $\mathbf{c}$ is replaced by $\bar{\mathbf{c}}$. Therefore, with the same procedure that was used in the development of (2.76), the controlled estimator for the mean response of interest is given by

$$\hat{\mu}_y = \bar{y} - \bar{c}\hat{\alpha} = \frac{1}{h}\mathbf{1}'[\bar{\mathbf{y}} - \bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{y}}] = \frac{1}{h}\mathbf{1}'[\mathbf{I}_h - \bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{P}]\bar{\mathbf{y}}, \tag{3.6}$$

where $\bar{y}$ and $\bar{c}$ are the sample means of $\bar{y}_i$ and $\bar{c}_i$ ($i = 1, 2, \ldots, h$), respectively, $\mathbf{1}$ is the $(h \times 1)$ column vector of 1's, $\mathbf{I}_h$ is the $(h \times h)$ identity matrix, and $\mathbf{P} = \mathbf{I}_h - \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$. Taking the operation of conditional expectation on (3.6) yields

$$\operatorname{Var}(\hat{\mu}_y \mid \bar{\mathbf{c}}) = \frac{1}{h^2}\mathbf{1}'[\mathbf{I}_h - \bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{P}]\operatorname{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{c}})[\mathbf{I}_h - \mathbf{P}\bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}']\mathbf{1} \tag{3.7}$$

Since the two-variate mean simulation output, $(\bar{y}_i, \bar{c}_i)$, of the ith pair of replications is independent of that of a different pair of replications, the joint distribution of $\bar{\mathbf{y}}$ and $\bar{\mathbf{c}}$ is given by

$$\begin{bmatrix} \bar{\mathbf{y}} \\ \bar{\mathbf{c}} \end{bmatrix} \sim N_{2h}\left[\begin{bmatrix} \mu_y \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \frac{1}{2}\begin{bmatrix} (1-\rho_1)\sigma_y^2 \mathbf{I}_h & \sigma_{yc}\mathbf{I}_h \\ \sigma_{yc}\mathbf{I}_h & \sigma_c^2 \mathbf{I}_h \end{bmatrix}\right], \tag{3.8}$$

where $\mathbf{1}$ and $\mathbf{I}_h$ are given in (3.6), and $\mathbf{0}$ is the $(h \times 1)$ column vector of 0's. From Theorem 2.5.1 in Anderson (1984), the conditional variance of $\bar{\mathbf{y}}$, given $\bar{\mathbf{c}}$, is given by

$$\operatorname{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{c}}) = \frac{1}{2}[(1-\rho_1)\sigma_y^2 \mathbf{I}_h - (\sigma_{yc}\mathbf{I}_h)(\sigma_c^2 \mathbf{I}_h)^{-1}(\sigma_{yc}\mathbf{I}_h)] = \frac{1}{2}[(1-\rho_1)\sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2}]\mathbf{I}_h. \tag{3.9}$$

This equation shows that, given $\bar{c}_i$, the conditional variance of $\bar{y}_i$ (that is, the ith diagonal element of $\operatorname{Var}(\mathbf{y} \mid \mathbf{c})$) obtained by antithetic replications is less than that of $\bar{y}_i$ resulting from two independent replications (see (2.68)). Plugging (3.9) into (3.7) yields

$$\operatorname{Var}(\hat{\mu}_y \mid \bar{\mathbf{c}}) = \frac{[(1-\rho_1)\sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2}]}{2h^2} \times$$

$$[\mathbf{1}'\mathbf{1} - \mathbf{1}'\bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{P}\mathbf{1} - \mathbf{1}'\mathbf{P}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}\mathbf{1} + \mathbf{1}'\bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{1}]$$

$$= \frac{[(1-\rho_1)\sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2}]}{2h^2}[h + \mathbf{1}'\bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{1}], \tag{3.10}$$

since $\mathbf{P1} = \mathbf{1}'\mathbf{P} = \mathbf{0}$. From Theorem 2.2, $(\mathbf{1}'\bar{\mathbf{c}})$ and $(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})$ are independent. Also, we have

$$\frac{\mathbf{1}'\bar{\mathbf{c}}\bar{\mathbf{c}}'\mathbf{1}}{\sigma_c^2} = \frac{h^2\bar{c}^2}{\sigma_c^2} \sim h\chi^2(1), \tag{3.11}$$

and

$$\frac{\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}}}{\sigma_c^2} \sim \chi^2(h-1), \tag{3.12}$$

where $\mathbf{P}$ is given in (3.6). Thus, we have

$$\frac{\mathbf{1}'\bar{\mathbf{c}}\bar{\mathbf{c}}'\mathbf{1}}{\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}}/(h-1)} \sim hF(1, h-1). \tag{3.13}$$

Since the expectation of the random variable with distribution given by $F(1, h-1)$ is $(h-1)/(h-3)$, we have

$$\mathrm{E}\left[\frac{\mathbf{1}'\bar{\mathbf{c}}\bar{\mathbf{c}}'\mathbf{1}}{\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}}}\right] = \frac{h}{h-3}. \tag{3.14}$$

Therefore, taking expectation on (3.10) gives the unconditional variance of $\hat{\mu}_y$ as follows:

$$\mathrm{Var}(\hat{\mu}_y) = \mathrm{E}[\mathrm{Var}(\hat{\mu}_y \mid \bar{\mathbf{c}})] = \frac{(1-\rho_1)\sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2}}{2h^2}\mathrm{E}[h + \mathbf{1}'\bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{1}]$$

$$= \frac{(1-\rho_1)\sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2}}{2h^2}[h + \frac{h}{h-3}] = \frac{(1-\rho_1)\sigma_y^2 - \sigma_{yc}^2\sigma_c^{-2}}{2h}(\frac{h-2}{h-3})$$

$$= \frac{\sigma_y^2}{2h}(1-\rho_1-\rho_{yc}^2)(\frac{h-2}{h-3}), \tag{3.15}$$

where ρ_{yc} is the correlation coefficient between y_i and c_i ($i = 1, 2, \ldots, 2h$). As we see in (3.15), the minimum variance ratio is decreased by the amount of ρ_1, but the loss factor is increased ($(h-2)/(h-3) > (2h-2)/(2h-3)$).

Next we consider the combined method for the case of multiple control variates. The results of a single control variate can be straight forwardly extended to the case of s control variates. Instead of a single control variate corresponding to y_i, the simulation response, we assume that we have a collection of control variates, from the ith run, given by $\mathbf{c}_i = (c_{i1}, c_{i2}, \ldots, c_{is})'$.

In addition to the assumptions (1)-(2) made before, we assume the following:

3. $\mathrm{Cov}(y_i, \mathbf{c}_i) = \boldsymbol{\sigma}_{yc}$, for $i = 1, 2, \ldots, 2h$ (homogeneity of control variates-response covariance across replicates), and $\mathrm{Cov}(y_i, \mathbf{c}_j) = \mathbf{0}'$, for $i \neq j$,

4. $\mathrm{Cov}(\mathbf{c}_i) = \boldsymbol{\Sigma}_c$, for $i = 1, 2, \ldots, 2h$ (homogeneity of control variates covariance structure across replicates), and

5. $\mathrm{Cov}(\mathbf{c}_i, \mathbf{c}_j) = \mathbf{O}_{s \times s}$, for $i \neq j$ (independence of control variates between replicates).

Under these assumptions, similar to (3.2) and (3.3), we have

$$\mathrm{Cov}(\bar{\mathbf{c}}_i) = \frac{1}{4}\left[\mathrm{Cov}(\mathbf{c}_{2i-1}) + \mathrm{Cov}(\mathbf{c}_{2i}) + 2\mathrm{Cov}(\mathbf{c}_{2i-1}, \mathbf{c}_{2i})\right] = \frac{1}{2}\boldsymbol{\Sigma}_c, \tag{3.16}$$

and

$$\mathrm{Cov}(\bar{y}_i, \bar{\mathbf{c}}_i) = \frac{1}{4}\mathrm{Cov}(y_{2i-1} + y_{2i}, \mathbf{c}_{2i-1} + \mathbf{c}_{2i}) = \frac{1}{2}\boldsymbol{\sigma}'_{yc}, \tag{3.17}$$

where $\bar{y}_i$ is given by (3.6), and $\bar{\mathbf{c}}_i = (\mathbf{c}_{2i-1} + \mathbf{c}_{2i})/2$. Similar to (3.4), the joint normality assumption of the response and control variates gives the joint distribution of $\bar{y}_i$ and $\bar{\mathbf{c}}_i$ as follows:

$$\begin{bmatrix} \bar{y}_i \\ \bar{\mathbf{c}}_i \end{bmatrix} \sim N_{s+1}\left[\begin{bmatrix} \mu_y \\ \mathbf{0} \end{bmatrix}, \frac{1}{2}\begin{bmatrix} (1-\rho_1)\sigma_y^2 & \boldsymbol{\sigma}'_{yc} \\ \boldsymbol{\sigma}_{yc} & \boldsymbol{\Sigma}_c \end{bmatrix}\right]. \tag{3.18}$$

From this joint distribution, the conditional variance of $\bar{y}_i$, given $\bar{\mathbf{c}}_i$, is as follows

$$\mathrm{Var}(\bar{y}_i \mid \bar{\mathbf{c}}_i) = \frac{1}{2}[(1-\rho_1)\sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}] \tag{3.19}$$

(see Theorem 2.5.1 in Anderson (1984)). As with the case of a single control variate, the vector of the mean responses within a pair, $\bar{\mathbf{y}}$, is represented as the linear model in (2.91). Regression analysis on this linear model yields the controlled estimator for the mean response as

$$\hat{\mu}_y = \frac{1}{h}\mathbf{1}'[\bar{\mathbf{y}} - \mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\bar{\mathbf{y}}] = \frac{1}{h}\mathbf{1}'[\mathbf{I}_h - \mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}]\bar{\mathbf{y}}, \tag{3.20}$$

where $\bar{\mathbf{y}}$ is given in (3.6), $\mathbf{C}$ is a $(h \times s)$ control variates matrix whose ith row is $\bar{\mathbf{c}}'_i$, and $\mathbf{P}$ is defined in (3.6). Given $\mathbf{C}$, taking operation of variance on (3.20) yields

$$\mathrm{Var}(\hat{\mu}_y \mid \mathbf{C}) = \frac{1}{h^2}\mathbf{1}'[\mathbf{I}_h - \mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}]\mathrm{Var}(\bar{\mathbf{y}} \mid \mathbf{C})[\mathbf{I}_h - \mathbf{P}\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}']\mathbf{1}. \tag{3.21}$$

By extending the results in (3.9) to the s control variates case, the conditional variance $\bar{\mathbf{y}}$, given $\mathbf{C}$, is

$$\mathrm{Var}(\bar{\mathbf{y}} \mid \mathbf{C}) = \frac{1}{2}[(1-\rho_1)\sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}]\mathbf{I}_h \tag{3.22}$$

since the $(s+1)$ variate mean response, $(\bar{y}_i, \bar{\mathbf{c}}_i)$, of the ith pair of replications is independent of that of a different pair of replications. Substituting for $\mathrm{Var}(\bar{\mathbf{y}} \mid \mathbf{C})$ into (3.21) gives

$$\mathrm{Var}(\hat{\mu}_y \mid \mathbf{C}) = \frac{[(1-\rho_1)\sigma^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}]}{2h^2} \times$$

$$[\mathbf{1}'\mathbf{1} - \mathbf{1}'\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\mathbf{1} - \mathbf{1}'\mathbf{P}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}\mathbf{1} + \mathbf{1}'\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{1}]$$

$$= \frac{[(1-\rho_1)\sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}]}{2h^2}[h + \mathbf{1}'\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{1}], \qquad (3.23)$$

since $\mathbf{P1} = \mathbf{1}'\mathbf{P} = 0$. From the result of (3.18), the $(h \times s)$ random matrix $\mathbf{C}$ has the matrix normal distribution: $\mathbf{C} \sim N_{h,s}(\mathbf{O}, \mathbf{I}_s, \boldsymbol{\Sigma}_c/2)$, where $\mathbf{O}$ is a $(h \times s)$ matrix of zeroes (see (2.38)). By definition of the Wishart distribution (see Chapter 17 in Arnold (1981)),

$$\mathbf{C}(\boldsymbol{\Sigma}_c/2)^{-1}\mathbf{C}' \sim W_h(s, \mathbf{I}_h), \qquad (3.24)$$

and, by Theorem 17.7a in Arnold (1981), the $(s \times s)$ random matrix $(\mathbf{C}'\mathbf{P}\mathbf{C})$ follows the Wishart distribution:

$$(\mathbf{C}'\mathbf{P}\mathbf{C}) \sim W_s(h-1, \tfrac{1}{2}\boldsymbol{\Sigma}_c) \qquad (3.25)$$

since $\mathbf{P}$ is the idempotent matrix with rank $(h-1)$. We note that $(\mathbf{1}'\mathbf{C})$ and $(\mathbf{C}'\mathbf{P}\mathbf{C})$ are independent (see Theorem 4.5.1 in Graybill (1976)). Thus, the expectation of the conditional variance in (3.23) can be written as

$$\mathrm{Var}(\hat{\mu}_y) = \mathrm{E}[\mathrm{Var}(\hat{\mu}_y \mid \mathbf{C})] = \frac{[(1-\rho_1)\sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}]}{2h^2}\mathrm{E}[h + \mathbf{1}'\mathbf{C}\mathrm{E}[(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}]\mathbf{C}'\mathbf{1}]. \quad (3.26)$$

Theorems 17.6a and 17.15d in Arnold (1981) gives, respectively,

$$E[\mathbf{C}(\boldsymbol{\Sigma}_c/2)^{-1}\mathbf{C}'] = s\mathbf{I}_h, \tag{3.27}$$

and

$$E[(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}] = \frac{(\boldsymbol{\Sigma}_c/2)^{-1}}{h-s-2} \quad \text{if } \ h > s+1. \tag{3.28}$$

Therefore, plugging (3.28) into (3.26) finally yields

$$\mathrm{Var}(\hat{\mu}_y) = \frac{[(1-\rho_1)\sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}]}{2h^2}[h + \frac{E[\mathbf{1}'\mathbf{C}(\boldsymbol{\Sigma}_c/2)^{-1}\mathbf{C}'\mathbf{1}]}{h-s-2}]$$

which further reduce to, by equation (3.27),

$$= \frac{(1-\rho_1)\sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}}{2h^2}[h + \frac{hs}{h-s-2}]$$

$$= \frac{\sigma_y^2}{2h}(1-\rho_1-R_{yc}^2)(\frac{h-2}{h-s-2}), \tag{3.29}$$

where R_{yc} is the multiple correlation coefficient between y_i and $\mathbf{c}_i$ ($i = 1, 2, \ldots, 2h$). The result in (3.29) indicates that the minimum variance ratio of this method is $(1-\rho_1-R_{yc}^2)$, and the loss factor is $(h-2)/(h-s-2)$.

3.2. Combined Method II

This section considers the second combined method of control variates and antithetic variates, which is based on correlated replications through the control variates for the two cases addressed in Section 3.1.

As before, we divide the set of g random number streams used to drive the simulation model, $\mathbf{R}_i$, into two mutually exclusive and exhaustive subsets $\mathbf{R}_{i1}$ and $\mathbf{R}_{i2}$. The first subset, $\mathbf{R}_{i1}$, consists of $(g - s^*)$ random number streams and is used to drive the stochastic components of the model other than the control variates. The second subset, $\mathbf{R}_{i2}$, consists of s^* random number streams and is used to drive the control variates. Contrary to the random number assignment in Section 3.1, we now use correlated replications on the control variate components and independent replicates on all other stochastic components in the model. Through the $2h$ replications, assignment of $\mathbf{R}_i = (\mathbf{R}_{i1}, \mathbf{R}_{i2})$ $(i = 1, 2, ..., 2h)$ is given in Table 2, where $\mathbf{R}_{2i-1,2}$ $(i = 1, 2, ..., h)$ are sets of randomly selected random number streams, and $\overline{\mathbf{R}}_{2i-1,2}$ $(i = 1, 2, ..., h)$ are sets of random number streams antithetic to $\mathbf{R}_{2i-1,2}$.

With this replication strategy, we induce negative correlations between the responses, between the control variates, and between the response and the control variate(s) within h pairs of the responses and the control variates, respectively, obtained from $(\mathbf{R}_{2i-1,1}, \mathbf{R}_{2i-1,2})$ and $(\mathbf{R}_{2i}, \overline{\mathbf{R}}_{2i-1,2})$ $(i = 1, 2, ..., h)$. However, across pairs of replications, we get independent outputs.

For the case of a single control variate, we analyze the combined method under the following assumptions:

1. $\mathrm{Var}(y_i) = \sigma_y^2$, for $i = 1, 2, ..., 2h$ (homogeneity of response variances across replicates),

2. $\mathrm{Cov}(y_i, y_j) = -\rho_2\sigma_y^2$ $(\rho_2 > 0)$, if $j = i + 1$ $(i = 1, 3, ..., 2h - 1)$ (homogeneity of induced negative correlations across replicates pairs). Otherwise, $\mathrm{Cov}(y_i, y_j) = 0$,

Table 2. Random Number Assignment of Correlated Replication II

Replication	Control Variates	Response
1	$c_1(\mathbf{R}_{12})$	$y_1(\mathbf{R}_{11}, \mathbf{R}_{12})$
2	$c_2(\overline{\mathbf{R}}_{12})$	$y_2(\mathbf{R}_{21}, \overline{\mathbf{R}}_{12})$
3	$c_3(\mathbf{R}_{22})$	$y_1(\mathbf{R}_{31}, \mathbf{R}_{22})$
4	$c_2(\overline{\mathbf{R}}_{22})$	$y_2(\mathbf{R}_{41}, \overline{\mathbf{R}}_{22})$
.	.	.
.	.	.
.	.	.
2h-1	$c_{2h-1}(\mathbf{R}_{h2})$	$y_{2h-1}(\mathbf{R}_{2h-1,1}\mathbf{R}_{h2})$
2h	$c_{2h}(\overline{\mathbf{R}}_{h2})$	$y_{2h}(\mathbf{R}_{2h,1}, \overline{\mathbf{R}}_{h2})$

3. $\text{Cov}(y_i, c_i) = \sigma_{yc} > 0$, for $i = 1, 2, ..., 2h$ and $\text{Cov}(y_i, c_j) = \sigma_{yc}^{(2)} < 0$ $(-\sigma_{yc} < \sigma_{yc}^{(2)} < 0)$ if $j = i + 1$ $(i = 1, 3, ..., 2h - 1)$ (homogeneity of control variates-response covariances across replicates and homogeneity of the induced control variates-response covariances between replicates pair). Otherwise, $\text{Cov}(y_i, c_j) = 0$,

4. $\text{Var}(c_i) = \sigma_c^2$, for $i = 1, 2, ..., 2h$ (homogeneity of control variate variances across replicates),

5. $\text{Cov}(c_i, c_j) = -\rho_c\sigma_c^2$ $(\rho_c > 0)$ if $j = i + 1$ $(i = 1, 3, ..., 2h - 1)$ (homogeneity of control variates variances across replicates). Otherwise, $\text{Cov}(c_i, c_j) = 0$.

First we identify the joint distribution of $\bar{y}_i$ and $\bar{c}_i$ $(i = 1, 2, ..., h)$, the mean response and the mean control variate of the ith pair of replications, respectively. Under the assumptions (1)-(5) stated above, the variance of $\bar{y}_i$ is of the same form as in (3.1) if ρ_2 replaces ρ_1, but note that, in general, $\rho_1 \neq \rho_2$. The variance of $\bar{c}_i$ and covariance between $\bar{y}_i$ and $\bar{c}_i$ are given by, respectively:

$$\text{Var}(\bar{c}_i) = \frac{1}{4}[\text{Var}(c_{2i-1}) + \text{Var}(c_{2i}) + 2\text{Cov}(c_{2i-1}, c_{2i})] = \frac{1}{2}(1 - \rho_c)\sigma_c^2, \qquad (3.30)$$

and

$$\text{Cov}(\bar{y}_i, \bar{c}_i) = \frac{1}{4}\text{Cov}(y_{2i-1} + y_{2i}, c_{2i-1} + c_{2i})$$

$$= \frac{1}{4}[\text{Cov}(y_{2i-1}, c_{2i-1}) + \text{Cov}(y_{2i-1}, c_{2i}) + \text{Cov}(y_{2i}, c_{2i-1}) + \text{Cov}(y_{2i}, c_{2i})]$$

$$= \frac{1}{2}[\sigma_{yc} + \sigma_{yc}^{(2)}]. \qquad (3.31)$$

Similarly to (3.4) and (3.5), the joint distribution of $\bar{y}_i$ and $\bar{c}_i$ is given by

$$\begin{bmatrix} \bar{y}_i \\ \bar{c}_i \end{bmatrix} \sim N_2 \left[\begin{bmatrix} \mu_y \\ 0 \end{bmatrix}, \ \frac{1}{2} \begin{bmatrix} (1-\rho_2)\sigma_y^2 & \sigma_{yc} + \sigma_{yc}^{(2)} \\ \sigma_{yc} + \sigma_{yc}^{(2)} & (1-\rho_c)\sigma_c^2 \end{bmatrix} \right]. \tag{3.32}$$

and, given $\bar{c}_i$, the conditional variance of $\bar{y}_i$ is

$$\mathrm{Var}(\bar{y}_i \mid \bar{c}_i) = \frac{1}{2} [(1-\rho_2)\sigma_y^2 - \frac{(\sigma_{yc} + \sigma_{yc}^{(2)})^2 \sigma_c^{-2}}{1-\rho_c}] \tag{3.33}$$

(see Theorem 2.5.1 in Anderson (1984)). Based on the results of (3.32) and (3.33), we can represent the mean vector of the response pairs, $\bar{\mathbf{y}} = (\bar{y}_1, \ \bar{y}_2, \ \ldots, \ \bar{y}_h)'$, as the linear model in (2.70) if $\bar{\mathbf{c}} = (\bar{c}_1, \ \bar{c}_2, \ \ldots, \bar{c}_h)'$ replaces $\mathbf{c}$. Hence, from (2.76), the controlled estimator for the mean response of interest is given by

$$\hat{\mu}_y = \bar{y} - \bar{c}\hat{\alpha} = \frac{1}{h}\mathbf{1}'[\bar{\mathbf{y}} - \bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{y}}] = \frac{1}{h}\mathbf{1}'[\mathbf{I}_h - \bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{P}]\bar{\mathbf{y}}, \tag{3.34}$$

where $\bar{y}, \bar{c}$, and $\mathbf{P}$ are as defined in (3.6). Since $(\bar{y}_i, \ \bar{c}_i)$ and $(\bar{y}_k, \ \bar{c}_k)$ $(i \neq k)$ are independent, the normality assumption on the response and control variates ensures

$$\begin{bmatrix} \bar{\mathbf{y}} \\ \bar{\mathbf{c}} \end{bmatrix} \sim N_{2h} \left[\begin{bmatrix} \mu_y \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \ \frac{1}{2} \begin{bmatrix} (1-\rho_2)\sigma_y^2 \mathbf{I}_h & (\sigma_{yc} + \sigma_{yc}^{(2)})\mathbf{I}_h \\ (\sigma_{yc} + \sigma_{yc}^{(2)})\mathbf{I}_h & (1-\rho_c)\sigma_c^2 \mathbf{I}_h \end{bmatrix} \right]. \tag{3.35}$$

Similar to (3.9), by Theorem 2.5.1 in Anderson (1984), we have the conditional variance of $\bar{\mathbf{y}}$, given $\bar{\mathbf{c}}$, as follows:

$$\begin{aligned} \mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{c}}) &= \frac{1}{2} [(1-\rho_2)\sigma_y^2 \mathbf{I}_h - (\sigma_{yc} + \sigma_{yc}^{(2)})\mathbf{I}_h [(1-\rho_c)\sigma_c^2 \mathbf{I}_h]^{-1} (\sigma_{yc} + \sigma_{yc}^{(2)})\mathbf{I}_h] \\ &= \frac{1}{2} [(1-\rho_2)\sigma_y^2 - (\sigma_{yc} + \sigma_{yc}^{(2)})^2 [(1-\rho_c)\sigma_c^2]^{-1}]\mathbf{I}_h. \end{aligned} \tag{3.36}$$

With the same procedure in (3.9), if we take the conditional variance on (3.34) with respect to $\bar{\mathbf{y}}$, then

$$\mathrm{Var}(\hat{\mu}_y \mid \bar{\mathbf{c}}) = \frac{1}{h}\mathbf{1}'[\mathbf{I}_h - \bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{P}]\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{c}})\frac{1}{h}\mathbf{1}[\mathbf{I}_h - \mathbf{P}\bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}],$$

which reduces to, by substituting for $\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{c}})$ in this equation,

$$= \frac{1}{2h^2}[(1-\rho_2)\sigma_y^2 - \frac{(\sigma_{yc} + \sigma_{yc}^{(2)})^2\sigma_c^{-2}}{1-\rho_c}][h + \mathbf{1}'\bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{1}] \tag{3.37}$$

Since $\bar{c}_i$ $(i = 1, 2, \ldots, h)$ are independent, by equation (3.14), we find

$$\mathrm{Var}(\hat{\mu}_y) = \mathrm{E}[\mathrm{Var}(\hat{\mu} \mid \bar{\mathbf{c}})] = \frac{1}{2h}[(1-\rho_2)\sigma_y^2 - \frac{(\sigma_{yc} + \sigma_{yc}^{(2)})^2\sigma_c^{-2}}{1-\rho_c}](h + \frac{1}{h-3})$$

$$= \frac{(1-\rho_2)\sigma_y^2}{2h}(1-\bar{\rho}_{yc}^2)(\frac{h-2}{h-3}), \tag{3.38}$$

where

$$\bar{\rho}_{yc}^2 = [(1-\rho_2)\sigma_y^2]^{-1}(\sigma_{yc} + \sigma_{yc}^{(2)})^2[(1-\rho_c)\sigma_c^2]^{-1}. \tag{3.39}$$

By definition of the correlation coefficient, $\bar{\rho}_{yc}$ can be interpreted as the correlation coefficient between $\bar{y}_i$ and $\bar{c}_i$ $(i = 1, 2, \ldots, h)$. Equation (3.39) indicates that the correlation coefficient between $\bar{y}_i$ and $\bar{c}_i$ increases by reducing the variances of both y_i and c_i, but it also decreases by the covariance between $\bar{y}_i$ and $\bar{c}_i$ $(\sigma_{yc}^{(2)} < 0)$. That is, the trade-off effect to the correlation coefficient is a major factor for determining the efficiency of this method. Inspection of equation (3.38) also shows that the minimum variance ratio is $(1-\rho_2)(1-\bar{\rho}_{yc}^2)$ and the loss factor is $(h-2)/(h-3)$, which is equal to that in (3.14).

For the case of s control variates, we can extend these results. Let $\mathbf{c}_i$ be the vector of s control variates at the ith replication: $\mathbf{c}_i = (c_{i1}, c_{i2}, \ldots, c_{is})$, which corresponds to y_i ($i = 1, 2, \ldots, 2h$). In addition to the assumptions (1)-(2), we assume that

3. $\mathrm{Cov}(y_i, \mathbf{c}_i) = \boldsymbol{\sigma}'_{yc}$, for $i = 1, 2, \ldots, 2h$ and $\mathrm{Cov}(y_i, \mathbf{c}_j) = \boldsymbol{\sigma}'^{(2)}_{yc}$ if $j = i + 1$ ($i = 1, 3, \ldots, 2h - 1$). (homogeneity of control variates-response covariances across replicates, and homogeneity of induced control variates covariances across replicates pair). Otherwise, $\mathrm{Cov}(y_i, \mathbf{c}_j) = \mathbf{0}$,

4. $\mathrm{Var}(\mathbf{c}_i) = \boldsymbol{\Sigma}_c$, for $i = 1, 2, \ldots, 2h$ (homogeneity of control variates covariance structure across replicates), and

5. $\mathrm{Cov}(\mathbf{c}_i, \mathbf{c}_j) = \boldsymbol{\Sigma}_c^{(2)}$ if $j = i + 1$ ($i = 1, 3, \ldots, 2h - 1$) (homogeneity of the induced control variates covariancs between replicates pair). Otherwise, $\mathrm{Cov}(\mathbf{c}_i, \mathbf{c}_j) = \mathbf{O}_{s \times s}$.

Under these assumptions, as with the case of a single control variate, we get

$$\mathrm{Var}(\bar{\mathbf{c}}_i) = \frac{1}{4}\,[\mathrm{Var}(\mathbf{c}_{2i-1}) + \mathrm{Var}(\mathbf{c}_{2i}) + 2\mathrm{Cov}(\mathbf{c}_{2i-1}, \mathbf{c}_{2i})] = \frac{1}{2}\,(\boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}_c^{(2)}), \qquad (3.40)$$

and

$$\mathrm{Cov}(\bar{y}_i, \bar{\mathbf{c}}_i) = \frac{1}{4}\,\mathrm{Cov}(y_{2i-1} + y_{2i}, \mathbf{c}_{2i-1} + \mathbf{c}_{2i})$$

$$= \frac{1}{4}\,[\mathrm{Cov}(y_{2i-1}, \mathbf{c}_{2i-1}) + \mathrm{Cov}(y_{2i-1}, \mathbf{c}_{2i}) + \mathrm{Cov}(y_{2i}, \mathbf{c}_{2i-1}) + \mathrm{Cov}(y_{2i}, \mathbf{c}_{2i})]$$

$$= \frac{1}{2}\,[\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(2)}]', \qquad (3.41)$$

where $\bar{\mathbf{c}}_i = (\mathbf{c}_{2i-1} + \mathbf{c}_{2i})/2$ $(i = 1, 2, ..., h)$. As before, based on the normality assumption on the response and control variates, we find the joint distribution of $\bar{y}_i$ and $\bar{\mathbf{c}}_i$ as follows:

$$\begin{bmatrix} \bar{y}_i \\ \bar{\mathbf{c}}_i \end{bmatrix} \sim N_{(s+1)}\left[\begin{bmatrix} \mu_y \\ \mathbf{0} \end{bmatrix}, \frac{1}{2}\begin{bmatrix} (1-\rho_2)\sigma_y^2 & \boldsymbol{\sigma}'_{yc} + \boldsymbol{\sigma}'^{(2)}_{yc} \\ \boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}^{(2)}_{yc} & \boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}^{(2)}_c \end{bmatrix}\right]. \tag{3.42}$$

Theorem 2.5.1 in Anderson (1984) gives the conditional variance of $\bar{y}_i$, given $\bar{\mathbf{c}}_i$, as follows:

$$\mathrm{Var}(\bar{y}_i \mid \bar{\mathbf{c}}_i) = \frac{1}{2}[(1-\rho_2)\sigma_y^2 - (\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}^{(2)}_{yc})'(\boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}^{(2)})^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}^{(2)}_{yc})]. \tag{3.43}$$

Equations (3.42) and (3.43) imply that the mean vector of the response pairs, $\bar{\mathbf{y}}$, can be written as the linear model in (2.91). Hence, the controlled estimator of the mean response of interest is given by

$$\hat{\mu}_y = \frac{1}{h}\mathbf{1}'[\bar{\mathbf{y}} - \mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}\bar{\mathbf{y}}] = \frac{1}{h}\mathbf{1}'[\mathbf{I}_h - \mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{P}]\bar{\mathbf{y}}, \tag{3.44}$$

where $\mathbf{C}$ is the $(h \times s)$ matrix of control variates whose ith row is $\bar{\mathbf{c}}_i$ and $\mathbf{P}$ is defined in (3.6). In the same manner as in (3.22), we have the conditional variance of $\hat{\mu}_y$ as

$$\mathrm{Var}(\hat{\mu}_y \mid \bar{\mathbf{c}}) = \frac{1}{2h^2}[(1-\rho_2)\sigma_y^2 - (\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}^{(2)}_{yc})'(\boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}^{(2)}_c)^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}^{(2)}_{yc})] \times$$

$$[h + \mathbf{1}'\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{1}]. \tag{3.45}$$

Since $\bar{\mathbf{c}}_i$ $(i = 1, 2, ..., h)$ are independent, in the same way for finding the unconditional variance of $\hat{\mu}_y$ in (3.29), we obtain

$$\mathrm{Var}(\hat{\mu}_y) = \frac{1}{2h}\left[(1-\rho_2)\sigma_y^2 - (\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(2)})'(\boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}_c^{(2)})^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(2)})\right]\left(\frac{h-2}{h-s-2}\right)$$

$$= \frac{(1-\rho_2)\sigma_y^2}{2h}(1-\overline{R}_{yc}^2)\left(\frac{h-2}{h-s-2}\right), \tag{3.46}$$

where

$$\overline{R}_{yc}^2 = [(1-\rho_2)\sigma_y^2]^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(2)})'(\boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}_c^{(2)})^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(2)}). \tag{3.47}$$

Similar to (3.39), by extending the dimension of control variates from 1 to s, $\overline{R}_{yc}$ can be interpreted as the multiple correlation coefficient between $\overline{y}_i$ and $\overline{\mathbf{c}}_i$. Also equation (3.46) shows that the minimum variance ratio of the combined method II is $(1-\rho_2)(1-\overline{R}_{yc}^2)$ and its loss factor is the same as that given in (3.29).

3.3. Combined Method III

This section presents Combined Method III of control variates and antithetic variates based on correlated replications through all stochastic components in the simulation model for the two cases addressed in Section 3.1.

In the same way as considered in Section 3.2, we separate the set of g random number streams, $\mathbf{R}_i$, into two mutually exclusive and exhaustive subsets $\mathbf{R}_{i1}$ and $\mathbf{R}_{i2}$ which consist of $(g - s^*)$ and s^* random number streams respectively. We assign $\mathbf{R}_{i1}$ to the $(g - s^*)$ stochastic components of the non-control variates and $\mathbf{R}_{i2}$ to the control variates. Unlike the random number assignment strategies discussed in Section 3.1 and 3.2, we apply antithetic variates to all stochastic components of control variates and non-control type variates in the model. The assignment rule of $\mathbf{R}_i = (\mathbf{R}_{i1}, \mathbf{R}_{i2})$ through the $2h$ replications is given in Table 3, where $\mathbf{R}_{2i-1,1}$ and $\mathbf{R}_{2i-1,2}$ are sets of randomly selected random number streams, and $\overline{\mathbf{R}}_{2i-1,2}$ and $\overline{\mathbf{R}}_{2i-1,2}$ are sets of random number streams antithetic to $\mathbf{R}_{2i-1,1}$ and $\mathbf{R}_{2i-1,2}$, respectively ($i = 1,\ 2,\ \ldots,\ h$).

This assignment strategy induces correlations across h pairs of the responses of interest and control variates similar to those considered in Section 3.2. That is, negative correlations are induced between the responses, between the control variates, and between the response and the control variate(s) within h pairs of the responses and the control variates. The induced correlation between the responses is greater than that obtained by Combined Method i and II. However, the induced correlation between the response and control variates is different from that of Combined Method I and II, respectively. In contrast, across the h pairs of replications, the mean response and the mean control variates (within a pair of replication) are independently observed by the assignment of different sets of randomly chosen random number streams.

For the case of a single control, we assume that

1. $\mathrm{Var}(y_i) = \sigma_y^2$, for $i = 1,\ 2, \ldots,\ 2h$ (homogeneity of response variances across replications),

2. $\mathrm{Cov}(y_i, y_j) = -\rho_3\sigma_y^2\ (\rho_3 > 0)$, if $j = i + 1\ (i = 1,\ 3,\ \ldots, 2h - 1)$ (homogeneity of induced negative correlations across replicate pairs). Otherwise, $\mathrm{Cov}(y_i, y_j) = 0$,

3. $\mathrm{Cov}(y_i, c_i) = \sigma_{yc} > 0$, for $i = 1,\ 2,\ \ldots,\ 2h$ and $\mathrm{Cov}(y_i,\ c_j) = \sigma_{yc}^{(3)} < 0\ (-\sigma_{yc} < \sigma_{yc}^{(3)} < 0)$ if $j = i + 1\ (i = 1,\ 3,\ \ldots, 2h - 1)$ (homogeneity of control variates-response covariances across replicates and homogeneity of control variates-response covariances between replicates pair). Otherwise, $\mathrm{Cov}(y_i, c_j) = 0$,

Table 3. Random Number Assignment of Correlated Replication III

Replication	Control Variates	Response
1	$c_1(\mathbf{R}_{12})$	$y_1(\mathbf{R}_{11}, \mathbf{R}_{12})$
2	$c_2(\overline{\mathbf{R}}_{12})$	$y_2(\overline{\mathbf{R}}_{11}, \overline{\mathbf{R}}_{12})$
3	$c_3(\mathbf{R}_{22})$	$y_3(\mathbf{R}_{21}, \mathbf{R}_{22})$
4	$c_4(\overline{\mathbf{R}}_{22})$	$y_4(\overline{\mathbf{R}}_{21}, \overline{\mathbf{R}}_{22})$
.	.	.
.	.	.
.	.	.
2h-1	$c_{2h-1}(\mathbf{R}_{h2})$	$y_{2h-1}(\mathbf{R}_{2h-1,1}\mathbf{R}_{h2})$
2h	$c_{2h}(\overline{\mathbf{R}}_{h2})$	$y_{2h}(\overline{\mathbf{R}}_{2h,1}, \overline{\mathbf{R}}_{h2})$

4. $\mathrm{Var}(c_i) = \sigma_c^2$, for $i = 1, 2, \ldots, 2h$ (homogeneity of control variates variances across replicates), and

5. $\mathrm{Cov}(c_i, c_j) = -\rho_c\sigma_c^2$ $(\rho_c > 0)$ if $j = i + 1$ $(i = 1, 3, \ldots, 2h - 1)$ (homogeneity of induced negative control variates covariances across replicates). Otherwise, $\mathrm{Cov}(c_i, c_j) = 0$.

Under these assumptions, the variance of the mean response of the ith pair of replications, $\bar{y}_i$, is given by the same form as in (3.1) with ρ_3 replacing ρ_1. Generally, we expect that ρ_3 is greater than either ρ_1 or ρ_2 since we apply antithetic variates to all stochastic variables in the model. However, the variance of the mean control variate of the ith pair of replications, $\bar{c}_i$, is the same as in (3.30), and the covariance between $\bar{y}_i$ and $\bar{c}_i$ is given by

$$\mathrm{Cov}(\bar{y}_i, \bar{c}_i) = \frac{1}{4}\mathrm{Cov}(y_{2i-1} + y_{2i}, c_{2i-1} + c_{2i})$$

$$= \frac{1}{4}[\mathrm{Cov}(y_{2i-1}, c_{2i-1}) + \mathrm{Cov}(y_{2i-1}, c_{2i}) + \mathrm{Cov}(y_{2i}, c_{2i-1}) + \mathrm{Cov}(y_{2i}, c_{2i})]$$

$$= \frac{1}{2}[\sigma_{yc} + \sigma_{yc}^{(3)}]. \tag{3.48}$$

Under the normality assumption of the response and control variate, the joint distribution of $\bar{y}_i$ and $\bar{c}_i$ $(i = 1, 2, \ldots, h)$ is given by

$$\begin{bmatrix} \bar{y}_i \\ \bar{c}_i \end{bmatrix} \sim N_2\left[\begin{bmatrix} \mu_y \\ 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} (1-\rho_3)\sigma_y^2 & \sigma_{yc} + \sigma_{yc}^{(3)} \\ \sigma_{yc} + \sigma_{yc}^{(3)} & (1-\rho_c)\sigma_c^2 \end{bmatrix}\right]. \tag{3.49}$$

Therefore, given $\bar{c}_i$, the conditional variance of $\bar{y}_i$ is

$$\mathrm{Var}(\bar{y}_i \mid \bar{c}_i) = \frac{1}{2}\left[(1-\rho_3)\sigma_y^2 - \frac{(\sigma_{yc} + \sigma_{yc}^{(3)})^2 \sigma_c^{-2}}{1-\rho_c}\right] \tag{3.50}$$

(see Theorem 2.5.1 in Anderson (1984)). This equation shows that Combined Method III can reduce the variances of the response and control variate, but it also reduces the covariance between the response and control control variate (note that $\sigma_{yc}^{(3)}$ in (3.49) is negative). This trade-off effect is a major factor in reducing the variance of the estimator for the mean response.

As before, equations (3.49) and (3.50) enable the mean vector of the response pairs, $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_h)'$, to be represented as a linear model. Hence, the least squares estimator of the linear model (the controlled estimator for the mean response) is the same form given in (3.6). Since the random observation of $(\bar{y}_i, \bar{c}_i)$ obtained by Combined Method III are also independent as the previous two methods, the conditional and unconditional variances of the controlled estimator for μ_y can be found by the same procedures shown in Section 3.2 with replacing the $\mathrm{Var}(\bar{y}_i \mid \bar{c}_i)$ in (3.50) with that in (3.33). That is, as is the case with (3.37) and (3.38), the conditional variance of $\hat{\mu}_y$ and its unconditional variance are given by, respectively:

$$\mathrm{Var}(\hat{\mu}_y \mid \bar{\mathbf{c}}) = \frac{1}{2h^2}\left[(1-\rho_3)\sigma_y^2 - \frac{(\sigma_{yc} + \sigma_{yc}^{(3)})^2 \sigma_c^{-2}}{1-\rho_c}\right]\left[h + \mathbf{1}'\bar{\mathbf{c}}(\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{1}\right] \tag{3.51}$$

and

$$\mathrm{Var}(\hat{\mu}_y) = \mathrm{E}[\mathrm{Var}(\hat{\mu}_y)] = \frac{1}{2h}\left[(1-\rho_3)\sigma_y^2 - \frac{(\sigma_{yc} + \sigma_{yc}^{(3)})^2 \sigma_c^{-2}}{1-\rho_c}\right]\left(h + \frac{1}{h-3}\right)$$

$$= \frac{(1-\rho_3)\sigma_y^2}{2h}\left[1 - ((1-\rho_3)\sigma_y^2)^{-1}(\sigma_{yc} + \sigma_{yc}^{(3)})^2((1-\rho_c)\sigma_c^2)^{-1}\right]\left(\frac{h-2}{h-3}\right)$$

$$= \frac{(1-\rho_3)\sigma_y^2}{2h}[1-\bar{\bar{\rho}}_{yc}^2](\frac{h-2}{h-3}), \qquad (3.52)$$

where

$$\bar{\bar{\rho}}_{yc}^2 = [(1-\rho_3)\sigma_y^2]^{-1}(\sigma_{yc}+\sigma_{yc}^{(3)})^2[(1-\rho_c)\sigma_c^2]^{-1}, \qquad (3.53)$$

which is the correlation coefficient between $yabr_i$ and $\bar{c}_i$ by definition. In applying this method, the minimum variance ratio is $(1-\rho_3)(1-\bar{\bar{\rho}}_{yc}^2)$, and the loss factor is $(h-2)/(h-3)$, which is equal to that in either (3.15) or (3.38).

The results of a single control variate can be directly extended for the case of s control variates. We let $\mathbf{c}_i = (c_{i1}, c_{i2}, ..., c_{is})$ be the vector of s control variates observed at the ith replication ($i = 1, 2, ..., 2h$). In addition to assumptions (1)-(2), we assume that

3. $\mathrm{Cov}(y_i, \mathbf{c}_i) = \boldsymbol{\sigma}'_{yc}$, for $i = 1, 2, ..., 2h$ and $\mathrm{Cov}(y_i, \mathbf{c}_j) = \boldsymbol{\sigma}_{yc}^{(3)\prime}$ if $j = i+1$ ($i = 1, 3, ..., 2h-1$) (homogeneity of control variates-response covariances across replicates and homogeneity of induced control variates-response covariances between replicates pair). Otherwise, $\mathrm{Cov}(y_i, \mathbf{c}_j) = \mathbf{0}$,

4. $\mathrm{Var}(\mathbf{c}_i) = \boldsymbol{\Sigma}_c$, for $i = 1, 2, ..., 2h$ (homogeneity of control variates covariance structure across replicates), and

5. $\mathrm{Cov}(\mathbf{c}_i, \mathbf{c}_j) = \boldsymbol{\Sigma}_c^{(3)}$ if $j = i+1$ ($i = 1, 3, ..., 2h-1$) (homogeneity of the induced covariances between replicates pairs). Otherwise, $\mathrm{Cov}(\mathbf{c}_i, \mathbf{c}_j) = \mathbf{O}_{s\times s}$.

Under these assumptions, similar to (3.40) and (3.41), we have

$$\mathrm{Var}(\bar{\mathbf{c}}_i) = \frac{1}{4}\,[\mathrm{Var}(\mathbf{c}_{2i-1}) + \mathrm{Var}(\mathbf{c}_{2i}) + 2\mathrm{Cov}(\mathbf{c}_{2i-1}, \mathbf{c}_{2i})] = \frac{1}{2}\,(\mathbf{\Sigma}_c + \mathbf{\Sigma}_c^{(3)}), \qquad (3.54)$$

and

$$\mathrm{Cov}(\bar{y}_i, \bar{\mathbf{c}}_i) = \frac{1}{4}\,\mathrm{Cov}(y_{2i-1} + y_{2i}, \mathbf{c}_{2i-1} + \mathbf{c}_{2i})$$

$$= \frac{1}{4}\,[\mathrm{Cov}(y_{2i-1}, \mathbf{c}_{2i-1}) + \mathrm{Cov}(y_{2i-1}, \mathbf{c}_{2i}) + \mathrm{Cov}(y_{2i}, \mathbf{c}_{2i-1}) + \mathrm{Cov}(y_{2i}, \mathbf{c}_{2i})]$$

$$= \frac{1}{2}\,[\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(3)}]', \qquad (3.55)$$

where $\bar{\mathbf{c}}_i = (\mathbf{c}_{2i-1} + \mathbf{c}_{2i})/2$ $(i = 1, 2, \ldots, h)$. Under the normality assumption of the response and control variates, the joint distribution of $\bar{y}_i$ and $\bar{\mathbf{c}}_i$ is given by

$$\begin{bmatrix} \bar{y}_i \\ \bar{\mathbf{c}}_i \end{bmatrix} \sim N_{(s+1)}\left[\begin{bmatrix} \mu_y \\ \mathbf{0} \end{bmatrix}, \ \frac{1}{2}\begin{bmatrix} (1-\rho_3)\sigma_y^2 & \boldsymbol{\sigma}'_{yc} + \boldsymbol{\sigma}_{yc}^{(3)\prime} \\ \boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(3)} & \mathbf{\Sigma}_c + \mathbf{\Sigma}_c^{(3)} \end{bmatrix}\right]. \qquad (3.56)$$

Thus, given $\bar{\mathbf{c}}_i$, the conditional variance of $\bar{y}_i$ is

$$\mathrm{Var}(\bar{y}_i \mid \bar{\mathbf{c}}_i) = \frac{1}{2}\,[(1-\rho_3)\sigma_y^2 - (\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(2)})'(\mathbf{\Sigma}_c + \mathbf{\Sigma}^{(3)})^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(3)})]. \qquad (3.57)$$

(see Theorem 5.2.1 in Anderson (1984)). Based on the results of (3.55) and (3.56), we can represent the mean vector of the response pairs, $\bar{\mathbf{y}}$, as a linear model. Therefore, the controlled estimator of the mean response of interest is the same as in (3.20) if $\bar{\mathbf{y}}$ and the $(h \times s)$ matrix of control variates $\mathbf{C}$ are substituted by those of the simulation outputs obtained by this combined method. Thus, with the same in procedures (3.21)-(3.23), we get the conditional variance of $\hat{\mu}_y$ as

$$\mathrm{Var}(\hat{\mu}_y \mid \overline{\mathbf{C}}) = \frac{1}{2h^2}[(1-\rho_3)\sigma_y^2 - (\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(3)})'(\boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}_c^{(3)})^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(3)})] \times$$

$$[h + \mathbf{1}'\mathbf{C}(\mathbf{C}'\mathbf{P}\mathbf{C})^{-1}\mathbf{C}'\mathbf{1}]. \tag{3.58}$$

Also using the procedures in (3.24)-(3.29), we find the unconditional variance of the estimator $\hat{\mu}_y$ by replacing the covariance matrix in (3.22) with (3.57):

$$\mathrm{Var}(\hat{\mu}_y) = \frac{1}{2h}[(1-\rho_3)\sigma_y^2 - (\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(3)})'(\boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}_c^{(3)})^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(3)})](\frac{h-2}{h-s-2}).$$

$$= \frac{(1-\rho_3)\sigma_y^2}{2h}(1 - \overline{\overline{R}}{}_{yc}^2)(\frac{h-2}{h-s-2}), \tag{3.59}$$

where

$$\overline{\overline{R}}{}_{yc}^2 = [(1-\rho_3)\sigma_y^2]^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(3)})'(\boldsymbol{\Sigma}_c + \boldsymbol{\Sigma}_c^{(3)})^{-1}(\boldsymbol{\sigma}_{yc} + \boldsymbol{\sigma}_{yc}^{(3)}). \tag{3.59}$$

By definition, $\overline{\overline{R}}_{yc}$ is the multiple correlation coefficient between $\bar{y}_i$ and $\overline{\mathbf{c}}_i$. Equation (3.59) shows that the minimum variance ratio of this method is $(1-\rho_3)(1-\overline{\overline{R}}{}_{yc}^2)$ and its loss factor is the same as that in (3.46).

3.4. Comparison of Combined Methods

In this section, we compare the three methods developed in the previous sections and the method of control variates with respect to the unconditional variances of the estimators for the mean response, and summarize these results. We present comparisons of these methods as follows: (a) three combined methods for a single control variates case, (b) three combined method for multiple control variates case, and

(c) three combined methods with the method of control variates, respectively, for multiple control variates case.

For the case of a single control variate, comparing Combined Methods I and II via equations (3.15) and (3.38) yields that Combined Method I is preferred to Combined Method II if:

$$(1-\rho_1-\rho_{yc}^2) < (1-\rho_2)(1-\bar{\rho}_{yc}^2). \tag{3.61}$$

Similarly, working with (3.15) and (3.52) yields that Combined Method I is better than Combined Method III, provided

$$(1-\rho_1-\rho_{yc}^2) < (1-\rho_3)(1-\bar{\bar{\rho}}_{yc}^2). \tag{3.62}$$

Also, working with (3.38) and (3.52) yields that Combined Method III is better than Combined Method II if

$$(1-\rho_3)(1-\bar{\bar{\rho}}_{yc}^2) < (1-\rho_2)(1-\bar{\rho}_{yc}^2). \tag{3.63}$$

As we discussed earlier, the loss factors for the three combined methods are the same and given by $(h-2)/(h-s-2)$. Thus, the preference of the three methods is determined according to their minimum variance ratios in (3.15), (3.38) and (3.52), respectively. Generally, we conjecture that (a) ρ_3 is greater than ρ_1 and ρ_2 since the more stochastic components used for implementing antithetic variates, the higher the correlation of the response within a pair of replicates, and (b) ρ_2 is greater than ρ_1 if the control variate stochastic components are highly correlated with the response; that is, antithetic variates through the strongly correlated stochastic components yields greater correlation in the responses than antithetic variates through the non-control variates components. Based on this conjecture, we can assume that $\rho_1 < \rho_2 < \rho_3$. However, it is not easy to identify an ordered relationship among ρ_{yc}^2, $\overline{\rho}_{yc}^2$, and $\overline{\overline{\rho}}_{yc}^2$ since these terms include the unknown factors: σ_c^2, σ_{yc}, $\sigma_{yc}^{(2)}$, $\sigma_{yc}^{(3)}$, $\rho_{c,}\rho_2$ and ρ_3. Our empirical simulation results show that, generally, ρ_{yc} is greater than $\overline{\rho}_{yc}$ and $\overline{\overline{\rho}}_{yc}$. Based on the conditions in (3.61)-(3.63), and the above discussion, it is conjectured that (a) if $\rho_{yc}^2 >> \rho_3$, then Combined Method I is preferred to Combined Method III, and (b) Combined Method II may be comparable to Combined Method I if $\rho_2 + \overline{\rho}_{yc}^2 \cong \rho_1 + \rho_{yc}^2$, and if $\rho_2 + \overline{\rho}_{yc}^2 \cong \rho_3 + \overline{\overline{\rho}}_{yc}^2$, then it may be comparable to Combined Method III since the terms $\rho_2\overline{\rho}_{yc}^2$ and $\rho_3\overline{\overline{\rho}}_{yc}^2$ are small comparing to the other terms in equations (3.61)-(3.63).

Second, we consider the more general case of s control variates. Comparisons of the unconditional variances of $\hat{\mu}_y$ in (3.29), (3.46), and (3.60) show that Combined Method I is better than Combined Methods II and III, provided

$$(1-\rho_1-R_{yc}^2) < (1-\rho_2)(1-\overline{R}_{yc}^2). \qquad (3.64)$$

and

$$(1-\rho_1-R_{yc}^2) < (1-\rho_3)(1-\overline{\overline{R}}_{yc}^2). \qquad (3.65)$$

Similarly, in comparing (3.46) with (3.60), we get that Combined Method III is better than Combined Method II if

$$(1-\rho_3)(1-\overline{\overline{R}}_{yc}^2) < (1-\rho_2)(1-\overline{R}_{yc}^2). \qquad (3.66)$$

As in the case of a single control variate, it is conjectured that (a) if $R_{yc} >> \rho_3$, then Combined Method I is preferred to Combined Method III, and (b) Combined Method II may be comparable to Combined Methods I and III under the conditions that

$\rho_2 + \overline{R}^2_{yc} \cong \rho_1 + R^2_{yc}$ and $\rho_2 + \overline{R}^2_{yc} \cong \rho_3 + \overline{\overline{R}}^2_{yc}$. Empirically, our simulation studies for several models show that (a) $\rho_1 + R^2_{yc}$ and $\rho_3 + \overline{\overline{R}}^2_{yc}$ are greater than $\rho_2 + \overline{R}^2_{yc}$, and (b) $\rho_1 + R^2_{yc}$ is greater than $\rho_3 + \overline{\overline{R}}^2_{yc}$ except for one case.

Finally, we compare the three combined methods with the method of control variates for the case of s control variates. A comparison of equations (3.29) and (2.97) yields that Combined Method I is better than the control variates method if

$$(1 - \rho_1 - R^2_{yc})(\frac{h-2}{h-s-2}) < (1 - R^2_{yc})(\frac{2h-2}{2h-s-2}). \tag{3.67}$$

Also comparing equations (3.46), (3.60) and (2.97) shows that Combined Method II is better than the control variates method if

$$(1 - \rho_2)(1 - \overline{R}^2_{yc})(\frac{h-2}{h-s-2}) < (1 - R^2_{yc})(\frac{2h-2}{2h-s-2}); \tag{3.68}$$

and Combined Method III yields a better result than the control variates method, provided

$$(1 - \rho_3)(1 - \overline{\overline{R}}^2_{yc})(\frac{h-2}{h-s-2}) < (1 - R^2_{yc})(\frac{2h-2}{2h-s-2}). \tag{3.69}$$

As shown in equations (3.67)-(3.69), the loss factor of each combined method is greater than that of the control variates method: $(h-2)/(h-s-2) > (2h-2)/(2h-s-2)$. Hence, for preference of each combined method to the control variates method, the minimum variance ratio of each combined method should, at least, compensate for an increase in the associated loss factor. The effects of antithetic variates and control varaites to the minimum variance ratio for Combined Method I is represented by an additive form in reducing the variance of the estimator for the mean response. Also, the minimum variance ratios for Combined Methods II and III can be described as trade-off effects of the correlation between the paired responses and that between the response and control variates: (a) an increase of the correlations between the responses by antithetic streams through the control variates, and (b) a decrease of the correlation between the response and control variates (the reduced variances of the mean response and mean control variate in a pair of replications increase the correlation between the response and control variate, and the reduced covariance between the mean response and mean control variate decreases the correlation between the response and control variate. Totally these effects result in a decrease of the correlation between the response and control variates). Our simulation studies indicate that, for the case where good control variates can be defined, it seems to be undesirable to apply antithetic variates only through control variates (Combined Method II) for all cases. Also it shows that (a) Combined Method I yields better results than the method of control variates for most cases, and (b) Combined Method III yields better results than the method of control variates for some cases.

CHAPTER 4. APPLICATIONS OF COMBINED METHODS TO SIMULATION MODELS

In this chapter, we apply the three combined methods developed in Chapter 3 as well as methods of control variates and antithetic variates to various simulation models. Through experimentation on these selected models, we explore the efficiency gains of the combined methods which are expected to show additive effects of both antithetic variates and control variates in reducing the variance of the estimator for the mean response. The examples we consider are: (a) a closed and mixed machine-repair network (Wilson and Pritsker (1984a, 1984b)), (b) an open machine-repair network, (c) a hospital resource allocation model (Schruben and Margolin (1978)), and (d) a port operations model (Schriber (1974), and Pritsker (1986)). We conduct a set of simulation experiments on these models to evaluate the performance of the variance reduction techniques proposed earlier. To this end, for each example, we offer a summary and analysis of the results obtained by employing antithetic variates, control variates and each of the three combined methods.

The remainder of this chapter consists of five sections: Section 5.1 applies Combined Method I to the closed machine-repair network. Section 5.2 applies Combined Method I to the mixed machine-repair system. Section 5.3 implements Combined Methods I and III for an open machine-repair network which is a variation of the mixed machine-repair system. Section 5.4 applies the three combined methods to the hospital resource allocation model. Finally, Section 5.5 applies the three combined methods to the port operations model. In each section, we give a detailed description of the system and the simulation environment in which it is modeled. We also summarize the experimental results and provide inferences about these results obtained by each of the applied methods.

4.1. Closed Machine-Repair Network

This section conducts a simulation experiment on a closed machine-repair network. We summarize the simulation results and present inferences as to these results.

4.1.1. System and model description

A diagram of the closed machine-repair system is depicted in Figure 1. Initially, there are s_1 machines operating at Station 1 with a backup queue of s'_1 spares. The times to failure for operating machines are exponentially distributed with mean μ_1. With probability p_1, a failed unit requires a major repair at Station 2; there it joins a FIFO queue to wait for service from one of s_2 repairmen having exponential service time with mean μ_2. A minor repair has probability $(1-p_1)$, and it is performed on a FIFO basis at Station 3 by s_3 repairmen having exponential service times with mean μ_3. Both types of repaired units then proceed to Station 4 where s_4 inspectors test the units in FIFO order. Inspection times are exponentially distributed with mean μ_4. A unit fails the test with probability $(1-p_2)$. Then it is sent back to Station 3 for an additional minor repair. A repaired unit has probability p_2 of passing the test. A passed unit returns to the pool of spares and goes directly into service if less than s_1 units

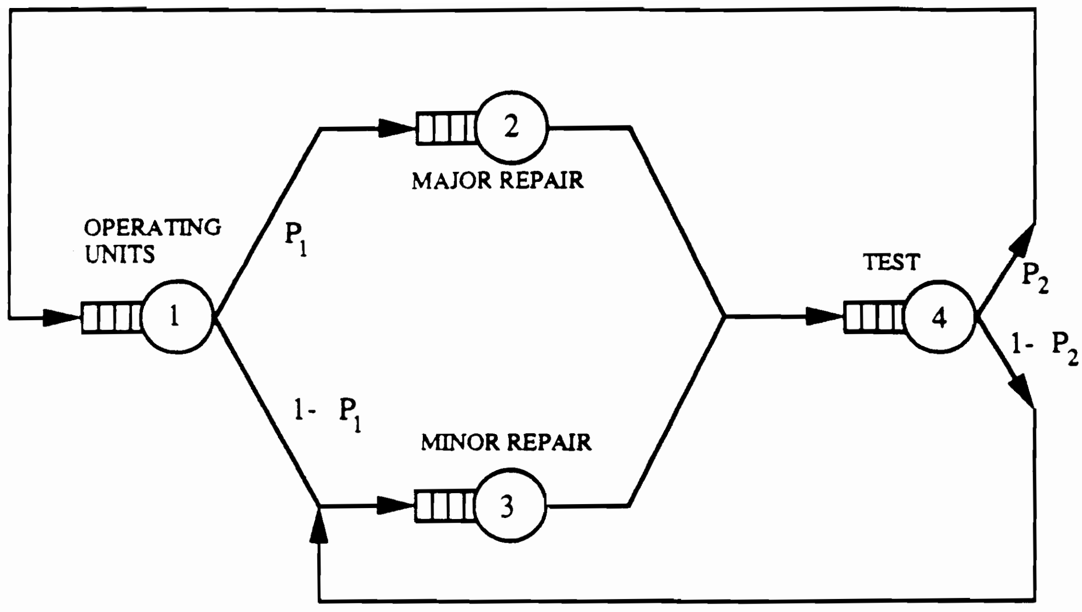


Figure 1. Closed Machine-Repair System

are operating. Otherwise, it waits for operation at Station 1 in FIFO order (For a more detailed description of this system, see Wilson and Pritsker (1984b)).

In this system, the following system responses are to be estimated: (a) the average number of operating units at Station 1; (b) the average time for a failed unit to return to Station 1; and (c) the average number of busy servers at Stations 2, 3 and 4. We conducted 200 simulation runs of this system with the same parameters as given in Wilson and Pritsker (1984a), which are presented in Table 4.

The closed machine-repair network consists of six stochastic components including two probabilistic branches. For the ith replication, we used six separate random number streams, that is,

$$\mathbf{R}_i = (\mathbf{r}_{i1}, \mathbf{r}_{i2}, \ldots, \mathbf{r}_{i6}),$$

for driving each stochastic component as indicated in Table 5. For a replication of each simulation method, six selected seeds are given for the six random number streams, $\mathbf{R}_i$, used by the SLAM II program. In all, the simulation program uses 200 sets of six randomly selected random number seeds. Of course these random number seeds are selected differently across replications according the methodological prescriptions given in Chapter 3.

Direct simulation (independent replications across replicates with no use of control variates) uses the 200 different sets of randomly selected set of six random number streams, $\mathbf{R}_i$. Antithetic variates uses randomly selected random number streams, $\mathbf{R}_i$, and its complementary random number streams, $\overline{\mathbf{R}}_i$ within a pair of replicates. However, antithetic variates uses randomly selected six streams across the

Table 4. Parameter Specification for Closed Machine Repair System

Parameter	Station 1	Station 2	Station 3	Station 4
Number of Servers:	$s_1 = 5$; $s'_1 = 2$	1	1	1
Mean Service Time:	10.0	1.5	1.0	0.5
Branching Probability:	$p_1 = 0.25$		$p_2 = 0.9$	

Table 5. Random Number Assignment for Closed Machine-Repair Network

Stream Number	Stochastic Process to be Sampled
1	Operating Time at Station 1
2	Branching Probability at Station 1
3	Service Time at Station 2
4	Service Time at Station 3
5	Test Time at Station 4
6	Branching Probability at Station 4

100 pairs of replications. The method of control variates applies the same assignment strategy as direct simulation.

As we discussed earlier, the efficiency of the control variates method is dependent on the selection of good control variates. For the closed machine-repair network, we can consider four control variates associated with the observed service times at each of the four stations. Usually, we select an appropriate set of control variates for each response of interest based on regression analysis on the pilot runs (see Wilson and Pritsker (1984b) and Nozari, Arnold and Pegden (1984)). Wilson and Pritsker (1984b) performed stepwise regression analysis on this set of four control variates and identified which set of yielded the optimal efficiency gain for estimating each response of interest (see Table 10 of Wilson and Pritsker (1984b)).

In implementing the method of control variates, we only consider two control variates based on regression analysis since the purpose of this example is to study the efficiency gain of control variates with the Combined Method I. First, we used the two best control variates with respect to the average time for a failed unit to return to Station 1. Second, we used the two worst control variates for the same response. That is, the first experiment uses the control variates of the machine operating times at Station 1 and the service times at Station 3, and the second uses the control variates of the service times at Stations 2 and 4. For both cases, we constructed the following standardized control variates defined, for the ith replication and the kth stochastic service time component as:

$$c_{ik} = \frac{1}{\sqrt{a_i(k, t)}} \sum_{j=1}^{a_i(k,t)} \frac{(s_{ij}(k) - \mu_k)}{\sigma_k} \quad \text{for } k = 1, 2, 3, 4; \tag{4.1}$$

where μ_k and σ_k are the known mean and standard deviation of the service time at station k; $s_{ij}(k)$ ($j = 1, 2, \ldots, a_i(k, t)$) are the random observations of the kth station service time in the ith replication; and $a_i(k, t)$ is the number of observations of the kth station service time in the ith replication during the simulation time (0, t) with $t = 950$.

Next, we discuss how Combined Method I was implemented for this system. In conducting the simulations based on Combined Method I, we separated the set of six random number streams, $\mathbf{R}_i$, into two mutually exclusive and exhaustive subsets ($\mathbf{R}_{i1}$, $\mathbf{R}_{i2}$). The first subset, $\mathbf{R}_{i1}$, consists of four random number streams used for the non-control variates components in the model. The second subset, $\mathbf{R}_{i2}$, of two random number streams is used for driving the control variate components. Combined Method I uses randomly selected $\mathbf{R}_i = (\mathbf{R}_{i1}, \mathbf{R}_{i2})$ across the 100 pairs of replications. However, within each pair of replications, it uses randomly selected random number streams ($\mathbf{R}_{i1}$, $\mathbf{R}_{i2}$) for the first replicate, and employs random number streams ($\overline{\mathbf{R}}_{i1}$, $\mathbf{R}_{i2}$) for the second replicate, where $\overline{\mathbf{R}}_{i1}$ is antithetic to $\mathbf{R}_{i1}$ and $\mathbf{R}_{i2}$ is randomly selected. For instance, consider the case where the standardized control variates of the major repair time and test time are used. Then, we randomly select seeds for random number streams 3 and 5 across the 200 replications. The other four streams used to drive the non-control variates have randomly selected seeds for the first replicate in each of 100 pairs of replications, and its antithetic seed selected for the second replicate in each of the 100 pairs of replicates.

The simulation model of the closed machine-repair system was coded in SLAM II and run on the IBM 3090 computer at Virginia Polytechnic Institute and State University. The code is given in Appendix B-1. One replication consists of simulating the machine repair process for 1000 time units. To eliminate the initial condition bias, the necessary statistics are collected after a warm-up period of length 50 time units.

4.1.2. Experimental results

To provide an assessment of the efficiency gain obtained by Combined Method I, we calculated performance statistics on it as well as the control variates method and the antithetic variates method based on: (a) the percentage reduction in variance and (b) the percentage reduction in width of a nominal 90% confidence interval. Before we present an experimental evaluation of Combined Method I, we address the computational procedure for obtaining the estimator of the mean response and its variance, and the confidence interval of the mean response based on the simulation outputs in the context of (1) direct simulation (simple estimator), (2) antithetic variates, (3) control variates, and (4) Combined Method I.

First, we considered the case where we estimated the mean response of interest without using control variates. Let $2h$ be the number of independent replications and y_i be the observation of the response on the ith replication. When we perform independent runs through $2h$ replications, the mean response μ_y is estimated by its sample mean, $\bar{y} = \sum_{i=1}^{2h} y_i/2h$, and the $(1-a)$-level confidence interval of μ_y is given by

$$\bar{y} \pm t_{1-a/2}(2h-1)\sqrt{\hat{V}_1/2h}\,, \tag{4.2}$$

where $\hat{V}_1$ is the sample variance of y_i $(i = 1, 2, \ldots, 2h)$ and $t_{1-a/2}(2h-1)$ is the upper $a/2$-percentile point of the t-distribution with $(2h-1)$ degrees of freedom. In the context of antithetic variates, the mean response is also estimated by the sample mean response $\bar{y}$ and the confidence interval of μ_y with confidence level $(1-a)$ is given by

$$\bar{y} \pm t_{1-a/2}(h-1)\sqrt{\hat{V}_2/h}\,, \tag{4.3}$$

where $\hat{V}_2$ is the sample variance of the h mean pair responses, $\bar{y}_i$ $(i = 1,\ 2,\ \ldots,\ h)$, and $t_{1-a/2}(h-1)$ is the upper $a/2$-percentile point of the t-distribution with $(h-1)$ degrees of freedom.

Second, we calculate the estimator of the mean response based on the control variates method. Let $\mathbf{c}_i$ be a vector of s control variates observed at the ith independent replication: $\mathbf{c}'_i = (c_{i1},\ c_{i2},\ \ldots,\ c_{is})$. Also let $\mathbf{C}$ be the $(2h \times s)$ matrix of control variates whose ith row consists of $\mathbf{c}'_i$. Based on regression analysis, the controlled estimator of the mean response is

$$\hat{\mu}_y = \bar{y} - \hat{\boldsymbol{\alpha}}'\bar{\mathbf{c}} = \bar{y} - \mathbf{S}'_{yc}\mathbf{S}_c^{-1}\bar{\mathbf{c}}, \tag{4.4}$$

where $\bar{y}$ and $\bar{\mathbf{c}}$ are the sample means of y_i and $\mathbf{c}_i$ respectively; $\hat{\boldsymbol{\alpha}}$ is the estimator for the coefficient vector of control variates; $\mathbf{S}_c$ is the sample covariance of $\mathbf{c}_i$; and $\mathbf{S}_{yc}$ is the sample covariance matrix between y_i and $\mathbf{c}_i$ $(i = 1, 2,\ \ldots,\ 2h)$. In terms of the residual mean square of the linear model in (2.70),

$$\hat{\sigma}^2_{\varepsilon^*} = \frac{\sum_{i=1}^{2h}[y_i - \hat{\mu}_y]^2}{2h - s - 1}, \tag{4.5}$$

the estimator for the variance of $\hat{\mu}_y$ is given by

$$\hat{V}_3 = s_{11}\hat{\sigma}^2_{\varepsilon^*},$$

where s_{11} denotes the first row entry in the first column of $(\mathbf{G}'\mathbf{G})^{-1}$, with $\mathbf{G} = (\mathbf{1}_{2h}, \mathbf{C})$ (see equation (2.95)). Then the $(1-a)$-level confidence interval of μ_y is given by

$$\hat{\mu}_y \pm t_{1-a/2}(2h - s - 1) \times \sqrt{s_{11}}\, \hat{\sigma}_{\varepsilon^*}, \tag{4.6}$$

where $t_{1-a/2}(2h - s - 1)$ is the upper $a/2$-percentile of the t-distribution with $(2h - s - 1)$ degrees of freedom.

Now we present computational procedures on the sample estimator for the variance of the mean response and the $(1-a)$-level confidence interval of μ_y when Combined Method I is applied. The estimators for the coefficients of control variates, $\boldsymbol{\alpha}$, and the mean response, μ_y, are given by, respectively,

$$\hat{\boldsymbol{\alpha}} = \mathbf{S}_c^{-1}\mathbf{S}_{yc} \tag{4.7}$$

and

$$\hat{\mu}_y = \bar{y} - \hat{\boldsymbol{\alpha}}'\bar{\mathbf{c}}, \tag{4.8}$$

where $\bar{y}$ and $\bar{\mathbf{c}}$ are the sample mean of h pairs of $\bar{y}_i$ and $\bar{\mathbf{c}}_i$. Note that $\bar{\mathbf{c}}_i$ is the ith sample pair mean control vector described in Section 3.1. Also,

$$\mathbf{S}_c = \frac{1}{h-1}\sum_{i=1}^{h}(\bar{\mathbf{c}}_i - \bar{\mathbf{c}})(\bar{\mathbf{c}}_i - \bar{\mathbf{c}})'; \tag{4.9}$$

and

$$\mathbf{S}_{yc} = \frac{1}{h-1}\sum_{i=1}^{h}(\bar{y}_i - \bar{y})(\bar{\mathbf{c}}_i - \bar{\mathbf{c}}). \tag{4.10}$$

Based on regression analysis, the residual mean square of this method is given by

$$\hat{\sigma}^2_{\varepsilon^*} = \frac{\sum_{i=1}^{h} [\bar{y}_i - \hat{\mu}_y]^2}{h - s - 1} \tag{4.11}$$

(see Myers (1986), p. 53), and the variance estimator of $\hat{\mu}_y$ is given by

$$\hat{V}_4 = s_{11} \hat{\sigma}^2_{\varepsilon^*}, \tag{4.12}$$

where s_{11} denotes the first-row and the first-column element of $(\mathbf{D'D})^{-1}$, with

$$\mathbf{D} = \begin{bmatrix} 1 & \bar{\mathbf{c}}'_1 \\ 1 & \bar{\mathbf{c}}'_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \bar{\mathbf{c}}'_h \end{bmatrix}. \tag{4.13}$$

Thus, the $(1 - a)$-level confidence interval of μ_y is given by

$$\hat{\mu}_y \pm t_{1-a/2}(h - s - 1) \times \sqrt{s_{11}}\, \hat{\sigma}_{\varepsilon^*}, \tag{4.14}$$

where $t_{1-a/2}(h - s - 1)$ is the upper $a/2$-percentile of the t-distribution with $(h - s - 1)$ degrees of freedom.

We measure the performance of each method by percentage reductions in variance of the estimator $\hat{V}_m$ $(m = 1, 2, 3)$, and half-length of the $(1 - a)$-level confidence interval of μ_y with respect to those obtained by direct simulation (that is, independent streams across all replications with no use of control variates). For the kth response in a given model, we let

$\hat{V}_{m,k}$ = the sample estimator of the variance of $\hat{\mu}_y$, and

$\hat{H}_{m,k}$ = half-length of the $(1-a)$-level confidence interval corresponding to $\hat{V}_{m,k}$.

With respect to this notation, we have

$$\text{variance reduction (\%)} = 100 \times \frac{[\hat{V}_{1,k} - \hat{V}_{m,k}]}{\hat{V}_{1,k}} \tag{4.15}$$

and

$$\text{confidence interval half-length reduction (\%)} = 100 \times \frac{[\hat{H}_{1,k} - \hat{H}_{m,k}]}{\hat{H}_{1,k}}. \tag{4.16}$$

Now we summarize the final results on two sets of simulation experiments obtained by direct simulation, antithetic variates, control variates and Combined Method I. Note that antithetic variates is applied through Station 1 and 3 service times for the first experiment (Table 5), and through Station 2 and 4 service times for the second experiment (Table 7). For the case where the two controls of Station 1 and 3 service times are used, (a) Table 6 presents the results on the estimators of the mean responses of interest, their variance estimators and percentage reductions of the variance estimators, respectively, (b) Table 7 gives the results on half-length of the 90% confidence intervals and their percentage reductions. Table 8 and 9 summarize the simulation results on the same statistics as given in Table 6 and 7, respectively, when the service times of Stations 2 and 4 are used as the two control variates.

Table 6. Percentage Reduction in Variance for Closed Machine-Repair Network: use two control variates of operating time and minor-repair time.

	Direct Simulation		Antithetic variates			Control Variates			Combined Method I		
Estimator of Parameter	Mean	Variance	Mean	Variance	Variance Reduction	Mean	Variance	Variance Reduction	Mean	Variance	Variance Reduction
Response time	2.665	0.467×10^{-1}	2.679	0.466×10^{-1}	0.09	2.646	0.362×10^{-1}	22.52	2.654	0.449×10^{-1}	3.87
Utilization of Station 1	4.780	0.195×10^{-2}	4.776	0.220×10^{-2}	-12.75	4.788	0.995×10^{-3}	49.04	4.786	0.102×10^{-2}	47.59
Utilization of Station 2	0.180	0.552×10^{-3}	0.179	0.387×10^{-3}	31.52	0.178	0.517×10^{-3}	0.92	0.177	0.380×10^{-3}	31.16
Utilization of Station 3	0.413	0.777×10^{-3}	0.416	0.804×10^{-3}	-3.47	0.408	0.161×10^{-3}	79.62	0.409	0.168×10^{-3}	78.38
Utilization of Station 4	0.266	0.251×10^{-3}	0.265	0.173×10^{-3}	31.08	0.264	0.171×10^{-3}	32.09	0.263	0.070×10^{-3}	72.11

Table 7. Percentage Reduction in 90% Confidence Interval and for Closed Machine-Repair Network: use two control variates of operating time and minor-repair time.

	Half-Length of Confidence Interval			Reduction (%)	
Estimator of Parameter	Direct Simulation	Control Variates	Combined Method I	Control Variates	Combined Method I
Response Time	0.0251	0.0221	0.0249	11.93	0.88
Utilization of Station 1	0.0051	0.0037	0.0038	28.60	26.81
Utilization of Station 2	0.0027	0.0026	0.0023	3.22	16.12
Utilization of Station 3	0.0032	0.0015	0.0015	54.48	52.99
Utilization of Station 4	0.0018	0.0015	0.0010	17.46	46.61

Table 8. Percentage Reduction in Variance for Closed Machine-Repair Network: use two control variates of major-repair time and test time.

	Direct Simulation		Antithetic variates			Control Variates			Combined Method I		
Estimator of Parameter	Mean	Variance	Mean	Variance	Variance Reduction	Mean	Variance	Variance Reduction	Mean	Variance	Variance Reduction
Response Time	2.659	0.385×10^{-1}	2.687	0.275×10^{-1}	28.66	2.658	0.356×10^{-1}	7.67	2.687	0.262×10^{-1}	31. 91
Utilization of Station 1	4.780	0.205×10^{-2}	4.778	0.101×10^{-3}	50.95	4.780	0.203×10^{-2}	1.22	4.779	0.925×10^{-3}	54.90
Utilization of Station 2	0.179	0.536×10^{-3}	0.178	0.497×10^{-3}	7.28	0.180	0.287×10^{-3}	46.48	0.179	0.193×10^{-3}	64.00
Utilization of Station 3	0.411	0.804×10^{-3}	0.411	0.334×10^{-3}	58.46	0.411	0.812×10^{-3}	-0.98	0.411	0.339×10^{-3}	58.70
Utilization of Station 4	0.266	0.268×10^{-3}	0.267	0.200×10^{-3}	25.37	0.265	0.123×10^{-3}	54.18	0.266	0.061×10^{-3}	77.24

Table 9. Percentage Reduction in 90% Confidence Interval for Closed Machine-Repair Network: use two control variates of major-repair time and test time.

	Half-Length of Confidence Interval			Reduction (%)	
Estimator of Parameter	Direct Simulation	Control Variates	Combined Method I	Control Variates	Combined Method I
Response Time	0.0228	0.0219	0.0191	3.91	16.58
Utilization of Station 1	0.0053	0.0052	0.0036	0.61	32.11
Utilization of Station 2	0.0027	0.0020	0.0016	28.83	39.34
Utilization of Station 3	0.0033	0.0033	0.0021	-0.50	35.04
Utilization of Station 4	0.0019	0.0013	0.0009	32.25	52.17

4.1.3. Inferences

When antithetic variates is applied through the two stochastic variates of the major repair time and test time, the variance reduction is in the range of -10% to 30%. On the other hand, when antithetic variates is implemented through the operating time and the minor service time, the variance is reduced from 5% to 60%. For the method of control variates, we observed the following: (a) with two controls consisting of the operating time and the minor-repair time, variance reduction of the estimators is in the range from 20% to 80% and confidence interval reduction is from 10% to 55% for the mean response time and the three utilizations of major-repair, minor-repair and test stations; and (b) with two controls consisting of the major-repair and test time, variance reduction in the range from 7% to 55% and confidence reduction in the range from 4% to 30% for the mean response time, and two utilizations of the major-repair and minor-repair stations.

As we expected, we observe that the variance reduction of the Combined Method I shows the additive effect of both antithetic variates and control variates. The only exception to case is the estimator of the mean response time when the two controls of the operating time and minor-repair service time are used. Except this case, the reduction ranges in variance and confidence interval of Combined Method I are respectively, from 30% to 80% and from 16% to 50%.

4.2. Mixed Machine-Repair Network

To gain more insight into Combined Method I, this section treats two sets of simulation runs on a mixed machine-repair network under the same scenarios as in Section 4.1. We present the summary of experimental results and their inferences.

4.2.1. System and model description

Consider a situation in which the closed machine-repair system in Section 4.1.1 starts to receive orders for major repairs from the exogenous system in addition to the repairs of regular operation units. Also, suppose the exogenous orders have higher priority than the regular units. A mixed machine-repair system arises by superimposing an exogenous stream of higher priority units on the closed machine-repair model (see Figure 2). Priority orders arrive at Station 2 for major repair with their interarrival times exponentially distributed with mean 8.0. At each station they visit, the priority units are served ahead of any regular units that are waiting in queue. After testing at Station 4, a priority unit has probability 0.1 of going to Station 3 for a minor adjustment. Otherwise, it leaves the system. The process of regular units and the system parameters are the same as in the closed machine-repair network in Section 4.1.1 (see the description of Wilson and Pritsker (1984b)).

For the mixed machine-repair network, our interest focused on the mean sojourn time of priority units in addition to the utilizations of the four stations, and the mean response time of regular units. To generate observations from the eight stochastic variables in this system, we used eight separate random number streams in the SLAM II program. Assignment of the corresponding random number seeds is given in Table 10. Direct simulation and the method of control variates use the different 200 sets of randomly selected eight random number streams. Antithetic variates uses random number streams across the 100 pairs of replications as the same way in direct simulation, but it uses antithetic streams within a pair of replicates.

In applying the method of control variates, we collected five control variates comprised of the service times at the four stations and the arrival process of priority

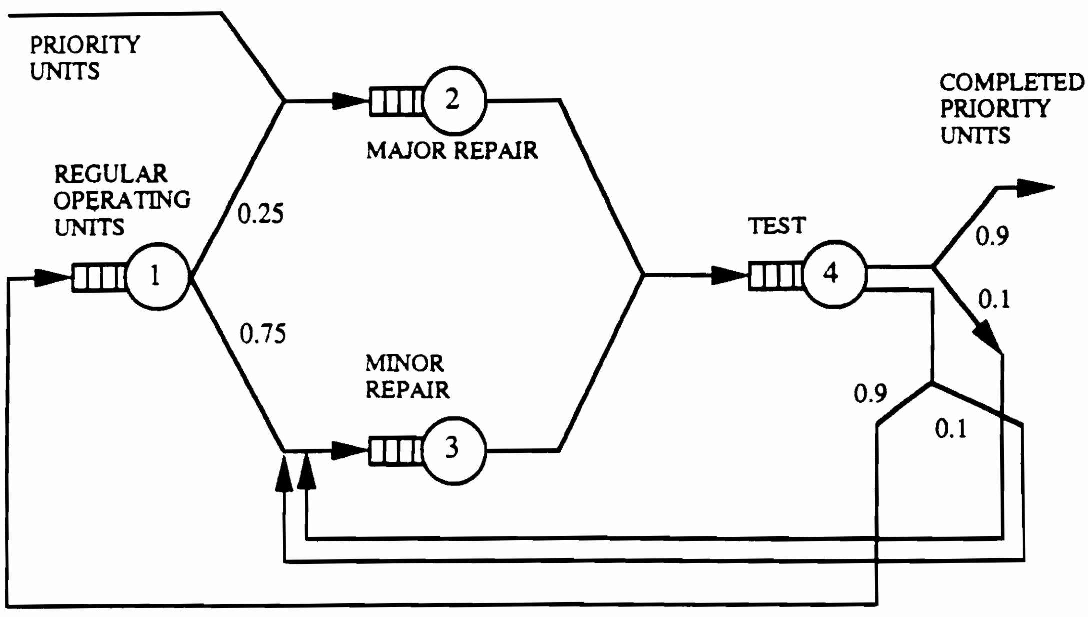


Figure 2. Mixed Machine-Repair System

Table 10. Random Number Assignment for Mixed Machine-Repair Network

Stream Number	Stochastic Process to be Sampled
1	Regular Unit Operating Time at Station 1
2	Branching Probability at Station 1
3	Service Time at Station 2
4	Service Time at Station 3
5	Test Time at Station 4
6	Branching Probability of Regular Unit at Station 4
7	Arrival Process of Priority Unit
8	Branching Probability of Priority Unit at Station 4

units, and chose the best subset of control variates for each response of interest. To see the effect of Combined Method I in comparison to the effects of control variates and antithetic variates, we employed the same sets of control variates used in Section 4.1.1. That is, the first experiment used the two standardized service times control variates at Stations 1 and 3, and the second used the two standardized service times control variates at Stations 2 and 4. Thus, for both experiments, we built the same type of control variates as in (4.1).

In using Combined Method I, we separated the set of eight random number streams into two mutually exclusive subsets including six streams and two streams, respectively. We then assigned the first set of six streams to each non-control variates stochastic process and the second set of two streams to the random processes generating the two control variates. Across 100 pairs of replicates, this method randomly selects eight random number streams. Within a pair of replications, the first replicate uses eight randomly selected random number streams for the corresponding stochastic components, but the second replicate uses random number streams antithetic to those used in the first replicate for the non-control variates and randomly selects streams for the control variates.

SLAM II program for this model was run on the IBM 3090 computer at Virginia Polytechnic Institute and State University. The program code is given in Appendix B-2. We simulated this model for the same period as in the experimentation of the closed machine-repair model for each method. We also started the collection of statistics after 50 time units in an effort to reduce the effect of initial bias.

4.2.2. Experimental results

Based on the computational procedures in Section 4.1.2, we obtained the estimators of the mean responses described before, their sample variance, and the 90% confidence intervals of the mean responses. For the case of the two service time control variates at Stations 2 and 4, Table 11 presents the final results on the estimators of the mean responses, their variance estimators and percentage reductions in variances; Table 12 provides the results on half-length of the 90% confidence intervals and the percentage reduction of each method. Table 13 and 14 summarize the results for the same statistics as shown in Tables 11 and 12, respectively, when the two service time control variates for Stations 1 and 3 were used.

4.2.3. Inferences

As shown in Tables 11 and 13, Combined Method I shows the additive effect of antithetic and control variates in reducing the variance of the estimator for the mean response. Notably, Table 11 indicates that the percentage reduction of the combined method I seems to be close to the sum of percentage reductions of both antithetic variates and control variates individually.

Antithetic variates through the two service time variates at Stations 2 and 4 reduces the variance of the estimator in the range from 10% to 55% except for the mean response time of priority unit. Also antithetic variates through the two service time variates at Stations 1 and 3 gives percentage reduction in variance from 10% to 45% for the response times of priority and regular units, and the utilizations at Stations 2 and 4. However, the variances of utilizations at Stations 1 and 3 are in-

Table 11. Percentage Reduction in Variance for Mixed Machine-Repair Network: use two control variates of major-repair time and test time.

	Direct Simulation		Antithetic variates			Control Variates			Combined Method I		
Estimator of Parameter	Mean	Variance	Mean	Variance	Variance Reduction	Mean	Variance	Variance Reduction	Mean	Variance	Variance Reduction
Response Time of Regular Unit	2.961	0.514×10^{-1}	2.978	0.454×10^{-1}	11.75	2.958	0.477×10^{-1}	7.34	2.974	0.416×10^{-1}	19.15
Response Time of Priority Unit	3.106	0.920×10^{-1}	3.090	0.104	-13.15	3.101	0.466×10^{-1}	49.42	3.098	0.586×10^{-1}	36.30
Utilization of Station 1	4.733	0.293×10^{-2}	4.732	0.230×10^{-2}	21.51	4.733	0.271×10^{-2}	7.71	4.733	0.209×10^{-2}	28.73
Utilization of Station 2	0.366	0.109×10^{-2}	0.363	0.813×10^{-3}	25.34	0.366	0.525×10^{-3}	51.75	0.364	0.301×10^{-3}	72.36
Utilization of Station 3	0.422	0.815×10^{-3}	0.422	0.366×10^{-3}	55.09	0.422	0.824×10^{-3}	-1.11	0.422	0.380×10^{-3}	53.37
Utilization of Station 4	0.333	0.363×10^{-3}	0.334	0.264×10^{-3}	22.27	0.332	0.144×10^{-3}	60.18	0.333	0.074×10^{-3}	79.61

Table 12. Percentage Reduction in 90% Confidence Interval for Mixed Machine-Repair Network: use two control variates of major-repair time and test time.

	Half-Length of Confidence Interval			Reduction (%)	
Estimator of Parameter	Direct Simulation	Control Variates	Combined Method I	Control Variates	Combined Method I
Response Time of Regular Unit	0.0264	0.0254	0.0240	3.74	9.10
Response Time of Priority Unit	0.0353	0.0251	0.0258	28.88	19.32
Utilization of Station 1	0.0063	0.0061	0.0054	3.93	14.66
Utilization of Station 2	0.0038	0.0027	0.0023	30.57	40.20
Utilization of Station 3	0.0033	0.0033	0.0023	-0.55	30.97
Utilization of Station 4	0.0022	0.0014	0.0010	37.02	54.36

Table 13. Percentage Reduction in Variance for Mixed Machine-Repair Network: use two control variates of operating time and minor repair time.

	Direct Simulation		Antithetic variates			Control Variates			Combined Method I		
Estimator of Parameter	Mean	Variance	Mean	Variance	Variance Reduction	Mean	Variance	Variance Reduction	Mean	Variance	Variance Reduction
Response Time of Regular Unit	2.991	0.593×10^{-1}	2.998	0.537×10^{-1}	9.36	2.976	0.544×10^{-1}	8.18	2.967	0.398×10^{-1}	32.83
Response Time of Priority Unit	3.089	0.874×10^{-1}	3.107	0.768×10^{-1}	12.12	3.088	0.925×10^{-1}	-5.82	3.109	0.908×10^{-1}	-3.90
Utilization of Station 1	4.729	0.248×10^{-2}	4.725	0.284×10^{-2}	-14.57	4.737	0.136×10^{-2}	44.92	4.736	0.151×10^{-2}	39.16
Utilization of Station 2	0.366	0.103×10^{-2}	0.364	0.559×10^{-3}	45.57	0.365	0.103×10^{-2}	-0.49	0.363	0.646×10^{-3}	37.10
Utilization of Station 3	0.422	0.837×10^{-3}	0.426	0.900×10^{-3}	-7.53	0.417	0.188×10^{-3}	77.58	0.419	0.176×10^{-3}	78.97
Utilization of Station 4	0.333	0.309×10^{-3}	0.332	0.188×10^{-3}	39.81	0.331	0.243×10^{-3}	21.28	0.330	0.102×10^{-3}	66.99

Table 14. Percentage Reduction in 90% Confidence Interval for Mixed Machine-Repair Network: use two control variates of operating time and minor-repair time.

	Half-Length of Confidence Interval			Reduction (%)	
Estimator of Parameter	Direct Simulation	Control Variates	Combined Method I	Control Variates	Combined Method I
Response Time of Regular Unit	0.0283	0.0271	0.0235	4.18	17.15
Response Time of Priority Unit	0.0344	0.0354	0.0354	-2.87	-3.05
Utilization of Station 1	0.0058	0.0043	0.0046	25.79	21.15
Utilization of Station 2	0.0037	0.0037	0.0009	-0.24	25.38
Utilization of Station 3	0.0034	0.0016	0.0016	52.61	53.64
Utilization of Station 4	0.0020	0.0018	0.0012	11.31	41.92

creased by around 10%. The method of control variates reduces the variances in the range from 40% to 80%, except for the response time of regular unit, when an effective set of two control variates for each response of interest is selected. Variance reduction due to Combined Method I is in the range from 20% to 80% for the two service time control variates at Stations 2 and 4, and from 30% to 80% for the two control variates at Stations 1 and 3 (except for the response time of priority unit). With reference to the confidence intervals, we see a similar trend of reduction as shown in variance reduction.

We now compare the performance of each method for this model with that resulting in the closed model. We first note that the performance of each method for both models is similar in estimating the utilizations of the four stations. The method of control variates yields a better result for the closed model than for the mixed model in estimating the response time of the regular unit. However, Combined Method I shows better results for the mixed model.

4.3. Open Machine-Repair Network

This section conducts simulation experiments on an open machine-repair network for two different arrival processes using the variance reduction techniques considered in Sections 4.1 and 4.2 and Combined Method III. We summarize the simulation results and provide inferences as to these results.

4.3.1. System and model description

Consider a mixed machine-repair system which only deals with the orders for major repair from the exogenous system. An open machine-repair system arises in

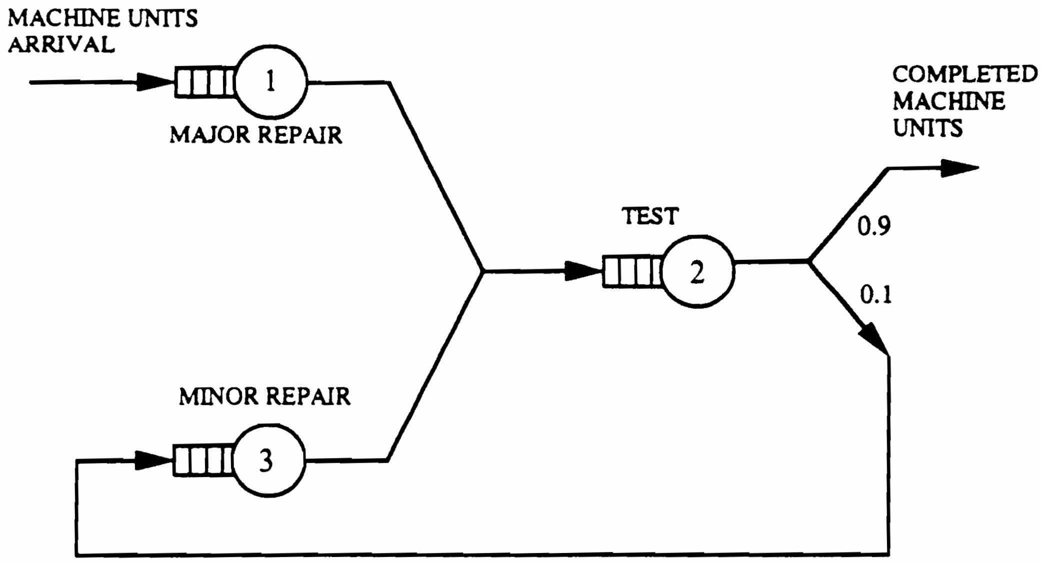


Figure 3. Open Machine-Repair System

this situation, whose diagram is presented in Figure 3. Machines which require major repair arrive at Station 1 with exponentially distributed interarrival times having mean 8.0 time units. The service time at the major repair station is exponentially distributed with mean 1.5 time units. At this station, one repairman performs repair service on a FIFO basis. After completing the major repair, a machine unit proceeds to Station 2 where one inspector tests units in a FIFO order. Inspection time is exponentially distributed with mean 0.5 time units. After test, a unit leaves the system with probability 0.9. If it fails test, it goes to Station 3 for the minor repair, where one repairman adjusts the failed unit with a service duration that is exponentially distributed with mean 1.0 time units. An adjusted machine unit at Station 3 is sent back to test station for the additional test.

Based on 200 simulation runs conducted on this system, the average mean sojourn time of the machine unit is to be estimated in addition to the average service utilizations of the three service stations. For implementing each replication, we used five separate random number streams for driving stochastic components as shown in Table 14. 200 independent sets of randomly selected streams were used in the direct simulation. Antithetic variates uses randomly selected five streams for odd replications, and complementary streams for even replicates, respectively. Thus, across 100 pairs of replications, antithetic variates selects a set of five random number streams in the same way as direct simulation. In using the control variates method, we can select a effective set of control variates associated with the random observations of the interarrival times, and the service times at the three stations by regression analysis on the pilot run. However, we simply use the three standardized service-time control variates in experimentation of this model.

Table 15. Random Number Assignment for Open Machine-Repair Network

Stream Number	Stochastic Process to be Sampled
1	Arrival Process of Priority Unit
2	Service Time at Station 1
3	Service Time at Station 2
4	Test Time at Station 3
5	Branching Probability at Station 3

In applying Combined Methods I and III, we employed the same set of control variates used in the control variate method. We divided the five random number streams into two mutually exclusive subsets; the first subset consists of random number streams 1 and 5 which are assigned to the two non-control variates random components, and the second set consists of random number streams 2, 3 and 4 which are assigned to the three control variates (see Table 15). Within a pair of replications, the first replicate of either Combined Method I or Combined Method III uses randomly selected random number streams, the second replicate of Combined Method I uses the antithetic streams to streams 1 and 5 used in the first replicate, and randomly selects streams 2, 3 and 4. The second replication of Combined Method III uses streams that are antithetic to those used in the first replicate. However, across the 100 pairs of replicates, both methods use randomly selected random number streams.

This model was also coded in SLAM II and run on the IBM 3090 at Virginia Polytechnic Institute and State University. The program code is given in Appendix B-3. Simulation run time and the collection period of statistics are the same as in Section 4.1.

As an embellishment of this model, we considered a case where Station 1 has a high utilization. Note that the original model has a small utilization factor at station 1 by excluding the regular units from the mixed machine-repair model (queue length of at each station will be small). We explore the performance of each method for the different arrival rate of the machine units to the system. The embellished system changes only the mean interarrival time of major repair orders to the system from 8.0

to 2.0 time units. Using the methods described above, we simulated this model under the same conditions as in the original model.

4.3.2. Experimental results

In this section, we summarize the simulation results on both the original and embellished models described above. Using the computational procedures in Section 4.2.1, we estimated the responses of interest, their sample variances and the 90% confidence intervals for the responses. For Combined Method III, we obtained the sample estimator of the variance of $\hat{\mu}_y$ and constructed the confidence interval of μ_y with the same procedure as in Combined Method I. The estimator of each response is the same as in (4.8) if we substitute y_i, $\mathbf{c}_i$ and $\hat{\alpha}$ with those obtained by Combined Method III. Similar to (4.12) and (4.14), we have the variance estimator of $\hat{\mu}_y$ and the $(1-a)$-level confidence interval of μ_y is given by

$$\hat{V}_6 = s_{11}\hat{\sigma}^2_{\varepsilon^*}, \tag{4.17}$$

where $\hat{\sigma}^2_{\varepsilon^*}$ is the residual mean square under Combined Method III and s_{11} denotes the first-row and first-column element of $(\mathbf{D}'\mathbf{D})^{-1}$ with replacement of $\bar{\mathbf{c}}_i$ in (4.13) with that obtained by this method; and

$$\hat{\mu}_y \pm t_{1-a/2}(h-s-1) \times \sqrt{s_{11}}\,\hat{\sigma}_{\varepsilon^*}, \tag{4.18}$$

where $t_{1-a/2}(h-s-1)$ is the upper $a/2$-percentile of the t-distribution with $(h-s-1)$ degrees of freedom.

Table 16 presents the results on the estimator of each response and its sample variance. Tables 17 and 18 summarize percentage reductions in variance and half-length of the 90% confidence interval with respect to the original model. Tables 19, 20 and 21 summarize the results on the same statistics as presented in Tables 15, 16 and 17, respectively, for the embellished model.

4.3.3. Inferences

From the simulation results of both models, we observe the following: (a) the performance of control variates in estimating the mean response time and utilization of Station 2 is better than that of antithetic variates. On the contrary, in estimating utilizations of Stations 1 and 3, antithetic variates yields better results; (b) the efficiency of Combined Method I shows the additive effects of antithetic variates and control variates, and its performance generally is better than either antithetic variates or control variates; (c) Combined Method III reduces the variance of each estimator more than antithetic variates in the range from 10% to 50%, and the 90% confidence interval in the range from 7% to 30%; and (d) the performances of Combined Methods I and III are similar in estimating utilizations of Stations 1, 2 and 3. However, in reducing the variances of the estimators for the response times of the original and embellished models, Combined Method I is superior to Combined Method III.

The performance of each method for the embellished model is better than that for the original model except for the control variates in estimating the mean system time. The reason for these results is considered as follows: for the embellished model, (a) the effect of the service time control variates at each station to the system response time is less than that for the original model, and (b) Combined Methods I and III take advantage of the stronger synchronization effect of random number streams for the arrival process than that for the original model. Also, compared with the previous two

Table 16. Mean and Variance of the Estimator for Open Machine-Repair Network: interarrival time = exponential with mean 8.0.

	Direct Simulation		Antithetic Variates		Control Variates		Combined Method I		Combined Method II		Combined Method III	
Estimator of Parameter	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
Response Time	2.552	0.721	2.557	0.411	2.551	0.220	2.562	0.211	NA	NA	2.553	0.275
Utilization of Station 1	0.188	0.010×10^{-2}	0.188	0.257×10^{-3}	0.187	0.298×10^{-3}	0.188	0.135×10^{-3}	NA	NA	0.188	0.145×10^{-3}
Utilization of Station 2	0.014	0.035×10^{-3}	0.014	0.026×10^{-3}	0.014	0.019×10^{-3}	0.014	0.014×10^{-3}	NA	NA	0.014	0.015×10^{-3}
Utilization of Station 3	0.069	0.092×10^{-3}	0.069	0.040×10^{-3}	0.069	0.044×10^{-3}	0.070	0.021×10^{-3}	NA	NA	0.070	0.024×10^{-3}

Table 17. Percentage Reduction in Variance for Open Machine-Repair Network: interarrival time = exponential with mean 8.0.

Estimator of Parameter	Antithetic Variates	Control Variates	Combined Method I	Combined Method II	Combined Method III
Response Time	43.02	69.47	70.76	NA	61.80
Utilization of Station 1	57.87	51.08	77.87	NA	76.23
Utilization of Station 2	25.71	44.96	60.00	NA	57.14
Utilization of Station 3	56.52	51.62	77.17	NA	73.91

Table 18. **Percentage Reduction in 90% Confidence Interval for Open Machine-Repair Network: interarrival time = exponential with mean 8.0**

Estimator of Parameter	Antithetic Variates	Control Variates	Combined Method I	Combined Method II	Combined Method III
Response Time	23.71	44.75	45.32	NA	37.51
Utilization of Station 1	34.40	30.11	52.44	NA	50.71
Utilization of Station 2	12.89	26.32	36.05	NA	33.81
Utilization of Station 3	33.36	30.84	51.69	NA	48.36

Table 19. Mean and Variance of the Esimator for Open Machine-Repair Network: interarrival time = exponential with mean 2.0.

	Direct Simulation		Antithetic Variates		Control Variates		Combined Method I		Combined Method II		Combined Method III	
Estimator of Parameter	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
Response Time	6.765	2.681	6.780	2.259	6.791	1.682	6.796	1.177	NA	NA	6.810	1.935
Utilization of Station 1	0.751	0.222×10^{-2}	0.751	0.894×10^{-3}	0.752	0.171×10^{-3}	0.751	0.436×10^{-3}	NA	NA	0.751	0.433×10^{-3}
Utilization of Station 2	0.056	0.154×10^{-3}	0.056	0.138×10^{-3}	0.055	0.075×10^{-3}	0.056	0.066×10^{-3}	NA	NA	0.056	0.066×10^{-3}
Utilization of Station 3	0.279	0.329×10^{-3}	0.277	0.139×10^{-3}	0.278	0.177×10^{-3}	0.278	0.081×10^{-3}	NA	NA	0.278	0.080×10^{-3}

Table 20. Percentage Reduction in Variance for Open Machine-Repair Network: Interarrival time = exponential with mean 2.0.

Estimator of Parameter	Antithetic Variates	Control Variates	Combined Method I	Combined Method II	Combined Method III
Response Time	15.74	37.26	56.11	NA	27.84
Utilization of Station 1	59.72	47.24	80.36	NA	80.50
Utilization of Station 2	10.39	51.56	57.14	NA	57.14
Utilization of Station 3	57.75	46.25	75.38	NA	75.68

Table 21. Percentage Reduction in 90% Confidence Interval for Open Machine-Repair Network: interarrival time = exponential with mean 2.0

Estimator of Parameter	Antithetic Variates	Control Variates	Combined Method I	Combined Method II	Combined Method III
Response Time	7.20	20.79	33.02	NA	14.11
Utilization of Station 1	35.87	27.37	55.19	NA	55.35
Utilization of Station 2	4.33	30.21	33.81	NA	33.81
Utilization of Station 3	34.31	26.65	49.83	NA	50.14

machine-repair models, generally, antithetic variates shows better performance in estimating the responses of interest. Thus, Combined Method I seems to show good performance in estimating the parameters of this model compared to its performances obtained in the previous two models.

4.4. Hospital Resource Allocation Model

In this section, we consider the application of all three combined methodologies to the hospital resource allocation model first considered by Schruben and Margolin (1978). For this model having a different queueing discipline, we explore the efficiency of each combined method. Also, we present the simulation results and their inferences.

4.4.1. System and model description

Figure 4.1 shows the operation of the hospital unit in terms of patient paths and types of resource (see Figure A in Schruben and Margolin (1978)). In this model, the hospital unit consists of three types of resources that are devoted to specialized care: intensive care, coronary care and intermediate care. Patients arrive at the hospital unit according to a poisson process with an arrival rate of 3.3 per day. Upon entering the hospital, 75% of the patients need intensive care, and 25% need coronary care. The service time distribution at intensive care is lognormal with mean 3.4 days and standard deviation 3.5 days, that of coronary care is lognormal with mean 3.8 days and standard deviation 1.6 days. After intensive care, 27% of the patients leave the hospital and 73% go to the intermediate care unit. Also, completing the coronary care, 20% of the patients leave the system and 80% go to the intermediate care unit. Intermediate care stay for intensive care patients is distributed lognormaly with mean

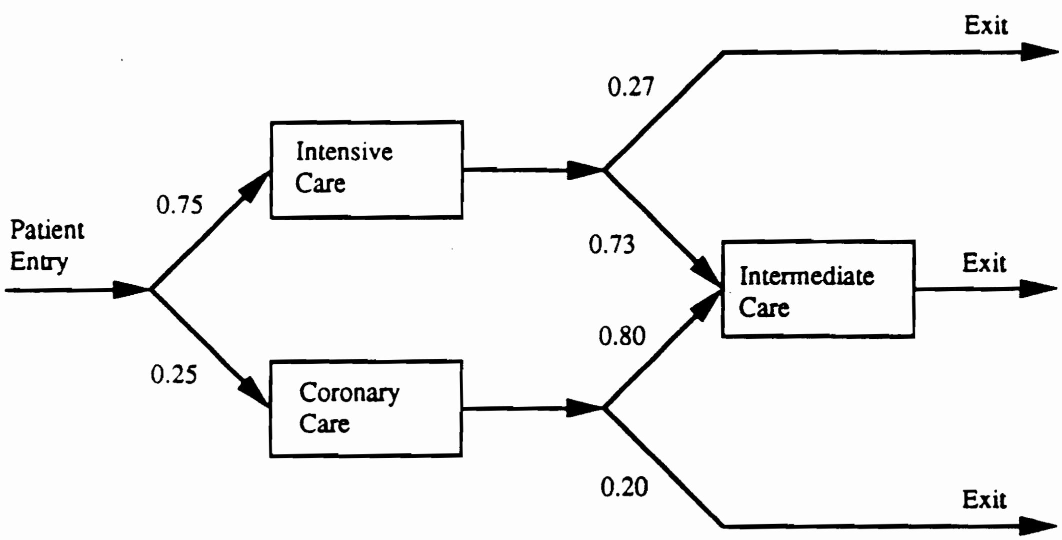


Figure 4. Hospital Resource Allocation Model

15.0 days and standard deviation 7.0 days. Finally, the length of intermediate care for coronary patients is distributed lognormaly with mean 17.0 days and standard deviation 3.0 days. When the patients request admission to special care units which are unavailable, they can not be accommodated and balk from the system.

Schruben and Margolin (1978) simulated this system and studied the effects of the numbers of resources (beds) of three different types to the mean failure rate (number per month) of the patients who can not be accommodated into system. They used a 2^3 factorial design and explored the response of interest (mean failure rate) in the region of eight design points. However, in this example, we are interested in the mean failure rate of the current system where the number of resources for intensive care, coronary care and intermediate care are, respectively, 15, 6 and 17 (considered as one design point). Additionally, we estimate the mean system time of the patients who complete any type(s) of service at special care units.

The simulation of this system was performed under direct simulation, antithetic variates, control variates and the three combined methods. We conduct 200 simulation runs under each method. A single replication uses eight separate random number streams for driving stochastic model components as presented in Table 22. Direct simulation uses 200 sets of randomly selected random number streams. Antithetic variates applies the same strategy as direct simulation across 100 paired replications. However, within a pair of replications, the first replicate uses randomly selected random number streams and the second employs the streams that are antithetic to these. In applying the method of control variates, we collected the four control variates: average interarrival times of the patients, and the average observed service times at the three hospital units. We expected that these control variates

Table 22. Random Number Assignment for Hospital Resource Allocation Model

Stream Number	Stochastic Process to be Sampled
1	Arrival Process of Patients to Hospital
2	Random Path Selection upon Entering the Hospital
3	Intensive Care Stay of Patients
4	Coronary Care Stay of Patients
5	Random Path Selection for Intensive Care Patients
6	Random Path Selection for Coronary Care Patients
7	Intermediate Care Stay for Intensive Care Patients
8	Intermediate Care Stay for Coronary Care Patients

would be correlated with the response of interest since it is expected that the mean failure rate will be high when most interarrival times of the patients are low and most service times of each specific unit are high. Further, we also expected the interarrival control variate would be strongly correlated with the mean failure rate. Based on this reasoning, we used a single control variate of interarrival times of the patients to system in using the three combined methods.

The three combined methods apply the same assignment rule of random number streams as direct simulation across 100 pairs of replications. For the first replicate within a given pair of replications, the combined methods randomly select eight random streams. For the second replicate, (a) Combined Method I randomly selects random number stream 1 (used to drive the interarrival time process), and assigns antithetic streams to the other streams, (b) Combined Method II used random number streams 1 antithetic to those used for the first replicate, and randomly selected the other streams for the stochastic processes of the non-control components in the model, and (c) Combined Method III employed random number streams antithetic to those of the first replicate.

The simulation program of this model was coded in SLAM II and run on the IBM 3090 computer at Virginia Polytechnic Institute and State University. The program code is given in Appendix B-4. In applying each method, we simulated this system for 1500 days, and begin collecting statistics after a warm-up period of 300 days to reduce the initial bias.

4.4.2. Experimental results

In computing the efficiency of the method of control variates, regression analysis on four control variates shows that all service time control variates are very poor ones. It is considered that the failure rate depends upon the capacity of each specialized care unit rather than the service times of each specific unit. Thus, we considered a interarrival time control variate in estimating the performance of control variates. For Combined Method II, we obtain the sample estimator for each response of interest, its variance, $\hat{V}_5$, and the 90% confidence interval of the mean response in the same way as in Combined Method III (see Section 4.3.2). When we apply the other methods mentioned above, appropriate statistics are obtained by the computational procedures outlined in Sections 4.1.2 and 4.3.2.

Table 23 summarizes the simulation results on the mean responses of interest and their sample variances. Also Tables 24 and 25 present a summary of the simulation results on percentage reductions in variance and half-length of the 90% confidence interval, respectively.

4.4.3. Inferences

From the simulation results of this model, we note the following: (a) Combined Methods I and III are more effective in reducing the variance of the estimator for each response than antithetic variates and control variates even if the latter two methods are effective, whereas the performance of Combined Method II is similar to that of control variates; (b) percentage reductions in variance and 90% confidence interval of Combined Method I are more than those obtained by the control variates method

Table 23. Mean and Variance of the Estimator for Hospital Resource Allocation Model

	Direct Simulation		Antithetic Variates		Control Variates		Combined Method I		Combined Method II		Combined Method III	
Estimator of Parameter	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
Failure Rate	43.233	2.008	43.086	0.753	43.093	0.505	43.065	0.387	43.117	0.492	43.055	0.360
Sojourn Time in System	12.230	0.242×10^{-1}	12.221	0.180×10^{-1}	12.237	0.206×10^{-1}	12.225	0.178×10^{-1}	12.236	0.208×10^{-1}	12.223	0.170×10^{-1}

Table 24. Percentage Reduction in Variance for Hospital Resource Allocation Model

Estimator of Parameter	Antithetic Variates	Control Variates	Combined Method I	Combined Method II	Combined Method III
Failure Rate	60.50	74.87	80.72	75.50	82.06
Sojourn Time in System	25.64	14.99	26.59	16.24	29.87

Table 25. Percentage Reduction in 90% Confidence Interval for Hospital Resource Allocation Model.

Estimator of Parameter	Antithetic Variates	Control Variates	Combined Method I	Combined Method II	Combined Method III
Failure Rate	38.11	49.87	55.62	49.97	57.18
Sojourn Time in System	12.85	7.80	13.40	6.27	15.35

by around 6%, respectively, in terms of the failure rate. These results imply that the efficiency gain of Combined Method I has the additive effects of antithetic variates and control variates, Combined Method III reduces the variance of the estimator for the failure rate, over antithetic variates by 20%, and the 90% confidence interval by 20%, and among the three combined methods, Combined method III yields marginally better results than Combined method I. Overall, both Combined Method I and III yield superior results to those by Combined Method II.

This model is considered as a little complicate one comparing to all the machine-repair models because of the service times of the intermediate care unit depending on the patients paths, and balking of the patients at each care unit. As we discussed earlier, using one more control variates does not show a greater reduction in the variance of the estimator. In this case, by applying the correlated replications through either the non-control variates (Combined Method I) or all stochastic components in the model (Combined Method III), we may have better results than by using the control variates or antithetic variates separately.

4.5. Port Operations Model

In this section, we conduct a simulation experiment on the port operations model given by Pritsker (1986) in Chapter 6. This model is more complex than the previous four models. For a complex model, generally, finding the best control variates is more difficult comparing to a simple model. We expect that the method(s) using antithetic variates and control variates would yield better results than the methods of either antithetic variates or control variates in this example. We summarize the simulation results and provide inferences as to these results.

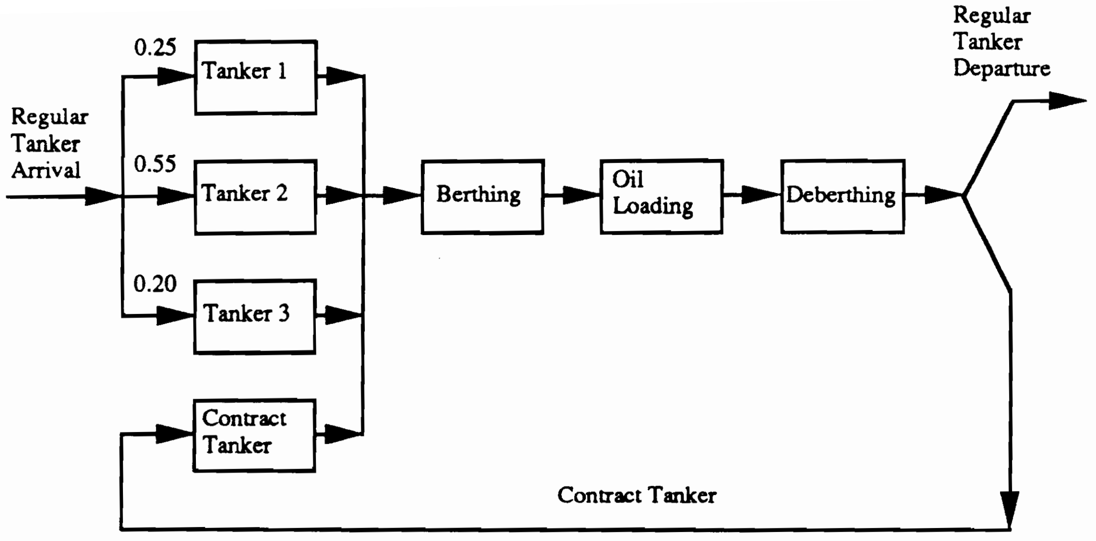
4.5.1. System and model description

The SLAM II network model for the port operation is presented in Figure 2 (see p. 197 in Pritsker (1986)). A port in Africa is used to load tankers with crude oil for overwater shipment. The port has facilities for loading as many as three tankers simultaneously. The tankers, which arrive at the port according to a uniform distribution with range [4, 18] hours, are of three different types. The relative frequency of the various types, their loading time requirements, and their distributions of loading time are as follows:

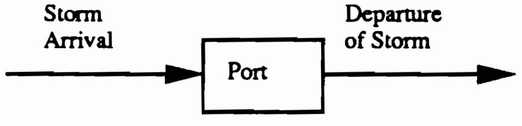
type	Relative Frequency	loading Time (Hours)	distribution
1	0.25	18 ± 2	uniform
2	0.55	24 ± 3	uniform
3	0.20	36 ± 4	uniform

There is one tug at the port. Tankers of all types require the services of this tug to move into a berth, and later to move out of a berth. When the tug is available, any berthing or deberthing activity takes about one hour. Top priority is given to the berthing activity.

A shipper is considering bidding on a contract to transfer oil from the port to the United Kingdom. He has determined that 5 tankers of a particular type would have to be committed to this task to meet contract specifications. These tankers would require 21 ± 3 hours, uniformly distributed, to load oil at the port. After loading and deberthing, they would travel to the United Kingdom, offload the oil, and return to the



(a) Tanker Arrival and Port Operation Segment



(b) Storm Segment

Figure 5. Port Operations Model

port for reloading. Their round-trip travel time, including offloading, is estimated to be 240 ± 24 hours with a uniform distribution. A complicating factor is that the port experiences storms. The time between the onset of storms is exponentially distributed with a mean of 48 hours, and a storm lasts 4 ± 2 hours, uniformly distributed. No tug can start an operation until a storm is over.

Before the port authorities can commit themselves to accommodating the proposed 5 tankers, the effect of the additional port traffic on the in-port residence time of the current port users must be determined. It is desired to simulate the operation of the port over a two-year period (19,280 hours) under the proposed new commitment to measure in-port residence time of the proposed additional tankers, as well as the three types of tankers which already use the port.

Applying all methods considered in Section 4.4, we conducted a simulation of this system 200 times for each method. The port operations model includes nine stochastic components to which nine separate random number streams are assigned. The assignment of these random number streams is given in Table 26. A single replication of each method uses a set of nine randomly selected random number streams. Direct simulation and the control variates method used randomly selected sets of nine random number streams. Also, antithetic variates used randomly selected sets of nine random number streams for the first replications within a pair of replications, but it used nine random number streams antithetic to those of the first replication for the second replication.

In using the control variates method, seven possible control variates present themselves. That is, interarrival times of tankers of three different types which are already in the system, oil loading times of each tanker (three regular types tankers

Table 26. Random Number Assignment for Port Operations Model

Stream Number	Stochastic Process to be Sampled
1	Interarrival Times of Tankers of Three Types to Port
2	Random Path Selection of Takers of Three Types upon Arriving to Port
3	Oil Loading Times of Type 1 Tanker
4	Oil Loading Times of Type 2 Tankers
5	Oil Loading Times of Type 3 Tankers
6	Oil Loading Times of Tankers on Contract
7	Round-Trip Travel Times of Tankers on Contract
8	Interarrival Times of Storm
9	Duration of Storm

Table 27. Correlation Matrix between the Responses and Control Variates.

	c_1	c_2	c_3	c_4	c_5	c_6
y_1	-0.689	0.133	0.288	-0.049	-0.029	-0.040
y_2	-0.675	0.113	0.278	-0.039	-0.015	-0.038
y_3	-0.639	0.108	0.252	-0.040	-0.028	-0.033
y_4	-0.698	0.114	0.267	-0.059	-0.011	-0.042

Note:

y_1 = in-port residence time of type 1 tankers

y_2 = in-port residence time of type 2 tankers

y_3 = in-port residence time of type 3 tankers

y_4 = in-port residence time of contract tankers

c_1 = interarrival time control variates for regular tankers

c_2 = oil loading time control variates for type 1 tankers

c_3 = oil loading time control variates for type 2 tankers

c_4 = oil loading time control variates for type 3 tankers

c_5 = oil loading time control variates for contract tankers

c_6 = round travel time control variates for contract tankers and tankers on a contract), round trip travel times of tankers on a contract, and duration of storm. We collected six control variates except the storm duration control variates since we expected that the frequency of storm is low and its in-port residence time is small. Table 27 shows the correlation matrix between the four responses of interest and the six collected control variates obtained by 200 independent replications. As we see in this table, (a) the interarrival time control variate is strongly correlated with each response of interest (correlation coefficients are in the range from 0.64 to 0.70), (b) each response is correlated with two control variates based on oil loading times of tankers 1 and 2 with the correlation coefficients in the range from 0.11 to 0.13 and from 0.25 to 0.29, respectively, and (c) the other control variates have little correlations with each response. Regression analysis on all six control variates indicates reduction in variance for each response in the range from 40% to 50%. When we chose the three most effective control variates (c_1, c_2, c_3 in Table 27), regression analysis shows an increment of reduction in variance for each response of interest by around 3%. Based on these results, we employed the three control variates of interarrival times of tankers already in system and oil loading times of tankers of type 1 and 2 for implementing the control variates method and the three combined methods.

Similarly as before, for the first replicate within each pair of replications, the three combined methods employ the same assignment rule as direct simulation. For the second replication, (a) Combined Method I uses a set of nine streams, those that correspond to the control variates (stream 1, 2 and 3) are randomly selected, and the others are set antithetic to their counterparts in the first replication, (b) Combined Method II randomly selects streams corresponding to the non-control variates and uses the other streams (1, 2 and 3) antithetic to those in the first replicate, and (c)

Combined Method III applies the same assignment strategy as antithetic variates. However, across the 100 pairs of replications, each of these methods randomly selects a set of nine random number streams.

A simulation model of this system was coded in SLAM II and run on the IBM 3090 computer at Virginia Polytechnic Institute and State University. The code is presented in Appendix B-5. For each method, we simulated the model for 21000 hours, and collected statistics after clearing data for the first 1000 hours to reduce the initialization bias.

4.5.2. Experimental results

Next, we summarize the simulation results in applying all variance reduction methods mentioned in Section 4.5.1. Table 28 presents the results on the estimators for the port residence times of the four types of tankers, and their variances. Tables 29 and 30, respectively, summarize the results on percentage reductions in variance and 90% half-length confidence intervals for each response of interest.

4.5.3. Inferences

Based on the simulation results of this model, we provide inferences in applying variance reduction techniques as follows: (a) antithetic variates and control variates reduce the variance of the estimator for each response in the range from 45% to 55%, and their performances are similar; (b) the efficiency gain of Combined Method I shows the additive effects of antithetic variates and control variates, and reduces the variance of each estimator more than antithetic variates and control variates in the range from 5% to 8%, and the 90% confidence interval in the range from 3% to 6%;

Table 28. Mean and Variance of the estimator for Port Operations Model.

	Direct Simulation		Antithetic Variates		Control Variates		Combined Method I		Combined Method II		Combined Method III	
Estimator of Parameter	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance	Mean	Variance
Sojourn Time of Tanker 1	40.958	20.186	40.979	9.768	40.825	9.441	40.888	8.063	41.252	15.775	40.979	9.385
Sojourn Time of Tanker 2	46.926	21.095	46.987	10.304	46.792	10.470	46.759	9.113	47.259	16.493	46.987	10.325
Sojourn Time of Tanker 3	58.823	22.714	58.917	12.289	58.693	12.594	58.731	11.334	59.225	18.414	58.912	12.236
Sojourn Time of Tanker on Contract	43.277	16.651	43.352	7.680	43.158	7.821	43.210	6.414	43.599	12.793	43.357	7.602

Table 29. **Percentage Reduction in Variance for Port Operations Model.**

Estimator of Parameter	Antithetic Variates	Control Variates	Combined Method I	Combined Method II	Combined Method III
Sojourn Time of Tanker 1	51.63	53.23	60.06	21.85	53.50
Sojourn Time of Tanker 2	51.16	50.37	56.80	21.82	51.05
Sojourn Time of Tanker 3	45.90	44.55	50.10	18.93	46.13
Sojourn Time of Tanker on Contract	54.00	53.03	61.15	23.17	54.35

Table 30. Percentage Reduction in 90% Confidence Interval for Port Operations Model.

Estimator of Parameter	Antithetic Variates	Control Variates	Combined Method I	Combined Method II	Combined Method III
Sojourn Time of Tanker 1	29.70	31.61	36.10	10.62	31.06
Sojourn Time of Tanker 2	29.37	29.55	33.55	10.60	29.26
Sojourn Time of Tanker 3	25.66	25.54	28.58	8.97	25.79
Sojourn Time of Tanker on Contract	29.21	29.31	35.58	8.59	29.54

(c) performance of Combined Method III is similar to that obtained by antithetic variates; and (d) among the three combined methods, Combined Method I yields better results than Combined Methods II and III, and performance of Combined method II yields inferior results to those of Combined Methods I and III. For instance, the percentage reduction in 90% confidence interval of each response using Combined Method II is less than other two combined methods by 20% in most cases.

As we expected, Combined Method I yields better results than the methods of control variates and antithetic variates. Generally, for a complex model, an effective set of control variates is small. Also, the marginal effect of including one more control variate is very small when there is a strong correlation between a set of control variates already used in the system and the control variates to be added (see the discussion of Beja (1967)). Thus, the combined method which is based on using the effective control variates and additionally trying to reduce the variance of the estimator by the correlated replicates (Combined Method I) may yield better results than applying either the control variates or antithetic variates separately.

From the simulation experiments on the five selected models, we note that (a) Combined Method I shows the additive effects of antithetic variates and control variates in reducing the variance of the estimator, (b) Combined Methods I and II are superior to Combined Method II, and (c) Combined Method III shows a good performance when the use of random number streams in the paired runs is synchronized. Also, we consider that, in general, the performance of Combined Method I would be better than that of control variates for a complex model.

CHAPTER 5. COMBINED CORRELATION METHODS FOR MULTIPOPULATION MODEL

This chapter develops three variance reduction techniques for improving the estimation of the model parameters in the multipopulation context. The first and second developments are extensions of Combined Method I to the multipopulation environment, respectively, with independent replications and the correlated replication strategy of Schruben and Margolin (1978) across the design points. The third approach is for improving Schruben-Margolin method by combining it with the method of control variates.

In the context of designed simulation experiments for the multipopulation model, several authors have developed procedures that improve the reliability of the estimators for the model parameters. Schruben and Margolin (1978) developed a method for combining the use of common random numbers and antithetic variates in one simulation experiment designed to estimate the parameters for the first-order multipopulation model in (2.3). Nozari, Arnold and Pegden (1984) added control variates to the model in (2.3), and evaluated the simulation efficiency of control variates for the general linear model in (2.5). Tew and Wilson (1989) proposed a combined approach using the Schruben-Margolin correlation induction strategy in conjunction with control variates to improve the estimation of the parameters in the first-order linear model in (2.5).

These studies exploited the correlations between the responses of the different design points, and between the response and control variates from the same design point. In the same spirit, we consider a way of extending Combined Method I to the multipopulation model. The results of Combined Method I in the previous chapter show that we may improve the estimator for the mean response in a single population model when the combined method of control variates and antithetic variates is applied. Based on these results, we first explore a method of extending Combined Method I to the estimation of the model parameters in the general linear model in (2.5). Second, we extend Combined Method I to the multipopulation context in conjunction with Schruben-Margolin method to improve the estimation of the parameters in a first-order linear model.

Next, we propose a new approach for improving the Schruben-Margolin method in estimating the parameters of the first-order linear model in (2.3). For the linear model which admits orthogonal blocking into two blocks, the Schruben-Margolin method assigns common random number streams to the design points in the first block, and their antithetic streams to the design points in the second block. Consider a 2^n factorial design together with a model that excludes the highest-order interaction effect of the factor variables (the design matrix admits orthogonal blocking into two blocks). Note that the levels of each factor variable can be represented as -1 and 1 by an appropriate reparameterization. In this case, the Schruben-Margolin method uses the same random number streams for the design point in which the level of the excluded highest-order interaction term is 1 (first block), and uses antithetic streams for the design point in which the level of this term is -1 (second block). For a more detailed discussion, see p. 514 of Schruben and Margolin (1978).

The assignment of the same random number streams to the design points in the same block (either first or second) allows the observed control variates (assumed to be independent of the factor variables) at these design points to be the same if we conduct the simulation experiment of each design point for the same simulation time. Also, by antithetic streams for the different block, the control variates observed at the first (second) block are negatively correlated with those observed at the second (first) block. Similar to the factor variables in the model, by the reparameterization of the control variates the levels of each control variate are given by, respectively, 1 for the first block, -1 for the second block. Hence, adding the control variates to the considered linear model gives the same effect as including the highest-order term to the given model. However, the highest-order term is negligible in the assumed model. In this situation, we present a new approach using the information of control variates under Schruben-Margolin strategy for improving the accuracy of the estimators for the parameters.

The remainder of this chapter is organized as follows: Section 5.1 provides an extension of Combined Method I to the multipopulation context. Section 5.2 presents a brief review of the Schruben-Margolin correlation induction strategy. Section 5.3 develops a method utilizing both Combined Method I and the Schruben-Margolin method. Section 5.4 proposes a new approach for applying the Schruben-Margolin method to the multipopulation model with control variates. Finally, Section 5.5 compares the simulation efficiency of the methods considered in this chapter.

5.1. Extension of Combined Method I

This section extends Combined Method I for a single population model to a multipopulation context with independent replications across the design points.

Consider an experimental design that specifies the combination of m factor settings in the multipopulation simulation model. Suppose we estimate the mean response at a single design point i by the sample mean, $\bar{y}_i$, of $2h$ replicates simulation runs. Let $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)'$ be the mean response vector across the m design points. As reviewed earlier in Chapter 2, the relationship between the responses and the function of factor settings across all m design points can be written as the linear model given in (2.3):

$$\bar{\mathbf{y}} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{5.1}$$

where $\mathbf{X}$, $\boldsymbol{\beta}$ and $\boldsymbol{\varepsilon}$ are given in (2.3). During the simulation experiment, often we observe control variates that are highly correlated with the response of interest. Let $\bar{\mathbf{c}}_i$ be the vector of control variates corresponding to $\bar{y}_i$ at design point i. Adding the control variates to (5.1), we have the following linear model with factor variables and control variates given in (2.5):

$$\bar{\mathbf{y}} = \mathbf{X}\boldsymbol{\beta} + \bar{\mathbf{C}}\boldsymbol{\alpha} + \boldsymbol{\varepsilon}^*, \tag{5.2}$$

where $\bar{\mathbf{y}}$, $\mathbf{X}$ and $\boldsymbol{\beta}$ are given in (5.1); $\bar{\mathbf{C}}$ is a $(m \times s)$ matrix whose ith row consists of $\mathbf{c}'_i$; and $\boldsymbol{\varepsilon}^*$ is a $(m \times 1)$ vector which represents the inability of the postulated model to determine $\bar{\mathbf{y}}$.

The correlated replication rule of Combined Method I (for a single population model) in Section 3.1 can be straight forwardly extended to the multipopulation model in (5.2). As in Section 3.1, let $\mathbf{R}_{ij}$ be the set of g random number streams for the jth replication of simulation run at the ith design point ($j = 1, 2, ..., 2h$, $i = 1, 2, ..., m$). We separate $\mathbf{R}_{ij}$ into two mutually exclusive subsets, $(\mathbf{R}_{ij1}, \mathbf{R}_{ij2})$. We use the first subset of $(g - s^*)$ streams, $\mathbf{R}_{ij1}$, for driving the non-control variate stochastic components, the second subset of s^* streams, $\mathbf{R}_{ij2}$, for driving the control variate stochastic components. We employ Combined Method I for the $2h$ replications at each design point, and use independent streams across the m design points.

Under this replication rule, we can easily find the simulation efficiency of this method. Let y_{ij} and $\mathbf{c}_{ij}$ be the response of interest and a vector of control variates, respectively, at the jth replication and the ith design point. To specify the joint distribution of the mean responses and mean control variates across the m design points, we extend Assumptions 1-5 in Section 3.1 to m design points as follows:

1. $\text{Var}(y_{ij}) = \sigma_y^2$, for $i = 1, 2, ..., m, j = 1, 2, ..., 2h$ (homogeneity of response variances across design points and replicates),

2. $\text{Cov}(y_{ij}, y_{ik}) = -\rho_y \sigma_y^2$ $(\rho_y > 0)$ for $i = 1, 2, ..., m$ if $k = j + 1$ $(j = 1, 3, ..., 2h - 1)$ (homogeneity of induced negative correlations across design points and replicates pairs). Otherwise, $\text{Cov}(y_{ij}, y_{ik}) = 0$,

3. $\text{Cov}(y_{ij}, \mathbf{c}_{kl}) = \boldsymbol{\sigma}'_{yc}$ if $i = k$ and $j = l$. (homogeneity of control variates-response covariance across design points and replicates). Otherwise, $\text{Cov}(y_{ij}, \mathbf{c}_{kl}) = \mathbf{0}'$,

4. $\mathrm{Cov}(\mathbf{c}_{ij}) = \boldsymbol{\Sigma}_c$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, 2h$. (homogeneity of control variates covariance structure across design points and replicates), and

5. $\mathrm{Cov}(\mathbf{c}_{ij}, \mathbf{c}_{kl}) = \mathbf{O}_{s \times s}$ for $i \neq k$ and $j \neq l$ (independence of control variates across design points and replicates).

Under these assumptions, we identify the joint distribution of $\bar{\mathbf{y}}$ and $\bar{\mathbf{C}}$: $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)'$ and $\bar{\mathbf{C}} = (\bar{\mathbf{c}}_1, \bar{\mathbf{c}}_2, \ldots, \bar{\mathbf{c}}_m)'$. The variances of the mean response and mean control variates, and the covariance between the mean response and mean control variates can be obtained by similar procedures in Section 3.1. First, we consider just one design point. At the ith design point, from Corollary 5.2.1 in Mood, Graybill and Boes (1974), the variance of $\bar{y}_i$ is given by

$$\mathrm{Var}(\bar{y}_i) = \mathrm{Var}\left(\frac{1}{2h}\sum_{j=1}^{2h} y_{ij}\right) = \frac{1}{4h^2}\left[\sum_{j=1}^{2h}\mathrm{Var}(y_{ij}) + 2\sum_{j<k}\mathrm{Cov}(y_{ij}, y_{ik})\right]$$

$$= \frac{1}{4h^2}\left[\sum_{j=1}^{2h}\mathrm{Var}(y_{ij}) + 2\sum_{j=1}^{h}\mathrm{Cov}(y_{i,2j-1}, y_{i,2j})\right] = \frac{1}{4h^2}\left[2h\sigma_y^2 - 2h\rho_y\sigma_y^2\right]$$

$$= \frac{1}{2h}(1-\rho_y)\sigma_y^2 \tag{5.3}$$

since $\mathrm{Cov}(y_{ij}, y_{ik}) = 0$ if either $k \neq j$ or $k \neq j+1$ by Assumption 2. Similarly, we get

$$\mathrm{Cov}(\bar{\mathbf{c}}_i) = \mathrm{Var}\left(\frac{1}{2h}\sum_{j=1}^{2h}\mathbf{c}_{ij}\right) = \frac{1}{4h^2}\left[\sum_{j=1}^{2h}\mathrm{Var}(\mathbf{c}_{ij}) + 2\sum_{j<k}\mathrm{Cov}(\mathbf{c}_{ij}, \mathbf{c}_{ik})\right] = \frac{1}{2h}\boldsymbol{\Sigma}_c \tag{5.4}$$

by Assumptions 4 and 5. Also the covariance between $\bar{y}_i$ and $\bar{\mathbf{c}}_i$ is given by

$$\mathrm{Cov}(\bar{y}_i, \bar{\mathbf{c}}_i) = \mathrm{Cov}(\frac{1}{2h}\sum_{j=1}^{2h} y_{ij}, \frac{1}{2h}\sum_{j=1}^{2h} \mathbf{c}_{ik}) = \frac{1}{4h^2}[\sum_{j=1}^{2h} \mathrm{Cov}(y_{ij}, \mathbf{c}_{ij}) + \sum_{j \neq k} \mathrm{Cov}(y_{ij}, \mathbf{c}_{ik})]$$

$$= \frac{1}{2h}\boldsymbol{\sigma}'_{yc} \tag{5.5}$$

by Assumptions 3. Thus, from (5.3)-(5.5), under the normality assumption of the response and control variates, the joint distribution of $\bar{y}_i$ and $\bar{\mathbf{c}}_i$ is given by

$$\begin{bmatrix} \bar{y}_i \\ \bar{\mathbf{c}}_i \end{bmatrix} \sim N_{s+1}\left[\begin{bmatrix} \mathbf{x}'_i\boldsymbol{\beta} \\ \mathbf{0} \end{bmatrix}, \frac{1}{2h}\begin{bmatrix} (1-\rho_y)\sigma_y^2 & \boldsymbol{\sigma}'_{yc} \\ \boldsymbol{\sigma}_{yc} & \boldsymbol{\Sigma}_c \end{bmatrix}\right], \tag{5.6}$$

where $\mathbf{x}'_i$ is the ith row of $\mathbf{X}$, and $\mathbf{x}'_i\boldsymbol{\beta}$ is the mean response at the ith design point.

The application of independent streams across the m design points allows that the $(s+1)$-variates simulation output, $(\bar{y}_i, \bar{\mathbf{c}}'_i)$, at the ith design point is independent of $(\bar{y}_j, \bar{\mathbf{c}}'_j)$ obtained at the different design point $(i \neq j)$. Therefore, under the joint normality assumption of the responses and control variates, from equation (5.6), we find the joint distribution of $\bar{\mathbf{y}}$ and $\bar{\mathbf{C}}$ as follows:

$$\begin{bmatrix} \bar{\mathbf{y}} \\ \mathrm{Vec}(\bar{\mathbf{C}}) \end{bmatrix} \sim N_{m(s+1)}\left[\begin{bmatrix} \mathbf{X}\boldsymbol{\beta} \\ \mathbf{0}_{ms} \end{bmatrix}, \boldsymbol{\Sigma}\right]; \tag{5.7}$$

where $\mathrm{Vec}(\bar{\mathbf{C}})$ denotes the operation that the columns of $\bar{\mathbf{C}}$ are stacked into a single ms-dimensional vector;

$$\boldsymbol{\Sigma} = \frac{1}{2h}\begin{bmatrix} \sigma_y\mathbf{I}_m & \boldsymbol{\sigma}'_{yc}\otimes\mathbf{I}_m \\ \boldsymbol{\sigma}_{yc}\otimes\mathbf{I}_m & \boldsymbol{\Sigma}_c\otimes\mathbf{I}_m \end{bmatrix}, \tag{5.8}$$

where $\otimes$ denotes a Kronecker operation of two matrices. From Theorem 2.5.1 in Anderson (1984), the conditional variance of $\bar{\mathbf{y}}$ given $\bar{\mathbf{C}}$ is as follows:

$$\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}}) = \frac{1}{2h}[\sigma_y \mathbf{I}_m - (\boldsymbol{\sigma}'_{yc} \otimes \mathbf{I}_m)(\boldsymbol{\Sigma}_c \otimes \mathbf{I}_m)^{-1}(\boldsymbol{\sigma}_{yc} \otimes \mathbf{I}_m)]$$

$$= \frac{1}{2h}[\sigma_y \mathbf{I}_m - (\boldsymbol{\sigma}'_{yc} \otimes \mathbf{I}_m)(\boldsymbol{\Sigma}_c^{-1} \otimes \mathbf{I}_m)(\boldsymbol{\sigma}_{yc} \otimes \mathbf{I}_m)] = \frac{1}{2h}[\sigma_y \mathbf{I}_m - (\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1} \otimes \mathbf{I}_m)(\boldsymbol{\sigma}_{yc} \otimes \mathbf{I}_m)]$$

$$= \frac{1}{2h}[\sigma_y \mathbf{I}_m - (\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} \otimes \mathbf{I}_m)] = \frac{1}{2h}[\sigma_y \mathbf{I}_m - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}\mathbf{I}_m]$$

$$= \frac{\sigma_y^2}{2h}(1 - \rho_y - R_{yc}^2)\mathbf{I}_m, \tag{5.9}$$

where R_{yc} is the multiple correlation coefficient between y_{ij} and $\mathbf{c}_{ij}$. The least squares estimator of $\boldsymbol{\beta}$ in (5.2) is given by

$$\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \bar{\mathbf{C}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{P}]\bar{\mathbf{y}} \tag{5.10}$$

(see (2.23)). Taking the operation of variance on (5.10) gives

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \bar{\mathbf{C}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{P}]\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}})[\mathbf{I}_m - \mathbf{P}\bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}']\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1},$$

which is developed into, by substituting for $\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}})$ with (5.9),

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \bar{\mathbf{C}}) = \frac{\sigma_y^2}{2h}(1 - \rho_y - R_{yc}^2)(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{P}][\mathbf{I}_m - \mathbf{P}\bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}']\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

$$= \frac{\sigma_y^2}{2h}(1 - \rho_y - R_{yc}^2)[(\mathbf{X}'\mathbf{X})^{-1} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}] \tag{5.11}$$

since $\mathbf{X}'\mathbf{P} = \mathbf{P}\mathbf{X} = \mathbf{0}$. Since the least squares estimator $\hat{\boldsymbol{\beta}}_{\mathbf{G}}$ is an unbiased estimator conditionally on $\bar{\mathbf{C}}$ from (2.41), the unconditional variance of $\hat{\boldsymbol{\beta}}_{\mathbf{G}}$ is given by

$$\text{Var}(\hat{\beta}_G) = E[\text{Var}(\hat{\beta}_G \mid \bar{C})] = \frac{\sigma_y^2}{2h}(1 - \rho_y - R_{yc}^2)\left(\frac{m-p-2}{m-p-s-2}\right)(X'X)^{-1} \quad (5.12)$$

(see the proof of (5.12) in Appendix A). This equation indicates that this method reduces the variance of the estimator for β_i ($i = 0, 1, \ldots, p$) by $(\rho_y + R_{yc}^2)\sigma_y^2/2h$ and its loss factor is $(m - p - 2)/(m - p - s - 2)$ due to the estimation of α in (5.2), respectively, compared with the results obtained by independent streams across the $2h$ replications and m design points.

5.2. Schruben-Margolin Correlation Induction Strategy

This section presents a brief summary of the Schruben-Margolin method for multipopulation simulation experiments.

Consider an experimental design together with the linear model given in (5.1). By a reparameterization of the function of the factor levels, the design matrix **X** may be chosen as an orthogonal matrix in the experimental design. For the design matrix **X** admitting orthogonal blocking into two blocks, Schruben and Margolin (1978) exploited the random number assignment rule which uses a combination of common random numbers and antithetic streams across m design points. Their assignment rule uses the same set of random number streams **R** for all m_1 design points in the first block, and uses the same set of antithetic random number streams $\bar{\mathbf{R}}$ for all m_2 design points in the second block ($m = m_1 + m_2$) within a replication. Based on the empirical results of simulation and the standard assumptions in statistical modeling, they assumed the following:

1. When the same set of random number streams is used at two design points, a positive correlation is induced between two observations of the responses, i.e., $\mathrm{Cov}(y_i, y_j) = \rho_{-}\sigma_y^2$.

2. Using the antithetic set of streams at two design points induces a negative correlation between two responses, i.e., $\mathrm{Cov}(y_i, y_j) = \rho_{-}\sigma_y^2$.

3. When two observations are made with different, randomly selected streams, the responses have zero correlation, i.e., $\mathrm{Cov}(y_i, y_j) = 0$.

4. ρ_{-} and ρ_{-} are constant and $\rho_{+} \geq -\rho_{-} > 0$.

Under these assumptions, if we assign the random number streams $\mathbf{R}_i$ for the first m_1 points of the design and the random number streams $\overline{\mathbf{R}}_i = \mathbf{1} - \mathbf{R}_i$ for the second m_2 design points, and randomly select $\mathbf{R}_i$ ($i = 1, 2, \ldots, 2h$) through the $2h$ replications at the ith design point, then we have the covariance matrix of the mean responses $\overline{\mathbf{y}}$ as follows:

$$\mathrm{Cov}\Big(\frac{1}{2h}\sum_{i=1}^{2h}\mathbf{y}_i\Big) = \mathrm{Cov}(\overline{\mathbf{y}}) = \frac{1}{2h}\sigma_y^2 \begin{bmatrix} 1 & \rho_+ & & \rho_+ & \rho_- & \rho_- & & \rho_- \\ \rho_+ & 1 & \cdots & \rho_+ & \rho_- & \rho_- & \cdots & \rho_- \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \rho_+ & \rho_+ & \vdots & 1 & \rho_- & \rho_- & \vdots & \rho_- \\ \rho_- & \rho_- & \vdots & \rho_- & 1 & \rho_+ & \vdots & \rho_+ \\ \rho_- & \rho_- & \vdots & \rho_- & \rho_+ & 1 & \vdots & \rho_+ \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \rho_- & \rho_- & & \rho_- & \rho_+ & \rho_+ & & 1 \end{bmatrix}, \quad (5.13)$$

(this covariance matrix is also given in Hussey, Myers and Houck (1987)); which can be represented as

$$\mathrm{Cov}(\overline{\mathbf{y}}) = \frac{\sigma_y^2}{2h}[\frac{1}{2}(\rho_+ + \rho_-)\mathbf{X}\mathbf{G}_{p+1}\mathbf{X}' + \frac{1}{2}(\rho_+ + \rho_-)\mathbf{z}_m\mathbf{z}'_m + (1 - \rho_+)\mathbf{I}_m], \quad (5.14)$$

where $\mathbf{G}_{p+1}$ is a $((p + 1) \times (p + 1))$ matrix whose first row and first column entry is 1 with all other entries 0, and $\mathbf{z}_m$ is a $(m \times 1)$ vector whose first m_1 elements are 1's and remaining elements are -1's. For the dispersion matrix given in (5.13), it is known that the ordinary least squares (OLS) and the weighted least squares (WLS) estimators of $\boldsymbol{\beta}$ in (5.1) are identical (see equation (63) in Rao (1967)). Taking the variance operation on the OLS estimator,

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{y}}, \quad (5.15)$$

and substituting in the form for $\mathrm{Cov}(\overline{\mathbf{y}})$ given in (5.13) yields

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{Cov}(\overline{\mathbf{y}})\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

$$= \frac{\sigma_y^2}{2h}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\frac{1}{2}(\rho_y + \rho_-)\mathbf{X}\mathbf{G}_{p+1}\mathbf{X}' + \frac{1}{2}(\rho_+ + \rho_-)\mathbf{z}_m\mathbf{z}' + (1 - \rho_+)\mathbf{I}_p]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

$$= \frac{\sigma_y^2}{2h}[\frac{1}{2}(\rho_+ + \rho_-)\mathbf{G}_{p+1} + (1 - \rho_+)(\mathbf{X}'\mathbf{X})^{-1}] \quad (5.16)$$

since $\mathbf{X}'\mathbf{z}_m = \mathbf{z}_{m'}\mathbf{X} = \mathbf{0}$. Comparison with the result in (2.28) (divided by $2h$) indicates the value of the diagonal element corresponding to β_i ($i = 1, 2, ..., p$) decreases by ρ_+, but the variance of β_0 increases by $(\rho_+ + \rho_-)/2 - \rho_+/m$.

With respect to the design criterion of D-optimality (that is, the determinant of the dispersion matrix), Schruben and Margolin (1978) showed their assignment rule yields the OLS estimator with a smaller D-value than (a) the assignment of one common **R** to all design points, or (b) the assignment of a different **R** to each design point, in the latter case, provided

$$[1 + (m - 1)\rho_+ - 2m^{-1}m_1m_2(\rho_+ + \rho_-)](1 - \rho_+)^p < 1. \tag{5.17}$$

5.3. Extended Combined Method I with Schruben-Margolin Strategy

This section extends Combined Method I for a single population model to a multipopulation model in conjunction with Schruben-Margolin Method, and identifies the simulation efficiency of this method with respect to the unconditional variance of the estimator.

In Section 5.1, we extended Combined Method I to the multipopulation experiments with the general linear model. Here, we consider a way of extending Combined Method I in Section 3.1 to the multipopulation model in (5.2) where a design matrix **X** admits orthogonal blocking into two blocks. Basically, this extension to the multipopulation model involves combining in an additive manner the Schruben-Margolin correlation induction strategy and Combined Method I. Instead of directly applying Schruben-Margolin method across m design points, we first partition a set of the stochastic components in the model into two subsets of the non-control variate components and control variate components. Then we use correlation methods of common random numbers and antithetic variates partially through the non-control variate stochastic components in the model. Even though this correlation induction strategy may weaken the desired correlation of the responses at two design points, it allows the control variates to be observed independently at each design point. This last point is critical in order to achieve an additive effect from the two methods.

In this extension of Combined Method I to m design points, parallel to the work of Schruben and Margolin (1978), we partition m design points into two orthogonal blocks consisting of m_1 and m_2 design points respectively. At the ith design point, similar to the method given in Section 3.1, suppose that the jth replication of the simulation run has been structured so that the set of g random number streams $\mathbf{R}_{ij}$ driving the simulation can be partitioned into two mutually exclusive subsets $(\mathbf{R}_{ij1}, \mathbf{R}_{ij2})$: the first subset of $(g - s^*)$ streams, $\mathbf{R}_{ij1}$, generates the random processes of the $(g - s^*)$ non-control stochastic components and the second subset of s^* random number streams, $\mathbf{R}_{ij2}$, completely determines the observations of the s control variates under consideration. For the ith design point in each block, the first set of streams $\mathbf{R}_{ij1}$ is selected according to the Schruben-Margolin assignment rule, and the second set of streams, $\mathbf{R}_{ij2}$, are randomly selected through the $2h$ replications in the experiment ($j = 1, 2, \ldots, 2h$). On the other hand, for the $2h$ replications at the ith design point, Combined Method I is employed. For instance, within the first pair of replications, two different design points i and k in the same block use (a) $(\mathbf{R}_{i11}, \mathbf{R}_{i12})$ and $(\overline{\mathbf{R}}_{i11}, \mathbf{R}_{k12})$ respectively for the first replication; and (b) $(\mathbf{R}_{i11}, \mathbf{R}_{i22})$ and $(\overline{\mathbf{R}}_{i11}, \mathbf{R}_{k22})$, respectively, for the second replication; where $\mathbf{R}_{i11}$, $\mathbf{R}_{il2}$, and $\mathbf{R}_{kl2}$ ($l = 1, 2$) are randomly selected, but $\overline{\mathbf{R}}_{i11}$ is antithetic to $\mathbf{R}_{i11}$.

For extending Combined Method I to the multipopulation model, Table 31 presents the complete assignment of random number streams for the $2h$ replications at m design points: the first m_1 design points are in the first block, and the second m_2 design points are in the second block; $\mathbf{R}_{1j1}$ consists of $(g - s^*)$ random number streams used

Table 31. Random Number Assignment Rule of Correlated Replications for m design points.

Design	Replication				
Point	1	2		$2h-1$	$2h$
1	$y_{11}(R_{111}, R_{112})$	$y_{12}(\bar{R}_{111}, R_{122})$	...	$y_{1,2h-1}(R_{1h1}, R_{12h-1,2})$	$y_{12h}(\bar{R}_{1h1}, R_{12h2})$
2	$y_{21}(R_{111}, R_{212})$	$y_{22}(\bar{R}_{111}, R_{222})$	...	$y_{2,2h-1}(R_{1h1}, R_{22h-1,2})$	$y_{22h}(\bar{R}_{1h1}, R_{22h2})$
.	.	.	...	.	.
.	.	.	...	.	.
.	.	.	...	.	.
m_1	$y_{m_11}(R_{111}, R_{m_112})$	$y_{m_12}(\bar{R}_{111}, R_{m_122})$	...	$y_{m_12h-1}(R_{1h1}, R_{m_12h-1,2})$	$y_{m_12h}(\bar{R}_{1h1}, R_{m_12h2})$
m_1+1	$y_{m_1+1,1}(\bar{R}_{111}, R_{m_1+1,12})$	$y_{m_1+1,2}(R_{111}, R_{m_1+1,22})$	...	$y_{m_1+1,2h-1}(\bar{R}_{1h1}, R_{m_1+1,2h-1,2})$	$y_{m_1+1,2h}(R_{1h1}, R_{m_1+1,2h2})$
.	.	.	...	.	.
.	.	.	...	.	.
.	.	.	...	.	.
m	$y_{m1}(\bar{R}_{111}, R_{m12})$	$y_{m2}(R_{111}, R_{m22})$	...	$y_{m2h-1}(\bar{R}_{1h1}, R_{m2h-1,2})$	$y_{m2h}(R_{1h1}, R_{m2h2})$

for the non-control stochastic components in the model ($j = 1, 2, ..., h$); $\mathbf{R}_{ij2}$ consists of s^* random number streams used for the control variates in the model ($i = 1, 2, ..., m, j = 1, 2, ..., h$); $\mathbf{R}_{1j1}$ is a set of randomly selected random number streams for the $(2j - 1)$th replication ($j = 1, 2, ..., h$); $\overline{\mathbf{R}}_{1j1}$ is a set of streams antithetic to $\mathbf{R}_{1j1}$ ($i = 1, 2, ..., h$); and $\mathbf{R}_{ij2}$ is a set of randomly selected random number streams for the jth replication at the ith design point ($i = 1, 2, ..., m, j = 1, 2, ..., 2h$).

The assignment rule in Table 31 forces the mean responses at the m design points to have a covariance structure different from that obtained by the Schruben-Margolin method. Let y_{ij} and $\mathbf{c}_{ij}$ be the response of interest and a vector of control variates, respectively, at the jth replication and the ith design point. We first specify the covariance matrix of the responses and control variates obtained by the assignment procedure described above. To this end, we extend Assumptions 1-5 in Section 3.1 to m design points and adopt Assumptions 1-4 of Schruben and Margolin (1978) in Section 5.2. Since the induced correlations by partial common random numbers and antithetic variates (for the non-control stochastic components in the model) across design points will be different (common experience suggests smaller than ρ_+ and ρ_- in Section 5.2), we use notation ρ_+^* and ρ_-^*, respectively, for analogs to ρ_+ and ρ_-. We now establish the assumptions as follows:

1. $\mathrm{Var}(y_{ij}) = \sigma_y^2$, for $i = 1, 2, ..., m, j = 1, 2, ..., 2h$ (homogeneity of response variances across design points and replicates).

2. $\mathrm{Cov}(y_{ij}, y_{ik}) = -\rho_y \sigma_y^2$ ($\rho_y > 0$) for $i = 1, 2, ..., m$ if $k = j + 1$ ($j = 1, 3, ..., 2h - 1$) (homogeneity of induced negative correlations across design points and replicates pairs). Otherwise, $\mathrm{Cov}(y_{ij}, y_{ik}) = 0$.

3. $\text{Cov}(y_{ij}, y_{kl}) = \rho_+^* \sigma_y^2$ if two design points i and k are in the same block, and $l = j$; $\text{Cov}(y_{ij}, y_{kl}) = \rho_-^* \sigma_y^2$ if two design points i and k are in the same block, and $l = j + 1$ ($j = 1, 3, \ldots, 2h - 1$); (homogeneity of induced correlations across design points: adopted from Schruben and Margolin (1978)). Otherwise, $\text{Cov}(y_{ij}, y_{kl}) = 0$.

4. $\text{Cov}(y_{ij}, y_{kl}) = \rho_-^* \sigma_y^2$ if two design point i and k are in two different blocks, and $j = l$; $\text{Cov}(y_{ij}, y_{kl}) = \rho_-^* \sigma_y^2$ if two design points i and k are in two different blocks, and $l = j + 1$ ($j = 1, 3, \ldots, 2h - 1$); (homogeneity of induced correlations across design points: adopted from Schruben and Margolin (1978)). Otherwise, $\text{Cov}(y_{ij}, y_{kl}) = 0$.

5. ρ_+^* and ρ_-^* are constant and $\rho_+^* \geq -\rho_-^* \geq 0$ (standard statistical assumption and empirical simulation results: adopted from Schruben and Margolin (1978)).

6. $\text{Cov}(y_{ij}, \mathbf{c}_{kl}) = \boldsymbol{\sigma}'_{yc}$ if $i = k$ and $j = l$ (homogeneity of control variates-response covariance across design points and replicates). Otherwise, $\text{Cov}(y_{ij}, \mathbf{c}_{kl}) = \mathbf{0}'$.

7. $\text{Cov}(\mathbf{c}_{ij}) = \boldsymbol{\Sigma}_c$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, 2h$ (homogeneity of control variates covariance structure across design points and replicates).

8. $\text{Cov}(\mathbf{c}_{ij}, \mathbf{c}_{kl}) = \mathbf{O}_{s \times s}$ for $i \neq k$ and $j \neq l$ (independence of control variates across design points and replicates).

Assumptions 3, 4 and 5 are from Schruben and Margolin (1978) in Section 5.2, the other assumptions are the same as those in Section 5.1. Under these assumptions, we identify the conditional distribution of $\bar{\mathbf{y}}$ given $\bar{\mathbf{C}}$, where $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)'$ and $\bar{\mathbf{C}} = (\bar{\mathbf{c}}_1, \bar{\mathbf{c}}_2, \ldots, \bar{\mathbf{c}}_m)'$.

Since this extension uses the same random number strategy as that considered in Section 5.1 at the ith design point, the variance of $\bar{y}_i$, the covariance of $\bar{\mathbf{c}}_i$, and the covariance between $\bar{y}_i$ and $\bar{\mathbf{c}}_i$ are, respectively, equivalent to those given in (5.3), (5.4), and (5.5). Thus, under the joint normality assumption of the responses and control variates, the joint distribution of $\bar{y}_i$ and $\bar{\mathbf{c}}_i$ is same as that in (5.6).

Next we specify the covariance of the mean responses between two different design points. When $\bar{y}_i$ and $\bar{y}_k$ are the mean responses observed at two design points in the same block, we find (see Theorem 5.2 in Mood, Graybill and Boes (1974)),

$$\mathrm{Cov}(\bar{y}_i, \bar{y}_k) = \mathrm{Cov}\Big(\frac{1}{2h}\sum_{j=1}^{2h} y_{ij}, \frac{1}{2h}\sum_{l=1}^{2h} y_{kl}\Big) = \frac{1}{4h^2}\Big[\sum_{j=1}^{2h}\sum_{l=1}^{2h}\mathrm{Cov}(y_{ij}, y_{kl})\Big]$$

$$= \frac{1}{4h^2}\sum_{j=1}^{h}\sum_{l=1}^{2h}[\mathrm{Cov}(y_{i,2j-1}, y_{kl}) + \mathrm{Cov}(y_{i,2j}, y_{kl})]$$

$$= \frac{1}{4h^2}\sum_{j=1}^{h}[\mathrm{Cov}(y_{i,2j-1}, y_{k,2j-1}) + \mathrm{Cov}(y_{i,2j-1}, y_{k,2j}) + \sum_{l \neq 2j-1,\ 2j}\mathrm{Cov}(y_{i,2j-1}, y_{kl})]$$

$$+ \frac{1}{4h^2}\Big[\sum_{j=1}^{h}\mathrm{Cov}(y_{i,2j}, y_{k,2j}) + \mathrm{Cov}(y_{i,2j}, y_{k,2j-1}) + \sum_{l \neq 2j-1,\ 2j}\mathrm{Cov}(y_{i,2j}, y_{kl})\Big]$$

$$= \frac{1}{4h^2}\Big[\sum_{j=1}^{2h}\mathrm{Cov}(y_{ij}, y_{kj}) + \sum_{i=1}^{h}\{\mathrm{Cov}(y_{i,2j-1}, y_{k,2j}) + \mathrm{Cov}(y_{i,2j}, y_{k,2j-1})\}\Big]$$

$$= \frac{1}{4h^2}[2h\rho_+^* \sigma_y^2 + 2h\rho_-^* \sigma_y^2] = \frac{1}{2h}(\rho_+^* + \rho_-^*)\sigma_y^2 \tag{5.18}$$

by Assumption 3. Also, for the mean response $\bar{y}_i$ and $\bar{y}_k$ in two different blocks, we have the same result as in (5.18):

$$\mathrm{Cov}(\bar{y}_i,\ \bar{y}_k) = \frac{1}{2h}(\rho_+^* + \rho_-^*)\sigma_y^2. \tag{5.19}$$

Next, the covariance between $\bar{y}_i$ and $\bar{\mathbf{c}}_k$ for $i \neq k$ is given by

$$\mathrm{Cov}(\bar{y}_i, \bar{\mathbf{c}}_k) = \mathrm{Cov}(\frac{1}{2h}\sum_{j=1}^{2h} y_{ij}, \frac{1}{2h}\sum_{l=1}^{2h} \mathbf{c}_{kl}) = \frac{1}{4h^2}[\sum_{j=1}^{2h}\sum_{l=1}^{2h} \mathrm{Cov}(y_{ij}, \mathbf{c}_{kl})] = 0. \tag{5.20}$$

by Assumption 6. Finally, we have

$$\mathrm{Cov}(\bar{\mathbf{c}}_i, \bar{\mathbf{c}}_k) = \mathrm{Cov}(\frac{1}{2h}\sum_{j=1}^{2h} \mathbf{c}_{ij}, \frac{1}{2h}\sum_{l=1}^{2h} \mathbf{c}_{kl}) = \frac{1}{4h^2}[\sum_{j=1}^{2h}\sum_{l=1}^{2h} \mathrm{Cov}(\mathbf{c}_{ij}, \mathbf{c}_{kl})] = \mathbf{O}_{s \times s} \tag{5.21}$$

by Assumption 8.

Under the joint normality assumption of the response and control variates, from equations (5.3)-(5.5) in Section 5.1, and (5.18)-(5.21), we find the joint distribution of $\bar{\mathbf{y}}$ and $\bar{\mathbf{C}}$ as follows:

$$\begin{bmatrix} \bar{\mathbf{y}} \\ \mathrm{Vec}(\bar{\mathbf{C}}) \end{bmatrix} \sim N_{m(s+1)}\left[\begin{bmatrix} \mathbf{X}\beta \\ \mathbf{0}_{ms} \end{bmatrix}, \ \mathbf{\Sigma}\right]; \tag{5.22}$$

where Vec($\mathbf{C}$) denotes the operation that the columns of $\bar{\mathbf{C}}$ are stacked into a single ms-dimensional vector;

$$\boldsymbol{\Sigma} = \frac{1}{2h}\begin{bmatrix} \boldsymbol{\Sigma}_y & \boldsymbol{\sigma}'_{yc}\otimes\mathbf{I}_m \\ \boldsymbol{\sigma}_{yc}\otimes\mathbf{I}_m & \boldsymbol{\Sigma}_c\otimes\mathbf{I}_m \end{bmatrix}, \tag{5.23}$$

where $\otimes$ denotes a Kronecker operation of two matrices; and

$$\boldsymbol{\Sigma}_y = \sigma_y^2 \begin{bmatrix}
1-\rho_y & \rho^*_+ + \rho^*_- & \cdots & \rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & \cdots & \rho^*_+ + \rho^*_- \\
\rho^*_+ + \rho^*_- & 1-\rho_y & \cdots & \rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & \cdots & \rho^*_+ + \rho^*_- \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & \cdots & 1-\rho_y & \rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & \cdots & \rho^*_+ + \rho^*_- \\
\rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & \cdots & \rho^*_+ + \rho^*_- & 1-\rho_y & \rho^*_+ + \rho^*_- & \cdots & \rho^*_+ + \rho_- \\
\rho^*_+ + \rho_- & \rho^*_+ + \rho_- & \cdots & \rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & 1-\rho_y & \cdots & \rho^*_+ + \rho_- \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & & \rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & \rho^*_+ + \rho^*_- & & 1-\rho_y
\end{bmatrix}$$

$$= \frac{\sigma_y^2}{2h}[(1-\rho_y-\rho^*_+-\rho^*_-)\mathbf{I}_m + (\rho^*_+ + \rho^*_-)\mathbf{1}_m\mathbf{1}'_m]. \tag{5.24}$$

From Theorem 2.5.1 in Anderson (1984), the conditional variance of $\bar{\mathbf{y}}$ given $\bar{\mathbf{C}}$ is as follows:

$$\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}}) = \frac{1}{2h}[\boldsymbol{\Sigma}_y - (\boldsymbol{\sigma}'_{yc}\otimes\mathbf{I}_m)(\boldsymbol{\Sigma}_c\otimes\mathbf{I}_m)^{-1}(\boldsymbol{\sigma}_{yc}\otimes\mathbf{I}_m)]$$

$$= \frac{1}{2h}[\boldsymbol{\Sigma}_y - (\boldsymbol{\sigma}'_{yc}\otimes\mathbf{I}_m)(\boldsymbol{\Sigma}_c^{-1}\otimes\mathbf{I}_m)(\boldsymbol{\sigma}_{yc}\otimes\mathbf{I}_m)] = \frac{1}{2h}[\boldsymbol{\Sigma}_y - (\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\otimes\mathbf{I}_m)(\boldsymbol{\sigma}_{yc}\otimes\mathbf{I}_m)]$$

$$= \frac{1}{2h}[\boldsymbol{\Sigma}_y - (\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}\otimes\mathbf{I}_m)] = \frac{1}{2h}[\boldsymbol{\Sigma}_y - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}\mathbf{I}_m]$$

$$= \frac{\sigma_y^2}{2h}[(1-\rho_y-\rho^*_+-\rho^*_-)\mathbf{I}_m + (\rho^*_+ + \rho^*_-)\mathbf{1}\mathbf{1}' - \sigma_y^{-2}\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}\mathbf{I}_m]$$

$$= \frac{\sigma_y^2}{2h}[(1 - \rho_y - \rho_+^* - \rho_-^* - R_{yc}^2)\mathbf{I}_m + (\rho_+^* + \rho_-^*)\mathbf{1}\mathbf{1}'] = \gamma\mathbf{I}_m + \delta\mathbf{1}\mathbf{1}', \qquad (5.25)$$

where R_{yc} is the multiple correlation coefficient between y_{ij} and $\mathbf{c}_{ij}$. A sufficient condition for the equivalence of OLS and WLS estimators is that the dispersion matrix in (5.25) is representable as

$$\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}}) = \mathbf{X}\Xi\mathbf{X}' + \mathbf{z}\theta\mathbf{z}' + \sigma^2\mathbf{I}_m, \qquad (5.26)$$

where $\mathbf{z}$ is an $(m \times 1)$ vector such that $\mathbf{z}'\mathbf{X} = \mathbf{0}$, and Ξ, θ, and σ^2 are arbitrary (see equation (63) in Rao (1967), and equation (3.6) in Schruben and Margolin (1978)). Clearly, the covariance matrix in (5.25) is of the form in (5.26) since

$$\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}}) = \frac{1}{2h}(1 - \rho_y - \rho_+^* - \rho_-^* - R_{yc}^2)\sigma_y^2\mathbf{I}_m + \frac{1}{2h}(\rho_+^* + \rho_-^*)\sigma_y^2\mathbf{X}\mathbf{G}_{p+1}\mathbf{X}', \qquad (5.26)$$

where $\mathbf{G}_{p+1}$ is defined in (5.14). The OLS estimator of $\boldsymbol{\beta}$ in (5.2) is given by

$$\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \bar{\mathbf{C}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{P}]\bar{\mathbf{y}} \qquad (5.28)$$

(see (2.23)). Taking the operation of variance on (5.28) gives

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \bar{\mathbf{C}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{P}]\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}})[\mathbf{I}_m - \mathbf{P}\bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}']\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1},$$

which is developed into, by substituting for $\mathrm{Var}(\bar{\mathbf{y}} \mid \bar{\mathbf{C}})$ with (5.25),

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \bar{\mathbf{C}}) = \gamma(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{P}][\mathbf{I}_m - \mathbf{P}\bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}']\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

$$+ \delta(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}'\mathbf{P}]\mathbf{1}_m\mathbf{1}'_m[\mathbf{I}_m - \mathbf{P}\bar{\mathbf{C}}(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}\bar{\mathbf{C}}']\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}. \qquad (5.29)$$

The first term in this equation reduces to

$$\gamma(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{P}][\mathbf{I}_m - \mathbf{P}\overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}']\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

$$= \gamma(\mathbf{X}'\mathbf{X})^{-1} + \gamma(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \qquad (5.30)$$

since $\mathbf{X}'\mathbf{P} = \mathbf{P}\mathbf{X} = \mathbf{0}$. After some computations, the second term is equivalent to

$$\delta(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\mathbf{I}_m - \overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{P}]\mathbf{1}_m\mathbf{1}'_m[\mathbf{I}_m - \mathbf{P}\overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}']\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \delta\mathbf{G}_{p+1}, \qquad (5.31)$$

where $\mathbf{G}_{p+1}$ is defined in (5.14) (see the proof of (5.31) in Appendix A). Substituting (5.30) and (5.31) into (5.29), we find the conditional variance of $\hat{\boldsymbol{\beta}}_{\mathbf{G}}$, given $\overline{\mathbf{C}}$, as follows:

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \overline{\mathbf{C}}) = \gamma[(\mathbf{X}'\mathbf{X})^{-1} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}] + \delta\mathbf{G}_{p+1}. \qquad (5.32)$$

Since the least squares estimator $\hat{\boldsymbol{\beta}}_{\mathbf{G}}$ is an unbiased estimator conditionally on $\overline{\mathbf{C}}$ from (2.41), the unconditional variance of $\hat{\boldsymbol{\beta}}_{\mathbf{G}}$ is given by

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}}) = \mathrm{E}[\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \overline{\mathbf{C}})] = \gamma(\frac{m-p-2}{m-p-s-2})(\mathbf{X}'\mathbf{X})^{-1} + \delta\mathbf{G}_{p+1}. \qquad (5.33)$$

(see the proof of (5.33) in Appendix A).

5.4. Extended Schruben-Margolin Method

This section proposes a new approach utilizing the Schruben-Margolin method and control variates method simultaneously in a designed simulation experiment. As Schruben and Margolin (1978) noted, the efficiency gain of their method highly depends on the induced correlation between two responses in the same block. Also the simulation efficiency of the control variates method is determined by the correlation between the response and a set of selected control variates. Thus, in combining both methods, a key issue is how to maintain the correlation coefficient between two controlled responses (having smaller variances) in the same block.

During simulation of the jth replication at the ith design point, in addition to the response variable y_{ij}, suppose we collect a $(s \times 1)$ vector of control variates $\mathbf{c}_{ij}$ ($i = 1, 2, ..., m$, $j = 1, 2, ..., 2h$). The control variates are assumed to be independent of the factor variables (see the discussion in Section 2.1). The Schruben-Margolin method yields the same control variates at all design points in the same block by common random number streams if we perform simulation replication during the same period at each design point: $\mathbf{c}_{ij} = \mathbf{c}_{kj}$ ($j = 1, 2, ..., 2h$) if two design points i and k are in the same block. Also the control variates observed from two different blocks are negatively correlated by antithetic streams. That is, the Schruben-Margolin correlation induction strategy allows the observation of the control variates across m design points for the $2h$ replications as follows:

$$\overline{\mathbf{C}} = \begin{bmatrix} \bar{\mathbf{c}}'_1 \\ \vdots \\ \bar{\mathbf{c}}'_{m_1} \\ \bar{\mathbf{c}}'_{m_1+1} \\ \vdots \\ \bar{\mathbf{c}}'_m \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{c}}'_1 \\ \vdots \\ \bar{\mathbf{c}}'_1 \\ \bar{\mathbf{c}}'_{m_1+1} \\ \vdots \\ \bar{\mathbf{c}}'_{m_1+1} \end{bmatrix}, \tag{5.34}$$

where the first m_1 design points are in the first block, and the second $m_2 = (m - m_1)$ design points are in the second block. A reparameterization of each control variate in (5.34) gives that each control variate in the first block of the design is at low (high) level and that in the second block is at high (low) level. Therefore, adding the matrix of the mean control variates $\overline{\mathbf{C}}$ (number of control variates is s) to the linear model in (5.1) gives the same effect as including the variable with its high level in the first block and its low level in the second block. In applying the Schruben-Margolin method, the effect of the variable described above is assumed to be negligible since this effect is compounded with the blocking effect of random number streams which needs to be incorporated into the model for decomposing the random error term ε in (5.1). Thus, regression analysis on the linear model with the factor variables and the control variates is not desirable although we can identify the control variates which are strongly correlated with responses, and independent of the factor variables.

We now consider a different way of utilizing the control variates under the Schruben-Margolin method: instead of including the matrix of the mean control variates of the $2h$ replicates to the linear model (5.1), and conducting regression analysis on the response with both factor variables and control variates (on the linear model in (5.2)), we first adjust the ith mean response using the control variates $\mathbf{c}_{ij}$ ($j = 1, 2, \ldots, 2h$) obtained at the corresponding design point, and then apply regression analysis on the controlled responses with factor variables. Since the controlled response at a single design point follows the normal distribution under the joint normality assumption of the response and control variates (see p. 36 and also equation (2.39)), the controlled responses across the m design points have the m-variates normal distribution. Also, conditioning on the control variates, the controlled mean response at each design point is an unbiased estimator of the mean response (see the discussion on p. 32). Therefore, the controlled response can be written as follows:

$$\bar{y}_i(\hat{\alpha}_i) = \bar{y}_i - \bar{\mathbf{c}}'_i\hat{\alpha}_i = \mathbf{x}'_i\beta + \varepsilon^{*}_i \quad \text{for } i = 1, 2, \ldots, m, \tag{5.35}$$

where $\bar{y}_i$ and $\bar{\mathbf{c}}_i$ are the mean response and a $(s \times 1)$ mean vector of control variates at the ith design point, respectively, $\hat{\alpha}_i$ is a $(s \times 1)$ coefficient vector of control variates, and $\varepsilon_i^* \sim N(0,\ \kappa\sigma_y^2)$, where κ will be specified lately in this section. In matrix form, we can write the linear model in (5.35) as follows:

$$\bar{\mathbf{y}}(\hat{\mathbf{A}}) = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}^*, \tag{5.36}$$

where $\mathbf{X}$, $\boldsymbol{\beta}$ and $\boldsymbol{\varepsilon}^*$ are in (5.2), $\bar{\mathbf{y}}(\hat{\mathbf{A}}) = (\bar{y}_1(\hat{\boldsymbol{\alpha}}_1),\ \bar{y}_2(\hat{\boldsymbol{\alpha}}_2),\ \ldots,\ \bar{y}_m(\hat{\boldsymbol{\alpha}}_m))'$.

In this model, since we adjust the response based on the control variates obtained at each design point (considered as a single population), this approach allows using more control variates than the approach of Nozari, Arnold and Pegden (1984) for the multipopulation simulation model with control variates. The number of control variates depends not on the number of design points but the efficiency of control variates in a single population model (see equation (2.108) and the ensuing discussion of the number of control variates in a single population model).

If we use independent random number streams across the m design points, then the controlled responses, $\bar{y}_i(\hat{\boldsymbol{\alpha}}_i)$ $(i = 1,\ 2,\ \ldots, m)$, are observed independently. Thus, the estimator for $\boldsymbol{\beta}$ can be obtained by the ordinary least squares method. However, in applying the Schruben-Margolin method, the control variates obtained at a design point are same as those at the other design points in the same block (either the first block or the second block) by common random number streams (see (5.34)). Also, the control variates obtained at any two design points not in the same block are negatively correlated with an equal amount since the Schruben-Margolin method uses random number streams antithetic to those used in the first (second) for the second (first) block, and the same random number streams for either the first block or the second block. Of course, the Schruben-Margolin method induces the correlation between the responses across all design points. Thus, a covariance matrix of the controlled response is different from that of the response given in (5.13). In the remainder of this section, we first specify the covariance matrix of the controlled responses, then develop a procedure to estimate the model parameters, and finally find the covariance matrix of the estimators.

For the responses and control variates obtained by the Schruben-Margolin correlation induction strategy across the m design points for each replication, and independent random number streams through the $2h$ replications, in addition to the assumptions established by Schruben and Margolin (1978), we can assume the relationships between the response variable and control variates, and between the control variate across the design points and replicates as follows:

1. $\mathrm{Cov}(y_{ij}, \mathbf{c}_{kl}) = \boldsymbol{\sigma}'_{yc}$ if two design points i and k are in the same block, and $j = l$ (homogeneity of control variates-response covariance across design points in the same block and replicates). Otherwise, $\mathrm{Cov}(y_{ij}, \mathbf{c}_{kl}) = \mathbf{0}'$.

2. $\mathrm{Cov}(y_{ij}, \mathbf{c}_{kl}) = \boldsymbol{\sigma}^{*\prime}_{yc}$ if two design points i and k are not in the same block, and $j = l$ (homogeneity of control variate-response covariance across design points in the different blocks and replicates). Otherwise, $\mathrm{Cov}(y_{ij}, \mathbf{c}_{kl}) = \mathbf{0}'$.

3. $\mathrm{Cov}(\mathbf{c}_{ij}) = \boldsymbol{\Sigma}_c$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, 2h$ (homogeneity of control variates covariance structure across design points and replicates).

4. $\mathrm{Cov}(\mathbf{c}_{ij}, \mathbf{c}_{kl}) = \boldsymbol{\Sigma}_c$ if two design points i and k are in the same block, and $j = l$ (homogeneity of control variates covariance structure across design points in the same block and replicates); $\mathrm{Cov}(\mathbf{c}_{ij}, \mathbf{c}_{kl}) = \boldsymbol{\Sigma}^*_c$ if two design points i and k are not in the same block, and $j = l$ (homogeneity of control variates covariance structure

across design points in the different blocks and replicates); Otherwise, $\mathrm{Cov}(\mathbf{c}_{ij},\ \mathbf{c}_{kl}) = \mathbf{O}_{s \times s}$ (independence of control variates across design design points and replicates).

Under these assumptions, we find the covariance structure of the controlled responses for the two cases that the optimal coefficient vector of control variates, $\boldsymbol{\alpha}$, is known and unknown.

When the optimal value of $\boldsymbol{\alpha} = \boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}$ is known, the variance of the mean controlled estimator at the ith design point is given by

$$\mathrm{Var}(\bar{y}_i - \bar{\mathbf{c}}'_i\boldsymbol{\alpha}) = \mathrm{Var}(\frac{1}{2h}\sum_{j=1}^{2h} y_{ij} - \frac{1}{2h}\sum_{i=1}^{2h} \mathbf{c}'_{ij}\boldsymbol{\alpha}) = \frac{1}{4h^2}\sum_{j=1}^{2h}\mathrm{Var}(y_{ij} - \mathbf{c}'_{ij}\boldsymbol{\alpha}) \qquad (5.37)$$

by Assumption 1. If we develop this equation according to the formula of Corollary 5.2.1 (Mood, Graybill and Boes (1974)), and replace $\boldsymbol{\alpha}$ with $\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}$, then we find, by Assumption 1,

$$\mathrm{Var}(\bar{y}_i - \bar{\mathbf{c}}'_i\boldsymbol{\alpha}) = \frac{1}{4h^2}\sum_{j=1}^{2h}[\mathrm{Var}(y_{ij}) - 2\mathrm{Cov}(y_{ij},\ \mathbf{c}'_{ij}\boldsymbol{\alpha}) + \boldsymbol{\alpha}'\mathrm{Cov}(\mathbf{c}_{ij})\boldsymbol{\alpha}]$$

$$= \frac{1}{4h^2}\sum_{j=1}^{2h}[\sigma_y^2 - 2\boldsymbol{\sigma}'_{yc}\boldsymbol{\alpha} + \boldsymbol{\alpha}'\boldsymbol{\Sigma}_c\boldsymbol{\alpha}] = \frac{1}{2h}\sigma_y^2(1 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc})$$

$$= \frac{1}{2h}\sigma_y^2(1 - R_{yc}^2), \qquad (5.38)$$

where R_{yc} is the multiple correlation coefficient between y_{ij} and $\mathbf{c}_{ij}$ (or $\mathbf{c}_{kj}$). This equation shows that the response adjusted by the control variates with known coefficient vector $\boldsymbol{\alpha}$ has a variance reduced by $(1 - R^2_{yc})$ over that given in (5.13). By Corollary 5.2.1 in Mood, Graybill and Boes (1974), the covariance between two controlled responses at design points i and k in the same block is given by

$$\mathrm{Cov}(\bar{y}_i - \bar{\mathbf{c}}'_i\boldsymbol{\alpha}, \bar{y}_k - \bar{\mathbf{c}}'_k\boldsymbol{\alpha}) = \mathrm{Cov}(\bar{y}_i, \bar{y}_k) - \mathrm{Cov}(\bar{y}_i, \bar{\mathbf{c}}'_k\boldsymbol{\alpha}) - \mathrm{Cov}(\bar{y}_k, \bar{\mathbf{c}}'_i\boldsymbol{\alpha}) + \boldsymbol{\alpha}'\mathrm{Cov}(\bar{\mathbf{c}}_i, \bar{\mathbf{c}}_k)\boldsymbol{\alpha}, \tag{5.39}$$

where each term is developed as follows:

$$\mathrm{Cov}(\bar{y}_i, \bar{y}_k) = \frac{1}{2h}\rho_+\sigma_y^2 \tag{5.40}$$

from equation (5.13); Corollary 5.2.1 in Mood, Graybill and Boes (1974) gives

$$\mathrm{Cov}(\bar{y}_i, \bar{\mathbf{c}}'_k\boldsymbol{\alpha}) = \mathrm{Cov}\left(\frac{1}{2h}\sum_{j=1}^{2h} y_{ij}, \ \frac{1}{2h}\sum_{l=1}^{2h}\mathbf{c}'_{kl}\boldsymbol{\alpha}\right) = \frac{1}{4h^2}\sum_{j=1}^{2h}\sum_{l=1}^{2h}\mathrm{Cov}(y_{ij}, \mathbf{c}'_{kl}\boldsymbol{\alpha})$$

which reduces to, by the assumption 1 and replacement of $\boldsymbol{\alpha}$ into this equation,

$$= \frac{1}{4h^2}\sum_{j=1}^{2h}\mathrm{Cov}(y_{ij}, \ \mathbf{c}'_{kj}\boldsymbol{\alpha}) = \frac{2h}{4h^2}\boldsymbol{\sigma}'_{yc}\boldsymbol{\alpha} = \frac{1}{2h}\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc}; \tag{5.41}$$

similar to (5.41), we have

$$\mathrm{Cov}(\bar{y}_k, \ \bar{\mathbf{c}}'_i\boldsymbol{\alpha}) = \frac{1}{2h}\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} \tag{5.42}$$

by Assumption 1; Using the formula of Corollary 5.2.1 in Mood, Graybill and Boes (1974), we develop

$$\alpha' \mathrm{Cov}(\bar{\mathbf{c}}_i, \bar{\mathbf{c}}_k)\alpha = \alpha' \mathrm{Cov}(\frac{1}{2h}\sum_{j=1}^{2h}\mathbf{c}_{ij}, \ \frac{1}{2h}\sum_{l=1}^{2h}\mathbf{c}_{kl})\alpha = \frac{1}{4h^2}\alpha' \sum_{j=1}^{2h}(\mathbf{c}_{ij}, \ \mathbf{c}_{kj})\alpha$$

which reduces to, by Assumptions 3 and 4, and substitution of α with $\Sigma_c^{-1}\sigma_{yc}$,

$$= \frac{2h}{4h^2}\alpha' \Sigma_c \alpha = \frac{1}{2h}\sigma'_{yc}\Sigma_c^{-1}\sigma_{yc}. \tag{5.43}$$

Plugging (5.40)-(5.43) into (5.39) gives

$$\mathrm{Cov}(\bar{y}_i - \bar{\mathbf{c}}'_i\alpha, \bar{y}_k - \bar{\mathbf{c}}'_k\alpha) = \frac{1}{2h}(\rho_+\sigma_y^2 - \sigma'_{yc}\Sigma_c^{-1}\sigma_{yc}) = \frac{1}{2h}\sigma_y^2(\rho_+ - R_{yc}^2), \tag{5.44}$$

where R_{yc} is defined in (5.38). As we see in this equation, the covariance between two controlled responses in the same block also decreases by the same amount as in the variance reduction of the controlled response in (5.38).

If two controlled responses i and k are not in the same block, in a similar procedure given as above, we find each term in (5.39) as follows:

$$\mathrm{Cov}(\bar{y}_i, \bar{y}_k) = \frac{1}{2h}\rho_-\sigma_y^2 \tag{5.45}$$

from equation (5.13);

$$\mathrm{Cov}(\bar{y}_i, \bar{\mathbf{c}}'_k\alpha) = \mathrm{Cov}(\frac{1}{2h}\sum_{j=1}^{2h}y_{ij}, \ \frac{1}{2h}\sum_{l=1}^{2h}\mathbf{c}'_{kl}\alpha) = \frac{1}{4h^2}\sum_{j=1}^{2h}\mathrm{Cov}(y_{ij}, \mathbf{c}'_{kj}\alpha)$$

$$= \frac{2h}{4h^2}\sigma_{yc}^{*\prime}\alpha = \frac{1}{2h}\sigma_{yc}^{*\prime}\Sigma_c^{-1}\sigma_{yc} \tag{5.46}$$

by Assumption 2 and Corollary 5.2.1 in Mood, Graybill and Boes (1974); similar to (5.46), we find

$$\text{Cov}(\bar{y}_k, \bar{\mathbf{c}}'_i\boldsymbol{\alpha}) = \frac{1}{2h}\,\boldsymbol{\sigma}^{*\prime}_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} \tag{5.47}$$

by Assumption 2;

$$\boldsymbol{\alpha}'\text{Cov}(\bar{\mathbf{c}}_i, \bar{\mathbf{c}}_k)\boldsymbol{\alpha} = \boldsymbol{\alpha}'\text{Cov}(\frac{1}{2h}\sum_{j=1}^{2h}\mathbf{c}_{ij},\ \frac{1}{2h}\sum_{l=1}^{2h}\mathbf{c}_{kl})\boldsymbol{\alpha} = \frac{1}{4h^2}\,\boldsymbol{\alpha}'\sum_{j=1}^{2h}(\mathbf{c}_{ij},\ \mathbf{c}_{kj})\boldsymbol{\alpha}$$

$$= \frac{2h}{4h^2}\,\boldsymbol{\alpha}'\boldsymbol{\Sigma}^*_c\boldsymbol{\alpha} = \frac{1}{2h}\,\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\Sigma}^*_c\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} \tag{5.48}$$

by substitution of $\boldsymbol{\alpha}$ and Assumptions 3 and 4. Substitution of each term in (5.39) with equations (5.45)-(5.48) yields

$$\text{Cov}(\bar{y}_i - \bar{\mathbf{c}}_i'\boldsymbol{\alpha},\ \bar{y}_k - \bar{\mathbf{c}}_k'\boldsymbol{\alpha}) = \frac{1}{2h}\,(\rho_-\sigma_y^2 - 2\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}^*_{yc} + \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\Sigma}^*_c\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc})$$

$$= \frac{1}{2h}\,\sigma_y^2(\rho_- - R^*_{yc}), \tag{5.49}$$

where $R^*_{yc} = \sigma_y^{-2}(2\boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}^*_{yc} - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\Sigma}^*_c\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc})$. The term $\sigma_y^2R^*_{yc}$ can be interpreted as a difference between the covariance of the two responses and that of the two responses adjusted by the control variates when there exist the correlations among $\bar{y}_i$, $\bar{y}_k$, $\bar{\mathbf{c}}_i$, and $\bar{\mathbf{c}}_k$ with the known $\boldsymbol{\alpha}$. For the case of a single control variate ($s = 1$), the term R^*_{yc} in (5.49) can be written as

$$R^*_{yc} = \sigma_y^{-2}\sigma_c^{-2}[2\sigma_{yc}\sigma^*_{yc} - \sigma^2_{yc}(- \rho_c\sigma_c^2)\sigma_c^{-2}] = 2\,\frac{\sigma_{yc}}{\sigma_y\sigma_c}\,\frac{\sigma^*_{yc}}{\sigma_y\sigma_c} + \rho_c\,\frac{\sigma^2_{yc}}{\sigma_y^2\sigma_c^2}$$

$$= 2\rho_{yc}\rho^{*}_{yc} + \rho_c\rho^2_{yc}; \tag{5.50}$$

where ρ_{yc} is the correlation coefficient between y_{ij} and c_{ij}; ρ^{*}_{yc} is the correlation coefficient between y_{ij} and c_{kj}, where y_{ij} and c_{kj} are in two different blocks; and $-\rho_c$ $(\rho_c > 0)$ is the correlation coefficient between c_{ij} and c_{kj} in the two different blocks. Instead of identifying the relationship among ρ_{yc}, ρ^{*}_{yc}, and ρ_c analytically, we computed this relationship based on the data set obtained from the simulation run for the hospital example (we will discuss this problem in detail in the next chapter). The computational results show that

$$\rho^{*}_{yc} \cong -\rho_c\rho_{yc}. \tag{5.51}$$

(see Table 40). Under this relationship, R^{*}_{yc} in (5.50) reduces to

$$R^{*}_{yc} \cong -2\rho_c\rho^2_{yc} + \rho_c\rho^2_{yc} = -\rho_c\rho^2_{yc} < 0, \tag{5.52}$$

which implies the negative correlation between the two controlled response in the two different blocks is reduced by approximately $\rho_c\rho^2_{yc}$ for the case of a single control variate.

We now consider the covariance matrix of the controlled responses across the m design points for the case of the known $\boldsymbol{\alpha}$. From equations (5.38), (5.44) and (5.49), if we divide the covariance matrix by the variance of the controlled response in (5.38), then we obtain the covariance matrix of the m controlled responses as follows:

$$\mathrm{Cov}(\bar{\mathbf{y}}(\alpha)) = \frac{(1 - R_{yc}^2)\sigma_y^2}{2h} \begin{bmatrix} 1 & r & \cdots & r & q & q & \cdots & q \\ r & 1 & \cdots & r & q & q & \cdots & q \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ r & r & \cdots & 1 & q & q & \cdots & q \\ q & q & \cdots & q & 1 & r & \cdots & r \\ q & q & \cdots & q & r & 1 & \cdots & r \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ q & q & \cdots & q & r & r & \cdots & 1 \end{bmatrix}, \tag{5.53}$$

where $\bar{\mathbf{y}}(\alpha) = \bar{\mathbf{y}} - \bar{\mathbf{C}}\alpha$,

$$r = \frac{\rho_+ - R_{yc}^2}{1 - R_{yc}^2} \quad \text{and} \quad q = \frac{\rho_- - R_{yc}^{*}}{1 - R_{yc}^2}. \tag{5.54}$$

This covariance matrix can be written in another form given by

$$\mathrm{Cov}(\bar{\mathbf{y}}(\alpha)) = \frac{(1 - R_{yc}^2)\sigma_y^2}{2h} [\frac{1}{2}(r + q)\mathbf{X}\mathbf{G}_{p+1}\mathbf{X}' + \frac{1}{2}(r + q)\mathbf{z}_m\mathbf{z}'_m + (1 - r)\mathbf{I}_m], \tag{5.55}$$

where $\mathbf{G}_{p+1}$ and $\mathbf{z}_m$ are as defined in (5.14). The covariance matrix in (5.53) has a structure similar to that in (5.13). For a dispersion matrix having the above pattern, the WLS estimator for $\boldsymbol{\beta}$ is equal to the OLS estimator (see equation (63) in Rao (1967)), which is given by

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\bar{\mathbf{y}}(\alpha)). \tag{5.56}$$

Taking the variance operation on this equation and substituting $\mathrm{Cov}(\bar{\mathbf{y}}(\alpha))$ with (5.53) yields

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{Cov}(\bar{\mathbf{y}}(\alpha))\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

$$= \frac{(1 - R_{yc}^2)\sigma_y^2}{2h} (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\frac{1}{2}(r + q)\mathbf{X}\mathbf{G}_{p+1}\mathbf{X}' + \frac{1}{2}(r + q)\mathbf{z}_m\mathbf{z}'_m + (1 - r)\mathbf{I}_m]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

$$= \frac{(1 - R_{yc}^2)\sigma_y^2}{2h} [\frac{1}{2}(r + q)\mathbf{G}_{p+1} + (1 - r)(\mathbf{X}'\mathbf{X})^{-1}] \tag{5.57}$$

since $\mathbf{X}'\mathbf{z}_m = \mathbf{z}'_m\mathbf{X} = \mathbf{0}$. Substitution for r and q in (5.54) into (5.57) gives

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = \frac{(1 - R_{yc}^2)\sigma_y^2}{2h} [\frac{1}{2}(\frac{\rho_+ - R_{yc}^2}{1 - R_{yc}^2} + \frac{\rho_- - R_{yc}^*}{1 - R_{yc}^2})\mathbf{G}_{p+1} + (1 - \frac{\rho_+ - R_{yc}^2}{1 - R_{yc}^2})(\mathbf{X}'\mathbf{X})^{-1}]$$

$$= \frac{\sigma_y^2}{2h} [\frac{1}{2}\{(\rho_+ + \rho_-) - (R_{yc}^2 + R_{yc}^*)\}\mathbf{G}_{p+1} + (1 - \rho_+)(\mathbf{X}'\mathbf{X})^{-1}] \tag{5.58}$$

As we see in this equation, the variances of β_i ($i = 1, 2, .., p$) are same as those obtained by the Schruben-Margolin method (see (5.13)), but the variance of β_0 is less than that in (5.13) provided $R_{yc}^2 + R_{yc}^* > 0$. Under the relationship in (5.51), this condition holds since

$$R_{yc}^2 + R_{yc}^* = \rho_{yc}^2 - \rho_c\rho_{yc}^2 = (1 - \rho_c)\rho_{yc}^2 > 0 \tag{5.59}$$

for the case of $s = 1$.

Next we consider the case that the optimal value of $\boldsymbol{\alpha}$ is unknown. For the ith design point, we estimate it by (see the estimator of $\boldsymbol{\alpha}$ in Section 2.2.2 for a single population model)

$$\hat{\boldsymbol{\alpha}}_i = S_{\mathbf{c}_i}^{-1}S_{y_i\mathbf{c}_i} = [(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')']^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')(\mathbf{y}_i - \bar{y}_i\mathbf{1}); \tag{5.60}$$

where $\mathbf{C}_i$ is a $(s \times 2h)$ matrix of the control variates; $\mathbf{y}_i$ is a $(2h \times 1)$ response vector; $\bar{\mathbf{c}}_i$ is a $(s \times 1)$ mean vector of control variates; $\bar{y}_i$ is the mean response; respectively, at the ith design point; and $\mathbf{1}$ is a $(2h \times 1)$ column vector of ones. The sample estimator of $\boldsymbol{\alpha}$ at the ith design point can be represented as

$$\hat{\boldsymbol{\alpha}}_i = [(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')']^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')\mathbf{y}_i = \mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')\mathbf{y}_i \tag{5.61}$$

since

$$(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')\bar{y}_i\mathbf{1} = \bar{y}_i(\mathbf{C}_i\mathbf{1} - \bar{\mathbf{c}}_i\mathbf{1}'\mathbf{1}) = \bar{y}_i(2h\bar{\mathbf{c}}_i - 2h\bar{\mathbf{c}}_i) = \mathbf{0}. \tag{5.62}$$

From the result of a single population model (see (2.97)), the variance of the mean controlled estimator at the ith design point is given by

$$\mathrm{Var}(\bar{y}_i - \bar{\mathbf{c}}'_i\hat{\boldsymbol{\alpha}}_i) = \frac{1}{2h}(1 - R^2_{yc})\sigma^2_y\left(\frac{2h-2}{2h-s-2}\right). \tag{5.63}$$

We now consider the joint distribution of $\mathbf{y}_i$ and $\mathbf{y}_k$, and their corresponding control variates $\mathbf{C}_i$ and $\mathbf{C}_k$, where the two design points i and k are in the same block. The matrices of control variates $\mathbf{C}_i$ and $\mathbf{C}_k$ are same since both $\mathbf{C}_i$ and $\mathbf{C}_k$ are obtained by common random number streams across two design points i and k. For the jth replicate of simulation output, the joint distribution of the response and control variates is given by, according to Assumption 1 in this section and Assumption 1 of Schruben and Margolin (1978) (see Section 5.2),

$$\begin{bmatrix} y_{ij} \\ y_{kj} \\ \mathbf{c}_{ij} \end{bmatrix} \sim N_{(s+2)}\left[\begin{bmatrix} \mu_i \\ \mu_k \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \sigma^2_y & \rho_+\sigma^2_y & \boldsymbol{\sigma}'_{yc} \\ \rho_+\sigma^2_y & \sigma^2_y & \boldsymbol{\sigma}'_{yc} \\ \boldsymbol{\sigma}_{yc} & \boldsymbol{\sigma}_{yc} & \boldsymbol{\Sigma}_c \end{bmatrix}\right], \tag{5.64}$$

where $\mu_i = \mathbf{x}'_i\boldsymbol{\beta}$ and $\mu_k = \mathbf{x}'_k\boldsymbol{\beta}$. Thus, given $\mathbf{c}_{ij}$ (or $\mathbf{c}_{kj}$), the conditional distribution of the responses is as follows:

$$\begin{bmatrix} y_{ij} \mid \mathbf{c}_{ij} \\ y_{kj} \mid \mathbf{c}_{ij} \end{bmatrix} \sim N_2 \left[\begin{bmatrix} \mu_i + \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\mathbf{c}_{ij} \\ \mu_k + \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\mathbf{c}_{ij} \\ \mathbf{0} \quad . \end{bmatrix}, \begin{bmatrix} \sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} & \rho_+\sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} \\ \rho_+\sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} & \sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} \end{bmatrix} \right] \tag{5.65}$$

by Theorem 2.5.1 in Anderson (1984). The off-diagonal element of this covariance matrix indicates that

$$\mathrm{Cov}(y_{ij}, y_{kl} \mid \mathbf{c}_{ij}) = \rho_+\sigma_y^2 - \boldsymbol{\sigma}'_{yc}\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\sigma}_{yc} = (\rho_+ - R_{yc}^2)\sigma_y^2 \ \text{ if } j = l, \tag{5.66}$$

where R_{yc}^2 is defined in (5.38). Since we use independent random number streams for the jth and lth replications ($j \neq l$), we have

$$\mathrm{Cov}(y_{ij}, y_{kl} \mid \mathbf{c}_{ij},\ \mathbf{c}_{kl}) = 0 \ \text{ if } j \neq l, \tag{5.67}$$

by Assumption 1 of this section and Assumption 3 of Schruben and Margolin (1978). Equations (5.66) and (5.67) imply that the conditional variance between $\mathbf{y}_i$ and $\mathbf{y}_k$, given $\mathbf{C}_i$ or $\mathbf{C}_k$, is the diagonal matrix having its diagonal element as in (5.66):

$$\mathrm{Cov}(\mathbf{y}_i,\ \mathbf{y}_k \mid \mathbf{C}_i,\ \mathbf{C}_k) = \mathrm{Cov}(\mathbf{y}_i,\ \mathbf{y}_k \mid \mathbf{C}_i) = (\rho_+ - R_{yc}^2)\sigma_y^2\mathbf{I}_{2h}. \tag{5.68}$$

By substitution of $\hat{\boldsymbol{\alpha}}_i$ and $\hat{\boldsymbol{\alpha}}_k$ in (5.61) for the controlled responses i and k, respectively, we develop the covariance of the controlled responses as follows: (note that $\bar{y}_i = \mathbf{1}'\mathbf{y}_i/2h$)

$$\mathrm{Cov}(\bar{y}_i - \bar{\mathbf{c}}'_i\hat{\boldsymbol{\alpha}}_i, \bar{y}_k - \bar{\mathbf{c}}'_k\hat{\boldsymbol{\alpha}}_k)$$

$$= \mathrm{Cov}[\frac{\mathbf{1}'}{2h}\mathbf{y}_i - \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')\mathbf{y}_i,\ \frac{\mathbf{1}'}{2h}\mathbf{y}_k - \bar{\mathbf{c}}'_k\mathbf{M}_k^{-1}(\mathbf{C}_k - \bar{\mathbf{c}}_k\mathbf{1}')\mathbf{y}_k]$$

$$= \mathrm{Cov}[(\frac{\mathbf{1}'}{2h} - \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}'))\mathbf{y}_i, (\frac{\mathbf{1}'}{2h} - \bar{\mathbf{c}}'_k\mathbf{M}_k^{-1}(\mathbf{C}_k - \bar{\mathbf{c}}_k\mathbf{1}'))\mathbf{y}_k]$$

$$= \mathrm{E}[\mathrm{Cov}[(\frac{\mathbf{1}'}{2h} - \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}'))\mathbf{y}_i, (\frac{\mathbf{1}'}{2h} - \bar{\mathbf{c}}'_k\mathbf{M}_k^{-1}(\mathbf{C}_k - \bar{\mathbf{c}}_k\mathbf{1}'))\mathbf{y}_k] \mid \mathbf{C}_i, \mathbf{C}_k]$$

$$= \mathrm{E}[(\frac{\mathbf{1}'}{2h} - \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}'))\mathrm{Cov}(\mathbf{y}_i, \mathbf{y}_k \mid \mathbf{C}_i, \mathbf{C}_k)(\frac{\mathbf{1}}{2h} - \mathbf{M}_k^{-1}(\mathbf{C}_k - \bar{\mathbf{c}}_k\mathbf{1}')\bar{\mathbf{c}}'_k)]$$

which reduces to, by substituting (5.68) for $\mathrm{Cov}(\mathbf{y}_i, \mathbf{y}_k \mid \mathbf{C}_i, \mathbf{C}_k)$ into the above equation,

$$= (\rho_+ - R_{yc}^2)\sigma_y^2\mathrm{E}[(\frac{\mathbf{1}'}{2h} - \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}'))(\frac{\mathbf{1}'}{2h} - \bar{\mathbf{c}}'_k\mathbf{M}_k^{-1}(\mathbf{C}_k - \bar{\mathbf{c}}_k\mathbf{1}'))'],$$

which further reduces to

$$= (\rho_+ - R_{yc}^2)\sigma_y^2\mathrm{E}[\frac{\mathbf{1}'\mathbf{1}}{4h^2} - \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')\frac{\mathbf{1}}{2h} - (\bar{\mathbf{c}}'_i\mathbf{M}_k^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')\frac{\mathbf{1}}{2h})'$$

$$+ \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')'(\mathbf{C}_k - \bar{\mathbf{c}}_k\mathbf{1}')\mathbf{M}_k^{-1}\bar{\mathbf{c}}_k]$$

$$= (\rho_+ - R_{yc}^2)\sigma_y^2\mathrm{E}[\frac{1}{2h} + \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}\bar{\mathbf{c}}_i] \tag{5.69}$$

since

$$\bar{\mathbf{c}}_i = \bar{\mathbf{c}}_k \quad \text{and} \quad \mathbf{C}_i = \mathbf{C}_k \tag{5.70}$$

by the assignment of common random number streams for the two design points i and k;

$$\bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')\mathbf{1} = \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(\mathbf{C}_i\mathbf{1} - \bar{\mathbf{c}}_i\mathbf{1}'\mathbf{1}) = \bar{\mathbf{c}}'_i\mathbf{M}_i^{-1}(2h\bar{\mathbf{c}}_i - 2h\bar{\mathbf{c}}_i) = \mathbf{0}; \tag{5.71}$$

and by (5.70),

$$\mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')'(\mathbf{C}_k - \bar{\mathbf{c}}_k\mathbf{1}')\mathbf{M}_k^{-1} = \mathbf{M}_i^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')'(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}')\mathbf{M}_i^{-1} = \mathbf{M}_i^{-1}\mathbf{M}_i\mathbf{M}_i^{-1} = \mathbf{M}_i^{-1}. \tag{5.72}$$

We note that (see Theorem 5.2.2 in Anderson (1984)),

$$T^2 = 2h(2h-1)\bar{\mathbf{c}}'_i \mathbf{M}_i^{-1} \bar{\mathbf{c}}_i \tag{5.73}$$

is Hotelling's T^2 statistic. Also, Corollary 5.2.1 of Anderson (1984) gives

$$\left(\frac{T^2}{2h-1}\right)\left(\frac{2h-s}{s}\right) = 2h\bar{\mathbf{c}}'_i \mathbf{M}_i^{-1} \bar{\mathbf{c}}_i \left(\frac{2h-s}{s}\right) \sim F(s,\ 2h-s), \tag{5.74}$$

and Kenny and Keeping (1951) gives

$$\mathrm{E}[F(s,\ 2h-s)] = \frac{2h-s}{2h-s-2}. \tag{5.75}$$

Using (5.73)-(5.75), we find (5.69) as follows:

$$\mathrm{Cov}(\bar{y}_i - \bar{\mathbf{c}}'_i\hat{\alpha}_i,\ \bar{y}_k - \bar{\mathbf{c}}'_k\hat{\alpha}_k) = (\rho_+ - R^2_{yc})\sigma^2_y\left[\frac{1}{2h} + E[\bar{\mathbf{c}}_i \mathbf{M}_i^{-1} \bar{\mathbf{c}}_i]\right]$$

$$= \frac{1}{2h}(\rho_+ - R^2_{yc})\sigma^2_y\left(1 + \frac{s}{2h-s-2}\right) = \frac{1}{2h}(\rho_+ - R^2_{yc})\sigma^2_y\left(\frac{2h-2}{2h-s-2}\right). \tag{5.76}$$

When two controlled responses are not in the same block, we can obtain their covariance with a similar procedure to that given above. With the same reasoning as in (5.67), we get

$$\mathrm{Cov}(y_{ij},\ y_{kl} \mid \mathbf{c}_{ij},\ \mathbf{c}_{kl}) = 0 \qquad \text{if } j \neq l. \tag{5.77}$$

Also, under the normality assumption of the response and control variates, the joint distribution of y_{ij} and $\mathbf{c}_{kj}$ is given by, according to Assumption 1-4, and Assumption 2 of Schruben and Margolin (1978) (see Section 5.2),

$$\begin{bmatrix} y_{ij} \\ y_{kj} \\ \mathbf{c}_{ij} \\ \mathbf{c}_{kj} \end{bmatrix} \sim N_{2(s+1)}\left[\begin{bmatrix} \mu_i \\ \mu_k \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho_-\sigma_y^2 & \boldsymbol{\sigma}'_{yc} & \boldsymbol{\sigma}_{yc}^{*\prime} \\ \rho_-\sigma_y^2 & \sigma_y^2 & \boldsymbol{\sigma}_{yc}^{*\prime} & \boldsymbol{\sigma}'_{yc} \\ \boldsymbol{\sigma}_{yc} & \boldsymbol{\sigma}_{yc}^{*} & \boldsymbol{\Sigma}_c & \boldsymbol{\Sigma}_c^{*} \\ \boldsymbol{\sigma}_{yc}^{*} & \boldsymbol{\sigma}_{yc} & \boldsymbol{\Sigma}_c^{*} & \boldsymbol{\Sigma}_c \end{bmatrix}\right], \tag{5.78}$$

where μ_i and μ_k are given in (5.64). Thus, given $\mathbf{c}_{ij}$ and $\mathbf{c}_{kj}$, by Theorem 2.5.1 in Anderson (1984), the covariance of the responses is as follows:

$$\mathrm{Cov}(y_{ij}, y_{kl} \mid \mathbf{c}_{ij}, \mathbf{c}_{kl}) = (\rho_- - \xi)\sigma_y^2 \qquad \text{if } j = l, \tag{5.79}$$

where ξ is the first-row second-column entry of the following matrix:

$$\sigma_y^{-2}\begin{bmatrix} \boldsymbol{\sigma}'_{yc} & \boldsymbol{\sigma}_{yc}^{*\prime} \\ \boldsymbol{\sigma}_{yc}^{*\prime} & \boldsymbol{\sigma}'_{yc} \end{bmatrix}\begin{bmatrix} \boldsymbol{\Sigma}_c & \boldsymbol{\Sigma}_c^{*} \\ \boldsymbol{\Sigma}_c^{*} & \boldsymbol{\Sigma}_c \end{bmatrix}^{-1}\begin{bmatrix} \boldsymbol{\sigma}_{yc} & \boldsymbol{\sigma}_{yc}^{*} \\ \boldsymbol{\sigma}_{yc}^{*} & \boldsymbol{\sigma}_{yc} \end{bmatrix}. \tag{5.80}$$

Thus, from (5.77) and (5.79), we find

$$\mathrm{Cov}(\mathbf{y}_i, \mathbf{y}_k \mid \mathbf{C}_i, \mathbf{C}_k) = (\rho_- - \xi)\sigma_y^2\mathbf{I}_{2h}. \tag{5.81}$$

Similar to the development of equation (5.69), we have the covariance of two controlled responses in different blocks as follows:

$$\mathrm{Cov}(\bar{y}_i - \bar{\mathbf{c}}'_i\hat{\boldsymbol{\alpha}}_i, \bar{y}_k - \bar{\mathbf{c}}'_k\hat{\boldsymbol{\alpha}}_k) = \mathrm{E}[\mathrm{Cov}(\bar{y}_i - \bar{\mathbf{c}}'_i\hat{\boldsymbol{\alpha}}_i, \bar{y}_k - \bar{\mathbf{c}}'_k\hat{\boldsymbol{\alpha}}_k \mid \mathbf{C}_i, \mathbf{C}_k)]$$

$$= \mathrm{E}[(\frac{\mathbf{1}'}{2h} - \bar{\mathbf{c}}'_i\mathbf{M}_k^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}'))\mathrm{Cov}(\mathbf{y}_i, \mathbf{y}_k \mid \mathbf{C}_i, \mathbf{C}_k)(\frac{\mathbf{1}}{2h} - \mathbf{M}_k^{-1}(\mathbf{C}_k - \bar{\mathbf{c}}_k\mathbf{1}')\bar{\mathbf{c}}'_k)]$$

which reduces to, by substitution of (5.81) for $\mathrm{Cov}(\mathbf{y}_i, \mathbf{y}_k \mid \mathbf{C}_i, \mathbf{C}_k)$,

$$= (\rho_- - \xi)\sigma_y^2\mathrm{E}[(\frac{\mathbf{1}'}{2h} - \bar{\mathbf{c}}'_i\mathbf{M}_k^{-1}(\mathbf{C}_i - \bar{\mathbf{c}}_i\mathbf{1}'))(\frac{\mathbf{1}'}{2h} - \mathbf{M}_k^{-1}(\mathbf{C}_k - \bar{\mathbf{c}}_k\mathbf{1}')\bar{\mathbf{c}}'_k)]$$

which further reduces to, by equation (5.62),

$$= (\rho_{-} - \xi)\sigma_y^2 E[\frac{1}{2h} + \bar{\mathbf{c}}'_i \mathbf{M}_i^{-1}(\mathbf{C}_i - \mathbf{c}_i\mathbf{1}')'(\mathbf{C}_k - \mathbf{c}_k\mathbf{1}')\mathbf{M}_k^{-1}\bar{\mathbf{c}}_k]. \tag{5.82}$$

To represent (5.82) as a similar form in (5.76), we now define

$$R_{yc}^{**} = (\frac{2h - s - 2}{2h - 2})[\xi - 2h(\rho_{-} - \xi)E[\bar{\mathbf{c}}'_i \mathbf{M}_i^{-1}(\mathbf{C}_i - \mathbf{c}_i\mathbf{1}')'(\mathbf{C}_k - \mathbf{c}_k\mathbf{1}')\mathbf{M}_k^{-1}\bar{\mathbf{c}}_k]]. \tag{5.83}$$

(it is difficult to obtain the expectation in (5.82)). Then, we have

$$\text{Cov}(\bar{y}_i - \bar{\mathbf{c}}'_i\hat{\alpha}_i, \bar{y}_k - \bar{\mathbf{c}}'_k\hat{\alpha}_k) = \frac{1}{2h}\sigma_y^2(\frac{2h-2}{2h-s-2})(\rho_{-} - R_{yc}^{**}). \tag{5.84}$$

Without loss of generality, we now assume that $\mathbf{C}_i$ and $\bar{\mathbf{c}}_i$ are obtained at the first block, and $\mathbf{C}_k$ and $\bar{\mathbf{c}}_k$ are obtained at the second block. Since $\mathbf{C}_i$ and $\bar{\mathbf{c}}_i$ are same for any design point i in the first block, and $\mathbf{C}_k$ and $\bar{\mathbf{c}}_k$ are same for any design point k in the second block, the expectation of the function of random variates in (5.82) is same for any two design points i and k in the two different blocks. Therefore, the covariance between any two controlled responses in two different blocks is given by (5.84).

Thus, dividing the covariances in (5.76) and (5.84) by the variance in (5.63) yields the covariance matrix of the m controlled responses as follows:

$$\mathrm{Cov}(\overline{\mathbf{y}}(\hat{\mathbf{A}})) = \frac{(1-R_{yc}^2)\sigma_y^2}{2h}\left(\frac{2h-2}{2h-s-2}\right)\begin{bmatrix} 1 & r & \cdots & r & v & v & \cdots & v \\ r & 1 & \cdots & r & v & v & \cdots & v \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ r & r & \cdots & 1 & v & v & \cdots & v \\ v & v & \cdots & v & 1 & r & \cdots & r \\ v & v & \cdots & v & r & 1 & \cdots & r \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v & v & \cdots & v & r & r & \cdots & 1 \end{bmatrix}, \tag{5.85}$$

where r is defined in (5.54) and

$$v = \frac{(\rho_{-} - R_{yc}^{**})}{(1-R_{yc}^2)}. \tag{5.86}$$

We represent this covariance matrix in another form given by

$$\mathrm{Cov}(\overline{\mathbf{y}}(\hat{\mathbf{A}})) = \frac{(1-R_{yc}^2)\sigma_y^2}{2h}\left(\frac{2h-2}{2h-s-2}\right)\left[\frac{1}{2}(r+v)\mathbf{X}\mathbf{G}_{p+1}\mathbf{X}' + \frac{1}{2}(r+v)\mathbf{z}_m\mathbf{z}'_m + (1-r)\mathbf{I}_m\right], \tag{5.87}$$

where $\mathbf{G}_{p+1}$ and $\mathbf{z}_m$ are defined in (5.14). From equation (63) in Rao (1967), the WLS estimator for $\boldsymbol{\beta}$ in (5.35) is equal to the OLS estimator in (5.56) for a dispersion matrix having the above pattern. Similarly to (5.57), taking the variance operation on $\hat{\boldsymbol{\beta}}$,

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\overline{\mathbf{y}}(\hat{\mathbf{A}})), \tag{5.88}$$

yields

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathrm{Cov}(\overline{\mathbf{y}}(\hat{\mathbf{A}}))\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} = \left(\frac{2h-2}{2h-s-2}\right) \times$$

$$\frac{(1-R_{yc}^2)\sigma_y^2}{2h}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'[\frac{1}{2}(r+v)\mathbf{X}\mathbf{G}_{p+1}\mathbf{X}'+\frac{1}{2}(v+r)\mathbf{z}_m\mathbf{z}'_m+(1-r)\mathbf{I}_m]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

$$=\frac{(1-R_{yc}^2)\sigma_y^2}{2h}(\frac{2h-2}{2h-s-2})[\frac{1}{2}(r+v)\mathbf{G}_{p+1}+(1-r)(\mathbf{X}'\mathbf{X})^{-1}] \tag{5.89}$$

since $\mathbf{X}'\mathbf{z}_m = \mathbf{z}'_m\mathbf{X} = \mathbf{0}$. If we replace r and v with (5.57) and (5.87) for (5.89), respectively, then we have

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}})=\frac{\sigma_y^2}{2h}(\frac{2h-2}{2h-s-2})[\frac{1}{2}\{(\rho_+ + \rho_-)-(R_{yc}^2+R_{yc}^{**})\}\mathbf{G}_{p+1}+(1-\rho_+)(\mathbf{X}'\mathbf{X})^{-1}]. \tag{5.90}$$

When the optimal coefficient of control variates, $\boldsymbol{\alpha}$, is unknown, the variances of β_i ($i = 1, 2, ..., p$) is inflated by the loss factor $(2h-2)/(2h-s-2)$ due to the estimation of $\boldsymbol{\alpha}$. Thus, the performance of this method in reducing the variance of β_i ($i = 1, 2, ..., p$) is inferior to the Schruben-Margolin method. For the case where the number of replications $2h$ is not small compared with the number of control variates, s, the Schruben-Margolin method is marginally better than this method. On the other hand, this method is superior to the Schruben-Margolin method in estimating β_0 if the effect of control variates $(R_{yc}^2 + R_{yc}^{**})$ compensates the loss factor. The term R_{yc}^{**} is considered as the expectation of the sample analogue of R_{yc}^{*} in (5.49). Similar to the case of the known $\boldsymbol{\alpha}$, R_{yc}^{**} is conjectured as the multiple correlation between the response and control variates in two different blocks. Since R_{yc} is the maximum correlation between the response and control variates, under the above conjecture, this method yields better results than the Schruben-Margolin method.

5.5. Comparison of Methods

In this section, we compare the Schruben-Margolin method with (a) Extended Combined Method I in Section 5.3 and (b) Extended Schruben-Margolin Method in Section 5.4. This comparison is based on the variance of the estimator for $\beta_i \, (i = 0, \; 1, \; \ldots, \; p)$, and the D-value of the estimator covariance matrix.

Applying the Schruben-Margolin method requires the design matrix **X** to admit orthogonal blocking into two blocks. Interestingly, inspection of the covariance matrix of the responses induced by Extended Combined Method I in Section 5.3 indicates that only the orthogonality of the design matrix is a sufficient condition for additionally reducing the variance of the estimator provided ρ_+^* is greater than $-\rho_-^*$ in covariance matrix in (5.24). The reason is that the resulting covariance matrix in (5.24) is the same form induced by common random number streams through all design points (see the covariance matrix of the response obtained by common random numbers across all design points in Schruben and Margolin (1978)). However, as Extended Combined Method I basically focuses on reducing the variance of the response at each design point, the induced correlations ρ_+^* and ρ_-^* may be less than ρ_+ and ρ_-, respectively.

In comparing methods, without loss of generality, we let the variance of the mean response obtained from the $2h$ replicates, $\sigma_y^2/2h$, be a unit for convenience. We also use the following notation:

- $\text{Var}(\beta_i)_{ass}$ = variance of β_i obtained by the Schruben-Margolin method.

- $\text{Var}(\beta_i)_{com}$ = variance of β_i obtained by Extended Combined Method I.

- $\mathrm{Var}(\beta_i)_{ext}$ = variance of β_i obtained by Extended Schruben-Margolin Method.

- D_{ass} = D-value of the covariance matrix of $\hat{\boldsymbol{\beta}}$ obtained by the Schruben-Margolin method.

- D_{com} = D-value of the covariance matrix of $\hat{\boldsymbol{\beta}}$ obtained by Extended Combined Method I.

- D_{ext} = D-value of the covariance matrix of $\hat{\boldsymbol{\beta}}$ obtained by Extended Schruben-Margolin Method.

First, the covariance matrix of $\hat{\boldsymbol{\beta}}$ in (5.16) indicates that

$$\mathrm{Var}(\hat{\beta}_0)_{ass} = \frac{1}{2}(\rho_+ + \rho_-) + \frac{1}{m}(1-\rho_+); \tag{5.91}$$

$$\mathrm{Var}(\hat{\beta}_i)_{ass} = \frac{1}{m}(1-\rho_+) \quad \text{for } i = 1,\ 2,\ \ldots,\ p; \tag{5.92}$$

and

$$D_{ass} = [\frac{1}{2}(\rho_+ + \rho_-) + \frac{1}{m}(1-\rho_+)]\,|\,\mathbf{X}^{*\prime}\mathbf{X}^{*}\,|^{-1}$$

$$= \frac{1}{m}[\frac{m}{2}(\rho_+ + \rho_-) + (1-\rho_+)](1-\rho_+)^p\,|\,\mathbf{X}^{*\prime}\mathbf{X}^{*}\,|^{-1}, \tag{5.93}$$

where $\mathbf{X}^*$ is the $(m \times p)$ partitioned matrix of $\mathbf{X}$ such that $\mathbf{X} = (\mathbf{1}_m \mid \mathbf{X}^*)$.

Second, from the covariance matrix of $\hat{\boldsymbol{\beta}}$ in (5.33), we find that

$$\mathrm{Var}(\hat{\beta}_0)_{com} = (\rho_+^* + \rho_-^*) + \frac{1}{m}(\frac{m-p-2}{m-p-s-2})(1-\rho_y-\rho_+^*-\rho_-^*-R_{yc}^2); \tag{5.94}$$

$$\mathrm{Var}(\hat{\beta}_i)_{com} = \frac{1}{m}\left(\frac{m-p-2}{m-p-s-2}\right)(1-\rho_y-\rho_+^*-\rho_-^*-R_{yc}^2) \quad \text{for } 1 = 1,\ 2,\ \ldots,\ p; \tag{5.95}$$

and

$$D_{com} = [\delta + \frac{(m-p-2)}{m(m-p-s-2)}\gamma][\frac{\gamma(m-p-2)}{(m-p-s-2)}]^p \mid \mathbf{X}^{**\prime}\mathbf{X}^* \mid^{-1}$$

$$= \frac{1}{m}[m\delta + \frac{(m-p-2)}{(m-p-s-2)}\gamma][\frac{\gamma(m-p-2)}{(m-p-s-2)}]^p \mid \mathbf{X}^{**\prime}\mathbf{X}^* \mid^{-1},$$

$$= \frac{1}{m}[m\delta + l\gamma][l\gamma]^p \mid \mathbf{X}^{**\prime}\mathbf{X}^* \mid^{-1}, \tag{5.96}$$

where $\delta = \rho_+^* + \rho_-^*$, $\gamma = (1-\rho_y-\rho_+^*-\rho_-^*-R_{yc}^2)$ and $l = (m-p-2)/(m-p-s-2)$.

Also based on the covariance matrix of $\hat{\boldsymbol{\beta}}$ in (5.90), we find that (for the case of the unknown $\boldsymbol{\alpha}$):

$$\mathrm{Var}(\hat{\beta}_0)_{ext} = \left(\frac{2h-2}{2h-s-2}\right)[\frac{1}{2}\{(\rho_+ + \rho_-) - (R_{yc}^2 + R_{yc}^{**})\} + \frac{1}{m}(1-\rho_+)]; \tag{5.97}$$

$$\mathrm{Var}(\hat{\beta}_i)_{ext} = \left(\frac{2h-2}{2h-s-2}\right)\frac{1}{m}(1-\rho_+) \quad \text{for } i = 1,\ 2,\ \ldots,\ p; \tag{5.98}$$

and

$$D_{ext} = \left(\frac{2h-2}{2h-s-2}\right)^{p+1}[\frac{1}{2}\{(\rho_+ + \rho_-) - (R_{yc}^2 + R_{yc}^{**})\} + \frac{1}{m}(1-\rho_+)](1-\rho_+)^p \mid \mathbf{X}^{**\prime}\mathbf{X}^* \mid^{-1}$$

$$= \left(\frac{2h-2}{2h-s-2}\right)^{p+1}\frac{1}{m}[\frac{m}{2}\{(\rho_+ + \rho_-) - (R_{yc}^2 + R_{yc}^{**})\} + (1-\rho_+)](1-\rho_+)^p \mid \mathbf{X}^{**\prime}\mathbf{X}^* \mid^{-1},$$

(5.99)

We first compare methods with respect to the variance of $\hat{\beta}_0$. Comparison of equations (5.91), (5.94) and (5.97) indicates that (a) Extended Combined Method I yields better result than the Schruben-Margolin method if

$$m(\dot{\rho}_+ + \dot{\rho}_-) + [(1 - \rho_y - R^2_{yc}) - (\dot{\rho}_+ + \dot{\rho}_-)](\frac{m-p-2}{m-p-s-2}) < \frac{m}{2}(\rho_+ + \rho_-) + (1 - \rho_+), \tag{5.100}$$

and (b) Extended Schruben-Margolin Method gives a smaller variance than the Schruben-Margolin method provided

$$\frac{s}{2}(\rho_+ + \rho_-) + \frac{s}{m}(1 - \rho_+) < (h-1)(R^2_{yc} + \ddot{R}_{yc}). \tag{5.101}$$

In estimating β_i ($i = 1, 2, \ldots, p$), comparison of equations (5.92), (5.95), and (5.98) shows that (a) Extended Combined Method I yields better result than the Schruben-Margolin method provided

$$(1 - \rho_y - \dot{\rho}_+ - \dot{\rho}_- - R^2_{yc})(\frac{m-p-2}{m-p-s-2}) < (1 - \rho_+), \tag{5.102}$$

and (b) the Schruben-Margolin method yields better results than Extended Schruben-Margolin Method marginally.

With respect to the determinant of the estimator covariance matrix, comparison of the D-value in (5.93), (5.96), and (5.99) shows that (a) Extended Combined Method I yields better results than the Schruben-Margolin method if

$$[m\delta + l\gamma][l\gamma]^p < [\frac{m}{2}(\rho_+ + \rho_-) + (1 - \rho_+)][1 - \rho_+]^p, \tag{5.103}$$

and (b) Extended Schruben-Margolin Method yields better result than the Schruben-Margolin method under the following condition:

$$\left(\frac{2h-2}{2h-s-2}\right)^{p+1}\left[\frac{m}{2}\{(\rho_{+}+\rho_{-})-(R_{yc}^{2}+R_{yc}^{**})\}+(1-\rho_{+})\right]<\frac{m}{2}(\rho_{+}+\rho_{-})+(1-\rho_{+}).$$

(5.104)

Based on the comparison of the methods considered in this chapter, we now discuss the overall performance in estimating the parameters of interest and the applicability of these methods. The basic approaches of the Schruben-Margolin method and Extended Combined Method I are quite different. The former method utilizes the correlations between two responses at the different design points, and the latter method exploits the inherent correlation between the response and control variates in a single design point and tries to take advantage of the additional effect from the Schruben-Margolin correlation induction strategy. In estimating the overall mean response of the system, as shown in (5.100), the preference of these two methods is determined by the magnitudes of $m(\rho_{+}^{*}+\rho_{-}^{*})$ and $m(\rho_{+}+\rho_{-})/2$ since the other terms in (5.100) are small compared to these two terms. Thus, if $2(\rho_{+}^{*}+\rho_{-}^{*})<(\rho_{+}+\rho_{-})$, Extended Combined Method I would yield better results than the Schruben-Margolin method. Independent streams through the control variates highly correlated with the response may yield the correlations, ρ_{+}^{*} and ρ_{-}^{*}, between the responses (not controlled) across the design points much less than ρ_{+} and ρ_{-} obtained by the Schruben-Margolin method. (our experiments support this claim). In this case, Extended Combined Method I is preferred to the Schruben-Margolin method.

In estimating the main and interaction effects of the factor variables, the performance of Extended Combined Method I is represented as a product form of the minimum variance ratio and a loss factor. Hence, this method should compensate the loss factor by the reduced variance of the response to guarantee a variance reduction of the estimator. Thus, for the cases that the experimental model includes a small number of design points (compared with the number of the parameters), and synchronization of random number streams can be easily achieved in the model, the Schruben-Margolin method is preferred to Extended Combined Method I. In the contrary case, Extended Combined Method I may yield better results than the Schruben-Margolin Method.

As we noted before, Extended Schruben-Margolin Method shows the similar performance as the Schruben-Margolin method in estimating the main and interaction effects if the loss factor is small, and better performance in estimating the overall mean of the responses under the condition of (5.101). Our simulation results strongly suggest the above conjecture.

CHAPTER 6. APPLICATION OF COMBINED CORRELATION METHODS TO MULTIPOPULATION MODEL

In this chapter, we apply two of the combined methods developed in Sections 5.3 and 5.4 as well as the Schruben-Margolin correlation induction method to a multi-population model. Specifically, we perform two sets of simulation experiments on the hospital resource allocation model (Schruben and Margolin (1978)) to evaluate the experimental performances of these variance reduction techniques. We then summarize the simulation results and present inferences as to these results.

6.1. Description of System and Model

Consider the hospital resource allocation model given in Section 4.4, where we measured the performance of the current hospital system by the failure rate. Suppose that the hospital administration considers construction of a new facility to provide better service to the patients. The administration's decision is complicated by conflicting interest of several groups because no one knows how the numbers of each type of bed will affect the frequency with which the patients can not be accommodated. To help resolve this conflict, a statistically designed simulation experiment is conducted.

Schruben and Margolin (1978) illustrated this problem to investigate the simulation efficiency of their correlation induction strategy. For estimating the effect of the number of beds of each type to the failure rate of the patients, they implemented a 2^3 factorial design: three factor variables (three types of beds) having two levels for each factor. The experimental conditions for the eight design points in the 2^3 factorial design are given in Table 32. They also proposed a linear model which includes a overall mean and all main effects and pairwise interactions. Their simulation results showed that two factor interaction effects are negligible. Based on these results, in the application of each method to this model, we consider a linear model consisting only of the overall mean response and all main effects. Thus, for the applying Schruben-Margolin strategy, we assumed that the responses across the eight design points can be written as the following:

$$\bar{y}_i = \beta_0 + \sum_{j=1}^{3} \beta_j x_{ij} + \varepsilon_i, \quad i = 1,\ 2,\ \ldots,\ 8, \tag{6.1}$$

where $\bar{y}_i$ is the average failure rate (the response of interest) at the ith design point; β_0 is the overall mean; β_j is the main effect of the jth factor variable (number of the specialized care beds); x_{ij} is 1 (-1) if the jth factor is at the high (low) level for a design point i (by a reparameterization of the factor variables); and ε_i is the error for the ith observation.

Table 32. Experimental Design Points in 2^3 Factorial Design

	Experimental Design Point	Number of Beds (Intensive)	Number of Beds (Coronary)	Number of Beds (Intermediate)
	1	13 (-1)	4 (-1)	15 (-1)
Block 1	2	13 (-1)	6 (1)	17 (1)
	3	15 (1)	4 (-1)	17 (1)
	4	15 (1)	6 (1)	15 (-1)
	5	13 (-1)	4 (-1)	17 (1)
Block 2	6	13 (-1)	6 (1)	15 (-1)
	7	15 (1)	4 (-1)	15 (-1)
	8	15 (1)	6 (1)	17 (1)

Clearly, the (8 × 4) design matrix $\mathbf{X} = (x_{ij})$ given in (6.1) admits orthogonal blocking into two blocks. For applying the Schruben-Margolin strategy, we partitioned the eight design points of the design matrix $\mathbf{X}$ into two blocks: the first block includes the design points 1-4, and the second block includes the design points 5-8 (see Table 32). For a given design point, we conducted 200 runs independently using 200 different sets of random number streams. Each replication used eight randomly selected random number streams for driving the stochastic model components as given in Table 22. Across the design points, we used the same random number streams (common random number streams) for design points 1-4 in the first block, and their antithetic random number streams for design points 5-8 in the second block.

In applying Extended Combined Method I developed in Section 5.3 to this model, we used the single standardized control variate of interarrival time of the patients to the system. The interarrival times of the patients to the system would be independently observed at each level of the factor variables (service times at the three hospital units) by using different number streams for driving the arrival process of the patients to the system. Thus, we can assume that this control variate is independent of the three factor variables (see the discussion of concomitant variables in Section 2.1). Adding this control variate to the linear model in (6.1), we have

$$\bar{y}_i = \beta_0 + \sum_{j=1}^{3} \beta_j x_{ij} + \bar{c}_i + \varepsilon_i^*, \quad i = 1,\ 2,\ \ldots,\ 8, \tag{6.2}$$

where $\bar{y}_i$, β_0, β_j and x_{ij} are given in (6.1); $\bar{c}_i$ is the mean control variate at the ith design point; and ε_i^* is the error for the ith observation.

For a given design point, Extended Combined Method I was comprised of 200 replications according to the random number assignment rule given in Section 4.4.1 (for a single population model). As before, each replicate used randomly selected random number streams for driving the stochastic model components as in Table 22 in Chapter 4. Across the design points, this method used independent random number streams for generating the interarrival time process (control variate), but employed the Schruben-Margolin random number assignment rule for driving the non-control variates stochastic model components (see Table 31).

Extended Schruben-Margolin Method developed in Section 5.4 (method utilizing the control variate under Schruben-Margolin method) uses the same random number assignment strategy as that given in the Schruben-Margolin method. For this method, we used the same simulation output obtained by the Schruben-Margolin method and additionally collected the standardized control variate of the interarrival times of the patients to the system during the simulation. As in Extended Combined Method I, this control variate is independent of the three factor variables for a single design point. Note that we conducted the 200 replications independently at a given design point. Based on the methodology given in Section 5.4, we have the following model for this example:

$$\bar{y}_i(\hat{\alpha}_i) = \bar{y}_i - \bar{c}_i\hat{\alpha}_i = \beta_0 + \sum_{j=1}^{3} \beta_j x_{ij} + \varepsilon_i^{**}, \quad i = 1, 2, \ldots, 8, \tag{6.3}$$

where $\bar{y}_i$, β_j, $(j = 0, 1, \ldots, 3)$ and x_{ij} are given in (6.1); $\bar{c}_i$ is the mean control variate at the ith design point; ε_i^{**} is the error for the ith observation; and $\hat{\alpha}_i$ is the coefficient estimator of the control variate for the ith controlled response (see equation (2.75)).

We used the same simulation program coded in SLAM II as given in Appendix B-4 in applying each method. The simulation run for a given design point used the number of the hospital units corresponding to the given design point in SLAM II code. We simulated this system under the same conditions as given in Section 4.4.

6.2. Experimental Results

To provide an assessment of the efficiency gains of the three methods considered in Sections 5.2-5.4, we computed the performance statistics of the D-value of the estimated covariance matrix of the parameters, and the variances of the estimators for the parameters based on each method.

We first address the computational procedure for obtaining the covariance matrices of the (controlled) responses at the eight design points resulting from the three methods. Let $2h$ be the number of replications at each design point and $\mathbf{y}_j = (y_{1j}, y_{2j}, \ldots, y_{8j})'$ be the response vector of the eight design points for the jth replication. Also let $\mathbf{c}_j = (c_{1j}, c_{2j}, \ldots, c_{8j})'$ be the vector of control variates corresponding to $\mathbf{y}_j$. For the Schruben-Margolin method, the sample covariance matrix of the responses at the eight design points is given by

$$\mathbf{S}_y = \frac{1}{2h-1} \sum_{j=1}^{2h} (\mathbf{y}_j - \bar{\mathbf{y}})(\mathbf{y}_j - \bar{\mathbf{y}})', \tag{6.4}$$

where $\bar{\mathbf{y}} = \sum_{i=1}^{2h} \mathbf{y}_i / 2h$ is the mean response vector at the eight design points. Applying Extended Combined Method I to this model gives the adjusted mean responses for the eight design points as follows:

$$\bar{\mathbf{y}}(\hat{\alpha}) = \bar{\mathbf{y}} - \hat{\alpha}\bar{\mathbf{c}} \tag{6.5}$$

where $\bar{\mathbf{c}} = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_8)'$ is the mean vector of control variates at the eight design points, and $\hat{\alpha}$ is the least squares estimator of the linear model in (6.2), given by

$$\hat{\alpha} = (\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{c}})^{-1}\bar{\mathbf{c}}'\mathbf{P}\bar{\mathbf{y}}. \tag{6.6}$$

(see equation (2.22)). Thus, the covariance matrix of the adjusted responses at the eight design points is estimated by

$$\mathbf{S}_{y(\hat{\alpha})} = \frac{1}{2h-1}\sum_{j=1}^{2h}(\mathbf{y}_j(\hat{\alpha}) - \bar{\mathbf{y}}(\hat{\alpha}))(\mathbf{y}_j(\hat{\alpha}) - \bar{\mathbf{y}}(\hat{\alpha}))', \tag{6.7}$$

where $\mathbf{y}_j(\hat{\alpha}) = \mathbf{y}_j - \hat{\alpha}\mathbf{c}_j$. For Extended Schruben-Margolin Method, the adjusted mean response at the ith design point is given by

$$\bar{y}_i(\hat{\alpha}_i) = \bar{y}_i - \hat{\alpha}_i\bar{c}_i, \tag{6.8}$$

where $\bar{c}_i$ is the mean control variate at the ith design point, and

$$\hat{\alpha}_i = S_c^{-1}S_{yc} = [\sum_{j=1}^{2h}(c_{ij} - \bar{c}_i)(c_{ij} - \bar{c}_i)]^{-1}[\sum_{j=1}^{2h}(y_{ij} - \bar{y}_i)(c_{ij} - \bar{c}_i)], \tag{6.9}$$

where y_{ij} and c_{ij} are the response and control variate, respectively, obtained at the ith design point and the jth replication. Similar to (6.7), the sample covariance matrix of the adjusted responses at the eight design points is given by

$$S_{y(\hat{\alpha})} = \frac{1}{2h-1} \sum_{j=1}^{2h} (\mathbf{y}_j(\hat{\alpha}) - \bar{\mathbf{y}}(\hat{\alpha}))(\mathbf{y}_j(\hat{\alpha}) - \bar{\mathbf{y}}(\hat{\alpha}))', \tag{6.10}$$

where $\mathbf{y}_j(\hat{\alpha}) = (y_{1j}(\hat{\alpha}_1),\ y_{2j}(\hat{\alpha}_2), \ldots,\ y_{8j}(\hat{\alpha}_8))'$ and $\bar{\mathbf{y}}(\hat{\alpha})$ is the mean vector of the adjusted responses at the eight design points.

Next, we present the computational procedures for obtaining the sample covariance matrix of the estimators for the parameters. Note that the estimators for the parameters, $\boldsymbol{\beta} = (\beta_0,\ \beta_1, \ldots,\ \beta_3)'$, are equivalent to the least squares estimators of the linear models in (6.1)-(6.3), respectively, for each applied method. When the Scruben-Margolin method is implemented, the sample covariance matrix of the estimator for $\boldsymbol{\beta}$ is given by

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}_y\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \tag{6.11}$$

(see equation (5.16)). Also, given the control variates, substituting for the sample covariance matrix of the responses, $\mathbf{S}_y$ in (6.11), with those given in (6.7) yields the sample covariance matrix of $\hat{\boldsymbol{\beta}}$ for Extended Combined Method I:

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}_{y(\hat{x})}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}. \tag{6.12}$$

Similar to this equation, given the control variates, the sample covariance matrix of $\hat{\boldsymbol{\beta}}$ is given by

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}_{y(\hat{\alpha})}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}, \tag{6.13}$$

for Extended Schruben-Margolin Method.

Using the computational procedures mentioned above, we obtained appropriate statistics for each method. We now summarize the simulation results obtained for the three applied methods considered in Sections 5.2-5.4: (a) Tables 33, 35, and 37 present the sample covariance matrices of the responses at the eight design points, (b) Tables 34, 36, and 38 give the sample correlation matrices of the responses at the eight design points, (c) Table 39 provides the estimator for the parameters $\boldsymbol{\beta}$, and (d) Table 40 presents the covariance matrices of the estimators for the parameters, and their D-values.

6.3. Inferences

Based on the simulation results of this system presented in the previous section, we provide inferences in applying these three variance reduction techniques.

First, we inspect the three sample covariance matrices of the (controlled) responses for each applied method, which are given in Tables 33, 35 and 37, respectively. From Table 33 obtained by the Schruben-Margolin method, we note that the variance of the response of interest at each design point seems to be approximately 2.0. Table 35 indicates that the variances of the controlled responses obtained by Extended Combined Method I are in the range from 0.43 to 0.57. Also Table 37 shows that the variance of the controlled responses are in the range from 0.43 to 0.67 for Extended Schruben-Margolin Method. As we expected, these two latter methods substantially reduce the variances in estimating the mean responses across all design points.

Next, we explore the correlation matrix of the (controlled) responses obtained for each applied method. As shown in Table 34, the Schruben-Margolin method yields

Table 33. Covariance Matrix of Responses: Schruben-Margolin Method.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
y_1	1.962	1.925	1.911	1.884	-1.116	-1.094	-1.096	-1.075
y_2	1.925	1.941	1.883	1.889	-1.110	-1.084	-1.092	-1.069
y_3	1.911	1.883	1.901	1.865	-1.069	-1.049	-1.052	-1.031
y_4	1.884	1.889	1.865	1.880	-1.071	-1.049	-1.057	-1.035
y_5	-1.116	-1.110	-1.069	-1.071	2.176	2.161	2.123	2.119
y_6	-1.094	-1.084	-1.049	-1.049	2.161	2.188	2.120	2.139
y_7	-1.096	-1.092	-1.052	-1.057	2.123	2.120	2.106	2.094
y_8	-1.075	-1.069	-1.031	-1.035	2.119	2.139	2.094	2.124

Table 34. Correlation Matrix of Responses: Schruben-Margolin Method.

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
y_1	1.000	0.986	0.989	0.981	-0.540	-0.528	-0.539	-0.527
y_2	0.986	1.000	0.980	0.989	-0.540	-0.526	-0.540	-0.526
y_3	0.989	0.980	1.000	0.986	-0.525	-0.514	-0.526	-0.513
y_4	0.981	0.989	0.986	1.000	-0.530	-0.517	-0.531	-0.518
y_5	-0.540	-0.540	-0.525	-0.530	1.000	0.990	0.992	0.986
y_6	-0.528	-0.526	-0.514	-0.517	0.990	1.000	0.988	0.992
y_7	-0.539	-0.540	-0.526	-0.531	0.992	0.988	1.000	0.990
y_8	-0.527	-0.526	-0.513	-0.518	0.986	0.992	0.990	1.000

Table 35. Covariance Matrix of Adjusted Responses: Extended Combined Method I

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
y_1	0.478	0.288	0.267	0.286	0.279	0.291	0.298	0.314
y_2	0.288	0.431	0.308	0.287	0.274	0.320	0.327	0.361
y_3	0.267	0.308	0.436	0.298	0.245	0.291	0.325	0.332
y_4	0.286	0.287	0.298	0.441	0.268	0.270	0.279	0.323
y_5	0.279	0.274	0.245	0.268	0.428	0.308	0.306	0.359
y_6	0.291	0.320	0.291	0.270	0.308	0.431	0.340	0.346
y_7	0.298	0.327	0.325	0.279	0.306	0.340	0.497	0.371
y_8	0.314	0.361	0.332	0.323	0.359	0.346	0.371	0.569

Table 36. Correlation Matrix of Adjusted Responses: Extended Combined Method I

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
y_1	1.000	0.634	0.584	0.624	0.617	0.642	0.611	0.601
y_2	0.634	1.000	0.709	0.657	0.637	0.741	0.706	0.730
y_3	0.584	0.709	1.000	0.680	0.567	0.672	0.699	0.667
y_4	0.624	0.657	0.680	1.000	0.617	0.619	0.597	0.644
y_5	0.617	0.637	0.567	0.617	1.000	0.718	0.664	0.729
y_6	0.642	0.741	0.672	0.619	0.718	1.000	0.734	0.699
y_7	0.611	0.706	0.699	0.597	0.664	0.734	1.000	0.698
y_8	0.601	0.730	0.667	0.644	0.729	0.699	0.698	1.000

Table 37. Covariance Matrix of Adjusted Responses: Extended Schruben-Margolin Method

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
y_1	0.428	0.416	0.417	0.408	-0.110	-0.121	-0.117	-0.125
y_2	0.416	0.458	0.414	0.438	-0.101	-0.107	-0.111	-0.116
y_3	0.417	0.414	0.446	0.427	-0.107	-0.119	-0.116	-0.124
y_4	0.408	0.438	0.427	0.460	-0.112	-0.120	-0.124	-0.129
y_5	-0.110	-0.101	-0.107	-0.112	0.592	0.583	0.583	0.603
y_6	-0.121	-0.107	-0.119	-0.120	0.583	0.616	0.586	0.628
y_7	-0.117	-0.111	-0.116	-0.124	0.583	0.586	0.609	0.620
y_8	-0.125	-0.116	-0.124	-0.129	0.603	0.628	0.620	0.672

Table 38. Correlation Matrix of Adjusted Responses: Extended Schruben-Margolin Method

	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
y_1	1.000	0.941	0.954	0.920	-0.219	-0.236	-0.229	-0.234
y_2	0.941	1.000	0.916	0.954	-0.194	-0.201	-0.210	-0.209
y_3	0.954	0.916	1.000	0.943	-0.208	-0.226	-0.223	-0.226
y_4	0.920	0.954	0.943	1.000	-0.214	-0.226	-0.233	-0.231
y_5	-0.219	-0.194	-0.208	-0.214	1.000	0.965	0.971	0.955
y_6	-0.236	-0.201	-0.226	-0.226	0.965	1.000	0.957	0.976
y_7	-0.229	-0.210	-0.223	-0.233	0.971	0.957	1.000	0.969
y_8	-0.234	-0.209	-0.226	-0.231	0.955	0.976	0.969	1.000

Table 39. Estimators for Model Parameters

Parameter	Schruben-Margolin Method	Extended Combined Method	Extended Schruben-Margolin Method
β_0	45.722	45.588	45.627
β_1	-0.291	-0.309	-0.290
β_2	-0.378	-0.384	-0.378
β_3	-1.805	-1.809	-1.805

Table 40. **Covariance Matrix of Estimators for Model Parameters**

Schruben-Margolin Method: D-Value = 9.442×10^{-9}

	β_0	β_1	β_2	β_3
β_0	0.4720055	-0.0007993	0.0026397	0.0005962
β_1	-0.0007993	0.0032940	0.0002315	-0.0000107
β_2	0.0026397	-0.0002315	0.0050496	0.0000306
β_3	0.0005962	-0.0000107	0.0000306	0.0012114

Extended Combined Method I: D-Value = 1.922×10^{-7}

	β_0	β_1	β_2	β_3
β_0	0.3255563	0.0080371	0.0063543	0.0038048
β_1	0.0080371	0.0210071	0.0017162	0.0030887
β_2	0.0063543	0.0017162	0.0170880	0.0040829
β_3	0.0038048	0.0030887	0.0040829	0.0181698

Extended Schruben-Margolin: D-Value = 3.895×10^{-9}
Method

	β_0	β_1	β_2	β_3
β_0	0.2002069	0.0013042	0.0034576	0.0026241
β_1	0.0013042	0.0033179	-0.0002449	-0.0000350
β_2	0.0034576	-0.0002449	0.0050850	0.0000174
β_3	0.0026241	-0.0000350	0.0000174	0.0012103

correlation coefficients in the range from 0.98 to 0.99 between two responses in the same block, and from -0.54 to -0.51 between responses from different blocks. For Extended Combined Method I, Table 36 indicates that the correlation coefficients between two adjusted responses either in the same block or different blocks are in the range from 0.58 to 0.74. In comparing to the induced correlation matrix of the Schruben-Margolin method, it seems more difficult to obtain the correlation matrix structure (equal correlation between the two responses) given in equation (5.24) in applying this method. This result indicates that the assumptions on the equal correlations between the two responses in either the same block or different blocks (Assumptions 3 and 4 in Section 5.3) need the analytical and empirical validation although similar assumptions of Schruben and Margolin (1978) are generally accepted. We conjecture that this is due to the use of independent random number streams for driving the control variate across design points which reduced the synchronization effect of random number streams in applying this method. However, the extended method yields positive correlations between any two controlled responses with values not much less than those induced by the Schruben-Margolin method for the responses in the same block.

From Table 38, we note that the correlations between two controlled responses in the same block are in the range from 0.92 to 0.98, and those from two different blocks are in the range from -0.20 to -0.24 for Extended Schruben-Margolin Method. To explore the notion that the induced correlations are consistent with those developed in equation (5.53), we computed the correlation matrix between the responses and control variates for the eight design points. This correlation matrix is given in Table 41. We estimated ρ_+ in (5.13) and ρ_{yc} in (5.38) by their sample analogues, respectively, given in Tables 34 and 41: $\hat{\rho}_+ \cong 0.985$ and $\hat{\rho}_{yc} \cong -0.867$ (note that we used a single control variate in this example). Then, from equation (5.54), the correlation coefficient between the two controlled responses in the same block is estimated by

$$\frac{\hat{\rho}_{+} - \hat{\rho}_{yc}^{2}}{1 - \hat{\rho}_{yc}^{2}} = \frac{0.985 - 0.767}{1 - 0.767} \cong 0.94.$$

This result indicates that the simulation result in Table 36 is consistent with that given in equation (5.54) for two responses in the same block. However, the sort of conjecture given above is difficult to make for correlations between two controlled responses in the two different blocks since R_{yc}^{**} in equation (5.57) is represented in the expectation of a complex function of the control variates (see equation 5.57).

We now compare the performances of the three methods with respect to the sample variances of the estimators, and the determinant of the sample covariance matrix of the estimators (D-value). From Table 40, we note that (a) in estimating the overall mean response (β_0), Extended Combined Method I and Extended Schruben-Margolin Method are superior to the Schruben-Margolin method, and Extended Schruben-Margolin Method yields better results than Extended Combined Method I, (b) in estimating the main factor effects, (β_1, β_2, β_3), the Schruben-Margolin method and Extended Schruben-Margolin Method yield better results than Extended Combined Method I, and the performances of the former two methods are almost the same as theoretical developments given in (5.16) and (5.90), respectively, and (c) with respect to the design criteria of the D-value, the Schruben-Margolin Method is superior to Extended Combined Method I, and Extended Schruben-Margolin Method is superior to both.

Table 41. Correlation Matrix between Responses and Control Variates: Schruben-Margolin Method

Correlation Matrix between Responses and Control Variates:

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
y_1	-0.885	-0.885	-0.885	-0.885	0.512	0.512	0.512	0.512
y_2	-0.875	-0.875	-0.875	-0.875	0.517	0.517	0.517	0.517
y_3	-0.876	-0.876	-0.876	-0.876	0.496	0.496	0.496	0.496
y_4	-0.870	-0.870	-0.870	-0.870	0.499	0.499	0.499	0.499
y_5	0.571	0.571	0.571	0.571	-0.854	-0.854	-0.854	-0.854
y_6	0.552	0.552	0.552	0.552	-0.848	-0.848	-0.848	-0.848
y_7	0.565	0.565	0.565	0.565	-0.844	-0.844	-0.844	-0.844
y_8	0.546	0.546	0.546	0.546	-0.828	-0.828	-0.828	-0.828

Correlation Matrix between Control Variates:

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
c_1	1.000	1.000	1.000	1.000	-0.603	-0.603	-0.603	-0.603
c_2	1.000	1.000	1.000	1.000	-0.603	-0.603	-0.603	-0.603
c_3	1.000	1.000	1.000	1.000	-0.603	-0.603	-0.603	-0.603
c_4	1.000	1.000	1.000	1.000	-0.603	-0.603	-0.603	-0.603
c_5	-0.603	-0.603	-0.603	-0.603	1.000	1.000	1.000	1.000
c_6	-0.603	-0.603	-0.603	-0.603	1.000	1.000	1.000	1.000
c_7	-0.603	-0.603	-0.603	-0.603	1.000	1.000	1.000	1.000
c_8	-0.603	-0.603	-0.603	-0.603	1.000	1.000	1.000	1.000

For this example, if independent streams are employed across design points and replications, then each of the OLS estimators would have an estimated variance approximately equal to one-eighth the average estimated variance presented in Table 33. Note that the variance of the response at each design point is approximately 2.0. Thus, independent streams would yield the variance of the estimator for β_j ($j = 0,\ 1, \ldots,\ 3$) as follows:

$$\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = \hat{\sigma}_y^2(\mathbf{X}'\mathbf{X})^{-1} = \frac{2}{8}\,\mathbf{I}_8 = 0.25\mathbf{I}_8$$

(see equation (2.28)). In comparing to the method of independent streams, the Schruben-Margolin method is very effective in estimating the effects of main factors on the responses, and reduces the variance of the estimator for β_1, for instance, by approximately 98% (= 1-0.0033/0.25), but increases the variance of the estimator for β_0 by approximately 188% (= 0.47/0.25). Compared with independent streams, Extended Schruben-Margolin Method reduces the variance of the estimator for β_0 by around 20% (= 1-0.20/0.25). Also this method shows similar performance in reducing the variances of the estimators for the main effects as that obtained by the Schruben-Margolin method.

For the case that a single control variate is applied, R^*_{yc} in (5.50) can be written as

$$R^*_{yc} = \sigma_y^{-2}\sigma_c^{-2}(2\sigma_{yc}\sigma^*_{yc} - \rho_c\sigma^{*2}_{yc}) = (2\rho_{yc}\rho^*_{yc} - \rho_c\rho^{*2}_{yc}), \tag{6.14}$$

where σ_{yc} and ρ_{yc} are the covariance and correlation coefficient, respectively, between the response and control variate in the same block; σ^*_{yc} and ρ^*_{yc} are the covariance and correlation coefficient, respectively, between the response and control variate in the two different blocks; and ρ_c is the correlation coefficient between control variates in two different blocks. The results in Table 41 show that ρ^{*}_{yc} is less than both ρ_{yc} and ρ_{c}. Although we can not identify the relationship among these terms, the results in Table 41 also indicate that this relationship would be $\rho^{*}_{yc} \cong -\rho_{yc} \times \rho_{c}$ $(\rho_{c} > 0)$. Under this assumption, the condition in (5.101) indicating the preference of the combined method to Schruben-Margolin method can be written as

$$R^2_{yc} + R^{*}_{yc} = \rho^2_{yc} + (2\rho_{yc}\rho^{*}_{yc} + \rho_c\rho^2_{yc}) = \rho_{yc}[(1 + \rho_c)\rho_{yc} + 2\rho^{*}_{yc}]$$

$$= \rho_{yc}[(1 + \rho_c)\rho_{yc} - 2\rho_{yc}\rho_c] = \rho^2_{yc}(1 - \rho_c) > 0 \tag{6.15}$$

since $\rho_c < 1$ for a single control variate case. This equation implies that Extended Schruben-Margolin Method yields better results than the Schruben-Margolin method in estimating the overall mean response of the model. We conjecture that the condition $R^2_{yc} + R^{*}_{yc} > 0$ would hold for the multiple control variates case by extending the result discussed above. However, more work should be done in this area for completely analyzing this method.

In reducing the variances of the estimators for the parameters, Extended Combined Method I focuses on reduction of the variances of the mean responses at each design point by using the correlation between the response and a set of control variates across the design points. This method also tries to take advantage of the Schuben-Margolin method by inducing correlations between any two responses in the design after the control variate effect has been accounted for. In applying the Schruben-Margolin method, the magnitude of the correlation coefficient between two responses in the same block is critical to the efficiency of this method in reducing the variances of the estimators for the main (interaction) effects of the factor variables. As shown in this example, the Schruben-Margolin method inflates the variance of the estimator for the overall mean response substantially when the difference between ρ_+ and $-\rho_-$ is not small. Thus, if synchronization of the random number streams yields highly correlated responses across the design points, then it may be desirable to use Extended Schruben-Margolin Method. For the case that an effective set of control variates can be identified and synchronization of the random number streams is difficult to achieve in the model, Extended Combined Method I may yield better results than the Schruben-Margolin method and the Extended Schruben-Margolin Method.

CHAPTER 7. SUMMARY AND CONCLUSIONS

This chapter summarizes the contributions of this research and reviews the conclusions reached regarding the estimation of the parameters of interest for both a single population and multipopulation model in simulation experiments. In carrying out this research, many avenues for future research were uncovered.

This research consisted of two major directions: (a) developing variance reduction techniques combining antithetic variates and control variates for a single population model, and (b) developing variance reduction techniques utilizing all correlation methods for a designed experiment of a multipopulation model. Part (a) was treated in Chapters 3 and 4 and part (b) was treated in Chapters 5 and 6. A brief review and summary of this research is given in Section 7.1. Future research is discussed in Section 7.2.

7.1 Overview and Summary of research

Chapter 3 developed three variance reduction techniques for improving the estimation of the mean response of interest in a single population model. The efficiency of each developed method in reducing the variance of the estimator is dependent on the trade-off effect of the correlations between the paired responses, and between the response and control variates. Our simulation studies implemented in Chapter 4 indicate that Combined Method I generally yields better results than the other combined methods as well as the methods of antithetic variates and control variates. In combining antithetic variates and control variates in a simulation run, we consider that a strategy using independent streams for driving the control variates would be better than antithetic streams for driving the control variates except for the case that synchronization of random number streams is easily achieved in the model. For a complex model where an effective set of control variates is small, it is expected that Combined Method I (using independent streams for driving the control variates and antithetic streams for the non-control variate stochastic model components) is better than the other methods designed to reduce the variance of the estimator in a single population model. We expect this result may be useful in the design of a large-scale simulation.

Chapter 5 developed three variance reduction techniques in one simulation experiment whose purpose is to estimate the parameters of a first-order linear model. First, we extended Combined Method I to the multipopulation model with independent simulation runs across the design points. Second, we extended Combined Method I to the multipopulation model in conjunction with the Schruben-Margolin strategy. This method focuses on reducing the variance of the response at each design point, and additionally taking advantage of the effect of the Schruben-Margolin strategy across the design points. Under certain conditions, this method is shown to be better than the Schruben-Margolin method in the estimation of the unknown model coefficients. Third, we provided a new approach which utilizes the control variates obtained during the course of the simulation run under the Schruben-Margolin method. The performance of this method is shown to be similar in estimating the main (interaction) effects of the factor variables, and to be superior to the Schruben-Margolin method in estimating the overall mean response in the hospital simulation experiment. For the general case, if the selected control variates are highly correlated with the response at each design point, and the loss factor is small, Extended Schruben-Margolin Method may yield better results than the Schruben-Margolin method as illustrated in this example.

7.2 Future Research

The directions for future research stemming from the material studied in this dissertation pertain to how to combine correlation methods for improving the estimation of the system parameters in designed simulation experiments. We outline some of these ideas below.

Application of Combined Method I to a large-scale simulation experiment presents a future direction of developing a procedure to determine which random number streams should be used for control variates and which ones should be used for inducing correlations via antithetic streams to maximize the efficiency of this method. Also, another direction is to develop a statistical procedure for obtaining the estimator from the simulation output of each replication rather than regression analysis based on the paired independent simulation outputs.

Perhaps a significant direction for future research is to develop statistical validation procedures for the use of Extended Combined Method I and Extended Schruben-Margolin Method in simulation experiments for the multipopulation parameter estimation. The validation procedures may consist of two stages: validation for the multivariate normality assumptions on the responses and control variates across the design points and validation for the assumed covariance structure of the controlled responses across the design points (see Tew and Wilson (1990)).

Application of Extended Schruben-Margolin Method suggests a future research for identifying the effect of including control variates to the estimation of the overall mean response of the model. Generally, the correlation within a pair of random variables cannot be represented as a function of the correlations of each of these two variables with a third random variable. However, in the context of the control variate method, the response and control variates are assumed to be the multivariate normally distributed, and the correlation between the response and control variate obtained at the same design point is greater (with respect to the absolute value) than that between the response at a design point and control variate at different design point. Also, this method uses antithetic variates for design points in different blocks. That is, the control variates from two different blocks are negatively correlated. Under this situation, the relationship among the correlations between, respectively, the response and control variates from the same block, the response and control variates from different blocks, and the control variates from different blocks may show certain relationship. Our simulation results strongly indicate a multiplication form of the relationship among the correlations of the variables considered above for the single control variate case.

Also, two other additional directions for future research are extensions of variance reduction techniques developed in Chapter 5 to either a second-order linear simulation model or multipopulation multivariate response simulation experiments.

REFERENCES

Arnold, S. F. 1981. *The Theory of Linear Models and Multivariate Analysis.* John Wiley & Sons, New York.

Anderson, T. W. 1984. *An Introduction to Multivariate Statistical Analysis.* John Wiley & Sons, New York.

Anderson, T. W. and D. A. Darling. 1952. Asymptotic Theory of Certain "Goodness-of-Fit" Criteria Based on Stochastic Process. *Annals of Mathematical Statistics,* **23**, 193-212.

Bauer, K. W. 1987. Control Variate Selection for Multiresponse Simulation. Unpublished Ph.D. Dissertation. Department of Industrial Engineering, University of Purdue, Lafayette, Indiana.

Beja, A. 1969. Multiple Control Variates in Monte Calro Simulation with Application to Queueing Systems with Priorities. *Developments in Operations Research,* 255-264, B. Aviltzhak (Ed.), Gordon and Breach, New York.

Bickel, P. J. and K. A. Doksum. 1977. *Mathematical Statistics: Basic Ideas and Selected Topics.* Holden-Day, San Francisco, California.

Bratley, P., B. L. Fox and L. Schrage. 1983. *A Guide to Simulation.* Spring-Verlag, New York.

Cheng, R. C. H. 1987. Analysis of Simulation Experiments under Normality Assumptions. *Journal of the Operational Research* **29**, 493-497.

Cheng, R. C. H. and G. H. Feast. 1980. Control Variables with Known Mean and Variance. *Journal of the Operational Research* **31**, 51-56.

Cornell, J. A. 1981. *Experiments with Mixtures.* John Wiley & Sons, New York.

Graybill, F. A. 1976. *Theory and Application of the Linear Model.* Duxury Press, North Scituate, Massachusetts.

Hogg, R. V. and A. T. Craig. 1978. *Introduction to Mathematical Statistics.* Macmillan, New York.

Hussey, J. R., R. H. Myers and E. C. Houck. 1987. Correlated Simulation Experiments in First-Order Response Surface Design. *Operations Research* **35**, 744-758.

Kenney, J. F. and J. E. Keeping. 1951. *Mathematics of Statistics Part II*. Van Nostrand Co. Inc., New York.

Kleijnen, J. P. C. 1974. *Statistical Techniques in Simulation, Part I*. Marcel Dekker, New York.

Lavenberg, S. S., T. L. Moeller and C. H. Sauer. 1979. Concomitant Control Variables Applied to the Regenerative Simulation of Queuing Systems. *Operations Research* **27**, 134-160.

Lavenberg, S. S., T. L. Moeller and P. D. Welch. 1982. Statistical Results on Control Variates with Application to Queuing Network Simulation. *Operations Research* **30**, 182-202.

Lavenberg, S. S. and P. D. Welch. 1981. A Perspective on the Use of Control Variables to Increase the Efficiency of Monte Carlo Simulations. *Management Science* **27**, 322-335.

Law, A. M. and W. D. Kelton. 1982. *Simulation Modeling and Analysis*. McGraw-Hill, New York.

Lewis, P. A. W., R. Ressler and R. K. Wood. 1987. Smarter Control Variables: Regression-Adjusted Linear and Nonlinear Controls. *Proceedings of the Winter Simulation Conference*, 342-346.

Mood, A.M., F. A. Graybill and D. C. Boes. 1974. *Introduction to the Theory of Statistics*. McGraw-Hill, New York.

Morrison, D. F. 1976. *Multivariate Statistical Methods*. McGraw-Hill, New York.

Moy, W. A. 1971. Practical Variance-Reducing Procedures for Monte Carlo Simulations. *Computer Simulation Experiments with Models of Economic Experiments*, 269-289, T. H. Naylor. (Ed.), John Wiley & Sons, New York.

Muirhead, R. J. 1982. *Aspects of Multivariate Statistical Theory*. John Wiley & Sons, New York.

Myers, R. H. 1971. *Response Surface Methodology*. Allyn and Bacon, Boston, Massachusetts.

Myers, R. H. 1986. *Classical and Modern Regression with Applications*. Duxbury Press, Boston, Massachusetts.

Nozari, A., S. F. Arnold and C. D. Pegden. 1984. Control Variates for Multipopulation Simulation Experiments. *IIE Transactions* **16**, 159-169.

Nova, A. M. P. 1985. A Generalized Approach to Variance Reduction in Discrete-event Simulation Using Control Variables. Unpublished Ph.D. Dissertation. Department of Mechanical Engineering, The University of Texas, Austin, Texas.

Pratt, J. W. and R. Schlaifer. 1984. On the Nature and Discovery of Structure. *Journal of American Statistical Association* **79**, 9-21.

Pritsker, A. A. B. 1986. *Introduction to Simulation and SLAM II.* John Wiley & Sons, New York.

Rao, C. R. 1967. Least Squares Theory Using an Estimated Dispersion Matrix and Its Application to Measurement of Signals. *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability,* 355-372. University of California Press, Berkeley, California.

Rao, C. R. 1973. *Linear Statistical Inference and Its Applications.* John Wiley & Sons, New York.

Rubinstein, R. Y. and R. Marcus. 1985. Efficiency of the Multivariate Control Variates in Monte Carlo Simulation. *Operations Research* **33**, 661-677.

Schriber, T. 1974. *Using GPSS.* John Wiley & Sons, New York.

Schruben, L. W. and B. H. Margolin. 1978. Pseudorandom Number Assignment in Statistically Designed Simulation and Distribution Sampling Experiments. *Journal of the American Statistical Association* **73**, 504-525.

Searle, S. R. 1971. *Linear Models.* John Wiley & Sons, New York.

Seber, G. A. F. 1977. *Linear Regression Analysis.* John Wiley & Sons, New York.

Tew, J. D. 1989. Correlation Induction Techniques for Fitting Second-order Metamodels in Simulation Experiments. *Proceedings of the Winter Simulation Conference*, 538-546.

Tew, J. D. and J. R. Wilson. 1989. Estimating Simulation Metamodels Using Integrated Variance Reduction Techniques. Technical Report, Department of Industrial Engineering and Operations Research, Virginia Polytechnic Institute and State University, Blacksburg, Virginia.

Tew, J. D. and J. R. Wilson. 1990. Validation of Simulation Analysis Methods for the Schruben-Margolin Correlation-Induction Strategy. *Operations Research* (to appear).

Venkatraman, S. and J. R. Wilson. 1986. The Efficiency of Control Variates in Multiresponse Simulation. *Operations Research Letters* **5**, 37-42.

Wilson, J. R. 1982. Variance Reduction Techniques for the Simulation of Queueing Networks. Technical Report, Department of Mechanical Engineering, The University of Texas, Austin, Texas.

Wilson, J. R. 1984. Variance Reduction Techniques for Digital Simulation. *American Journal of Mathematical and Management Sciences* **4**, 227-312.

Wilson, J. R. and A. A. B. Pritsker. 1984a. Variance Reduction in Queuing Simulation Using Generalized Concomitant Variables. *Journal of Statistical Computation and Simulation* **19**, 129-153.

Wilson, J. R. and A. A. B. Pritsker. 1984b. Experimental Evaluation of Variance Reduction Techniques for Queuing Simulation Using Generalized Concomitant Variables. *Management Science* **30**, 1459-1472.

Appendix A. Derivation of Equations

Proof of Equation (5.12):

From Theorem 2.4.3 in Anderson (1984), the marginal distribution of $\bar{\mathbf{c}}_i$ in (5.6) is given by

$$\bar{\mathbf{c}}_i \sim N_s(\mathbf{0},\ \frac{1}{2h}\Sigma_c). \qquad (A1)$$

Since $\mathbf{c}_i$ ($i = 1,\ 2,\ \ldots,\ m$) are independent by Assumption 5 in Section 5.1, we have

$$\bar{\mathbf{C}} \sim N_{m,\ s}(\mathbf{O},\ \frac{1}{2h}\Sigma_c,\ \mathbf{I}_m). \qquad (A2)$$

and

$$\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}} \sim W_s(m-p-1,\ \frac{1}{2h}\Sigma_c). \qquad (A3)$$

(see p. 19). Since $\bar{\mathbf{C}}(\Sigma_c/2h)^{-1/2} \sim N_{m,\ s}(\mathbf{O},\ \mathbf{I}_s,\ \mathbf{I}_m)$ from (A2), we have

$$E[\bar{\mathbf{C}}(\Sigma/2h)^{-1}\bar{\mathbf{C}}'] = s\mathbf{I}_m \qquad (A4)$$

(see Theorem 17.6a in Arnold (1981)). Also from Theorem 17.15d in Arnold (1981),

$$E[(\bar{\mathbf{C}}'\mathbf{P}\bar{\mathbf{C}})^{-1}] = \frac{2h}{m-p-s-2}\Sigma_c^{-1} \ \text{ if } m > p-s-2. \qquad (A5)$$

Discussion on matrix **P** in Chapter 2 (see p. 13) indicates that **P** is a symmetric and idempotent matrix with rank $(m-p-1)$. Thus, **P** is a positive semi-definite matrix (see Theorem 1.7.1 in Graybill (1974)). In such a case, $\overline{\mathbf{C}}\mathbf{X}$ and $\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}}$ are independent since $\mathbf{PX} = \mathbf{0}$ (see Theorem 4.5.1 in Graybill (1974)). Therefore, by (A5),

$$\mathrm{E}[\mathbf{X}'\overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{X}] = \mathrm{E}[\mathbf{X}'\overline{\mathbf{C}}\mathrm{E}[(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}]\overline{\mathbf{C}}'\mathbf{X}]$$

$$= \frac{1}{m-p-s-2}\mathrm{E}[\mathbf{X}'\overline{\mathbf{C}}(\Sigma_c/2h)^{-1}\overline{\mathbf{C}}'\mathbf{X}], \qquad (A6)$$

which further reduces to

$$\mathrm{E}[\mathbf{X}'\overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{X}] = \frac{1}{m-p-s-2}\mathbf{X}'(s\mathbf{I}_m)\mathbf{X} = \frac{s}{m-p-s-2}\mathbf{X}'\mathbf{X} \qquad (A7)$$

by (A4). Therefore, taking the operation of expectation on (5.11) finally yields

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}}) = \mathrm{E}[\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \overline{\mathbf{C}})] = \frac{\sigma_y^2}{2h}(1\rho_y - R_{yc}^2)(\mathbf{X}'\mathbf{X})^{-1}[1 + \frac{s}{m-p-s-2}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}]$$

$$= \frac{\sigma_y^2}{2h}(1\rho_y - R_{yc}^2)(\frac{m-p-2}{m-p-s-2})(\mathbf{X}'\mathbf{X})^{-1}, \qquad (A8)$$

which is equivalent to (5.12).

Proof of Equation (5.31):

Note that $\mathbf{1}_m$ is the first column of **X** and $\mathbf{PX} = \mathbf{0}$. Therefore, we have

$$\mathbf{P1}_m = \mathbf{1}_m\mathbf{P} = \mathbf{0} \qquad (A9)$$

since $\mathbf{PX} = \mathbf{X}'\mathbf{P} = \mathbf{0}$. Developing the second term in (5.29) gives

$$\delta(\mathbf{X'X})^{-1}\mathbf{X}'[\mathbf{I}_m - \overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{P}]\mathbf{1}_m\mathbf{1}'_m[\mathbf{I}_m - \mathbf{P}\overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}']\mathbf{X}(\mathbf{X'X})^{-1}$$

$$= \delta(\mathbf{X'X})^{-1}\mathbf{X}'[\mathbf{1}_m\mathbf{1}'_m - \overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{P}\mathbf{1}_m\mathbf{1}'_m - \mathbf{1}_m\mathbf{1}'_m\mathbf{P}\mathbf{C}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'$$

$$+ \overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{P}\mathbf{1}_m\mathbf{1}'_m\mathbf{P}\mathbf{C}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}']\mathbf{X}(\mathbf{X'X})^{-1}$$

$$= \delta(\mathbf{X'X})^{-1}\mathbf{X}'\mathbf{1}_m\mathbf{1}'_m\mathbf{X}(\mathbf{X'X})^{-1} \qquad (A10)$$

by (A9). Since $\mathbf{X}$ is orthogonal, $\mathbf{X}'\mathbf{1}_m = (m,\ 0,\ \dots, 0)'$, which implies

$$\mathbf{X}'\mathbf{1}_m\mathbf{1}'\mathbf{X}_m = m^2\mathbf{G}_{p+1}, \qquad (A11)$$

where $\mathbf{G}_{p+1}$ is as defined in (5.14). Thus we have

$$(\mathbf{X'X})^{-1}\mathbf{X}'\mathbf{1}'_m\mathbf{1}_m\mathbf{X}(\mathbf{X'X})^{-1} = m^{-1}m^2\mathbf{G}_{p+1}m^{-1} = \mathbf{G}_{p+1}. \qquad (A12)$$

Substitution (A12) into (A10) finally yields (5.31).

Proof of Equation (5.33):

From equations (5.22) and (5.23), the marginal distribution of $\overline{\mathbf{C}}$ in (5.22) is same as given in (A2). Therefore, using the same procedures in (A3)-(A7), we find

$$\mathrm{E}[\mathbf{X}'\overline{\mathbf{C}}(\overline{\mathbf{C}}'\mathbf{P}\overline{\mathbf{C}})^{-1}\overline{\mathbf{C}}'\mathbf{X}] = \frac{s}{m-p-s-2}\mathbf{X'X}. \qquad (A13)$$

Therefore, taking the operation of expectation on (5.32) yields

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}}) = \mathrm{E}[\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathbf{G}} \mid \overline{\mathbf{C}})] = \gamma(\mathbf{X'X})^{-1}[1 + \frac{s}{m-p-s-2}(\mathbf{X'X})(\mathbf{X'X})^{-1}] + \delta\mathbf{G}_{p+1}$$

$$= \gamma\left(\frac{m-p-2}{m-p-s-2}\right)(\mathbf{X}'\mathbf{X})^{-1} + \delta \mathbf{G}_{p+1}, \qquad (A14)$$

which is equivalent to (5.33).

Appendix B. SLAM II Code and FORTRAN Program for Computing Estimator of Parameters

Appendix B-1

```
C
C THIS IS THE SLAM II NETWEORK CODE FOR THE CLOSED
C MACHINE-REPAIR NETWORK GIVEN BY WILSON AND PRITSKER (1984)
C
C MAIN PROGRAM
C
C CALL SLAM PROGRAM
C
      SUBROUTINE EVENT(I)
      INCLUDE (SLMSCOM1)
      GO TO (1),I
C
C WRITE THE RESPONSES AND CONTROL VARIATES:
C
C         Y1= TIME IN SYSTEM OF REGULAR UNIT
C         Y2= UTILIZATION OF STATION 1
C         Y3= UTILIZATION OF STATION 2
C         Y4= UTILIZATION OF STATION 3
C         Y5= UTILIZATION OF STATION 4
C         C1= STANDARD CONTROL VARIATE AT STATION 1
C         C2= STANDARD CONTROL VARIATE AT STATION 2
C         C3= STANDARD CONTROL VARIATE AT STATION 3
C         C4= STANDARD CONTROL VARIATE AT STATION 4 (QUEUE 4)
C
    1 Y1=XX(14)/XX(13)
      Y2=XX(9)/950.
      Y3=XX(10)/950.
      Y4=XX(11)/950.
      Y5=XX(12)/950.
      C1=XX(2)/XX(1)**.5/10.
      C2=XX(4)/XX(3)**.5/1.5
      C3=XX(6)/XX(5)**.5/1.0
      C4=XX(8)/XX(7)**.5/.5
      WRITE(6,2) Y1,Y2,Y3,Y4,Y5,C1,C2,C3,C4
    2 FORMAT(9F7.3)
      RETURN
      END
C
C SLAM II CODE
C
GEN,KWO,PRO,6/21/1990,20,NO,NO,,NO,NO;
LIM,4,5,100;
INTLC,XX(1)=0,XX(2)=0.0,XX(3)=0,XX(4)=0.0,
      XX(5)=0,XX(6)=0.0,XX(7)=0,XX(8)=0.0;
INTLC,XX(9)=0,XX(10)=0.0,XX(11)=0,XX(12)=0.0,
      XX(13)=0.0,XX(14)=0.0;
NETWORK;
;
;  THIS IS THE SLAM II NETWORK CODE FOR THE COLESD MACHINE
;  REPAIR PROBLEM OF SIMULATION GIVEN BY WILSON AND PRITSKER(1984).
;
      RESOURCE/MACH(5),1;
      RESOURCE/MAREP(1),2;
      RESOURCE/MIREP(1),3;
      RESOURCE/MTEST(1),4;

OPERA ASSIGN,ATRIB(1)=EXPON(10.0,1);
      ACT,,,Q1;
Q1    AWAIT(1),MACH/1;
      GOON,1;
      ACT,,TNOW+ATR(1).GT.1000.,G11;
      ACT,,,G12;
G11   ASSIGN,XX(9)=XX(9)+1000.-TNOW;
      ACT,,,G12;
G12   GOON,1;
```

```
      ACT/1,ATR(1);
      FREE,MACH/1;
      GOON,1;
      ACT,,TNOW.LE.50,RBRA;
      ACT,,,G13;
G13   ASSIGN,ATR(5)=TNOW;
      GOON,1;
      ACT,,TNOW-50..LE.ATR(1),G14;
      ACT,,,G15;
G14   ASSIGN,XX(9)=XX(9)+TNOW-50.;
      ACT,,,RBRA;
G15   ASSIGN, XX(1)=XX(1)+1,
              XX(2)=XX(2)+ATRIB(1)-10.0;
      ASSIGN, XX(9)=XX(9)+ATR(1);
      ACT,,,RBRA;
RBRA  GOON,1;
      ACT,,UNFRM(0.0,1.0,2).LE.0.25,MAJOR;
      ACT,,,MINOR;

MAJOR ASSIGN,ATRIB(2)=EXPON(1.5,3);
      ACT,,,Q2;
Q2    AWAIT(2),MAREP(1),1;
      GOON,1;
      ACT,,TNOW+ATR(2).GT.1000.,G21;
      ACT,,,G22;
G21   ASSIGN,XX(10)=XX(10)+1000-TNOW;
      ACT,,,G22;
G22   GOON,1;
      ACT/2, ATRIB(2);
      FREE,MAREP/1;
      GOON,1;
      ACT,,TNOW.LE.50.,TEST;
      ACT,,,G23;
G23   GOON,1;
      ACT,,TNOW-50..LE.ATR(2),G24;
      ACT,,,G25;
G24   ASSIGN,XX(10)=XX(10)+TNOW-50.;
      ACT,,,TEST;
G25   ASSIGN,XX(3)=XX(3)+1,
            XX(4)=XX(4)+ATRIB(2)-1.5,
            XX(10)=XX(10)+ATR(2);
      ACT,,,TEST;

MINOR ASSIGN,ATRIB(3)=EXPON(1.0,4);
      ACT,,,Q3;
Q3    AWAIT(3),MIREP/1;
      GOON,1;
      ACT,,TNOW+ATR(3).GT.1000.,G31;
      ACT,,,G32;
G31   ASSIGN,XX(11)=XX(11)+1000-TNOW;
      ACT,,,G32;
G32   GOON,1;
      ACT/2, ATRIB(3);
      FREE,MIREP/1;
      GOON,1;
      ACT,,TNOW.LE.50.,TEST;
      ACT,,,G33;
G33   GOON,1;
      ACT,,TNOW-50..LE.ATR(3),G34;
      ACT,,,G35;
G34   ASSIGN,XX(11)=XX(11)+TNOW-50.;
      ACT,,,TEST;
G35   ASSIGN,XX(5)=XX(5)+1,
             XX(6)=XX(6)+ATRIB(3)-1.0,
             XX(11)=XX(11)+ATR(3);
      ACT,,,TEST;
```

```
TEST  ASSIGN,ATRIB(4)=EXPON(0.5,5);
      ACT,,,Q4;
Q4    AWAIT(4),MTEST/1;
      GOON,1;
      ACT,,TNOW+ATR(4).GT.1000.,G41;
      ACT,,,G42;
G41   ASSIGN,XX(12)=XX(12)+1000.-TNOW;
      ACT,,,G42;
G42   GOON,1;
      ACT/4, ATRIB(4);
      FREE,MTEST/1;
      GOON,1;
      ACT,,TNOW.LE.50.,TBRA;
      ACT,,,G43;
G43   GOON,1;
      ACT,,TNOW-50..LE.ATR(4),G44;
      ACT,,,G45;
G44   ASSIGN,XX(12)=XX(12)+TNOW-50.;
      ACT,,,TBRA;
G45   ASSIGN,XX(7)=XX(7)+1,
             XX(8)=XX(8)+ATRIB(4)-0.5,
             XX(12)=XX(12)+ATR(4);
      ACT,,,TBRA;
TBRA  GOON,1;
      ACT,,UNFRM(0.0, 1.0,6).LE.0.9,G46;
      ACT,,,MINOR;
G46   GOON,1;
      ACT,,TNOW.LT.50.,OPERA;
      ACT,,,G47;
G47   ASSIGN,XX(13)=XX(13)+1,XX(14)=XX(14)+TNOW-ATR(5);
      COLCT,INT(5),REG TIME;
      ACT,,,OPERA;

      CREATE,,1000,,1,1;
      ACT,0.0;
      EVENT,1,1;
      TERMINATE;
      END;
SEEDS,1928027(1),6352891(2),3389027(3),2889047(4),6940321(5),4898215(6);
INIT,0,1000;
MONTR,CLEAR,50;
ENTRY/ 1,10.0/ 1,5.0 / 1,12.0/ 1,11.0 / 1,9.0 / 1,7.0/1,14.0;
SIMULATE;
FIN;
```

Appendix B-2

```
C
C THIS IS THE SLAM II NETWORK CODE FOR THE MIXED
C MACINE-REPAIR NETWORK GIVEN BY WILSON AND PRITSKER (1984)
C
C MAIN PROGRAM
C
C CALL SLAM PROGRAM
C
        SUBROUTINE EVENT(I)
        INCLUDE (SLMSCOM1)
        GO TO (1),I
C
C WRITE THE RESPONSES AND CONTROL VARIATES:
C
C      Y1= TIME IN SYSTEM OF REGULAR UNIT
C      Y2= TIME IN SYSTEM OF PRIORITY UNIT
C      Y3= UTILIZATION OF STATION 1
C      Y4= UTILIZATION OF STATION 2
C      Y5= UTILIZATION OF STATION 3
C      Y6= UTILIZATION OF STATION 4
C      C1= STANDARD CONTROL VARIATE AT STATION 1
C      C2= STANDARD CONTROL VARIATE AT STATION 2
C      C3= STANDARD CONTROL VARIATE AT STATION 3
C      C4= STANDARD CONTROL VARIATE AT STATION 4
C
     1 Y1=XX(14)/XX(13)
       Y2=XX(16)/XX(15)
       Y3=XX(9)/950.
       Y4=XX(10)/950.
       Y5=XX(11)/950.
       Y6=XX(12)/950.
       C1=XX(2)/XX(1)**.5/10.
       C2=XX(4)/XX(3)**.5/1.5
       C3=XX(6)/XX(5)**.5/1.0
       C4=XX(8)/XX(7)**.5/.5
       WRITE(6,2) Y1,Y2,Y3,Y4,Y5,Y6,C1,C2,C3,C4
     2 FORMAT(10F7.3)
       RETURN
       END
C
C SLAM PROGRAM
C
GEN,KWO,PRO,6/21/1990,50,NO,NO,,NO,NO;
LIM,4,16,100;
PRIORITY/2,LVF(2)/3,LVF(2)/4,LVF(2);
INTLC,XX(1)=0,XX(2)=0.0,XX(3)=0,XX(4)=0.0,
      XX(5)=0,XX(6)=0.0,XX(7)=0,XX(8)=0.0;
INTLC,XX(9)=0,XX(10)=0.0,XX(11)=0,XX(12)=0.0,
      XX(13)=0.0,XX(14)=0.0,XX(15)=0.0,XX(16)=0.0;
NETWORK;
;
;  THIS IS THE SLAM II NETWORK CODE FOR THE COLESD MACHINE
;  REPAIR PROBLEM OF SIMULATION GIVEN BY WILSON AND PRITSKER(1984).
;
      RESOURCE/MACH(5),1;
      RESOURCE/MAREP(1),2;
      RESOURCE/MIREP(1),3;
      RESOURCE/MTEST(1),4;

      CREATE,EXPON(8.0,7),0.0,15,,1;
      ASSIGN,ATR(2)=1;
      ACT,,,MAJOR;

OPERA ASSIGN,ATRIB(1)=EXPON(10.0,1),ATR(2)=2;
      ACT,,,Q1;
```

```
Q1    AWAIT(1),MACH/1;
      GOON,1;
      ACT,,TNOW+ATR(1).GT.1000.,G11;
      ACT,,,G12;
G11   ASSIGN,XX(9)=XX(9)+1000.-TNOW;
      ACT,,,G12;
G12   GOON,1;
      ACT/1,ATR(1);
      FREE,MACH/1;
      GOON,1;
      ACT,,TNOW.LE.50,RBRA;
      ACT,,,G13;
G13   ASSIGN,ATR(5)=TNOW;
      GOON,1;
      ACT,,TNOW-50..LE.ATR(1),G14;
      ACT,,,G15;
G14   ASSIGN,XX(9)=XX(9)+TNOW-50.;
      ACT,,,RBRA;
G15   ASSIGN, XX(1)=XX(1)+1,
              XX(2)=XX(2)+ATRIB(1)-10.0;
      ASSIGN, XX(9)=XX(9)+ATR(1);
      ACT,,,RBRA;
RBRA  GOON,1;
      ACT,,UNFRM(0.0,1.0,2).LE.0.25,MAJOR;
      ACT,,,MINOR;
;
MAJOR ASSIGN,ATRIB(14)=EXPON(1.5,3);
      ACT,,,Q2;
Q2    AWAIT(2),MAREP(1),1;
      GOON,1;
      ACT,,TNOW+ATR(14).GT.1000.,G21;
      ACT,,,G22;
G21   ASSIGN,XX(10)=XX(10)+1000-TNOW;
      ACT,,,G22;
G22   GOON,1;
      ACT/2, ATRIB(14);
      FREE,MAREP/1;
      GOON,1;
      ACT,,TNOW.LE.50.,TEST;
      ACT,,,G23;
G23   GOON,1;
      ACT,,TNOW-50..LE.ATR(14),G24;
      ACT,,,G25;
G24   ASSIGN,XX(10)=XX(10)+TNOW-50.;
      ACT,,,TEST;
G25   ASSIGN,XX(3)=XX(3)+1,
             XX(4)=XX(4)+ATRIB(14)-1.5,
             XX(10)=XX(10)+ATR(14);
      ACT,,,TEST;
MINOR ASSIGN,ATRIB(3)=EXPON(1.0,4);
      ACT,,,Q3;
Q3    AWAIT(3),MIREP/1;
      GOON,1;
      ACT,,TNOW+ATR(3).GT.1000.,G31;
      ACT,,,G32;
G31   ASSIGN,XX(11)=XX(11)+1000-TNOW;
      ACT,,,G32;
G32   GOON,1;
      ACT/2, ATRIB(3);
      FREE,MIREP/1;
      GOON,1;
      ACT,,TNOW.LE.50.,TEST;
      ACT,,,G33;
G33   GOON,1;
      ACT,,TNOW-50..LE.ATR(3),G34;
      ACT,,,G35;
G34   ASSIGN,XX(11)=XX(11)+TNOW-50.;
```

```
      ACT,,,TEST;
G35   ASSIGN,XX(5)=XX(5)+1,
             XX(6)=XX(6)+ATRIB(3)-1.0,
             XX(11)=XX(11)+ATR(3);
      ACT,,,TEST;

TEST  ASSIGN,ATRIB(4)=EXPON(0.5,5);
      ACT,,,Q4;
Q4    AWAIT(4),MTEST/1;
      GOON,1;
      ACT,,TNOW+ATR(4).GT.1000.,G41;
      ACT,,,G42;
G41   ASSIGN,XX(12)=XX(12)+1000.-TNOW;
      ACT,,,G42;
G42   GOON,1;
      ACT/4, ATRIB(4);
      FREE,MTEST/1;
      GOON,1;
      ACT,,TNOW.LE.50.,PRI;
      ACT,,,G43;
G43   GOON,1;
      ACT,,TNOW-50..LE.ATR(4),G44;
      ACT,,,G45;
G44   ASSIGN,XX(12)=XX(12)+TNOW-50.;
      ACT,,,PRI;
G45   ASSIGN,XX(7)=XX(7)+1,
             XX(8)=XX(8)+ATRIB(4)-0.5,
             XX(12)=XX(12)+ATR(4);
      ACT,,,PRI;
PRI   GOON,1;
      ACT,,ATR(2).EQ.1,PTBRA;
      ACT,,,RTBRA;
RTBRA GOON,1;
      ACT,,UNFRM(0.0, 1.0,6).LE.0.9,G46;
      ACT,,,MINOR;
G46   GOON,1;
      ACT,,TNOW.LT.50.,OPERA;
      ACT,,,G47;
G47   ASSIGN,XX(13)=XX(13)+1,XX(14)=XX(14)+TNOW-ATR(5);
      COLCT,INT(5),REG TIME;
      ACT,,,OPERA;

PTBRA GOON,1;
      ACT,,UNFRM(0.0, 1.0,8).LE.0.9,G48;
      ACT,,,MINOR;
G48   GOON,1;
      ACT,,TNOW.LT.50.,TRM;
      ACT,,,G49;
G49   ASSIGN,XX(15)=XX(15)+1,XX(16)=XX(16)+TNOW-ATR(15);
      COLCT,INT(15),PRI.TIME;
TRM   TERMINATE;
;
      CREATE,,1000,,1,1;
      ACT,0.0;
      EVENT,1,1;
      TERMINATE;
      END;
SEEDS,1723641(1),3029391(2),6029157(3),2039383(4),
      3029387(5),5927611(6),4958675(7),9483467(8);
INIT,0,1000;
MONTR,CLEAR,50;
ENTRY/ 1,10.0/ 1,5.0 / 1,12.0/ 1,11.0 / 1,9.0 / 1,7.0/1,14.0;
SIMULATE;
FIN;
```

Appendix B-3

```
C
C THIS IS THE SLAM II NETWORK CODE FOR THE OPEN
C MACHINE-REPAIR NETWORK
C
C MAIN PROGRAM
C
C CALL SLAM PROGRAM
C
        SUBROUTINE EVENT(I)
        INCLUDE (SLMSCOM1)
        GO TO (1),I
C
C WRITE THE RESPONSES AND CONTROL VARIATES:
C
C      Y1= TIME IN SYSTEM OF PRIORITY UNIT
C      Y2= UTILIZATION OF STATION 1
C      Y3= UTILIZATION OF STATION 2
C      Y4= UTILIZATION OF STATION 3
C      C1= STANDARD CONTROL VARIATE AT STATION 1
C      C2= STANDARD CONTROL VARIATE AT STATION 2
C      C3= STANDARD CONTROL VARIATE AT STATION 3
C
     1 Y1=XX(16)/XX(15)
       Y2=XX(10)/950.
       Y3=XX(11)/950.
       Y4=XX(12)/950.
       C1=XX(4)/XX(3)**.5/1.5
       C2=XX(6)/XX(5)**.5/1.0
       C3=XX(8)/XX(7)**.5/.5
       WRITE(6,2) Y1,Y2,Y3,Y4,C1,C2,C3
     2 FORMAT(10F7.3)
       RETURN
       END
C
C SLAM PROGRAM
C
GEN,KWO,PRO,6/21/1990,11,NO,NO,,NO,NO;
LIM,4,16,100;
INTLC,XX(1)=0,XX(2)=0.0,XX(3)=0,XX(4)=0.0,
      XX(5)=0,XX(6)=0.0,XX(7)=0,XX(8)=0.0;
INTLC,XX(9)=0,XX(10)=0.0,XX(11)=0,XX(12)=0.0,
      XX(13)=0.0,XX(14)=0.0,XX(15)=0.0,XX(16)=0.0;
NETWORK;
;
;  THIS IS THE SLAM II NETWORK CODE FOR THE OPEN MACHINE
;  REPAIR PROBLEM OF SIMULATION (VARIATION OF WILSON AND PRITSKER(1984))
;
     RESOURCE/MAREP(1),2;
     RESOURCE/MIREP(1),3;
     RESOURCE/MTEST(1),4;

     CREATE,EXPON(2.0,1),0.0,15,,1; ;  ASSIGN THE RANDOM SEED FOR STOCHASTIC COMPONENTS
     ASSIGN,ATRIB(2)=EXPON(1.5,2);  SERVICE TIME FOR MAJOR
     ASSIGN,ATRIB(3)=EXPON(1.0,3);  SERVICE TIME FOR MINOR
     ASSIGN,ATRIB(4)=EXPON(0.5,4);  SERVICE TIME FOR TEST
     ASSIGN,ATRIB(5)=UNFRM(0.0,1.0,5);  PROBABILITY FOR BRANCH
     ACT,,,Q2; ; Q2    AWAIT(2),MAREP(1),1;
     GOON,1;
     ACT,,TNOW+ATR(14).GT.1000.,G21;
     ACT,,,G22; G21   ASSIGN,XX(10)=XX(10)+1000-TNOW;
     ACT,,,G22; G22   GOON,1;
     ACT/2, ATRIB(2);
     FREE,MAREP/1;
     GOON,1;
     ACT,,TNOW.LE.50.,TEST;
```

```
      ACT,,,G23; G23   GOON,1;
      ACT,,TNOW-50..LE.ATR(2),G24;
      ACT,,,G25; G24   ASSIGN,XX(10)=XX(10)+TNOW-50.;
      ACT,,,TEST; G25   ASSIGN,XX(3)=XX(3)+1,
             XX(4)=XX(4)+ATR(2)-1.5,
             XX(10)=XX(10)+ATR(2);
      ACT,,,TEST; MINOR AWAIT(3),MIREP/1;
      GOON,1;
      ACT,,TNOW+ATR(3).GT.1000.,G31;
      ACT,,,G32; G31   ASSIGN,XX(11)=XX(11)+1000-TNOW;
      ACT,,,G32; G32   GOON,1;
      ACT/3, ATRIB(3);
      FREE,MIREP/1;
      GOON,1;
      ACT,,TNOW.LE.50.,TEST;
      ACT,,,G33; G33   GOON,1;
      ACT,,TNOW-50..LE.ATR(3),G34;
      ACT,,,G35; G34   ASSIGN,XX(11)=XX(11)+TNOW-50.;
      ACT,,,TEST; G35   ASSIGN,XX(5)=XX(5)+1,
             XX(6)=XX(6)+ATRIB(3)-1.0,
             XX(11)=XX(11)+ATR(3);
      ACT,,,TEST;

TEST  AWAIT(4),MTEST/1;
      GOON,1;
      ACT,,TNOW+ATR(4).GT.1000.,G41;
      ACT,,,G42; G41   ASSIGN,XX(12)=XX(12)+1000.-TNOW;
      ACT,,,G42; G42   GOON,1;
      ACT/4, ATRIB(4);
      FREE,MTEST/1;
      GOON,1;
      ACT,,TNOW.LE.50.,PTBRA;
      ACT,,,G43; G43   GOON,1;
      ACT,,TNOW-50..LE.ATR(4),G44;
      ACT,,,G45; G44   ASSIGN,XX(12)=XX(12)+TNOW-50.;
      ACT,,,PTBRA; G45   ASSIGN,XX(7)=XX(7)+1,
             XX(8)=XX(8)+ATRIB(4)-0.5,
             XX(12)=XX(12)+ATR(4);
      ACT,,,PTBRA;

PTBRA GOON,1;
      ACT,,ATR(5).LE.0.9,G48;
      ACT,,,MINOR; G48   GOON,1;
      ACT,,TNOW.LT.50.,TRM;
      ACT,,,G49; G49   ASSIGN,XX(15)=XX(15)+1,XX(16)=XX(16)+TNOW-ATR(15);
      COLCT,INT(15),PRI.TIME; TRM   TERMINATE; ;
      CREATE,,1000,,1,1;
      ACT,0.0;
      EVENT,1,1;
      TERMINATE;
      END;
 SEEDS,5098763(1),2546217(2),2355489(3),8103277(4),3486731(5);
 INIT,0,1000;
 MONTR,CLEAR,50;
 SIMULATE;
 FIN;
```

Appendix B-4

```
C
C     THIS IS THE SLAM II NETWORK CODE FOR THE PATIENT PATHS
C     IN HOSPITAL UNIT SIMULATION GIVEN BY SCHRUBEN AND MARGOLIN(1978)
C     AND ALSO BY HUSSEY, MYERS, AND HOUCK (1987)
C
C MAIN PROGRAM
C
C CALL SLAM II PROGRAM
C
      SUBROUTINE EVENT(I)
      INCLUDE (SLMSCOM1)
      GO TO (1),I
    1 C1=XX(4)/((XX(5)/0.303)**0.5)
      WRITE(6,2) XX(1),C1
    2 FORMAT(2F9.4)
      RETURN
      END
C
C SLAM PROGRAM
C
GEN,KWO,OR,9/20/1990,100,NO,NO,,NO,NO;
LIMITS,3,15,100;
INTLC,XX(1)=0,XX(2)=0,XX(3)=0,XX(4)=0,XX(5)=0;
NETWORK;
;
;     CREAT THE ARRIVING PATIENTS TO THE SYSTEM

      CREATE,EXPON(.303,1),0.0,13,,1;
      ASSIGN, ATR(9)=TNOW,XX(2)=TNOW-XX(3)-.303,XX(3)=TNOW;
      GOON,1;
      ACT,,TNOW .LT. 300,GC1;
      ACT,,,GC2; GC2   ASSIGN,XX(4)=XX(4)+XX(2),XX(5)=XX(5)+1;
;
;     ASSIGN ALL OF THE SEVICE TIMES TO THE ENTITY AS WELL AS THE
;     PATH PROBABILITIES
; GC1    ASSIGN,ATRIB(1)=UNFRM(0.0,1.0,2),
              ATRIB(2)=RLOGN(3.4,3.5,3),
              ATRIB(3)=RLOGN(3.8,1.6,4),
              ATRIB(4)=UNFRM(0.0,1.0,5),
              ATRIB(5)=UNFRM(0.0,1.0,6);
      ACT,0.;
      ASSIGN,ATRIB(6)=RLOGN(15.0,7.0,7),
              ATRIB(7)=RLOGN(17.0,3.0,8),
              ATRIB(8)=0.0,
              ATRIB(11)=15.0,
              ATRIB(12)=17.0;
      ACT,0.;
;
;     GO TO EITHER INTENSIVE CARE UNIT OR CORONARY UNIT
;
      GOON,1;
      ACT,0.0,ATRIB(1) .LE. .75,ICU;
      ACT,0.0,,CCU;
;
;     INTENSIVE CARE UNIT
; ICU   QUEUE(1),0,0,BALK(FAIL);
      ACT(15)/1,ATR(2);
      GOON,1;
      ACT,0.0,ATRIB(4) .LE. .27,T1;
      ACT,0.0;
      ASSIGN,ATRIB(8)=ATRIB(6),ATR(10)=ATR(11);
      ACT,0.0,,INTRC;
;
;     CORONARY CARE UNIT
; CCU   QUEUE(2),0,0,BALK(FAIL);
```

```
      ACT(6)/2,ATRIB(3);
      GOON,1;
      ACT,0.0,ATRIB(5) .LE. .20,T1;
      ACT,0.0;
      ASSIGN,ATRIB(8)=ATRIB(7),ATR(10)=ATR(12);
      ACT,0.0,,,INTRC;
;
;     INTERMEDIATE CARE UNIT
; INTRC QUEUE(3),0,0,BALK(FAIL);
      ACT(17)/3,ATRIB(8);
      ACT,,,T1;
;
;     TERMINATE PATIENTS WHO DID NOT BALK
;
T1    TERMINATE;
;
;     COUNT THE NUMBER OF PATIENTS WHO FAILED TO GAIN ADMISSION
;     IF NOT WITHIN THE FIRST 10 MONTHS OF OPERATION
; FAIL  GOON,1;
      ACT,0.0,TNOW .LE. 300,T2;
      ACT,0.0;
      ASSIGN, XX(1) = XX(1)+1.0;
      ACT,0.0; T2    TERMINATE;
;
;     WRITE THE DESIRED OUTPUT AT THE END OF THE SIMULATION RUN
;
      CREATE,,1500,,1,1;
      ACT,0.0;
      ASSIGN,XX(1) = XX(1)/40.;
      ACT,0.0;
      EVENT,1,1;
      TERMINATE;
      ENDNETWORK;
```

Appendix B-5

```
C
C THIS IS THE SLAM II NETWORK CODE FOR THE PORT
C OPERATIONS MODEL GIVEN IN PRITSKER (1986)
C
C MAIN PROGRAM
C
C CALL SLAM PROGRAM
C
      SUBROUTINE EVENT(I)
      INCLUDE (SLMSCOM1)
      GO TO (1),I
C
C WRITE THE RESPONSES AND CONTROL VARIATES:
C
C  Y1= IN-PORT TIME OF TANKER 1
C  Y2= IN-PORT TIME OF TANKER 2
C  Y3= IN-PORT TIME OF TANKER 3
C  Y4= IN-PORT TIME OF TANKER ON CONTRACT
C  C1= STA. CONTROL VAR. OF LOADING TIME OF OIL (TANKER 1)
C  C2= STA. CONTROL VAR. OF LOADING TIME OF OIL (TANKER 2)
C  C3= STA. CONTROL VAR. OF LOADING TIME OF OIL (TANKER 3)
C  C4= STA. CONTROL VAR. OF LOADING TIME OF OIL (TANKER ON CONTRACT)
C  C5= STA. CONTROL VAR. OF ROUND TRIP TRAVEL TIME OF CONTRCT TANKER
C  C6= STA. CONTROL VAR. OF INTER-ARRIVAL TIME OF REGULAR TANKERS
C
    1 Y1=XX(2)/XX(1)
      Y2=XX(4)/XX(3)
      Y3=XX(6)/XX(5)
      Y4=XX(8)/XX(7)
      C1=XX(10)/XX(9)**.5/SQRT(4./3.)
      C2=XX(12)/XX(11)**.5/SQRT(3.)
      C3=XX(14)/XX(13)**.5/SQRT(16./3.)
      C4=XX(16)/XX(15)**.5/SQRT(3.)
      C5=XX(18)/XX(17)**.5/SQRT(192.)
      C6=XX(24)/XX(23)**.5/4.04
      WRITE(6,2) Y1,Y2,Y3,Y4,C1,C2,C3,C4,C5,C6
    2 FORMAT(10F7.3)
      RETURN
      END
C
C SLAM PROGRAM
C
GEN,KWO,PRO,6/21/1990,1,NO,NO,,NO,NO;
LIM,3,10,30;
INTLC,XX(1)=0.0,XX(2)=0.0,XX(3)=0.0,XX(4)=0.0,
      XX(5)=0.0,XX(6)=0.0,XX(7)=0.0,XX(8)=0.0;
INTLC,XX(9)=0.0,XX(10)=0.0,XX(11)=0.0,XX(12)=0.0,
      XX(13)=0.0,XX(14)=0.0,XX(15)=0.0,XX(16)=0.0;
INTLC,XX(17)=0.0,XX(18)=0.0,XX(19)=0.0,XX(20)=0.0;
INTLC,XX(21)=0.0,XX(22)=0.0,XX(23)=0.0,XX(24)=0.0;
NETWORK;
      RESOURCE/BERTH(3),1;
      RESOURCE/TUG(1),2,3;
;
; TANKER ARRIVAL SEGMENT
;
      CREATE,UNFRM(4.,18.,1);
      ASSIGN,XX(22)=TNOW-XX(21),XX(21)=TNOW;
      GOON,1;
      ACT,,TNOW.LE.1000.,K1;
      ACT,,,K2;
K2    ASSIGN,XX(24)=XX(24)+XX(22)-11.,XX(23)=XX(23)+1;
K1    ASSIGN,ATRIB(4)=UNFRM(0.0,1.0,2);
      ACT,,,G1;
G1    GOON,1;
```

```
      ACT,,ATR(4).LE.0.25,ARV1;
      ACT,,ATR(4).LE.0.80,ARV2;
      ACT,,,ARV3;
ARV1  ASSIGN,ATR(1)=UNFRM(16.,20.,3),ATR(2)=1,ATR(5)=ATR(1)-18.;
      ACT,,,PORT;
ARV2  ASSIGN,ATR(1)=UNFRM(21.,27.,4),ATR(2)=2,ATR(5)=ATR(1)-24.;
      ACT,,,PORT;
ARV3  ASSIGN,ATR(1)=UNFRM(32.,40.,5),ATR(2)=3,ATR(5)=ATR(1)-36.;
      ACT,,,PORT;
      CREATE,48,0,,5;
ARV4  ASSIGN,ATR(1)=UNFRM(18.,24.,6),ATR(2)=4,ATR(5)=ATR(1)-21.;
      ACT,,,PORT;
;
; PORT OPERATION SEGMENT
;
PORT  ASSIGN,ATRIB(3)=TNOW;
      AWAIT(1),BERTH/1;            WAIT FOR AN AVAILABLE BERTH
      AWAIT(2),TUG/1;              WAIT FOR AN AVAILABLE TUG
      ACT,1;                       TRAVEL TO BERTH
      FREE, TUG/1;                 RELEASE THE TUG
      ACT, ATRIB(1);               TRANSFER THE CARGO
      GOON,1;
      ACT,,TNOW.LE.1000.,Q3;
      ACT,,,G3;
G3    GOON,1;
      ACT,,ATRIB(2).EQ.1,D1;
      ACT,,ATRIB(2).EQ.2,D2;
      ACT,,ATRIB(2).EQ.3,D3;
      ACT,,ATRIB(2).EQ.4,D4;
D1    ASSIGN,XX(9)=XX(9)+1, XX(10)=XX(10)+ATR(5);
      ACT,,,Q3;
D2    ASSIGN,XX(11)=XX(11)+1, XX(12)=XX(12)+ATR(5);
      ACT,,,Q3;
D3    ASSIGN,XX(13)=XX(13)+1, XX(14)=XX(14)+ATR(5);
      ACT,,,Q3;
D4    ASSIGN,XX(15)=XX(15)+1, XX(16)=XX(16)+ATR(5);
      ACT,,,Q3;
Q3    AWAIT(3),TUG/1;              WAIT FOR AN AVAILABLE TUG
      ACT,1;                       TRAVEL TO SEA
      FREE, BERTH/1;               RELEASE THE BERTH
      FREE, TUG/1;                 RELEASE THE TUG
      GOON,1;
      ACT,,TNOW.LE.1000. .AND. ATR(2).NE.4,G4;
      ACT,,,G5;
G5    GOON,1;
      ACT,,ATRIB(2).EQ.1,DPT1;
      ACT,,ATRIB(2).EQ.2,DPT2;
      ACT,,ATRIB(2).EQ.3,DPT3;
      ACT,,ATRIB(2).EQ.4,DPT4;
DPT1  ASSIGN,XX(1)=XX(1)+1, XX(2)=XX(2)+TNOW-ATR(3);
G4    TERM;
DPT2  ASSIGN,XX(3)=XX(3)+1, XX(4)=XX(4)+TNOW-ATR(3);
      TERM;
DPT3  ASSIGN,XX(5)=XX(5)+1, XX(6)=XX(6)+TNOW-ATR(3);
      TERM;
DPT4  GOON,1;
      ACT,,TNOW.LE.1000., G7;
      ACT,,,G8;
G8    ASSIGN,XX(7)=XX(7)+1, XX(8)=XX(8)+TNOW-ATR(3);
G7    ASSIGN, ATR(6)=UNFRM(216.,264.,7);
      ACT,ATR(6),,G10;
G10   GOON,1;
      ACT,,TNOW.LE.1000., G11;
      ACT,,,G12;
G12   ASSIGN,XX(17)=XX(17)+1,XX(18)=XX(18)+ATR(6)-240;
G11   GOON,1;
      ACT,,,ARV4;
```

```
;
; STORM SEGMENT
;
      CREATE;
STOR GOON,1
      ACT,EXPON(48.,8);
      ALTER,TUG/-1,1;
      ASSIGN,ATR(7)=UNFRM(2.,6.,9);
      ACT,ATR(7);
      ALTER,TUG/+1;
      ACT,,,STORM;
;
      CREATE,,21000,,1,1;
      ACT,0.0;
      EVENT,1,1;
      TERMINATE;
      END;
SEEDS,2198725(1),4928427(2),6649875(3),5043987(4),2384619(5),
       1098723(6),5039871(7),3894567(8),2854639(9),8272135(10);
INIT,0,21000;
MONTR,CLEAR,1000;
SIMULATE;
FIN;
```

Appendix B-6

```
C
C THIS PROGRAM IS FOR OBTAINING MEAN AND VARIANCE OF ESTIMATORS:
C (1) SIMPLE ESTIMATOR
C (2) INDEPENDENT CONTROLLED ESTIMATOR
C
C USING THE SIMULATION RESULTS OF FOLLOWING:
C (1) CLOSED MACHINE REPAIR PROBLEM IN WILSON AND PRITSCKER (1984);
C (2) MIXED MACHINE REPAIR PROBLEM IN WILSON AND PRITSCKER (1984);
C (3) OPEN MACHINE REPAIR PROBLEM;
C (4) HOSPITAL RESOURCE ALLOCATION PROBLEM (A SINGLE DESIGN POINT)
C     IN SCHRUBEN AND MARGOLIN (1978);
C (5) PORT OPERATIONS PROBLEM IN PRITSKER (1986).
C
C HERE WE HAVE: 200 REPLICATION RUNS;
C               NUMBER OF RESPONSES AND CONTROL VARIATES DEPEND
C               ON EACH PROBLEM.
C
      DOUBLE PRECISION CONT(200,5),C(200,1),Y(200,2),S(7),CBAR(3),
     *                 SSC(1,1),SSY(2,2),SSYC(1,2),CINV(1,1),ALP(1,2),
     *                 SCALP(5),CC(200,2),CONY(200,2),CYBAR(2),VARCY(2),
     *                 SSC1(1,1),SSY1(2,2),SSYC1(1,2),RED(2),YBAR(2),
     *                 AVARCY(2),ARED(2),X(200,2),XX(2,2),XXINV(2,2)
      INTEGER NR,NS,M,NC,KC(6)
C
C IMSL SUBROUTINE DLINRG IS FOR THE INVERSE MATIX.
C
      EXTERNAL DLINRG
C
C INITIAL CONDITION: M=NUMBER OF SIMULATION RUNS
C                    NR=NUMBER OF RESPONSES
C                    NC=NUMBER OF CONTROLS COLLECTED FROM SIMULATION
C                    NS=NUMBER OF CONTROLS USED FOR ANALYSIS
C                    KC(I)=INDEX OF CONTROL VARIATES USED FOR ANALYSIS
C
      M=200
      NR=2
      NC=5
C
C READ DATA SET
C
      DO 10 I=1,M
      READ (5,20) (Y(I,J),J=1,NR),(CONT(I,J),J=1,NC)
   20 FORMAT (F8.3,6F9.3)
   10 CONTINUE
C
C CHOOSE THE CONTROL VARIATES FOR ANALYSIS
C
      NS=2
      KC(1)=1
      KC(2)=2
C
C ADJUST THE INPUT MATRIX OF CONTOL VARIATES
C
      DO 1000 I=1,M
      DO 1000 J=1,NS
 1000 C(I,J)=CONT(I,KC(J))
      DO 1100 I=1,M
 1100 C(I,1)=C(I,1)/SQRT(0.303)
C
C COMPUTE MEANS OF CONTROL VARIATE AND RESPONSE
C
      DO 30 J=1,NS
      S(J)=0.0
      DO 30 I=1,M
      S(J)=S(J)+C(I,J)
```

```
   30 CONTINUE
      DO 40 I=1,NS
      CBAR(I)=S(I)/FLOAT(M)
   40 CONTINUE
      DO 50 J=1,NR
      S(J)=0.0
      DO 50 I=1,M
      S(J)=S(J)+Y(I,J)
   50 CONTINUE
      DO 60 J=1,NR
      YBAR(J)=S(J)/FLOAT(M)
   60 CONTINUE
C
C COMPUTE COVARIANCE MATRIX OF RESPONSE AND CONTROL VARIATES
C
      DO 70 J=1,NS
      DO 70 K=1,NS
      SSC(J,K)=0.0
      DO 70 I=1,M
      SSC(J,K)=SSC(J,K)+(C(I,J)-CBAR(J))*(C(I,K)-CBAR(K))
   70 CONTINUE
      DO 71 J=1,NS
      DO 71 K=1,NS
      SSC1(J,K)=SSC(J,K)/SQRT(SSC(J,J)*SSC(K,K))
   71 CONTINUE

      DO 80 J=1,NR
      DO 80 K=1,NR
      SSY(J,K)=0.0
      DO 80 I=1,M
      SSY(J,K)=SSY(J,K)+(Y(I,J)-YBAR(J))*(Y(I,K)-YBAR(K))
   80 CONTINUE
      DO 81 J=1,NR
      DO 81 K=1,NR
   81 SSY1(J,K)=SSY(J,K)/SQRT(SSY(J,J)*SSY(K,K))
      DO 90 J=1,NS
      DO 90 K=1,NR
      SSYC(J,K)=0.0
      DO 90 I=1,M
      SSYC(J,K)=SSYC(J,K)+(Y(I,K)-YBAR(K))*(C(I,J)-CBAR(J))
   90 CONTINUE
      DO 91 J=1,NS
      DO 91 K=1,NR
   91 SSYC1(J,K)=SSYC(J,K)/SQRT(SSC(J,J)*SSY(K,K))
C
C COMPUTE C-INVERSE MATRIX USING IMSL SUBROUTINE DLINRG
C
      CALL DLINRG(NS,SSC,NS,CINV,NS)
C
C COMPUTE THE COEFFICICENT OF CONTROL VARIATES (ESTIMATORS OF ALPHA)
C
      DO 95 I=1,NS
      DO 95 J=1,NR
      ALP(I,J)=0.0
      DO 97 K=1,NS
      ALP(I,J)=ALP(I,J)+CINV(I,K)*SSYC(K,J)
   97 CONTINUE
   95 CONTINUE
C
C COMPUTE THE CONTROLLED ESTIMATOR
C
      DO 100 J=1,NR
      SCALP(J)=0.0
      DO 100 I=1,NS
      SCALP(J)=SCALP(J)+CBAR(I)*ALP(I,J)
  100 CONTINUE
      DO 110 J=1,NR
```

```
      CYBAR(J)=YBAR(J)-SCALP(J)
  110 CONTINUE
C
C COMPUTE THE CONTROLLED RESPONSE
C
      DO 140 K=1,NR
      DO 130 I=1,M
      CC(I,K)=0.0
      DO 120 J=1,NS
      CC(I,K)=CC(I,K)+C(I,J)*ALP(J,K)
  120 CONTINUE
      CONY(I,K)=Y(I,K)-CC(I,K)
  130 CONTINUE
  140 CONTINUE
C
C COMPUTE THE VARIANCE OF CONTROLLED ESTIMATOR
C
      DO 160 K=1,NR
      VARCY(K)=0.0
      DO 160 I=1,M
      VARCY(K)=VARCY(K) +(CONY(I,K)-CYBAR(K))**2
  160 CONTINUE
      DO 170 K=1,NR
  170 VARCY(K)=VARCY(K)/FLOAT(M-NS-1)
C
C ADJUST THE VARIANCE USING THE INVERSE MATRIX OF (X'X)
C
      NX=NS+1
      DO 180 I=1,M
  180 X(I,1)=1.
      DO 185 J=1,NS
      DO 185 I=1,M
  185 X(I, J+1)=C(I,J)
      DO 190 I=1,NX
      DO 190 J=1,NX
      XX(I,J)=0.0
      DO 200 K=1,M
      XX(I,J)=XX(I,J)+X(K,I)*X(K,J)
  200 CONTINUE
  190 CONTINUE
      CALL DLINRG(NX,XX,NX,XXINV,NX)
C
C WRITE THE RESULTS OF ANALYSIS
C
C WRITE THE RAW DATA
C
      WRITE (6,400)
  400 FORMAT(//10X,'INDEPNDENT REPLICATION CASE: HOSPITAL')
      WRITE (6,405)
  405 FORMAT(//2X,'OBS',8X,'Y1',8X,'Y2',8X,'Y3',8X,'Y4',8X,
     *       'C1',8X,'C2')
      DO 410 I=1,M
      WRITE(6,420) I,(Y(I,J),J=1,NR),(CONT(I,J),J=1,NC)
  420 FORMAT(I5,10F10.4)
  410 CONTINUE
      WRITE(6,407) (KC(I),I=1,NS)
  407 FORMAT(//2X,'CONTROLS USED FOR ANLYSIS:',4I3)
C
C WRITE MEAN AND VARIANCE
C
      WRITE (6,430)
  430 FORMAT(///30X,'MEAN AND VARIANCE OF RESPONSES AND CONTROL VARIATES
     *')
      WRITE (6,440)
  440 FORMAT(//3X,'VARIABLE',5X,'OBS',4X,'MEAN',6X,'VARIANCE')
      DO 450 I=1,NR
      WRITE(6,455) I, M,YBAR(I),SSY(I,I)/FLOAT(M-1)
```

```
 455 FORMAT( 9X,'Y',I1,5X,I3,F11.4,F10.6)
 450 CONTINUE
     DO 460 I=1,NS
     WRITE(6,465)  KC(I), M,CBAR(I),SSC(I,I)/FLOAT(M-1)
 465 FORMAT( 9X,'C',I1,5X,I3,F11.4,F10.4)
 460 CONTINUE
C
C WRITE THE CORRELATION MATRIX
C
     WRITE (6,470)
 470 FORMAT(////5X,'CORRELATION MATRIX OF RESPONSE VARIABLES')
     WRITE (6,475)
 475 FORMAT(/10X,'Y1',8X,'Y2',8X,'Y3',8X,'Y4')
     DO 480 J=1,NR
     WRITE (6,485) J,(SSY1(J,K),K=1,NR)
 485 FORMAT ('Y', I1, 6F10.4)
 480 CONTINUE
     WRITE (6,490)
 490 FORMAT(//5X,'CORRELATION MATRIX OF CONTROL VARIATES')
     WRITE (6,500)
 500 FORMAT(/10X,'C1',8X,'C2',8X,'C3',8X,'C4',8X,'C5',8X,'C6')
     DO 510 J=1,NS
     WRITE (6,505) KC(J),(SSC1(J,K),K=1,NS)
 505 FORMAT ('C', I1,6F10.4)
 510 CONTINUE
     WRITE (6,520)
 520 FORMAT(//5X,'CORRELATION MATRIX OF RESPONSE BETWEEN CONTROLS')
     WRITE (6,475)
     DO 530 J=1,NS
     WRITE (6,540) KC(J),(SSYC1(J,K),K=1,NR)
 540 FORMAT ('C',I1,6F10.4)
 530 CONTINUE
C
C WRITE THE COVARIANCE MATRIX
C
     WRITE (6,770)
 770 FORMAT(////5X,'COVARIANCE MATRIX OF RESPONSE VARIABLES AND CONTROL
    *  VARIATES')
     WRITE (6,775)
 775 FORMAT(/5X,'Y1',8X,'Y2',8X,'Y3',8X,'Y4',8X,'C1',8X,'C2'
    *,8X,'C3',8X,'C4')
     DO 780 J=1,NR
     WRITE (6,785) J,(SSY(J,K)/FLOAT(M-1),K=1,NR)
    * ,(SSYC(K,J)/FLOAT(M-1),K=1,NS)
 785 FORMAT ('Y', I1, 10F10.4)
 780 CONTINUE
     DO 790 J=1,NS
     WRITE (6,795) J,(SSYC(J,K)/FLOAT(M-1),K=1,NR),
    * (SSC(J,K)/FLOAT(M-1),K=1,NS)
 795 FORMAT ('C', I1, 10F10.4)
 790 CONTINUE
C
C WRITE THE COEFFICIENTS OF CONTROL VARIATES ( ALP HAT )
C
     WRITE (6,550)
 550 FORMAT(//5X,'COEFFICIENTS OF CONTROL VARIATES (ALP HAT)')
     WRITE (6,475)
     DO 560 I=1,NS
     WRITE (6,570) KC(I), (ALP(I,J),J=1,NR)
 570 FORMAT('C',I1,6F10.4)
 560 CONTINUE
C
C WRITE THE MEAN AND VARIANCE OF CONTROLLED ESTOMATOR
C
     WRITE(6,600)
 600 FORMAT(//5X,'MEAN AND VARIANCE OF CONTROLLED ESTIMATOR')
     WRITE(6,610)
```

```
610 FORMAT(/7X,'RESPONSE',2X,'VARIANCE',2X,'VARIANCE REDUCTION',2X,
   *          'ADJUSTED VARIANCE',2X,'ADJUSTED VAR REDUCTION')
    DO 630 I=1,NR
    AVARCY(I)=VARCY(I)*XXINV(1,1)*200.
    RED(I)=(1.-VARCY(I)/(SSY(I,I)/FLOAT(M-1)))*100.
    ARED(I)=(1.-AVARCY(I)/(SSY(I,I)/FLOAT(M-1)))*100.
    WRITE (6,620) I,CYBAR(I),VARCY(I),RED(I),AVARCY(I),ARED(I)
620 FORMAT(3X,'Y', I1 ,F10.4,F10.6,F10.2,'%',F10.6,5X,F10.2,'%')
630 CONTINUE
    STOP
    END
```

Appendix B-7

```
C
C THIS PROGRAM IS FOR OBTAINING THE MEAN AND VARIANCE OF ESTIMATORS:
C (1) ANTITHETIC ESTIMATOR
C (2) ANTITHETIC CONTROLLED ESTIMATOR
C
C USING THE SIMULATION RESULTS OF FOLLOWING:
C (1) CLOSED MACHINE REPAIR PROBLEM IN WILSON AND PRITSCKER (1984);
C (2) MIXED MACHINE REPAIR PROBLEM IN WILSON AND PRITSCKER (1984);
C (3) OPEN MACHINE REPAIR PROBLEM;
C (4) HOSPITAL RESOURCE ALLOCATION PROBLEM (A SINGLE DESIGN POINT)
C     IN SCHRUBEN AND MARGOLIN (1978);
C (5) PORT OPERATIONS PROBLEM IN PRITSKER (1986).
C
C HERE WE HAVE: 200 REPLICATION RUNS-100 INDPENDENT RUNS AND 100
C               ANTITHETIC RUNS;
C               NUMBER OF RESPONSES AND CONTROL VARIATES DEPEND
C               ON EACH PROBLEM.
C
      DOUBLE PRECISION CONT(200,5),C(200,1),Y(200,2),S(5),
     *                 CBAR(5),YBAR(2),
     *                 SSC(1,1),SSY(2,2),SSYC(1,2),CINV(1,1),ALP(1,2),
     *                 SCALP(5),CC(200,2),CONY(200,2),CYBAR(2),VARCY(2),
     *                 SSC1(1,1),SSY1(2,2),SSYC1(1,2),RED(2),AY(100,2),
     *                 AC(100,1),
     *                 AVARCY(2),ARED(2),X(200,2),XX(2,2),XXINV(2,2)
      INTEGER NR,NC,NS,KC(6)
C
C IMSL SUBROUTINE DLINRG IS FOR THE INVERSE MATIX.
C
      EXTERNAL DLINRG
C
C INITIAL CONDITION: M=NUMBER OF SIMULATION RUNS NR=NUMBER OF RESPONSES
C                    NC=NUMBER OF CONTROLS COLLECTED FROM SIMULATION
C                    NS=NUMBER OF CONTROLS USED FOR ANLAYSIS
C
      M=200
      NR=2
      NC=5
C
C READ DATA SET
C
      DO 10 I=1,M
      READ (5,20) (Y(I,J),J=1,NR),(CONT(I,J),J=1,NC)
   20 FORMAT (F8.4,6F9.4)
   10 CONTINUE
C
C CHOOSE CONTROL VARIATES FOR ANALYSIS
C
      NS=2
      KC(1)=1
      KC(2)=3
C
C ADJUST INPUT MATRIX OF CONTROL VARIATES
C
      DO 1000 I=1,M
      DO 1000 J=1,NS
 1000 C(I,J)=CONT(I,KC(J))
      DO 1100 I=1,M
 1100 C(I,1)=C(I,1)/SQRT(0.303)
C
C ADJUST THE DATA SET OF INPUT MATRIX: M/2 PAIR OF INDEP. AND ANTI.
C
      M=100
      DO 5 I=1,M
      DO 4 J=1,NR
```

```
      K2=I+100
    4 AY(I,J)= (Y(I,J)+Y(K2,J))/2.
      DO 3 J=1,NS
    3 AC(I,J)=(C(I,J) +C(K2,J))/2.
    5 CONTINUE
      DO 17 I=1,M
      DO 15 J=1,NR
   15 Y(I,J)=AY(I,J)
      DO 16 J=1,NS
   16 C(I,J)=AC(I,J)
   17 CONTINUE
C
C COMPUTE MEANS OF CONTROL VARIATE AND RESPONSE
C
      DO 30 J=1,NS
      S(J)=0.0
      DO 30 I=1,M
      S(J)=S(J)+C(I,J)
   30 CONTINUE
      DO 40 I=1,NS
      CBAR(I)=S(I)/FLOAT(M)
   40 CONTINUE
      DO 50 J=1,NR
      S(J)=0.0
      DO 50 I=1,M
      S(J)=S(J)+Y(I,J)
   50 CONTINUE
      DO 60 J=1,NR
      YBAR(J)=S(J)/FLOAT(M)
   60 CONTINUE
C
C COMPUTE COVARIANCE MATRIX OF RESPONSE AND CONTROL VARIATES
C
      DO 70 J=1,NS
      DO 70 K=1,NS
      SSC(J,K)=0.0
      DO 70 I=1,M
      SSC(J,K)=SSC(J,K)+(C(I,J)-CBAR(J))*(C(I,K)-CBAR(K))
   70 CONTINUE
      DO 71 J=1,NS
      DO 71 K=1,NS
      SSC1(J,K)=SSC(J,K)/SQRT(SSC(J,J)*SSC(K,K))
   71 CONTINUE

      DO 80 J=1,NR
      DO 80 K=1,NR
      SSY(J,K)=0.0
      DO 80 I=1,M
      SSY(J,K)=SSY(J,K)+(Y(I,J)-YBAR(J))*(Y(I,K)-YBAR(K))
   80 CONTINUE
      DO 81 J=1,NR
      DO 81 K=1,NR
   81 SSY1(J,K)=SSY(J,K)/SQRT(SSY(J,J)*SSY(K,K))
      DO 90 J=1,NS
      DO 90 K=1,NR
      SSYC(J,K)=0.0
      DO 90 I=1,M
      SSYC(J,K)=SSYC(J,K)+(Y(I,K)-YBAR(K))*(C(I,J)-CBAR(J))
   90 CONTINUE
      DO 91 J=1,NS
      DO 91 K=1,NR
   91 SSYC1(J,K)=SSYC(J,K)/SQRT(SSC(J,J)*SSY(K,K))
C
C COMPUTE C-INVERSE MATRIX
C
      CALL DLINRG(NS,SSC,NS,CINV,NS)
C
```

```
C COMPUTE THE COEFFICICENT OF CONTROL VARIATES (ESTIMATORS OF ALPHA)
C
      DO 95 I=1,NS
      DO 95 J=1,NR
      ALP(I,J)=0.0
      DO 97 K=1,NS
      ALP(I,J)=ALP(I,J)+CINV(I,K)*SSYC(K,J)
   97 CONTINUE
   95 CONTINUE
C
C COMPUTE THE CONTROLLED ESTIMATOR
C
      DO 100 J=1,NR
      SCALP(J)=0.0
      DO 100 I=1,NS
      SCALP(J)=SCALP(J)+CBAR(I)*ALP(I,J)
  100 CONTINUE
      DO 110 J=1,NR
      CYBAR(J)=YBAR(J)-SCALP(J)
  110 CONTINUE
C
C COMPUTE THE CONTROLLED RESPONSE
C
      DO 140 K=1,NR
      DO 130 I=1,M
      CC(I,K)=0.0
      DO 120 J=1,NS
      CC(I,K)=CC(I,K)+C(I,J)*ALP(J,K)
  120 CONTINUE
      CONY(I,K)=Y(I,K)-CC(I,K)
  130 CONTINUE
  140 CONTINUE
C
C COMPUTE THE VARIANCE OF CONTROLLED ESTIMATOR
C
      DO 160 K=1,NR
      VARCY(K)=0.0
      DO 160 I=1,M
      VARCY(K)=VARCY(K) +(CONY(I,K)-CYBAR(K))**2
  160 CONTINUE
      DO 170 K=1,NR
  170 VARCY(K)=VARCY(K)/FLOAT(M-NS-1)
C
C ADJUST THE VARIANCE USING THE INVERSE MATRIX OF (X'X)
C
      NX=NS+1
      DO 180 I=1,M
  180 X(I,1)=1.
      DO 185 J=1,NS
      DO 185 I=1,M
  185 X(I, J+1)=C(I,J)
      DO 190 I=1,NX
      DO 190 J=1,NX
      XX(I,J)=0.0
      DO 200 K=1,M
      XX(I,J)=XX(I,J)+X(K,I)*X(K,J)
  200 CONTINUE
  190 CONTINUE
C
      CALL DLINRG(NX,XX,NX,XXINV,NX)
C
C WRITE THE RESULTS OF ANALYSIS
C
C WRITE THE RAW DATA
C
      WRITE (6,400)
  400 FORMAT(//20X,'ANTITHETIC REPLICATION CASE (BASED ON PAIRS):MIXED')
```

```
      WRITE (6,405)
 405  FORMAT(//2X,'OBS',8X,'Y1',8X,'Y2',8X,'Y3',8X,'Y4',8X,
     *                 'C1',8X,'C2',8X,'C3',8X,'C4',8X,'C5',8X,'C6')
      DO 410 I=1,M
      WRITE(6,420) I, (Y(I,J),J=1,NR),(CONT(I,J),J=1,NC)
 420  FORMAT(I3,1X,10F9.3)
 410  CONTINUE
      WRITE(6,407) (KC(I),I=1,NS)
 407  FORMAT(//2X,'CONTROL VARIATES USED FOR ANLAYSIS:',3I4)
C
C WRITE MEAN AND VARIANCE
C
      WRITE (6,430)
 430  FORMAT(///30X,'MEAN AND VARIANCE OF RESPONSES AND CONTROL VARIATES
     *')
      WRITE (6,440)
 440  FORMAT(//3X,'VARIABLE',5X,'OBS',4X,'MEAN',6X,'VARIANCE')
      DO 450 I=1,NR
      WRITE(6,455) I, M,YBAR(I),SSY(I,I)*2./FLOAT(M-1)
 455  FORMAT(9X,'Y',I1,5X,I3,F11.4,F10.6)
 450  CONTINUE
      DO 460 I=1,NS
      WRITE(6,465)  I, M,CBAR(I),SSC(I,I)/FLOAT(M-1)
 465  FORMAT(9X,'C',I1,5X,I3,F11.4,F10.6)
 460  CONTINUE
C
C WRITE THE COEFFICIENT MATRIX
C
      WRITE (6,470)
 470  FORMAT(////5X,'CORRELATION MATRIX OF RESPONSE VARIABLES')
      WRITE (6,475)
 475  FORMAT(/10X,'Y1',8X,'Y2',8X,'Y3',8X,'Y4')
      DO 480 J=1,NR
      WRITE (6,485) J,(SSY1(J,K),K=1,NR)
 485  FORMAT ('Y', I1, 6F10.4)
 480  CONTINUE
      WRITE (6,490)
 490  FORMAT(//5X,'CORRELATION MATRIX OF CONTROL VARIATES')
      WRITE (6,500)
 500  FORMAT(/10X,'C1',8X,'C2',8X,'C3',8X,'C4',8X,'C5',8X,'C6')
      DO 510 J=1,NS
      WRITE (6,505) KC(J),(SSC1(J,K),K=1,NS)
 505  FORMAT ('C', I1,6F10.4)
 510  CONTINUE
      WRITE (6,520)
 520  FORMAT(//5X,'CORREALTION MATRIX OF RESPONSE BETWEEN CONTROLS')
      WRITE (6,475)
      DO 530 J=1,NS
      WRITE (6,540) KC(J),(SSYC1(J,K),K=1,NR)
 540  FORMAT ('C',I1,6F10.4)
 530  CONTINUE
C
C WRITE THE COVARIANCE MATRIX
C
      WRITE (6,770)
 770  FORMAT(////5X,'COVARIANCE MATRIX OF RESPONSE VARIABLES AND CONTROL
     *  VARIATES')
      WRITE (6,775)
 775  FORMAT(/5X,'Y1',8X,'Y2',8X,'Y3',8X,'Y4',8X,'C1',8X,'C2'
     *,8X,'C3',8X,'C4')
      DO 780 J=1,NR
      WRITE (6,785) J,(SSY(J,K)/FLOAT(M-1),K=1,NR)
     * ,(SSYC(K,J)/FLOAT(M-1),K=1,NS)
 785  FORMAT ('Y', I1, 10F10.4)
 780  CONTINUE
      DO 790 J=1,NS
      WRITE (6,795) J,(SSYC(J,K)/FLOAT(M-1),K=1,NR),
```

```
     * (SSC(J,K)/FLOAT(M-1),K=1,NS)
 795 FORMAT ('C', I1, 10F10.4)
 790 CONTINUE
C
C WRITE THE COEFFICIENTS OF CONTROL VARIATES ( ALP HAT )
C
      WRITE (6,550)
 550 FORMAT(//5X,'COEFFICIENTS OF CONTROL VARIATES (ALP HAT)')
      WRITE (6,475)
      DO 560 I=1,NS
      WRITE (6,570) KC(I), (ALP(I,J),J=1,NR)
 570 FORMAT('C',I1,6F10.4)
 560 CONTINUE
C
C WRITE THE MEAN AND VARIANCE OF CONTROLLED ESTOMATOR
C
      WRITE(6,600)
 600 FORMAT(//5X,'MEAN AND VARIANCE OF CONTROLLED ESTIMATOR')
      WRITE(6,610)
 610 FORMAT(/7X,'RESPONSE',2X,'VARIANCE',2X,
     *            'ADJUSTED VARIANCE')
      DO 630 I=1,NR
      AVARCY(I)=VARCY(I)*XXINV(1,1)*100.
      WRITE (6,620) I,CYBAR(I),VARCY(I)*2.,AVARCY(I)*2.
 620 FORMAT(3X,'Y', I1 ,F10.4,F10.6,F10.6)
 630 CONTINUE
      STOP
      END
```

Appendix B-8

```
C
C THIS PROGRAM IS FOR OBTAINING THE COVARIANCE OF ESTIMATORS AND
C ITS DETERMINENT BASED ON SCHRUBEN-MARGOLIN METHOD FOR HOSPITAL
C MODEL (SCHRUBEN AND MARGOLIN (1978)):
C
C HERE WE HAVE : 8 DESIGN POINTS;
C                200 SIMULATION RUNS AT EACH DESIGN POINT;
C                1 RESPONSE OF INTEREST (FAILURE RATE).
C
      DOUBLE PRECISION SIM(1600,2),Y(8,200),C(8,200),S(8),SS(8),
     *                 YBAR(8),CBAR(8),SSY(8,8),SSY1(8,8),FAC(4,4),
     *                 X(8,4),XX(4,4),XS(4,8),XSX(4,4),BETA(4)
      INTEGER IPVT(4),ND,NP,NR,NS,M,NC
      DOUBLE PRECISION DET1,DET2
C
C IMSL SUBROUTINE DLINRG IS FOR THE INVERSE MATIX.
C IMSL SUBROUTINE DFTRG AND LFTRG ARE FOR THE DETERMINENT OF MATIX.
C
      EXTERNAL DLINRG,LFTRG,LFDRG
C
C INITIAL CONDITION: M=NUMBER OF SIMULATION RUNS
C                    NR=NUMBER OF RESPONSES
C                    NC=NUMBER OF CONTROLS COLLECTED FROM SIMULATION
C                    NS=NUMBER OF CONTROLS USED FOR ANALYSIS
C                    ND=NUMBER OF DESIGN POINTS
C                    NP=NUMBER OF PARAMETERS
C
      M=200
      NR=1
      NC=1
      NP=4
      ND=8
C
C READ DATA SET FROM SIMULATION OUTPUT
C
      DO 5  I=1,M*8
      READ (5,10) (SIM(I,J),J=1,2)
    5 CONTINUE
   10 FORMAT (F8.4,F9.4)
C
C ADJUST DATA SET AS MATRICES OF RESPONSE AND CONTROL VARIATES:
C Y= ( ND*M ); C=( ND*M );
C
      DO 15 I=1,ND
      DO 20 J=1,M
      K=200*(I-1)+J
      Y(I,J)= SIM (K,1)
      C(I,J)= SIM (K,2)
   20 CONTINUE
   15 CONTINUE
C
C SET UP THE DESIGN MATRIX
C
      DO 900 I=1,4
      X(2*I-1,1)=1.
      X(2*I,1)=1.
      X(2*I-1,3)=-1.
      X(2*I,3)=1.
  900 CONTINUE
      DO 905 I=1,2
      X(I,2)=-1.
      X(I+2,2)=1.
      X(I+4,2)=-1.
      X(I+6,2)=1.
  905 CONTINUE
```

```
      DO 930 I=1,8
      IF (I.GT.4) GO TO 910
      X(I,4)=-X(I,2)*X(I,3)
      GO TO 930
  910 X(I,4)=X(I,2)*X(I,3)
  930 CONTINUE
C
C CHOOSE THE CONTROL VARIATES FOR ANALYSIS
C
      NS=1
C
C COMPUTE MEANS OF RESPONSE AND CONTROL VARIATE
C
      DO 30 I=1,ND
      S(I)=0.0
      SS(I)=0.0
      DO 35 J=1,M
      S(I)=S(I)+Y(I,J)
      SS(I)=SS(I)+C(I,J)
   35 CONTINUE
      YBAR(I)=S(I)/FLOAT(M)
      CBAR(I)=SS(I)/FLOAT(M)
   30 CONTINUE
C
C COMPUTE COVARIANCE MATRIX OF RESPONSE
C
      DO 80 I=1,ND
      DO 80 J=1,ND
      SSY(I,J)=0.0
      DO 80 K=1,M
      SSY(I,J)=SSY(I,J)+(Y(I,K)-YBAR(I))*(Y(J,K)-YBAR(J))
   80 CONTINUE
C
C FIND CORRELATION MATRIX OF RESPONSE
C
      DO 85 J=1,ND
      DO 85 K=1,ND
   85 SSY1(J,K)=SSY(J,K)/SQRT(SSY(J,J)*SSY(K,K))
C
C FIND THE VECTOR OF ESTIMATOR: BETA = [ 1/8 * X' Y BAR ]
C
      DO 150 I=1,NP
      BETA(I)=0.
      DO 150 K=1,ND
      BETA(I)=BETA(I)+X(K,I)*YBAR(K)
  150 CONTINUE
C
C OBTAIN THE MATRIX OF [ X' COV(Y BAR ) X ] = [X'*SSY*X]
C
      DO 210 I=1,NP
      DO 210 J=1,ND
      XS(I,J)=0.
      DO 220 K=1,ND
      XS(I,J)=XS(I,J)+X(K,I)*SSY(K,J)
  220 CONTINUE
  210 CONTINUE

      DO 230 I=1,NP
      DO 230 J=1,NP
      XSX(I,J)=0.
      DO 240 K=1,ND
      XSX(I,J)=XSX(I,J)+XS(I,K)*X(K,J)
  240 CONTINUE
  230 CONTINUE
C
C FIND THE COVARIANCE MATRIX OF ESTIMNATOR BETA HAT:
C      (1/64 * XSX )
```

```
C
      DO 250 I=1,NP
      DO 260 J=1,NP
      XSX(I,J)=XSX(I,J)/(64.*200.)
 260  CONTINUE
 250  CONTINUE
C
C FIND THE DETERMINENT OF COVARIANCE MATRIX OF ESTIMNATOR:
C USING IMSL SUBROUTINE-LFTRG AND LFDRG
C
      CALL DLFTRG(NP,XSX,NP,FAC,NP,IPVT)
      CALL DLFDRG(NP,FAC,NP,IPVT,DET1,DET2)
C
C WRITE THE RESULTS OF ANALYSIS
C
      WRITE (6,400)
 400  FORMAT(//10X,'ANALYSIS OF SCHRUBEN-MARGOLIN METHOD: HOSPITAL'
     *//15X,'8 DESIGN POINTS;'
     */15X,'4 PARAMETERS;'
     */15X,'200 REPLICATIONS' )
C
C WRITE THE DESIGN MATRIX OF X
C
      WRITE (6,405)
 405  FORMAT(//10X,'DESIGN MATRIX X'/)
      DO 410 I=1,ND
      WRITE (6,415) I, (X(I,K),K=1,NP)
 415  FORMAT (10X,'X',I1,4F8.1)
 410  CONTINUE
C
C WRITE MEAN AND VARIANCE
C
      WRITE (6,430)
 430  FORMAT(//10X,'MEAN OF RESPONSE AT DESIGN POINT'/)
      DO 450 I=1,ND
      WRITE(6,455) I, YBAR(I)
 455  FORMAT(10X,'Y',I1,F10.3)
 450  CONTINUE
C
C WRITE THE CORRELATION MATRIX
C
      WRITE (6,470)
 470  FORMAT(//5X,'COVARIANCE MATRIX OF RESPONSE VARIABLES')
      WRITE (6,475)
 475  FORMAT(/19X,'Y1',7X,'Y2',7X,'Y3',7X,'Y4',7X,'Y5',7X,'Y6',7X,
     *'Y7',7X,'Y8')
      DO 480 J=1,ND
      WRITE (6,485) J,(SSY(J,K)/FLOAT(M),K=1,ND)
 485  FORMAT (10X,'Y', I1, 8F9.3)
 480  CONTINUE
      WRITE (6,490)
 490  FORMAT(//10X,'CORRELATION MATRIX OF RESPONSE VARIATES')
      WRITE (6,500)
 500  FORMAT(/19X,'Y1',7X,'Y2',7X,'Y3',7X,'Y4',7X,'Y5',7X,'Y6',7X,
     *'Y7',7X,'Y8')
      DO 510 J=1,ND
      WRITE (6,505) J, (SSY1(J,K),K=1,ND)
 505  FORMAT (10X,'Y', I1,8F9.3)
 510  CONTINUE
      WRITE (6,520)
 520  FORMAT(//10X,'COVARIANCE MATRIX OF ESTIMATORS')
      WRITE (6,524)
 524  FORMAT(/21X,'BETA 0',6X,'BETA 1',6X,'BETA 2',6X,'BETA 3')
      DO 530 J=1,NP
      WRITE (6,540) J-1,(XSX(J,K),K=1,NP)
 540  FORMAT (10X,'BETA',I1,4F12.7)
 530  CONTINUE
```

```
C
C WRITE THE ESTIMATOR VECTOR: BETA HAT
C
      WRITE (6,750)
  750 FORMAT(//10X,'THE ESTIMATORS OF PARAMETERS')
      DO 753 I=1,NP
      WRITE (6,755) I-1,BETA(I)/FLOAT(ND)
  755 FORMAT(10X,'BETA ', I1, ' =',F12.7)
  753 CONTINUE
C
C WRITE THE DETERMINENT OF ESTIMATOR COARIANCE MATRIX
C
      WRITE (6,770)
  770 FORMAT(//10X,'DETERMINENT OF COVARIANCE MATRIX OF ESTIMATOR')
      WRITE (6,775) DET1,DET2
  775 FORMAT(/10X,'DETERMINENT =',F8.5, '* 10**', F5.2)
      STOP
      END
```

Appendix B-9

```
C
C THIS PROGRAM IS FOR OBTAINING THE COVARIANCE OF ESTIMATORS AND
C ITS DETERMINENT BASED ON EXTENDED METHOD OF CORRELATED REPLICATIONS
C WITH CONTROL VARIATES FOR HOSPITAL MODEL
C (SCHRUBEN AND MARGOLIN (1978))
C
C HERE WE HAVE : 8 DESIGN POINTS;
C                200 SIMULATION RUNS AT EACH DESIGN POINT;
C                1 RESPONSE OF INTEREST (FAILURE RATE).
C
      DOUBLE PRECISION SIM(1600,2),Y(8,200),C(8,200),S(8),SS(8),
     *                 YBAR(8),CBAR(8),SSY(8,8),SSY1(8,8),BETA(4),
     *                 X(8,4),XX(4,4),XS(4,8),XSX(4,4),FAC(4,4)
      DOUBLE PRECISION SSC(8),SSYC(8),ALP(8),CONY(8,100),SSCONY(8,8),
     *                 CYBAR(8),AY(8,100),AC(8,100),SSCONY1(8,8)
      DOUBLE PRECISION CX(4),YX(4),A,CXY,CXC,SC,SCY
      INTEGER IPVT(4),ND,NP,NR,NS,M,NC
      DOUBLE PRECISION DET1,DET2
C
C IMSL SUBROUTINE DLINRG IS FOR THE INVERSE MATIX.
C IMSL SUBROUTINE DFTRG AND LFTRG ARE FOR THE MULTIPLICATION
C OF TWO MATICES.
C
      EXTERNAL DLINRG,LFTRG,LFDRG
C
C INITIAL CONDITION: M=NUMBER OF SIMULATION RUNS
C                    NR=NUMBER OF RESPONSES
C                    NC=NUMBER OF CONTROLS COLLECTED FROM SIMULATION
C                    NS=NUMBER OF CONTROLS USED FOR ANALYSIS
C                    ND=NUMBER OF DESIGN POINTS
C                    NP=NUMBER OF PARAMETERS
C
      M=200
      NR=1
      NC=1
      NS=1
      NP=4
      ND=8
C
C READ DATA SET FROM SIMULATION OUTPUT
C
      DO 5  I=1,M*8
      READ (5,10) (SIM(I,J),J=1,2)
    5 CONTINUE
   10 FORMAT (F8.4,F9.4)
C
C ADJUST DATA SET AS MATRICES OF RESPONSE AND CONTROL VARIATES:
C Y= ( ND*M ); C=( ND*M );
C
      DO 15 I=1,ND
      DO 20 J=1,M
      K=200*(I-1)+J
      Y(I,J)= SIM (K,1)
      C(I,J)= SIM (K,2)
   20 CONTINUE
   15 CONTINUE
C
C SET UP THE DESIGN MATRIX
C
      DO 900 I=1,4
      X(2*I-1,1)=1.
      X(2*I,1)=1.
      X(2*I-1,3)=-1.
      X(2*I,3)=1.
  900 CONTINUE
```

```
      DO 905 I=1,2
      X(I,2)=-1.
      X(I+2,2)=1.
      X(I+4,2)=-1.
      X(I+6,2)=1.
  905 CONTINUE
      DO 930 I=1,8
      IF (I.GT.4) GO TO 910
      X(I,4)=-X(I,2)*X(I,3)
      GO TO 930
  910 X(I,4)=X(I,2)*X(I,3)
  930 CONTINUE
C
C COMPUTE MEANS OF RESPONSE AND CONTROL VARIATE
C
      DO 30 I=1,ND
      S(I)=0.0
      SS(I)=0.0
      DO 35 J=1,M
      S(I)=S(I)+Y(I,J)
      SS(I)=SS(I)+C(I,J)
   35 CONTINUE
      YBAR(I)=S(I)/FLOAT(M)
      CBAR(I)=SS(I)/FLOAT(M)
   30 CONTINUE
C
C ADJUST THE DATA SET OF INPUT MATRIX AS 100 PAIRS:
C M/2 PAIRS OF INDEPENDENT AND ANTITHETIC
C
      M=100
      DO 45 I=1,ND
      DO 45 J=1,M
      K2=J+100
      AY(I,J)=(Y(I,J)+Y(I,K2))/2.
      AC(I,J)=(C(I,J) +C(I,K2))/2.
   45 CONTINUE
      DO 47 I=1,ND
      DO 47 J=1,M
      Y(I,J)=AY(I,J)
      C(I,J)=AC(I,J)
   47 CONTINUE
C
C OBTAIN THE MATRIX OF ALPHA: A
C
      SC=0.
      SCY=0.
      DO 213 I=1,ND
      SC=SC+CBAR(I)*CBAR(I)
      SCY=SCY+CBAR(I)*YBAR(I)
  213 CONTINUE
      DO 201 I=1,NP
      CX(I)=0.
      YX(I)=0.
      DO 202 K=1,ND
      CX(I)=CX(I)+CBAR(K)*X(K,I)
      YX(I)=YX(I)+YBAR(K)*X(K,I)
  202 CONTINUE
  201 CONTINUE
      CXC=0.
      CXY=0.
      DO 203 I=1,NP
      CXC=CXC+CX(I)*CX(I)
      CXY=CXY+CX(I)*YX(I)
  203 CONTINUE
      A=(SCY-CXY/8.)/(SC-CXC/8.)
      DO 117 I=1,ND
      CYBAR(I)=YBAR(I)-CBAR(I)*A
```

```
 117 CONTINUE
C
C COMPUTE THE COEFFICICENT OF CONTROL VARIATES (ESTIMATORS OF ALPHA)
C
C      DO 95 I=1,ND
C      ALP(I)=SSYC(I)/SSC(I)
C   95 CONTINUE
C
C COMPUTE THE CONTROLLED ESTIMATOR
C
      DO 110 I=1,ND
      CYBAR(I)=YBAR(I)-CBAR(I)*A
 110 CONTINUE
C
C WRITE THE MEAN RESPONSE
C
      WRITE (6,430)
 430 FORMAT(//10X,'CONTROLLED MEAN OF RESPONSE AT DESIGN POINT'/)
      DO 450 I=1,ND
      WRITE(6,455) I, CYBAR(I)
 455 FORMAT(10X,'Y',I1,F10.3)
 450 CONTINUE
C
C COMPUTE THE CONTROLLED RESPONSE
C
      DO 125 I=1,ND
      DO 125 J=1,M
      CONY(I,J)=Y(I,J)-C(I,J)*A
 125 CONTINUE
C
C COMPUTE COVARIANCE MATRIX OF ADJUSTED RESPONSES
C
      DO 186 I=1,ND
      DO 186 J=1,ND
      SSCONY(I,J)=0.0
      DO 180 K=1,M
      SSCONY(I,J)=SSCONY(I,J)+(CONY(I,K)-CYBAR(I))*(CONY(J,K)-CYBAR(J))
 180 CONTINUE
 186 CONTINUE
      DO 189 I=1,ND
      DO 189 J=1,ND
      SSCONY(I,J)=SSCONY(I,J)*2./FLOAT(M-NS-1)
 189 CONTINUE
C
C FIND CORRELATION MATRIX OF ADJUSTED RESPONSE
C
      DO 190 J=1,ND
      DO 190 K=1,ND
 190 SSCONY1(J,K)=SSCONY(J,K)/SQRT(SSCONY(J,J)*SSCONY(K,K))
C
C FIND THE VECTOR OF ESTIMATOR: BETA HAT = [ 1/8 * X' CONTROLLED Y ]
C
      DO 200 I=1,NP
      BETA(I)=0.
      DO 200 K=1,ND
      BETA(I)=BETA(I)+X(K,I)*CYBAR(K)
 200 CONTINUE
      DO 205 I=1,ND
      BETA(I)=BETA(I)/FLOAT(ND)
 205 CONTINUE
C
C OBTAIN THE MATRIX OF [ X' COV(Y BAR ) X ] = [X'*SSY*X]
C
      DO 210 I=1,NP
      DO 210 J=1,ND
      XS(I,J)=0.
      DO 220 K=1,ND
```

```
      XS(I,J)=XS(I,J)+X(K,I)*SSCONY(K,J)
  220 CONTINUE
  210 CONTINUE

      DO 230 I=1,NP
      DO 230 J=1,NP
      XSX(I,J)=0.
      DO 240 K=1,ND
      XSX(I,J)=XSX(I,J)+XS(I,K)*X(K,J)
  240 CONTINUE
  230 CONTINUE
C
C FIND THE COVARIANCE MATRIX OF ESTIMNATOR BETA HAT:
C      (1/(64*NUMBER OF REPLICATION) * XSX )
C
      DO 250 I=1,NP
      DO 260 J=1,NP
      XSX(I,J)=XSX(I,J)/FLOAT(ND)**2
  260 CONTINUE
  250 CONTINUE
C
C FIND THE DETERMINENT OF COVARIANCE MATRIX OF ESTIMNATOR:
C USING IMSL SUBROUTINE-LFTRG AND LFDRG
C
      CALL DLFTRG(NP,XSX,NP,FAC,NP,IPVT)
      CALL DLFDRG(NP,FAC,NP,IPVT,DET1,DET2)
C
C WRITE THE RESULTS OF ANALYSIS
C
C WRITE THE RAW DATA
C
      WRITE (6,400)
  400 FORMAT(//10X,'ANALYSIS OF EXTENDED METHOD USING 1 ALP: HOSPITAL'
     *//15X,'8 DESIGN POINTS;'
     */15X,'4 PARAMETERS;'
     */15X,'200 REPLICATIONS;')
C
C WRITE THE DESIGN MATRIX OF X
C
      WRITE (6,405)
  405 FORMAT(//10X,'DESIGN MATRIX '/)
      DO 410 I=1,ND
      WRITE (6,415) I, (X(I,K),K=1,NP)
  415 FORMAT (10X,'X',I1,4F8.3)
  410 CONTINUE
      WRITE (6,420)
  420 FORMAT(//10X,'COEFFICIENT OF CONTROL VARIATES'/)
      WRITE(6,423) A
  423 FORMAT(10X,'ALPHA=',F10.3)
      WRITE (6,431)
  431 FORMAT(//10X,'CONTROLLED MEAN OF RESPONSE AT DESIGN POINT'/)
      DO 443 I=1,ND
      WRITE(6,453) I, CYBAR(I)
  453 FORMAT(10X,'Y',I1,F10.3)
  443 CONTINUE
C
C WRITE THE CORRELATION MATRIX
C
      WRITE (6,470)
  470 FORMAT(//10X,'COVARIANCE MATRIX OF ADJUSTED RESPONSE VARIABLES')
      WRITE (6,475)
  475 FORMAT(/19X,'Y1',7X,'Y2',7X,'Y3',7X,'Y4',7X,'Y5',7X,'Y6',7X,
     *'Y7',7X,'Y8')
      DO 480 J=1,ND
      WRITE (6,485) J,(SSCONY(J,K),K=1,ND)
  485 FORMAT (10X,'Y', I1, 8F9.3)
  480 CONTINUE
```

```
      WRITE (6,490)
  490 FORMAT(//10X,'CORRELATION MATRIX OF ADJUSTED RESPONSE VARIATES')
      WRITE (6,500)
  500 FORMAT(/19X,'Y1',7X,'Y2',7X,'Y3',7X,'Y4',7X,'Y5',7X,'Y6',7X,
     *'Y7',7X,'Y8')
      DO 510 J=1,ND
      WRITE (6,505) J, (SSCONY1(J,K),K=1,ND)
  505 FORMAT (10X,'Y', I1,8F9.3)
  510 CONTINUE
      WRITE (6,520)
  520 FORMAT(//10X,'COVARIANCE MATRIX OF ESTIMATORS WITH CONTROL')
      WRITE (6,524)
  524 FORMAT(/21X,'BETA 0',6X,'BETA 1',6X,'BETA 2',6X,'BETA 3')
      DO 530 J=1,NP
      WRITE (6,540) J-1,(XSX(J,K),K=1,NP)
  540 FORMAT (10X,'BETA',I1,4F12.7)
  530 CONTINUE
C
C WRITE THE ESTIMATE VECTOR: BETA HAT
C
      WRITE (6,730)
  730 FORMAT(//10X,'THE ESTIMATOR OF PARAMETERS')
      DO 735 I=1,NP
      WRITE (6,740) I-1, BETA(I)
  740 FORMAT(10X,'BETA ',I1,' =',F12.7)
  735 CONTINUE
C
C WRITE THE DETERMINENT OF ESTIMATOR COVARIANCE MATRIX
C
      WRITE (6,770)
  770 FORMAT(//10X,'DETERMINENT OF COVARIANCE MATRIX OF ESTIMATOR')
      WRITE (6,775) DET1,DET2
  775 FORMAT(/10X,'DETERMINENT =',F8.5, '* 10**', F5.2)
      STOP
      END
```

Appendix B-10

```
C
C THIS PROGRAM IS FOR OBTAINING THE COVARIANCE OF ESTIMATORS AND
C ITS DETERMINENT BASED ON SCHRUBEN-MARGOLIN METHOD
C WITH CONTROL VARIATES FOR HOSPITAL MODEL
C (SCHRUBEN AND MARGOLIN (1978))
C
C HERE WE HAVE : 8 DESIGN POINTS;
C                200 SIMULATION RUNS AT EACH DESIGN POINT;
C                1 RESPONSE OF INTEREST (FAILURE RATE).
C
      DOUBLE PRECISION SIM(1600,2),Y(8,200),C(8,200),S(8),SS(8),
     *                 YBAR(8),CBAR(8),SSY(8,8),SSY1(8,8),BETA(4),
     *                 X(8,4),XX(4,4),XS(4,8),XSX(4,4),FAC(4,4)
      DOUBLE PRECISION SSC(8),SSYC(8),ALP(8),CONY(8,100),SSCONY(8,8),
     *                 CYBAR(8),AY(8,100),AC(8,100),SSCONY1(8,8)
      INTEGER IPVT(4),ND,NP,NR,NS,M,NC
      DOUBLE PRECISION DET1,DET2
C
C IMSL SUBROUTINE DLINRG IS FOR THE INVERSE MATIX.
C IMSL SUBROUTINE DFTRG AND LFTRG ARE FOR THE MULTIPLICATION
C OF TWO MATICES.
C
      EXTERNAL DLINRG,LFTRG,LFDRG
C
C INITIAL CONDITION: M=NUMBER OF SIMULATION RUNS
C                    NR=NUMBER OF RESPONSES
C                    NC=NUMBER OF CONTROLS COLLECTED FROM SIMULATION
C                    NS=NUMBER OF CONTROLS USED FOR ANALYSIS
C                    ND=NUMBER OF DESIGN POINTS
C                    NP=NUMBER OF PARAMETERS
C
      M=200
      NR=1
      NC=1
      NS=1
      NP=4
      ND=8
C
C READ DATA SET FROM SIMULATION OUTPUT
C
      DO 5  I=1,M*8
      READ (5,10) (SIM(I,J),J=1,2)
    5 CONTINUE
   10 FORMAT (F8.4,F9.4)
C
C ADJUST DATA SET AS MATRICES OF RESPONSE AND CONTROL VARIATES:
C Y= ( ND*M ); C=( ND*M );
C
      DO 15 I=1,ND
      DO 20 J=1,M
      K=200*(I-1)+J
      Y(I,J)= SIM (K,1)
      C(I,J)= SIM (K,2)
   20 CONTINUE
   15 CONTINUE
C
C SET UP THE DESIGN MATRIX
C
      DO 900 I=1,4
      X(2*I-1,1)=1.
      X(2*I,1)=1.
      X(2*I-1,3)=-1.
      X(2*I,3)=1.
  900 CONTINUE
      DO 905 I=1,2
```

```
      X(I,2)=-1.
      X(I+2,2)=1.
      X(I+4,2)=-1.
      X(I+6,2)=1.
  905 CONTINUE
      DO 930 I=1,8
      IF (I.GT.4) GO TO 910
      X(I,4)=-X(I,2)*X(I,3)
      GO TO 930
  910 X(I,4)=X(I,2)*X(I,3)
  930 CONTINUE
C
C COMPUTE MEANS OF RESPONSE AND CONTROL VARIATE
C
      DO 30 I=1,ND
      S(I)=0.0
      SS(I)=0.0
      DO 35 J=1,M
      S(I)=S(I)+Y(I,J)
      SS(I)=SS(I)+C(I,J)
   35 CONTINUE
      YBAR(I)=S(I)/FLOAT(M)
      CBAR(I)=SS(I)/FLOAT(M)
   30 CONTINUE
C
C COMPUTE COVARIANCE MATRIX OF RESPONSE
C
      DO 80 I=1,ND
      DO 80 J=1,ND
      SSY(I,J)=0.0
      DO 80 K=1,M
      SSY(I,J)=SSY(I,J)+(Y(I,K)-YBAR(I))*(Y(J,K)-YBAR(J))
   80 CONTINUE
C
C FIND CORRELATION MATRIX OF RESPONSE
C
      DO 81 J=1,ND
      DO 81 K=1,ND
   81 SSY1(J,K)=SSY(J,K)/SQRT(SSY(J,J)*SSY(K,K))
      WRITE (6,771)
  771 FORMAT(//10X,'CORRELATION MATRIX OF REPONSE VARIATES')
      WRITE (6,772)
  772 FORMAT(/19X,'Y1',7X,'Y2',7X,'Y3',7X,'Y4',7X,'Y5',7X,'Y6',7X,
     *'Y7',7X,'Y8')
      DO 773 J=1,ND
      WRITE (6,774) J, (SSY1(J,K),K=1,ND)
  774 FORMAT (10X,'Y', I1,8F9.3)
  773 CONTINUE
C
C COMPUTE COVARIANCE MATRIX OF CONTROL VARIATES
C
      DO 70 I=1,ND
      SSC(I)=0.0
      DO 70 J=1,M
      SSC(I)=SSC(I)+(C(I,J)-CBAR(I))*(C(I,J)-CBAR(I))
   70 CONTINUE
C
C COMPUTE COVARIANCE MATRIX OF BETWEEN RESPONSE AND CONTROL VARIATES
C
      DO 90 I=1,ND
      SSYC(I)=0.0
      DO 90 J=1,M
      SSYC(I)=SSYC(I)+(Y(I,J)-YBAR(I))*(C(I,J)-CBAR(I))
   90 CONTINUE
C
C COMPUTE THE COEFFICICENT OF CONTROL VARIATES (ESTIMATORS OF ALPHA)
C
```

```
      DO 95 I=1,ND
      ALP(I)=SSYC(I)/SSC(I)
   95 CONTINUE
C
C COMPUTE THE CONTROLLED ESTIMATOR
C
      DO 110 I=1,ND
      CYBAR(I)=YBAR(I)-CBAR(I)*ALP(I)
  110 CONTINUE
C
C WRITE THE MEAN RESPONSE
C
      WRITE (6,430)
  430 FORMAT(//10X,'CONTROLLED MEAN OF RESPONSE AT DESIGN POINT'/)
      DO 450 I=1,ND
      WRITE(6,455) I, CYBAR(I),I,CBAR(I),I,ALP(I)
  455 FORMAT(10X,'Y',I1,F10.3,5X,'C',I1,F10.3,5X,'ALP',I1,F10.3)
  450 CONTINUE
C
C COMPUTE THE CONTROLLED RESPONSE
C
      DO 125 I=1,ND
      DO 125 J=1,M
      CONY(I,J)=Y(I,J)-C(I,J)*ALP(I)
  125 CONTINUE
C
C COMPUTE COVARIANCE MATRIX OF ADJUSTED RESPONSES
C
      DO 186 I=1,ND
      DO 186 J=1,ND
      SSCONY(I,J)=0.0
      DO 180 K=1,M
      SSCONY(I,J)=SSCONY(I,J)+(CONY(I,K)-CYBAR(I))*(CONY(J,K)-CYBAR(J))
  180 CONTINUE
  186 CONTINUE
      DO 189 I=1,ND
      DO 189 J=1,ND
      SSCONY(I,J)=SSCONY(I,J)/FLOAT(M-NS-1)
  189 CONTINUE
C
C FIND CORRELATION MATRIX OF ADJUSTED RESPONSE
C
      DO 190 J=1,ND
      DO 190 K=1,ND
  190 SSCONY1(J,K)=SSCONY(J,K)/SQRT(SSCONY(J,J)*SSCONY(K,K))
C
C FIND THE VECTOR OF ESTIMATOR: BETA HAT = [ 1/8 * X' CONTROLLED Y ]
C
      DO 200 I=1,NP
      BETA(I)=0.
      DO 200 K=1,ND
      BETA(I)=BETA(I)+X(K,I)*CYBAR(K)
  200 CONTINUE
      DO 205 I=1,ND
      BETA(I)=BETA(I)/FLOAT(ND)
  205 CONTINUE
C
C OBTAIN THE MATRIX OF [ X' COV(Y BAR ) X ] = [X'*SSY*X]
C
      DO 210 I=1,NP
      DO 210 J=1,ND
      XS(I,J)=0.
      DO 220 K=1,ND
      XS(I,J)=XS(I,J)+X(K,I)*SSCONY(K,J)
  220 CONTINUE
  210 CONTINUE
```

```
      DO 230 I=1,NP
      DO 230 J=1,NP
      XSX(I,J)=0.
      DO 240 K=1,ND
      XSX(I,J)=XSX(I,J)+XS(I,K)*X(K,J)
  240 CONTINUE
  230 CONTINUE
C
C FIND THE COVARIANCE MATRIX OF ESTIMNATOR BETA HAT:
C      (1/(64*NUMBER OF REPLICATION) * XSX )
C
      DO 250 I=1,NP
      DO 260 J=1,NP
      XSX(I,J)=XSX(I,J)/FLOAT(ND)**2
  260 CONTINUE
  250 CONTINUE
C
C FIND THE DETERMINENT OF COVARIANCE MATRIX OF ESTIMNATOR:
C USING IMSL SUBROUTINE-LFTRG AND LFDRG
C
      CALL DLFTRG(NP,XSX,NP,FAC,NP,IPVT)
      CALL DLFDRG(NP,FAC,NP,IPVT,DET1,DET2)
C
C WRITE THE RESULTS OF ANALYSIS
C
C WRITE THE RAW DATA
C
      WRITE (6,400)
  400 FORMAT(//10X,'ANALYSIS OF EXTENDED METHOD: HOSPITAL'
     *//15X,'8 DESIGN POINTS;'
     */15X,'4 PARAMETERS;'
     */15X,'200 REPLICATIONS;')
C
C WRITE THE DESIGN MATRIX OF X
C
      WRITE (6,405)
  405 FORMAT(//10X,'DESIGN MATRIX '/)
      DO 410 I=1,ND
      WRITE (6,415) I, (X(I,K),K=1,NP)
  415 FORMAT (10X,'X',I1,4F8.3)
  410 CONTINUE
C
C WRITE THE CORRELATION MATRIX
C
      WRITE (6,470)
  470 FORMAT(//10X,'COVARIANCE MATRIX OF ADJUSTED RESPONSE VARIABLES')
      WRITE (6,475)
  475 FORMAT(/19X,'Y1',7X,'Y2',7X,'Y3',7X,'Y4',7X,'Y5',7X,'Y6',7X,
     *'Y7',7X,'Y8')
      DO 480 J=1,ND
      WRITE (6,485) J,(SSCONY(J,K),K=1,ND)
  485 FORMAT (10X,'Y', I1, 8F9.3)
  480 CONTINUE
      WRITE (6,490)
  490 FORMAT(//10X,'CORRELATION MATRIX OF ADJUSTED RESPONSE VARIATES')
      WRITE (6,500)
  500 FORMAT(/19X,'Y1',7X,'Y2',7X,'Y3',7X,'Y4',7X,'Y5',7X,'Y6',7X,
     *'Y7',7X,'Y8')
      DO 510 J=1,ND
      WRITE (6,505) J, (SSCONY1(J,K),K=1,ND)
  505 FORMAT (10X,'Y', I1,8F9.3)
  510 CONTINUE
      WRITE (6,520)
  520 FORMAT(//10X,'COVARIANCE MATRIX OF ESTIMATORS WITH CONTROL')
      WRITE (6,524)
  524 FORMAT(/21X,'BETA 0',6X,'BETA 1',6X,'BETA 2',6X,'BETA 3')
      DO 530 J=1,NP
```

```
      WRITE (6,540) J-1,(XSX(J,K),K=1,NP)
  540 FORMAT (10X,'BETA',I1,4F12.7)
  530 CONTINUE
C
C WRITE THE ESTIMATE VECTOR: BETA HAT
C
      WRITE (6,730)
  730 FORMAT(//10X,'THE ESTIMATOR OF PARAMETERS')
      DO 735 I=1,NP
      WRITE (6,740) I-1, BETA(I)
  740 FORMAT(10X,'BETA ',I1,' =',F12.7)
  735 CONTINUE
C
C WRITE THE DETERMINENT OF ESTIMATOR COVARIANCE MATRIX
C
      WRITE (6,770)
  770 FORMAT(//10X,'DETERMINENT OF COVARIANCE MATRIX OF ESTIMATOR')
      WRITE (6,775) DET1,DET2
  775 FORMAT(/10X,'DETERMINENT =',F8.5, '* 10**', F5.2)
      STOP
      END
```

VITA

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He taught statistics and operations research courses at Department of Applied Statistics in Dong-A University in Korea as a full-time instructor for three years. He is currently an assistant professor at Dong-A University in Korea.