Shape Design Using Intrinsic Geometry

by

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(ABSTRACT)

The present work outlines a methodology of shape synthesis using intrinsic geometry concepts for engineering design of two-dimensional and three-dimensional curves as well as three-dimensional surfaces. Using concepts of intrinsic geometry of curves, the shape of a curve can be defined in terms of intrinsic parameters such as the curvature and torsion as a function of the arc length. The method of shape synthesis proposed here consists of selecting a shape model, defining a set of shape design variables and then evaluating the Cartesian coordinates of a curve. It is assumed that the end-point coordinates and tangents are specified for design of curves. A shape model is conceived as a set of continuous piecewise linear segments of the curvature, each segment defined as a function of the arc length. The shape design variables are the values of curvature and/or arc length at some of the end-points of the linear segments. The proposed method of shape synthesis is general in nature. It has been shown how this method can be used to find the optimal shape of planar and spatial Variable Geometry Truss (VGT) manipulators for pre-specified position and orientation of the end-effectors. It is expected that the proposed methodology could be used for problems of shape optimization.

The shape design of a three-dimensional curve is accomplished by modeling it as a generalized helix. The base of the helix lies in a plane perpendicular to the skew direc-

tion between the end-point tangents. The base curve is designed as a planar curve consisting of a set of linear curvature segments. The gradient of the helix can be modeled by choosing any one of the following three curves: (i) a parabolically blended curve, (ii) a cubically blended curve and (iii) a pair of Bezier curves. The proposed method has been shown to be useful for designing the shape of a Variable Geometry Truss-type manipulator. The method can also be used for a variety of other applications such as the path of a manipulator end-effector, or the geometry of a highway clover loop.

A three-dimensional surface is considered as a surface swept by a generatrix curve when it moves along a directrix curve. In the present work a generatrix curve is considered to be a planar curve and it is defined using the intrinsic geometry concepts of shape models and shape variables. Four different types of surfaces have been proposed. (i) linearly swept surfaces, (ii) surfaces of revolution, (iii) generalized swept surfaces and (iv) transition surfaces. In each case, the generatrix can have a variable shape as it moves along the directrix. The proposed approach has been found suitable for modeling deformed geometries such as fabric drape surfaces. By controlling the variation of the shape design variables of the generatrix curve, it has been found that the proposed definition of surfaces can be used to design variable-shape three-dimensional surfaces.

The present work is an attempt to develop definitions of planar curves, space curves and surfaces which are based on the intrinsic geometry concepts. It has been found that in engineering analysis and design-optimization work, an engineer is able to represent and manipulate the shape effectively using an intrinsic form of geometry as compared to a parametric form.

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List of Symbols

$\mathbf{b}$	Binormal vector at a generic point P.
k	Constant denoting the length of a curvature vector.
$\mathbf{m}_1$	Projection of tangent vector $\mathbf{t}_A$ in the $O_1 - UV$ plane.
$\mathbf{n}$	Normal vector at a generic point P.
$\mathbf{p}$	Position vector of a parametric cubic curve.
$\mathbf{r}$	Position vector of a point.
$\mathbf{R}$	Remainder vector.
s	Arc length from a reference point to a generic point measured along the curve.
SDV	Shape design variable.
$\mathbf{t}$	Tangent vector at a generic point P.
T	Transformation matrix.
x, y, z	Cartesian coordinates of a generic point P.
w	Rise of a helix.
W	Objective function.

Greek Symbols

κ	Curvature of a generic point P.
$\boldsymbol{\kappa}$	Curvature Vector of a generic point P.
τ	Torsion of a generic point P.

ψ	Tangent angle.
γ	Tangent angle of a cubically blended curve.
η	Complex variable defining location of a point.

1. Introduction

1.1 Characterization of Shape

Engineering Design is a methodology involving several stages starting with the statement of a need and ending with the final presentation of a design. One can outline these stages in the form of a flowchart as shown in Fig. 1.1 (Sandor, 1964). It can be observed that the stages of synthesis, analysis and optimization constitute the core of any engineering design methodology.

The synthesis stage involves type synthesis, dimensional synthesis, material synthesis, synthesis of component tolerances as well as assembly clearances, and synthesis for manufacturability. The overall objective of dimensional synthesis is to design the geometry of the physical objects. The geometrical aspects of dimensional synthesis involve shape as well as size determination. The geometrical parameters could be described either as scalar variables or as continuous functions. The diameter of a shaft is an example of a sizing design variable, whereas, the profile of a cam and a fillet of a structural

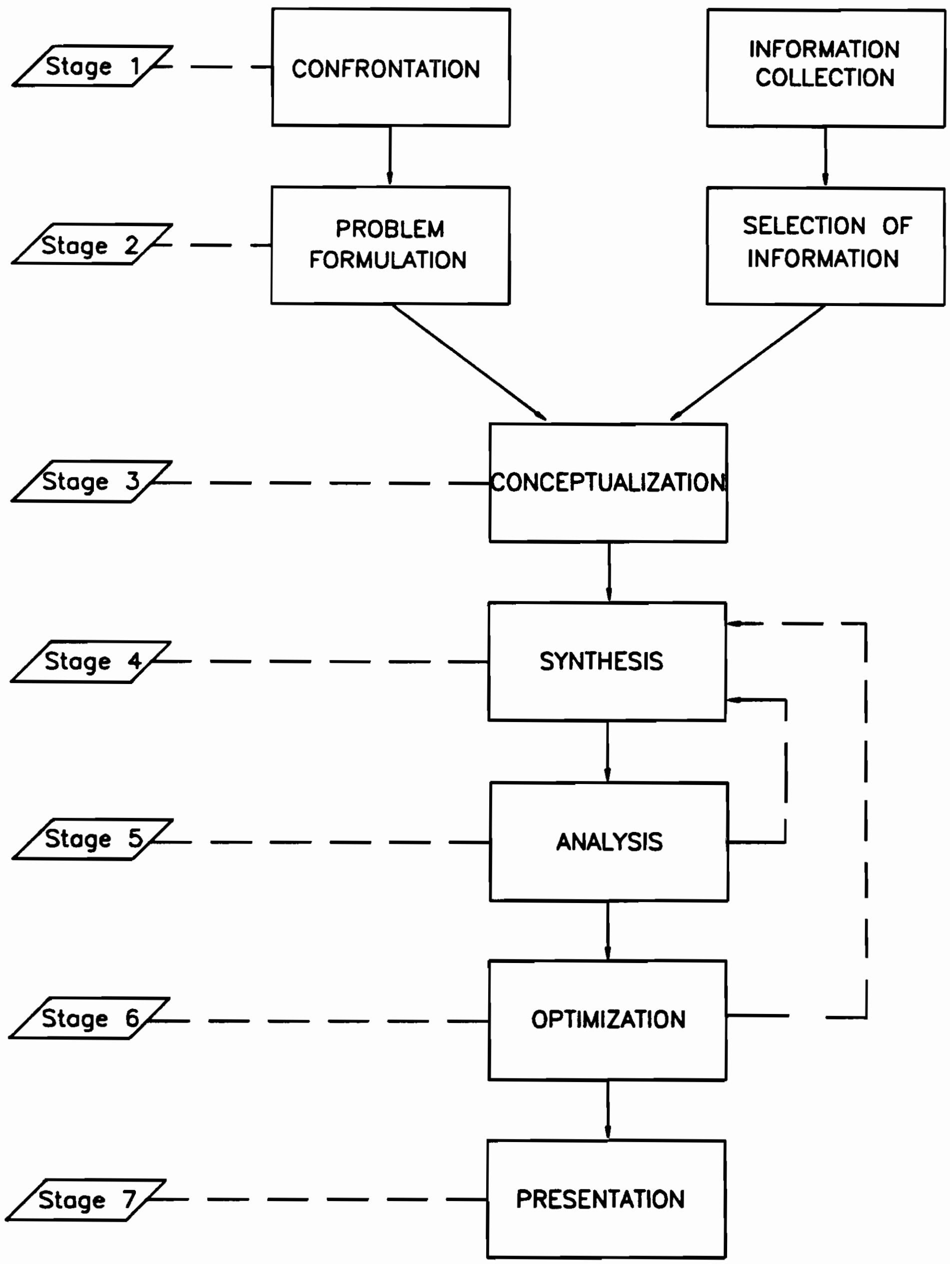


Figure 1.1. Seven Stages of Engineering Design.

member's cross-section are examples of shape design functions. The term shape generally refers to the intrinsic properties of geometry that define a curve or surface. Examples of shape synthesis properties of geometry for three-dimensional space curves are curvature and torsion.

Design optimization methods have been used successfully for optimization problems which involve selecting a set of optimal values for scalar design variables. The methods developed have been widely used for structural design as well as mechanical component design activities. Shape optimization is also a major consideration in many engineering design problems. The problem of shape optimization for a component can be described as deciding the geometry of a curve or a surface which will maximize or minimize an objective function as well as satisfy a set of constraints. One of the keys to success of any shape optimization method depends on how the shape synthesis module (Fig. 1.2) has been defined. The most frequently published approach is that of representing the shape in terms of an array of control points. This representation does not evaluate shape explicitly and does not indicate clearly how a shape should be modified for improving a design.

Although it is difficult to define, shape is still a basic property for design and manufacture. For example, the design of a highway path is the design of its geometry. The crucial decision is about the shape. Lord and Wilson (1984) define shape as "those aspects of geometrical form which have to do with the external aspect that an object presents to the world." It is difficult to model a mathematical representation of the shape of an object. Shape is an overall attribute or characteristic. Unlike microscopic properties of a geometry like coordinates and slopes, shape is a macroscopic attribute. One can also find aspects of geometrical form that deal with extrinsic and intrinsic geomet-

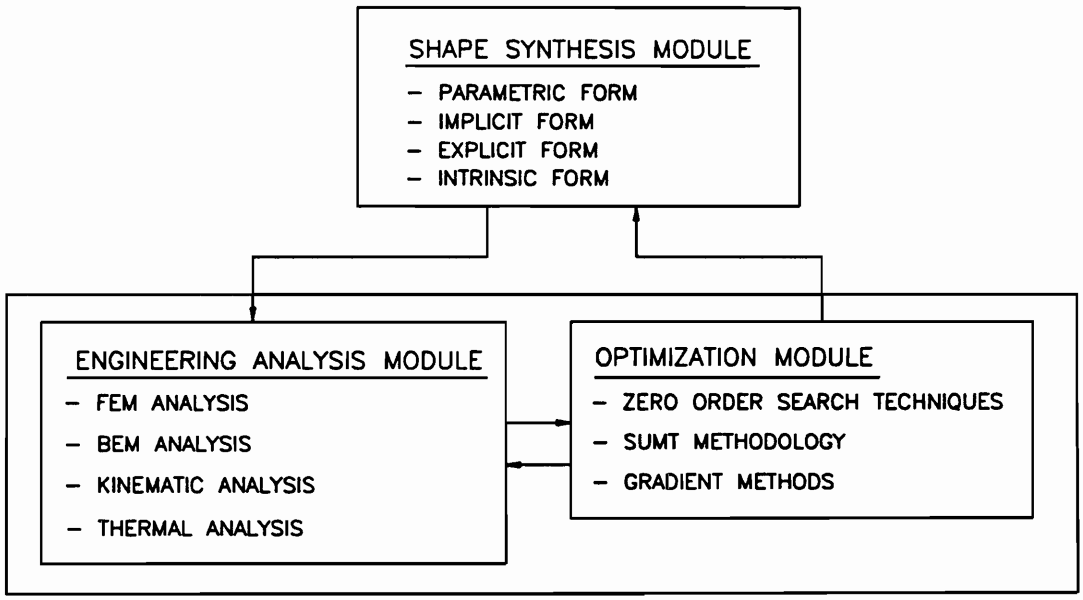


Figure 1.2. Schematic Block Diagram of the Shape Optimization Methodology. (Note that lists indicated in each module are examples and not complete lists.)

rical properties. The coordinates that represent the geometry are extrinsic shape properties and are analogous to mass and temperature in thermodynamics. Shape, on the other hand, is independent of scale, position and orientation and can be viewed as an intrinsic property of geometry much like entropy is an intensive thermodynamic property. Examples of shape intrinsic properties are the curvature and torsion of a three-dimensional space curve.

Figure 1.3 shows the relationship between the intrinsic geometry representation and the Cartesian geometry description required for engineering activities such as analysis, design, etc. Shape is synthesized using a mathematical model in the intrinsic geometry domain. This representation of shape is then mapped into a Cartesian domain, in which different analyses are performed based upon the particular application. The results are evaluated and updated till the optimum shape is found.

This work is an attempt to show how the concepts of intrinsic geometry can be used to represent shape in a manner to carry out the design and manufacturing activities effectively. A method of shape synthesis is outlined with the intent of using it for shape optimization problems. If one needs to define a curve passing through two points with specified end-point tangents, then it is proposed that the shape of the curve can be defined as a set of continuous piecewise linear segments, each segment having the curvature as a linear function of the arc length. The specific parameters of the curvature function are obtained by solving the constraints of end-point geometries. The Cartesian coordinates are obtained by solving the Serret-Frenet equations (Struik, 1950). Applications of intrinsic-geometry-based shape optimization to the configuration design of planar and spatial Variable Geometry Truss manipulators (VGT) is discussed in Chapters 3 and 4.

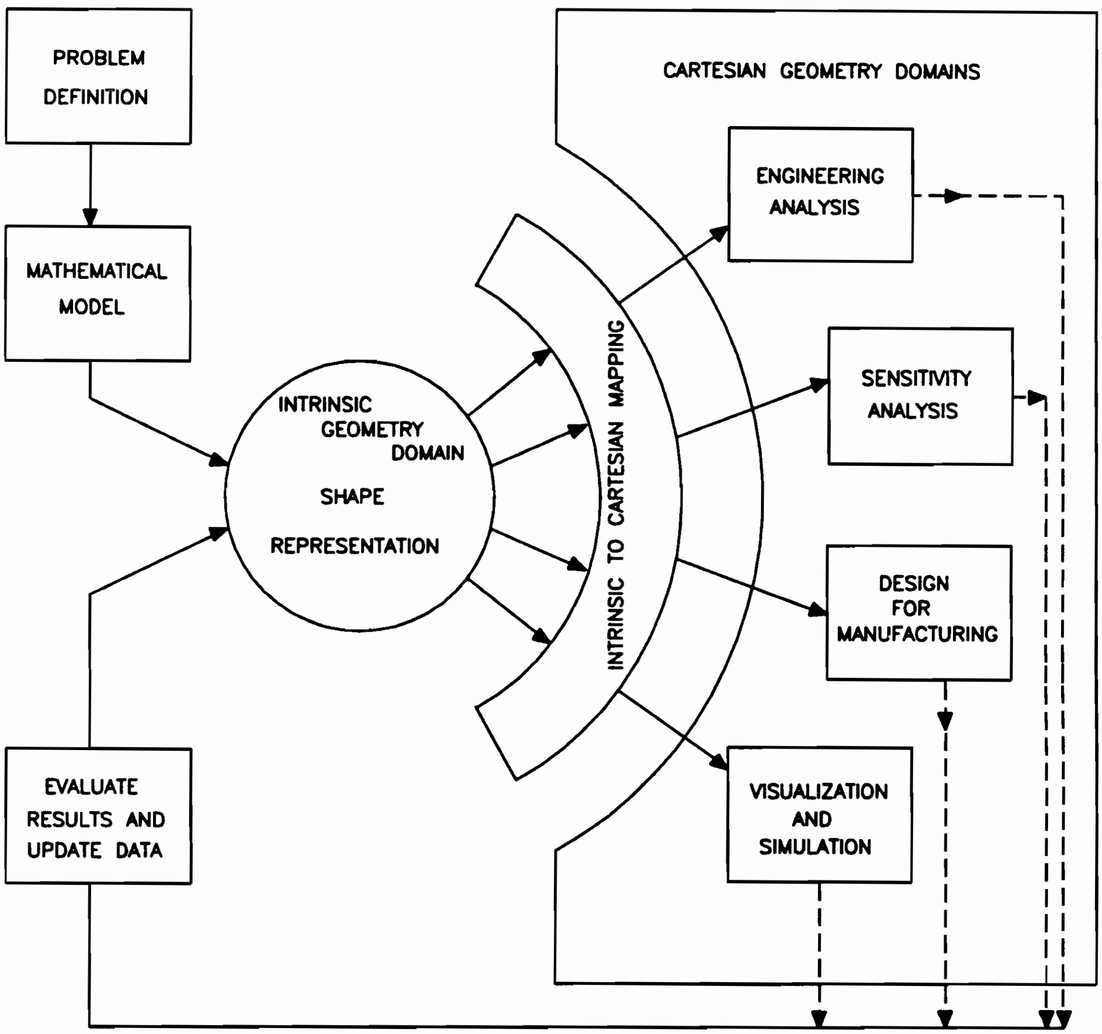


Figure 1.3. The Relationship Between Cartesian and Intrinsic Geometry.

1.2 Shape Synthesis in Design

Any shape synthesis starts with type synthesis and is followed with dimensional synthesis that involves size as well as shape parameters of geometry. The term shape generally refers to the geometry of a component that needs to be described as a curve or a surface or a solid. Synthesis of curves and surfaces can be carried out by representing these geometrical forms using any one of the following algebraic representations.

- Explicit form
- Implicit form
- Parametric form
- Intrinsic form

The explicit as well as implicit representations for the geometry of curves and surfaces have not been found to be suitable either from the viewpoint of synthesis/modeling or from the viewpoint of display. The methods of solid modeling involve set theoretic representation of points enclosed by bounding curves and surfaces. In turn, the curves and surfaces are defined either in parametric form or in intrinsic form. The parametric form has been found to be a suitable means for rendering a display of the geometry (Mortenson, 1985 , Faux and Pratt, 1983). The parametric form is also suitable for synthesizing geometries which need to fulfill a set of boundary constraints. However, the parametric form is not convenient for evaluating the intrinsic geometrical properties such as curvature and torsion. Moreover, the parametric form is not suitable when a certain behavior of curvature or torsion is to be built into the geometry of a component. The intrinsic form, on the other hand, deals with the properties such as curvature and torsion explicitly. Therefore, it is believed to have considerable potential for shape synthesis-optimization problems.

Once a geometrical model for a component has been conceived, it needs to be subjected to a cycle of analysis and optimization. It is important to define a set of geometrical design parameters which define the shape of a component and can also be used for analysis and optimization. Parametric forms do not define the shape explicitly and hence have been found to be inadequate for analysis and optimization activity. Also, in many cases, the parametric form results in a large number of design variables thus increasing the complexity of optimization-analysis procedures. Intrinsic forms, on the other hand, can be used with fewer design variables and can provide the shape information explicitly. However, in an intrinsic form, the Cartesian geometry of a curve or a surface is not explicitly available and needs to be obtained by integrating the Serret-Frenet equations (Struik, 1950).

In any design methodology, the optimization stage embodies a philosophy of selecting a set of values of design variables which maximize or minimize a design goal and simultaneously satisfy a set of design constraints. Techniques of mathematical programming have been found to be most suitable for solving problems of design optimization. If the design variables involve the shape of a component, besides its size, then the problem is referred to as a shape optimization problem. Figure 1.4 shows some examples of shape optimization problems for continuum structures. The problem of shape optimization for a component can be described as deciding the geometry of a curve or a surface which will maximize or minimize an objective function as well as satisfy a set of constraints. In other words, the entire shape or geometry of a curve or a surface constitutes a macro design variable. For instance, in the design of a turbine disk, the curve representing the cross-section of a blade is the design variable. (Fig. 1.4)

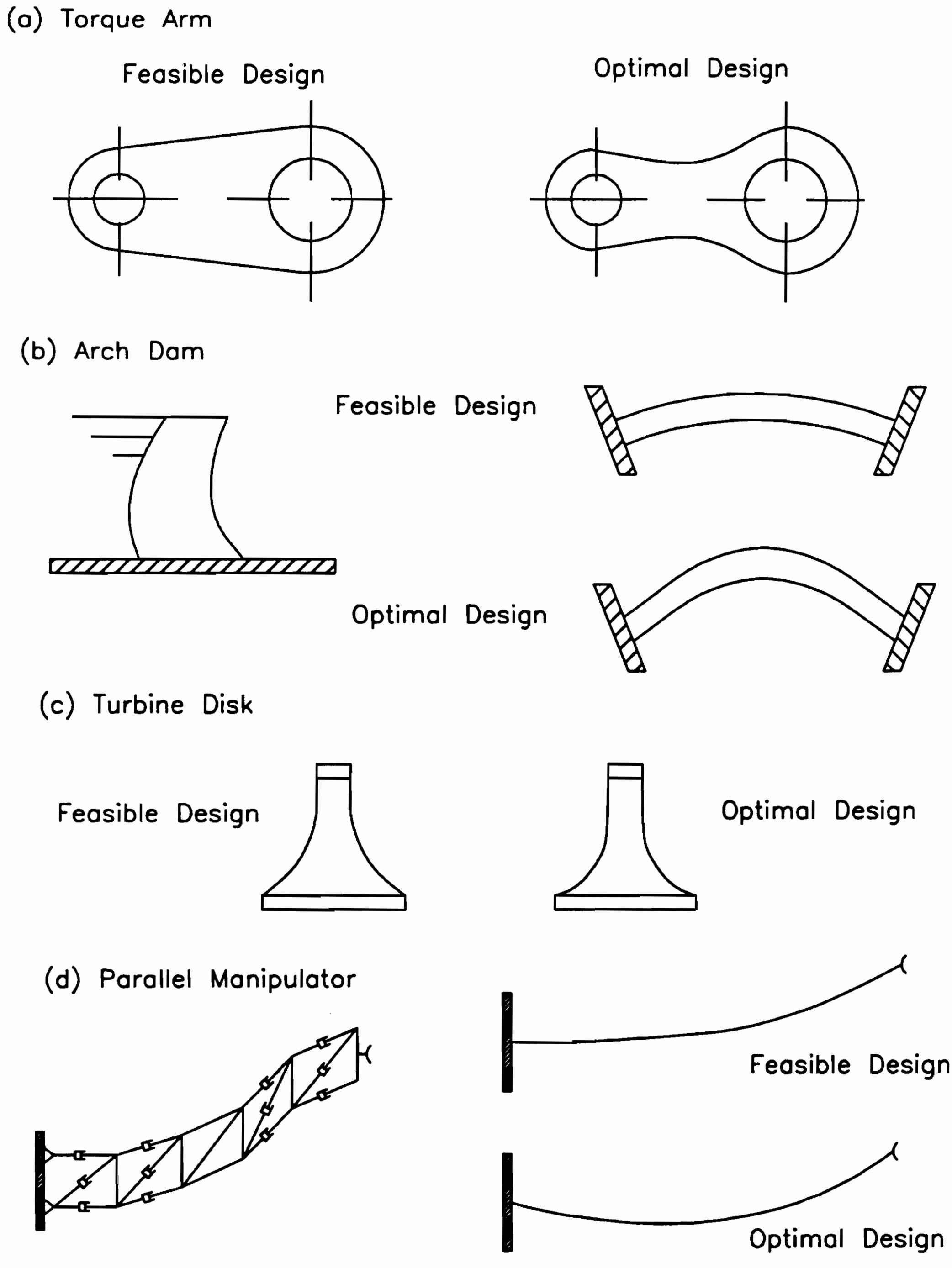


Figure 1.4. Typical Applications of Shape Optimization in Engineering Design.

Shape optimization is an area of active interest for designers, in general, and for structural designers, in particular. The most frequently published approach is that of using the FEM (Finite Element Method) or the BEM (Boundary Element Method) as the analysis method, applying nonlinear programming method as the optimization technique and representing the shape as an array of control points. Unfortunately, this representation does not evaluate shape explicitly, nor does it provide direct information on how a shape should be modified for an optimal design. The intrinsic form is clearly suitable for evaluating the shape information explicitly. It is also felt that the intrinsic geometry description may be helpful in modeling boundary elements in the BEM method.

1.3 Shape Synthesis in Manufacturing

While designing shapes, one needs to assess the feasibility of realizing that shape by means of a suitable manufacturing process. The means of describing shape for manufacturing has been either using projective geometry or descriptive geometry techniques. For example, curved surfaces and their development patterns are designed using methods of triangulation and orthographic projections. Such problems are required to be solved using a mathematical approach of isometric mapping. Unfortunately, these techniques are generally inadequate for shape design work. Shape design and manufacture is basically an interaction between three shape definitions (Fig. 1.5). Typically the original shape of the material is known. The desired shape of the material is conceived by the designer. The shape of the tool that accomplishes the link between the original and the

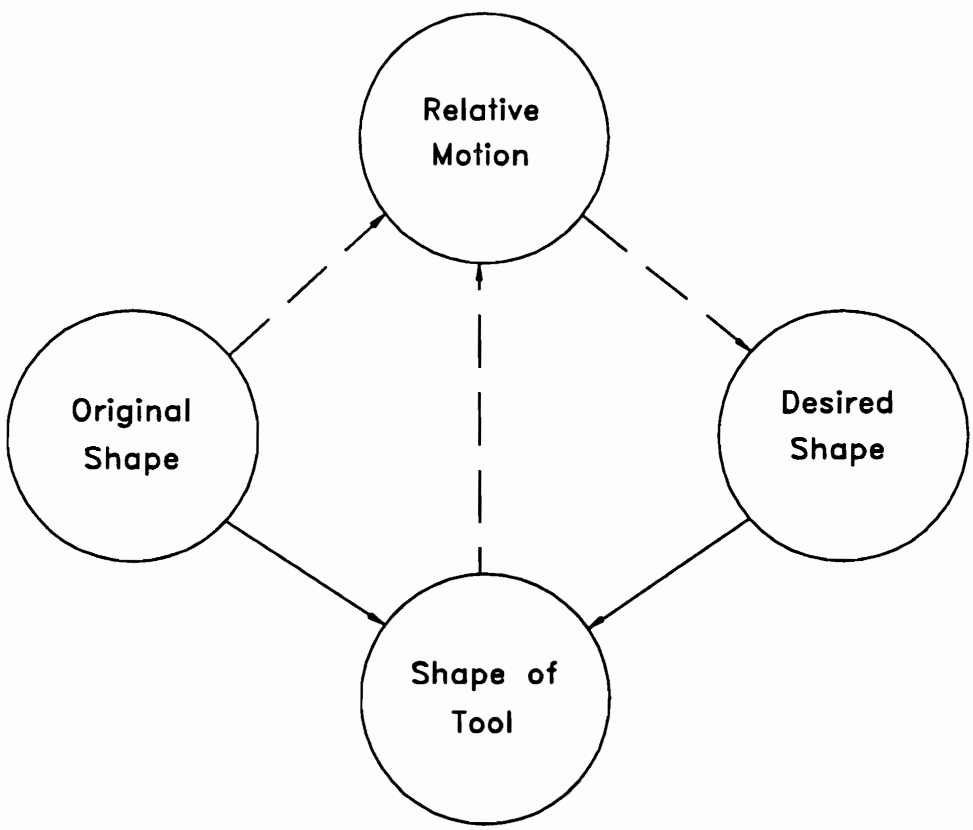


Figure 1.5. Shape Definition.

desired shape is to be determined. However, one needs to design the geometry of the tool based on the geometrical shape of the component as well as the scheme of manufacturing. Manufacturing of a shape involves either shape generation or shape formation techniques. Bending, stretching, and extrusion are examples of shape formation processes, whereas milling, and turning are examples of shape generation processes. The method of shape synthesis for a manufacturing process affects the accuracy of generating or forming a desired shape. This is where shape synthesis plays an important role.

A shape generation process involves removal of a volume from the original volume of the blank. The material removal (or volume subtraction) is achieved by the sweeping of the cutter with respect to the blank. The envelope surface generated by the cutter sweep defines the geometry of the shape generated by the manufacturing process. All metal cutting operations such as milling, shaping, planing and turning can be defined as shape generating processes. The theory of conjugate geometry can be used to define the shape generated by a process of this class (Chakraborty and Dhande, 1977). Voruganti (1990) describes how the manufacture of helically swept extruder screw surfaces can be modeled using the theory of conjugate geometry. This work has also shown the importance of using symbolic manipulation software, such as MACSYMA, to model the conjugate geometry problems encountered in shape geometry processes. Figure 1.6 shows the envelope conjugate plane surface being generated by a linear sweep of a horizontal side-mill in a typical manufacturing situation.

A shape forming process on the other hand involves a geometric transformation of a shape from its original state to a desired state. This can be achieved by using a manufacturing process such as extrusion, bending or deep-drawing. Figure 1.7 shows how a deep-drawing process can be geometrically modeled using intrinsic geometry parame-

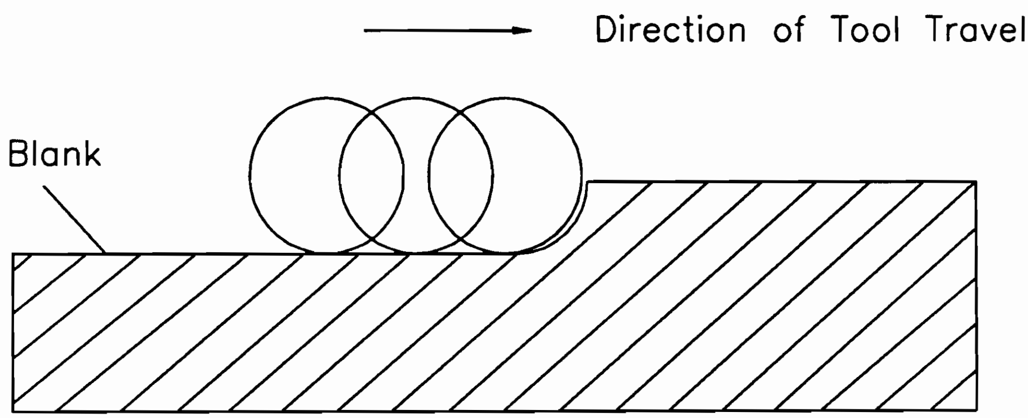


Figure 1.6. Generation of a Planar Surface.

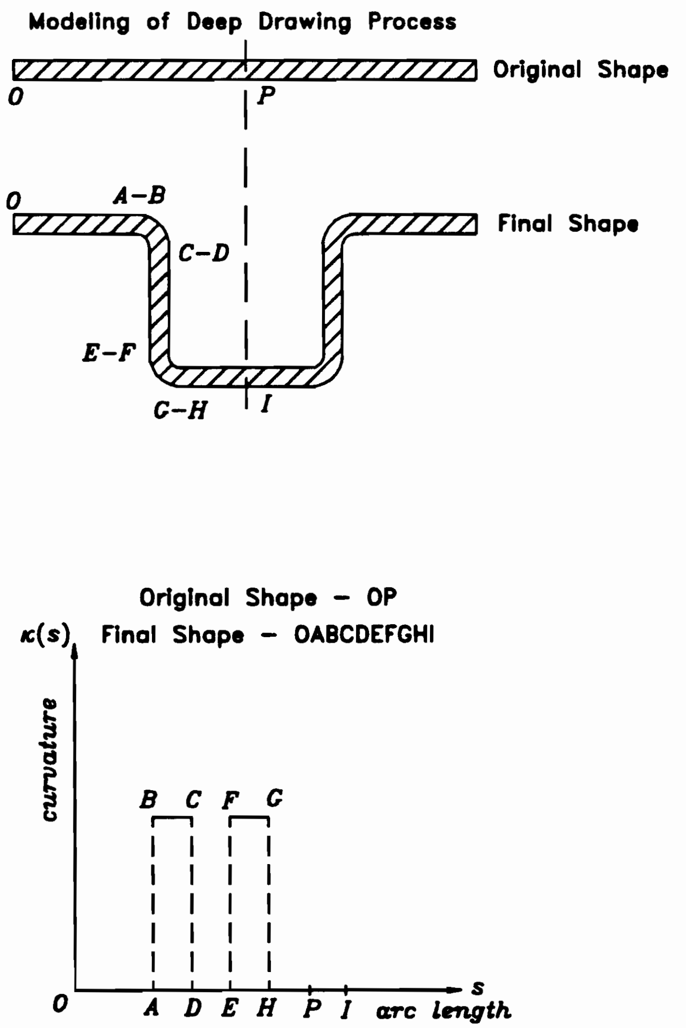


Figure 1.7. Shape Transformation.

ters. The transformation of a shape can be viewed as the affine transformation (i.e. translation and/or rotation) of a curve (representing the original shape) in the intrinsic plane.

In short, the intrinsic geometry representation is an appropriate conceptual framework to model shape forming as well as shape generation processes. This representation enables a designer to fulfill not only the functional requirements of a design but also satisfy the manufacturability requirements.

1.4 Review of Literature

The study of geometry is one of the most fascinating fields in the world of mathematics. Among the different branches of mathematics, geometry is probably the most appealing to our intuition. Historical reviews indicate that geometry was one of the fundamental branches of science developed by the Babylonians and the Egyptians. Euclid, one of the Greek philosophers, established the foundation of what is known as the Euclidean Geometry. In the Euclidean Geometry, one studies the actual shape of objects. More accurately, one studies those properties of objects that are unchanged by rotation, translation and reflections of objects.

Different branches of geometry have evolved in the past few centuries. Projective geometry and differential geometry are two examples. Both these fields are extensions of Euclidean Geometry. In projective geometry, one studies the way objects are seen in contrast to Euclidean geometry in which one studies metric properties of real objects.

Projective geometry has been of more interest to engineers than it has to mathematicians. It has become a tool for design engineers for conveying their ideas effectively. Car bodies and ship hull surfaces are designed and drawn using orthographic projections of a family of curves lying on these surfaces. Differential geometry on the other hand deals with the geometry of curves and surfaces studied by means of differential calculus. Concepts of intrinsic geometry have been described in well-known texts of differential geometry (Struik, 1950 , Pogorelov, 1970 , Do Carmo, 1976). In the literature on differential geometry, the Cornu spiral is probably the most widely illustrated example of a curve for intrinsic geometry. The intrinsic definition of a Cornu spiral is a linear curve in the curvature-arc length, $\kappa - s$, plane.

The developments in the area of computational geometry during the 70's and 80's gave an impetus for developing computational algorithms for many geometrical problems. In particular, curve and surface design became an active topic of research. One can now see a series of excellent texts which describe, in detail, how curves and surfaces can be designed using a variety of spline-type models. (Faux and Pratt, 1983, Mortenson, 1985, Farin, 1988). The major drawback of using a spline approach for shape synthesis is that the information about curvature and torsion is not explicitly available. This makes it difficult to provide a feedback from the analysis-optimization sequence to the shape synthesis algorithm.

The works of Adams (1975), Nutbourne (1977), Pal (1978a), Remashchandran (1982), and Dhande (1988) form an interesting section of the literature reviewed here. The concept of shape synthesis using linear curvature elements, termed as LINCE, had been developed by Nutbourne. This concept had been implemented in the form of a detailed procedure by Adams (1975) and Nutbourne and Pal (1977). There are two difficulties in reviewing the works of Nutbourne, Adams and Pal. The first one being that they do not solve the boundary condition constraints of Serret-Frenet equations explicitly. This makes the computational procedure lengthy. The second aspect is that they have not shown how the method can be used for analysis-optimization process. Remashchandran (1982) had used a spline curve to model the intrinsic shape definition. He had shown how a planar cam profile can be designed using this approach. It should be noted, however, that the method has not been extended for any optimization work.

Optimization as a philosophy of engineering structural design was reviewed by Schmit (1984). There is now a collection of excellent titles reviewing optimization methods in engineering design (Arora, 1989, Rao, 1984).

Traditionally, the optimization problem is cast as a minimization or maximization of an objective function subject to a set of constraints. A finite number of scalar variables are defined as design variables. However, if one needs to consider the shape geometry of a component as the design variable, then it no longer is a set of discrete variables. The problems of optimization dealing with continuum-type shape design variables are classified as shape optimization problems.

Shape optimization has attracted the attention of many researchers. Haftka and Grandhi have reviewed the work of several researchers in a review article (1986). Vanderplaats has discussed the numerical problems associated with shape optimization. A review of the current state of the art shows that the geometrical description of a component is taken into account either as a set of control points or as a set of points lying on the geometry of the curve or surface. Control points may not lie on the curve or surface, however, do define the shape of the curve or the surface. West and Sandgren (1989) have used the variational approach to solve shape optimization problems. The review of current literature shows that introducing the concept of intrinsic shape definition may provide a better foundation for solving shape design problems.

The concept of shape optimization has been applied to the design of many mechanical components. Some applications are mentioned here.

As an application of optimization in the automotive industry, Botkin (1982) used finite element analysis to design a torque arm for an optimum shape which would minimize the weight subject to strength limitations. However, due to geometrical changes during the design process, the finite element mesh had to be continually redefined during the optimization process. The optimum shape of rotating disks was studied by Bhavikatti and Ramakrishnan (1980) using a non-linear programming method. They used a 5th degree polynomial to define the shape of the cross section. They also performed a stress analysis of the disk by FEM using isoparametric elements. They had satisfactory results which did not need any smoothing of the shape after optimization.

The optimum shape of a gas turbine disk was obtained by Luchi, et al. (1980). They used an interactive procedure based on a method in which the disk profile was defined by spline interpolation. The shape of a gas turbine disk spinning at 10,000 rpm was obtained. Queau and Trompette (1983) also worked on the optimal shape design of turbine blades. The weight of the rotating turbine blades was minimized in optimizing the cross section shapes. They used polynomials to define the shape.

Sandgren and Wu (1988) investigated the optimum shape for the ladle hook using boundary element method. Seventh order B-spline curves with 9 control points were used to define the shape. As a second approach (West and Sandgren, 1989), the shape optimization problem was formulated as a variational statement. The optimal shape represented by the direct boundary element method was shown to be in good agreement with the analytic solution.

A fillet design problem is addressed as the final example in this section. The optimum design of a two dimensional fillet was studied by Wu (1986). Figure 1.8 shows the upper portion of a 2-D fillet under axial loading. The design objective is to find the optimum shape of the edge AB in order to minimize the total area of the fillet. Stress constraints were imposed in order to force the stress along the edge AB to be less than a specified value. A 4th order B-spline was used to represent the design curve. Wu obtained optimum shapes by specifying different values of the permissible maximum stress.

1.5 Objectives

The overall objective of this work is to develop a methodology of shape design using concepts of intrinsic geometry and to show how this methodology can be used in a variety of design and manufacturing applications. The specific objectives of this research can be stated as follows:

- To develop a robust numerical scheme of mapping of a curve from intrinsic space to Cartesian space.

  The Serret-Frenet equations take a central position in the theory of curves. The solution of these equations using robust numerical integration schemes will provide the necessary mapping.

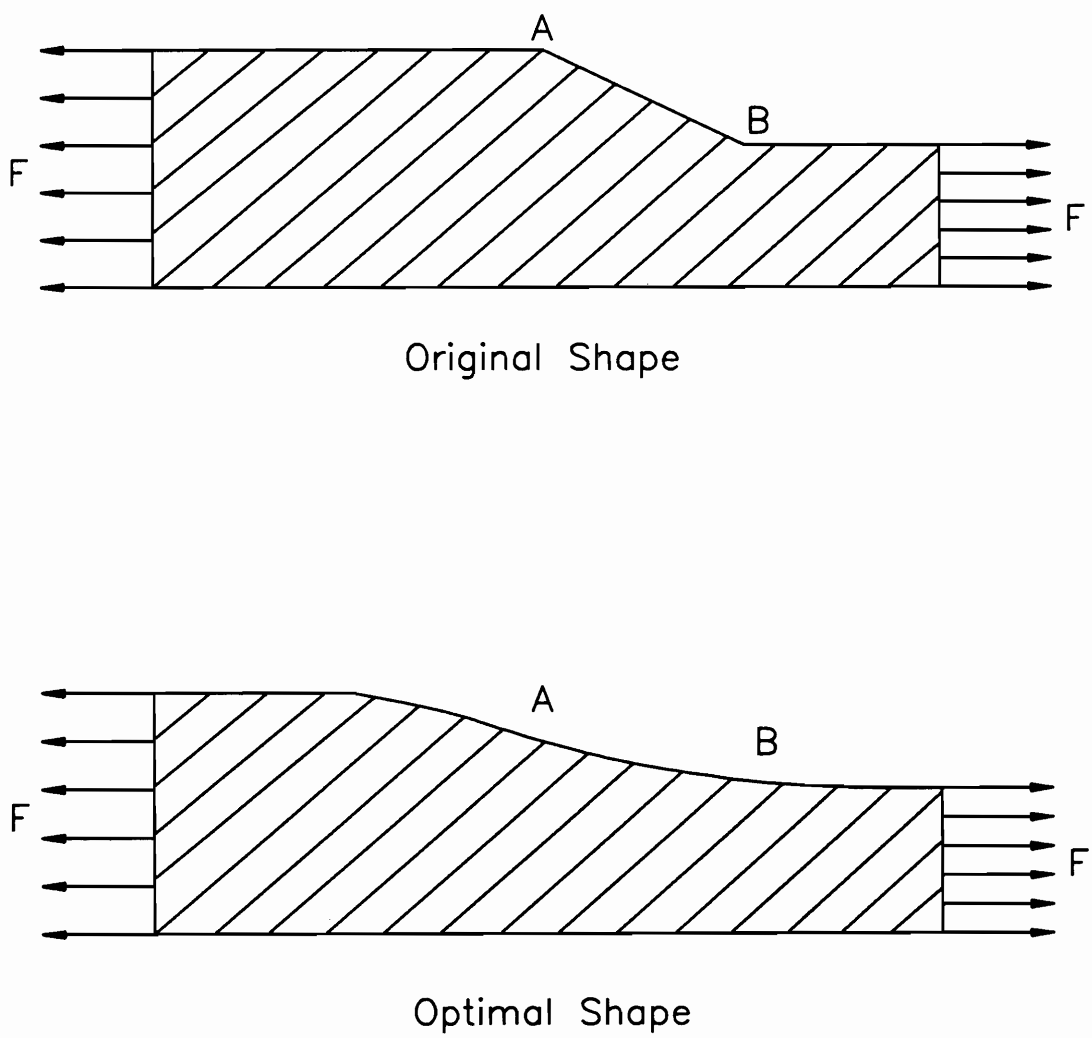


Figure 1.8. The Upper Portion of a 2-D Fillet Under Axial Loading.

- To develop a set of shape models.

Two-dimensional shape synthesis has been accomplished for linear shape models. Some higher order shape models will be investigated where appropriate. A set of shape models will be proposed, based on intrinsic geometry, that are computationally tractable and meaningful from an engineering design and manufacturing viewpoint.

- To develop a scheme of evaluating shape design variables based on the boundary constraints.

Curvature and arc length have been used as shape design variables in the proposed work. Computational procedures for shape synthesis will also be developed.

- To study the classical methods of synthesizing the geometry of three-dimensional curves.

The classical solutions of the Serret-Frenet equations for design of 3-D curves will be reviewed. These will establish a foundation for the shape synthesis of 3-D curves in this work.

- To develop either intrinsic or pseudo-intrinsic approaches to 3-D curve problems.

Linear shape models will be used in design of 3-D curves. A solution to the Serret-Frenet equation will be discussed and applied.

- To develop either intrinsic or pseudo-intrinsic approaches to 3-D surface problems.

The concepts of intrinsic geometry will be used to predict the shape of a biparametric swept surface. The proposed definition will be used to study the shape of a fabric drape surface. The input parameters for such a fabric drape surface should reflect the shear and bending properties of the fabric.

- To study the tools of engineering analysis and show how intrinsic geometry based shape definition can be used for engineering analysis.

Presently, analysis of engineering problems requires either FEM/BEM (finite element Method / Boundary Element Method) approach or either algebraic or numerical analysis methods. All these techniques presume that the Cartesian geometry is available. The intrinsic geometry approach should provide the information about the Cartesian geometry and any change in the shape made through the intrinsic variables should update the Cartesian geometry. In short, it is necessary to study how intrinsic geometry along with the Cartesian geometry will be used for engineering analysis.

- To indicate the use of methodologies of shape synthesis using intrinsic geometry in the optimization process.

The proposed methodologies of shape synthesis and optimization require an understanding of how a shape model and its associated shape design variables can affect the Cartesian geometry of a curve or a surface.

- To select applications in the design area to show how the intrinsic approach would be used for synthesis and analysis activities.

Several Applications are considered. However, efforts will be focused on the Variable Geometry Truss (VGT) applications. Various configurations in 2-D and 3-D space will be explored for the VGT.

- To select applications in the manufacturing area and show how the intrinsic approach would be used for activities of design for manufacturing.

1.6 Work Scope

The present work is aimed at developing the methodology of shape design using intrinsic geometry. Specific goals and objectives of this research are given in Section 1.5. Chapter 2 deals with the concepts of differential geometry. The basic concepts of differential geometry are presented in this chapter. The intrinsic geometry aspects of the planar and spatial curves are introduced. Some fundamental definitions such as curvature, torsion and arc length are discussed. The Serret-Frenet equations which are the central equations in the theory of curves are examined. For the case of planar curves, the differential equations relating the Cartesian coordinates as functions of arc length are derived for the Serret-Frenet equations. The solution approach for synthesizing 2-D and 3-D curves is introduced. Finally, the intrinsic geometry of the 3-D surfaces is presented.

Chapter 3 deals with the shape design of 2-D curves. The problem statement is discussed in detail and a methodology of synthesizing 2-D curves based on the intrinsic geometry is presented. Linear shape models are introduced along with a set of shape design variables. An algorithm for designing 2-D curves is developed for a given set of initial and final coordinates and orientations. This algorithm is then demonstrated using a variety of applications. Each application is discussed and concluding remarks are presented. The near optimal shape of a VGT is found using an exhaustive search method.

The shape design of 3-D curves is presented in Chapter 4. The statement of the problem is discussed and a methodology of synthesizing three-dimensional curves based on the solution approach shown in Chapter 3 is presented. The corresponding shape models for three-dimensional curves are introduced. An algorithm of designing three-dimensional curves is developed for a given set of initial and final coordinates and orientations. The developed algorithm is applied to design of a spatial VGT.

The shape design of 3-D surfaces is accomplished in Chapter 5. Using the concepts of intrinsic geometry, a predictive model to define the shape of a fabric drape surface is proposed. Computational results and illustrative examples are presented in this chapter.

The conclusions of this work is presented in Chapter 6. The use of intrinsic geometry as a tool in synthesizing 2-D and 3-D curves along with 3-D surfaces is evaluated.

• • •

2. Basic Concepts of Differential Geometry

2.1 Definitions

This section introduces the different algebraic representations of the geometrical forms for which synthesis of curves and surfaces can be carried out.

- Explicit form
- Implicit form
- Parametric form
- Intrinsic form

Geometric elements are represented by use of either parametric or non-parametric equations. For a plane curve, an explicit non-parametric equation takes the form $y = f(x)$, which is not a convenient form to represent closed or multiple-valued curves. On the other hand, an implicit non-parametric equation takes the form $f(x, y) = 0$ in which multiple-valued curves are possible. In either case both these forms are axis dependent. Moreover, the slope at a point may become infinitely large depending on the orientation of the curve with respect to the coordinate axes.

In parametric form, on the other hand, the slope values are computed in terms of the rate of change of the coordinates with respect to the parameter. For a plane curve, a set of two equations $x = x(u), y = y(u)$, $u_{min} \leq u \leq u_{max}$, with the parameter u, represent its parametric form. If a curve must be fit between a set of points, it is seen that the shape of the curve is dependent on the location of these points and it is not dependent on the orientation of the coordinate axes. In most geometric modeling problems the choice of the coordinate system should not affect the shape. For these reasons, the parametric representation seems more convenient for curve fitting routines. However, this representation does not explicitly affect the shape control of the generated curve. Let us briefly examine two parametric forms.

The formulation and application of parametric cubic (pc curves) and Bezier curves have been studied in detail by Mortenson (1985). Mortenson believes that "the parametric cubic form is the single mathematical form that fits identically any curve or surface useful to us in geometric modeling." Continuous piecewise cubic curves are connected between a set of n points in order to accomplish a curve fitting routine. The two pc curve segments meeting at a point have a common tangent and the curvature is continuous through this point, since the curve to be modeled must be continuous through all points. It should be noted that there is limited scope for shape control using parametric cubic curves, since the shape of the generated curve is a cubic curve with coefficients depending on the initial and final points. The algebraic form of a planar parametric cubic curve is given by the following equations.

$$x(u) = a_{3x}u^3 + a_{2x}u^2 + a_{1x}u + a_{0x}$$
$$y(u) = a_{3y}u^3 + a_{2y}u^2 + a_{1y}u + a_{0y} \tag{2.1}$$

where $0 \le u \le 1$.

The geometric form of a pc curve is given as:

$$\mathbf{p} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ k_0\mathbf{t}_0 \\ k_1\mathbf{t}_1 \end{bmatrix} \tag{2.2}$$

where $0 \le u \le 1$ and $\mathbf{p}_0$, $\mathbf{p}_1$ are position vectors and $\mathbf{t}_0$, $\mathbf{t}_1$ are tangent vectors.

Bezier curves, on the other hand, have indirect control over the shape through the use of control points. These control points represent the vertices of a characteristic polygon which are weighting coefficients to blending functions. The shape of the curve is not affected by reversing the sequence of the control points since the parametric functions of the Bezier curves are symmetric with respect to their parameter. The location of these control points cause the shape of the Bezier curve not to oscillate wildly away from the characteristic polygon. Neither one of the above parametric forms have the ability to control the intrinsic properties of a curve, which makes the intrinsic shape synthesis more attractive.

A curve requires two intrinsic equations $\kappa = \kappa(s)$ (curvature as a function of arc length) and $\tau = \tau(s)$ (torsion as a function of arc length) in order to determine its unique shape. For the case of a 2-D curve, $\tau = 0$, as explained in Section 2.2. The case $\tau = 0$

is a natural equation, since it is a property of all 2-D curves. It suffices to claim that for any given function $\kappa(s)$, it is possible to solve the Serret-Frenet second-order equations for $x(s)$ and $y(s)$. However, only ramp type curvature functions which correspond to spirals in the Cartesian coordinates have been studied. The selection of multi-ramp function curvature equations provide the designer with more free parameters and hence more flexibility of controlling the shape. The effects of second and higher order curvature equations is yet to be studied. These cases may allow the designer to explore the result of shape synthesis by means of these higher order forms.

2.2 Serret-Frenet Equations

Consider a curve C as shown in Figure 2.1. Let **r** be the position vector of a generic point P and let s be the arc length of P from a reference point A, be the parameter describing the curve as $\mathbf{r}(s)$. The unit tangent vector, the curvature, the torsion, the normal and the binormal of the curve C at the point P are given as follows:

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} \tag{2.3}$$

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n} \tag{2.4}$$

$$\frac{d\mathbf{n}}{ds} = -\kappa \mathbf{t} + \tau \mathbf{b} \tag{2.5}$$

$$\frac{d\mathbf{b}}{ds} = -\tau \mathbf{n} \tag{2.6}$$

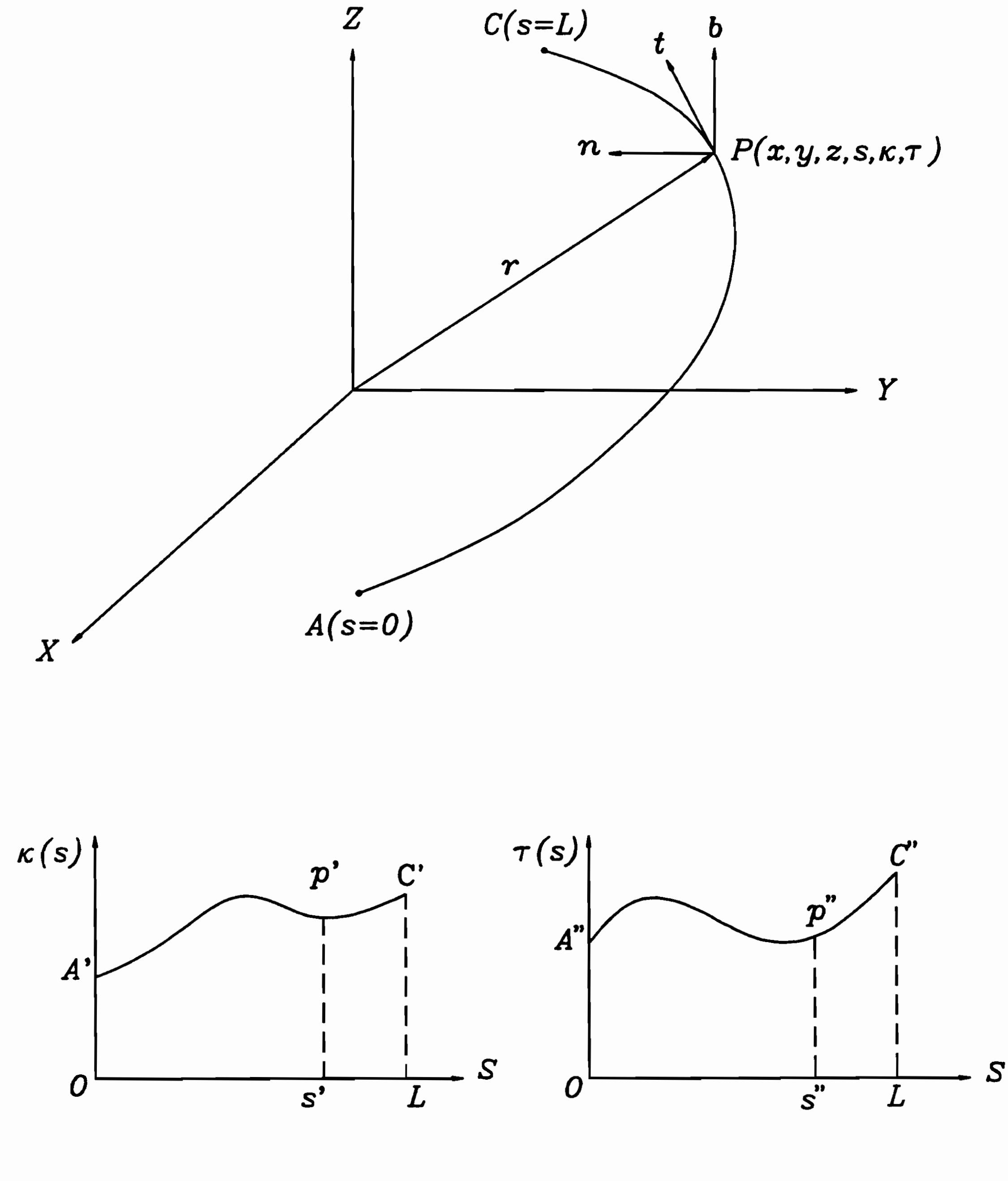


Figure 2.1. Intrinsic Geometry of a Three-dimensional Curve.

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \tag{2.7}$$

where **t** is the unit tangent vector, **n** is the unit normal vector, **b** is the unit binormal vector, κ is the curvature, and τ is the torsion. Equations (2.4), (2.5) and (2.6) are known as the formulas of Serret-Frenet (Faux and Pratt, 1983, Goetz, 1970 and Struik, 1950). These equations describe the location as well as the orientation of a moving trihedron consisting of the unit vectors **t**, **n** and **b** along the curve. It was mentioned by Struik (1950), that these equations take a central position in the theory of curves. Furthermore, these equations can also be used as the basis for shape synthesis.

If a curve is planar then $\tau = 0$ and

$$\mathbf{r} = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} \tag{2.8}$$

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} \quad \text{and} \quad \mathbf{n} \bullet \mathbf{t} = 0 \tag{2.9}$$

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n} \tag{2.10}$$

$$\frac{d\mathbf{n}}{ds} = -\kappa \mathbf{t} \tag{2.11}$$

The torsion and curvature can be directly related to the position vector **r** by the following equations (Faux and Pratt, 1983).

$$\kappa = \left| \frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2} \right| \tag{2.12}$$

$$\tau\kappa^2 = \frac{d\mathbf{r}}{ds} \bullet \left(\frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3} \right) \tag{2.13}$$

where

$$\frac{d\mathbf{r}}{ds} = \begin{bmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \end{bmatrix} \tag{2.14}$$

$$\frac{d^2\mathbf{r}}{ds^2} = \begin{bmatrix} \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \end{bmatrix} \tag{2.15}$$

and

$$\frac{d^3\mathbf{r}}{ds^3} = \begin{bmatrix} \frac{d^3x}{ds^3} & \frac{d^3y}{ds^3} & \frac{d^3z}{ds^3} \end{bmatrix} \tag{2.16}$$

The review of literature on the 2-D and 3-D curves indicate that different mathematical strategies have been utilized in order to find the general solution of these curves using the above equations. In the following sections, a brief review of a few of these methods is provided in order to establish a preliminary background.

2.3 Solution Approach for Two-dimensional Problems

Let us consider the problem of defining a curve passing through two points $P_0(x_0, y_0)$ and $P_n(x_n, y_n)$ in a two-dimensional space (Fig. 2.2). Let us also assume that the

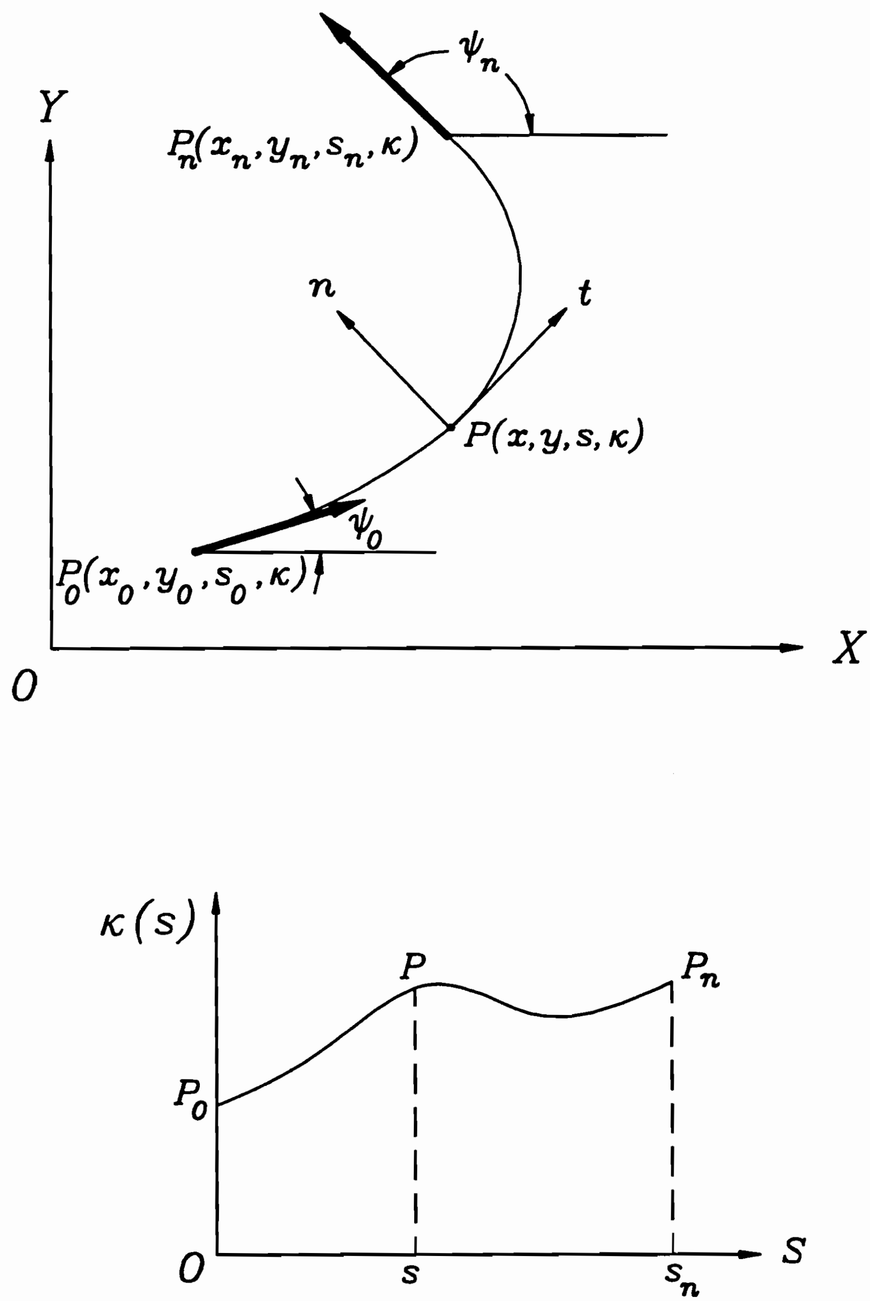


Figure 2.2. Curve Definition Using Cartesian Coordinates and Tangent Angles.

directions of tangents at P_0 and P_n have been specified as ψ_0 and ψ_n and the arc lengths from a reference point A have been specified as s_0 and s_n respectively. If κ is the curvature at any point P, then it is assumed that the variation of κ as a function of the arc length parameter $\kappa = \kappa(s)$ has been specified. It can be seen that $\kappa(s)$ defines the shape of the curve. The problem of finding the coordinates x and y as a function of the arc length parameter s can be solved by using the Serret-Frenet equations (2.10) and (2.11). It should be remembered that the variation of the intrinsic property of curvature, κ, gives flexibility to shape synthesis.

To begin with, let us rewrite Eqn. (2.9) as follows:

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \begin{bmatrix} \dfrac{dx(s)}{ds} \\ \dfrac{dy(s)}{ds} \end{bmatrix} \tag{2.17}$$

Since $\mathbf{n} \cdot \mathbf{t} = 0$ from Eqn. (2.9), then

$$\mathbf{n} = \begin{bmatrix} -\dfrac{dy(s)}{ds} \\ \dfrac{dx(s)}{ds} \end{bmatrix}. \tag{2.18}$$

On the other hand:

$$\frac{d\mathbf{t}}{ds} = \begin{bmatrix} \dfrac{d^2x(s)}{ds^2} \\ \dfrac{d^2y(s)}{ds^2} \end{bmatrix} \tag{2.19}$$

Substituting Eqns. (2.18) and (2.19) in Eqn. (2.10) yields:

$$\begin{bmatrix} \dfrac{d^2x(s)}{ds^2} \\ \dfrac{d^2y(s)}{ds^2} \end{bmatrix} = \kappa(s) \begin{bmatrix} -\dfrac{dy(s)}{ds} \\ \dfrac{dx(s)}{ds} \end{bmatrix} \tag{2.20}$$

Therefore,

$$\frac{d^2x}{ds^2} + \kappa(s)\frac{dy}{ds} = 0 \tag{2.21}$$

$$\frac{d^2y}{ds^2} - \kappa(s)\frac{dx}{ds} = 0 \tag{2.22}$$

where, at $s = s_0$, $x = x_0$, $y = y_0$, $\psi = \psi_0$ and at $s = s_n$, $x = x_n$, $y = y_n$, $\psi = \psi_n$.

In order to solve Eqns. (2.21) and (2.22), let us introduce the complex variable η such that $\eta = x + iy$. Then:

$$\eta' = x' + iy' \tag{2.23}$$

where $\eta' = \dfrac{d\eta}{ds}$, $x' = \dfrac{dx}{ds}$, and $y' = \dfrac{dy}{ds}$.

The governing Eqns. (2.21) and (2.22) can now be written as follows.

$$(\eta')' - ik(s)(\eta') = 0 \tag{2.24}$$

or

$$\eta' = e^{i\int k(s)ds} = e^{i\,\psi(s)} \tag{2.25}$$

Separating the real and imaginary parts, we get,

$$\frac{dx}{ds} = \cos[\psi(s)] \tag{2.26}$$

$$\frac{dy}{ds} = \sin[\psi(s)] \tag{2.27}$$

Integrating these equations with respect to the arc length s yields the parametric coordinates of the 2-D curve in terms of the intrinsic parameter s.

$$x(s) = \int_{s_0}^{s} \cos[\psi(\sigma)]d\sigma + x_0 \tag{2.28}$$

$$y(s) = \int_{s_0}^{s} \sin[\psi(\sigma)]d\sigma + y_0 \tag{2.29}$$

where

$$\psi(\sigma) = \int_{s_0}^{\sigma} \kappa(s)ds + \psi_0 \tag{2.30}$$

Equations (2.28) through (2.30) could be solved using numerical integration and numerical equation solving for the Cartesian coordinates $x(s)$ and $y(s)$ and the change in the tangent angle $\psi(s)$ along the 2-D curve once the curvature function $\kappa(s)$ is known.

2.4 *Solution Approach for Three-dimensional Problems*

The following section introduces a brief review of some of the analytical techniques of designing three-dimensional curves. These different methods of solutions are presented in order to establish a preliminary background. The interested reader can refer to the references for detailed explanations (Struik, 1950, Do Carmo, 1976, Pal, 1978b, Adams, 1975). These methods use the intrinsic definition of a three-dimensional curve. The solution procedures of these methods are however computationally cumbersome. After having reviewed these methods, a computational scheme of designing the geometry of a space curve using linear segments of curvature and a cubically blended linear segment for torsion has been proposed. The proposed method is presented in Chapter 4.

The problem of synthesizing the shape of a three-dimensional curve can be stated as follows. Given two end-points A and B and the end-point tangents and/or curvatures, it is required to find the parametric relation of the Cartesian coordinates x , y and z as functions of the arc length s. These relations can be obtained by solving the following three differential equations presented earlier, but repeated here for convenience.

$$\frac{d\mathbf{t}}{ds} = \kappa \mathbf{n} \tag{2.4}$$

$$\frac{d\mathbf{n}}{ds} = -\kappa \mathbf{t} + \tau \mathbf{b} \tag{2.5}$$

$$\frac{d\mathbf{b}}{ds} = -\tau \mathbf{n} \tag{2.6}$$

where $\kappa(s)$ and $\tau(s)$ are given as input functions describing the intrinsic geometry of the curve.

2.4.1 The Exact Approach

The solution approach described by Struik (1950) can be summarized as follows. Equations (2.4), (2.5) and (2.6) form a set of three first-order differential equations. These equations can be reduced to the Riccati equation (O'Neil, 1983). The desired solution for the parametric equation of the space curve can be stated as:

$$x = \int_0^s \alpha_1 ds, \qquad y = \int_0^s \alpha_2 ds, \qquad z = \int_0^s \alpha_3 ds, \tag{2.31}$$

where α_i's are given by:

$$\alpha_1 = \frac{(f_1^2 - f_3^2) - (f_2^2 - f_4^2)}{2(f_1 f_4 - f_2 f_3)}, \tag{2.32}$$

$$\alpha_2 = \left[\frac{(f_1^2 - f_3^2) + (f_2^2 - f_4^2)}{2(f_1 f_4 - f_2 f_3)} \right] i, \tag{2.33}$$

$$\alpha_3 = \frac{f_3 f_4 - f_1 f_2}{f_1 f_4 - f_2 f_3}. \tag{2.34}$$

The functions f_1, f_2, f_3 and f_4 are parametric relations of the arc length s. The Serret-Frenet eqns. (2.4) through (2.6) can be expressed as follows (Struik, 1950):

$$\frac{df}{ds} = -i\frac{\tau}{2} - i\kappa f + i\frac{\tau}{2}f^2. \tag{2.35}$$

The solution of this equation is:

$$f = \frac{cf_1 + f_2}{cf_3 + f_4}, \tag{2.36}$$

where $\kappa(s)$ and $\tau(s)$ are functions of the arc length and c is a constant of integration. Using these functions, it is required to find the expressions for α_1, α_2 and α_3. Integrating the functions α_1, α_2 and α_3, one can find the functions for x, y and z.

Let us take an example of a planar curve. In this case $\tau = 0$. Then the Riccati equation reduces to

$$\frac{df}{f} = -i\kappa ds. \tag{2.37}$$

The solution of this equation is

$$f = ce^{-i\phi}, \qquad \phi = \int \kappa ds. \tag{2.38}$$

Comparing $f = ce^{-i\phi}$ and $f = \dfrac{cf_1 + f_2}{cf_3 + f_4}$ we get:

$$f_1 = e^{-i\phi}, \quad f_2 = 0, \quad f_3 = 0, \quad f_4 = 1, \quad c = 1. \tag{2.39}$$

Substituting these expressions in equations for α_1, α_2 and α_3 we get:

$$\alpha_1 = \cos\phi \qquad \alpha_2 = \sin\phi \qquad \alpha_3 = 0, \tag{2.40}$$

where $\phi = \int \kappa ds$. Using the above expressions of α_1, α_2 and α_3 the expressions for x, y and z can be obtained as:

$$x = \int \cos\phi \, ds, \qquad y = \int \sin\phi \, ds, \qquad z = 0. \tag{2.41}$$

2.4.2 Local Canonical Approach

Another method to solve the governing equations has been proposed by Do Carmo (1976). It is based on finding the coordinates by studying the local properties of a curve in the neighborhood of a given point s_0. This is accomplished by using the natural coordinate system of the trihedron $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ at s_0. The parametric equation of the curve with reference to the trihedron coordinate system can be written as Taylor series expansion about s_0.

$$\mathbf{q}(s) = \mathbf{q}(s_0) + (s - s_0)\left(\frac{d\mathbf{q}}{ds}\right)_{s=s_0} + \frac{(s-s_0)^2}{2}\left(\frac{d^2\mathbf{q}}{ds^2}\right)_{s=s_0} + \frac{(s-s_0)^3}{6}\left(\frac{d^3\mathbf{q}}{ds^3}\right)_{s=s_0} + \mathbf{R} \tag{2.42}$$

Note that $\mathbf{R}$ is a remainder vector consisting of components along $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$. Furthermore,

$$\left(\frac{d\mathbf{q}}{ds}\right)_{s=s_0} = \mathbf{t}, \qquad \left(\frac{d^2\mathbf{q}}{ds^2}\right)_{s=s_0} = \kappa\mathbf{n}, \qquad \left(\frac{d^3\mathbf{q}}{ds^3}\right)_{s=s_0} = -\kappa^2\mathbf{t} - \kappa\tau\mathbf{b} + \kappa'\mathbf{n}. \tag{2.43}$$

Substituting the above expressions in Eqn. (2.42) and assuming that $s_0 = 0$ and $\mathbf{q}(s_0)$ coincides with the origin of the $\mathbf{t} - \mathbf{n} - \mathbf{b}$ coordinate system;

$$\mathbf{q}(s) = \mathbf{q}(0) + \left(s - \frac{\kappa^2 s^3}{6}\right)\mathbf{t} + \left(\frac{s^2\kappa}{2} + \frac{\kappa' s^3}{6}\right)\mathbf{n} - \left(\frac{s^3\kappa\tau}{6}\right)\mathbf{b} + \mathbf{R} \tag{2.44}$$

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_t \\ \mathbf{q}_n \\ \mathbf{q}_b \end{bmatrix} = \left[s - \frac{\kappa^2 s^3}{6} + \mathbf{R}_x \quad \frac{s^2\kappa}{2} + \frac{\kappa' s^3}{6} + \mathbf{R}_y \quad - \frac{s^3\kappa\tau}{6}\mathbf{b} + \mathbf{R}_z \right]^T \tag{2.45}$$

Equation (2.45) gives the coordinates of the points on the curve in the local neighborhood of a given point. One can transfer the local coordinates $\mathbf{q}_t$, $\mathbf{q}_n$ and $\mathbf{q}_b$ to the global coordinates x, y and z using the transformation between $t - n - b$ and $x - y - z$ axes. The process can be repeated by taking the end point of the local neighborhood of s_0 as the new starting point.

2.4.3 Solution Approach Based on Relative Curvature and Relative Torsion

Pal (1978b) has outlined a computational procedure for designing a three-dimensional curve between two prespecified points as well as the tangents at these end-points. It should be noted that this approach solves the problem of curve design as a boundary value problem as compared to the earlier two approaches of Struik and Do Carmo which were initial value problems. A second feature of Pal's procedure is the use of a series of piecewise continuous linear segments to model the rate of change of curvature and torsion. It also should be noted that Pal's formulation is valid in the neighborhood of a given point.

Consider the generic point P in the neighborhood of a given point A where the moving trihedron is denoted by **t** , **n** and **b**. The location of point P can be specified by

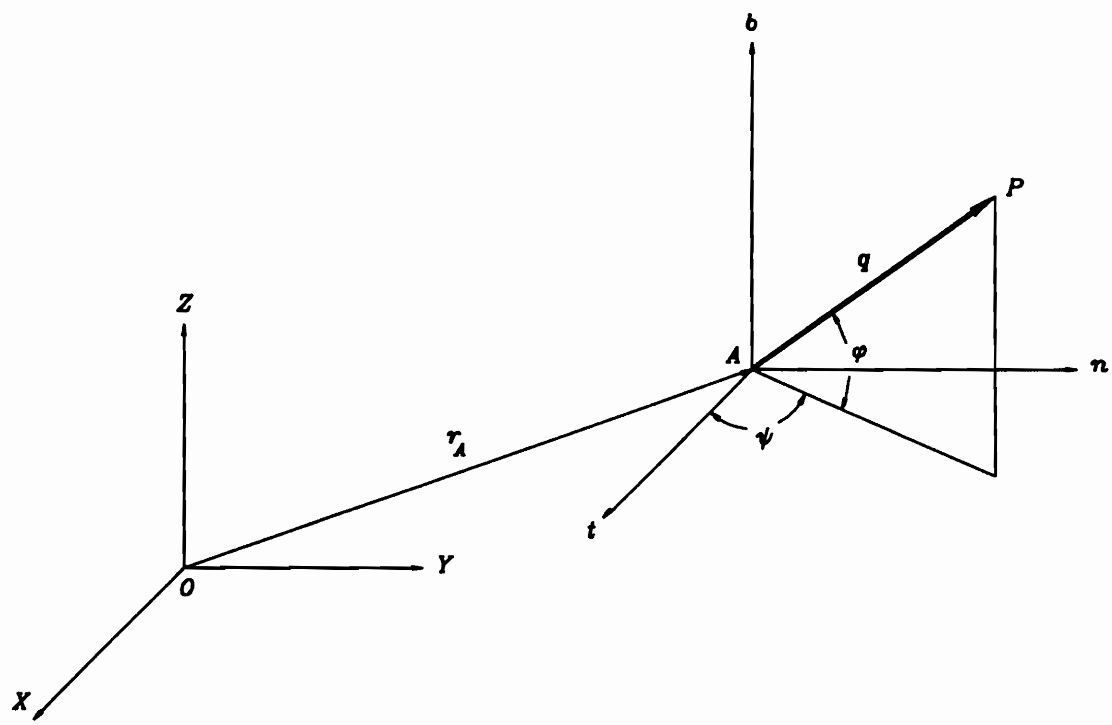


Figure 2.3. Local Geometry of a Curve Defined by Pal (1978b).

means of azimuthal angle ψ and the latitude angle ϕ as shown in Fig. 2.3. If functional variation of the rate of change of the angles ϕ and ψ are specified as piecewise continuous linear segments, then the following relation will hold.

$$\mathbf{q} = AP = (\cos\psi \ \cos\phi)\,\mathbf{t} + (\sin\psi \ \cos\phi)\,\mathbf{n} + (\sin\phi)\,\mathbf{b} \tag{2.46}$$

where: $\psi = \int_0^s \psi' ds$, $\phi = \int_0^s \phi' ds$ and

$$\mathbf{r} = \mathbf{r}_A + [M]\mathbf{q} \tag{2.47}$$

$$\mathbf{q} = \mathbf{q}_A + \left(\int_0^s \cos\psi \cos\phi ds\right)\mathbf{t} + \left(\int_0^s \sin\psi \cos\phi ds\right)\mathbf{n} + \left(\int_0^s \sin\phi ds\right)\mathbf{b} \tag{2.48}$$

Pal has outlined the details of the numerical integration scheme for obtaining the global Cartesian coordinates of a curve.

2.4.4 Iterative Numerical Solution

Adams (1975) uses a numerical iteration technique applied to the Serret-Frenet equations in order to generate space curves. The required information is the end-point conditions including the coordinates and the tangent vectors. Furthermore, the intrinsic properties of curvature and torsion as a function of arc length are also assumed to be specified. An iterative algorithm has been developed by Adams to establish the slopes of the two linear elements of the curvature and arc length profile and also the slopes of the torsion and arc length profile. The feasibility of this method was addressed by Adams even though no specific numerical example was presented.

Consider a pair of points A and B. It is required to define a curve such that the tangent at point A is along the $x-$ axis. The tangent at A and B is specified by means of angles ψ_f and ϕ_f shown in Fig. 2.4. By using the definition of linear curvature and torsion segments, it can be seen that the area under the $\kappa - s$ curve and $\tau - s$ curve should be equal to ψ_f and ϕ_f respectively. Adams employs an iterative technique to establish the corner points P and Q by evaluating the values of s_2 and s_3 respectively. Once the intrinsic profile is established, then the Serret- Frenet equations are used to find the Cartesian coordinates of the points lying on the curve. If the curve doesn't pass through the other end point B then the value of the end point tangent at point A is scaled appropriately and then the Serret-Frenet equations are integrated one more time.

2.5 *Intrinsic Geometry of Surfaces*

Consider a biparametric surface Σ shown in Fig. 2.5. A generic point p on this surface can be described by the radius vector $\mathbf{r}(u, v)$ where the parameters u and v vary between 0 and 1.

$$\mathbf{r}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix} \qquad 0 \le u, v \le 1 \tag{2.49}$$

The intrinsic properties of a surface patch can be described in terms of the unit normal vector at point P, the tangent plane at point P and the principal curvatures at point P. The intrinsic qualities of a surface in the neighborhood of a given point can

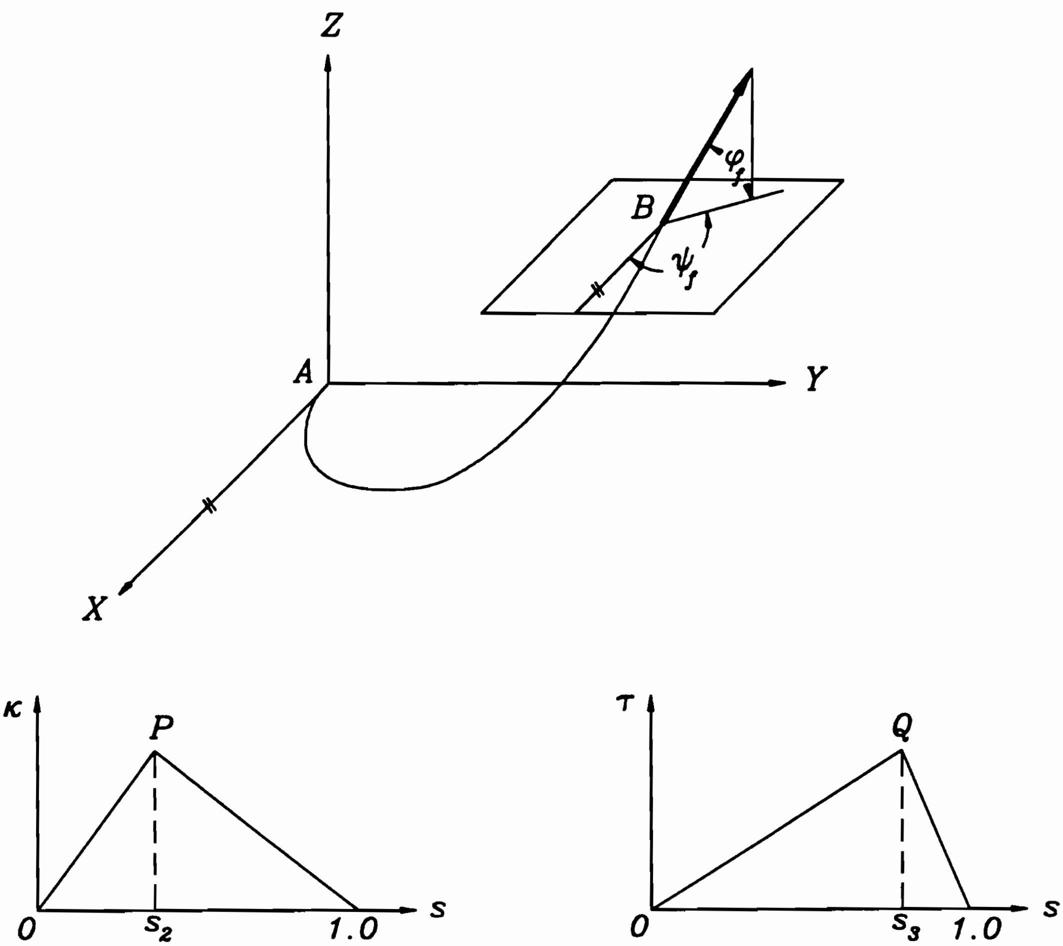


Figure 2.4. Cartesian Geometry, Curvature Profile and Torsion Profile of a Three-dimensional Curve Defined by Adams (1975).

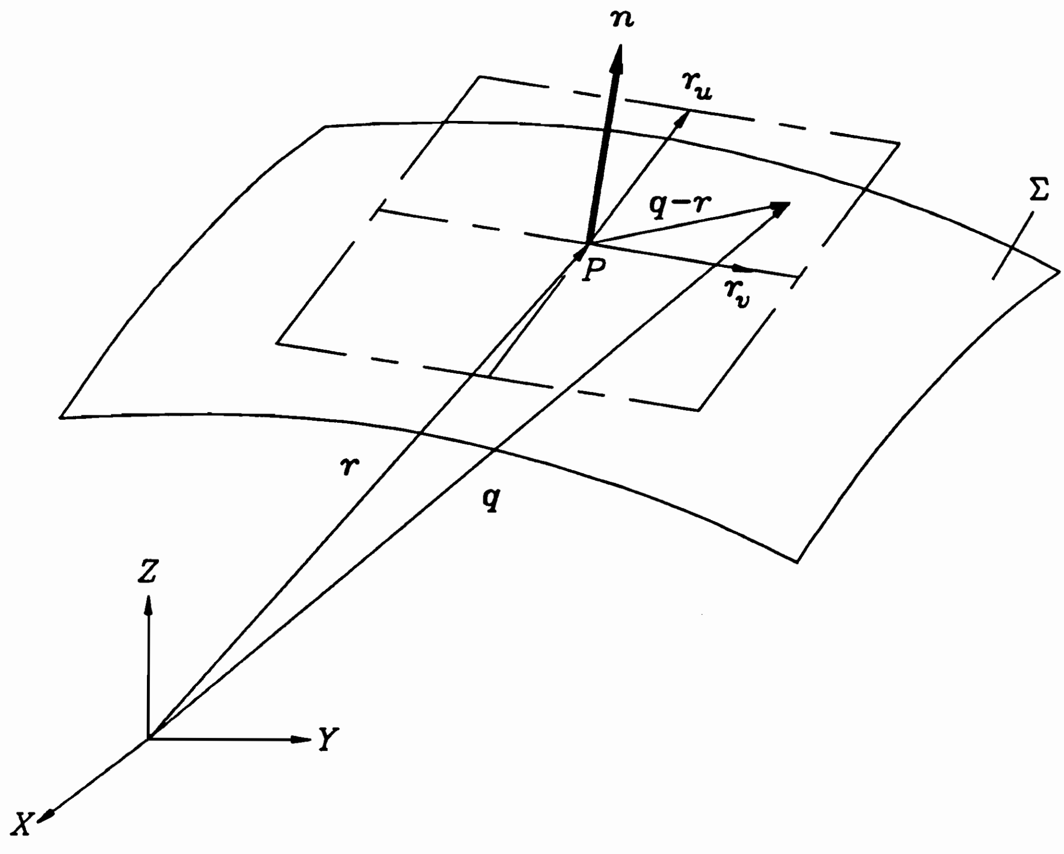


Figure 2.5. Tangent Plane and Normal of a Biparametric Surface.

also be described by the coefficients of the first and the second fundamental forms denoted by Form I and Form II respectively (Mortenson, 1985). A brief description of these intrinsic properties of a surface is given below.

The unit normal vector to the surface at point P is defined as

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}. \tag{2.50}$$

The tangent plane passing through the point P can be described by the following equation.

$$(\mathbf{q} - \mathbf{r}) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0 \tag{2.51}$$

where $\mathbf{q}$ is the position vector of a generic point lying on the tangent plane.

In order to define the curvature of a surface, it is necessary to define a specific curve lying on the surface and passing through the given point P. Let us consider a plane Π perpendicular to the tangent plane and passing through the normal $\mathbf{n}$. The curve of intersection between the surface Σ and the plane Π is denoted as σ shown in Fig. 2.6. The curvature of the curve σ at point P is defined as the normal curvature κ_n of the surface Σ at point P.

If there is a curve σ' obtained as the curve of intersection of an inclined plane Π' with the surface Σ, then the curvature of the curve σ' is denoted by κ. The curvature vectors $\boldsymbol{\kappa}_n$ and $\boldsymbol{\kappa}$ are related by the following equation.

$$\boldsymbol{\kappa}_n = (\boldsymbol{\kappa} \cdot \mathbf{n})\mathbf{n} \tag{2.52}$$

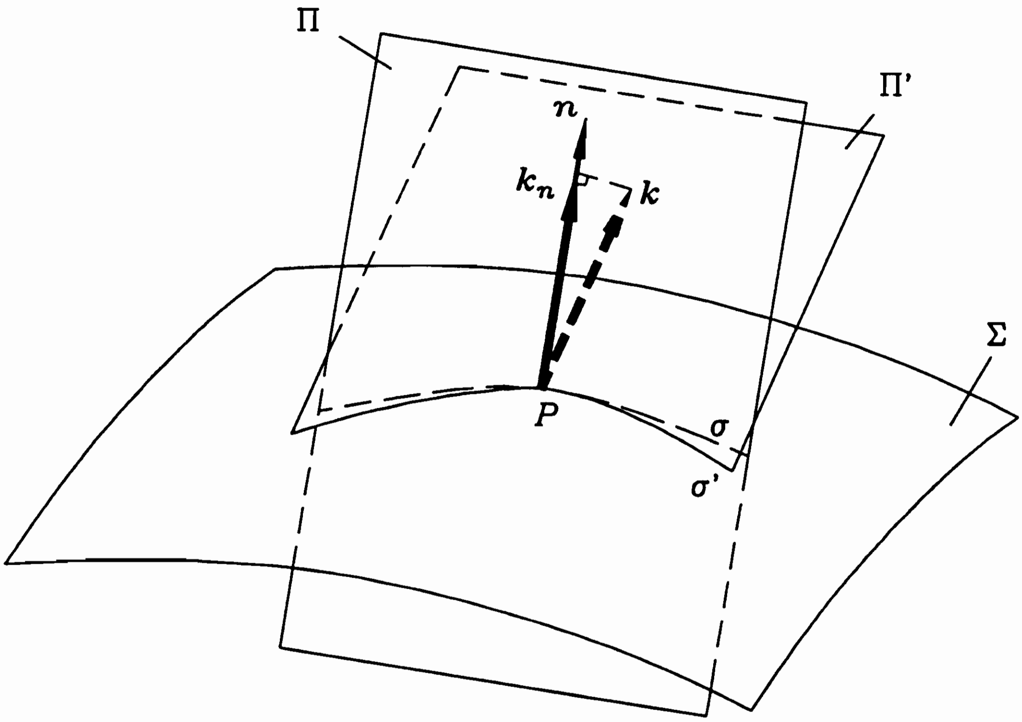


Figure 2.6. Normal Curvature of a Biparametric Surface.

Now consider the effect of the variation of the curvature κ_n as the orientation of the normal plane changes. Let α denote a reference angle in the tangent plane for describing the orientation of the plane Π. As α varies from 0 to 2π, the curvature κ_n changes its magnitude. The direction of the curvature vector $\boldsymbol{\kappa}_n$ is along $\mathbf{n}$ for all values of α. It has been shown in the literature that within the range of 0 to 2π the curvature κ_n has a maximum and a minimum value at values of α, say α^* and $\alpha^* + \frac{\pi}{2}$ (Struik, 1950). The extremum values of curvature are termed as the principal curvatures and the principal curvatures are denoted as κ_I and κ_{II}. It can also be mentioned that the curvature of a surface is a tensor and analogous to a quantity such as the stress at a point. In short, the properties such as the unit normal vector $\mathbf{n}$, the principal curvatures κ_I, κ_{II} and the angle α^* uniquely characterize the intrinsic geometry of the surface Σ at point P.

The intrinsic properties of the surface can also be described by means of the coefficients E, F, G and L, M, N where:

$$\begin{aligned} E &= \mathbf{r}_u \cdot \mathbf{r}_u \\ F &= \mathbf{r}_u \cdot \mathbf{r}_v \\ G &= \mathbf{r}_v \cdot \mathbf{r}_v \end{aligned} \tag{2.53}$$

and

$$\begin{aligned} L &= \mathbf{r}_{uu} \cdot \mathbf{n} \\ M &= \mathbf{r}_{uv} \cdot \mathbf{n} \\ N &= \mathbf{r}_{vv} \cdot \mathbf{n} \end{aligned} \tag{2.54}$$

These coefficients are used to express the well known first and second fundamental forms as follows:

$$\begin{aligned} \text{Form I } &= d\mathbf{r}.d\mathbf{r} = E\,du^2 + 2F\,du\,dv + G\,dv^2 \\ \text{Form II } &= -\,d\mathbf{r}.d\mathbf{n} = = L\,du^2 + 2M\,du\,dv + N\,dv^2 \end{aligned} \tag{2.55}$$

Furthermore, it has been shown that the principal curvatures are the roots of the following equation (Mortenson, 1985).

$$(EG - F^2)\,\kappa^2 - (EN + GL - 2FM)\,\kappa + (LN - M^2) = 0 \tag{2.56}$$

where E, F, G and L, M, N are the coefficients described in Eqns. (2.53) and (2.54).

The foregoing description about the intrinsic properties of a surface needs to be taken into account for designing or synthesizing the geometry using the intrinsic properties. Although little work has been reported in terms of developing a computational geometry technique for designing a surface using the intrinsic properties, it has been found appropriate to use the intrinsic properties for describing surfaces such as fabric drape. A detailed presentation of this approach is given in Chapter 5.

• • •

3. Shape Design of Planar Curves

3.1 Statement of Problem

Let us again consider the problem of defining a curve passing through two points $P_0(x_0, y_0)$ and $P_n(x_n, y_n)$ in a two-dimensional space already presented in Section 2.3. Figure 3.1 shows Fig. 2.2 of Chapter 2 for convenience. The directions of tangents at P_0 and P_n have been specified as ψ_0 and ψ_n and the arc lengths from a reference point as s_0 and s_n respectively. κ is the curvature at any point P. It is assumed that the variation of κ as a function of the arc length (i.e. the parameter $\kappa = \kappa(s)$) has been specified. Note that $\kappa(s)$ defines the shape of the curve.

The design of a 2-D curve passing through the given initial and final points with the specified end tangents can be accomplished using the following steps. (1) Define a model for the curvature as a function of the arc length. This step is called defining the shape model. (2) Based on the selected shape model, define a set of dependent or independent

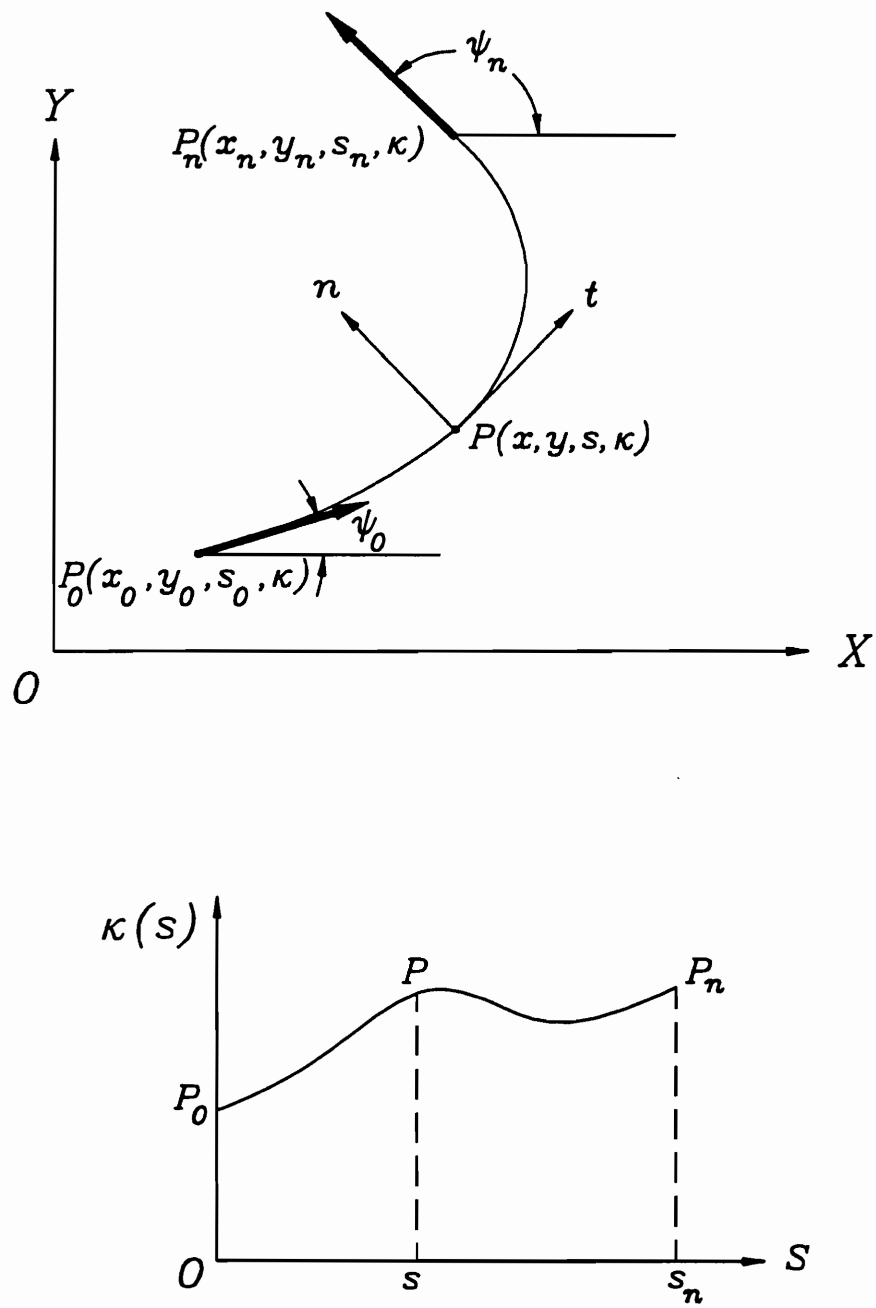


Figure 3.1. Curve Definition Using Cartesian Coordinates and Tangent Angles.

shape variables. These variables are a combination of curvatures and arc lengths at given points along the curvature versus arc length curve. (3) Knowing the initial and final coordinates and tangent angles, solve the equations (2.28), (2.29) and (2.30) of Chapter 2 simultaneously for the dependent shape variables. (4) Once the curvature function is known, the above equations are used to generate the Cartesian coordinates of the 2-D curve and its tangent angle as a function of the arc length.

3.2 *Shape Models*

In this section, linear equations are used as shape models. Let the curvature $\kappa(s)$ be defined as a series of piecewise continuous linear functions with curvatures κ_0 and κ_n at the initial and final points (Fig. 3.2). The intrinsic equation can be written as:

$$\kappa(s) = \left(\frac{\kappa_1 - \kappa_0}{s_1 - s_0} \right)(s - s_0) + \kappa_0 \qquad s_0 \leq s \leq s_1$$

$$\kappa(s) = \left(\frac{\kappa_2 - \kappa_1}{s_2 - s_1} \right)(s - s_1) + \kappa_1 \qquad s_1 \leq s \leq s_2 \tag{3.1}$$

or in general:

$$\kappa(s) = \left(\frac{\kappa_n - \kappa_{n-1}}{s_n - s_{n-1}} \right)(s - s_{n-1}) + \kappa_{n-1} \qquad s_{n-1} \leq s \leq s_n$$

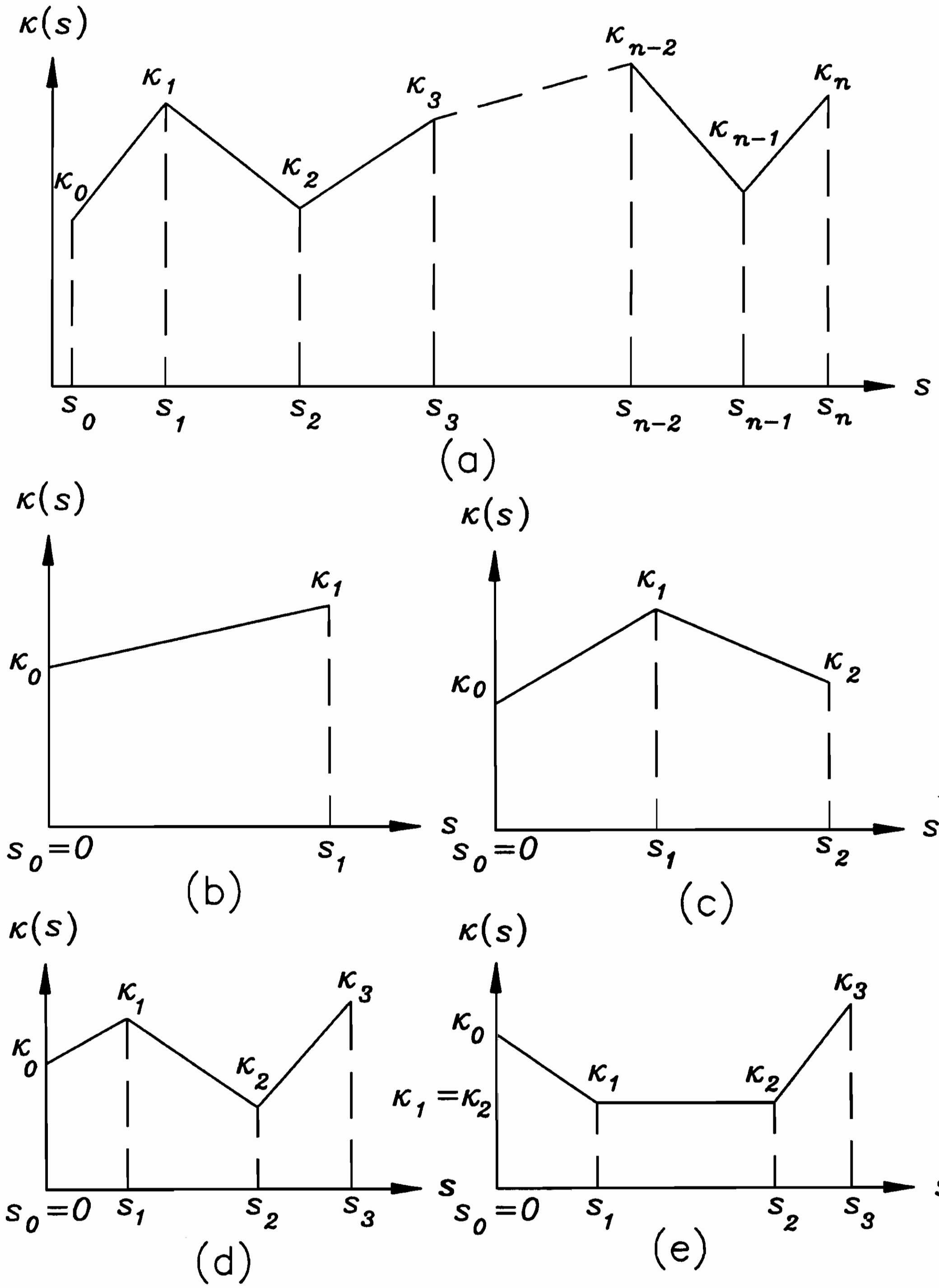


Figure 3.2. Shape Models (a) Generalized n-R Model (b) R Model n = 1, l = 0 (c) R-R Model n = 2, l = 0 (d) R-R-R Model n = 3, l = 0 (e) R-R-R Model n = 3, l = 1.

Each linear intrinsic segment corresponds to a spiral in Cartesian coordinates. The area under the curvature equation represents the change in the tangent angles of the initial and final points of the 2-D curve. Therefore:

$$\psi_n - \psi_0 = \left(\frac{\kappa_0 + \kappa_1}{2}\right)(s_1 - s_0) + \left(\frac{\kappa_1 + \kappa_2}{2}\right)(s_2 - s_1) + \cdots + \left(\frac{\kappa_{n-1} + \kappa_n}{2}\right)(s_n - s_{n-1}) \quad (3.2)$$

Upon substitution of the curvature Eqn. (3.1) in Eqn. (2.30) and subsequently in Eqns. (2.28) and (2.29) of Chapter 2 and considering the boundary conditions of the 2-D curve results in the following equations.

$$\begin{aligned} x_n - x_0 = & \int_{s_0}^{s_1} \cos\left[\left(\frac{\kappa_1 - \kappa_0}{s_1 - s_0}\right)\left(\frac{\sigma^2}{2} - s_0\sigma\right) + \kappa_0\sigma + C_1\right] d\sigma \\ & + \int_{s_1}^{s_2} \cos\left[\left(\frac{\kappa_2 - \kappa_1}{s_2 - s_1}\right)\left(\frac{\sigma^2}{2} - s_1\sigma\right) + \kappa_1\sigma + C_2\right] d\sigma \\ & + \ldots \\ & + \int_{s_{n-1}}^{s_n} \cos\left[\left(\frac{\kappa_n - \kappa_{n-1}}{s_n - s_{n-1}}\right)\left(\frac{\sigma^2}{2} - s_{n-1}\sigma\right) + \kappa_{n-1}\sigma + C_n\right] d\sigma \end{aligned} \quad (3.3)$$

and

$$
\begin{aligned}
y_n - y_0 = & \int_{s_0}^{s_1} \sin\left[\left(\frac{\kappa_1 - \kappa_0}{s_1 - s_0}\right)\left(\frac{\sigma^2}{2} - s_0\sigma\right) + \kappa_0\sigma + C_1\right] d\sigma \\
& + \int_{s_1}^{s_2} \sin\left[\left(\frac{\kappa_2 - \kappa_1}{s_2 - s_1}\right)\left(\frac{\sigma^2}{2} - s_1\sigma\right) + \kappa_1\sigma + C_2\right] d\sigma \\
& + \ldots \\
& + \int_{s_{n-1}}^{s_n} \sin\left[\left(\frac{\kappa_n - \kappa_{n-1}}{s_n - s_{n-1}}\right)\left(\frac{\sigma^2}{2} - s_{n-1}\sigma\right) + \kappa_{n-1}\sigma + C_n\right] d\sigma
\end{aligned}
\tag{3.4}
$$

where C_1 through C_n are constants of integration. Note that $C_1 = \psi_0$. The other constants can be expressed in terms of κ_i, s_i, $i = 0, \ldots, n$ and ψ_0 using the following set of equations.

$$
\begin{aligned}
C_j = C_{j-1} & + \frac{\kappa_{j-1} - \kappa_{j-2}}{s_{j-1} - s_{j-2}}\left(\frac{s_{j-1}^2}{2} - s_{j-1}s_{j-2}\right) + \kappa_{j-2}s_{j-1} \\
& + \frac{\kappa_j - \kappa_{j-1}}{s_j - s_{j-1}}\left(\frac{s_{j-1}^2}{2}\right) - \kappa_{j-1}s_{j-1}, \qquad j = 2, 3, \ldots, n
\end{aligned}
\tag{3.4.a}
$$

Examination of the right-hand side of Eqns. (3.2) through (3.4) indicates the existence of $(2n + 2)$ variables s_0 through s_n and κ_0 through κ_n. Since there are only three simultaneous equations to be satisfied, one is left with $(2n - 1)$ free parameters. Furthermore, if s_0 is assumed to be zero, then the number of free parameters reduces to $(2n - 2)$. These $(2n - 2)$ parameters can be termed as Shape Design Variables (SDV). One can select a feasible set of these $(2n - 2)$ shape design variables and evaluate the shape. In order to find the optimal shape, it will be necessary to search the feasible domain of these $(2n - 2)$ shape design variables.

The shape design process can now be thought of as a two step process. The first step involves deciding the number n . Here, one is selecting a shape model consisting of n linear segments in the intrinsic plane. Examples of shape models for $n = 1$, 2 and 3 are shown in Fig. 3.2. These models will be termed as R model, R-R model and R-R-R model for $n = 1$, 2 and 3 respectively. The second step of shape definition involves establishing the set of shape design variables for a given shape model.

The total number of the SDV's will be $(2n - 2 - l)$ where l could be the additional constraints relating the intrinsic variables κ_i's and s_i's. Notice that for $n = 1$ and $l = 0$ the number of SDV's is zero and for $n = 2$, $l = 0$ and $n = 3$, $l = 0$, the number of SDV's would be two and four respectively. Furthermore, consider an R-R-R model with $n = 3$. If an additional constraint of $\kappa_1 = \kappa_2$ is imposed, then the total number of SDV's would be 3 instead of 4, since l will be 1.

Once the total number of SDV's has been established, it is a matter of choice for the designer to select either all κ_i's or s_i's or a combination of some κ_i's and some s_i's as the shape design variables. It should be noted, however, that as the number n of linear segments increases, the feasible domain of SDV's may become smaller and difficult to evaluate. Computationally, one has to select stable schemes for solving Eqns. (3.3) and (3.4) as well as for evaluating integrals of Eqns. (2.28) and (2.29) of Chapter 2.

3.3 Algorithm

The proposed methodology of shape synthesis and optimization requires an understanding of how a shape model and its associated shape design variables can affect the Cartesian geometry of a curve. A computer program, written in FORTRAN, has been developed for shape synthesis. This shape synthesis module can be used as a subroutine in any optimization code. For shape synthesis, the type of shape model and the shape design variables are specified as the input. The program attempts to solve for the remaining variables of the curvature functions using the nonlinear equation solver NEQNF routine of the IMSL library. The numerical integration is accomplished using the QDAG routine of the IMSL library. Upon finding all the variables of the shape model, the program generates data about the curvature and the tangent angle at specified values of the arc length. As a next step, the Serret-Frenet equations are solved to obtain the Cartesian coordinates of the curve. Finally, the program uses the CA-DISSPLA package to plot the curvature, the tangent angle and the Cartesian coordinates, thus accomplishing shape synthesis. It is important to emphasize that the shape synthesis of a 2-D spiral curve connecting two arbitrary points in a plane has zero degrees of freedom. In other words, the designer has no choice in selecting any of the initial and final curvatures or the arc length between the selected points. There is a unique solution to this particular synthesis dictated by Eqns. (3.2) through (3.4).

A number of computer programs were developed on the VAX/VMS system for 2-D shape synthesis. The initial and final coordinates of a curve and the initial and final tangent angles of the corresponding points are specified. The type of shape model is selected first. The free variables are chosen depending on the shape model selected.

Subroutine NEQNF of IMSL is called to solve the three non-linear equations. NEQNF solves a system of non-linear equations using the Levenberg-Marquardt algorithm and a finite difference approximation to the Jacobian. The numerical integration routine is accomplished using QDAG of IMSL. QDAG is a general purpose integrator that uses a globally adaptive scheme based on Gauss-Kronrod rules in order to reduce the absolute error.

Upon determination of the curvature equation, the parametric equations of the 2-D curve are solved for and the Cartesian coordinates are obtained. The graphics subroutines of CA-DISSPLA are used to plot the graphics output results of the program. The curvature and the tangent angle curves are plotted as a function of the arc length. Finally the parametric coordinates $y(s)$ and $x(s)$ are plotted in the Cartesian coordinates, accomplishing the shape synthesis.

3.4 Applications

In this section, an overview of applications important to shape optimization is presented. These are only a few limited applications; the presentation is not intended to be comprehensive. It should be pointed out that the primary interest of this research is in mechanical applications involving shape synthesis and shape optimization. Nevertheless, the literature provides a large number of structural applications. Therefore, it is felt necessary to allocate a section to some of the structural applications involving shape optimization. It should also be pointed out that because of the ongoing research in the area of robotics in general, and the variable-geometry truss-type manipulators in particular, this work will be concerned with using the developed tools of shape optimization for improving the design of a VGT.

When shape synthesis is carried out using the intrinsic approach, the designer has to select (i) appropriate end conditions, (ii) a suitable shape model, and (iii) an appropriate set of independent shape design variables. This will be illustrated by means of two examples.

In Example 3.1, it is required to define a curve passing through points (0,0) and (10,10). The tangent angle at the first point is zero. The tangent angle at the end point is taken to be (i) 120 degrees, (ii) -60 degrees and (iii) 480 degrees. Moreover, for the first case, a single ramp (linear curvature) function is selected as the shape model while for the other two cases, a two ramp (two linear curvature segments) shape model is selected. The three cases produce different shapes of the curve as shown in Fig. 3.3. It should be noted that when a single ramp function is selected as a shape model, there are no free parameters. In such a case, Eqns. (3.2) through (3.4) result in a unique answer.

The selection of different free parameters is studied in Example 3.2. The R-R model is selected as the shape model in this example. The variables for this model are κ_0, κ_1, κ_2, s_1 and s_2 . Two cases are shown in Fig. 3.4. The solid curve is designed with κ_0 and κ_2 as free parameters having values 0.65 and -1.25 respectively. This causes the curvature plot to cross the s-axis producing a point of inflection. The dashed-line curve is designed with κ_1 and s_1 as free parameters having values -0.4 and 12.0 respectively. The end-point coordinates in both cases are the same as in the previous example with $\psi_0 = 0$, $\psi_2 = -70$ degrees.

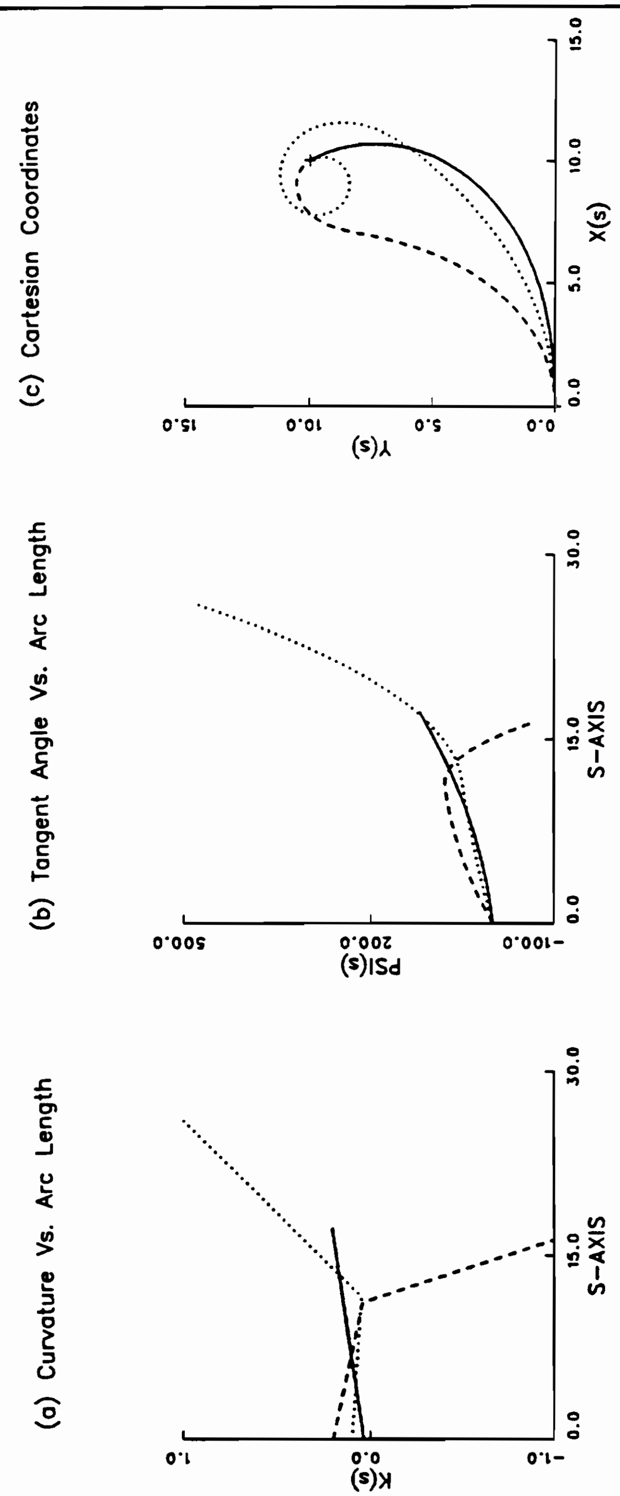


Figure 3.3. Graphical Output of κ-s, ψ-s and X-Y Curves of Example 3.1. The Solid Line is an Example of an R Model. The Other Two Curves Are Examples of R-R Models.

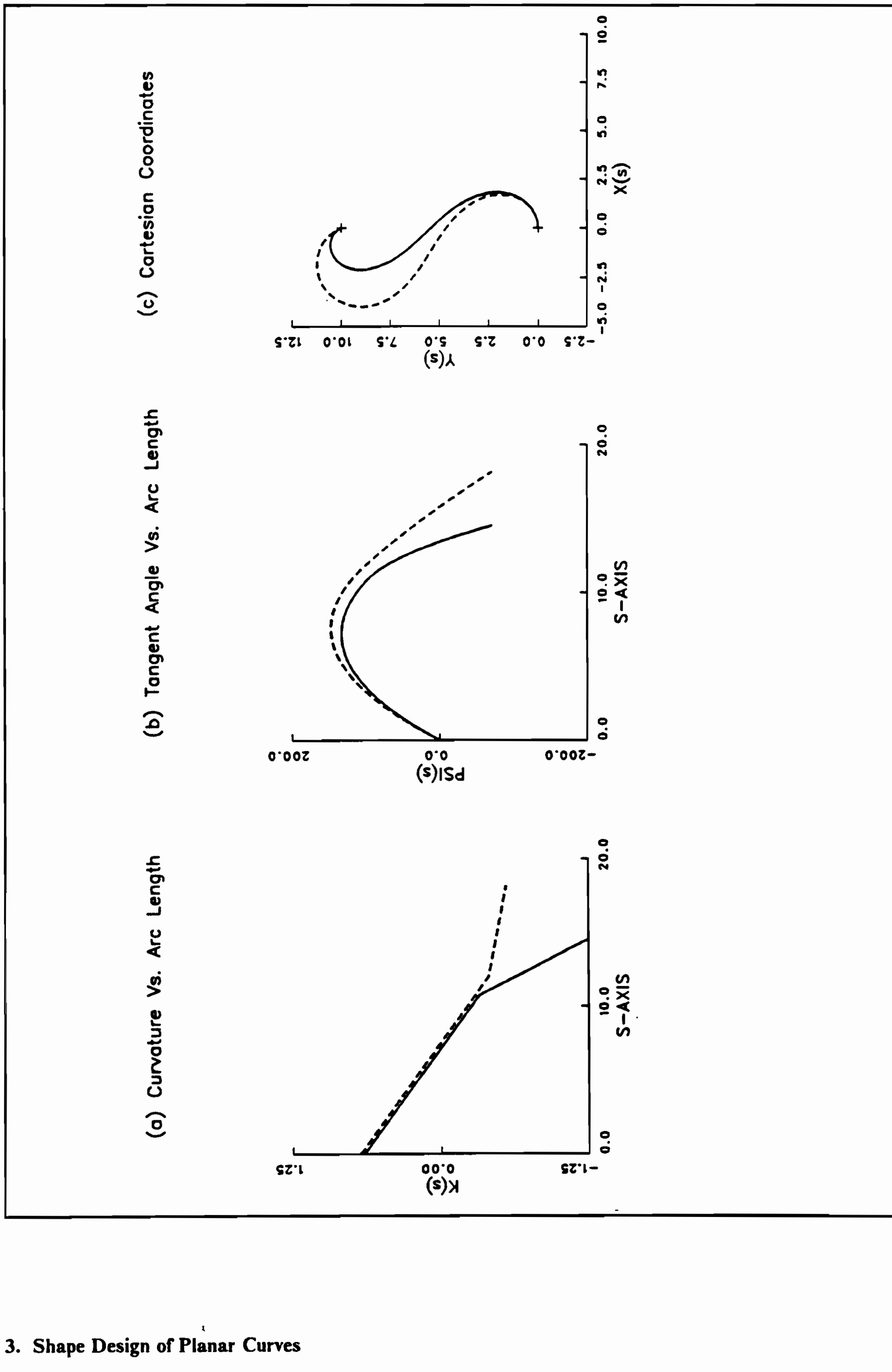


Figure 3.4. Graphical Output of κ-s, ψ-s and X-Y Curves of Example 3.2.

3.4.1 Optimal Shape of a Planar VGT

A Variable Geometry Truss (VGT) is a statically determinate structure consisting of a set of truss elements and has some extendible members (Fig. 3.5). VGT's have been used for collapsible, adaptive and actively damped structures (Reinholtz, 1987). Presently, VGT's have been explored as possible configuration for long chain robotic manipulators, antenna controllers and passive-legged crawling vehicles (Stulce, 1990). The VGT consists of a sequence of identical structural units called cells. The octahedron is widely used as a fundamental cell (Fig. 3.5). Each octahedral cell contains some variable length members. When all the cells are connected along a longitudinal axis, then a high degree of freedom variable geometry structural configuration is formed. The total number of variable length members in the chain is equal to the number of degrees of freedom of the VGT. One of the problems encountered in controlling the VGT as a robotic device concerns the design of the shape of the truss. In this case, it is required to actuate appropriately the variable-length members and achieve a pre-specified location as well as orientation of an end-effector connected to one end of the VGT assuming that the other end is fixed.

A shape control algorithm has been developed for long chain, high degree of freedom VGT manipulators (Salerno, 1988). Salerno has used parametric cubic curves in controlling the position and orientation of planar VGT manipulators. His algorithms are based on assuming a general shape for the manipulator and thus finding a suitable curve in order to attain the given end positions and orientations. Once an appropriate curve is found, the curve is partitioned using some curve partitioning such as equal arc length or equal chord length, and then the truss cells are fit along the curve. Some optimization

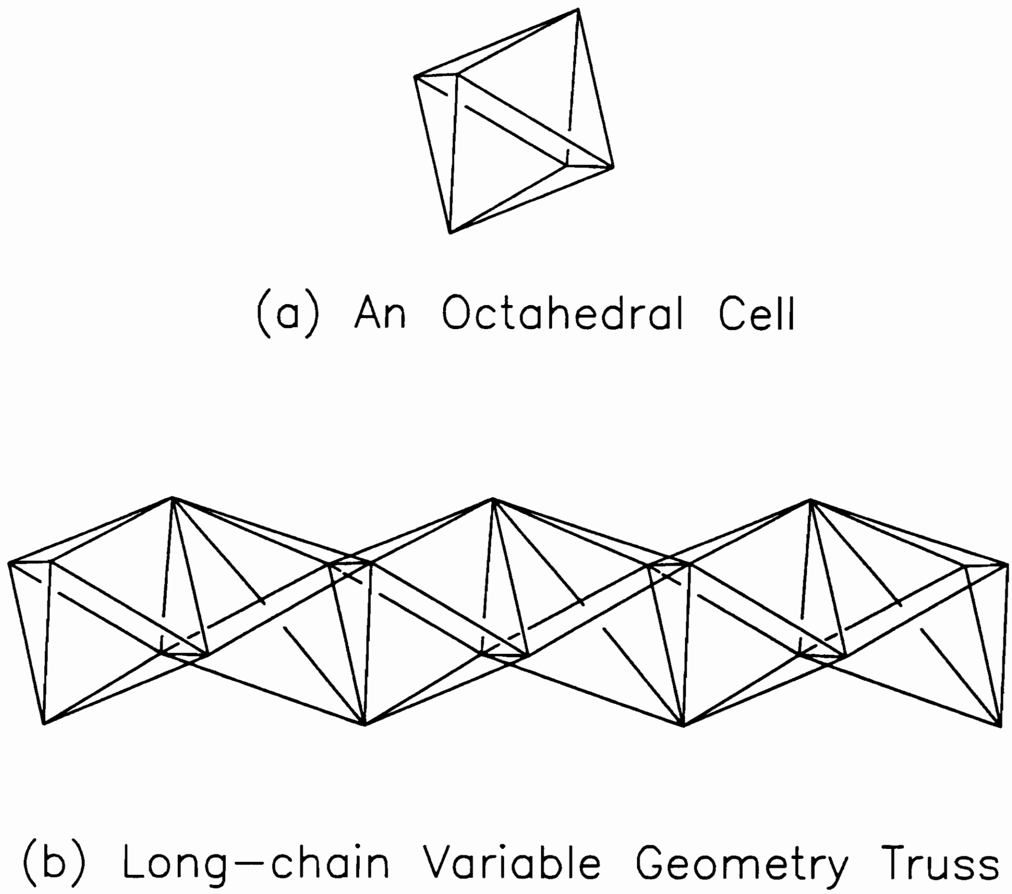


Figure 3.5. A Long-chain Variable Geometry Truss.

techniques such as tangent vector optimization and one-dimensional optimal distribution of nodal respacing of the partitioned curve are utilized in an attempt to produce cells that are equally extended.

It is important to note that in some situations the shape of a VGT is restricted by the curvature or the radius of curvature of its mean curve. The intrinsic geometry approach takes into account the curvature constraint explicitly. Consider the planar VGT as shown in Figure 3.6. The general shape of a VGT can be synthesized for given initial and final coordinates and orientations using the shape synthesis technique proposed here. Once the general shape of the VGT is determined, the 2-D curve is equally divided for a given number of bays. The VGT is then assembled for a given selected condition, and the shape design variables are varied in order to find the feasible optimum shape. The objective function is defined as:

$$W = \left[\sum_{i=1}^{n} \sum_{j=1}^{3} (L_{ij} - \overline{L}_{ij})^2 \right]^{1/2} \tag{3.5}$$

where L_{ij} 's are the lengths of the variable links in any position and $\overline{L}_{ij}$'s are the lengths of the variable links in the neutral position. Minimization of the objective function W denotes that minimum energy is expended in order to assemble the VGT in the required position. If for any i or j the following constraint is violated, then the design is infeasible and is rejected.

$$L_{ij}^{\min} \leq L_{ij} \leq L_{ij}^{\max} \tag{3.6}$$

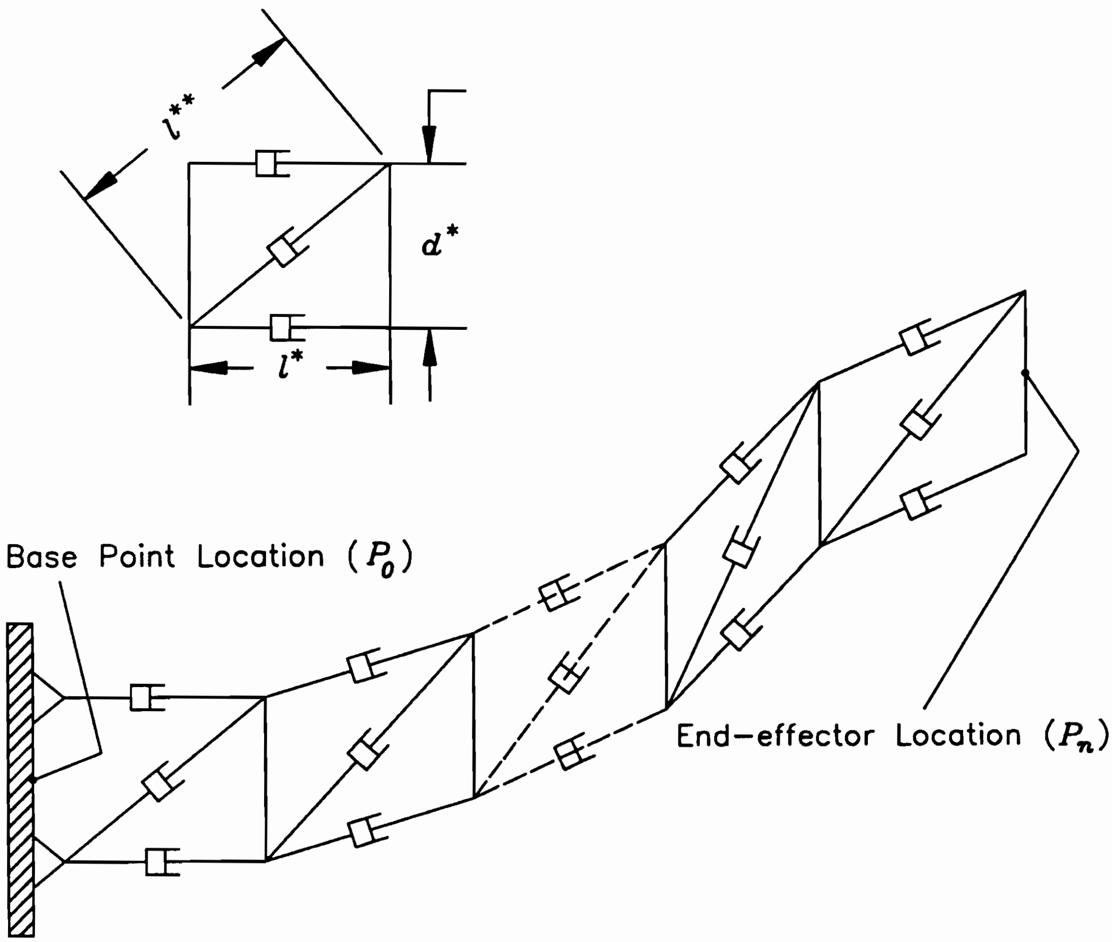


Figure 3.6. Schematic Diagram of a Planar n-Bay Variable Geometry Truss.

where $L_{ij}^{min} = 0.7143l^{*}$ and $L_{ij}^{max} = 1.2857l^{*}$ for the parallel variable links (Salerno,1988). Also $L_{ij}^{min} = 0.7143l^{**}$ and $L_{ij}^{max} = 1.2857l^{**}$ for the cross members. Values for l^{*} and l^{**} are prescribed in advance. The following are two examples of the planar VGT's in which the near optimal shapes are found through exhaustive search within a particular region of the shape design variables.

Example 3.3 is shown in Figure 3.7. In this example, the curvature is defined by two line segments which represent two spirals in the Cartesian coordinates. The initial and final coordinates are selected as (0,0) and (-10,10). The initial tangent angle is 0 and the final tangent angle is 240 degrees. The number of bays of the VGT is chosen to be 10. The fixed length of the VGT d^{*} is selected as 1.0 and l^{*} the parallel variable link length is chosen to be 2.0, both in the neutral position. κ_0 and κ_2 , the first and last curvatures are varied from (0.4,0.3) with an increment of 0.05. Table 1 on page 71 shows the results of the objective function W for different combinations of the curvatures. For the infeasible combinations, W is set to be negative since it is not acceptable. The optimum W is seen to be 1.16356 corresponding to $\kappa_0 = 0.55$ and $\kappa_2 = 0.40$. Figure 3.7(b) shows a feasible combination of $\kappa_0 = 0.40$ and $\kappa_2 = 0.35$. However, the value of the objective function is $W = 1.50783$, which is clearly not an optimum design in this case.

Example 3.4 is shown in Figure 3.8. The curvature model is selected to be a three line segment. The initial coordinates are at (0,0) with a zero degree tangent angle and the final coordinates are (10,10) corresponding to the final tangent angle of zero degree. κ_1 and κ_2 are kept constant at 0.48 and -0.48. The shape design variables are the first and last curvatures in this example. The initial values of κ_0 and κ_3 are chosen as 0.0 and 0.0 with an increment of -1.0 and 1.0 respectively. Table 2 on page 72 shows different values of W for given initial and final combinations of the curvature. Figure 3.8(b)

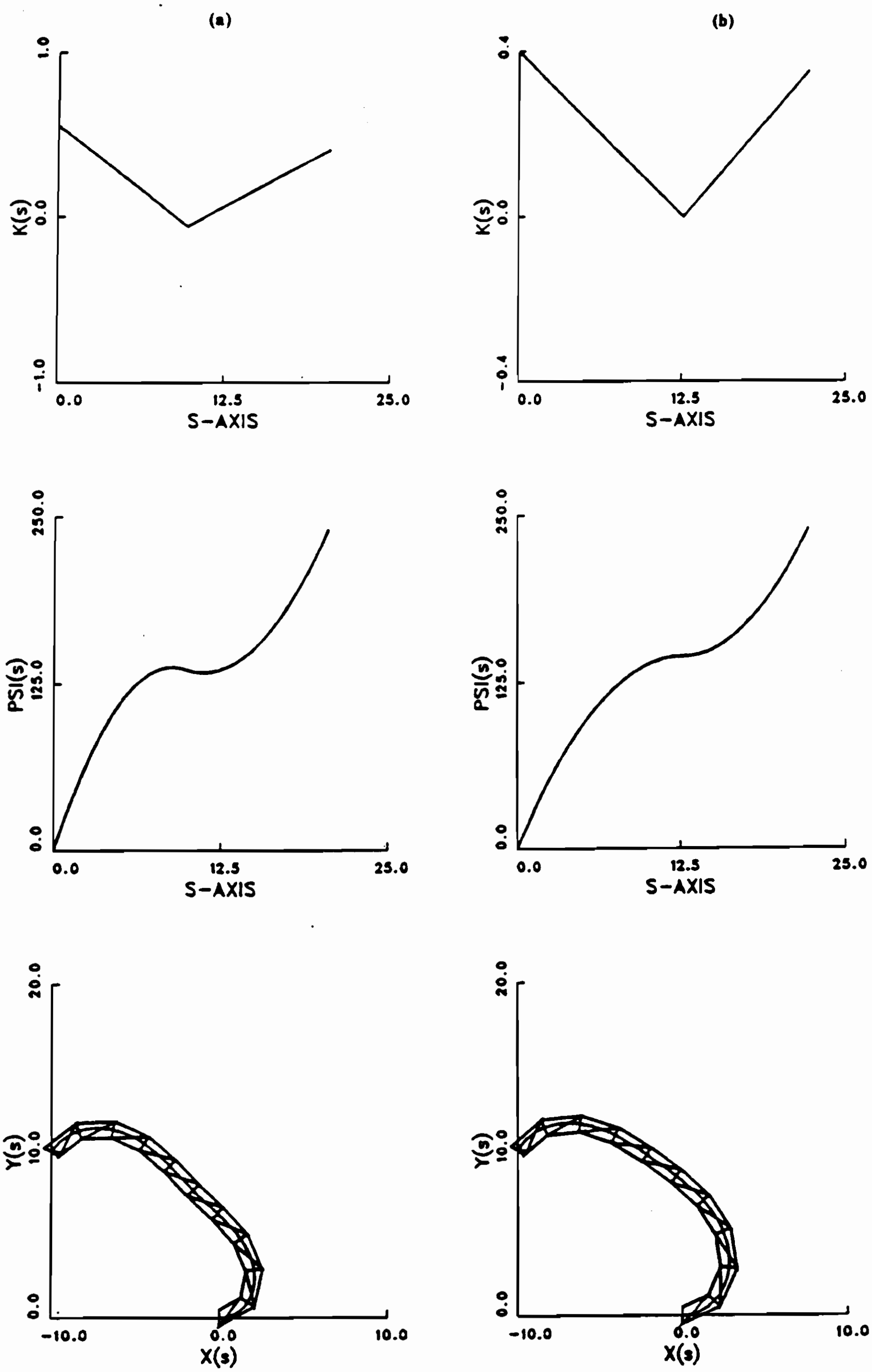


Figure 3.7. Shape Design of a Planar VGT, Example 3.3 (a) Optimal Configuration (b) Feasible Configuration.

shows a feasible design corresponding to $d^* = 1.2$, $l^* = 1.77$, $\kappa_0 = 0.0$, $\kappa_3 = 0.0$ and $W = 1.38186$. Figure 3.8(a) shows the optimal configuration with $W = 1.10145$, $\kappa_0 = -3.0$, and $\kappa_3 = 3.0$.

3.5 *Summary*

The present chapter is an attempt to show how shape synthesis as well as shape optimization can be carried out using concepts of intrinsic geometry. The shape synthesis of a two-dimensional curve can be accomplished by solving the three Eqns. (3.2), (3.3) and (3.4) provided that the curvature is given as a function of the arc length. The curvature can be represented using one of the proposed shape models. The shape models consist of a number of linear curvature segments. As the number of linear segments increases, the number of shape design variables and hence the complexity of the optimization problem, also increases. The linear models work well for engineering design problems tested to date.

Moreover, the intrinsic geometry approach seems to be able to provide direct control for manipulation of the shape of a curve. The Bezier and B-spline representations do not provide the capability to control the intrinsic properties of a curve directly. The intrinsic geometry approach, however, requires mapping the shape description from the intrinsic frame to the Cartesian frame. This requires solving a set of three non-linear equations and also a set of two coupled second-order differential equations. In the present case, the subroutines NEQNF and QDAG of IMSL library were used for numerical analysis work. In order to be able to use the intrinsic parameters as the shape

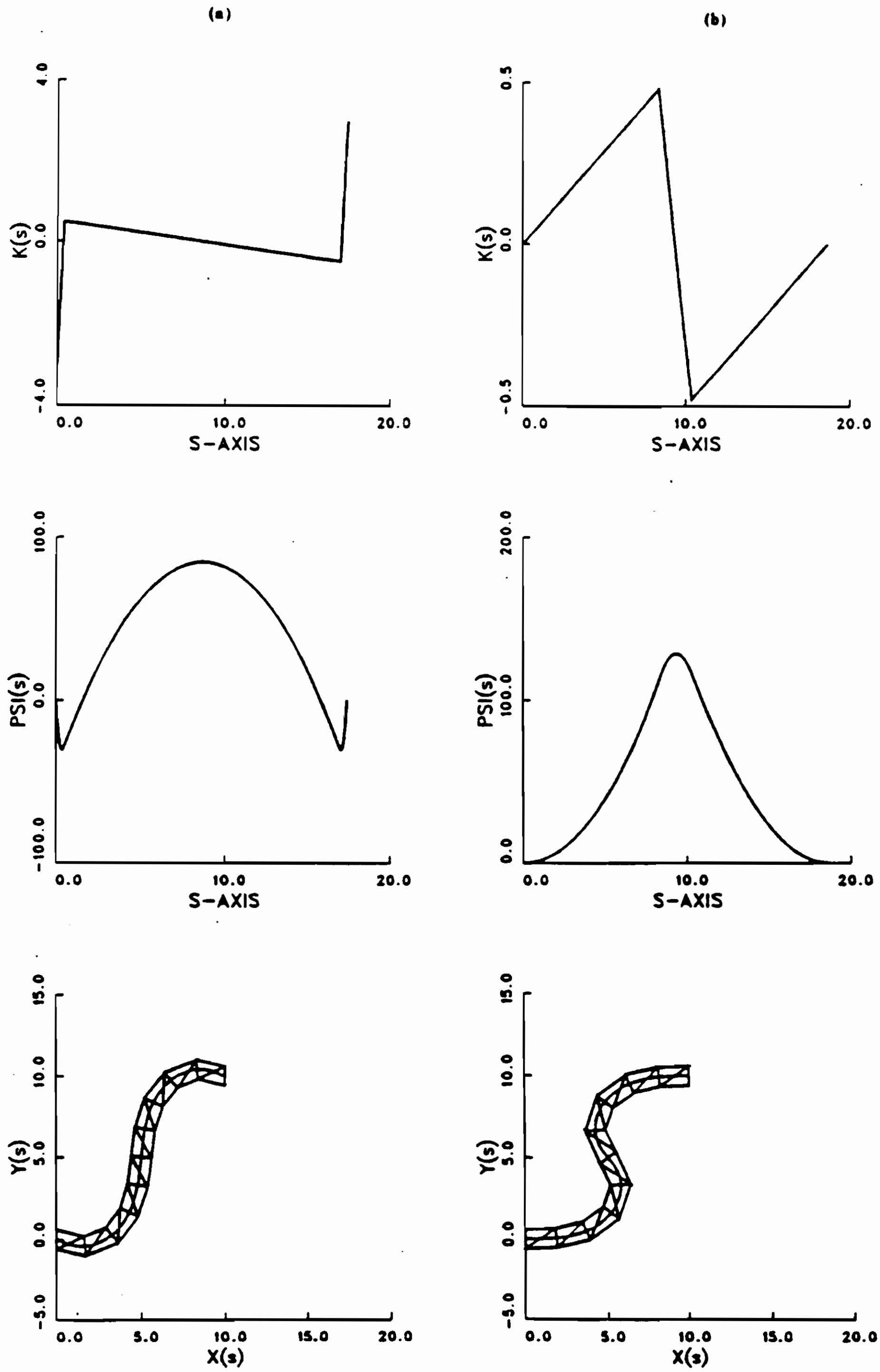


Figure 3.8. Shape Design of a Planar VGT, Example 3.4 (a) Optimal Configuration (b) Feasible Configuration.

design variables in any first-order or second-order optimization scheme, it is necessary to evaluate the sensitivity of the Cartesian coordinates with respect to the shape design variables. In the present work, a zero-order optimization method was used. It required the evaluations of constraints only. The optimization scheme selection was based on the nature of constraints and the nature of the objective function. In general, it is felt that the evaluation of gradients will be helpful. The objective function is now related to the intrinsic shape design variables instead of expressing it as a function of the Cartesian coordinates of the vertices of a control polygon; the latter being the case for spline-like shape definitions. The present investigation has shown that, once the mapping of geometry from the intrinsic plane to the Cartesian plane is completed, the evaluation of the objective function is relatively straight forward.

Any optimization methodology can be considered as a useful design technique if it can be applied to a broad spectrum of applications. In many cases, the evaluation of constraints may require the structural analysis of the component. Such an analysis is usually carried out using the FEM/BEM analysis techniques. These techniques require the geometry information, in terms of Cartesian parameters, as an input. It is necessary to investigate how the definition of geometry in terms of intrinsic parameters can be related to the FEM/BEM techniques.

• • •

Table 1

Values of the Objective Function, W, in the Design Space for Example 3.3

κ_0 \ κ_2	0.30	0.35	0.40	0.45	0.50
0.40	-1.67645	1.50783	1.40609	1.34587	1.30918
0.45	-1.47528	1.31875	1.24060	1.20449	1.19397
0.50	1.34791	1.21861	1.17210	1.16920	1.18606
0.55	1.27061	1.17387	1.16356	1.19097	1.23352
0.60	1.22515	1.16400	-1.18751	-1.24078	-1.30269

Table 2

Values of the Objective Function, W, in the Design Space for Example 3.4

κ_0 \ κ_3	0.0	1.0	2.0	3.0	4.0	5.0
0.0	1.38186	***	***	***	***	***
-1.0	***	1.13165	1.13825	1.14810	1.15374	1.15719
-2.0	***	1.13826	1.10163	1.10204	1.10316	1.10409
-3.0	***	1.14810	1.10201	1.10145	1.10225	1.10303
-4.0	***	1.15370	1.10316	1.10228	1.10307	1.10380
-5.0	***	1.15723	1.10406	1.10308	1.10385	1.10459

*** - For these values of κ_0, κ_3, a feasible curve satisfying constraints of Eqns. (3.2), (3.3) and (3.4) does not exit. Hence, the objective function W is not evaluated.

4. Shape Design of Spatial Curves

4.1 Statement of Problem

Consider the problem of defining a space curve passing through two end-points A and B (Fig. 4.1). Let $\mathbf{t_A}$ and $\mathbf{t_B}$ be two unit tangent vectors at A and B respectively. Let CD be the common perpendicular between $\mathbf{t_A}$ and $\mathbf{t_B}$ along the skew direction $A' - A''$. The common perpendicular is the shortest distance between two non-intersecting lines which is measured along the one and only one perpendicular that intersects both lines (French and Vierck, 1970). It is desired to create a space curve with known end-point coordinates and end-point tangents. The desired space curve is proposed to be a generalized helix having a base on the plane normal to the skew direction and a rise along the skew direction. The methodology of Chapter 3 is used in order to create the base curve of this generalized helix on its normal plane. The desired space curve will be defined in three steps. (i) Design a two dimensional curve in a plane perpendicular to $A' - A''$ containing $\mathbf{t_A}$. (ii) Design a two dimensional curve for defining the rise of the

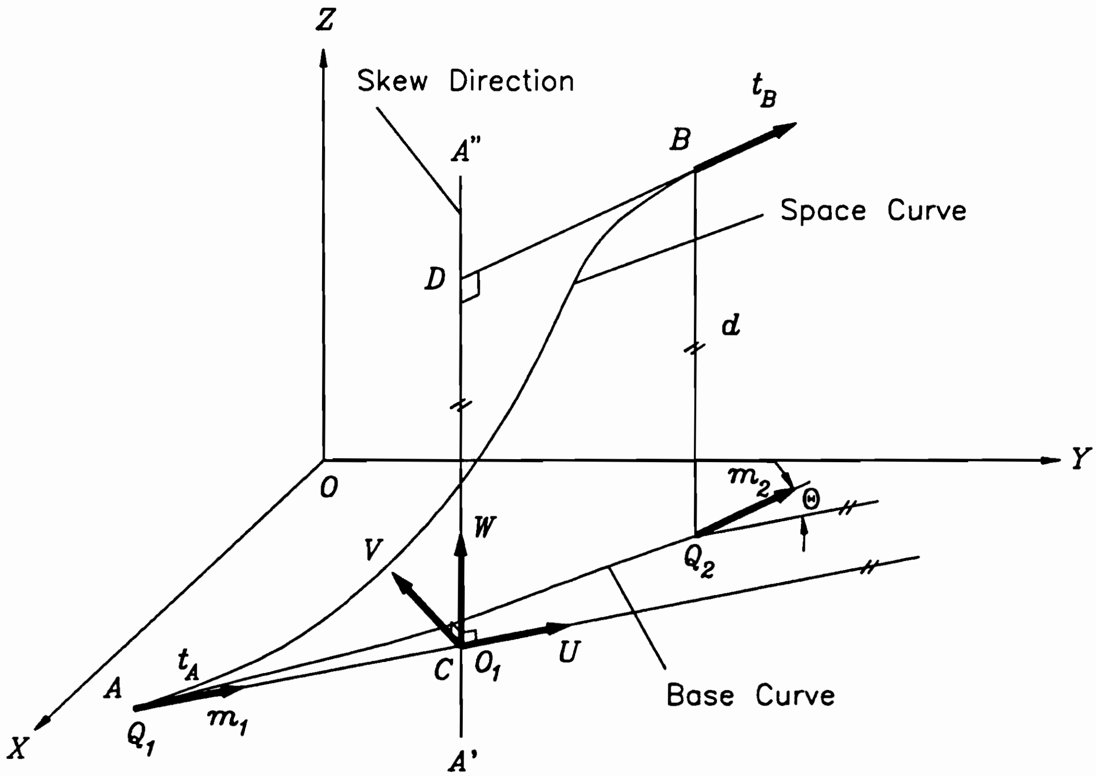


Figure 4.1. A Schematic Diagram of Input Conditions Skew Distance, Skew Direction and Base Helix Curve.

helix as a function of the arc length of the base curve. (iii) Use the definitions of these two curves to define the desired 3-D curve.

4.2 Proposed Approach

Construct a local coordinate system $O_1 - UVW$ such that O_1 is coincident with C, the U - axis is along the direction $\mathbf{t_A}$, the W - axis is along the direction $A' - A''$, and the V - axis is such that $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ (Fig. 4.1). Let Q_1 and Q_2 be the projection of the end-points A and B in the $O_1 - UV$ plane and let $\mathbf{m_1}$ and $\mathbf{m_2}$ be the projection of the unit tangent vectors $\mathbf{t_A}$ and $\mathbf{t_B}$ on the UV plane. Note that $\mathbf{t_A} \equiv \mathbf{m_1}$. The angle that $\mathbf{m_2}$ makes with $\mathbf{m_1}$ is denoted as θ where $\theta = \cos^{-1}(\mathbf{m_1} \bullet \mathbf{m_2})$. To begin with, it is necessary to define a planar curve with specified end-point coordinates and tangent vectors in the UV plane. This can be accomplished using the method proposed in Chapter 3. It consists of defining a set of linear curvature segments which will satisfy end-point geometry constraints (Fig. 4.2).

The method requires selection of values for a set of shape design variables. These shape design variables allow a designer the flexibility of manipulating the shape of the curve subject to the functional constraint or requirements. In general, a two segment linear curvature element model seems to be a good choice. The values of the u, v coordinates of the planar curve can be obtained by integrating the following equations.

$$\frac{d^2u}{ds'^2} + \kappa_{2D}(s')\,\frac{dv}{ds'} = 0 \tag{4.1}$$

$$\frac{d^2v}{ds'^2} - \kappa_{2D}(s')\,\frac{du}{ds'} = 0 \tag{4.2}$$

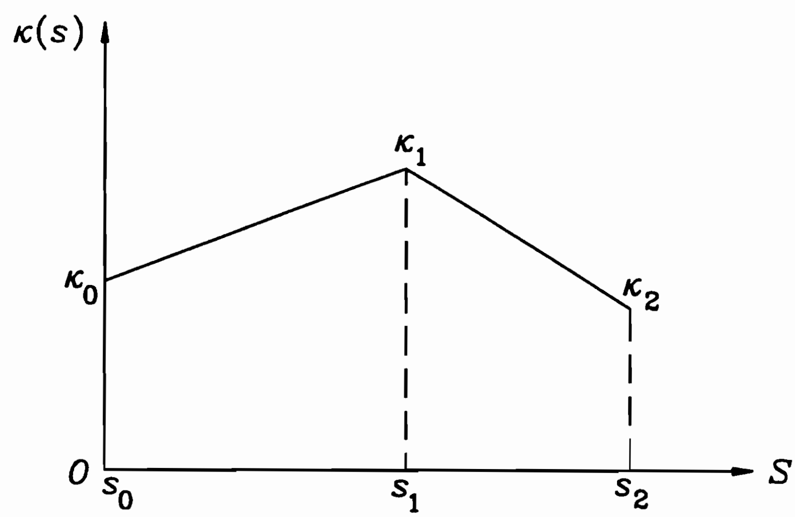


Figure 4.2. Linear Curvature Element Shape Model.

where $\kappa_{2D}(s')$ is the curvature of the planar curve in the $\kappa - s'$ plane.

Once the base curve is defined, it is necessary to describe the rise of the helix as a function of s', the arc length along the base curve. It should be noted that the rise of the helix is along the W-axis. In other words, it is now necessary to define another curve in the $W - S'$ plane. The two end-points of this curve corresponding to points A and B are $(s_A,\ 0)$ and $(s_B,\ d)$ where d is the skew distance between $\mathbf{t}_A$ and $\mathbf{t}_B$. Furthermore, this rise curve should be tangential to the horizontal S'-axis at both the end-points A and B. Without loss of generality, it can be assumed that $s_A = 0$ and $s_B = s^*$, where s^* is the total arc length of the base curve.

One can conceive the geometry of this rise curve in several different ways. These proposed approaches, which are computationally efficient, are described below.

- Parabolically Blended Curve
- Cubically Blended Curve
- Cubic Bezier Curve

Parabolically Blended Curve

Figure 4.3 shows the schematic concept of a parabolically blended curve. It consists of two parabolic segments at both end-points A and B and a linear segment in the middle. It should be emphasized that end-point tangents at A and B must be parallel to the S'-axis. The equations of three segments of Fig. 4.3 can be written as:

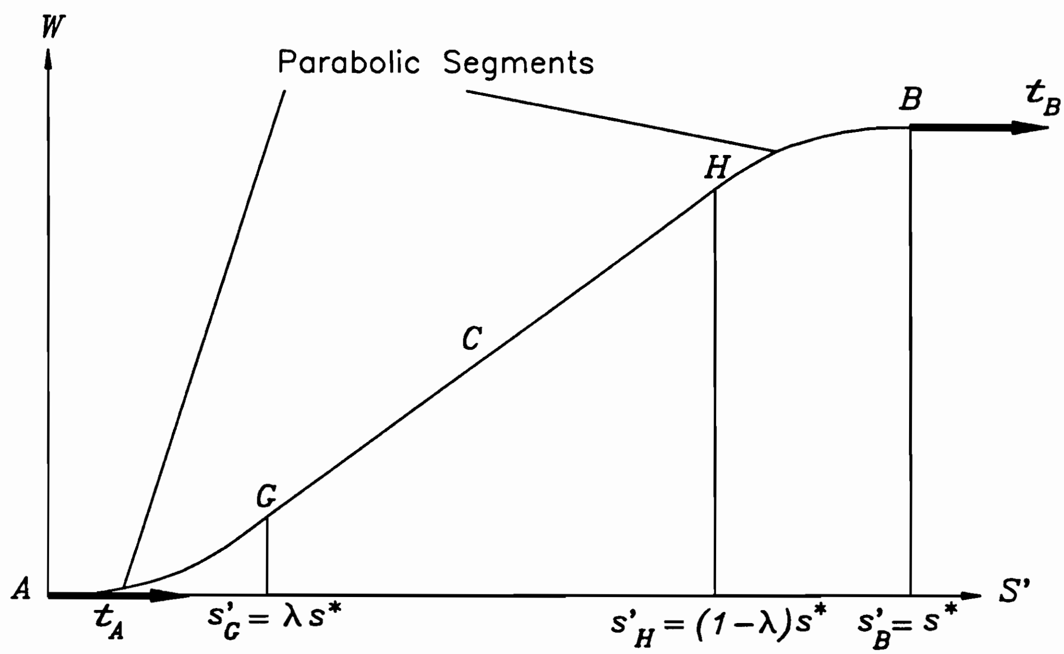


Figure 4.3. A Parabolically Blended Curve for Modeling of the Helix Rise.

$$w(s') = \frac{1}{2}\ddot{w}s'^2 \qquad 0 \le s' \le s'_G \tag{4.3}$$

$$w(s') = w_G + \left(\frac{s' - s'_G}{s'_H - s'_G}\right)(w_H - w_G) \qquad s'_G \le s' \le s'_H \tag{4.4}$$

$$w(s') = w_B - \frac{1}{2}\ddot{w}(s'_B - s')^2 \qquad s'_H \le s' \le s'_B \tag{4.5}$$

where the values of w_G, w_H and $\ddot{w}$ are obtained using the continuity conditions at points G and H and $w_B = d$ the common shortest distance between $\mathbf{t}_A$ and $\mathbf{t}_B$. The necessary expressions are as follows.

$$w_G = \frac{-\frac{1}{2}s'^2_G w_B}{\Delta} \tag{4.6}$$

$$w_H = \frac{\left(\frac{1}{2}s'^2_G - s'_G s'_H\right)w_B}{\Delta} \tag{4.7}$$

$$\ddot{w} = \frac{-w_B}{\Delta} \tag{4.8}$$

and

$$\Delta = \frac{1}{2}s'^2_G - \frac{1}{2}(s'_B - s'_H)^2 - s'_G s'_H \tag{4.9}$$

The parameter λ is defined such that $s'_G = \lambda s^*$ and $s'_H = (1 - \lambda)s^*$ where s^* is the length of the projection curve on the UV plane. λ is given as an input parameter.

Though the parabolically blended curve is a computationally simple model, it may produce discontinuity of curvature at points G and H.

Cubically Blended Curve

This model provides a curve consisting of cubically blended curves at the end-points A and B and a straight line portion in the middle range GH (Fig. 4.4). Consider a cubic curve required to be designed between points A and G. This can be accomplished as follows. Assume that point $C(\frac{d}{2}, \frac{s^*}{2})$ is known. Draw a line through C with a slope of $\tan(\gamma)$. Now the cubic curve should pass through point A and should be tangent to the S' −axis. at point A. Furthermore, at its other end ($u = 1$), the cubic curve should have zero curvature and should be tangential to the line passing through C and having a slope $\tan(\gamma)$. Let the equation of the cubic curve be:

$$\mathbf{r} = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3 \qquad 0 \le u \le 1 \tag{4.10}$$

The boundary condition can be specified as:

(i) End-point Condition. At $u = 0$, $\mathbf{r} = 0$

(ii) End-point Tangent Condition. At $u = 0$, $\frac{d\mathbf{r}}{du} = [k_0 \quad 0]^T$

(iii) End-point Tangent Condition. At $u = 1$, $\frac{d\mathbf{r}}{du} = [k_1 \cos\gamma \quad k_1 \sin\gamma]^T$

(iv) End-point Zero Curvature Condition. At $u = 1$, $\frac{d\mathbf{r}}{du} \times \frac{d^2\mathbf{r}}{du^2} = 0$

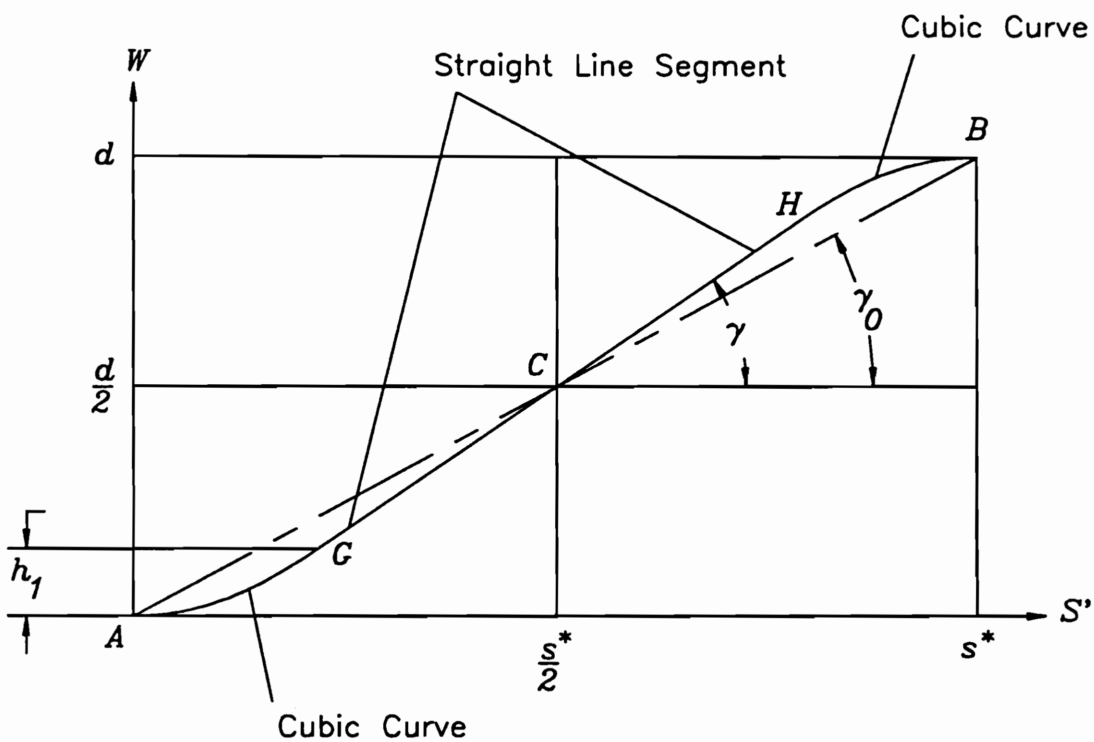


Figure 4.4. Cubically Blended Curve for Modeling the Rise of a 3-D Curve.

where k_0 and k_1 are constants to be evaluated along with $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$.

Let h_1 be the height where the end-point G of the cubic curve blends with the straight line segment solving the boundary condition equation for the unknown coefficients. One gets,

$$
\begin{aligned}
&\mathbf{a}_0 = 0 \\
&\mathbf{a}_1 = [k_0 \quad 0]^T \\
&\mathbf{a}_2 = -\mathbf{a}_1 \\
&\mathbf{a}_3 = \left[\frac{1}{3}(k_1 \cos\gamma + k_0) \quad \frac{1}{3}(k_1 \sin\gamma) \right]^T \\
&k_0 = 3\left(\frac{s^*}{2} - \frac{\frac{d}{2}}{\tan\gamma} \right) \\
&k_1 = \frac{3h_1}{\sin\gamma} \\
&\gamma > \gamma_0 = \tan^{-1}\left(\frac{d}{s^*} \right)
\end{aligned}
\tag{4.11}
$$

Finally, the cubic curve AG is defined as:

$$
\mathbf{r} = \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3 \tag{4.12}
$$

It should be noted that the curve HB can be defined in a similar way. The rise curve will now consist of four segments AG, GC, CH and HB. One can compute the points of the segments AG and GC. The points on the segments CH and HB can be obtained using transformations of mirror reflections around W and S' axes. The cubically blended curve is definitely superior to the parabolically blended curve since it has a continuity of slope and curvature at points G and H respectively.

Cubic Bezier Curve

If the requirement of having a straight line segment is not essential, then one can define a cubic Bezier curve passing through point A and C such that it is tangential to the S' −axis at point A and is tangential to a line of slope $\tan\gamma$ at C (Fig. 4.5). Furthermore, one can specify the curvature at point C to be zero. The control points for the cubic Bezier curve will be A, K, L and C. It should be noted that K, L and C will be collinear because the curvature at C is zero. Also point K will be on the S' −axis, because of the tangency condition at point A. The distance KL will be governed by the magnitude of curvature at point A. The equation of the Bezier curve is specified as:

$$\mathbf{r}_u = (1-u)^3\,\mathbf{r}_A + 3(1-u)^2 u\,\mathbf{r}_K + 3(1-u)u^2\,\mathbf{r}_L + u^3\,\mathbf{r}_C \qquad 0 \le u \le 1. \qquad (4.13)$$

The cubic Bezier curve between points C and B can be defined in a similar manner. In the present work the cubic blended model has been used for the construction of three-dimensional curves.

4.3 *Algorithm*

Based on the shape models discussed in Section 4.2, it is now possible to state the algorithm for geometrical synthesis of space curves based on an intrinsic geometry approach. It is necessary to point out that the proposed algorithm doesn't define the intrinsic properties of the space curve such as curvature and torsion explicitly. It defines

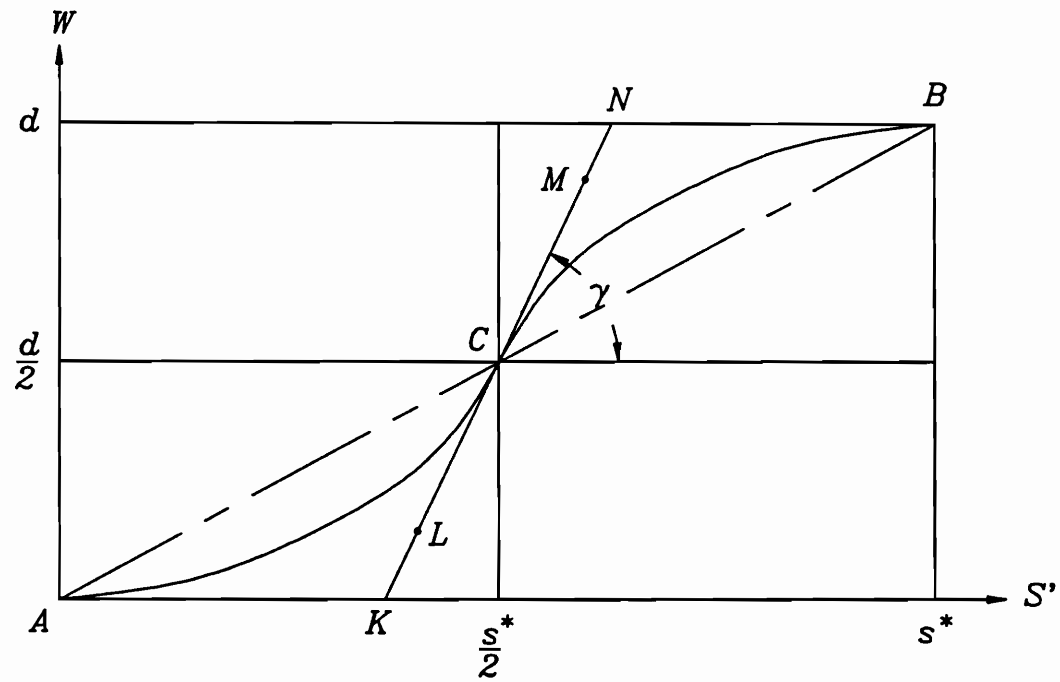


Figure 4.5. Cubic Bezier Model.

a space curve by means of two planar curves. Out of these two planar curves, one curve is defined using the intrinsic properties. One can therefore term the proposed approach of defining the shape of a three-dimensional curve as a pseudo-intrinsic approach.

It should be pointed out that, ideally one would like to solve the problem of three-dimensional curve design using the intrinsic approach. Such an approach will take into account the curvature and torsion profiles of the desired curve. However, the resulting Serret-Frenet equations have been found to be extremely difficult to solve either analytically or numerically as a boundary value problem. Hence, a pseudo-intrinsic approach has been outlined as follows.

Give two end-points A and B and the unit tangent vectors at these points of $\mathbf{t}_A$ and $\mathbf{t}_B$, it is required to define the geometry of a curve passing through these points and satisfying the end-point boundary conditions. The algorithm for this purpose can be stated as follows.

Step 1: Input Data and Check for Planarity

Read in the values of $\mathbf{r}_A$, $\mathbf{r}_B$, $\mathbf{t}_A$ and $\mathbf{t}_B$. Compute the distance d between $\mathbf{t}_A$ and $\mathbf{t}_B$ (Fig. 4.6), using

$$d = \| (\mathbf{r}_B + l_2\mathbf{t}_B) - (\mathbf{r}_A + l_1\mathbf{t}_A) \| \tag{4.14}$$

and

$$l_1 = \frac{[\mathbf{t}_A \cdot (\mathbf{r}_B - \mathbf{r}_A)] - (\mathbf{t}_A \cdot \mathbf{t}_B)[\mathbf{t}_B \cdot (\mathbf{r}_B - \mathbf{r}_A)]}{[1 - (\mathbf{t}_A \cdot \mathbf{t}_B)^2]} \tag{4.15}$$

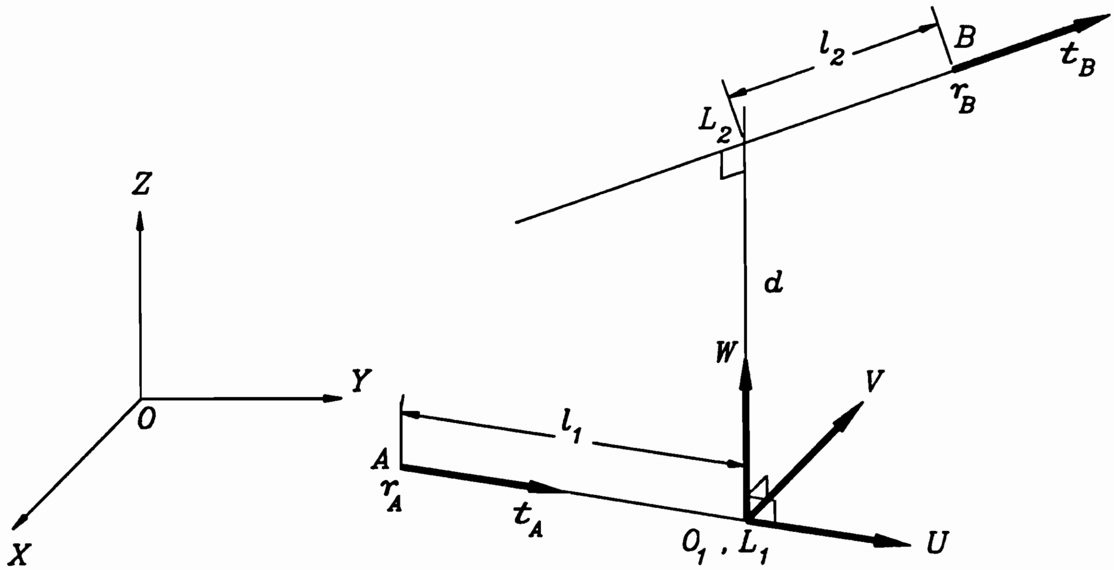


Figure 4.6. Skew Distance.

$$l_2 = \frac{(\mathbf{t_A} \cdot \mathbf{t_B})[\mathbf{t_A} \cdot (\mathbf{r_B} - \mathbf{r_A})] - [\mathbf{t_B} \cdot (\mathbf{r_B} - \mathbf{r_A})]}{[1 - (\mathbf{t_A} \cdot \mathbf{t_B})^2]} \tag{4.16}$$

Check if $\mathbf{t_A}$ and $\mathbf{t_B}$ are parallel by comparing the components of $\mathbf{t_A}$ and $\mathbf{t_B}$. If so the tangents are coplanar. Then design a planar curve passing through A and B using the method outlined in Chapter 3. If $\mathbf{t_A}$ and $\mathbf{t_B}$ are not parallel, then check the value of d. If $d = 0$ then $\mathbf{t_A}$ and $\mathbf{t_B}$ are intersecting and coplanar. Once again use the method outlined in Chapter 3. If neither of these cases is present, then it is required to design a pseudo-intrinsic 3-D curve. For this, proceed to step 2.

Step 2: Set up $O_1 - UVW$ Coordinate System

Construct the local coordinate system $O_1 - UVW$ such that O_1 is coincident with L_1, the U−axis is along the $\mathbf{t_A}$ direction, the W−axis is along the direction L_1L_2 and the V−axis is such that $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ and

$$\begin{aligned} \mathbf{r}_{L_1} &= \mathbf{r_A} + l_1\mathbf{t_A} \\ \mathbf{r}_{L_2} &= \mathbf{r_B} + l_2\mathbf{t_B} \end{aligned} \tag{4.17}$$

The relationship between the local coordinate system $O_1 - UVW$ and $O - XYZ$ can be described using the following relations (Fig. 4.6).

$$\begin{aligned} \mathbf{u} &= \mathbf{t_A} \\ \mathbf{w} &= \frac{1}{d}[\mathbf{r}_{L_2} - \mathbf{r}_{L_1}] \\ \mathbf{w} &= \mathbf{u} \times \mathbf{v} \end{aligned} \tag{4.18}$$

$$[T]_L^G = \begin{bmatrix} u_x & v_x & w_x & x_{L_1} \\ u_y & v_y & w_y & y_{L_1} \\ u_z & v_z & w_z & z_{L_1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.19}$$

Step 3: Generate the Base Curve

Define the base curve in the $O_1 - UV$ plane using Eqns. (4.1) and (4.2). Note that it is necessary to select the shape model and associated set of shape design variables for the base curve. Let s^* be the total arc length of this base curve.

Step 4: Generate the Rise Curve

In order to define a generalized helix, it is necessary to define a base curve and a rise curve. The base curve is defined in the $O_1 - UV$ plane and the rise curve will be defined in the $W - S'$ space. Select any one of the three options for the shape model of a rise curve. The cubically blended curve is recommended. For this shape model, h_1 will be a shape design variable. Select a value of h_1 and define the rise curve.

Step 5: Define the Space Curve

The base curve and the rise curve defined in Steps (3) and (4) together define a parametric equation of space curve (Fig. 4.7). The parameter here is s' where $0 \leq s' \leq s^*$. Note that the space curve needs to be defined in the XYZ system. This can be done using the transformation matrix $[T]_L^G$ defined in Step (2). Both the shape model selected for the base curve as well as the rise curve are continuous in their zeroth, first and

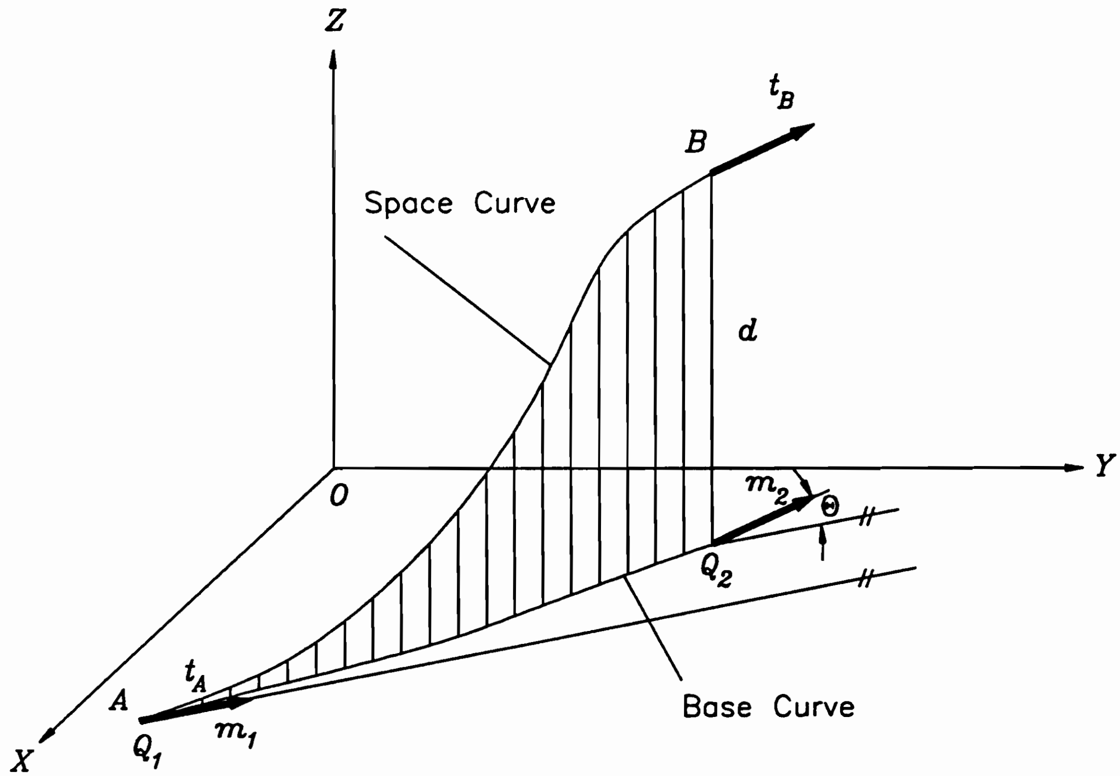


Figure 4.7. A Schematic Diagram of a Three-dimensional Curve Defined Using its Base Curve and the Rise Curve.

second derivatives. Hence, it can be ensured that the generated space curve has a continuous curvature.

Step 6: Output the Results

The generated space curve can be displayed either in orthographic or axonometric views. Points on the curve can be tabulated. The tangent, the curvature and the torsion can be computed using the following relations.

$$\begin{aligned} \mathbf{t} &= \dot{\mathbf{r}} \\ \kappa^2 &= \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \bullet (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})}{(\dot{\mathbf{r}} \bullet \dot{\mathbf{r}})^3} \\ \tau &= \frac{\dot{\mathbf{r}} \bullet (\ddot{\mathbf{r}} \times \dddot{\mathbf{r}})}{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \bullet (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})} \end{aligned} \tag{4.20}$$

where, $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{ds'}$, $\ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{ds'^2}$, $\dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{ds'^3}$, $0 \le s' \le s^*$.

Step 7: Shape Manipulation Using Shape Design Variables

Steps (2) through (6) provide a methodology of designing a curve that fulfills the boundary conditions. Note that the shape of the curve is dependent on the values of the shape design variables of the rise curve and the base curve. If the shape model selected for the base curve consists of two linear segments, then it provides two shape design variables and the cubically blended curve for the rise of the helix provides one shape design variable. In all a total of three shape design variables define the space curve. It can be seen that one can generate a variety of three-dimensional curves by changing the values of three shape design variables. All these curves will fulfill the same end-point boundary conditions.

4.4 *Applications*

4.4.1 Design of a Three-dimensional Path

This example illustrates the geometrical design of a three-dimensional path that connects one point at a lower level to another point at a higher level. The two points have different tangent directions. This example can be treated as a mathematical model for designing a portion of a three-dimensional path of a robot end-effector or a segment of a roller coaster path or the linking segment between two highways such as a clover loop. The example also illustrates that the intrinsic geometry approach not only allows the designer to evaluate the parameters such as curvature explicitly but also provides the flexibility to change the shape of the path while fulfilling the end-point constraints.

In this example, the end-points are selected as (1,0,0) and (1,0,1). The unit tangent vectors at these end-points are specified as (0,1,0) and (1,0,0) respectively. The shortest distance in this example happens to be unity and the values of l_1 and l_2 are zero. The $O-XY$ plane is coincident with the O_1-UV plane. The base curve in the O_1-UV plane is the projected curve whose end-points are (0,0) and (0,0). Figure 4.8(a) shows the $\kappa - s$ specification for the projected curve. The projected curve serves as the base of the helix. In the present case it is designed using two linear curvature segments which provide two shape design variables. By altering the values of the shape design variables, one can manipulate the shape of the curve. The u and v coordinates of this projected curve are displayed in Fig. 4.8(b). The rise of the helix is modeled using a cubically blended curve shown in Fig. 4.9. Note that there is one additional shape design variable

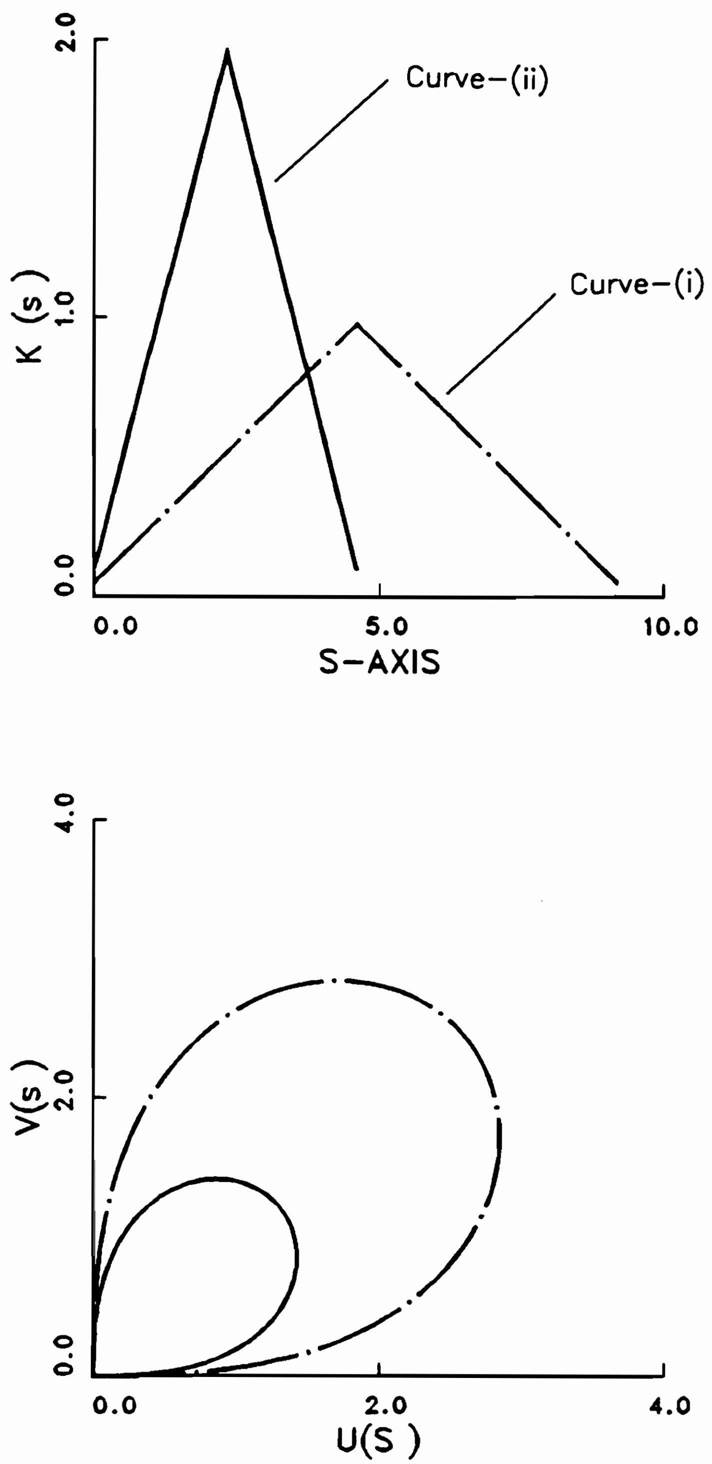


Figure 4.8. (a) κ-s Plot of the Projected Curve. (b) U-V Plot of the Projected Curve for the Base of the Helix. The Infeasible Design is the Solid and the Feasible Design is the Dashed Line.

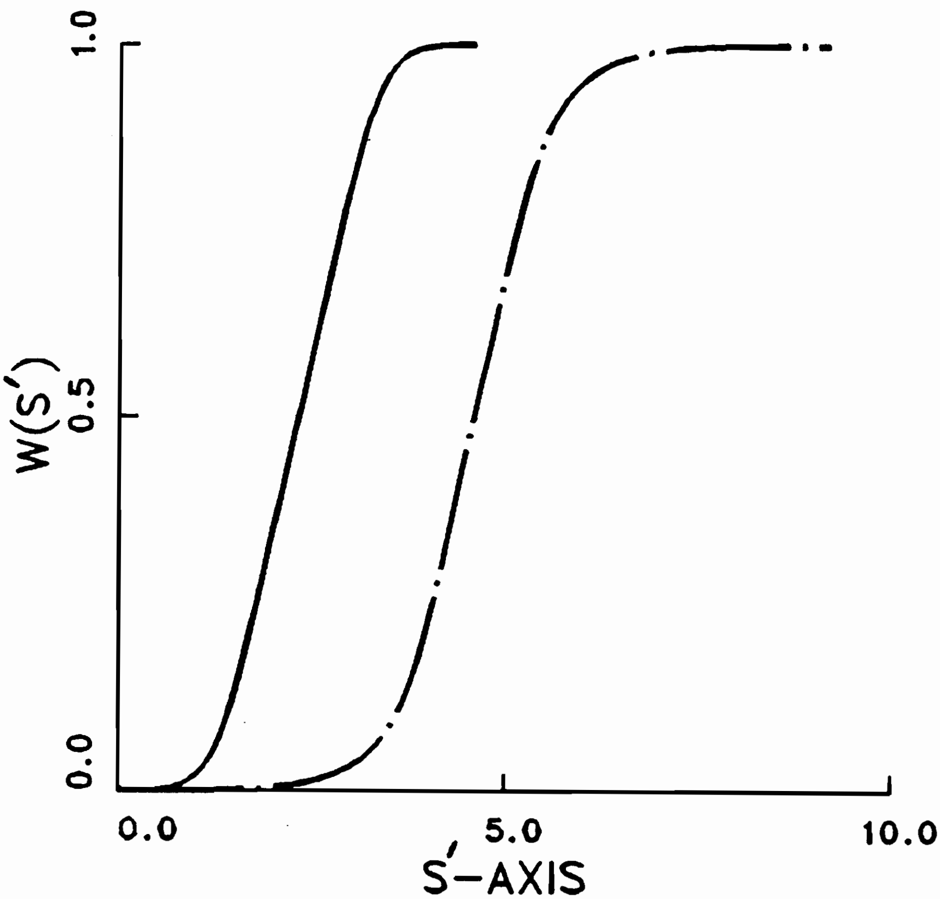


Figure 4.9. W-s′ Cubically Blended Curve for the Rise of the Helix.

that can be appropriately selected for designing the shape of the 3-D curve. The desired 3-D curve is displayed in Fig. 4.10. It can be seen that the curve-(i) corresponds to one model of $\kappa - s$ profile. If the constraint specified indicates that the curvature should not be greater than a specified value κ^*, then the curve-(ii) will be an infeasible design while curve-(i) can be treated as a feasible design.

If it is necessary to optimize a certain objective function, one can generate a variety of feasible shapes by varying the magnitudes of the shape design variables.

4.4.2 Design of the Shape of a 3-D Variable Geometry Truss

The shape control of a two-dimensional Variable Geometry Truss manipulator based on an intrinsic geometry definition was presented in Chapter 3. An example of an assembled VGT for a two-line segment curvature profile was shown.

Consider a spatial VGT as shown in Fig. 4.11. The general shape of this VGT is synthesized using the shape synthesis technique proposed here for given initial and final coordinates. Once the general shape of the VGT is determined, the 3-D curve is equally divided for a given number of bays. The bays are double-octahedral-cells. The VGT is then assembled for the given end conditions. A critical constraint to be checked for is the length of each link of the VGT in order to meet a pre-specified range. In the case of an infeasible design, the shape design variables are varied in order to adjust for the link length constraint. The initial and final coordinates are (1,0,0) and (1,0,1). The corresponding unit tangent vectors at the end-points are (0,0.7071,0.7071) and (-0.7071,0,0.7071) respectively. The number of bays is selected to be 5. Figure 4.12 shows the shape of the projected curve in the U-V plane and its $\kappa - s$ profile. The rise

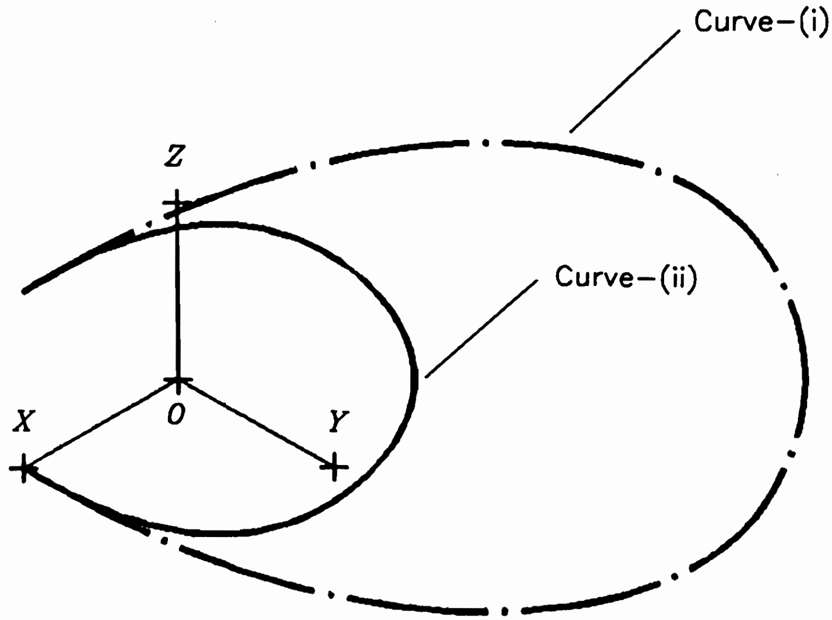


Figure 4.10. An Isometric View of the Pseudo-intrinsic Three-dimensional Curve.

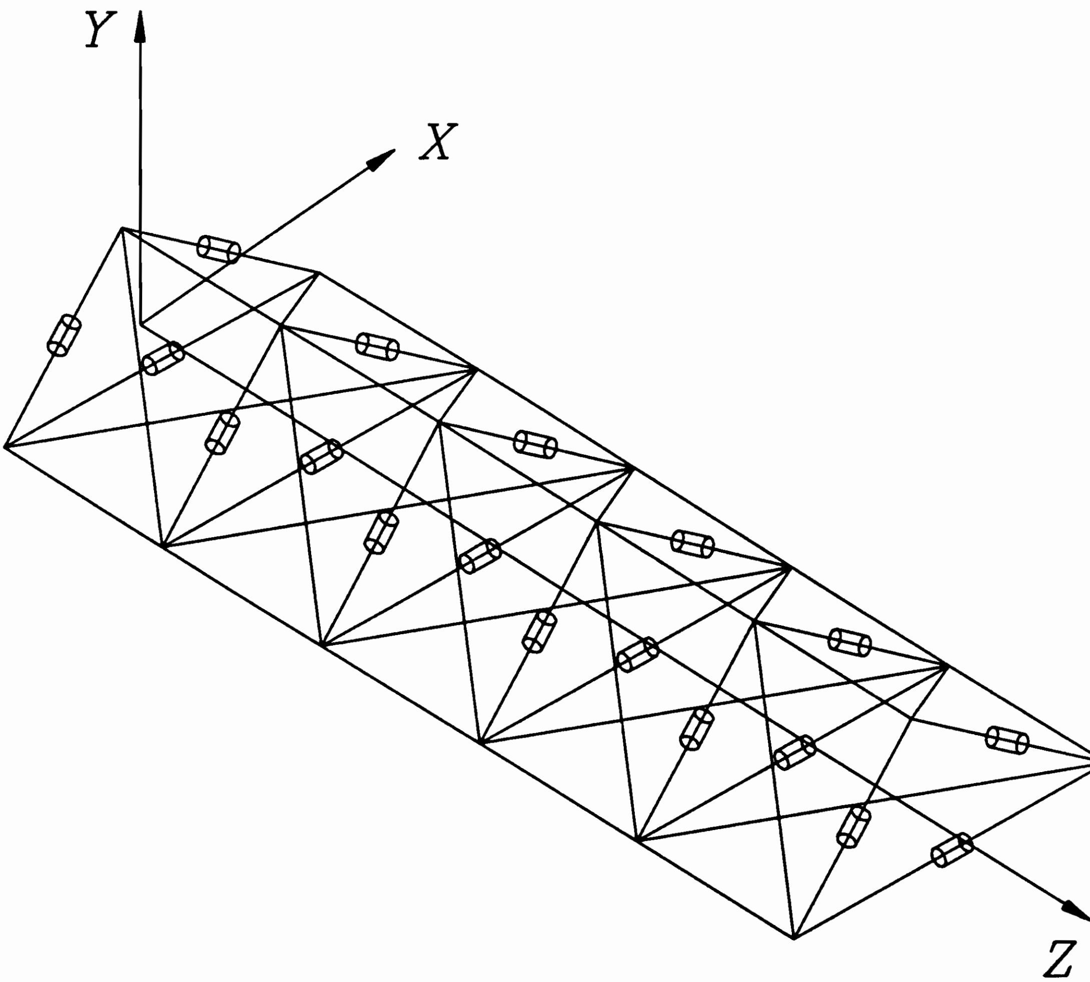


Figure 4.11. Schematic Diagram of a Three-dimensional Variable Geometry Truss-based Manipulator.

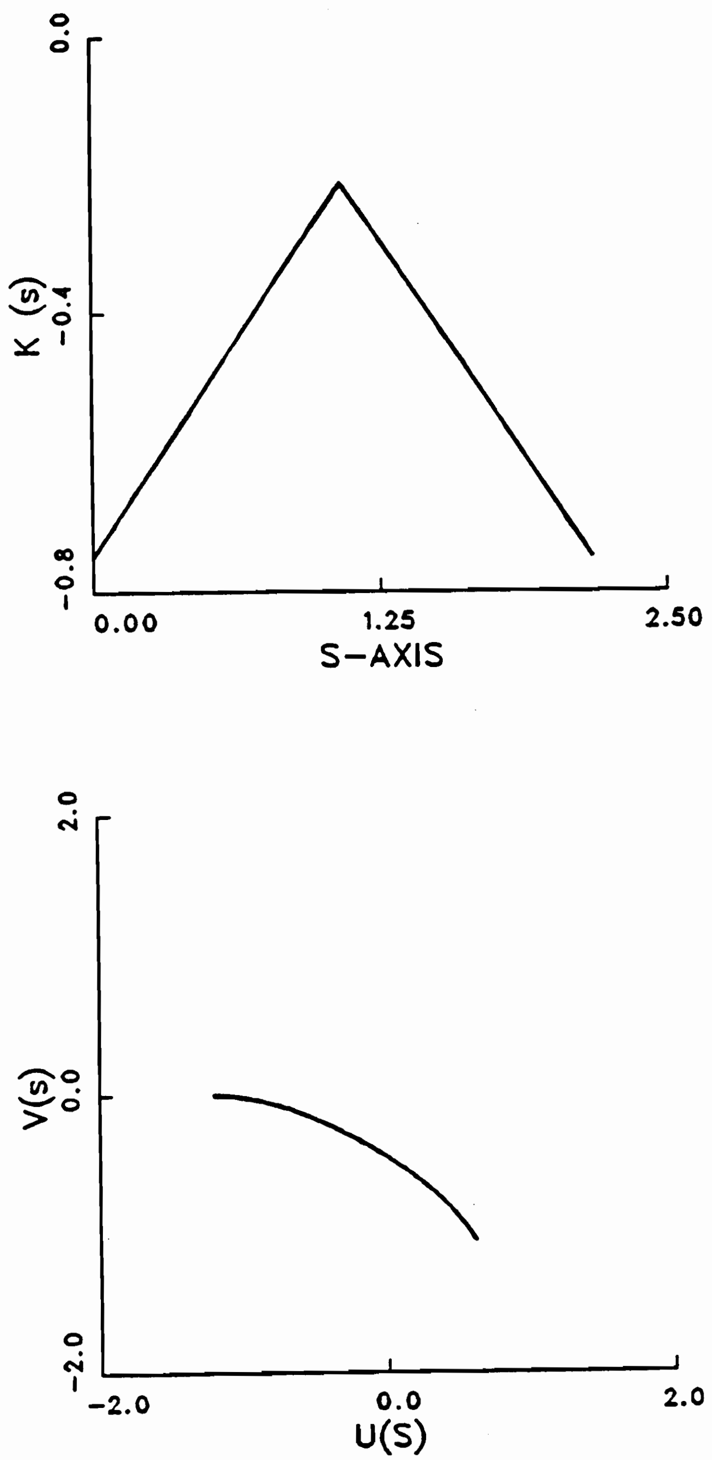


Figure 4.12. $\kappa(s)$ and U-V Profiles of a Three-dimensional Variable Geometry Truss.

of the three-dimensional curve is shown in Fig. 4.13 for cubically blended ends. The assembled three-dimensional VGT is shown in Fig. 4.14.

4.5 Summary

A methodology of designing the shape of a three-dimensional curve using intrinsic geometry has been presented in this chapter. Some classical methods of shape design synthesis as reported in the literature on differential geometry have been reviewed in Chapter 2. These methods seem to provide either an approximate solution or computationally exhaustive procedures.

The proposed method defines the shape of a three-dimensional curve by means of two planar curves. The first planar curve can be considered to be a base curve of a helix. This curve is defined in its intrinsic form. The second curve defines the rise of the helix. In short, a three-dimensional curve passing through two end-points and end-point tangents can be considered to be a generalized helix. It should be noted that the proposed approach cannot be termed to be truly based on intrinsic geometry. The three-dimensional curve is finally defined in the parametric form. The arc length of the base curve is the parameter. One can find the curvature and torsion of the three-dimensional curve in terms of the parametric representation of the base curve and the rise curve.

The proposed definition has several distinct advantages. If some range or some part of the curve has to have a specific curvature behavior, then it can be accomplished by appropriately modeling the base and the rise curve. For example, if a straight line path

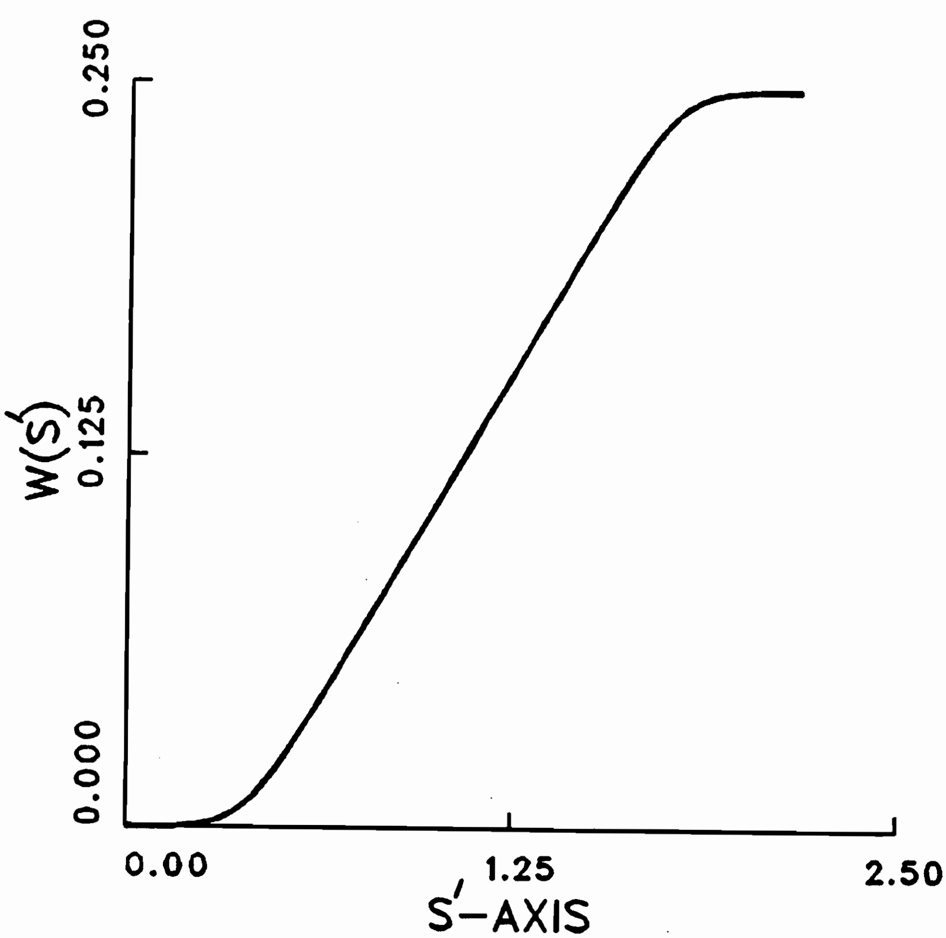


Figure 4.13. W-s′ profile of a Three-dimensional Variable Geometry Truss.

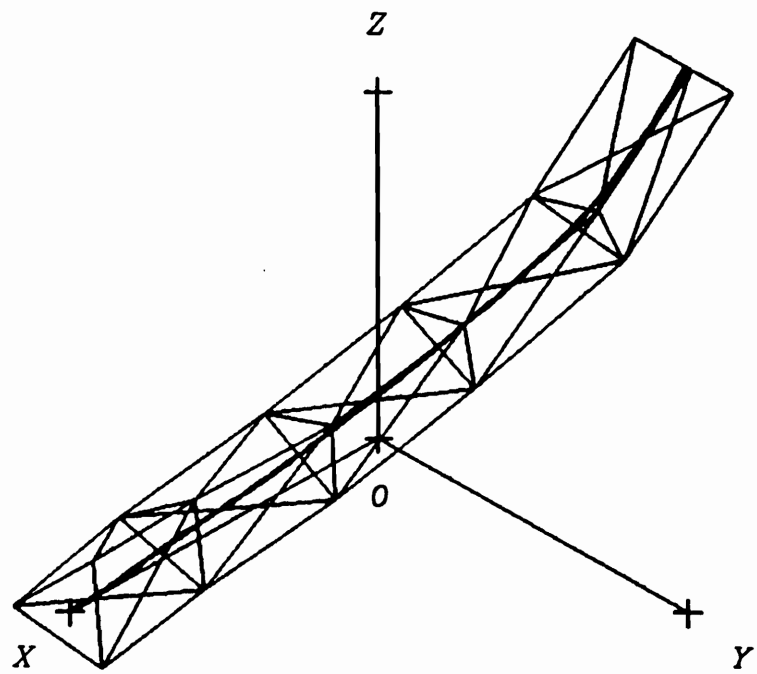


Figure 4.14. An Isometric View of a Feasible Assembly of a Three-dimensional Variable Geometry Truss Manipulator Based on a Pseudo-intrinsic Geometry Definition.

is required in between the end-points, then it can be obtained by choosing appropriate segments of the base curve as well as the rise curve to be straight line segments.

• • •

5. Shape Design of Surfaces

5.1 Methodologies of Surface Design

The problem of designing geometry of three-dimensional surfaces is frequently encountered in engineering design. It is often required to design aerodynamically or hydrodynamically streamlined surfaces, transition patches and blends for automobile bodies, ship hulls, aircraft fuselages and wings, turbine blades, extruder screws and many other engineering components. In order to appreciate the role of intrinsic geometry for the design of surfaces, it will be appropriate to review the classification of surfaces and some of the salient methods of designing free-form surfaces.

A surface may be considered to be generated by a line/curve element, called the generatrix which moves accordingly to some specified law with reference to a line/curve element called the directrix. Broadly, the surfaces are classified as ruled surfaces and double-curved surfaces. In case of ruled surfaces the generatrix is a straight line and in case of double-curved surfaces, the generatrix is a curve (Fig. 5.1). Ruled surfaces are

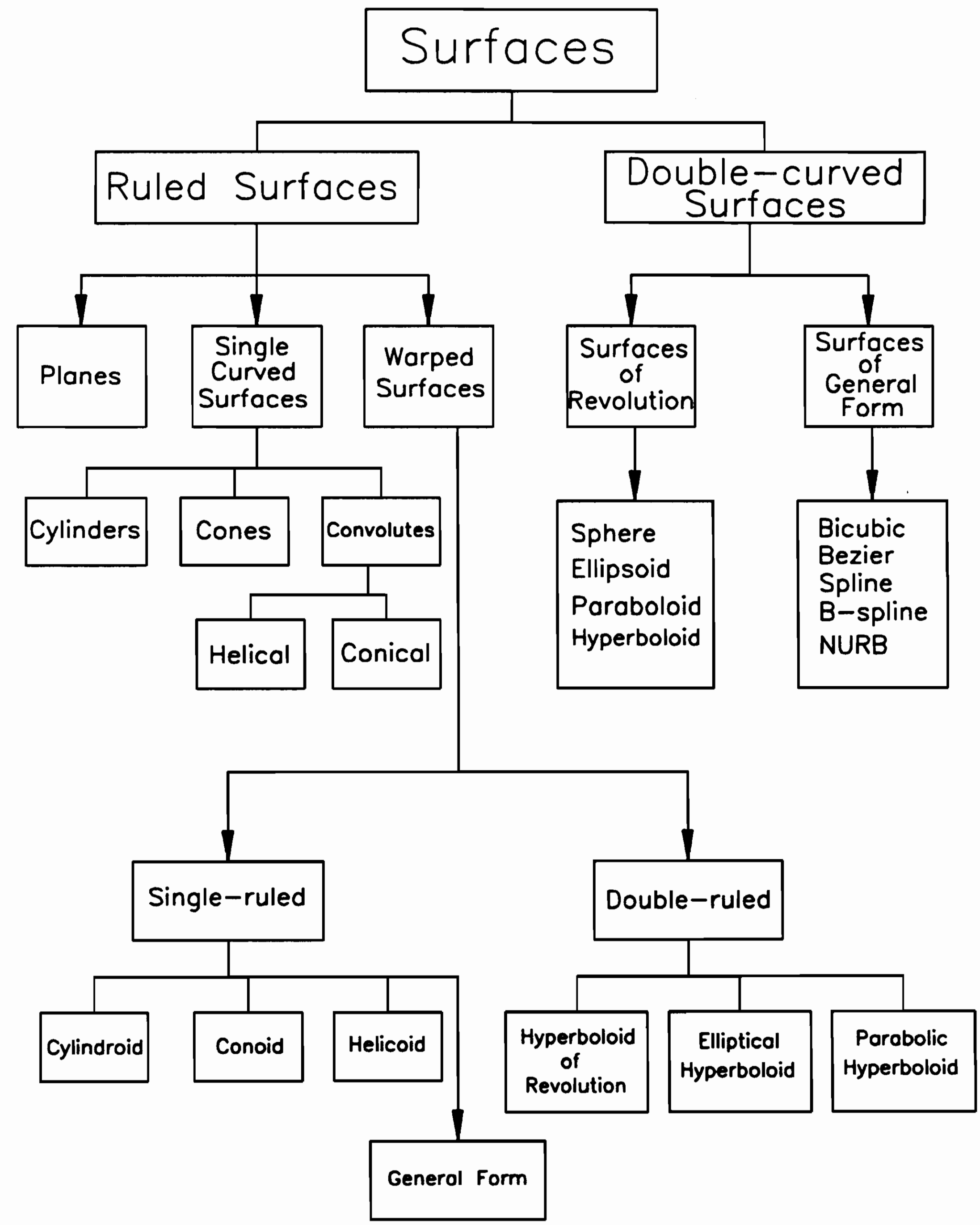


Figure 5.1. **Classification of Surfaces.**

further classified as planar surfaces, single-curved surfaces and warped surfaces. A plane may be generated by a straight line generatrix moving in such a way as to touch two intersecting or parallel straight lines or a straight line and a point. Single-curved surfaces have the consecutive positions of the generatrix elements either parallel (cylinder), intersecting in a unique point (cone) or intersecting in a point which is shifting (convolute) (Fig. 5.2). Warped surfaces have two consecutive elements neither parallel nor intersecting. They are generated by a straight line generatrix which is always parallel to a plane called the director besides touching the directrix curve(s). Warped surfaces are further classified as single-ruled surfaces and double-ruled surfaces (Fig. 5.3). Cylindroid, conoid, helicoid as well as air-foil surfaces are some well known examples of single-ruled surfaces. A double-ruled surface could be a hyperboloid of revolution or an elliptical hyperboloid or a parabolic hyperboloid. Manufacturing of ruled surfaces either by means of a shape generation process or by means of a shape forming process is relatively easy as compared to the manufacturing of double-curved surfaces.

Double-curved surfaces are classified as surfaces of revolution or surfaces of general form. Surfaces generated by revolving a curved-line generatrix about a straight line as an axis are called surfaces of revolution (Fig. 5.4). Spheres, prolate/oblate ellipsoid, paraboloid, hyperboloid and torus are some examples of this class of surfaces. Surfaces of general form are surfaces generated by moving a constant shape or a variable shape curved-line generatrix along another curved path as a directrix.

Although a variety of ruled as well as double curved surfaces are well defined and can be used for engineering design applications, it has been found that the shapes of these surfaces cannot be manipulated as freely as required by a designer. Details of these geometrically well-defined surfaces are given in the literature (French and Vierck, 1970).

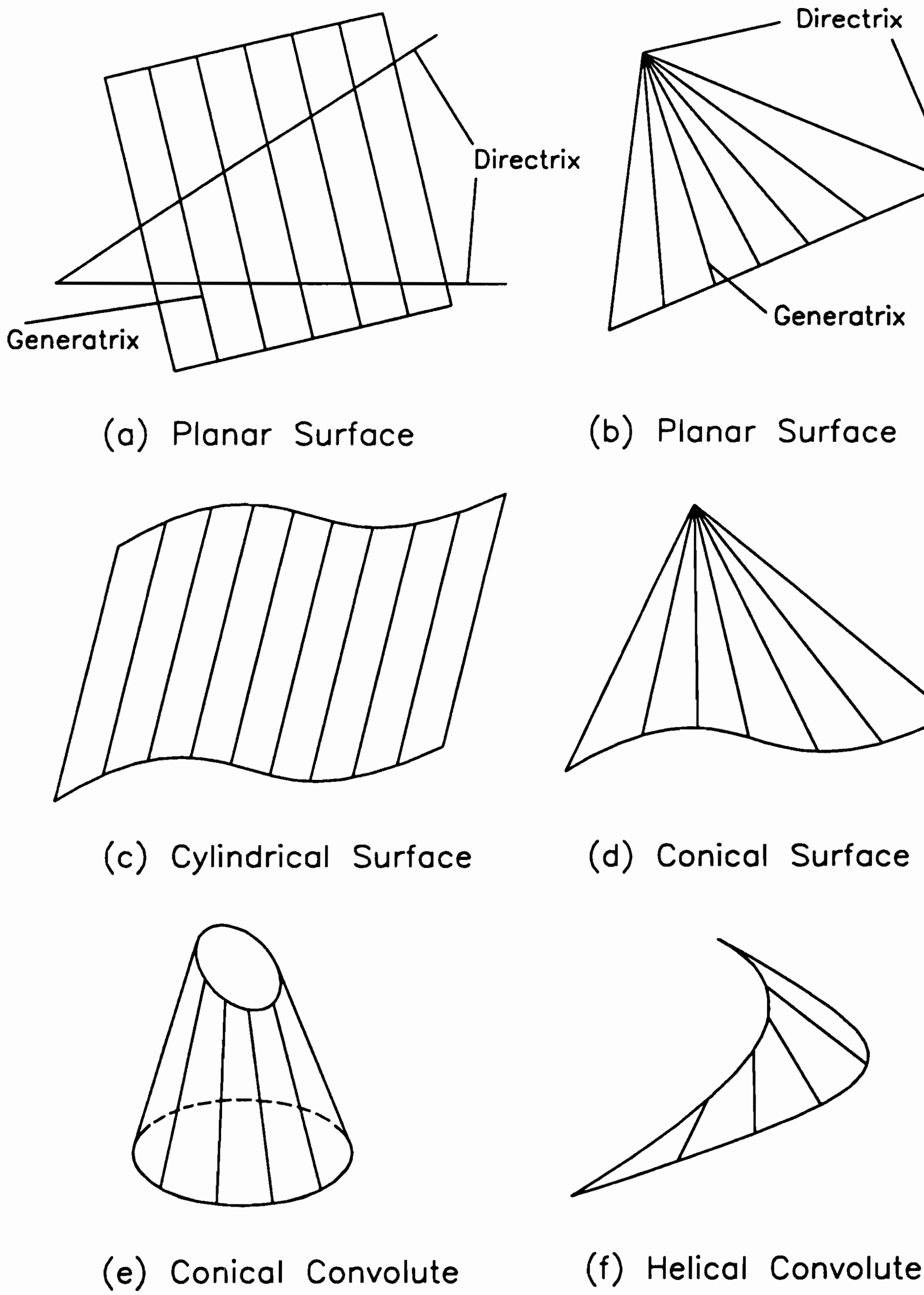


Figure 5.2. Single-curved and Planar Surfaces.

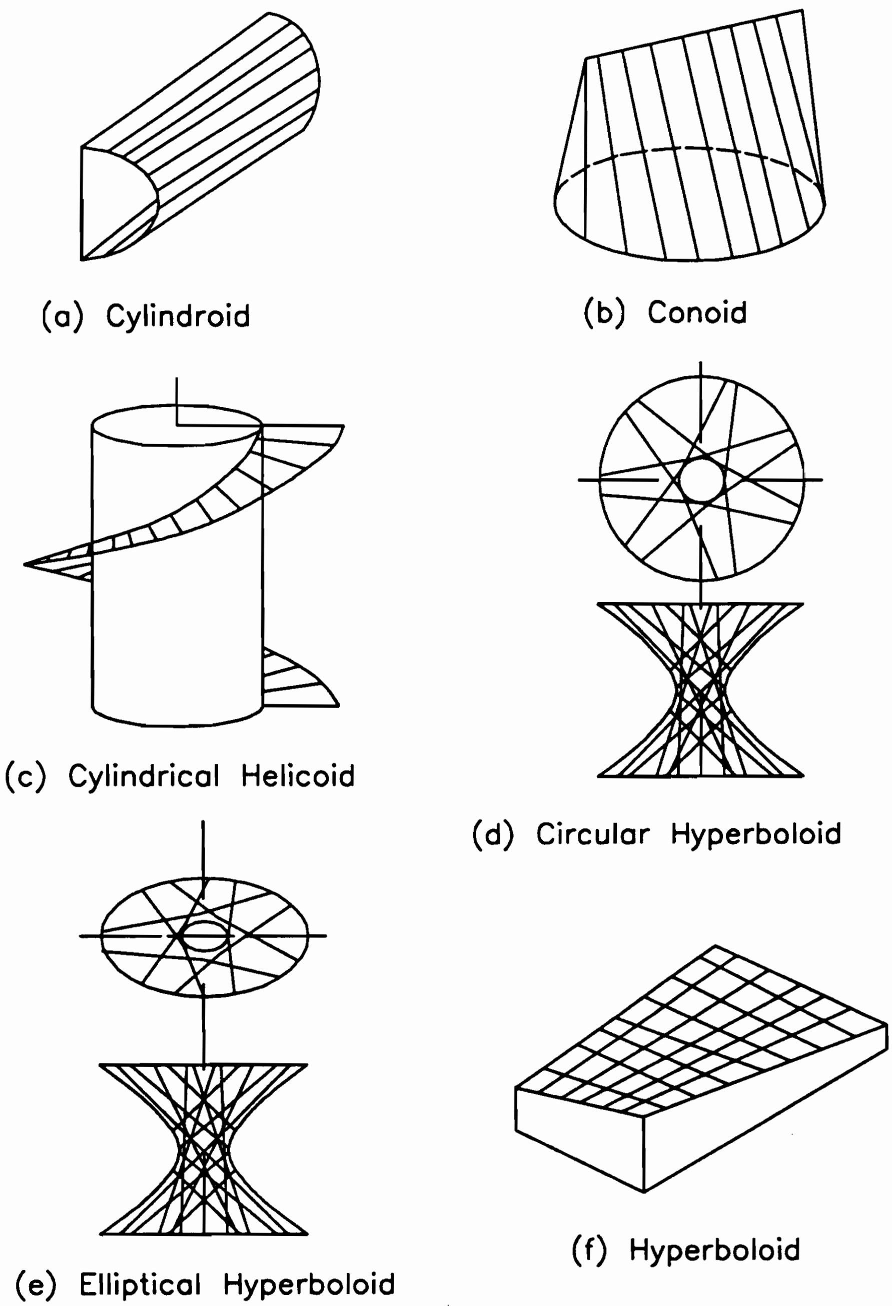


Figure 5.3. Some Examples of Warped Surfaces.

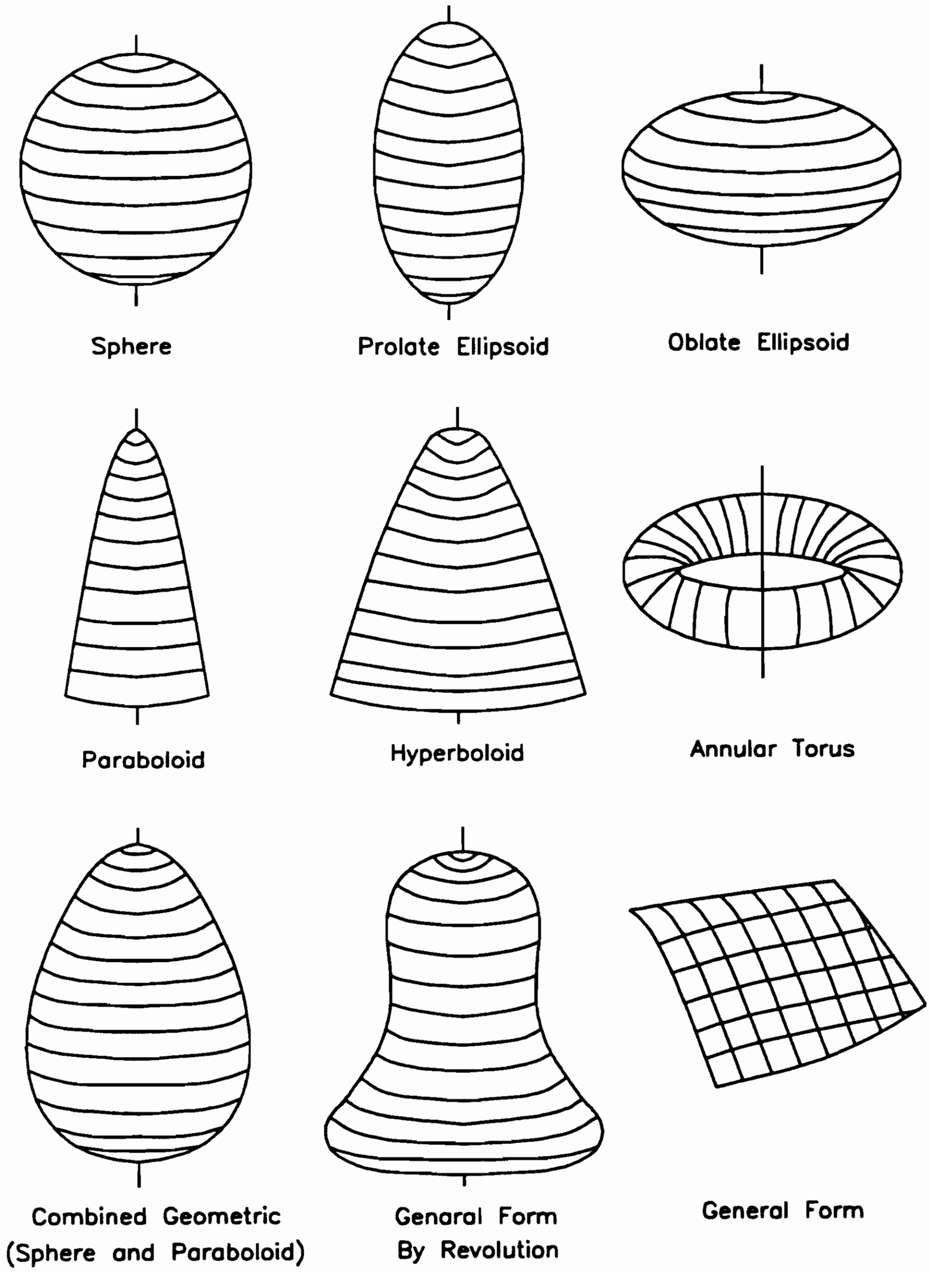


Figure 5.4. Double-curved Surfaces.

During the past two decades, the rapid growth of computer-controlled display techniques as well as the mathematical development of elegant spline-like curve definitions, it is now possible to define as well as manipulate free-form surface patches. Bilinear surfaces, Coon's linear surfaces, bicubic surface patches, Bezier surfaces, B-spline surfaces, non-uniform rational B-spline surfaces are some of the well-known surface definitions that are now available for designing engineering surfaces (Mortenson, 1985). Although it is possible to define a surface patch using these techniques, it has been found that there are still some drawbacks while using these spline-like surface definitions. The shape definition is not explicit but it is simply controlled by the locations of the boundary points, the boundary curves and the auxiliary control points. A designer has to explore extensively the appropriate locations of these auxiliary points so as to achieve a certain shape description over a broad region of a surface patch. This process is in many cases time consuming. The other drawback of these spline-like definitions is that it is not possible to achieve a certain shape specification in the middle of a surface patch by controlling the intrinsic properties such as the principle curvatures of the surface.

The endeavor in the present chapter will be to introduce methodologies of surface design which will incorporate the concepts of intrinsic geometry and will be based on the curve design techniques presented earlier in Chapter 3.

5.2 *Surface Design Using Intrinsic Geometry Concepts*

The present section outlines the specification of various types of surface patches that can be defined using the concepts of intrinsic geometry. It will be assumed that a surface is generated by a generatrix curve C. The curve C will be a planar curve and will be defined in a plane Π and its associated $O_1 - UV$ coordinate system. The intrinsic geometry of the curve C will be defined by means of a shape model consisting of a set of piecewise continuous linear curvature segments and associated shape design variables. As outlined in Chapter 3, it is possible to define the uv coordinates of the curve in terms of the shape design variables and the end-point locations as well as tangents in the $U - V$ plane. In order to generate a surface patch, the $U - V$ plane will be moved through the Cartesian space $O - XYZ$. If the plane Π containing the UV curve is moved along a linear path maintaining the same orientation, then a linearly swept surface will be generated (Fig. 5.5(a)). If the plane Π is revolved around an axis, then a surface of revolution will be generated (Fig. 5.5(b)). If the plane Π is moved using a directrix curve C' then it will be defined as a generalized swept surface (Fig. 5.5(c)). In this case the curve C' may be specified in a parametric form or in an intrinsic form. The movement of the generatrix can be specified as follows. For a given value of the parameter of the directrix curve, a point P' on the curve C' will be located and the moving trihedron of the unit tangent, normal and binormal vectors, **t-n-b**, will be established. The plane Π will be located such that it is coincident with the **b-n** plane, and the origin O_1 is coincident with the point P'. The orientation of the UV axes with respect to **b-n** axes is also

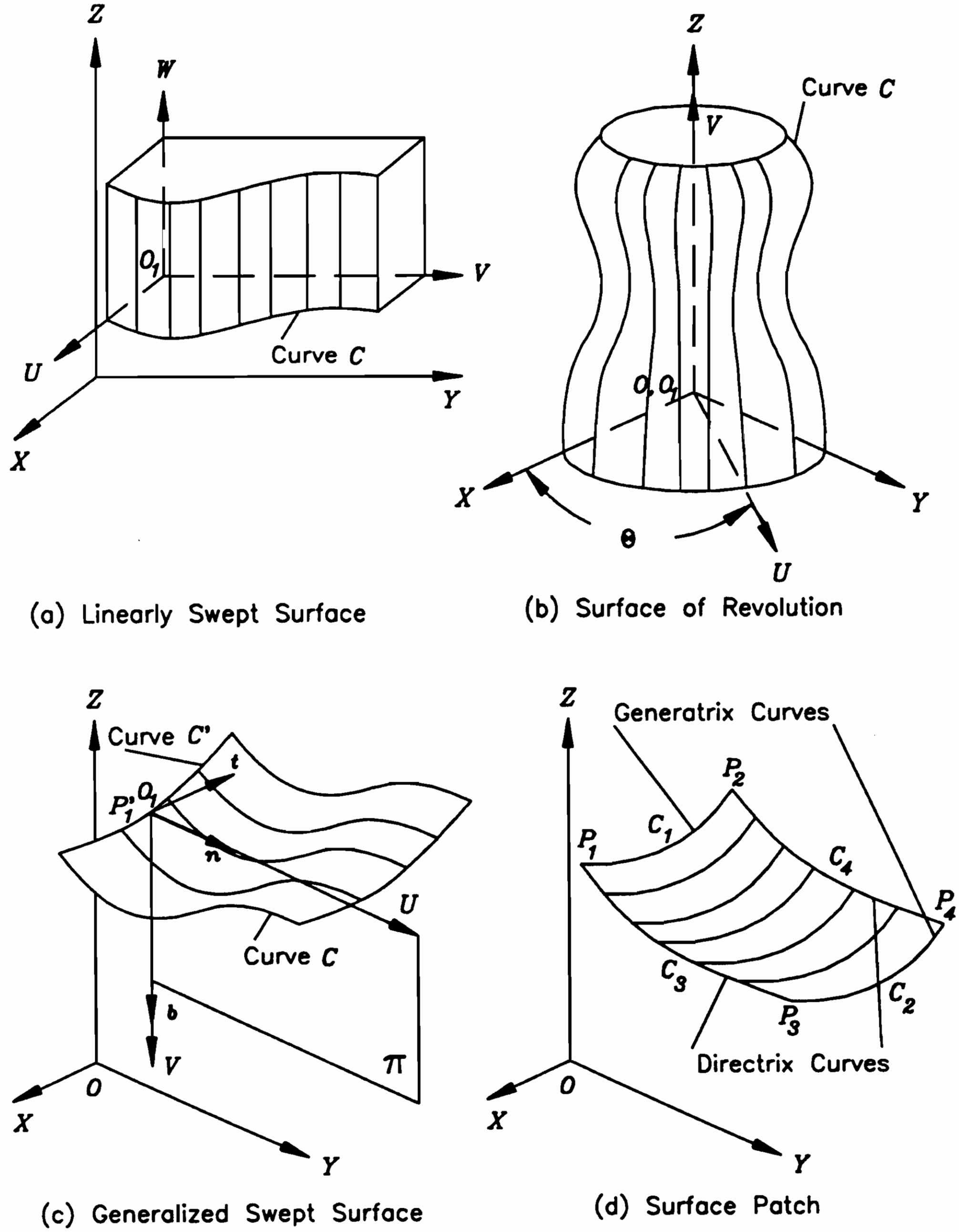


Figure 5.5. Classification of Surfaces Using Intrinsic Geometry Definitions.

prespecified. By varying the location of the point P' along its path and placing the generatrix curve in the manner described above the desired swept surface is defined.

All the foregoing three surface definitions can have a constant shape generatrix or a variable shape generatrix. If the generatrix is of a variable shape type, then it will be assumed that the shape model of the generatrix curve is the same but the values of the shape design variables are changing as a function of the sweep parameter. The sweep parameter will either be a linear distance in case of linearly swept surfaces or the angle of rotation in case of a revolved surface. In case of generalized swept surfaces, the sweep parameter will be the parameter defining the curve C'.

Another class of surfaces that has been proposed here can be termed as transition surfaces. This definition will be suitable if a surface patch is defined by means of its four boundary curves. It will be assumed that a pair of opposite curves C_1 and C_2 shown in Fig. 5.5(d) are defined in terms of their intrinsic geometry. In other words, the shape models and the associated shape design variables have been defined for C_1 and C_2. The other two curves C_3 and C_4 will be treated as the directrix curves. These curves may be defined either in a parametric form or in an intrinsic form. The surface definition will define a series of in-between planes by appropriately selecting a pair of points on the curves C_3 and C_4. Once a plane has been defined, then the shape of the curve in that plane will be defined as a linear combination of the shape model of the curve C_1 as well as the shape model of the curve C_2. The law of this linear combination will be such that at the two extremes, one can get the specified curves C_1 and C_2.

Sections 5.3, 5.4 and 5.5 describe the mathematical definitions of swept surfaces whose generatrix curve may have either a constant shape or a variable shape. Section

5.6 outlines the mathematical formulation of the transition surface. Illustrative examples are given in Section 5.7.

5.3 Linearly Swept Surface

Consider the problem of defining a linearly swept surface using the concepts of intrinsic geometry of a planar curve. The generatrix is a planar curve defined in the $O_1 - UV$ plane. The end-points P_0 and P_1 as well as the end-point tangents $\mathbf{t}_0$ and $\mathbf{t}_1$ are considered to be given as the input (Fig. 5.6). Furthermore, it is assumed that the local coordinate system $\{L\}$ consisting of $O_1 - UVW$ axes is specified in terms of its location and orientation with respect to the global coordinate frame $\{G\}$ consisting of $O - XYZ$ axes (Fig. 5.7). In other words, the transformation matrix $[T]_L^G$ is assumed to be known. Furthermore, it is assumed that the surface is swept along the W −axis and the sweep parameter along the W −axis, say t, varies from 0 to 1.

To begin with, a designer has to select the shape model for the uv curve. A set of linear curvature elements are selected as shown in Fig. 5.6. If two linear elements are selected, then one gets two shape design variables. By fulfilling the necessary constraints and selecting the appropriate values of the shape design variables, one can define the generatrix curve in the $O_1 - UV$ plane. The coordinates u and v will be functions of the arc length s assuming that the shape design variables have been appropriately selected. Let SDV_j $(j = 1, \dots, k)$ be the shape design variables. If the SDV_j's are independent constants and are not functions of t, then the generatrix will be a constant shape gener-

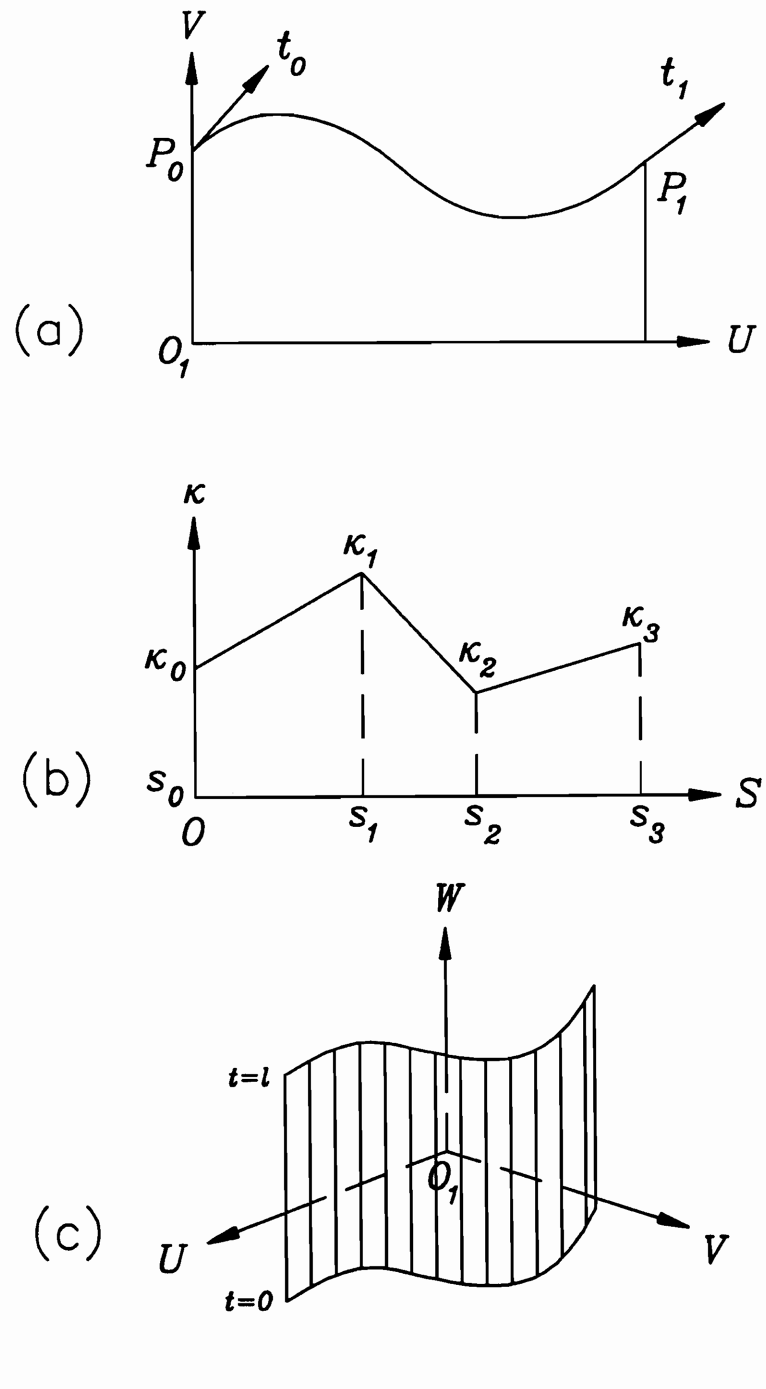


Figure 5.6. Linearly Swept Surface Defined Using Intrinsic Geometry.

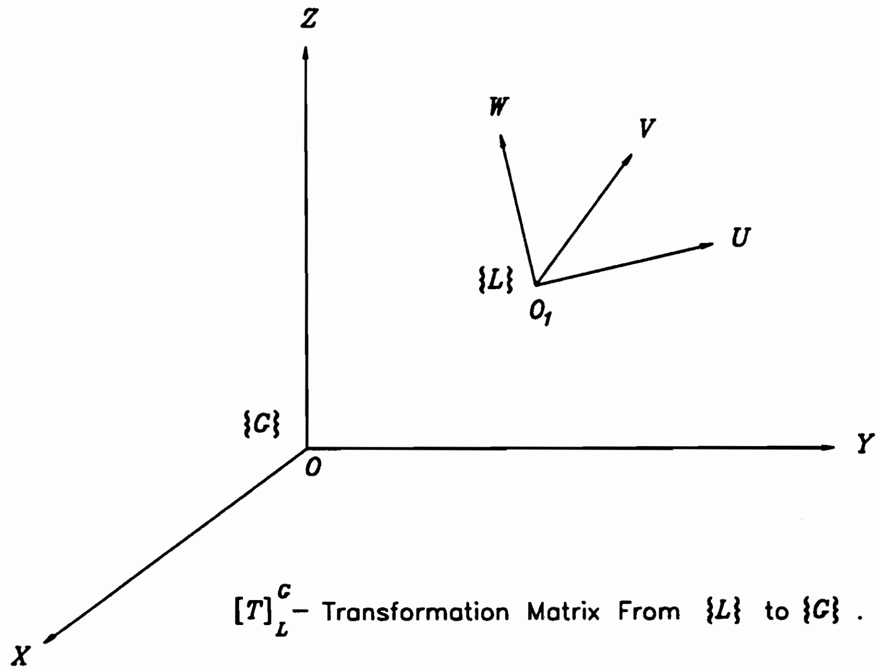


Figure 5.7. Local and Global Coordinate Systems.

atrix. If SDV_j's are functions of t then the generatrix is a variable shape generatrix. The parametric equation of the generatrix is symbolically specified as:

$$\begin{aligned} u &= u(s, SDV_j(t),\ \ j = 1, \ldots, k) \\ & \qquad\qquad\qquad 0 \le s \le s^* \quad 0 \le t \le l \\ v &= v(s, SDV_j(t),\ \ j = 1, \ldots, k) \end{aligned} \tag{5.1}$$

When the generatrix is swept along the W –axis through a distance t, then the coordinates of a generic point P can be specified as:

$$\begin{aligned} u &= u(s, SDV_j(t),\ \ j = 1, \ldots, k) \\ v &= v(s, SDV_j(t),\ \ j = 1, \ldots, k) \qquad 0 \le s \le s^* \\ w &= t \end{aligned} \tag{5.2}$$

Equation (5.2) describes the biparametric surface in the local coordinate system. One can obtain the global coordinates x, y, z by using the following equation.

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = [T]_L^G \begin{bmatrix} u \\ v \\ w \\ 1 \end{bmatrix} \tag{5.3}$$

The biparametric equation of this linearly swept surface will generate a constant shape or a variable shape surface depending on whether the shape design variables are constants or functions of t.

5.4 *Surface of Revolution*

Consider a generatrix curve defined in the $O_1 - UV$ plane. Using the end-point locations of P_0 and P_1 and end point tangents $\mathbf{t}_0$ and $\mathbf{t}_1$, this generatrix curve can be defined in exactly the same way as described in the preceding section (Fig. 5.8). The coordinates u and v will be functions of the arc length s and the shape design variables. It was mentioned earlier that the surface generated by this generatrix curve will be either of constant shape or of variable shape depending on whether the shape design variables are constants or functions of the parameter of rotation, θ in this case. Symbolically one can write:

$$\begin{aligned} u &= u(s, SDV_j(\theta),\ \ j = 1, \dots, k) \\ & \qquad\qquad\qquad 0 \le s \le s^* \qquad 0 \le \theta \le 2\pi \\ v &= v(s, SDV_j(\theta),\ \ j = 1, \dots, k) \end{aligned} \tag{5.4}$$

In order to generate the axi-symmetric surface, it is assumed that the plane $O_1 - UV$ is placed in the $O - XYZ$ coordinate system such that the points O_1 and O coincide and the U and Z axes are collinear. Let θ be an angle that, at any given instant, the $O_1 - UV$ plane makes with $O - XZ$ plane. The coordinates x, y, z of a given point can be specified as:

$$\begin{aligned} x &= v \cos\theta \\ y &= v \sin\theta \\ z &= u, \end{aligned} \tag{5.5}$$

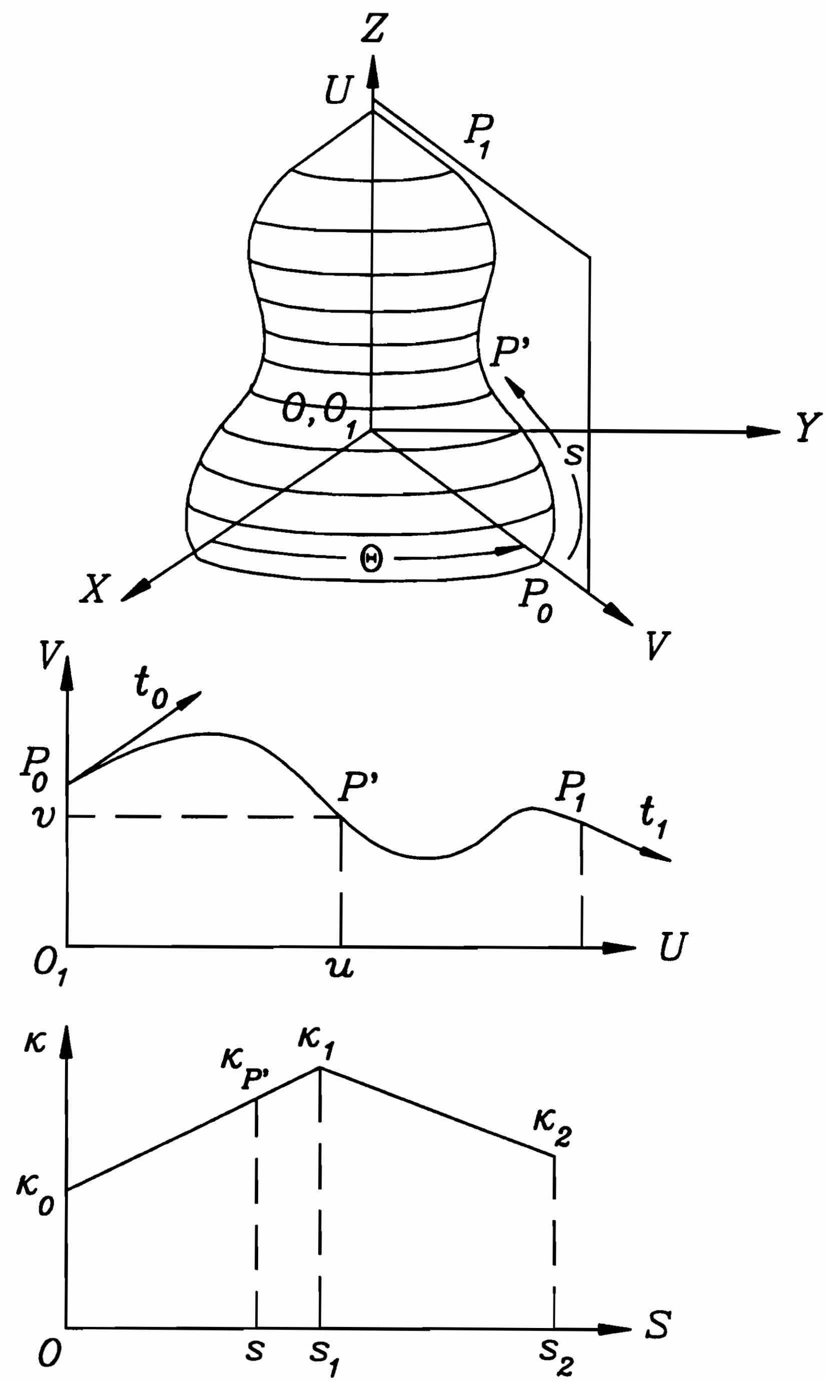


Figure 5.8. Surface of Revolution Defined Using Intrinsic Geometry.

where u and v are the coordinates given in Eqn. (5.4). Equation (5.5) defines the revolved surface. If the $O - XYZ$ coordinate system is not the global frame of reference then the global coordinates can be obtained from the local coordinate system using the transformation matrix $[T]_L^G$ described in the preceding section.

5.5 *Generalized Swept Surface*

Let us consider the problem of defining a swept surface along a directrix specified in the form of a space curve. In this case, the location and the orientation of the generatrix curve needs to be specified in terms of the geometry of the directrix curve. The generatrix curve, however, is considered to be given in its intrinsic form in the same way considered for the linearly swept surfaces or the surfaces of revolution in Sections 5.3 and 5.4 respectively.

Consider a curve C as the generatrix curve (Fig. 5.9). The curve C is defined in a planar coordinate system $O_1 - UV$. The end-point locations P_0 and P_1 as well as the end-point tangents $\mathbf{t}_0$ and $\mathbf{t}_1$ are assumed to be given as the input. To begin with, a shape model of a linear curvature element is selected and the associated shape design variables are specified.

The directrix curve C' is a space curve. It can be specified either in a parametric form or in an intrinsic form using the method outlined in Chapter 4. For discussion here, it will be assumed that the curve C' is specified as a parametric curve. The coordinates x, y, z of a generic point P' on the curve C' will be functions of the parameter

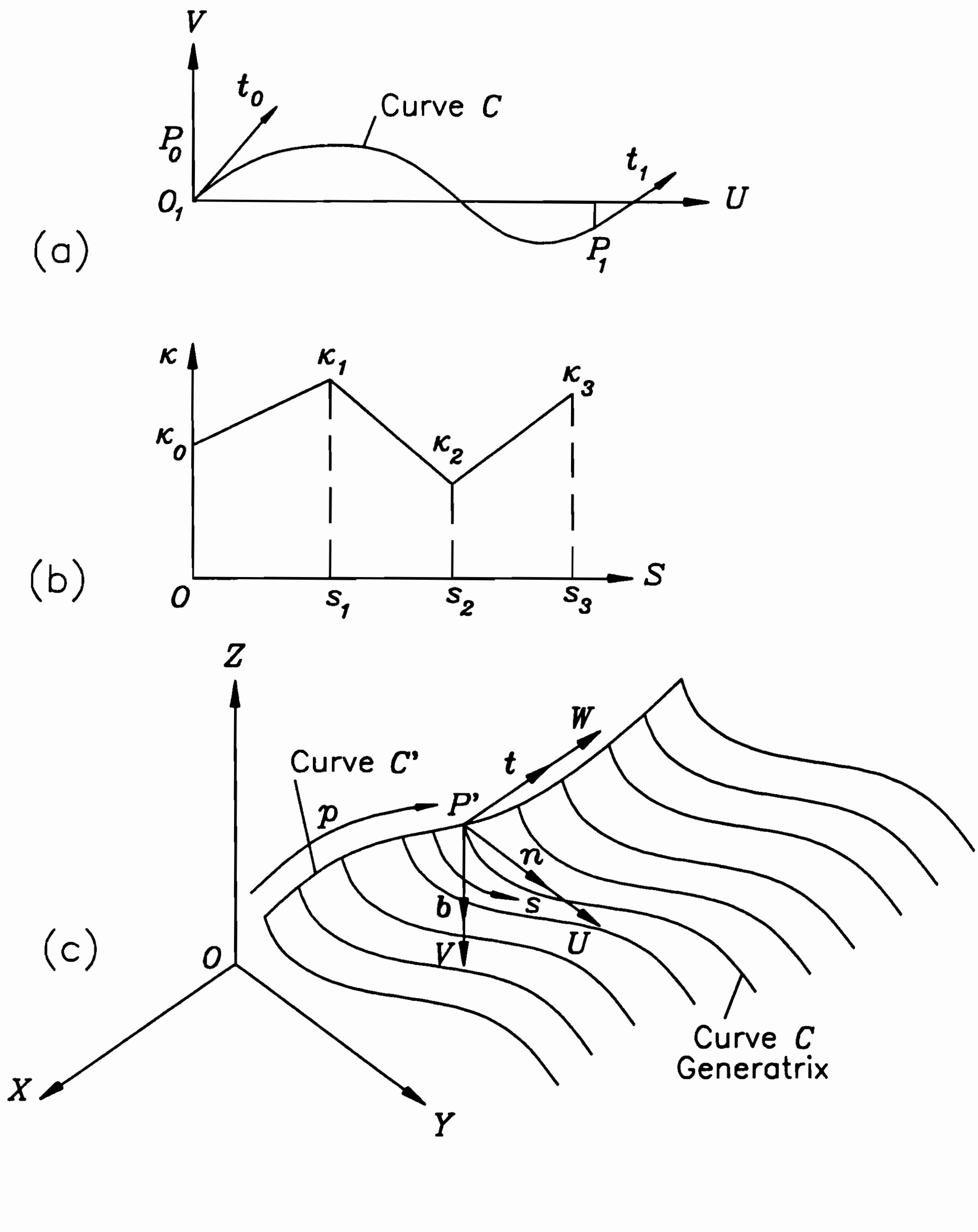


Figure 5.9. Generalized Swept Surface Using Intrinsic Geometry.

p, $(0 \leq p \leq p^*)$. The moving trihedron of the vectors **t**, **n**, **b** can be obtained using the following equations (Faux and Pratt, 1983).

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{\dot{s}}$$
$$\kappa\mathbf{b} = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}})}{\dot{s}^3} \tag{5.6}$$
$$\mathbf{n} = \mathbf{b} \times \mathbf{t}$$

where $\dot{s} = \left| \frac{d\mathbf{r}}{dp} \right|$ and $\dot{\mathbf{r}} = \left[\frac{dx(p)}{dp} \quad \frac{dy(p)}{dp} \quad \frac{dz(p)}{dp} \right]$.

Let $[T]_L^G$ be the transformation matrix that will transform the coordinate of a point from the moving trihedron coordinate system to the Cartesian global coordinate system. The generatrix curve defined in the $O_1 - UV$ plane is now placed in the moving trihedron coordinate system such that O_1 coincides with the point P', the U−axis coincides with the **n**-axis and the V−axis is along the **b**-axis. In other words, for a generic point P on the generatrix curve the coordinates in the $\mathbf{t} - \mathbf{n} - \mathbf{b}$ system would be $[0 \; u \; v \; 1]^T$, where:

$$u = u(s, SDV_j(p), \; j = 1, \dots, k)$$
$$0 \leq s \leq s^* \qquad 0 \leq p \leq p^* \tag{5.7}$$
$$v = v(s, SDV_j(p), \; j = 1, \dots, k)$$

It can be seen that if the shape design variables SDV_j's are functions of the parameter p, then a variable shape surface would be generated. If the SDV_j's are constants, then a constant shape surface will be generated.

The global coordinates of the generalized swept surface can be obtained using the equation:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = [T]_L^G \begin{bmatrix} 0 \\ u \\ v \\ 1 \end{bmatrix} \tag{5.8}$$

5.5.1 Case Study: Fabric Drape Surface

In apparel design, it is required to evaluate how the drape surface of a selected fabric will be formed. This information allows a designer to evaluate the appearance of a dress aesthetically. The problem of predicting the geometry of the drape surface of a fabric when hung over a specified support geometry is of great interest in the area of textile design. As outlined here, this problem can be solved by defining the geometry of the drape surface using concepts of intrinsic geometry. As an illustrative example, consider the shape of a fabric drape surface. One can show that a fabric drape can be modeled using the concept of a variable shape generalized swept surface. Let us consider the case of a fabric piece in the form of a circle of radius r draped concentrically around a circular table of radius r_0 where $r_0 < r$ (Fig. 5.10(a)).

The directrix C' in the present case will be a circle of radius r_0 and the center at the origin O. The equation of a generic point P' on this directrix C' will be:

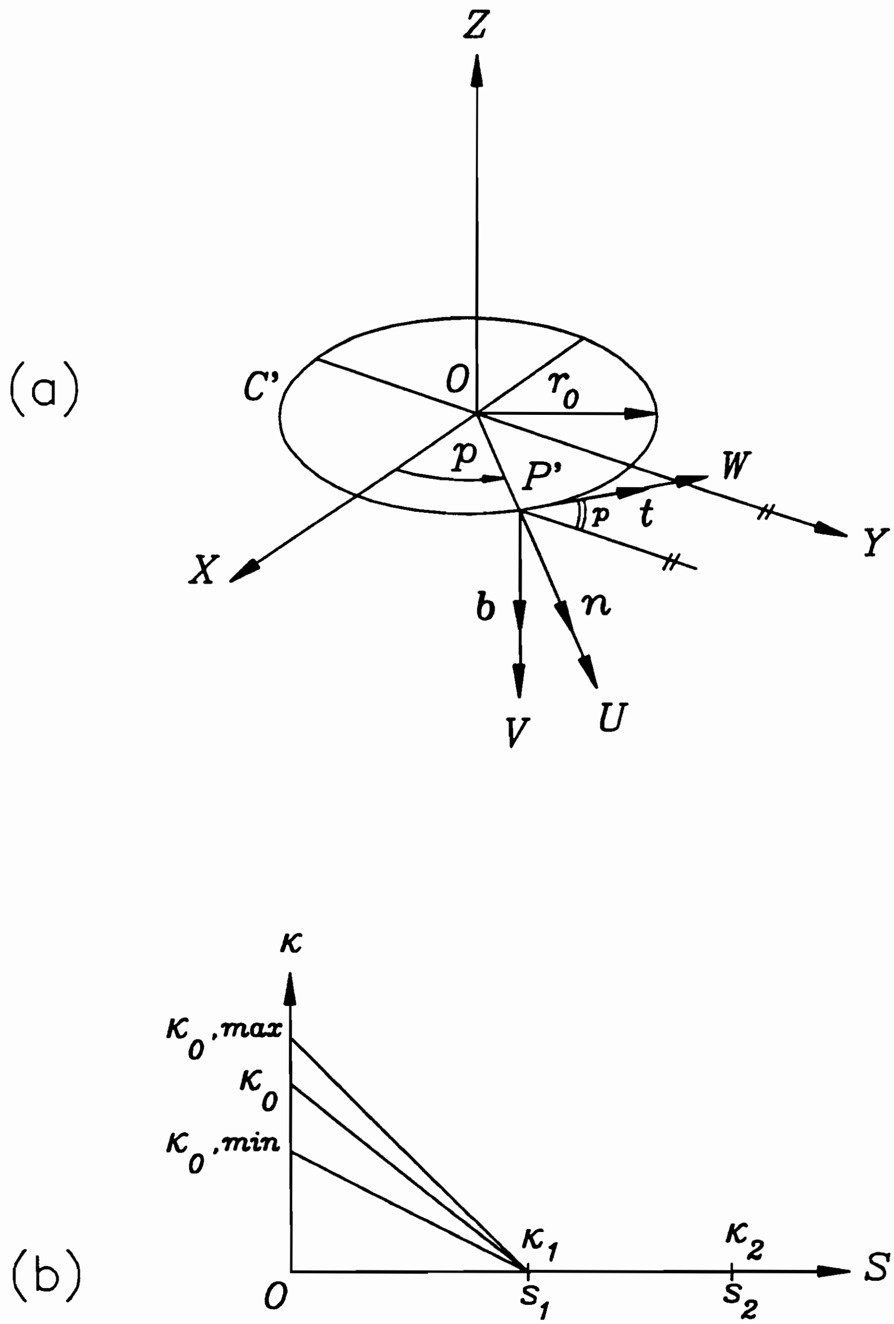


Figure 5.10. Intrinsic Definition of a Draped Surface.

$$\mathbf{r}'(p) = \begin{bmatrix} x'(p) \\ y'(p) \\ z'(p) \\ 1 \end{bmatrix} = \begin{bmatrix} r_0 \cos p \\ r_0 \sin p \\ 0 \\ 1 \end{bmatrix} \qquad 0 \le p \le 2\pi \tag{5.9}$$

The unit tangent vector **t**, the unit normal vector **n** and the unit binormal vector **b** are as follows.

$$\mathbf{t} = \begin{bmatrix} -\sin p \\ \cos p \\ 0 \end{bmatrix} \qquad \mathbf{n} = \begin{bmatrix} \cos p \\ \sin p \\ 0 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{5.10}$$

The transformation matrix from the $U - V - W$ system to the global coordinate system can be written as:

$$[T]_L^G = \begin{bmatrix} \cos p & 0 & -\sin p & r_0 \cos p \\ \sin p & 0 & \cos p & r_0 \sin p \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.11}$$

Now the generatrix is a curve defined in the intrinsic plane. It can be seen that the fabric bends initially due to the weight of the overhang and as the length of the overhang decreases the curvature reduces to zero. It should also be noted that because of the shear effects in the yarn, the curvature κ_0 is not a constant, but it is a function of the sweep parameter p. The variable shape intrinsic form of the generatrix curve (Fig. 5.10(b)) can be specified as:

$$\kappa_0(p) = \kappa_{0,\,\min}\left[1 - \sin\left(\frac{mp}{2}\right)\right] + \kappa_{0,\,\max}\left[\sin\left(\frac{mp}{2}\right)\right]$$
$$\kappa_1 = \kappa_2 = 0$$
$$s_0 = 0 \tag{5.12}$$
$$s_1 = \lambda(r - r_0) \qquad 0 \le \lambda \le 1$$
$$s_2 = (r - r_0)$$

where $\kappa_{0,\,\min}$, $\kappa_{0,\,\max}$ and λ are the parameters that characterize the quality of a given fabric. Normally the experimental data available from the literature is used to define these geometric draped coefficients (Dowlen, 1976). The parameter m indicates the number of lobes that a given drape fabric forms around a given support geometry. Depending on the value of the overhang $(r - r_0)$ the number m can be selected to be an integer between 3 to 6. Once the intrinsic definition is specified by Eqn. (5.12), the local coordinates u and v can be obtained by integrating the Serret-Frenet equations. The initial boundary conditions are as follows.

$u = 0,\ v = 0,\ \frac{du}{ds} = 1,\ \frac{dv}{ds} = 0$ at $s = 0$. The uv curve of the generatrix will be a spiral followed by a tangential straight line shown in Fig. 5.11. The x, y, z coordinates of the surface are obtained by calculating the uv curve for a given value of the sweep parameter p and using the transformation equation:

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = [T]_L^G \begin{bmatrix} 0 \\ u \\ v \\ 1 \end{bmatrix} \tag{5.13}$$

The resulting surface is shown in Figs. 5.12(a) and (b).

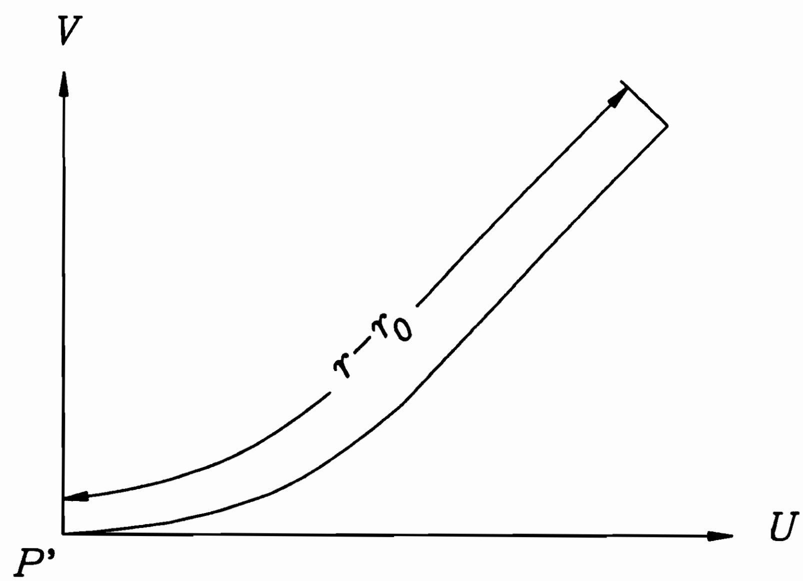


Figure 5.11. Generatrix Curve of a Variable Shape Fabric Drape Surface.

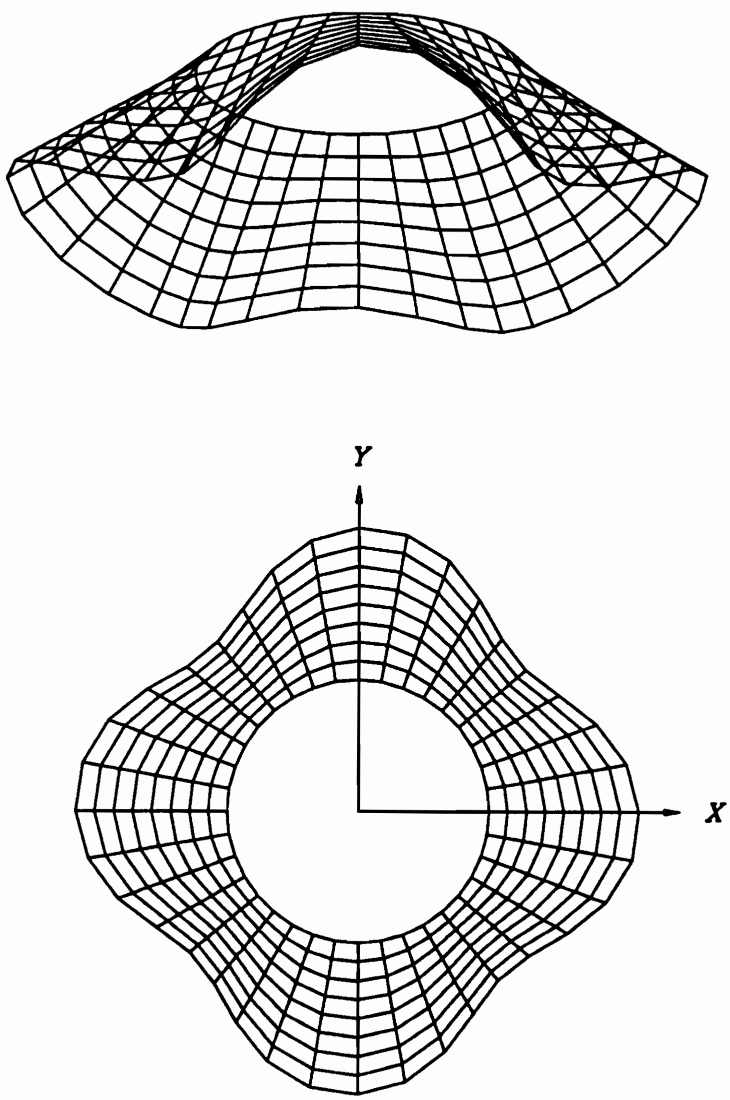


Figure 5.12. Isometric and Top Views of a Fabric Drape Surface.

5.6 Surface Patch Design

In the present section, a methodology of designing a surface patch using the concepts of intrinsic geometry has been outlined. Let us consider the problem of defining a surface patch using the four boundary points P_1, P_2, P_3 and P_4 (Fig. 5.13(a)). Consider the boundary curve C_1 passing through points P_1 and P_2 and lying in a vertical plane Π_1 parallel to the Z-axis. The curve C_1 is specified in terms of its intrinsic shape model and its associated shape design variables shown in Fig. 5.13(b). Furthermore, the end-point tangents $\mathbf{t}_{00}$ and $\mathbf{t}_{10}$ at points P_1 and P_2 respectively are assumed to be specified. Using the methodology outlined in Chapter 3, one can get the local coordinates as well as the global coordinates of the curve C_1 lying in the plane Π_1. The desired surface has also a boundary curve C_2 at the other end passing through points P_3 and P_4. Consider a vertical plane Π_2 passing through points P_3 and P_4 and parallel to the Z-axis. The curve C_2 lies in the plane Π_2. The end-point tangents at P_3 and P_4 are specified as input, say $\mathbf{t}_{01}$ and $\mathbf{t}_{11}$ respectively. The shape model and the associated shape design variables for the curve C_2 are selected as shown in Fig. (5.13 (c)).

The problem of surface design is now to define a surface passing through C_1 and C_2. In order to define such a surface patch, the directrix curve C_3 and C_4 are also defined either in a parametric form or in an intrinsic form. Let u be the parameter defining the curve C_3 and the curve C_4. For a given value of u, where $0 \leq u \leq 1$, $\mathbf{r}_3(u)$ defines the point P_5 on curve C_3 and $\mathbf{r}_4(u)$ defines the point P_6 on the curve C_4. Let Π_u be a plane passing through P_5 and P_6 and parallel to the Z-axis.

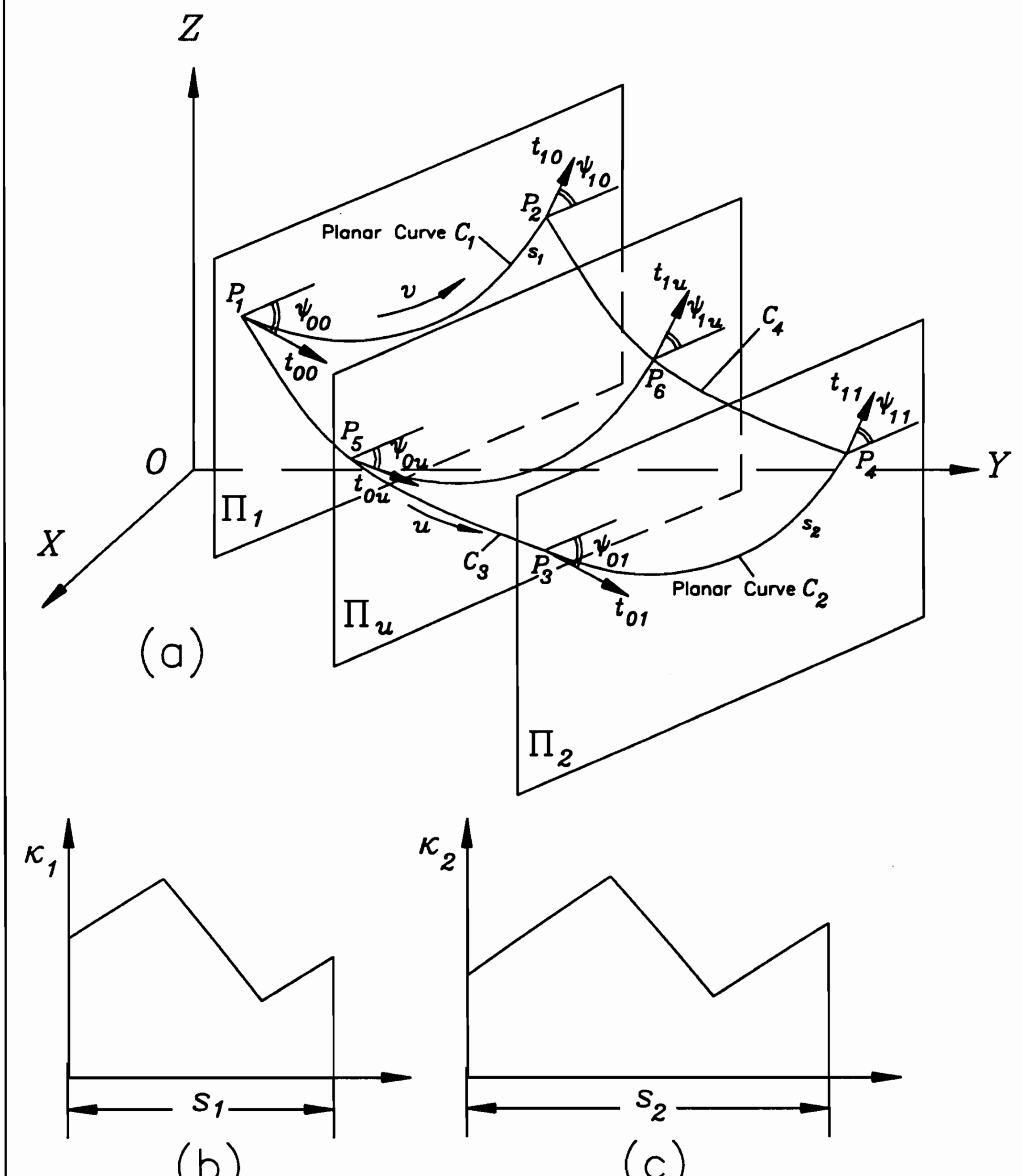


Figure 5.13. Surface Patch Definition Using Intrinsic Geometry.

The proposed surface patch definition, based on the intrinsic geometry, now reduces to the problem of defining a curve passing through P_5 and P_6 as well as lying in the plane Π_u. Let $\mathbf{t}_{0u}$ and $\mathbf{t}_{1u}$ be the end-point tangents at points P_5 and P_6 respectively.

$$\begin{aligned} \mathbf{t}_{0u} &= (1-u)\mathbf{t}_{00} + u\,\mathbf{t}_{01} \\ \mathbf{t}_{1u} &= (1-u)\mathbf{t}_{10} + u\,\mathbf{t}_{11} \end{aligned} \tag{5.14}$$

The intrinsic profile for the curve lying in the plane Π_u can be specified as follows:

$$\begin{aligned} \kappa(u, v) &= (1-u)\kappa_1[(1-v)s_1 + vs_1] + u\kappa_2[(1-v)s_2 + vs_2] \\ &\qquad 0 \le u \le 1, \quad 0 \le v \le 1 \\ s(u, v) &= (1-u)[(1-v)s_1 + vs_1] + u[(1-v)s_2 + vs_2] \end{aligned} \tag{5.15}$$

where s_1 and s_2 are the arc lengths along C_1 and C_2 respectively.

For a given value of u, the above set of equations provide the intrinsic definition of a curve lying in the plane Π_3. Using the methodology outlined in Chapter 3, one can obtain the local coordinates of the curve. The global coordinates can be obtained by transforming the local coordinates to the global frame of reference $O - XYZ$.

5.7 *Illustrative Examples*

A set of examples are discussed in the present section. Example 5.1 illustrates a linearly swept surface with variable shape. Example 5.2 illustrates a surface of revolution with constant shape. Example 5.3 shows how a fabric surface can be modeled. This example is an illustration of a generalized swept surface. Example 5.4 shows how a blend surface can be designed between two end curves. All these examples use the theory described in Sections 5.3, 5.4, 5.5 and 5.6. The generatrix curve in Examples 5.1, 5.2 and 5.3 is defined using a two-line segment model of a linear curvature element.

Example 5.1: Linearly Swept Surface

Figure 5.14 shows the intrinsic definition as well as the Cartesian geometry of the generatrix curve of a linearly swept surface. The surface is to be generated by sweeping the generatrix curve along the W-axis from $w = 0$ to $w = 10$. The generatrix curve lies in the UV plane. It should be noted that the shape of the generatrix is changing as it moves from $w = 0$ to $w = 10$. The local and the global coordinate systems defined in Section 5.3 coincide. In short, the transformation matrix $[T]_L^G$ in Eqn. (5.3) will be an identity matrix. Figure 5.15 shows three orthographic views as well as an isometric view of the surface patch.

Example 5.2: Surface of Revolution

The generatrix curve of a surface of revolution has again been defined using an R-R (two-segment) shape model. This model will have two shape design variables. The intrinsic definition as well as the Cartesian coordinates in the $O_1 - UV$ plane are shown in Fig. 5.16. In the present example, the surface is considered to be a constant shape surface. The surface of revolution is generated by using Eqn. 5.5 where θ varies from 0 to 150 degrees. The orthographic and the isometric views of the surface patch are shown in Fig. 5.17.

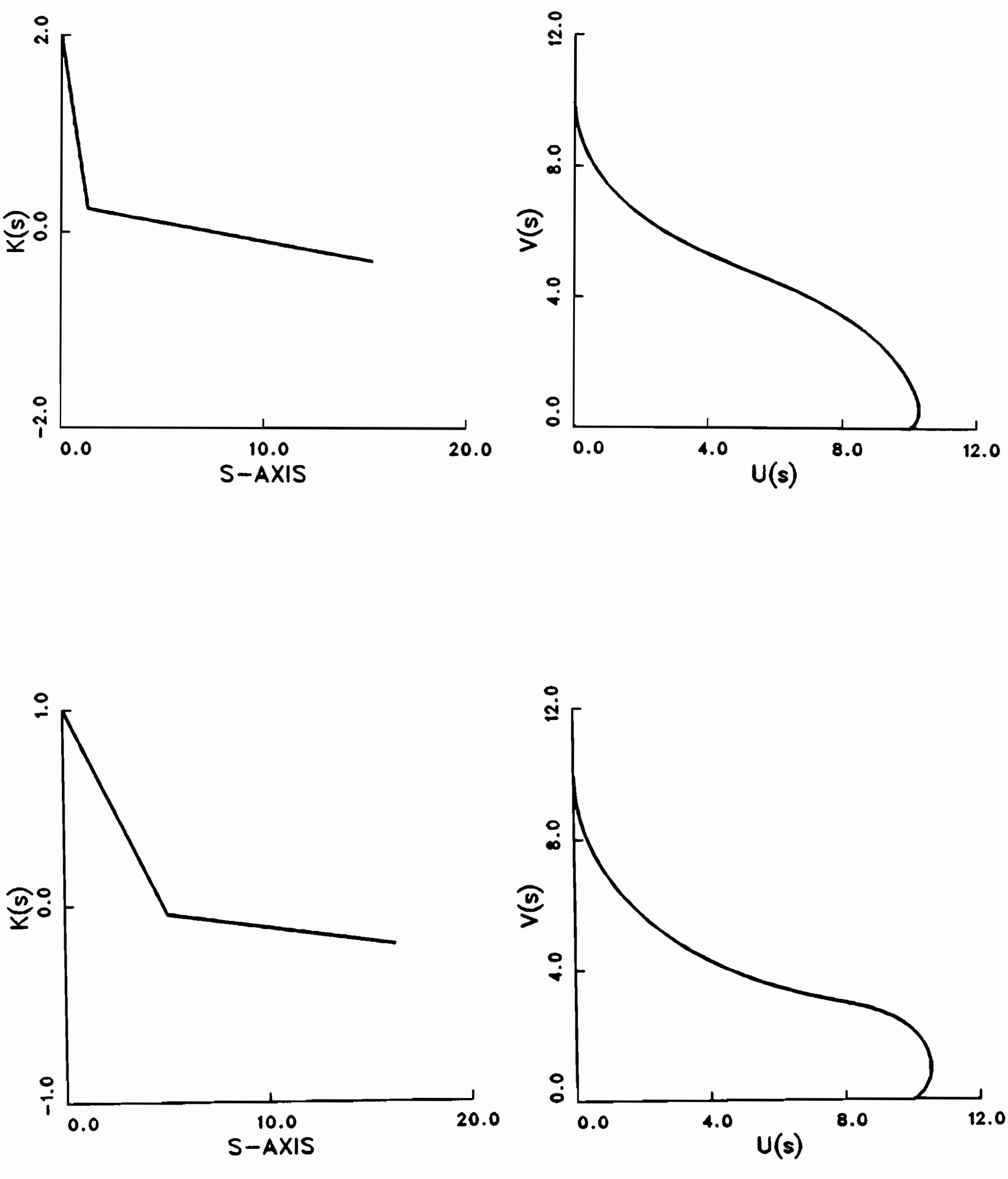


Figure 5.14. Intrinsic Profiles of the Top and Bottom Curves of a Linearly Swept Surface, Example 5.1.

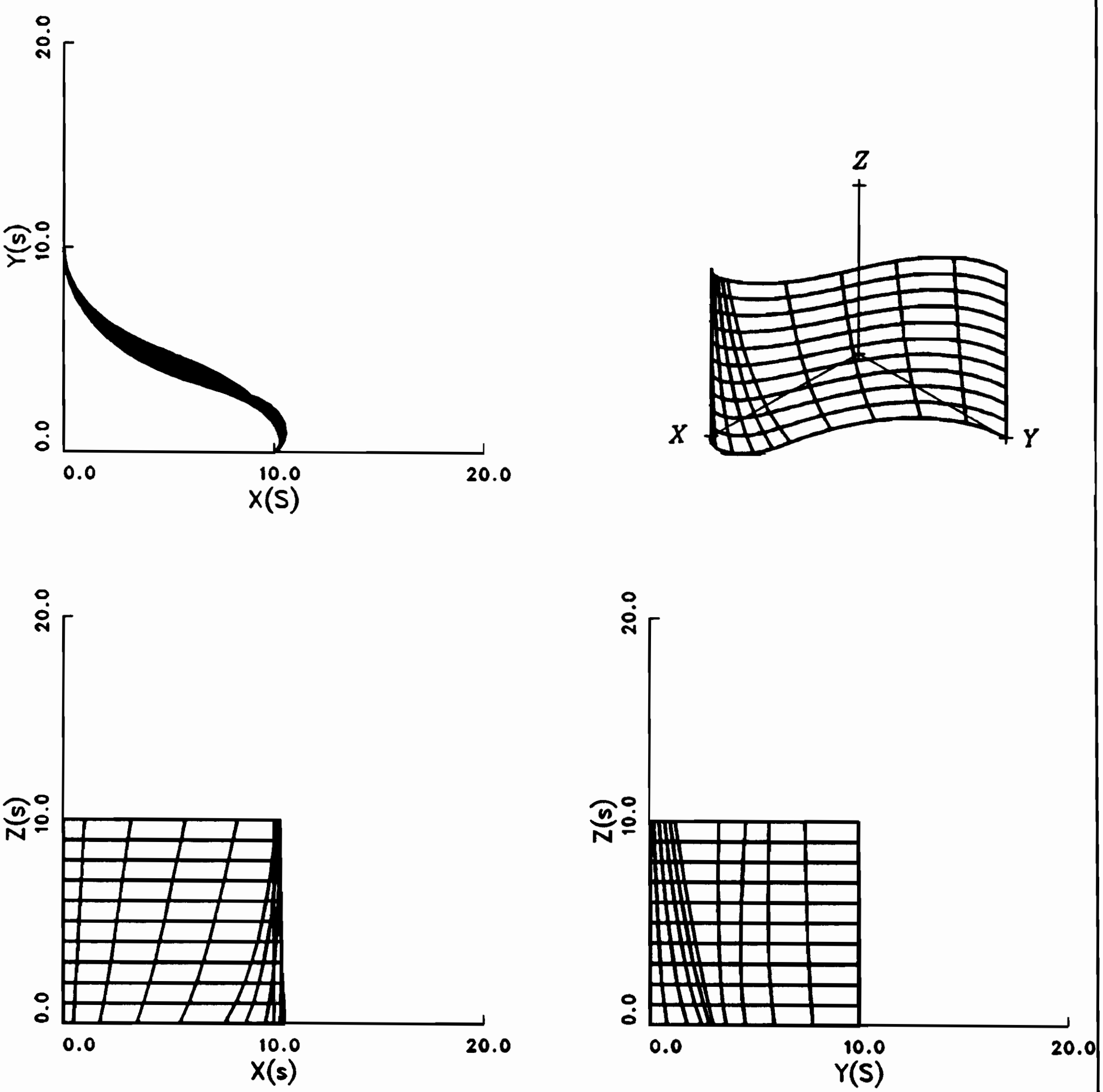


Figure 5.15. Orthographic and Isometric Views of a Linearly Swept Surface, Example 5.1.

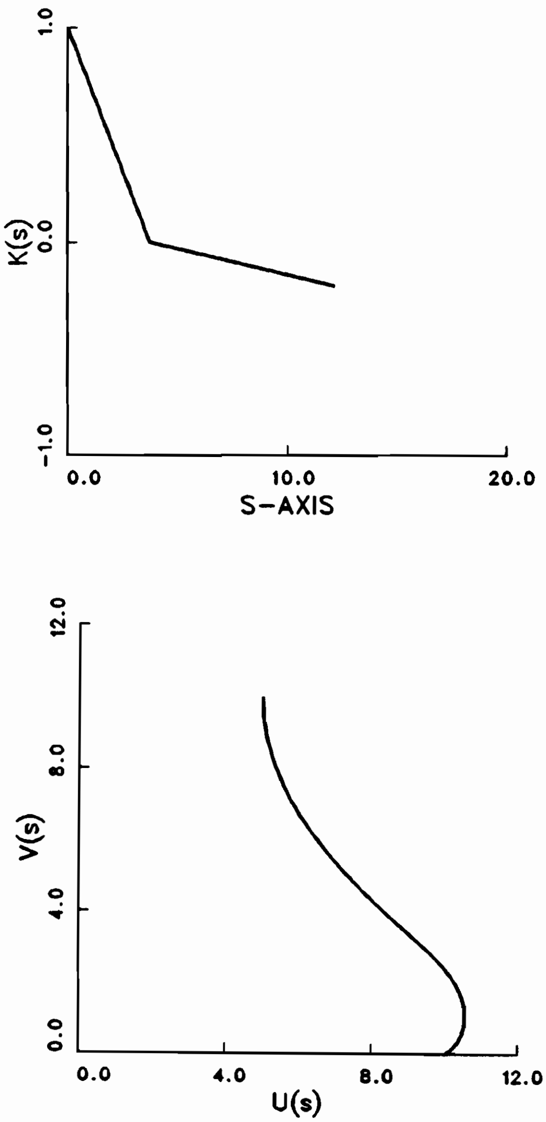


Figure 5.16. Intrinsic Profile and Geometry of a Surface of Revolution, Example 5.2.

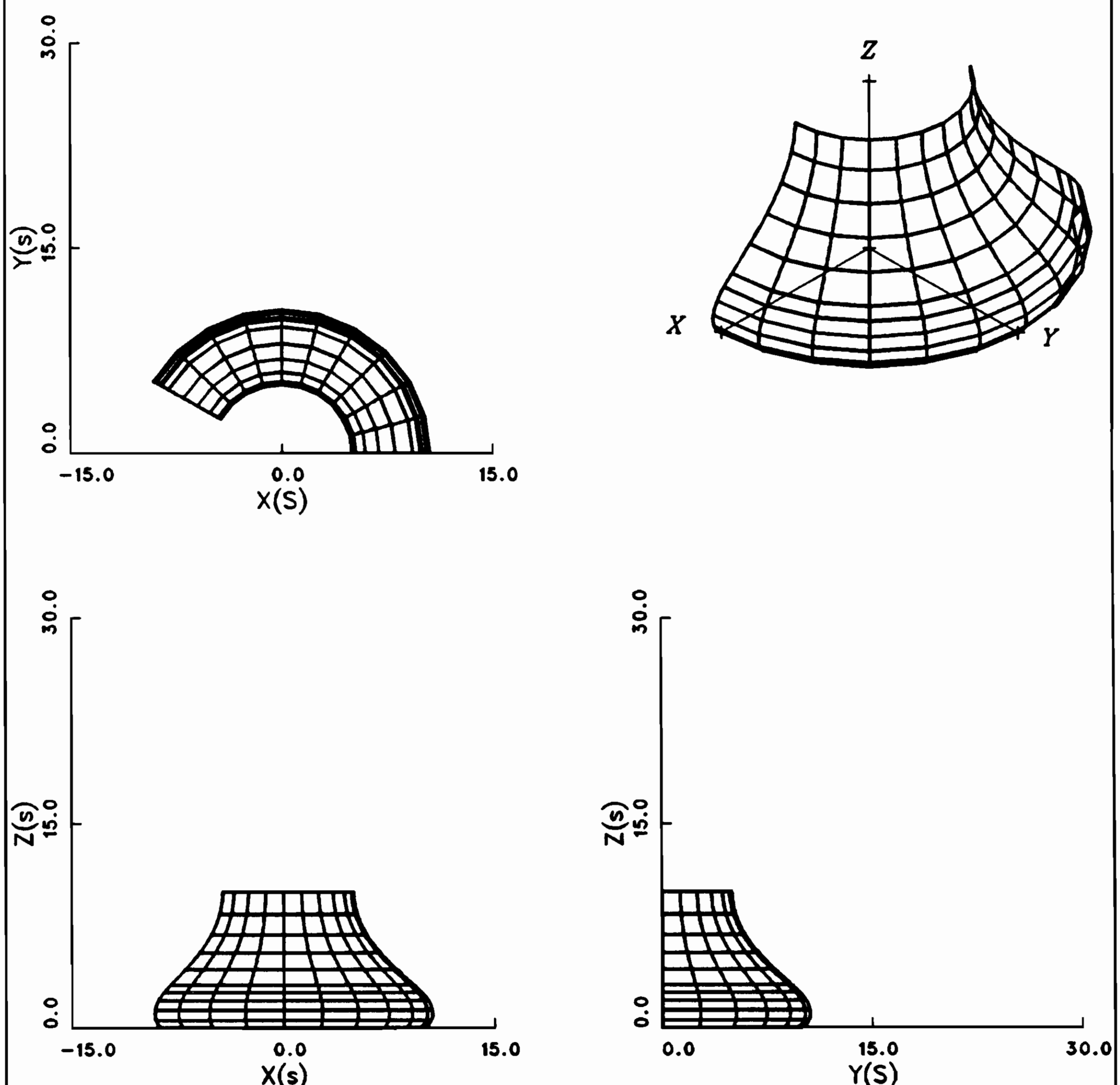


Figure 5.17. Orthographic and Isometric Views of a Surface of Revolution, Example 5.2.

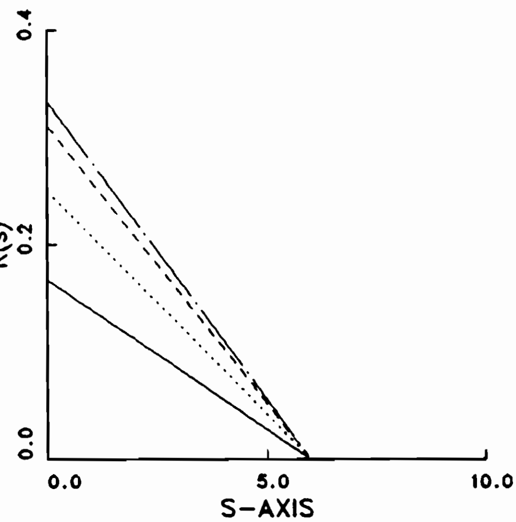
Example 5.3: Fabric Drape Surface

This example illustrates how a generalized swept surface with varying shape of the generatrix can be defined using Eqns. 5.9 through 5.12. Imagine a piece of cloth in the form of a quadrant of a circle of radius $r = 15$. This piece is kept on a flat circular table of radius 5. The centers of the circular table and the cloth coincide. The deformed shape of the cloth is shown in Fig. 5.19. This surface is generated using the definitions of the generatrix curve as shown in Fig. 5.18. It can be seen that the generatrix curve changes its shape from $p = 0$ to $p = 45$ degrees. When p changes from 45 to 90 degrees, the generatrix changes the shape in a symmetric fashion. The shape of the generatrix at $p = 0$ and $p = 90$ degrees is identical. If the cloth is a complete circular piece, then the geometry described in this example will be repeated four times. The resulting shape will be as shown in Fig. 5.12. Depending on the quality of the fabric, the number of lobes formed and the values of $\kappa_{0,\max}$ and $\kappa_{0,\min}$ will change. A research effort is underway to establish values of m, s, $\kappa_{0,\max}$ and $\kappa_{0,\min}$ for commonly used fabrics.

C_2.

Example 5.4: Generalized Surface Patch

This example illustrates the methodology described in Section 5.6. The curve C_1 and C_2 are defined as circular arcs (Fig. 5.20). These two curves are lying in planes $z = 0$ and $z = 10$ respectively. The directrix curves C_3 and C_4 are taken to be straight lines joining the end-points of the curves C_1 and C_2. The four boundary curves of the surface patch are shown in Fig. 5.21. The intrinsic definition of a curve between C_1 and C_2 is given



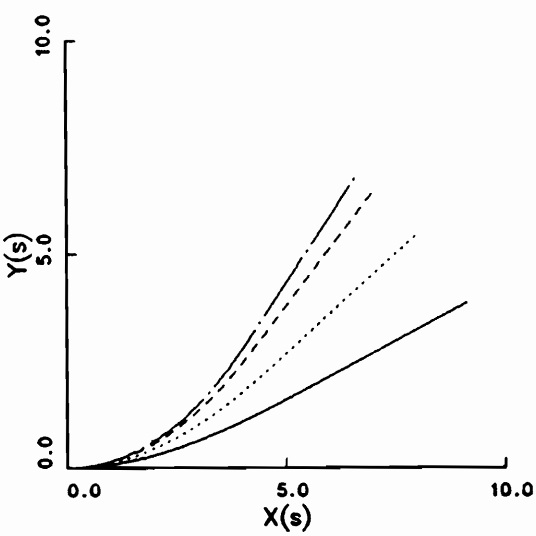


Figure 5.18. Intrinsic Profiles and Geometry Variation of the Generatrix of a Fabric Drape Surface, Example 5.3. The Different Lines Indicate the Generatrix Curves at p = 0, 15, 30 and 45 Degrees.

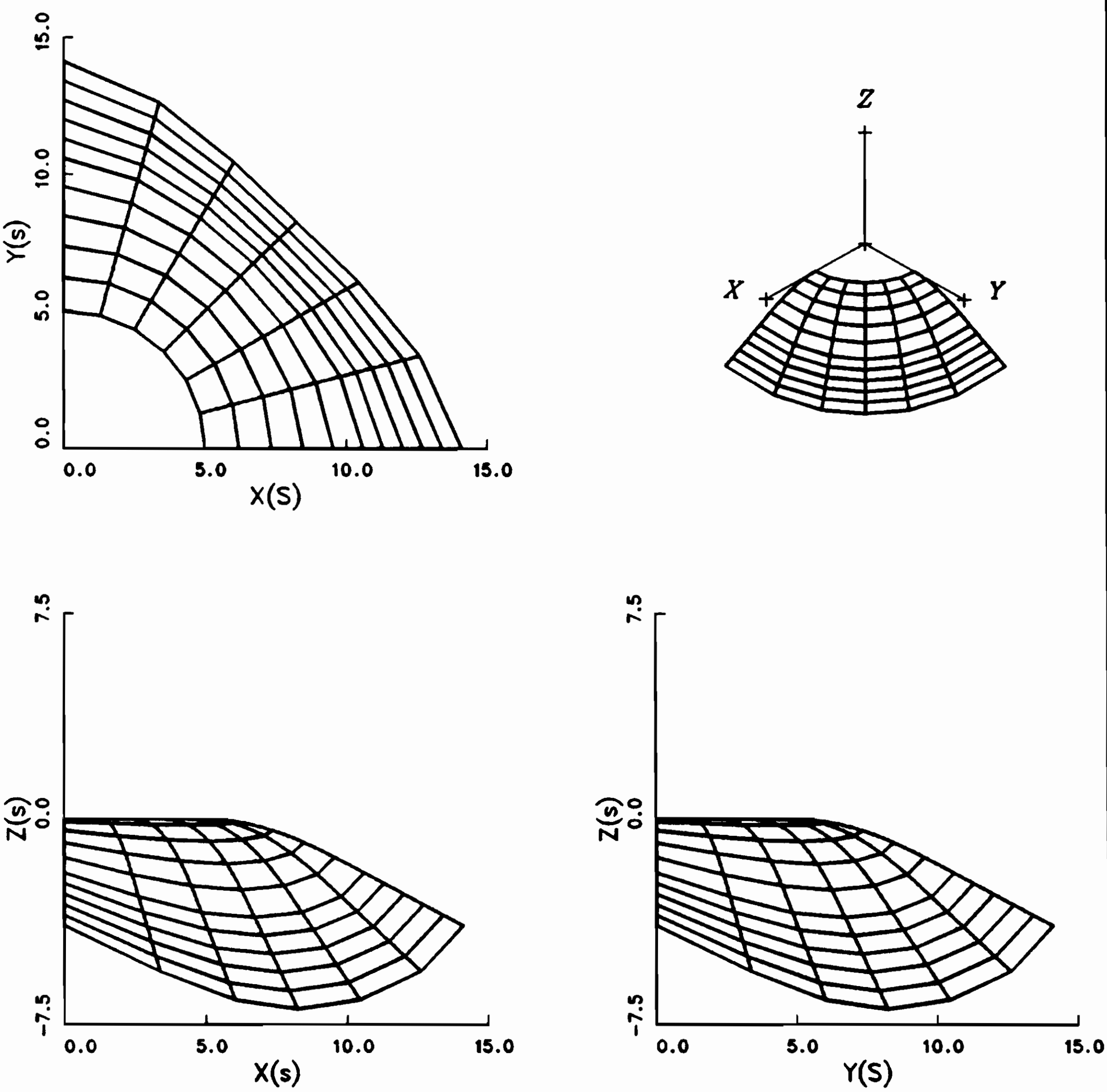


Figure 5.19. Orthographic and Isometric Views of a Fabric Drape Surface, Example 5.3.

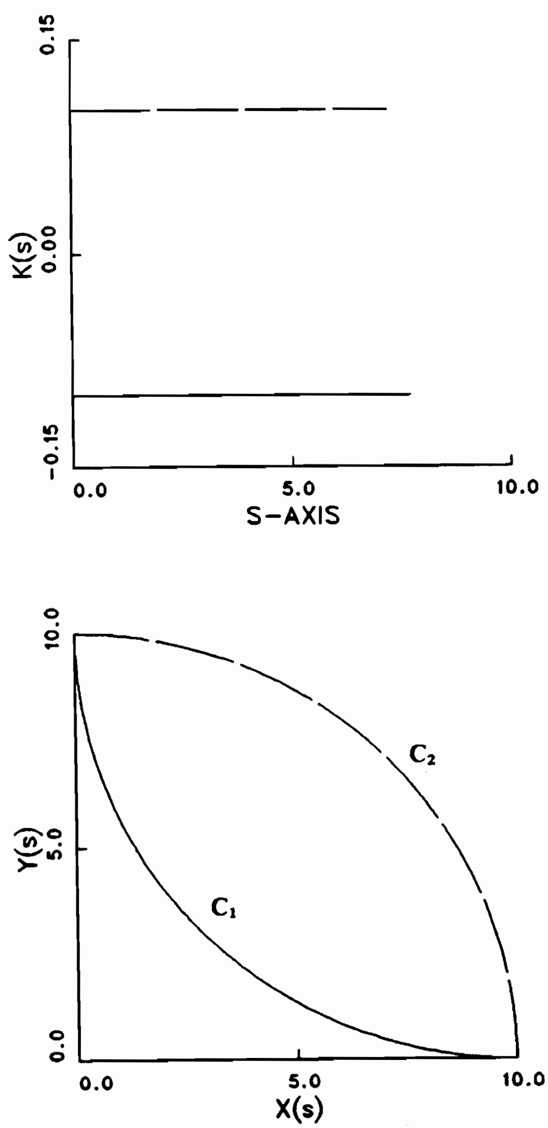


Figure 5.20. Intrinsic Profiles of the Generatrix Curves C_1 and C_2, Example 5.4.

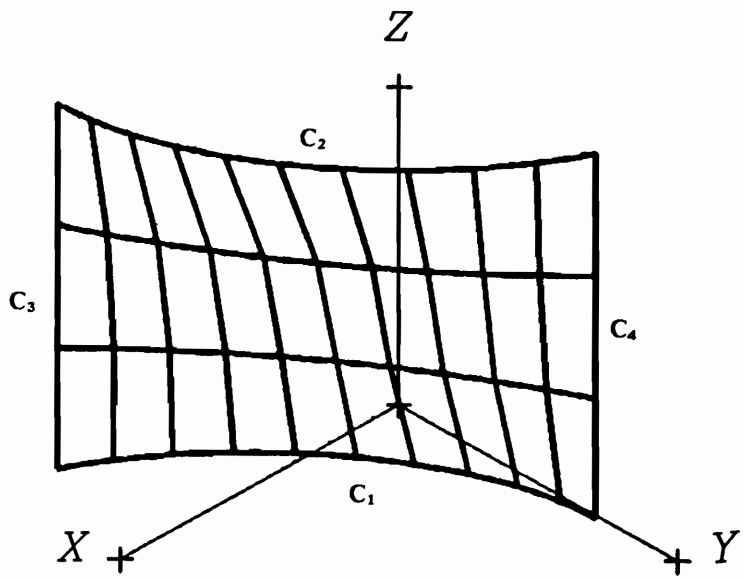


Figure 5.21. Axonometric View of Curves C_1, C_2, C_3 and C_4, Example 5.4.

by Eqn. (5.15). The Cartesian geometry of such a curve is obtained using Eqns. (5.14), (5.15), (2.28) and (2.29). The resulting surface patch is shown in Fig. 5.22. It can be observed that interpolating the intrinsic properties such as the curvature provides a gradual transition for C_1 to C_2.

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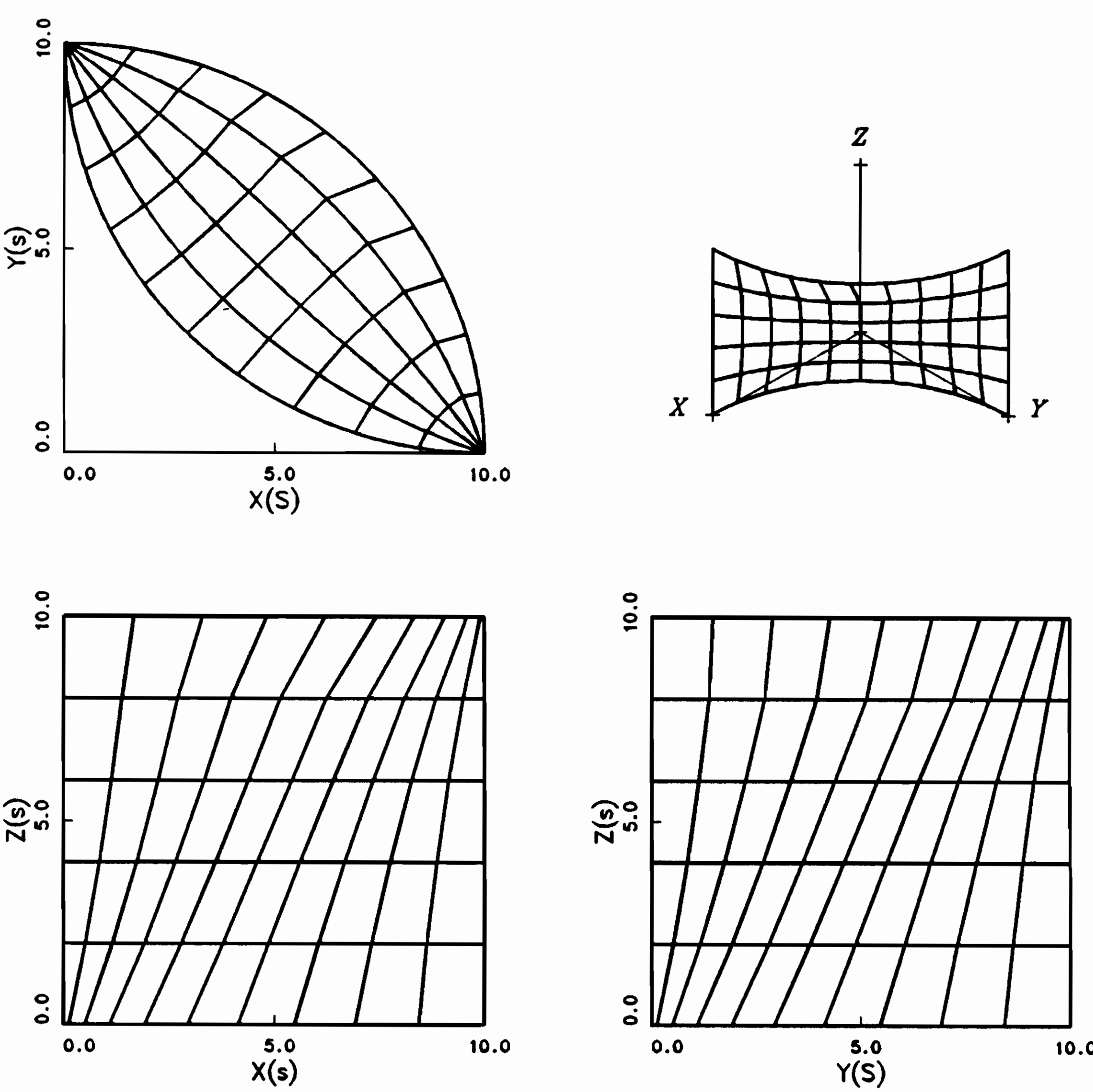

Figure 5.22. Orthographic and Isometric Views of a Surface Patch Defined Using Intrinsic Geometry Concepts, Example 5.4.

6. Conclusions

6.1 Technical Summary

6.1.1 General Observations

Traditionally projective geometry and analytical geometry have been used for developing mathematical models to describe size and shape variables of engineering design and manufacturing problems. Perhaps the main reason these geometries were used in the past is that the computational tools and the display technologies needed for other approaches were not yet available. For instance, the curve of intersection between a cone and a cylinder would be computed and drawn manually on a drawing sheet using an elaborate construction procedure based on projective geometry. Any computational work required for determining the design variables was carried out using simple tools like logarithmic tables and calculators.

During the past two decades, the display technologies and computational tools have undergone a rapid change. Moreover, a unique integration of computational and display devices has occurred in the form of engineering workstations. As the means of computation and display have changed, it has become imperative to re-examine the geometrical theories used for modeling design and manufacturing problems. It can be seen that research in the area of computational geometry during the past two decades has brought out many interesting techniques of geometric modeling such as, for example, solid modeling.

The parametric form of modeling geometry has been found most attractive from two viewpoints. In the first place, it is suitable for rendering the geometry in a parametric form on computer-controlled display devices. Secondly, by changing the position of some control points or auxiliary points, one can manipulate the shape of the geometrical entities such as a curve or a surface. Extensive use of the parametric form has also brought into focus some drawbacks, especially from the engineering design viewpoint. In particular, when a geometrical shape is to be manipulated for better or improved performance, the relationship between the performance indices and the locations of the control points is not explicit. It is in this context that the present work has explored the role of one other form of representation of geometry, namely, the intrinsic form. It has been found that if the shape of a geometrical entity is presented in an intrinsic form, then it can be used for display purposes as well as for engineering analysis, optimization and manufacturing considerations. The present work proposes a methodology of geometric modeling of curves and surfaces by selecting a shape model in the intrinsic space. For the selected shape model, one can characterize a set of shape design variables by assigning suitable numerical values to these shape design variables. In this way, it is possible to define the geometry of curves and surfaces completely. Any change in the values of design variables reflects in the corresponding change of the shape of the geometrical entity.

A major part of the present work elaborates the computational procedures of how this intrinsic geometry approach can be used to design planar curves, three-dimensional curves and surfaces. Though the intrinsic geometry approach provides an elegant way for shape synthesis and manipulations, it requires mapping the given data from the intrinsic space to the Cartesian space. This requires more computations than when curves and surfaces are represented in a parametric form. The design of planar curves has been completely accomplished in the intrinsic form. However, the design of the three-dimensional curves as well as surfaces has been accomplished in pseudo-intrinsic form.

The objective of the present endeavor has been to develop the computational methodology of shape synthesis using intrinsic geometry. There are many aspects that have not been dealt with in this work. Software tools need to be developed to integrate the proposed computational tools in conventional design methods. The proposed method needs to be applied to a broad range of applications to evaluate its effectiveness compared with the parametric and other forms.

6.1.2 Planar Curves

In practice, one often encounters the problem of designing the shape of a planar curve. The method of shape design using a series of linear curvature elements has proven to be effective in this study. The number of shape design variables will depend on the number of elements and the number of constraints. The basic constraints of end-point coordinates and end-point tangents are to be satisfied in every case. The choice of the value of the total arc length is crucial. This value should be greater than the chordal length. However, if it is too large then the shape of the curve will have many bends and turns.

The shape of the curve can be changed by changing the tangent directions, end-point locations and the values of the shape design variables. Numerical examples show that these parameters control the shape to a considerable extent. It has also been shown by means of a numerical example that one can find an optimal shape by varying the values of the shape design variables and evaluating an objective function at every stage. The zero-order sequential search method has been used for arriving at an optimal solution. While simple, the example shows that the proposed method of shape synthesis is amenable for shape optimization work. However, further research along this direction is required.

6.1.3 Three-dimensional Curves

A review was undertaken to find the classical methods in differential geometry for synthesizing the shape of a three-dimensional curve based on a formulation of curvature and torsion. It was concluded that the classical methods are not suitable for solving problems of shape synthesis as encountered in design and manufacturing problems.

The proposed method exploits some of the intrinsic properties such as the skew distance and the skew direction between the end-point tangents. The three-dimensional curve is synthesized as a generalized helix. The helix model seems to provide a solution where it is composed of two two-dimensional curves, namely, the base curve and the rise curve. In practice, designers seem to have a better "feel" for the shape in terms of the base curve and the rise curve variations. An important advantage of the concept of the generalized helix is that the total three-dimensional curve can be segmented as three-dimensional spirals, three-dimensional circles and straight lines with the curvature continuity between segments.

Two numerical examples have been presented. The computational procedures seem to be robust and efficient. For three-dimensional geometries one has to provide a viewing scheme. In the present work, the three-dimensional curves are rendered in four views. The three orthographic views and one isometric view depicts the geometry of the curve. Depending on the requirements one can render other views as well.

6.1.4 Surfaces

Design of three-dimensional surfaces is a critical task in many engineering industries. The prevalent techniques of surface design use the parametric forms extensively. The present work shows that one can exercise effective shape control by defining the generatrix curve in an intrinsic form.

The proposed approach of considering the generatrix in an intrinsic form and the directrix in either a parametric or in an intrinsic form allows us to define not only constant shape surfaces but also variable shape surfaces.

It has been found that, of all surface definitions, the concept of a swept surface is used in practice extensively. This is partly because surfaces so defined can be machined using one of the well-established manufacturing processes such as milling, shaping, turning, etc. The present work outlines an approach of defining linearly swept surfaces, axi-symmetric surfaces, and generalized swept surfaces. In every case, the generatrix is assumed to be a planar curve defined in an intrinsic form. This allows us to build the definition of a surface using the intrinsic definition of a curve as developed in Chapter 3.

A method of defining a transition surface has also been proposed. This method can be used to define a surface patch. The gradual variation of curvature from one end to the other is a salient feature of this definition. Since the curvature of the generatrix curve varies gradually, it also provides a smooth surface having a gradual variation of its principle curvature. The swept surface definitions as well as the transition surface covers a broad spectrum of possible surface definition problems encountered in design and manufacturing applications.

The rendering of surfaces is done by means of showing only one family of curves. The other family of curves is not shown so as to keep the curve mesh clean and simple. An isometric view as well as two orthographic views of a surface are essential to get a clear idea about the shape of a surface. The problem of removal of hidden lines or hidden geometry needs to be considered while displaying three-dimensional surfaces. This has not been dealt with in the present work. The topics of shading and shadows are not considered while displaying the surfaces.

The case study of the fabric drape surface is an interesting application. This example shows that deformed geometries can be modeled using the intrinsic form. In a sense, the shape model selected to define a fabric drape surface represents the bending moment diagram of a long cantilever beam subjected to a uniformly distributed load. If a shape model is analogous to the bending moment diagram of a component, then the Cartesian geometry will represent a deformed shape of that component. This is believed to be an interesting approach for geometrical modeling of deformed shapes.

6.2 *Future Work*

6.2.1 Computational Techniques

Intrinsic geometry approach for designing curves and surfaces is a useful tool for geometric modeling of engineering components. Though this approach is elegant in conceiving the shape of a geometrical entity, it requires mapping the data from the intrinsic space to the Cartesian space. This mapping is accomplished through the solution of the Serret-Frenet equations. The Serret-Frenet equations, in turn, are a set of coupled differential equations. This problem of solving a set of coupled differential equations as a boundary value problem is the computational task involved in the proposed approach. The present work uses the QDAG routine of the IMSL library. It is however desirable to develop a computationally efficient algorithm or procedure for the solution of the Serret-Frenet equation.

In Chapter 3, the $\kappa(s)$ function is approximated as a set of contiguous linear curvature elements. For such a shape model, some parameters are considered as shape design variables and some are considered as dependent variables. These dependent and independent parameters are related through the constraint equations (3.2), (3.3) and (3.4). The solution of the constraint equations has been obtained using the NEQNF routine of the IMSL library. It is required to develop an algorithm that will be computationally efficient for this purpose also.

In the present work, the method for designing three-dimensional curves was found to be computationally more exhaustive than the methods proposed by Struik (1950), Pal (1978b) and Adams (1975). The pseudo-intrinsic approach proposed is useful in dealing with design and manufacturing applications. However, it is an indirect way of designing a three-dimensional curve. The problem of designing a three-dimensional curve using its curvature and torsion information still remains to be tackled. The major difficulty in solving this problem seems to be the lack of a computationally efficient method of solving the three-dimensional Serret-Frenet equations and simultaneously mapping the points of the curve from the moving trihedron $\mathbf{t} - \mathbf{n} - \mathbf{b}$ to the global frame $O - XYZ$.

The intrinsic definition of surfaces is a topic that has not received wide attention of researchers so far. The intrinsic properties of surfaces are useful in many engineering analysis and manufacturing activities. However, very little work has been carried out to synthesize surface geometries based on the intrinsic properties. The proposed approach is an attempt to develop a computationally simple technique of synthesizing surfaces using the intrinsic properties of the generatrix curve. This approach does not address the issue of intrinsic properties of surfaces directly. What is necessary in this area is to solve a problem of the following type: Given the specification of the principal curvatures of a surface patch over a parametric domain, develop a computational algorithm to find the Cartesian coordinates of the points lying on the surface patch in the given parametric domain.

6.2.2 Software Development

A mathematician appreciates the description of geometry primarily in the algebraic form while, for an engineer, perception of shape of a geometrical entity is primarily through the Cartesian geometry and the parametric representation. The intrinsic definition seems to suffer from a handicap that one doesn't get the "feel" of the geometry from the intrinsic definition of a curve or a surface. Fortunately, this problem can be overcome by using a multi-window platform of computer-controlled display techniques. As the future work, it is strongly suggested that software should be developed for design of curves and surfaces using a multi-window user interface for input-output results. One can visualize at least two different windows; one for the Cartesian space and one for the intrinsic space. By inputting the data of a user's choice about the shape model and shape design variables in the intrinsic window, it should be possible to look at the effect of such an input in the Cartesian window immediately.

6.2.3 Shape Optimization

One of the advantages of the proposed approach of shape synthesis is believed to be its potential usefulness in solving shape optimization problems. The literature in the area of shape optimization has indicated that the parametric form of representing shape is inadequate while carrying out the optimization work. In Chapter 3, a simple exercise of optimizing the shape of a Variable Geometry Truss manipulator was carried out. This limited experience shows that considerable work is required to develop a methodology of shape optimization using intrinsic geometry representation. Such a development looks feasible and interesting based on the work reported here.

6.2.4 Design and Manufacturing Applications

Development of any technique of geometric modeling can be evaluated on the basis of its applications as well as on the basis of its usefulness in a wide range of situations. The proposed method of geometric modeling using intrinsic form needs to be evaluated by applying it in various cases related to shape synthesis in design and manufacturing applications. Some examples have been worked out and discussed in Chapters 3, 4 and 5. However, it is felt that some future work can be undertaken to apply the intrinsic geometry approach for the following cases.

- Design of a Cam Profile

Consider the case of a cam profile shown in Fig. 6.1. In practice, it is required to define the shape of a profile between the base radius and the nose radius of a cam profile. A simple design, as in case of a tangent cam, is that of a straight line tangential to the base circle and the nose circle. Such a shape however is detrimental for the performance of mechanism. This is because such a design produces discontinuities of curvatures at points A, B, C and D. Subsequently, these discontinuities of curvature produce discontinuities in the acceleration of the follower resulting in large magnitudes of jerk.

The problem here is to design the shape of the cam such that the acceleration profile of the follower is as close to the desired one as possible.

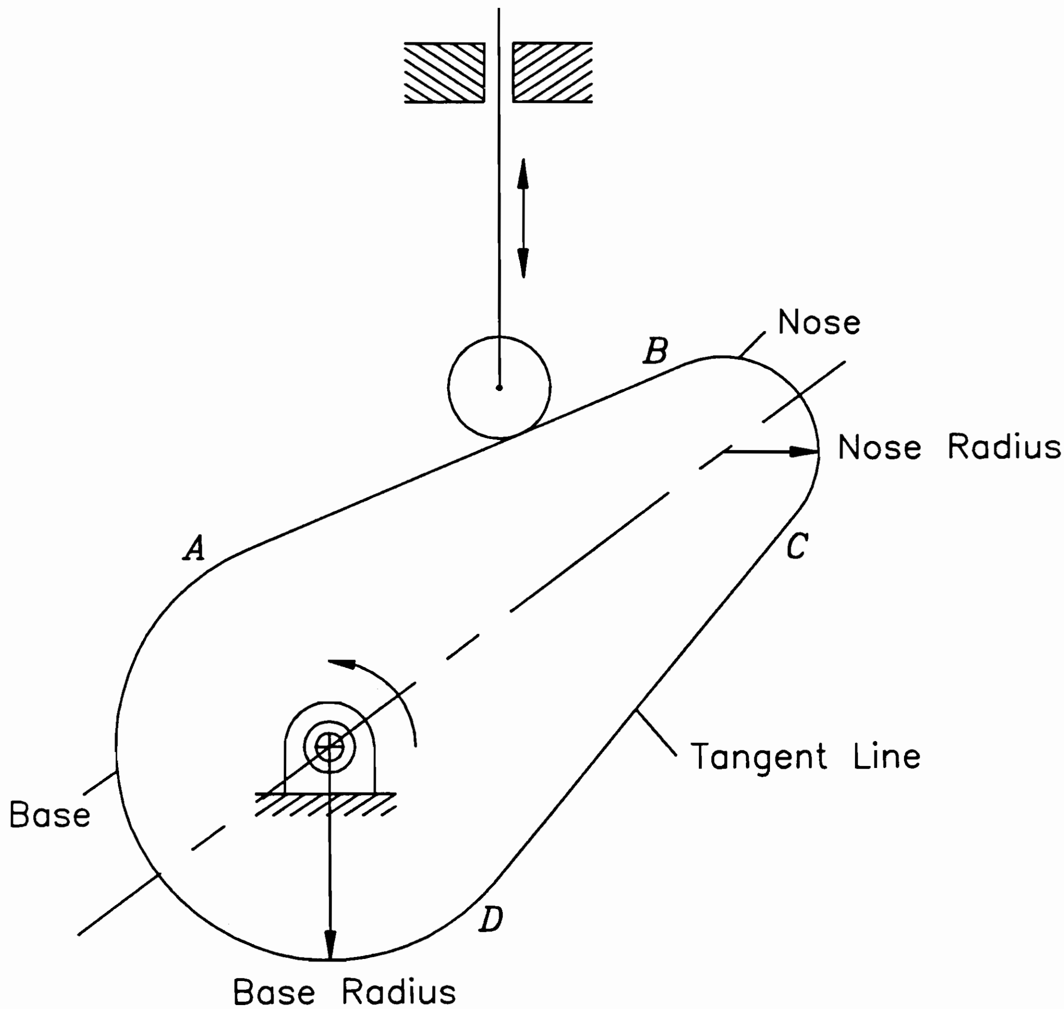

Figure 6.1. Schematic Diagram of a Common Tangent Cam with a Translating Roller Follower.

- Design of an End-effector Path

Design of the path of a mobile robot or an end-effector of a stationary robot requires that it should pass through a given set of points and fulfill certain tangency conditions. The literature in robotics shows that these paths are designed as parabolically blended straight line segments as shown in Fig. 6.2. In case of three-dimensional situations these paths are designed as spline curves. Neither one of these approaches is convenient. In case of the parabolic blend, the discontinuity of curvature produces undesirable dynamic behavior. In the case of spline functions, the generated path has a shape not necessarily desirable because of undesirable changes in the sign of the curvature function. The problem of shape synthesis of the path of an end-effector requires that between consecutive via points, the curve should have continuity of curvature and should have as much of a straight line segment for cruising as possible. It is believed, this problem can be solved using the intrinsic approach presented in Chapters 3 and 4.

- Design of the Path of a Spacecraft for Orbital Change

When a spacecraft is moving from one orbit to another orbit, the geometry of the path in space needs to be defined. There are several constraints to be taken into account. However, it is believed that this transition path, if defined in the intrinsic form, can provide a better dynamic performance of the spacecraft.

- Design of Blend Curves, Surfaces and Fillets

Locations such as the root of a turbine blade or the section of an aircraft wing using the fuselage surface or an abrupt change in the cross-section of a component are illustrative examples of undesirable shape geometries in engineering. Such situations are al-

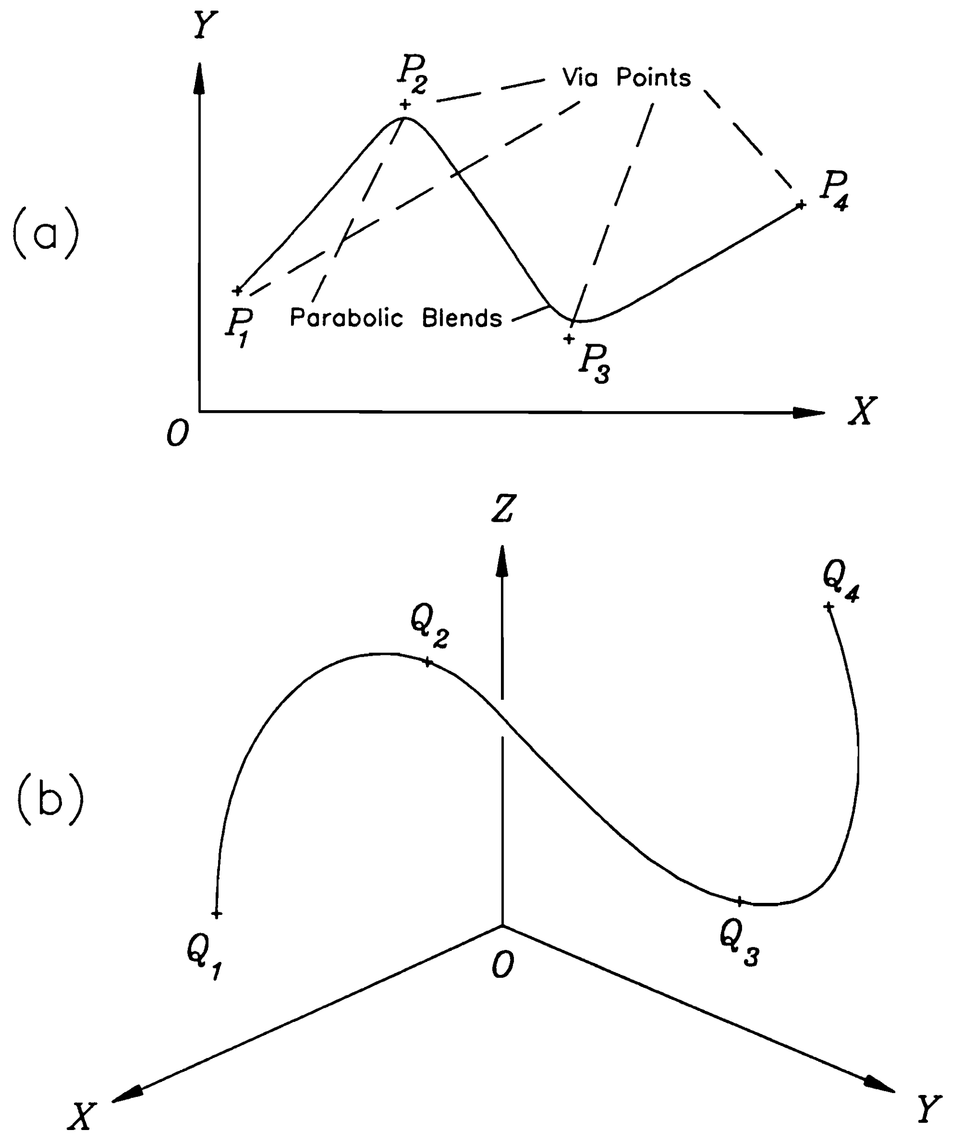

Figure 6.2. (a) A Parabolically Blended Curve. (b) A Spline Curve Passing Through Points Q_1, Q_2, Q_3 and Q_4.

leviated by designing some type of a blended curve or a surface. It will provide a smooth transition from one shape to another.

Given a pair of curves C_1 and C_2 and given corresponding points P_1 and P_2 on C_1 and C_2 respectively, a blend curve such as σ, as shown in Fig. 6.3, needs to be designed. The blend curve should not only have the continuity of position, tangency and curvature at points P_1 and P_2 but also has a certain pre-specified variation of the curvature from one end to the other. It is recommended that such problems be solved using the methods proposed in Chapters 3 and 5.

- Modeling of Shape Deformations

The dual of representation of a shape in terms of its Cartesian and intrinsic forms can be effectively used for modeling deformations such as bending and stretching. It can be seen from the sequence of diagrams (i) through (v) of Fig. 6.4 that affine transformation of a curve represented in the intrinsic space corresponds to a shape deformation in the intrinsic space.

This approach needs to be investigated once the shape deformation and their relations with the shape models of the intrinsic geometry are well understood. It is believed that this information can be used for design of tooling such as extrusion dies, deep-drawing dies and press tools.

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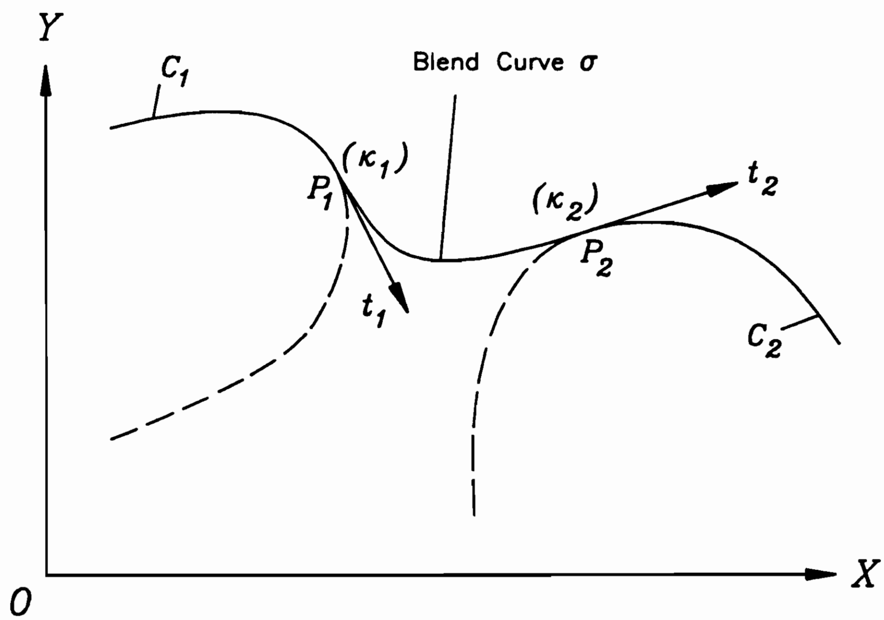

Figure 6.3. A Blend Curve Between C_1 and C_2.

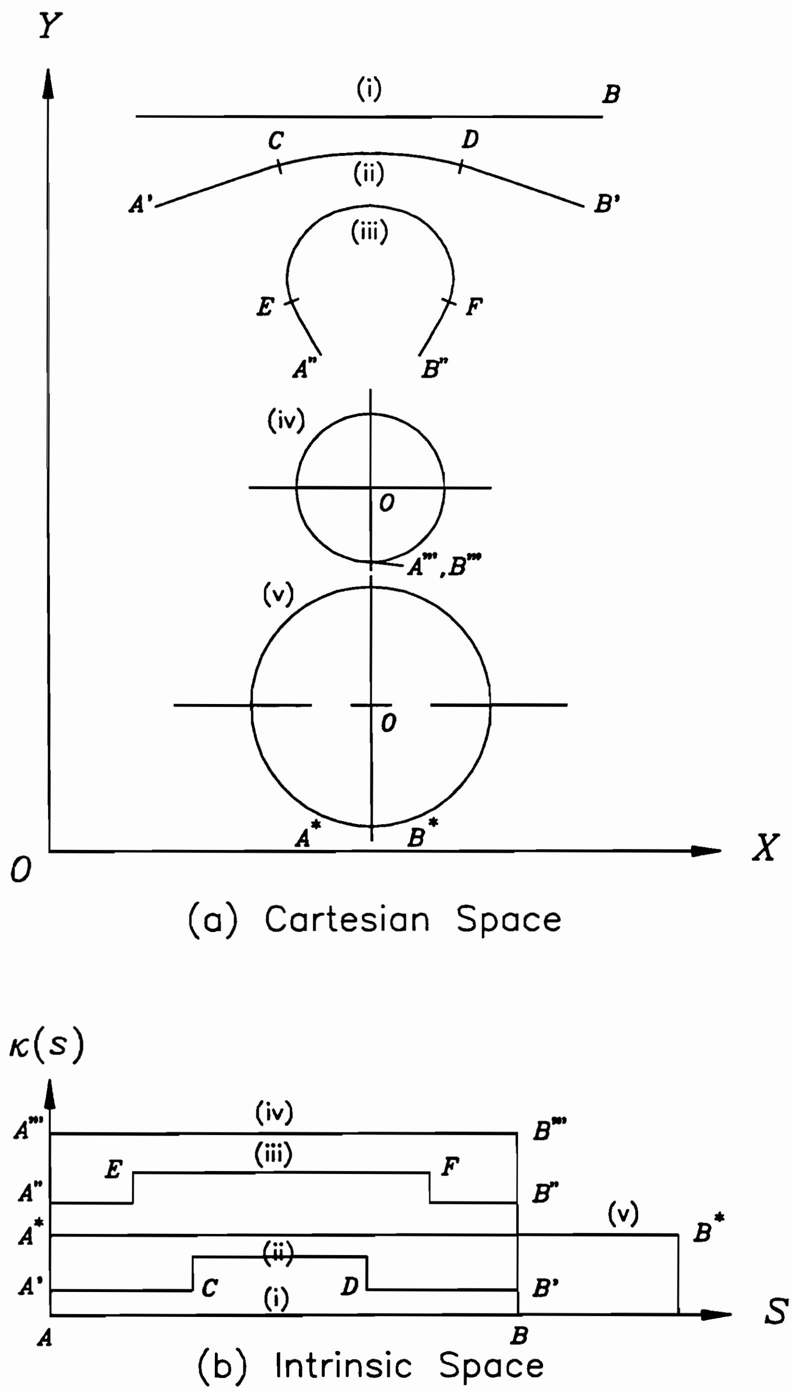

Figure 6.4. Modeling of Deformations in the Cartesian and Intrinsic Spaces.

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Shahriar Tavakkoli