

**PIECEWISE-CONSTANT CONTROL STRATEGIES FOR USE IN MINIMUM
FUEL AEROASSISTED ORBITAL TRANSFERS**

by

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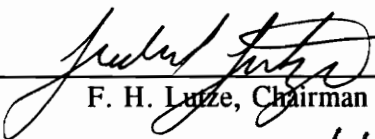
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APPROVED:


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(ABSTRACT)

The use of aerodynamic forces to assist in certain orbital transfers can greatly reduce the fuel consumption as compared with corresponding all-propulsive transfers. Therefore, in seeking minimum fuel trajectories, aeroassisted transfers need to be investigated. A review of the current literature indicates that such problems have been solved almost exclusively via optimal control theory formulations that result in continuously varying control laws. The use of a piecewise-constant strategy allows the controls to vary to a degree necessary to affect changes in the desired state dependent parameters while simplifying the optimization process.

In the process of searching for a tool to produce numerical results, the current research investigates three candidate methods of solving the parameter optimization problem of minimum fuel aeroassisted orbital transfer with piecewise-constant controls. A method based on implicitly integrating the state trajectory is chosen over methods which analytically and explicitly integrate the state trajectory. The implicit method offers improved performance over the explicit method while presenting a more correct solution than the analytic method. The analytic method is shown to suffer from approximations that lead to undesirable solutions.

Analytic expressions for the characteristic velocities of Hohmann and idealized aeroassisted transfers are presented and compared. For a large number of transfers from high Earth orbit to low Earth orbit, the aeroassisted mode requires less fuel. Numerical results are presented for minimum fuel transfer from geosynchronous Earth orbit to low earth orbit for a variety of control strategies. The piecewise-constant strategies are seen to provide solutions which are comparable to those found via optimal control theory.

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List of Symbols

C	Active constraint
C_D	Drag coefficient
C_{D_0}	Zero lift drag coefficient
C_L	Lift coefficient
C_Y	Side force coefficient
c	Constraint
E	Event
E^*	Maximum lift to drag ratio
F	Force applied to vehicle
G	Nonlinear constraint
g	Local acceleration due to gravity or Gradient of the objective function
H	Quasi-Newton approximation to the Hessian
H_R	Heating rate
h	Angular momentum or Altitude
h_{atm}	Altitude of the atmospheric boundary
J	Objective Function
K	Induced drag factor
k^2	Atmospheric constant
L	Augmented Lagrangian function
L/D	Lift to drag ratio
m	Vehicle mass
n	Exponent in drag polar
P	Vector of independent variables
p	Search direction
q	Dynamic pressure

R	Rotation matrix
R_{atm}	Radial distance to atmospheric boundary
r	Radial distance from center of Earth
r_a	Apogee radius
S	Vehicle reference area
T	Segment length (time) or Transformation matrix
t	Time
t_{sw}	Switching time
u	Vector of controls
V	Velocity
V_c	Circular speed
v	Nondimensional velocity
W	Hessian of augmented Lagrangian
x	Vector of states
Z	Nondimensional altitude
z	Null space of Jacobian of active constraint vector
α	Angle of attack or Step length
β	Nondimensional target radius or Inverse scale height or Sideslip angle
γ	Flight path angle
Δ	Defect at segment center
δ	Jump in state vector or Perturbation
$\mathcal{E}$	Energy
Θ_R	Relative pitch angle
θ	Longitude
λ	Scaled lift coefficient or Lagrange multiplier
μ	Gravitational parameter
ρ	Atmospheric density
σ	Bank angle
τ	Specified fraction of phase time length
Φ_R	Relative roll angle
ϕ	Latitude

Ψ_R	Relative Yaw angle
ψ	Heading angle
Ω	Angular rate
ω	Rate of rotation of Earth
GEO	Geosynchronous Earth orbit
HEO	High Earth orbit
LEO	Low Earth orbit
OTIS	Optimal Trajectories by Implicit Integration
OTV	Orbital Transfer Vehicle

1 Introduction

Since the pioneering work of London [12], it has been known that the use of aerodynamic forces to assist in orbital transfers can significantly reduce fuel consumption as compared with all-propulsive transfers. Since then, optimal aeroassisted transfers have been widely researched. A survey paper by Mease [14] published in 1988 provides a concise yet comprehensive overview of the body of work dedicated to optimization of aeroassisted orbital transfers.

An aeroassisted transfer can be defined as one in which part of the transfer trajectory penetrates a planet's sensible atmosphere. An optimal transfer is one in which a particular cost function is either minimized or maximized, subject to a set of constraints, by selecting the appropriate control laws. Cost functions that have been investigated include: minimum fuel, minimum peak heating rate, and maximum plane change. Solved problems range in scope from transfer between coplanar circular orbits to transfer between elliptical orbits with a plane change. A brief description of some of the categories of solved problems is given in Chapter 2 along with applicable references.

It is evident from the literature to date that the problems have been solved almost exclusively via optimal control theory formulations that result in continuously varying control laws. However, as a matter of practical implementation, constant controls are preferable. On the other hand, the controls must be allowed to vary to some extent to cope with uncertainties in the atmosphere and to provide solutions which approach those found by optimal control theory. Therefore, the current research focuses on the use of piecewise-constant control laws in the solution of the optimal aeroassisted transfer problem. More specifically, the problem of minimum fuel transfer between circular coplanar orbits is thoroughly investigated.

2 Background

The optimal aeroassisted transfer problem can be formulated and solved in a variety of manners. Section 2.1 identifies the major assumptions that are prevalent throughout the current literature and adopted in the current research. The remaining sections summarize the major differences in problem specifications and solution methods.

2.1 Common Assumptions

The greatest body of work concentrates on transfers that take place around a single nonrotating planet. Furthermore, the boundary between the vacuum of space and the sensible atmosphere occurs at a predefined radius at which the acceleration due to atmospheric forces accounts for a small percentage of the total acceleration [20]. All motion outside the atmosphere is considered to be Keplerian with the conservation laws for energy and angular momentum completely defining the motion. The propulsive forces required to affect changes during exoatmospheric flight are considered to act impulsively in the form of an instantaneous velocity vector change commonly referred to as a delta-V (ΔV). The fuel consumption due to all exoatmospheric maneuvering is then characterized by the sum of the delta-Vs. This sum is often referred to as the characteristic velocity of the transfer. Reference [2] contains detailed information about Keplerian motion.

2.2 Atmospheric Control Parameters

Most studies assume a transfer vehicle that is capable of varying the angle of attack (α) and the bank angle (σ) during the atmospheric portion of the trajectory. Ordinarily the variation in α is

modeled as a direct variation in the lift coefficient (C_L) which is bounded by upper and lower limits [16]. The lift control allows the vehicle to alter its altitude and thus to alter its energy dissipation rate. The lower the altitude, the greater the atmospheric density which yields increased drag leading to increased energy dissipation and larger heating rates. The bank angle control is used to alter the vehicle heading and is employed primarily during plane change maneuvers.

Most analyses consider vehicles that do not thrust during the atmospheric portion of the transfer trajectory. Such a nonthrusting pass in which energy is inevitably dissipated is termed *aeroglide* or is referred to as an *atmospheric skip*. However, some studies allow for vehicles with thrusting capabilities in the atmosphere to cancel the drag and thereby maintain or even increase their energy [15]. These thrusting modes are commonly referred to as *aerocruise*.

2.3 Geometric Considerations

The geometry of the optimal transfer trajectory is dictated by the geometry of the problem as stated by the analyst. A major distinction in problem specification is whether the transfer trajectory is to take place between orbits that lie in a single plane or between orbits that are at some relative inclination to one another. Also of importance are the size and shape of the initial and target orbits.

2.3.1 Coplanar Transfers

The class of coplanar transfers includes all transfers between orbits that lie in a single plane. The Orbital Transfer Vehicle (OTV), a reusable upper stage that would, for example, transfer payloads from the shuttle to geosynchronous Earth orbit (GEO) will likely use an aeroassisted trajectory for its return to low Earth orbit (LEO) [14]. The OTV is the motivation for much of the study of coplanar aeroassisted transfers. The following two subsections pertain specifically to the minimum fuel transfer problem between circular coplanar orbits about the Earth. The problem of minimum fuel is important because it establishes a lower bound on fuel usage for

other optimal transfers (such as transfers with heating rate constraints).

2.3.1.1 *Optimal Modes*

There are four potentially optimal modes for minimum fuel transfer between coplanar circular orbits using impulsive thrust, two all propulsive and two aeroassisted [16]. The two all propulsive modes are the Hohmann and the biparabolic. The two aeroassisted modes are the aero-elliptic and the aero-parabolic. More practically, elongated elliptical arcs are used in place of the parabolic arcs discussed below.

Consider transfer between two orbits where the initial orbit has a radius greater than that of the target orbit. The four transfers can then be described as follows: The Hohmann transfer begins with a single tangential decelerative burn to lower perigee to the target orbit radius. The vehicle then coasts along the elliptical decent orbit until a tangential decelerative burn is applied at perigee to circularize the orbit at the target radius. The biparabolic transfer consists of a two burn deorbit. The first is a tangential accelerative burn to place the vehicle in a parabolic escape orbit. An infinitesimal burn is then applied at "infinity" to put the vehicle on a parabolic orbit with perigee at the radius of the target orbit. A decelerative burn is applied at perigee to circularize the orbit. The aero-elliptic transfer begins with a tangential decelerative burn to lower perigee to a radius inside the radius of the atmosphere. The aero-parabolic transfer consists of an accelerative burn to place the vehicle in a parabolic escape orbit. An infinitesimal burn is then applied at "infinity" to put the vehicle on a parabolic orbit with perigee inside the radius of the atmosphere. In both of the two aeroassisted modes, the vehicle leaves the atmosphere on an elliptical arc with an apogee radius greater than or equal to the radius of the target orbit. A circularization burn is applied when the vehicle reaches the radius of the target orbit.

As Mease and Vinh [16] point out, the savings in using one of the parabolic modes for transfer from high Earth orbit (HEO) to LEO is small while the time to complete the transfer is greatly increased. For this reason, and keeping in mind that an interesting and relatively uninvestigated class of problems are those with time constraints, the parabolic transfers are not considered in the remainder of this paper.

2.3.1.2 Idealized Transfer

The idealized minimum fuel aeroassisted transfer considered here is the aero-elliptic transfer consisting of a tangential deorbit burn to lower perigee to the radius of the atmosphere. The idealization is that the vehicle can modulate its lift so as to maintain flight at the radius of the atmosphere and dissipate energy until apogee is lowered to the radius of the target orbit. The vehicle then leaves the atmosphere with zero flight path angle and coasts to the target orbit where a tangential accelerative burn is applied to circularize the orbit. Figure 1 illustrates the geometry of the idealized transfer where: r_1 is the initial orbit radius, ΔV_1 is the deorbit or deboost burn, r_2 is the target orbit radius, ΔV_2 is the circularization burn, R_{atm} is the radius of the atmosphere, and ΔV_{atm} is the change in the vehicle's velocity due to atmospheric drag. The fuel usage, or more precisely the characteristic velocity, can then be calculated as the sum of the magnitudes of the exoatmospheric delta-Vs ($\Delta V_1 + \Delta V_2$).

The characteristic velocity from the idealized transfer is now compared to that of a realistic transfer. Any realistic transfer would require a larger ΔV_1 to ensure deep enough penetration into the atmosphere to prevent the vehicle from skipping out prematurely. Thus, the ΔV_1 for the idealized transfer represents a lower bound on ΔV_1 for any realistic transfer. Furthermore, for a single impulse transfer from atmospheric exit to the target orbit, exit with a zero flight path angle into an elliptical orbit with apogee equal to the target orbit radius, leads to the minimum circularizing ΔV_2 [16]. Therefore, the characteristic velocities for the idealized transfer represent lower bounds for the characteristic velocities for any realistic transfer.

The velocities for the idealized transfer can be given as follows:

Let

$$\beta = \frac{r_2}{R_{atm}}, \quad \chi = \frac{r_1}{r_2}, \quad \Delta v_i = \Delta V_i \left(\frac{\mu}{R_{atm}} \right)^{-\frac{1}{2}}$$

The deorbit impulse for the idealized transfer is then given by

$$\Delta v_1 = \sqrt{\frac{1}{\beta}} \left(\sqrt{\frac{1}{\chi}} - \sqrt{\frac{2}{\chi(\beta\chi+1)}} \right) \quad (1)$$

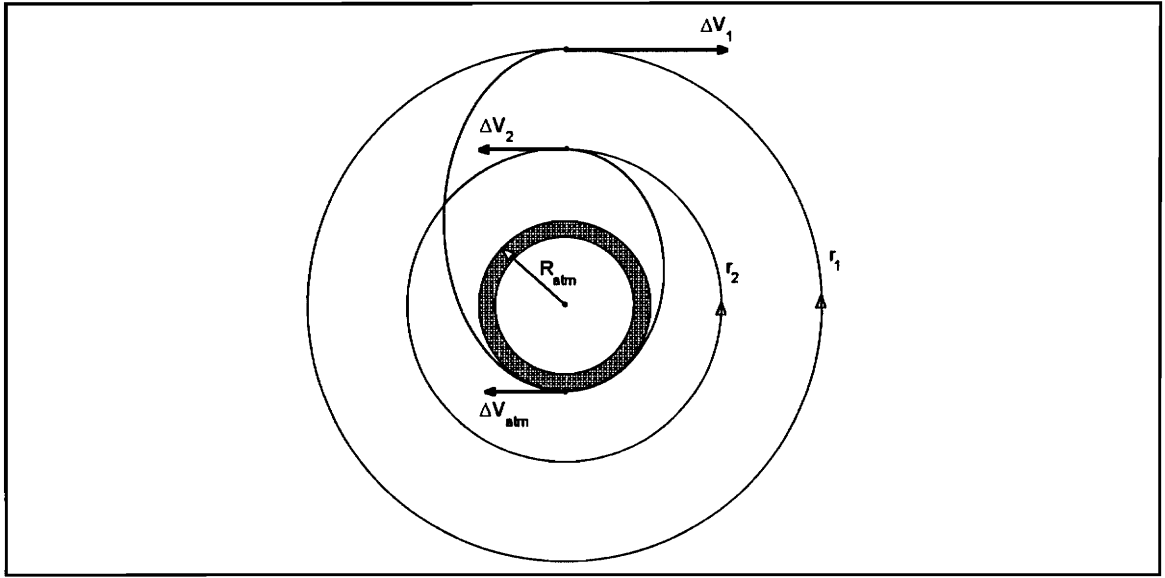


Figure 1. Idealized Aeroassisted Coplanar Transfer

The second impulse used to circularize the orbit at r_2 is

$$\Delta v_2 = \sqrt{\frac{1}{\beta}} \left(1 - \sqrt{\frac{2}{\beta+1}} \right) \quad (2)$$

Thus the total characteristic velocity for the aeroassisted transfer is given by

$$\Delta v_{\text{aero}} = \Delta v_1 + \Delta v_2 \quad (3)$$

$$\Delta v_{\text{aero}} = \sqrt{\frac{1}{\beta}} \left[\sqrt{\frac{1}{\chi}} + 1 - \sqrt{\frac{2}{\beta+1}} - \sqrt{\frac{2}{\chi(\beta\chi+1)}} \right] \quad (4)$$

The above can now be compared to the all propulsive Hohmann transfer which has a characteristic velocity given by

$$\Delta v_{\text{Hoh}} = \sqrt{\frac{1}{\beta}} \left[\sqrt{\frac{1}{\chi}} - 1 + \sqrt{\frac{2\chi}{1+\chi}} \left(1 - \frac{1}{\chi} \right) \right] \quad (5)$$

Figure 2 illustrates the boundary between the two prospective optimal transfers. The solid line indicates those pairs (χ, β) for which $\Delta v_{\text{aero}} = \Delta v_{\text{Hoh}}$. Also plotted are dotted lines indicating an

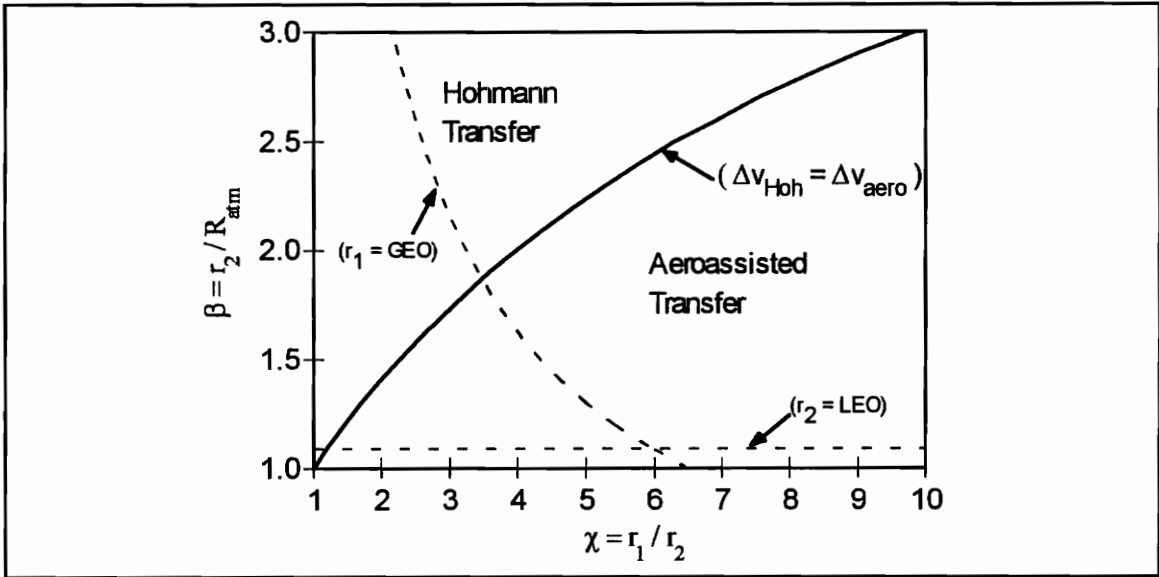


Figure 2. Locus of Equal Characteristic Velocities for Idealized Aeroassisted and Hohmann Transfers

initial geosynchronous Earth orbit (GEO) and an upper bound on low Earth orbit (LEO) target orbits for manned spaceflight [2]. For values below the curve the idealized transfer requires less fuel than the corresponding Hohmann transfer. The plot illustrates that the aeroassisted mode is preferred for many transfers from GEO to lower orbits. It should be noted that the plot does not contain either of the parabolic modes for the reasons outlined in Section 2.3.1.1 .

2.3.2 Noncoplanar Transfers

The class of noncoplanar transfers range from pure plane change maneuvers [9,19,22] to transfer from HEO to LEO with a plane change [18,21]. In the process of using aerodynamic forces to affect a given plane change, apogee can be lowered below the radius of the target orbit. In order to boost apogee to the radius of the target orbit, a reorbit burn (ΔV) can be applied upon atmospheric exit. Such reorbit burns are commonly included in the sequence of maneuvers for noncoplanar transfers. The current research focuses on coplanar transfers.

2.4 Constant Control Solutions

Some work has been dedicated to piecewise-constant control solutions. The work of Martell [13] is the direct motivation for the current research. Martell derives closed form solutions containing between two and six control parameters for the atmospheric portion of the transfer trajectory. Numerical results are obtained for transfers between coplanar and non-coplanar orbits for only the two parameter case. The two parameters being a constant lift coefficient and a constant bank angle. The problem is formulated and solved as a parameter optimization problem with the objective being a minimum characteristic velocity for the transfer. The solutions compare favorably with those obtained via optimal control theory.

Concurrent to much of this research, Zhao [23] tackles the guidance problem with piecewise-constant controls. The guidance problem as stated is to find piecewise-constant values for lift to match as closely as possible exit conditions from the atmosphere that are known to produce the minimum circularizing ΔV (as in the ideal case). The solutions obtained use energy as the independent variable and allow switching the control value only at equally spaced energy points. Numerical solutions are obtained for the case with four segment constant controls.

3 Solution Methods

As mentioned previously, the current research deals primarily with the problem of minimum fuel aeroassisted transfer between circular coplanar orbits while employing piecewise-constant controls. The problem naturally lends itself to a parameter optimization formulation which can be stated as

$$\begin{aligned} \min_{u_i} \quad & J = \|\Delta \vec{V}_1\| + \|\Delta \vec{V}_2\| \\ \text{subject to} \quad & c(x) \geq 0 \end{aligned} \tag{6}$$

where u_i represents the time invariant parameters necessary to specify the deorbit burn as well as those necessary to specify the controls through the atmosphere, and $c(x)$ represents the problem constraints. In order to solve the minimization problem a variety of methods are explored. The remaining sections of this chapter present a chronology of the evolution toward the solution method used to produce the numerical results of the later chapters.

3.1 Analytic

The method developed by Martell [13] is one in which the trajectory through the atmosphere is solved in closed form. Martell derives the equations of motion for a nonthrusting point mass in flight over a spherical nonrotating Earth with an exponential atmosphere. The set of six scalar equations are simplified by introducing a dimensionless arc length as the independent variable as proposed by Vinh [20]. The arc length, s , is defined as

$$s = \int_0^t \frac{V}{r} \cos \gamma \, dt \quad (7)$$

where V is the magnitude of the vehicle velocity, r is the distance from the center of the Earth to the vehicle, γ is the flight path of the vehicle, and t is the time of flight. Further simplification is attained by introduction of the modified Chapman variables, also proposed by Vinh [20]. In particular, the radial position r and velocity V are replaced by the variables

$$Z = \frac{\rho S C_L^*}{2m} \sqrt{\frac{r}{\beta}} \quad (8)$$

$$v = \frac{V^2}{gr} \quad (9)$$

where ρ is the atmospheric density, S is a characteristic area, C_L^* is the value of the lift coefficient at maximum lift to drag ratio, m is the vehicle mass, V and r are as in equation (7), g is the local acceleration due to gravity, and β is the inverse scale height for the exponential density model. Under this model the density is characterized by

$$\rho = \rho_o \left(\frac{r}{r_o} \right)^{-\beta r} \quad (10)$$

where βr is considered a constant ($k^2 = \beta r$) and ρ_o and r_o denote reference values. The complete set of dimensionless equations with arc length as the independent variable are then given as

$$\frac{dZ}{ds} = -k^2 Z \tan \gamma \quad (11)$$

$$\frac{dv}{ds} = \frac{-2vZk}{E^* \cos \gamma} \left[\frac{n-1+\lambda^n}{n} \right] - (2-v) \tan \gamma \quad (12)$$

$$\frac{d\gamma}{ds} = \frac{Z \lambda k \cos \sigma}{\cos \gamma} + \left(1 - \frac{1}{v} \right) \quad (13)$$

$$\frac{d\theta}{ds} = \frac{\cos\psi}{\cos\phi} \quad (14)$$

$$\frac{d\phi}{ds} = \sin\psi \quad (15)$$

$$\frac{d\psi}{ds} = \frac{Z\lambda k \sin\sigma}{\cos^2\gamma} - \cos\psi \tan\phi \quad (16)$$

where E^* is the maximum lift to drag ratio, θ is the longitude, ϕ is the latitude, ψ is the vehicle heading angle (due east), σ is the bank angle control, λ is the scaled lift coefficient control ($\lambda = C_L/C_L^*$ where C_L is the lift coefficient and C_L^* is the minimum drag lift coefficient), and n is the exponent in the usual drag model ($C_D = C_{D_0} + KC^n$).

Martell then makes three simplifying assumptions about the vehicle's trajectory. (1) Since, as Martell states, "the component of gravity tangent to the flight path is small compared to the aerodynamic forces generated," the "gravity" term $(2 - v) \tan\gamma$ of equation 12 is neglected. (2) The gravitational acceleration is said to "nearly cancel the centrifugal relief term" resulting in the $(1 - 1/v)$ term being neglected in equation 13. (3) The vehicle is assumed to cover a small cross range resulting in the small angle approximation of $\cos\phi = 1$ being used in equation 14 and $\tan\phi = 0$ being used in equation 16.

After making the simplifications stated above, the independent variable is changed to γ . Closed form solutions are then found for the new set of partially decoupled state equations. The closed form solutions are demonstrated to be accurate by comparison with direct integration of the simplified equations. It should be noted that the closed form solutions were never compared with integration of the equations resulting from changing the independent variable of equations 11 through 16 to γ without applying the simplifications stated in the above paragraph.

Martell successfully integrated his atmospheric equations into a program by Kelley et al [11] that solved the parameter optimization problem of arbitrary transfer between orbits. More precisely, the routine allows any number of ΔV s separated by Keplerian coasting arcs to be used to solve the problem of an interceptor vehicle that wishes to rendezvous with a passive target. With Martell's modification, the transfer between orbits was formulated as follows: The interceptor

flies along an initial coasting arc. A single deboost ΔV is then used to place the vehicle on a coasting arc that reaches the atmosphere. The closed form equations are evaluated based on the current states and the atmospheric control parameters (λ and σ) to return a change in position, velocity, and time. Such a change in position and time as well as velocity is termed a "generalized impulse". A final coasting arc follows the generalized impulse with a rendezvous ΔV applied at the end of the arc. The cost function is taken as the sum of the two exoatmospheric ΔV s subject to the constraint that the interceptor's position and velocity vectors matched those of the target after the final (rendezvous) ΔV .

Numerical solutions are presented for both coplanar and noncoplanar transfers. Of interest is to note that boundary between the vacuum of space and the sensible atmosphere is specified to occur at an altitude of 60.96 km ($h_{atm} = 60.96$ km). This value for atmospheric altitude leads to large entrance dynamic pressures. The normal g-load calculated as

$$g_{load} = \frac{\rho V^2 S C_L}{2mg} \quad (17)$$

is correspondingly large at atmospheric entry. Martell gives values of three to four times the acceleration due to gravity which is much greater than the near zero value required for a smooth transition between exoatmospheric and atmospheric flight [20].

3.1.1 Advantages

There are several advantages to the analytic generalized impulse formulation. The analytic solution for the atmospheric phase of the transfer greatly improves the speed with which a single transfer can be calculated as compared with formulations that require numerical integration of the equations of motion. This improvement in speed is magnified when coupled with an optimization routine that must interrogate the trajectory routine multiple times while computing the gradients necessary during the solution of the problem. The generalized impulse formulation as implemented in conjunction with the routine of Kelley et al is well suited to solve time constrained problems.

3.1.2 Shortcomings

As mentioned previously, the atmospheric altitude chosen by Martell is unacceptably low. The value can be simply raised to 120 km to alleviate this problem. The Martell optimization code was then modified to allow solution with the piecewise-constant control formulation, with four control parameters, developed but not implemented by Martell. The formulation allows a parameterization of the atmospheric portion of the trajectory consisting of four parameters, a lift coefficient and a bank angle for the descent down to zero flight path ($C_{L, \text{descent}}$ and σ_{descent}) and lift coefficient and bank angle for the ascent to atmospheric exit ($C_{L, \text{ascent}}$ and σ_{ascent}). When this formulation started producing out of plane transfers between coplanar orbits, the major shortcoming of the Martell formulation was discovered.

At this point the simplifications used in the closed form solution were investigated. In order to validate his analytic solution, Martell poses four heading change problems. One of which has the initial conditions given in Table 1 where the subscript e represents conditions upon atmospheric entrance. Martell compares his analytic solution, given in row 1 of Table 2, to numerical integration of the simplified equations of motion, given in row 2 of Table 2. The subscript f refers to conditions upon atmospheric exit. These solutions are seen to compare exactly (to the precision given) indicating that his closed form solutions are an excellent approximation to the exact solutions of the simplified equations.

The third row of Table 2 shows numerical integration results of equations 11 through 16 with only the $(1 - 1/v)$ term of equation 13 being neglected. The flight path angle was used as the independent variable. Again the results compare favorably with the analytic results of row 1. However, when the $(1 - 1/v)$ term is not neglected, the numerical integration routine terminates with an error at a flight path of slightly more than -0.2 degrees. The integration fails because the flight path angle is not monotonic when the $(1 - 1/v)$ term is included. In particular, at a flight path of -0.2°

$$\frac{Z \lambda k \cos \sigma}{\cos \gamma} = 0.243 \quad \text{and} \quad \left(1 - \frac{1}{v}\right) = -0.195$$

which shows the $(1 - 1/v)$ term is 80% as large as the term to which in comparison it is supposed

Table 1. Initial Conditions for a 10° Heading Change

γ_e (deg)	σ (deg)	C_L	t_e (s)	V_e (km/s)	ψ_e (deg)	ϕ_e (deg)	θ_e (deg)
-1.000	78.6	0.13	0.0	7.9107	0.0	0.0	0.0

Table 2. Comparison of Solutions to Heading Change Problem

	γ_f (deg)	t_f (s)	V_f (km/s)	ψ_f (deg)	ϕ_f (deg)	θ_f (deg)
Analytic Solution (Full Approximation)	1.000	249.42	6.923	9.919	1.432	16.507
Numerical Solution (Full Approximation)	1.000	249.42	6.923	9.919	1.432	16.507
Numerical Solution (only $(1 - 1/v)$ neglected)	1.000	249.38	6.920	9.794	1.425	16.515

to be negligible. At a slightly increased value of the flight path the two terms become equal in magnitude yielding a zero derivative for the flight path angle. It is at this point that the flight path angle is no longer monotonic and the integration routine fails.

An examination of two terms of equation 13 reveals when the $(1 - 1/v)$ term cannot be neglected. Given transfer from HEO to LEO with an orbit ratio of 3, yields a value for v of approximately 1.5 at entrance to the atmosphere. The value of 1.5 is fairly insensitive to the value of the atmospheric altitude. This in turn yields values of $(1 - 1/v)$ of around 0.3. The $(1 - 1/v)$ term can therefore only be neglected when the magnitude of the other term is significantly greater than 0.3. That is, when

$$\left| \frac{Z \lambda k \cos \sigma}{\cos \gamma} \right| > 0.3 \tag{18}$$

where k is a constant with a value of 30 for the Earth [13]. The nature of the transfer always yields entrance into the atmosphere with a small negative flight path angle, so that $\cos \gamma$ is

approximately unity. Therefore, the term may only be neglected when the product $Z\lambda\cos\sigma$ is much greater than 0.01 in magnitude. For a given vehicle and atmospheric density model, the Z Chapman variable only depends on the radial distance r . Using the values of Martell yields Z values of 0.008 and 0.000002 for altitudes of 60.96 and 120.0 kilometers respectively. Furthermore, since $\lambda = 7.7C_L$, λ approaches zero as C_L approaches zero. Also $\cos\sigma$ approaches zero for bank angles near $\pm 90^\circ$. Therefore, it is seen that the term cannot be neglected at the high altitudes necessary for smooth transition between atmospheric and exoatmospheric flight. It is also noted that the simplification does not hold for small lift coefficients or large bank angles. The optimization routine was exploiting these facts when it produced the out of plane results for coplanar transfers as those solutions exhibit small values for the lift coefficient coupled with near 90 degree bank angles. The mechanism by which the out of plane transfers yield a savings in total characteristic velocity is unclear. However, the mechanism is observed to be a result of the $(1 - 1/v)$ term being neglected as only solutions found with the term neglected produce out of plane results.

Unfortunately it is the neglecting of the $(1 - 1/v)$ term that allows the partial decoupling of the equations which lead to the closed form solution. Without this simplification a closed form solution is not obtainable (well not obviously obtainable). Since the above analytic method for solution of the atmospheric equations breaks down for all problems of interest, another solution method is sought.

As a final note to this section, the results that Martell obtains from his optimal transfer problems are correct. That is, the neglected term is indeed negligible for the problems he solves. All the solutions exhibit large values for the lift coefficient, small values for the bank angle and all atmospheric flight occurs at low altitudes.

3.2 Explicit Integration

Due to the shortcomings of the method of the previous section, an alternative formulation is sought. However, much time and effort has been exhausted in the refinement of the Martell routine as coupled with that of Kelley et al. Since the code functions well, it is desirable to retain

as much of it as possible. To this end, only the routine that calculates the generalized impulse needs modification. Since an analytic solution is no longer readily obtainable, a numerically integrated one is used.

The modified Chapman variable formulation of Martell is deemed suitable for implementation in the new routine. An equation involving time is obtained from Equation 7 and is given by

$$\frac{ds}{dt} = \frac{V}{r} \cos \gamma \quad (19)$$

Equations 11 through 16 are then divided by Equation 19 yielding six scalar differential equations with time as the independent variable. A routine that numerically integrated the resulting equations replaced the analytic routine leaving the rest of the above code intact. A variable step as well as a fixed step Runge-Kutta integrator are implemented.

3.2.1 Advantages

The advantages of this formulation is two-fold. First, it leaves the vast majority of the optimization code intact. As mention in Section 3.1.1, the code is well suited to solve the orbital transfer problem. Secondly, since the equations are being numerically integrated there is no advantage gained by making any of the simplifying assumptions that lead to the demise of the analytic method. Therefore the atmospheric equations used in the numerically integrated scheme were a better representation of the physical realities of the problem.

3.2.2 Disadvantages

Similarly there are two major problems with this formulation. The first is speed. Numerical integration takes considerably more computer time than do analytic solutions. A fact that is multiplied considering that the optimization routine must evaluate the equations multiple times for gradient calculations. Secondly, the rendezvous ΔV is very sensitive to the atmospheric exit conditions. When the atmospheric equations are numerically integrated over an atmospheric skip, the resulting exit conditions often are not smooth with respect to the control parameters. This

lack of smoothness inhibits convergence of the optimization routine. These two drawbacks, when taken together, make solutions with this formulation few and far between.

3.3 Implicit Integration

In order to speed up performance and improve convergence, another alternative is sought. A method in which the states as well as the controls are parameterized was used by Enright and Conway [3] to solve optimal finite thrust spacecraft trajectories. With the states being represented by a set of parameters (generally piecewise Hermite cubics), some sort of implicit integration scheme is required to satisfy the equations of motion. Detail of such a scheme is presented in Chapter 4 and in Reference [7].

3.3.1 Advantages

The implicit integration scheme directly addresses the two disadvantages of the explicit integration routine. By parameterizing the states, an increase in speed is obtained. Also, as Enright points out, this method avoids the inherent sensitivity of other methods and demands less accurate initial guesses for successful solution. Since control functions are sought, a solution is obtained that is said to approximate that found from optimal control theory without having to derive (or satisfy) the necessary conditions for optimality using calculus of variations techniques. Therefore, comparison between piecewise-constant control solutions and the so called approximate optimal control solutions can be easily acquired.

3.3.2 Disadvantages

Inherent in this formulation is a tradeoff between speed and accuracy. From a speed standpoint, the fewer parameters used to describe the states, the faster the evaluation of a trajectory. However, with fewer parameters, the trajectory will be less accurate and may tend to deviate from the physical realities of the specified problem. Nonetheless, this method, as described in Chapter 4, is used to solve the optimal transfer problem that is stated explicitly in Chapter 5.

4 Software (OTIS)

Optimal Trajectories by Implicit Simulation (OTIS) is the title given a general purpose trajectory optimization program developed by Boeing Aerospace for the United States Air Force [8]. OTIS incorporates flexible data handling and modeling capabilities and would prove relatively simple to use if not for certain errors and omissions in the documentation. The sections in this chapter present an overview of some of the major features of the code. While this chapter presents information necessary to understand the underlying mathematics used for the optimization of the trajectory, only those features pertinent to the current research are shown in any detail. The information in this chapter is taken largely from the Formulation Manual accompanying the Version of OTIS released November 1988.

4.1 *Modes of Solution*

OTIS operates in four distinct modes.

- Mode 1: Generates a discrete explicitly integrated trajectory from a defined set of controls and a fixed set of initial conditions.
- Mode 2: Solves targeting problem by varying a set of independent variables to provide a trajectory which satisfies a given set of boundary conditions.
- Mode 3: Solves parameter optimization problem where a set of independent variables is varied to minimize a cost function while satisfying given problem constraints.
- Mode 4: Solves optimization problem by generating control functions that minimize a cost function while satisfying given problem constraints. This is the only mode that incorporates an implicitly integrated trajectory.

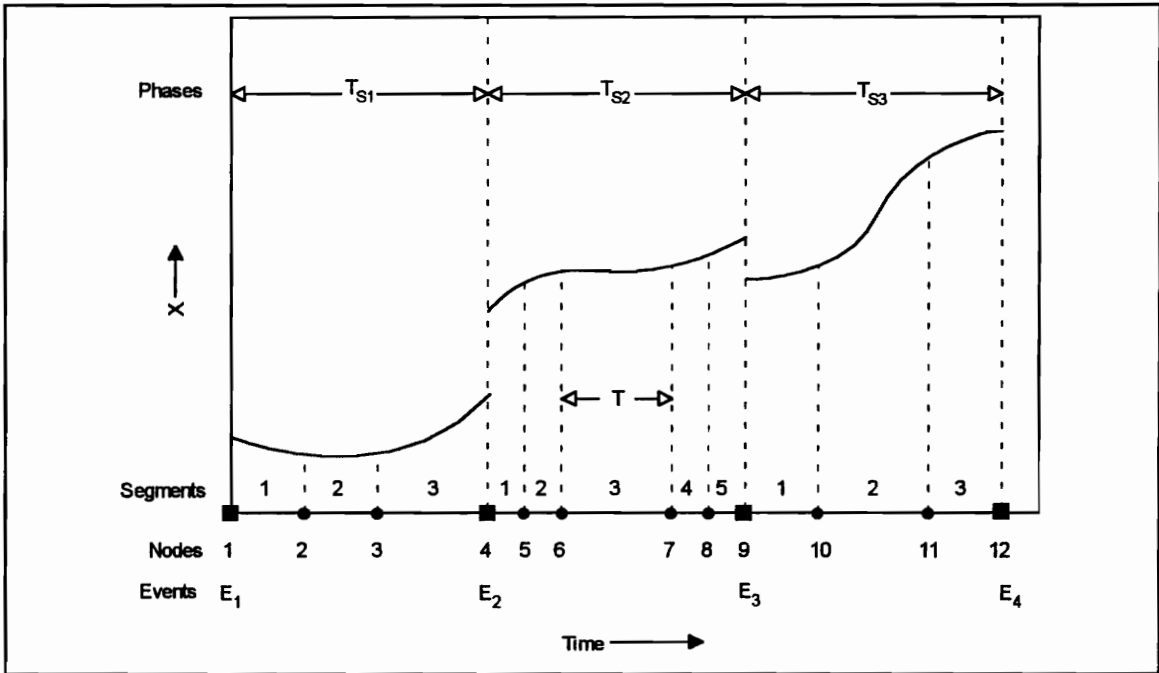


Figure 3. Relationship among Phases, Segments, and Nodes

In order to reap the benefits of the implicit integration scheme, mode 4 is used for all trajectory optimization solutions presented in the current research. Once a mode 4 solution is generated, a mode 1 (explicitly integrated) trajectory is often produced as a check of the accuracy of the implicitly integrated trajectory. In particular, the fixed step fourth order Runge-Kutta integrator is used to produce the explicitly integrated trajectories of the later chapters.

4.2 Phase Definition

The trajectory consists of a sequence of arcs called phases. Phases define the boundaries at which the data needed to describe the various models may be changed. For example, an instantaneous change in mass caused when a rocket stage is jettisoned would occur at a phase boundary. Also, the control and constraint sets may change at phase boundaries. The start and end times for each phase are defined as events. For mode 4, each phase is subdivided into a sequence of segments. The endpoints of the segments are called nodes. Figure 3 illustrates the relationship between phases, events, segments, and nodes.

4.3 Optimization with Implicit Integration

The states and controls within each segment are represented by polynomials in time. The differential equations of motion

$$\frac{dx}{dt} = f(x, u, t) \quad (20)$$

where x is the state vector, u is the vector of controls, t is the time, and f is a vector valued function, are always satisfied at the nodes (Hermite Interpolation). Implicit integration is then used to enforce the differential equations at the segment centers. The length of each segment (T) is a given fraction of the phase time length (T_p). That is, $T = \tau T_p$.

Let the states, x , over a given segment be represented by cubics of the form

$$x = c_0 + c_1 s + c_2 s^2 + c_3 s^3 \quad (21)$$

where $t = sT$ with $s \in [0, 1]$. Now, denoting conditions at the beginning of the segment with the subscript 1 and conditions at the end of the segment with the subscript 2, let

$$x(0) = x_1, \quad x(1) = x_2, \quad x'(0) = x'_1, \quad x'(1) = x'_2$$

where the primes indicate differentiation with respect to s ($x' = dx/ds$). Differentiating equation 21 with respect to s and evaluating at 0 and 1 gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x'_1 \\ x_2 \\ x'_2 \end{bmatrix} \quad (22)$$

Inverting the 4x4 matrix to solve for the coefficients yields

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{bmatrix} \quad (23)$$

Substituting the coefficients from equation 23 into equation 21 and evaluating at the segment center ($s=1/2$) gives

$$x_c = \frac{(x_1 + x_2)}{2} + \frac{(x_1' - x_2')}{8} \quad (24)$$

Transforming to segment time, $x' = T \, dx/dt$, yields

$$x_c = \frac{(x_1 + x_2)}{2} + \frac{T(f_1 - f_2)}{8} \quad (25)$$

where f_i is obtained from equation 20 by evaluating at x_i . In a similar manner, the derivative of the cubic polynomial at the center of the segment is

$$x_c' = \frac{-3(x_1 - x_2)}{2T} - \frac{(f_1 + f_2)}{4} \quad (26)$$

where x_c' now represents the derivative (dx/dt) of the cubic polynomial of equation 21 evaluated at the center of the segment. Evaluating the differential equations of motion (equation 20) at x_c yields f_c . The difference between the derivative found from evaluation of the equations of motion and that found from the cubic polynomial is termed the defect and is given by

$$\begin{aligned} \Delta &= f_c - x_c' \\ &= f_c + \frac{3(x_1 - x_2)}{2T} + \frac{(f_1 + f_2)}{4} \end{aligned} \quad (27)$$

The defect must be driven to zero (by selecting x_1 , x_2 , and T_s) in order for the cubic polynomial to produce an accurate solution to equation 20.

The above relationships are illustrated in Figure 4. The defects for each state are evaluated and

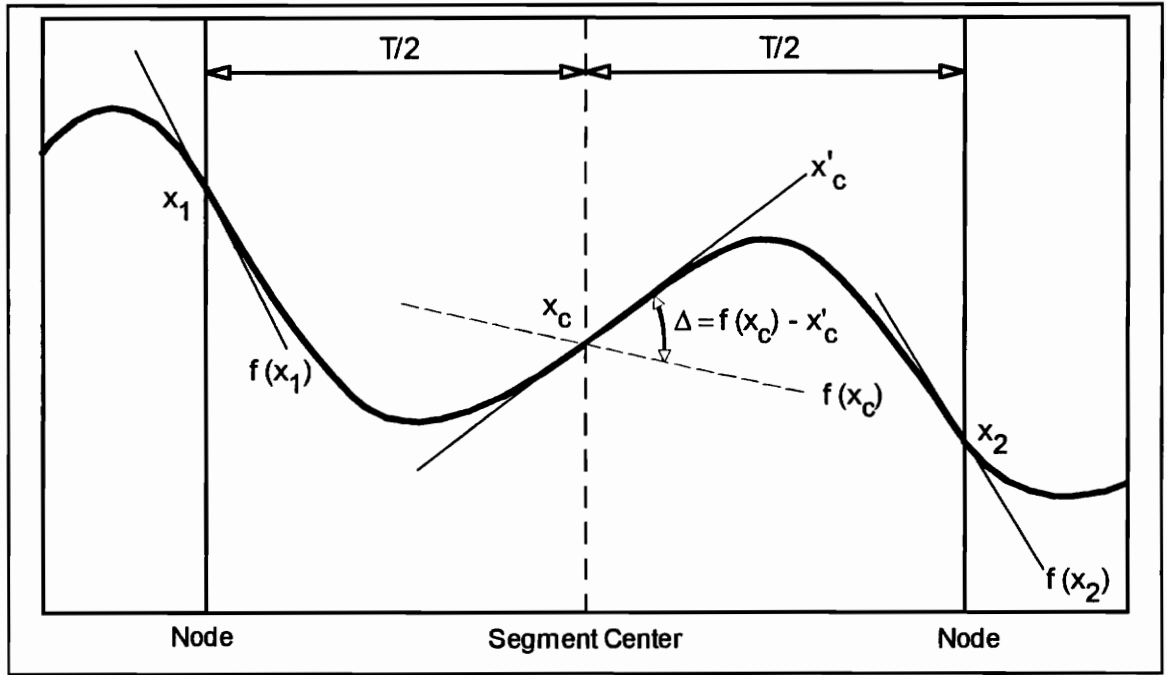


Figure 4. Implicit Integration

equated to zero. This constitutes a set of nonlinear algebraic equations that are functions of the controls and states at each node and the phase time lengths. The values and slopes of the controls at each node are free. Cubic interpolation, analogous to that of equation 24, is used to give the control values at the segment centers. The second derivatives of the controls are constrained to be continuous at interior nodes and zero at the phase boundaries. Boundary conditions of the form

$$l_{BC} \leq BC(x, u, E) \leq u_{BC} \quad (28)$$

are evaluated at the phase boundaries and path constraints of the form

$$l_{PC} \leq g(x) \leq u_{PC} \quad (29)$$

are evaluated at the nodes and segment centers. Equations 28 and 29 provide additional constraints to be satisfied. A set of linear equations that allow discontinuities in the states at the events (i.e. between phases) connect all the phases and are given by

$$x(E_j^+) = x(E_j) + \delta_j \quad (30)$$

where δ_j represents a vector jump in the states at event j .

All of the independent variables are collected into a single vector P which is given by

$$P^T = [Z^T, E^T] \quad (31)$$

where

$$Z^T = (x_1^T, u_1^T, \dots, x_n^T, u_n^T)$$

and where the superscript T denotes the matrix transpose and x_i and u_i represent the state and control vectors at node i and n indicates the total number of nodes. Collecting the nonlinear equations yields

$$G^T = [\Delta^T, C_N^T, B_N^T] \quad (32)$$

where

$$\Delta^T = (\Delta_{11}, \Delta_{12}, \dots, \Delta_{ij}, \dots)$$

and

Δ_{ij} is the defect for the i th state at the j th segment center.

C_N is the collection of all nonlinear constraints from Equation 29.

B_N is the collection of all nonlinear boundary conditions from Equation 28.

Now, the trajectory optimization problem can be stated as

$$\text{minimize } J = \phi(P)$$

$$\text{subject to } l \leq \begin{bmatrix} P \\ AP \\ G(P) \end{bmatrix} \leq u \quad (33)$$

where AP is composed of all the linear relationships from equations 28 and 29. l and u are lower and upper bounds. Equality constraints are imposed by setting l and u to the same value. Equation 33 represents a problem in nonlinear programming the solution to which is discussed in Section 4.6. Note the objective function J , which is normally a function of the states, controls and time, is solely a function of the independent variables P . It is also interesting to note that implicit integration yields an accurately integrated trajectory only when the optimization code has converged and driven all the defects to zero.

4.4 Finite Difference Derivative Computation

The partial derivatives of the objective function and the constraints with respect to the independent variables are required in the solution of Equation 33. These partial derivatives are computed by finite differences and are calculated as follows: Using the current value of the independent variable vector P , the objective ϕ and all constraints C , are calculated. Then each component of P is perturbed individually ($P_j = P_j + \delta P_j$) and new values for the objective function and constraints are obtained (ϕ_p and C_p). The partial derivatives are then given by

$$\frac{\partial \phi}{\partial P_j} = \frac{\phi - \phi_p}{\delta P_j} \quad \text{and} \quad \frac{\partial C}{\partial P_j} = \frac{C - C_p}{\delta P_j} \quad (34)$$

For an explicit integration routine, each finite difference calculation requires numerical integration of the differential equations. However with implicit integration, the defects depend only on states and controls at adjacent nodes and the phase time length. The constraints at the nodes depend only on the controls at that node and the constraints at the segment centers depend only on states and controls at adjacent node. Therefore, the finite difference computations for the implicit integration routine are completely algebraic and require no numerical integration. The computing cost associated with the derivative calculation is thereby reduced and the accuracy improved.

4.5 Equations of Motion

In an inertial reference frame, the motion of a body of mass is given by Newton's equation

$$\vec{F}_I = m \frac{d^2 \vec{r}_I}{dt^2} \quad (35)$$

where r_I is the position vector drawn from the origin of the inertial reference frame to the point mass and F_I represents the total vector of forces acting on the constant mass. The remainder of this chapter presents the details necessary to give the equations for the vehicle motion as expressed in flight path coordinates.

4.5.1 Coordinate Systems

The use of a variety of reference frames is necessary to simplify the derivation of the equations. The reference frames are described below. The letter in parenthesis is used subsequently to refer to the given reference frame.

Earth Centered Inertial (I): The origin is at the center of the Earth. The z-axis points north along the Earth rotation axis. The x-axis lies along the vernal equinox with the y-axis completing the right hand system. The x and y axes lie in the equatorial plane.

Earth Centered Rotating (E): The origin is at the center of the Earth. The z-axis points north along the Earth rotation axis. The x-axis lies in the equatorial plane and passes through the prime meridian. The y-axis completes the right hand system.

Vehicle Carry (V): The origin lies at the vehicle center of gravity. The z-axis points downward to the center of the Earth. The x-axis points north. The y-axis completes the right hand system. The x and y axes lie in the local horizontal plane.

Vehicle Tracking (T): The tracking frame is coincident with the vehicle carry frame at any given instant. However, the vehicle tracking system is fixed to the Earth and therefore rotates with the same angular velocity as the Earth Centered Rotating frame.

Aerodynamic (A): The origin lies at the vehicle center of gravity. The x-axis lies along the Earth relative velocity vector of the vehicle. The y-axis lies in the horizontal plane and points right when looking out the x-axis. The horizontal plane is perpendicular to the radius vector from the center of the Earth to the vehicle. The z-axis completes the system.

Body (B): The origin lies at the vehicle center of gravity. The x-axis points out the nose of the vehicle. The z-axis lies in the plane of symmetry of the vehicle and points toward the bottom of the vehicle. The y-axis completes the right hand system. If there is no plane of symmetry, the y and z-axes are defined arbitrarily.

4.5.2 Coordinate Transformations

A vector expressed in one reference frame may be expressed in another by application of a linear transformation of the form

$$\vec{V}_B = T_{BA} \vec{V}_A \quad (36)$$

where T_{BA} is the 3x3 transformation matrix that transforms a vector represented in system A to the same vector represented in system B. The transformation matrix is commonly expressed by a combination of simple rotations. A simple rotation is a rotation about a single axis. Each rotation is taken about the current axis system which may differ from either the original or final system. The three possible simple rotations occur about the x, y, and z-axes and are given by

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}$$

$$R_y(\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix}$$

$$R_z(\theta_z) = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (37)$$

For a sequence of rotations given, in order, by Ψ about the initial z-axis, Θ about the intermediate y-axis, and Φ about the final x-axis, the transformation is given by

$$\vec{V}_B = R_x(\Phi) R_y(\Theta) R_z(\Psi) \vec{V}_A = T_{BA} \vec{V}_A \quad (38)$$

where T_{BA} is found by multiplying the three simple rotation matrices.

Since the rotations matrices are orthogonal, the inverse transformation is simply the transpose

$$T_{AB} = T_{BA}^{-1} = T_{BA}^T \quad (39)$$

The transformation from the Vehicle Carry frame to the Aerodynamic frame is given by

$$T_{AV} = R_y(\gamma) R_z(\psi) \quad (40)$$

where γ is the flight path angle and ψ is the heading angle. The flight path angle is the angle between the local horizontal plane and the Earth relative velocity vector of the vehicle and is positive up. The heading angle is the angle between the projection of the Earth relative velocity vector onto the local horizontal plane and the vehicle carry x-axis (north). The heading angle is measured positive east.

The transformation from the Earth centered rotating frame to the Vehicle tracking system is

$$T_{TE} = R_y(-(\frac{\pi}{2} + \phi)) R_z(\theta) \quad (41)$$

where θ is the geographic longitude and ϕ is the geographic latitude of the point on the earth surface through which a radius vector from the origin of the Earth centered system to the origin of the Vehicle tracking system must pass.

The transformation from the Aerodynamic frame to the Body frame is given by

$$T_{BA} = R_y(\alpha) R_z(-\beta) R_x(\sigma) \quad (42)$$

where σ is the bank angle and is a positive rotation about the velocity vector, β is the sideslip and is positive nose left, and α is the angle of attack and is positive nose up.

The transformation from the vehicle carried system to the body system depends on the relative Euler angles and is given by

$$T_{BV} = R_x(\Phi_R) R_y(\Theta_R) R_z(\Psi_R) \quad (43)$$

where Ψ_R is the relative yaw angle and represents the angle between the projection of the body x-axis onto the horizontal plane and the Vehicle carried x-axis and is measured positive clockwise from the vehicle carried x-axis. Θ_R is the relative pitch angle and is the elevation of the body

x-axis above the horizontal plane. Φ_R , the relative roll angle, is the angle between the horizontal plane and the body y-axis and is measured positive down.

4.5.3 Flight Path Equations of Motion

Newton's law, which only holds in an inertial reference frame, must be translated to a rotating frame to be cast in the flight path equations. The state variables for the flight path equations are:

- V = Magnitude of Earth relative velocity
- ψ = Heading (clockwise from North)
- γ = Flight path angle (positive up)
- r = Magnitude of radius vector from center of the Earth to the vehicle
- θ = Longitude (East from prime meridian)
- ϕ = Latitude (positive north of the equator)

Section 4.5.3.1 presents some vector mechanics necessary in the derivations of the equations. Kinematic expressions involving the time derivative of r , θ , and ϕ and expressions involving the time derivatives of V , ψ , and γ are derived in Section 4.5.3.2.

4.5.3.1 Necessary Relationships

The relationship between the time derivative of a vector as measured in a fixed system ($d[\]/dt$) and the time derivative of the same vector as measured in a rotating system ($\delta[\]/\delta t$) is given in reference [17] as

$$\left. \frac{d[\]}{dt} \right|_A = \left. \frac{\delta[\]}{\delta t} \right|_B + \vec{\Omega} \times [\] \quad (44)$$

where Ω is the angular velocity of reference frame B relative to reference frame A. Note that reference system A is considered fixed but need not be considered inertial. Applying the vector operator (equation 44) to itself, the following expression is obtained for the second derivative

$$\left. \frac{d^2[\]}{dt^2} \right|_A = \left. \frac{\delta^2[\]}{\delta t^2} \right|_B + \left. \frac{\delta \vec{\Omega}}{\delta t} \right|_B \times [\] + \vec{\Omega} \times (\vec{\Omega} \times [\]) + 2\vec{\Omega} \times \left. \frac{\delta[\]}{\delta t} \right|_B \quad (45)$$

Also needed is a relationship between the relative angular velocity (Ω) of reference frame B to reference frame A, and the associated Euler angle rates. Given a rotation from frame A to frame B of $T_{BA} = R_x(\Theta_x) R_y(\Theta_y) R_z(\Theta_z)$, the desired relationship is given in reference [4] as

$$\vec{\Omega}_B = \begin{bmatrix} 1 & 0 & -\sin\Theta_y \\ 0 & \cos\Theta_x & \sin\Theta_x \cos\Theta_y \\ 0 & -\sin\Theta_x & \cos\Theta_x \cos\Theta_y \end{bmatrix} \begin{bmatrix} \dot{\Theta}_x \\ \dot{\Theta}_y \\ \dot{\Theta}_z \end{bmatrix} = R_\omega(\Theta_x, \Theta_y) \begin{bmatrix} \dot{\Theta}_x \\ \dot{\Theta}_y \\ \dot{\Theta}_z \end{bmatrix} \quad (46)$$

where Ω_B is the relative angular velocity represented in reference frame B. The dots represent time derivatives to be taken term by term.

4.5.3.2 Derivation of the Differential Equations of Motion

The equations of motion are the six scalar equations necessary to specify the time derivatives of the state variables given in Section 4.5.3. Letting V be the Earth relative velocity vector and using subscripts to denote in which reference frame a vector is represented, kinematic expressions can be found as follows:

The Earth relative velocity expressed in the vehicle carried system is

$$\vec{V}_V = T_{VA} \vec{V}_A = T_{VA} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} = V \begin{bmatrix} \cos\gamma \cos\psi \\ \cos\gamma \sin\psi \\ -\sin\psi \end{bmatrix} \quad (47)$$

Also, the Earth relative velocity can be expressed as the time rate of change of the position vector

$$\vec{V} = \left. \frac{d\vec{r}}{dt} \right|_E = \left. \frac{\delta \vec{r}}{\delta t} \right|_V + \vec{\Omega}^{VE} \times \vec{r} \quad (48)$$

where Ω^{VE} is the angular rate of the vehicle carried system relative to the Earth centered rotating frame. The angular rate can be expressed in the vehicle carried system using equation 46

$$\vec{\Omega}_V^{VE} = R_\omega(0, -(\frac{\pi}{2} + \phi)) \begin{bmatrix} 0 \\ -\dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \cos\phi \\ -\dot{\phi} \\ -\dot{\theta} \sin\phi \end{bmatrix}_V \quad (49)$$

The radius vector and its time derivative as measured in the vehicle carried frame are

$$\vec{r}_v = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix}_v, \quad \frac{\delta \vec{r}_v}{\delta t} \Big|_v = \begin{bmatrix} 0 \\ 0 \\ -\dot{r} \end{bmatrix}_v \quad (50)$$

Substituting the expressions of equations 47, 49 and 50 into equation 48 and evaluating yields the kinematic relationships given by

$$\begin{bmatrix} r\dot{\phi} \\ r\dot{\theta}\cos\phi \\ \dot{r} \end{bmatrix} = V \begin{bmatrix} \cos\gamma\cos\psi \\ \cos\gamma\sin\psi \\ \sin\gamma \end{bmatrix} \quad (51)$$

and constitutes three of the six differential equations needed.

The remaining three equation are found by evaluating Newton's law in a rotating reference system. Letting $\vec{F}$ be the vector of all forces acting on the vehicle, and $\vec{r}$ be the inertial radius vector from the center of the Earth, the vector force equation can be written

$$\frac{\vec{F}}{m} = \frac{d^2\vec{r}}{dt^2} \Big|_I \quad (52)$$

The second derivative of the radius vector as measured in the inertial frame can be related to that measured in the vehicle tracking frame by

$$\frac{d^2\vec{r}}{dt^2} \Big|_I = \frac{\delta^2\vec{r}}{\delta t^2} \Big|_T + \frac{\delta \vec{\Omega}^{\eta I}}{\delta t} \Big|_T \times \vec{r} + \vec{\Omega}^{\eta I} \times (\vec{\Omega}^{\eta I} \times \vec{r}) + 2\vec{\Omega}^{\eta I} \times \frac{\delta \vec{r}}{\delta t} \Big|_T \quad (53)$$

The angular rate of the vehicle tracking frame relative to the inertial frame and represented in the vehicle tracking system is found by

$$\vec{\Omega}_T^{\eta I} = \vec{\Omega}_T^{EI} = T_{TE} \vec{\Omega}_E^{EI} = T_{TE} \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}_E = \omega \begin{bmatrix} \cos\phi \\ 0 \\ -\sin\phi \end{bmatrix}_T \quad (54)$$

where ω is the mean rotation rate of the Earth about its spin axis. The rate of rotation is assumed constant and therefore the time derivative of the angular rate as seen from the vehicle tracking frame is identically zero.

The radius vector and its time derivatives as measured in the vehicle tracking system and expressed in the vehicle tracking system are given by

$$\vec{r}_T = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix}, \quad \frac{\delta \vec{r}_T}{\delta t} \Big|_T = \vec{V}_T = \vec{V}_V = V \begin{bmatrix} \cos \gamma \cos \psi \\ \cos \gamma \sin \psi \\ -\sin \gamma \end{bmatrix}, \quad \frac{\delta^2 \vec{r}_T}{\delta t^2} \Big|_T = \frac{\delta \vec{V}_T}{\delta t} \Big|_T \quad (55)$$

where the the fact that the time rate of change of the radius vector as seen from any Earth fixed system is the Earth relative velocity vector ($\vec{V}$) and the fact that the vehicle tracking coordinate system is parallel to the vehicle carried system are used in evaluating the first derivative.

Now, substituting expression given in equations 52, 54, and 55 into equation 53 , evaluating and replacing δ by d in the differential operator yields

$$\frac{d\vec{V}_T}{dt} \Big|_T = \frac{\vec{F}_T}{m} - r \omega^2 \cos \phi \begin{bmatrix} \sin \phi \\ 0 \\ \cos \phi \end{bmatrix} - 2 \omega V \begin{bmatrix} \sin \phi \cos \gamma \sin \psi \\ \cos \phi \sin \gamma - \sin \phi \cos \gamma \cos \psi \\ \cos \phi \cos \gamma \sin \psi \end{bmatrix} \quad (56)$$

The time derivative of the Earth relative velocity vector as measured in the vehicle tracking system can now be related to that measured in the vehicle carried system by

$$\frac{d\vec{V}}{dt} \Big|_T = \frac{\delta \vec{V}}{\delta t} \Big|_V + \vec{\Omega}^{V/T} \times \vec{V} \quad (57)$$

where, since the vehicle tracking frame does not rotate with respect to the Earth centered rotating frame, $\vec{\Omega}^{V/T} = \vec{\Omega}^{V/E}$ and is given as expressed in the vehicle carried system by equation 49. The Earth relative velocity vector expressed in the vehicle carried system is given in equation 47.

The above cross product is then evaluated to be

$$\vec{\Omega}^{V/T} \times \vec{V}_V = V \begin{bmatrix} \dot{\phi} \sin \gamma + \dot{\theta} \sin \phi \cos \gamma \sin \psi \\ -\dot{\theta} \sin \phi \cos \gamma \cos \psi + \dot{\theta} \cos \phi \sin \gamma \\ \dot{\theta} \cos \phi \cos \gamma \sin \psi + \dot{\phi} \cos \gamma \cos \psi \end{bmatrix}_V = \frac{V^2 \cos \gamma}{r} \begin{bmatrix} \sin \gamma \cos \psi + \sin^2 \psi \cos \gamma \tan \phi \\ \sin \psi (\sin \gamma - \cos \gamma \cos \psi \tan \phi) \\ \cos \gamma \end{bmatrix}_V \quad (58)$$

where equation 51 is used to eliminate the time derivatives of θ and ϕ .

Differentiating V directly gives

$$\begin{aligned}
 \left. \frac{\delta \vec{V}_V}{\delta t} \right|_V &= \dot{V} \begin{bmatrix} \cos\gamma \cos\psi \\ \cos\gamma \sin\psi \\ -\sin\gamma \end{bmatrix}_V + \dot{\psi} V \begin{bmatrix} -\cos\gamma \sin\psi \\ \cos\gamma \cos\psi \\ 0 \end{bmatrix}_V - \dot{\gamma} V \begin{bmatrix} \sin\gamma \cos\psi \\ \sin\gamma \sin\psi \\ \cos\gamma \end{bmatrix}_V \\
 &= \begin{bmatrix} \cos\gamma \cos\psi & -\sin\gamma & \sin\gamma \cos\psi \\ \cos\gamma \sin\psi & \cos\psi & \sin\gamma \sin\psi \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{bmatrix} \dot{V} \\ V\cos\gamma\dot{\psi} \\ -V\dot{\gamma} \end{bmatrix}_V = T_{AV}^T \begin{bmatrix} \dot{V} \\ V\cos\gamma\dot{\psi} \\ -V\dot{\gamma} \end{bmatrix}_V
 \end{aligned} \tag{59}$$

Equating equations 56 and 57 and substituting in equations 58 and 59 yields the following differential equations for V , ψ , and γ :

$$\begin{bmatrix} \dot{V} \\ V\cos\gamma\dot{\psi} \\ -V\dot{\gamma} \end{bmatrix} = T_{AV} \frac{\vec{F}_V}{m} - T_{AV} \vec{V}_k \tag{60}$$

where

$$\begin{aligned}
 \vec{V}_k &= r\omega^2 \cos\phi \begin{bmatrix} \sin\phi \\ 0 \\ \cos\phi \end{bmatrix} + 2\omega V \begin{bmatrix} \sin\phi \cos\gamma \sin\psi \\ \cos\phi \sin\gamma - \sin\phi \cos\gamma \cos\psi \\ \cos\phi \cos\gamma \sin\psi \end{bmatrix} \\
 &\quad + \frac{V^2 \cos\gamma}{r} \begin{bmatrix} \sin\gamma \cos\psi + \sin^2\psi \cos\gamma \tan\phi \\ \sin\psi (\sin\gamma - \cos\gamma \cos\psi \tan\phi) \\ \cos\gamma \end{bmatrix}
 \end{aligned}$$

The force vector, $\vec{F}_V$, represents the net force, represented in the vehicle carried coordinate system, acting on the vehicle. The forces are normally categorized as either aerodynamic, propulsive, or gravitational and are given as

$$\vec{F}_V = T_{VA} \vec{F}_A^{aero} + T_{VB} \vec{F}_B^{prop} + \vec{F}_V^{grav} \tag{61}$$

where

$$\vec{F}_A^{aero} = -\frac{1}{2} \rho V^2 S \begin{bmatrix} C_D \\ C_Y \\ C_L \end{bmatrix}, \quad \vec{F}_B^{prop} = \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix}, \quad \vec{F}_V^{grav} = m \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

where C_D is the drag coefficient, C_Y is the side force coefficient, C_L is the lift coefficient, T is the thrust along the vehicle body x-axis, and g is the local acceleration due to gravity ($g = \mu/r^2$).

Equations 51 and 60 along with all the associated equations specify the time derivatives of the six states and are referred to as the equations of motion. The six scalar equations can be cast as a vector equation of the form of equation 20 so as to accommodate the numerical solution methods as described in this chapter.

4.6 Nonlinear Programming

The nonlinear programming problem stated in equation 33 is solved in OTIS with the use the NPSOL routines developed by the Stanford Optimization Laboratory [6]. NPSOL assumes a nonlinear programming problem stated in the following form

$$\begin{aligned} &\text{minimize} \quad F(x) \\ &\quad \quad \quad x \in \mathbb{R}^n \\ &\text{subject to} \quad l \leq \begin{bmatrix} x \\ A_L x \\ c(x) \end{bmatrix} \leq u \end{aligned} \tag{62}$$

where $F(x)$ is a smooth nonlinear function, A_L is a constant matrix, and $c(x)$ is a vector of smooth nonlinear constraint functions. Upper and lower bounds are specified for all the variables and constraints. Equality constraints are enforced by setting the upper and lower bounds to the same value. The absence of a bound is enforced by setting the bound to a large value ($\pm \infty$).

A sequential quadratic programming method is used to solve the nonlinear programming problem. Letting x_k denote the current estimate of the solution, a new estimate is defined by

$$x_{k+1} = x_k + \alpha_k p_k \quad (63)$$

where the vector p_k is the solution to the quadratic programming problem stated below. The positive scalar α_k is the step length parameter and is chosen to produce a sufficient decrease in an augmented Lagrangian merit function. The augmented Lagrangian is given by

$$L = F(x) - \lambda^T C(x) + \frac{\rho}{2} C(x)^T C(x) \quad (64)$$

where $C(x)$ is a vector of the active constraints, ρ is a positive penalty parameter, and λ is a vector of Lagrange multipliers.

The Quadratic programming subproblem that defines p_k is of the form

$$\begin{aligned} & \underset{p \in \mathbb{R}^n}{\text{minimize}} && g^T p + \frac{1}{2} p^T H p \\ & \text{subject to} && l_{QP} \leq \begin{bmatrix} p \\ A_L p \\ A_N p \end{bmatrix} \leq u_{QP} \end{aligned} \quad (65)$$

where g is the gradient of F at x_k , the matrix H is a positive definite quasi-Newton approximation to the Hessian of the Lagrangian function, A_L is as given above, and A_N is the Jacobian matrix of $c(x)$ evaluated at x_k . The solution of the quadratic programming problem is discussed by Gill et al [5].

The necessary conditions for a strong local minimum, x^* are:

- 1) x^* is a feasible point (i.e. no constraints are violated).
- 2) The gradient of the Lagrangian function is zero, or equivalently, $z^T g = 0$, where z is a basis for the null space of the Jacobian matrix of $C(x)$ evaluated at x^* and g is the gradient of F .
- 3) The Lagrange multiplier corresponding to an active inequality constraint must be non-negative if the constraint is at its lower bound, and non-positive if the constraint is at its upper bound.
- 4) The matrix, $z^T W z$, is positive semi definite. W is the Hessian of the augmented Lagrangian function.

For more detailed information on the solution methods employed by NPSOL consult the user guide of reference [6].

5 Problem Formulation

As noted previously, minimum fuel transfer between circular coplanar orbits is considered. This chapter contains the specifications of the problem formulation. The geometry of the assumed optimal trajectory is outlined in Section 5.1. The vehicle data is given in Section 5.2. Section 5.3 gives the atmospheric model formulation. Section 5.4 presents available control sets. The cost function of the optimization problem is presented in Section 5.5 with the problem constraints given in Section 5.6. The numerical values are summarized in Section 5.7.

5.1 *Assumed Trajectory*

The transfer trajectory is assumed to be an aero elliptic trajectory as described in Section 2.3.1.1. The transfer is assumed to occur between circular coplanar orbits about the Earth. Namely, the vehicle, which is initially in a circular GEO ($r_1 = 42,241$ km), applies a single impulse onto an elliptical descent orbit with a perigee inside the radius of the atmosphere. The vehicle enters the atmosphere with velocity, V_e , and flight path angle, γ_e . Piecewise-constant controls as given in Section 5.4 are applied through the atmosphere. The vehicle then leaves the atmosphere with velocity, V_f , and flight path angle, γ_f . After exiting the atmosphere, the vehicle coasts along an elliptical ascent orbit with an apogee greater than or equal to the radius of the LEO circular target orbit ($r_2 = 6,578.7$ km). At the radius of the target orbit a single impulse is used to circularize the vehicle orbit to match the target orbit. Figure 5 shows the geometry of a realistic transfer. The ΔV s are not necessarily assumed to be applied tangentially.

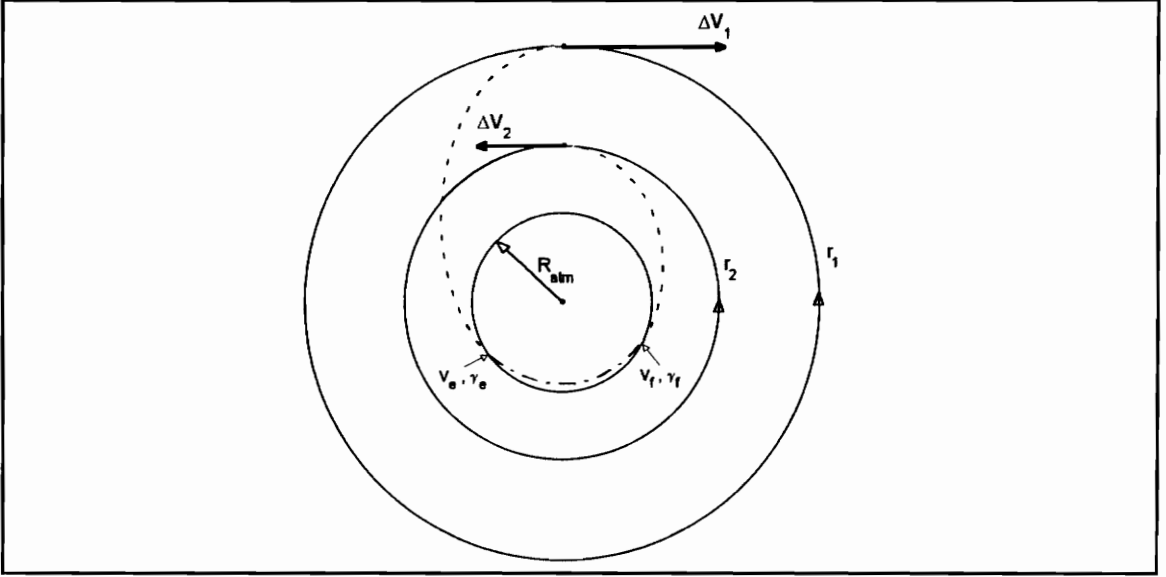


Figure 5. Realistic Aeroassisted Coplanar Transfer

5.2 Vehicle Model

In order to compare results obtained from this research with those obtained from optimal control theory, the vehicle properties are taken from Mease and Vinh [16]. Specifically, the properties of the moderate L/D vehicle considered by Mease and Vinh are used for this research. The vehicle has a maximum lift to drag ratio of 1.5 and a corresponding maximum lift coefficient of 0.9. The vehicle mass to surface area is given as 300 kg/m². Since OTIS requires separate inputs for the mass and the surface area these values are taken as 4,898.8 kg and 16.35 m² respectively. The drag model is assumed to be the normal parabolic drag polar given as

$$C_D = C_{D_0} + KC_L^2 \quad (66)$$

At hypersonic velocities the drag coefficient can be shown to be Mach independent [1]. Although the values of C_{D_0} and K are still functions of the Reynolds number, constant values will be used for simplicity and comparison with Mease and Vinh. For precise calculations, variations in C_{D_0} and K would need to be modeled. The remaining procedure is unchanged and applicable to problems with varying coefficients. The values used for C_{D_0} and K are 0.10 and 0.11 respectively.

5.3 Atmospheric Model

Once again comparison with Mease and Vinh is sought. Mease and Vinh model the atmospheric density between the altitudes of 40 and 120 km using a fifth degree Chebyshev polynomial whose coefficients are determined by a least-squares fit to the 1976 U.S. Standard Atmosphere. A similar atmospheric model based on the 1976 U.S. Standard atmosphere was constructed using OTIS. The 1976 U.S. Standard Atmosphere was interrogated to produce one hundred equally spaced (in altitude) density values from 40 to 120 km. The resulting array was then interpolated using the OTIS quintic spline package to produce a smooth atmospheric model which should be similar to that used by Mease and Vinh.

The radius of the atmosphere is taken as the radius of the Earth (6378.15 km) plus 120 km. At this altitude, the accelerations of the vehicle produced by the atmosphere are small but detectable as recommended by Vinh [20].

5.4 Controls

Two sets of atmospheric controls are investigated. The first set is termed *aerodynamic controls* and includes the aerodynamic lift coefficient (C_L) and bank angle (σ). The lift coefficient is replaced by the aerodynamic angle of attack (α) in the solution of the problem since OTIS does not recognize C_L as a control. The lift coefficient is then modeled with a one to one correspondence to angle of attack ($C_L = \alpha$) yielding the effect of direct lift coefficient control. The second set can be referred to as *attitude controls* and includes the relative euler pitch angle (Θ_R) and the relative roll angle (Φ_R). Numerical results were only obtained for coplanar transfers in which case only the lift coefficient or pitch angle is used as the control variable. The controls are taken to be piecewise-constant in time. Both fixed switching time as well as a free switching time problems are solved.

The initial ΔV is the only control to be applied outside of the atmosphere. Since OTIS (Version 2.0) does not allow analytic propagation of state (i.e. Kepler's equations) for exoatmospheric flight, it is convenient to use the entrance conditions to the atmosphere as the control parameters.

Given the entrance velocity and flight path angle, the velocity and flight path angle of the vehicle just after the initial impulse (i.e. at the initial radius) can be determined from the conservation laws for energy and angular momentum. The energy and angular momentum of an orbit are defined as

$$\mathcal{E} = \frac{V_j^2}{2} - \frac{\mu}{r_j} \quad (67)$$

$$h = r_j V_j \cos \gamma_j \quad (68)$$

where the subscript j refers to the conditions at any particular point along the orbit. In particular, the quantities can be calculated at entrance to the atmosphere using V_e , γ_e , and R_{atm} . Since the energy and angular momentum are constant along the orbit, the conditions just after the initial impulse are given by

$$V_1 = \sqrt{2 \left(\mathcal{E}_1 + \frac{\mu}{r_1} \right)} \quad (69)$$

$$\cos \gamma_1 = \frac{h_1}{V_1 r_1} \quad (70)$$

where the energy and angular momentum are as calculated at entrance to the atmosphere. The magnitude of the initial impulse, illustrated in Figure 6, can then be determined from the law of cosines as

$$\Delta V_1 = \left[V_1^2 + V_{c_1}^2 - 2 V_1 V_{c_1} \cos \gamma_1 \right]^{\frac{1}{2}} \quad (71)$$

Where V_{c_1} is the circular speed at the initial radius and is given by

$$V_{c_1} = \sqrt{\frac{\mu}{r_1}} \quad (72)$$

Therefore, the aeroassisted coplanar transfer with two phase piecewise-constant controls employed during the atmospheric skip only requires five control parameters. The parameters for aerodynamic control being: V_e , γ_e , C_{L1} , t_{sw} , and C_{L2} where t_{sw} denotes the time to switch from

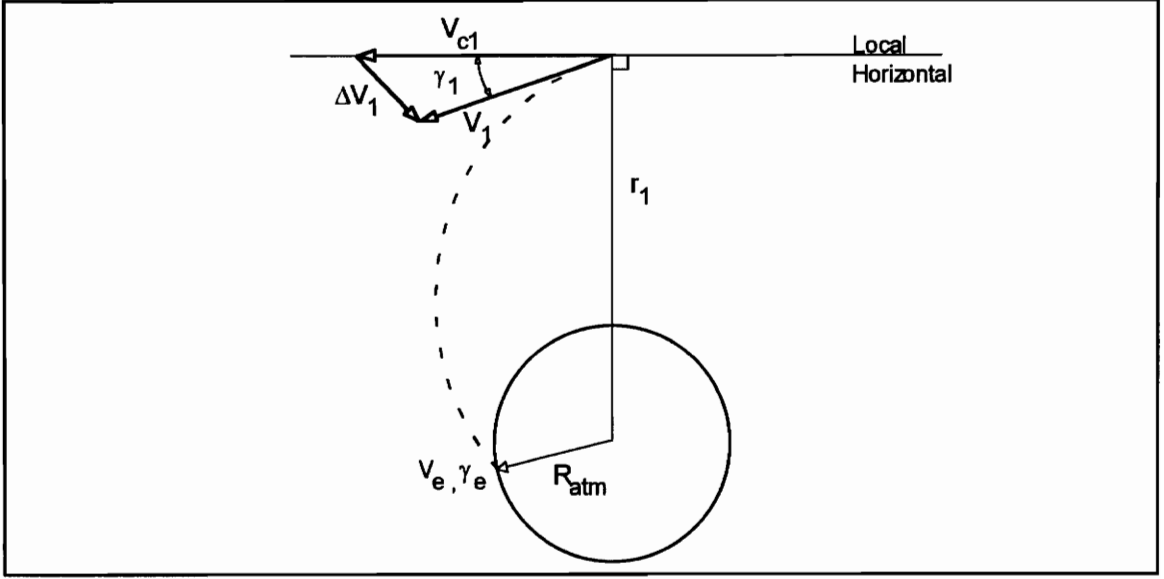


Figure 6. Initial Impulse

C_{L1} to C_{L2} . For attitude control C_L is replaced by Θ .

5.5 Objective Function

Assuming a high thrust vehicle, the fuel consumption can be characterized by sum of the instantaneous changes in velocity. Referring to Figure 5, the objective function for a minimum fuel transfer is given by

$$J = \|\Delta \vec{V}_1\| + \|\Delta \vec{V}_2\| = \Delta V_1 + \Delta V_2 \quad (73)$$

Since Version 2.0 of OTIS does not allow for analytic propagation of state for exoatmospheric flight, the above changes in velocity are calculated from the entrance and exit conditions from the atmosphere. The magnitude of the initial impulse (ΔV_1) is found as described in Section 5.4 and is given by Equation 71. In a completely analogous manner, the magnitude of the rendezvous impulse can be shown to be

$$\Delta V_2 = \left[V_2^2 + V_{c2}^2 - 2 V_2 V_{c2} \cos \gamma_2 \right]^{\frac{1}{2}} \quad (74)$$

where

$$V_2 = \sqrt{2 \left(\mathcal{E}_2 + \frac{\mu}{r_2} \right)} \quad (75)$$

$$\cos \gamma_2 = \frac{h_2}{V_2 r_2} \quad (76)$$

$$V_{c_2} = \sqrt{\frac{\mu}{r_2}} \quad (77)$$

with the energy and angular momentum calculated at exit from the atmosphere using Equations 67 and 68 with V_f , γ_f , and R_{atm} .

5.6 Constraints

To ensure the solution adheres to physical reality certain constraints must be placed on the minimization problem. Since the ΔV 's are calculated from conditions at the atmospheric boundary, constraints must be added to impose the conditions that the transfer orbit extends to the initial and target orbits. Therefore, the radius at apogee of the descent orbit must be greater than or equal to the initial orbit radius. Similarly, the radius at apogee of the ascent orbit must be greater than or equal to the radius of the target orbit. The apogee radius (r_a) of an elliptical orbit can be shown to be

$$r_a = -\frac{\mu}{2 \mathcal{E}} \left[1 + \sqrt{1 + \frac{2 \mathcal{E} h^2}{\mu^2}} \right] \quad (78)$$

Therefore the above constraints can be written as

$$r_{a_1} = -\frac{\mu}{2 \mathcal{E}_1} \left[1 + \sqrt{1 + \frac{2 \mathcal{E}_1 h_1^2}{\mu^2}} \right] \geq r_1 \quad (79)$$

and

$$r_{a_2} = -\frac{\mu}{2\mathcal{E}_2} \left[1 + \sqrt{1 + \frac{2\mathcal{E}_2 h_2^2}{\mu^2}} \right] \geq r_2 \quad (80)$$

During the atmospheric portion of the transfer, the vehicle must stay within the boundary of the atmosphere. That is, the altitude must not exceed that of the atmosphere nor may it drop below the minimum altitude used in the atmospheric density model. Denoting the vehicle altitude by h , yields the following constraint

$$120\text{km} \geq h \geq 40\text{km} \quad (81)$$

during the atmospheric skip. The condition that the altitude at the end of the skip be equal to the atmospheric altitude is stated as

$$h_f = 120\text{km} \quad (82)$$

The controls used during the atmospheric skip are limited by simple bounds. The bounds on the aerodynamic controls are given by

$$|C_{L_i}| \leq C_{L_{\max}} \quad (83)$$

where $C_{L_{\max}}$ is 0.9 and i ranges from one to the number of phases. Since the physical limits on the vehicle are described in terms of the aerodynamics (maximum lift capability), these same constraints are used when attitude controls are employed.

5.7 Summary

The problem of minimum fuel aeroassisted transfer between coplanar circular orbits with two phase piecewise-constant controls can now be stated as

$$\min_{V_o, \gamma_o, C_{L1}, t_{sw}, C_{L2}} J = \Delta V_1 + \Delta V_2 \quad (84)$$

subject to the constraints given in Section 5.6. A reduced problem, where the first ΔV is fixed,

can also be stated. In that case, V_e and γ_e are no longer controls and the problem can be given by

$$\min_{C_{L_1}, t_{sw}, C_{L_2}} J = \Delta V_2 \quad (85)$$

subject to the same constraints as Equation 39 minus that pertaining to the apogee radius of the descent orbit. The fixed ΔV_1 problem, which is also referred to as the fixed deboost perigee problem, is given more attention in the coming chapters.

The numerical values given in this chapter are summarized in Table 3.

Table 3. Physical Data

Symbol	Description	Value
R_e	Radius of the Earth	6,378.15 km
h_{atm}	Altitude of the Atmosphere	120.00 km
R_{atm}	Radius of the Atmosphere	6,498.15 km
μ	Gravitational Parameter	$3.986012 \times 10^5 \text{ km}^3/\text{s}^2$
m	Vehicle Mass	4,898.8 kg
S	Vehicle Reference Area	16.35 m^2
C_{D0}	Zero Lift Drag Coefficient	0.10
K	Induced Drag Factor	0.11
$C_{L \max}$	Maximum Lift Coefficient	0.90
r_1	Radius of Initial Orbit	42,241.0 km
r_2	Radius of Target Orbit	6,578.7 km

6 Validation of OTIS Results

Before results from OTIS are accepted, the use of the code to solve aeroassisted transfer problems must be validated. First, an approximate continuous control solution obtained with OTIS is compared with a published optimal control solution. Then, an approximate continuous control solution is compared with an explicitly integrated trajectory. The first comparison will indicate whether OTIS can solve the optimization problem as stated. The second comparison will show how "good" the approximation is in comparison with a more "exact" trajectory.

6.1 *Comparison With Optimal Control Theory*

Mease and Vinh [16] solve the problem of minimum fuel coplanar aeroassisted transfer between GEO and LEO using modulated lift control. The problem is solved in an optimal control fashion yielding a continuously varying control law for the lift coefficient. For the numerical computations, only the second ΔV is considered in the cost function. That is, the perigee of the deboost orbit is fixed. The given value for the deboost perigee is 6400.0 km which yields entrance conditions to the atmosphere of $V_e = 10.3$ km/s and $\gamma_e = -6.5^\circ$. Even though ΔV_1 is the larger of the two burns, the difference required to target perigee at the atmospheric boundary ($\Delta V_1 = 1,485.6$ m/s) and that required to target perigee at the surface of the Earth ($\Delta V_1 = 1,498.4$ m/s) is only 13 m/s. On the other hand, a change in the exit flight path angle from zero to one degree increases the second V by 114 m/s, from 24 m/s to 138 m/s. Therefore, ΔV_2 is seen to be the more sensitive parameter in the minimum fuel problem.

Using the physical data given in Chapter 5, and assuming no bounds on the lift coefficient or heating rate, Mease and Vinh produced the numerical results given in the first row of Table 4.

Table 4. Comparison Between Optimal Control Theory and OTIS Solutions

	γ_e (deg)	ΔV_2 (m/s)	min h (km)	max q (kN/m ²)	max H_R (W/cm ²)	max g-load (g's)
Optimal Control Theory	0.4	31.0	58.2	18.6	222.8	2.7
OTIS approximate optimal	0.4	30.5	58.4	18.1	220.0	2.7

The heating rate (H_R) is taken as the convective heating rate for a sphere with a 1 m radius under conditions of laminar flow. The heating rate, in W/cm², is calculated by

$$H_R = (3.08 \times 10^{-4}) \rho^{\frac{1}{2}} V^{3.08} \quad (86)$$

where ρ is the atmospheric density in kg/km³ and V is the speed in km/s. Solving the same problem using the approximate optimal control capabilities of OTIS gives the results shown in the second row of Table 4. The two solutions compare favorably indicating OTIS indeed provides an approximate solution of the optimal control problem.

Figures 7 through 14 show the time histories of lift coefficient, altitude, velocity, flight path angle, dynamic pressure, heating rate, g-load, and lift to drag ratio. The solid lines indicate results obtained from the implicitly integrated approximate optimal control routines of OTIS. The OTIS results compare very favorably with the results of Mease and Vinh which are unfortunately not reproduced in this paper. The only notable difference is seen in the first two minutes of the lift coefficient control. The results from optimal control theory indicate the lift coefficient starting at a positive value slightly less than one and decreasing to a minimum value of slightly less than negative 2.5. The OTIS result shows a lift coefficient that starts at negative 0.5 and increases for the first minute at which point it begins to decrease and follow a similar path as that found from optimal control theory. The initial difference in lift coefficient is not of consequence since the final conditions upon exit from the atmosphere wholly determine the the value of the cost function. Therefore, any path that reaches the same exit conditions will produce a minimum fuel result.

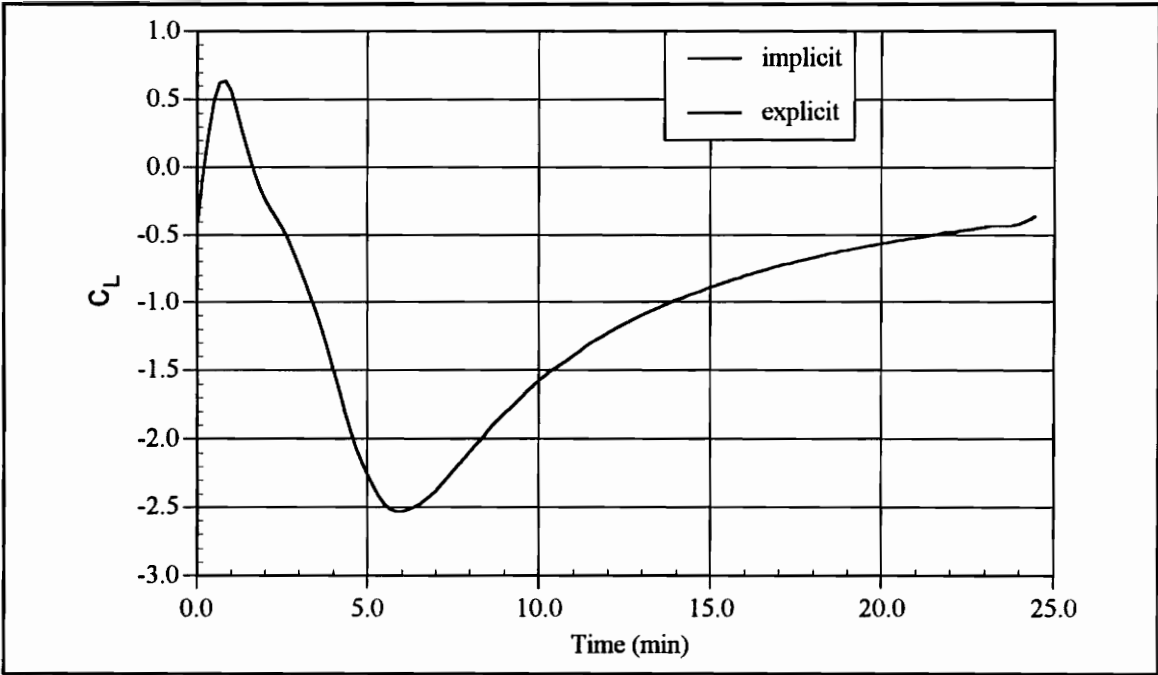


Figure 7. Lift Coefficient vs. Time for Approximate Optimal Control Solution

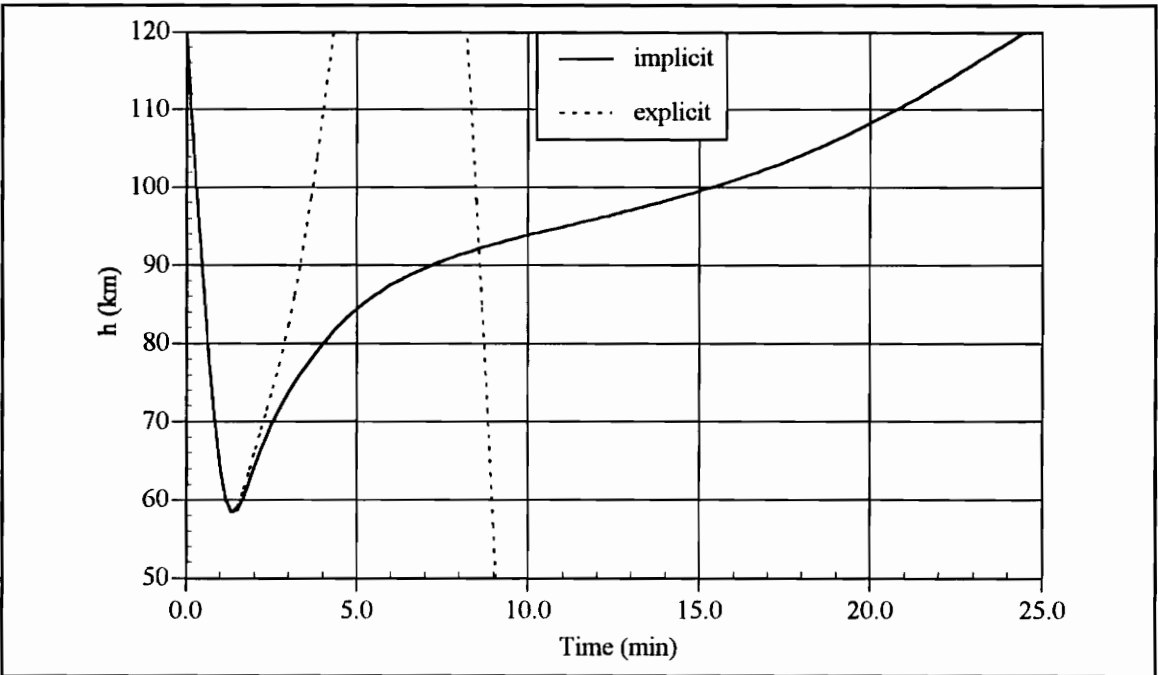


Figure 8. Altitude vs. Time for Approximate Optimal Control Solution

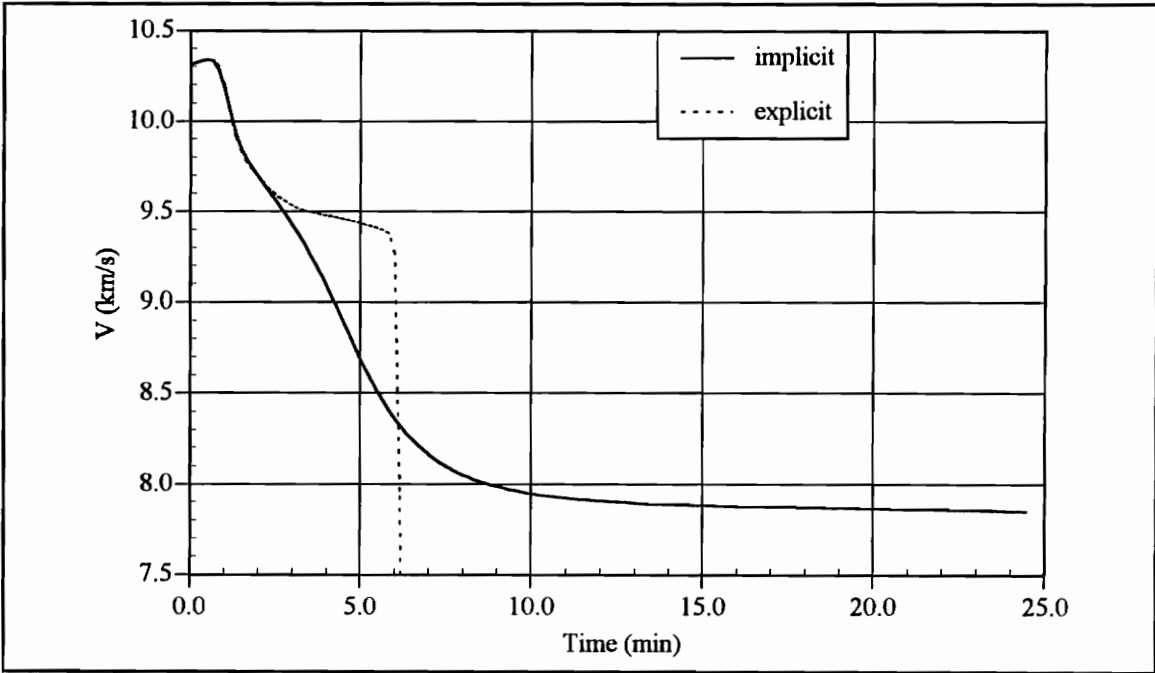


Figure 9. Velocity vs. Time for Approximate Optimal Control Solution

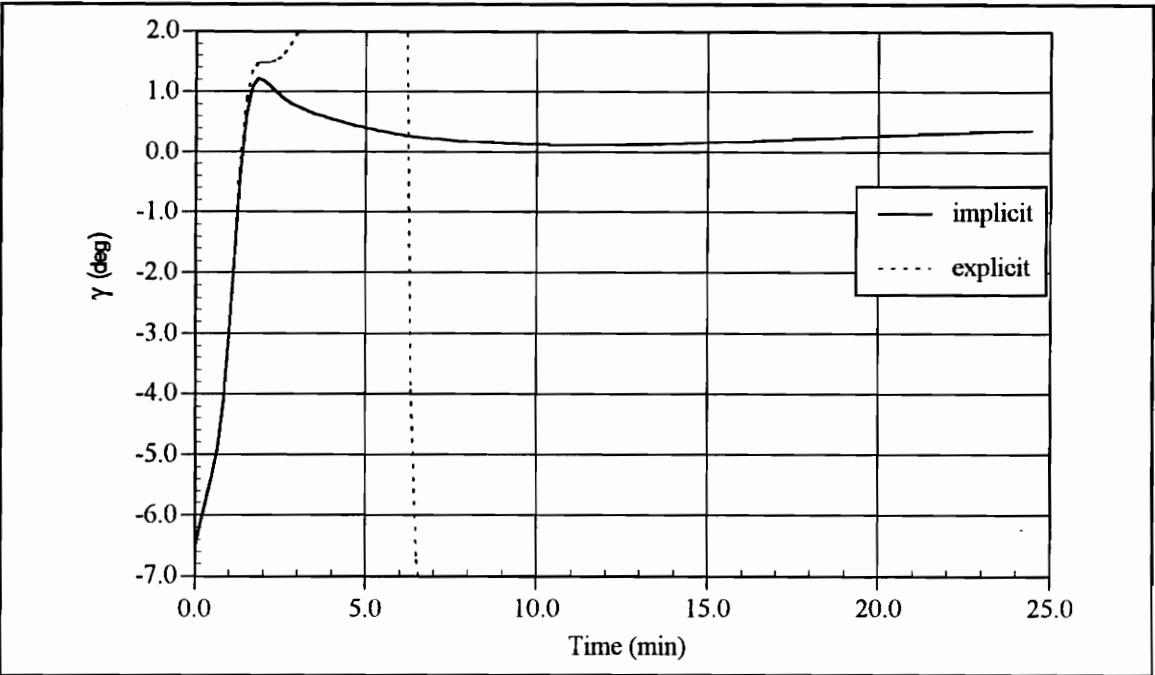


Figure 10. Flight Path Angle vs. Time for Approximate Optimal Control Solution

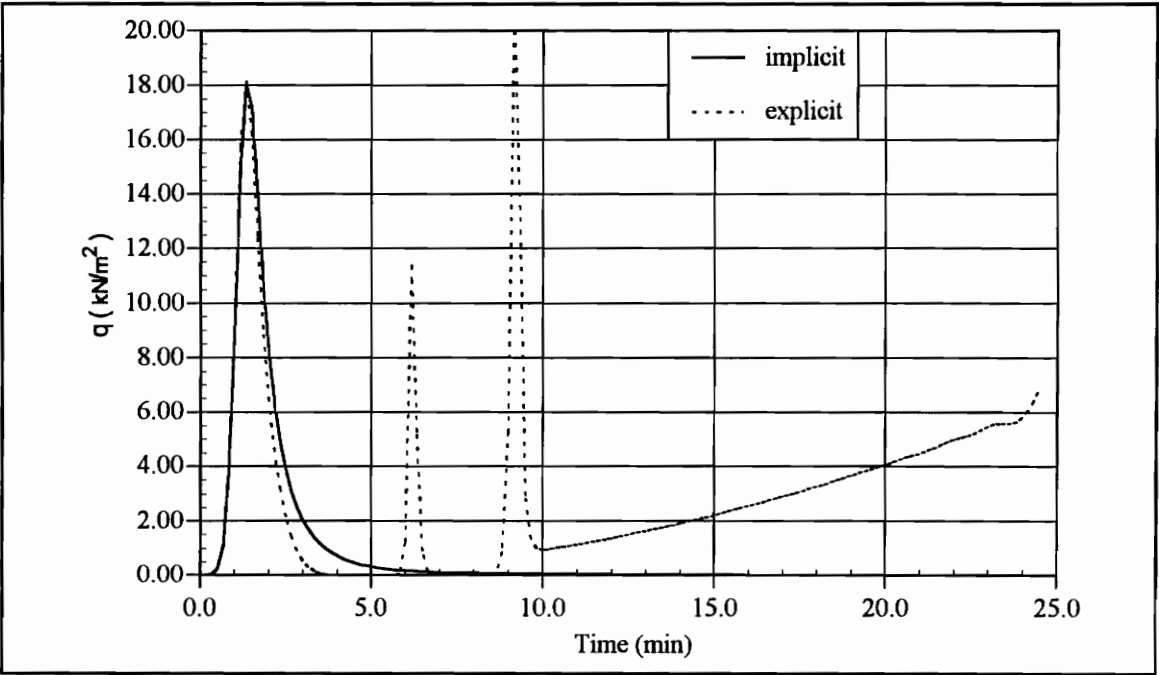


Figure 11. *Dynamic Pressure vs. Time for Approximate Optimal Control Solution*

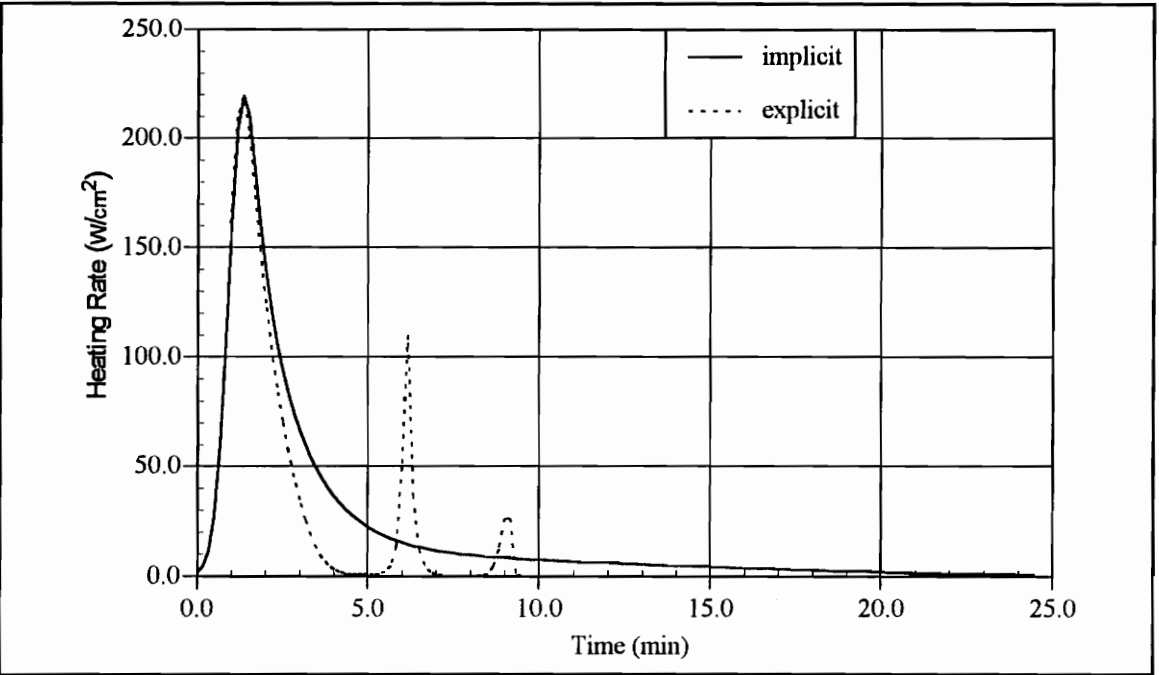


Figure 12. *Convective Heating Rate vs. Time for Approximate Optimal Control Solution*

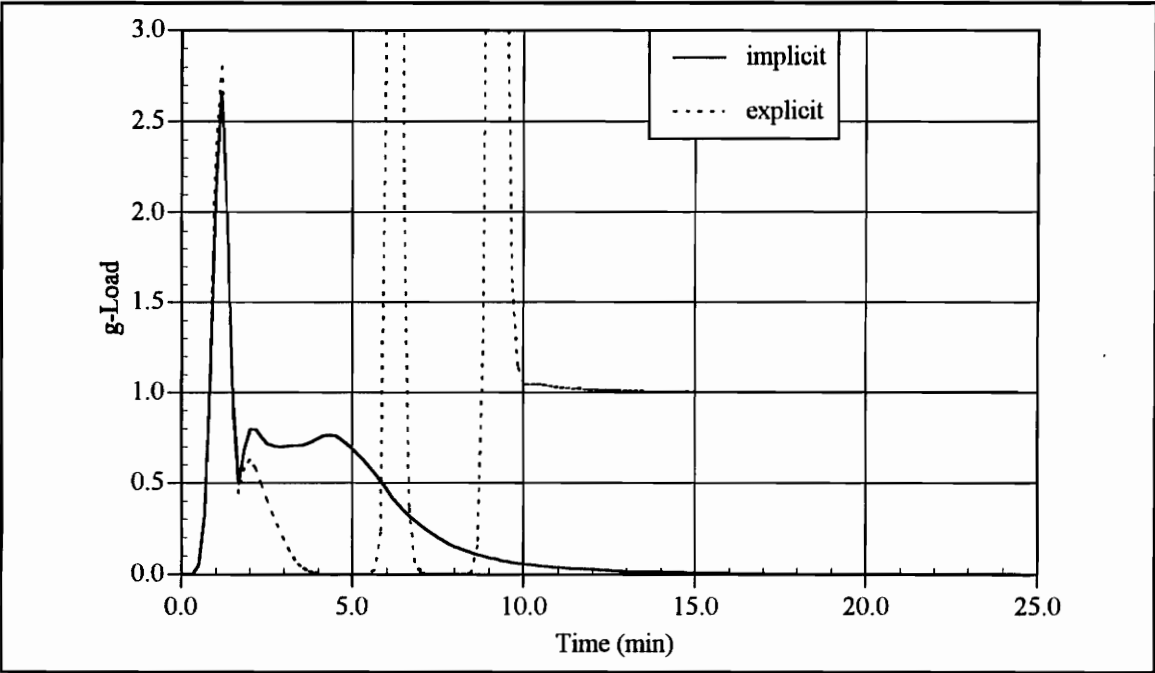


Figure 13. *G-Load vs. Time for Approximate Optimal Control Solution*

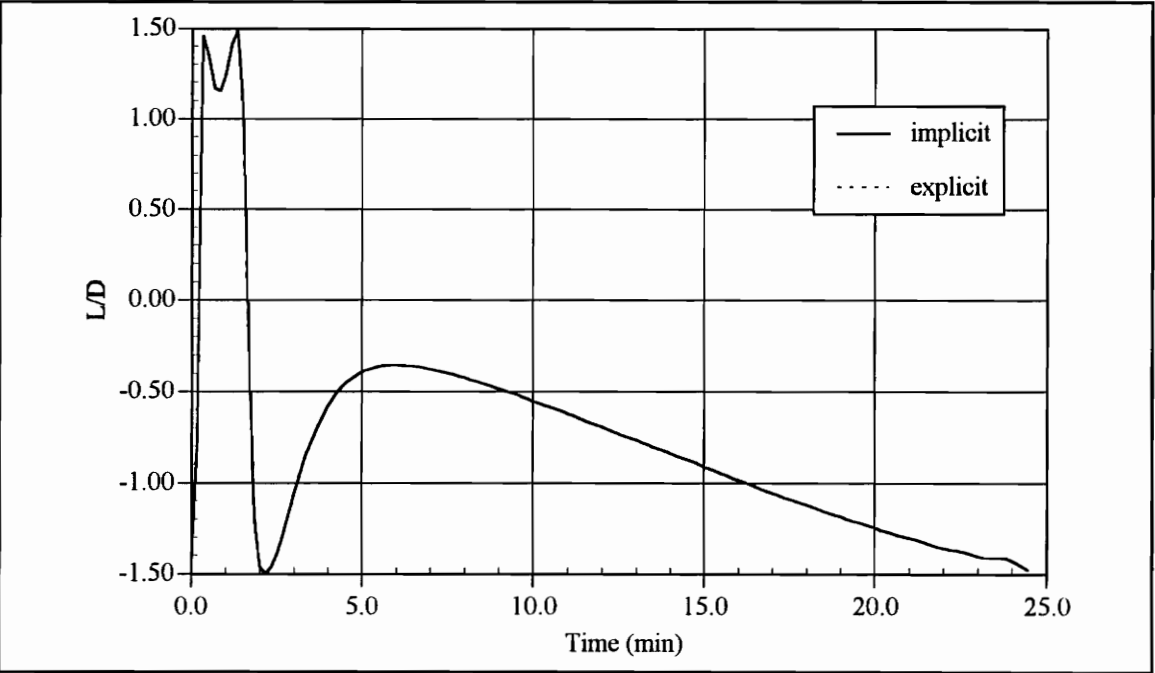


Figure 14. *Lift to Drag Ratio for Approximate Optimal Control Solution*

The dotted lines in the figures are obtained using the explicit integration routines of OTIS to integrate the trajectory using the control history obtained from the approximate optimal solution. The explicit trajectory can be seen to depart from the implicit trajectory shortly after one minute into the skip. The second ΔV can be calculated when the explicit trajectory leaves the atmosphere ($h=120$ km) just short of the five minute mark. Based on those exit conditions, the rendezvous ΔV , and hence the cost function, has a value of 1,866 m/s. This value is some sixty times that obtained from the implicit trajectory. This discrepancy is the motivation for the comparison of the next section.

6.2 Comparison Between Implicit and Explicit Modes

The departure of the explicitly integrated trajectory from the implicitly integrated trajectory is cause for concern. A lack of agreement between the two indicates in some sense that at least one of the trajectories is not adhering to the physics of the problem. Ordinarily, explicitly integrated trajectories are believed to approach "exact" solutions as long as the integration step size is taken as an appropriately small value. Using the fixed step Runge-Kutta integrator with a step size of one second has been found to produce accurate results in a reasonable amount of time. Experience has shown that reducing the step size to one tenth of a second produces less than a one percent change in the rendezvous ΔV for skips of approximately twenty minutes in duration.

Figures 7 through 14 show that the implicit and explicit trajectories found using the same control history are very different. However, the important fact is not how different the trajectories using the same control law are, but how different must the control law be to produce similar trajectories. That is, how good a job does OTIS do of approximating the control law to produce the given trajectory. To pursue this idea, a two phase piecewise-constant control law was found using the implicit integration routines of OTIS. The minimization problem solved is nearly identical to that of the previous Section. Since the entrance conditions to the atmosphere are fixed, there are only three control parameters: C_{L1} , t_{tw} , and C_{L2} .

Figure 15 shows the resulting control law. Figures 16 through 18 show the time histories for the altitude, velocity, and flight path angle. Once again, the solid lines indicate the implicitly

integrated trajectory. The dotted lines indicate the explicitly integrated trajectory using the control law obtained from the implicit optimization. From the altitude plot it is evident that the explicit trajectory departs from the implicit one at a time of approximately five minutes. After eleven minutes, the explicit trajectory leaves the atmosphere. Calculating the rendezvous ΔV from the associated exit conditions yields a value of 315.3 m/s. The value obtained from the implicit trajectory, which stays in the atmosphere for a total of twenty six minutes, is 30.94 m/s.

Now, an explicitly integrated trajectory is sought that is similar to the implicit trajectory but uses a modified version of the control law of Figure 15. The switching time is chosen as the variable to modify. The switching time obtained from the implicit optimization is 147.202440 seconds. The switching time is varied until the apogee of the ascent orbit becomes close to that obtained from the implicitly integrated trajectory. A switching time of 147.166884 seconds is found to bring the apogee to an appropriate value. The dash-dot line in the above figures indicate the trajectory found with the modified switching time. The dash-dot lines are however not visible because they lie directly below the implicit trajectory lines. Decreasing the switching time by a mere 0.035556 seconds brings the modified explicitly integrated trajectory almost exactly in line with the implicit trajectory. The rendezvous ΔV found from the modified explicit trajectory is only 0.06 m/s greater than that found from the implicit trajectory. Also, the total skip times are within 0.4 seconds of each other.

Since only a small modification to the control law was necessary to produce a trajectory identical to that given as the optimal trajectory by the implicit integration optimization, the solution obtained by implicit integration closely approximates the control law necessary to produce the given trajectory. Therefore, the results of this section show that solutions obtained using the OTIS implicit integration routines closely approximate the trajectory and the associated control law.

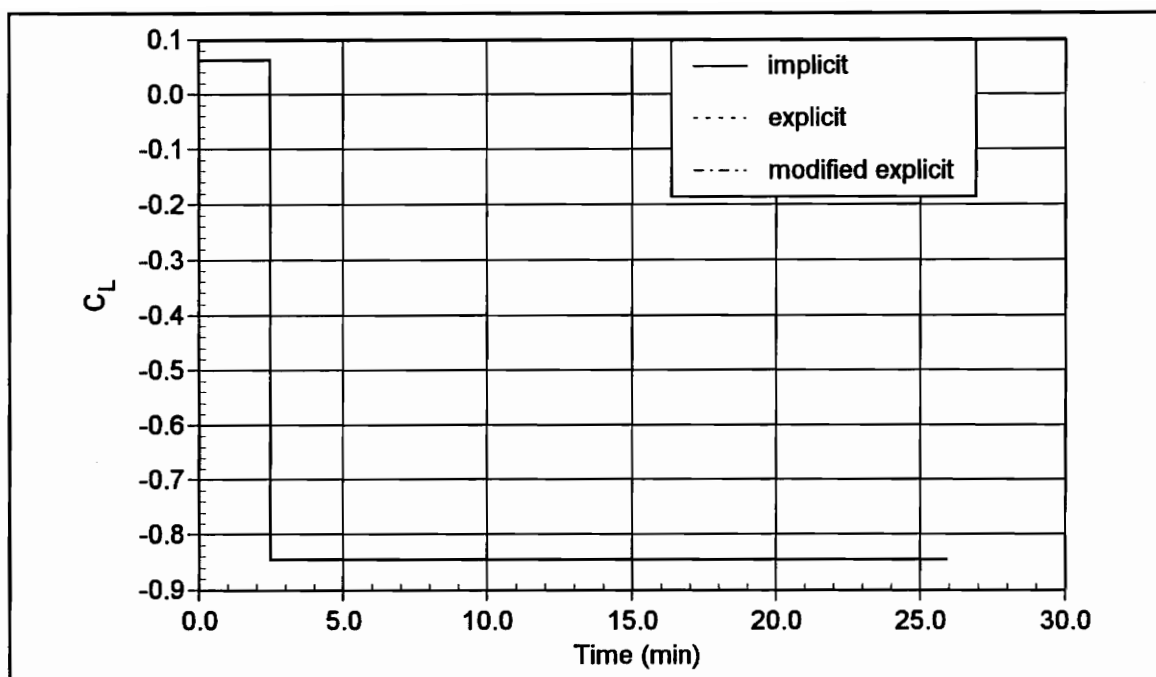


Figure 15. Comparison of Lift Coefficient vs. Time for Implicitly and Explicitly Integrated Trajectories

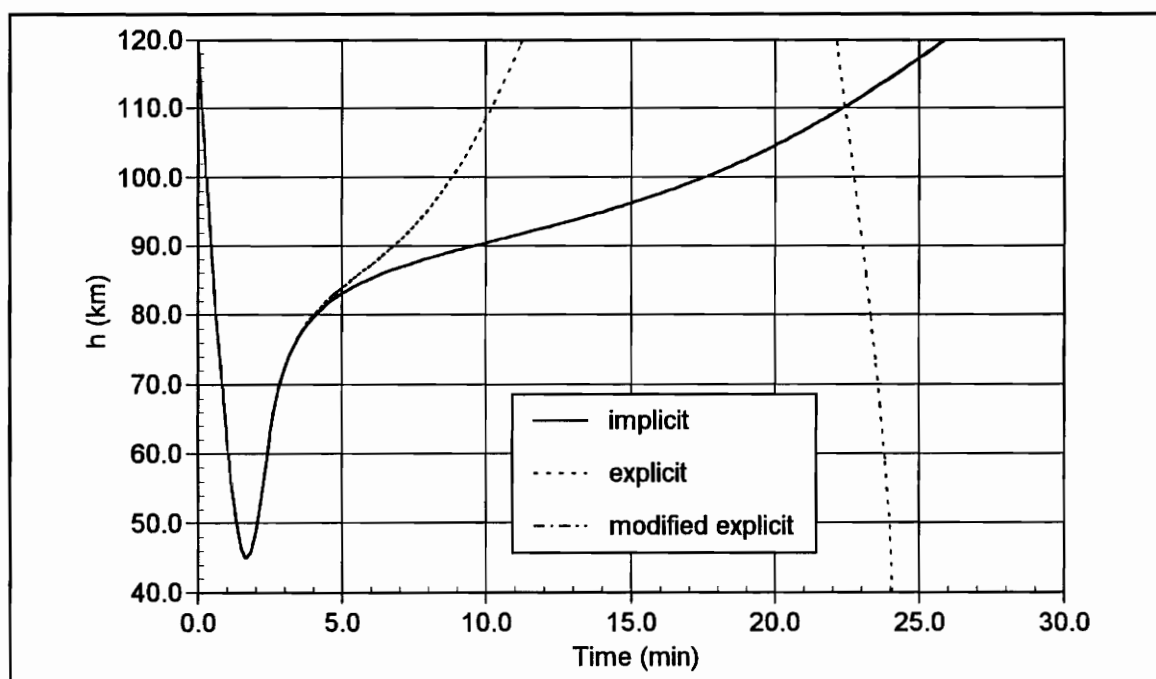


Figure 16. Comparison of Altitude vs. Time for Implicitly and Explicitly Integrated Trajectories

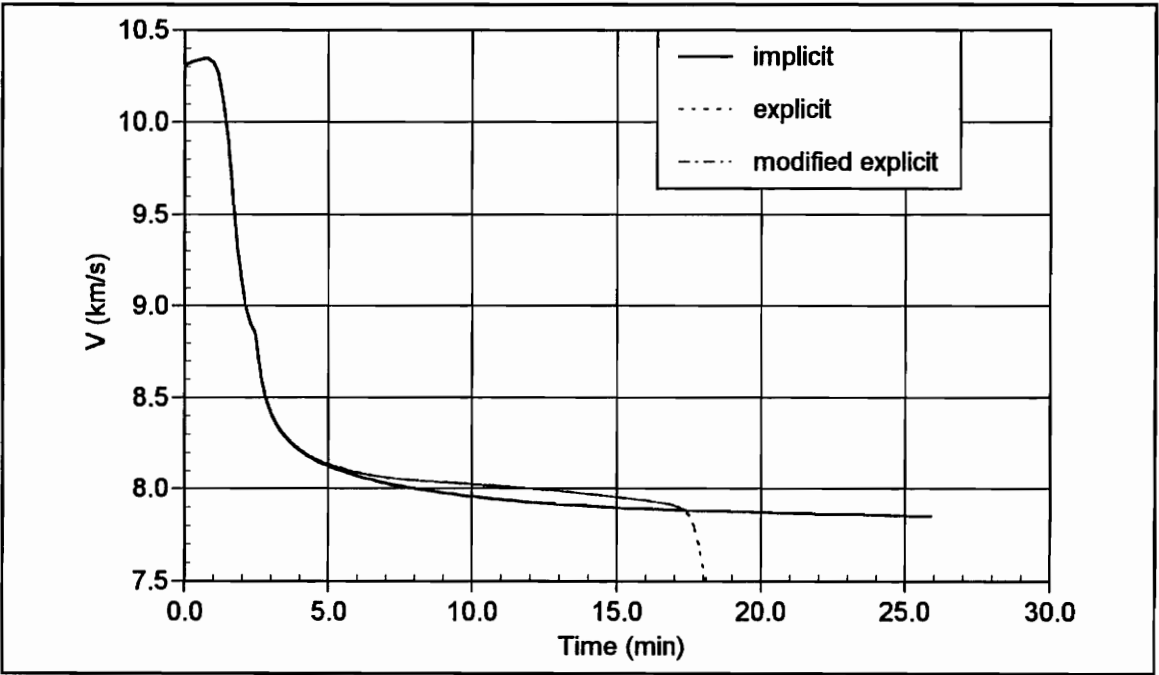


Figure 17. Comparison of Velocity vs. Time for Implicitly and Explicitly Integrated Trajectories

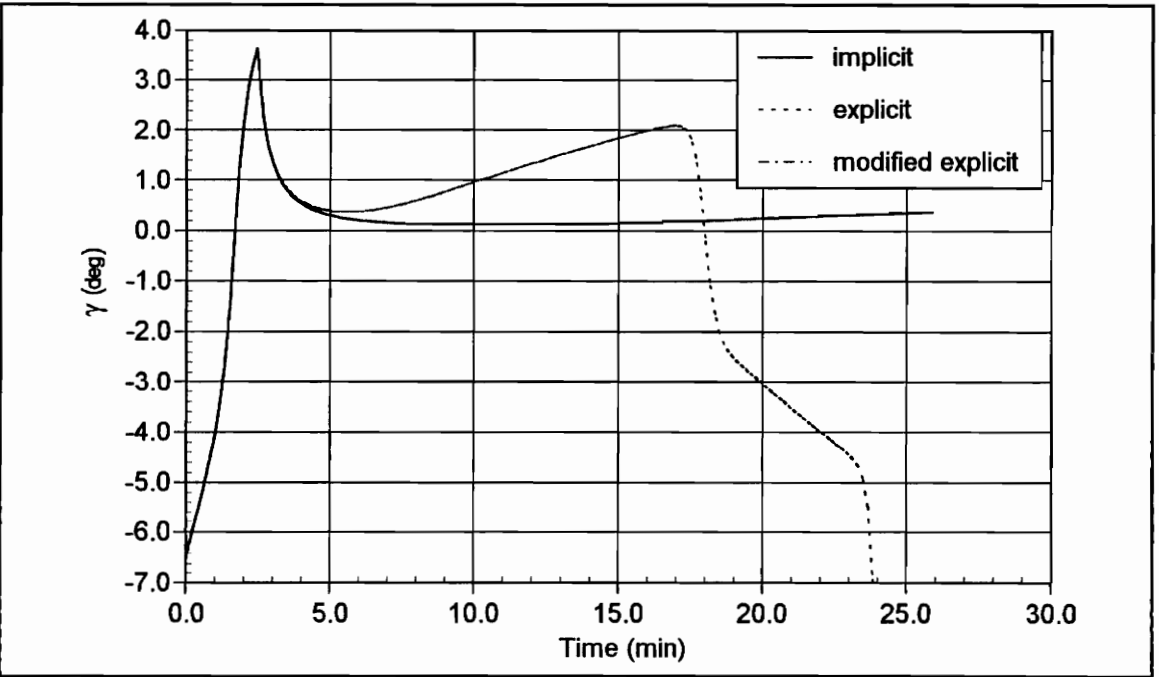


Figure 18. Comparison of Flight Path Angle vs. Time for Implicitly and Explicitly Integrated Trajectories

7 Comparison of Control Parameterization Schemes

A parameterization of the control law is sought that simplifies the calculation of optimal aeroassisted transfers while yielding solutions comparable to those obtained from optimal control theory. The problem specified in Chapter 5 is solved with the following control schemes: approximate continuous lift coefficient, piecewise-constant lift coefficient, and piecewise-constant pitch angle. Comparisons of the solutions are made.

7.1 *Approximate Optimal Lift Coefficient Control*

With a bounded lift coefficient taken as the control, OTIS is used to produce a continuously varying control law for the minimum fuel transfer. A trajectory consisting of forty nodes is used to characterize the atmospheric portion of the transfer. Conceivably, increasing the number of nodes increases the accuracy of the solution. However, forty nodes are sufficient to produce adequate results without placing an extreme burden on computer resources. Both fixed deboost perigee and free deboost perigee problems are solved.

Figures 19 through 24 show the time histories of the lift coefficient, altitude, velocity, flight path angle, heating rate, and g-load for representative solutions with fixed deboost perigee (solid lines) and free deboost perigee (dotted lines). As mentioned earlier, the minimum fuel problem as stated has no unique solution since all control laws reaching the same condition at atmospheric exit yield the same value for the second ΔV . The approximate optimal solutions obtained all exhibited some degree of oscillation in the lift coefficient control as seen in the first five minute of Figure 19. This oscillation is a consequence of the parameterization of the control law, and is not found in the results produced by optimal control theory [16]. The lift coefficient is against

the lower bound for much of the rest of the skip but increases slightly as the vehicle nears atmospheric exit. Except for the initial oscillations, the approximate optimal control law agrees with that of Mease and Vinh [16].

The problem with fixed deboost perigee is the same as that solved in Chapter 6 except here a bound is placed on the lift coefficient. Namely, the atmospheric entry conditions are taken as $V_e = 10.31$ km/s and $\gamma_e = -6.5^\circ$ which gives a deboost perigee of 6400.0 km and a characteristic velocity $\Delta V_1 = 1496.2$ m/s. The control law produces atmospheric exit conditions such that $\Delta V_2 = 30.8$ m/s. When compared to the unbounded lift solution of Section 6.1, this bounded lift case is seen to exhibit the same cost ($\Delta V_1 + \Delta V_2$). Furthermore, the free deboost perigee case also reaches the same exit conditions and hence has the same ΔV_2 . However, the entrance conditions are found to be $V_e = 10.31$ km/s and $\gamma_e = -4.5^\circ$ which leads to a higher deboost perigee. The initial ΔV_1 is calculated to be 1490.9 m/s which is a savings of 5.3 m/s over the fixed deboost perigee case.

The atmospheric trajectories of the two cases presented in Figures 19 through 24 are largely the same. The major difference being a spike in the g-load for the free deboost perigee solution. The spike is caused by a relatively large value of the control being applied with the vehicle at its lowest altitude. The greater lift coefficient combined with the large dynamic pressure yields a larger force and a higher acceleration. The trajectories of the two cases are indistinguishable for the last twenty minutes of the skip.

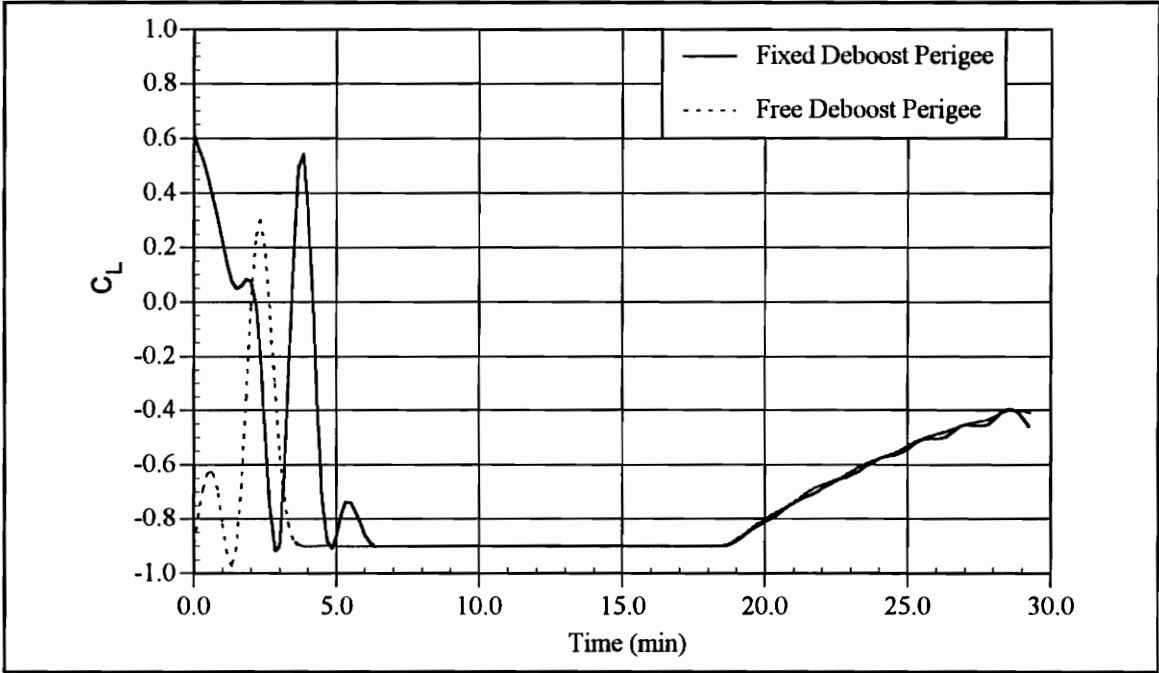


Figure 19. Lift Coefficient vs. Time for Approximate Optimal Control with Bounded Lift Coefficient

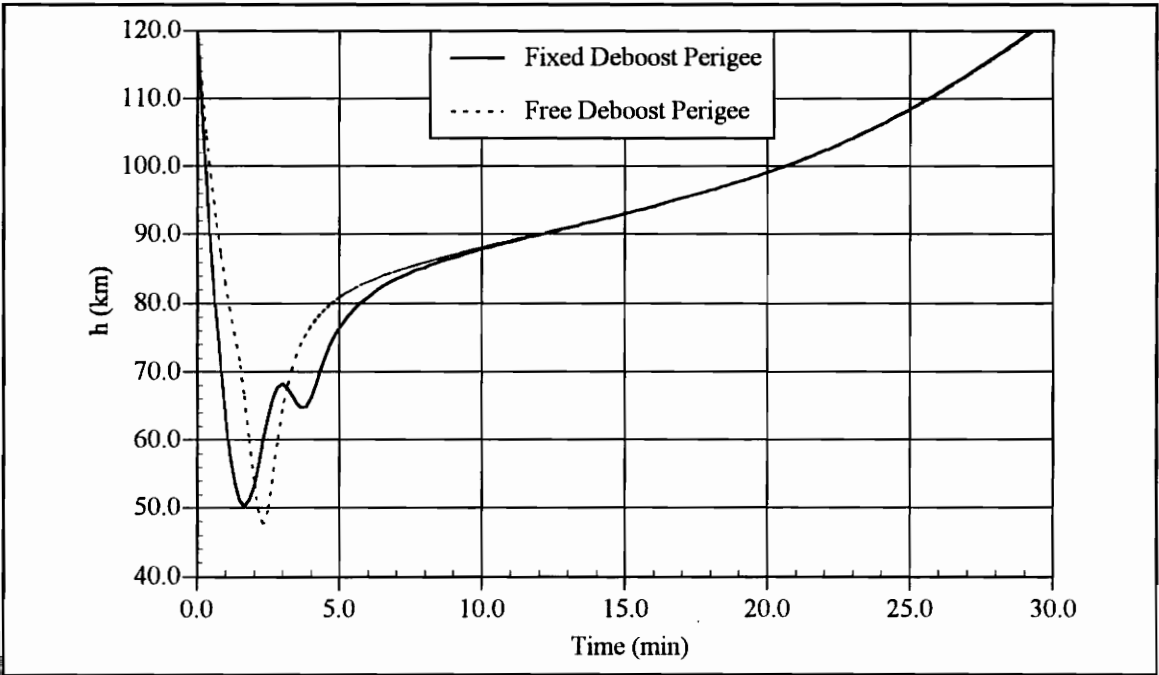


Figure 20. Altitude vs. Time for Approximate Optimal Control Solution

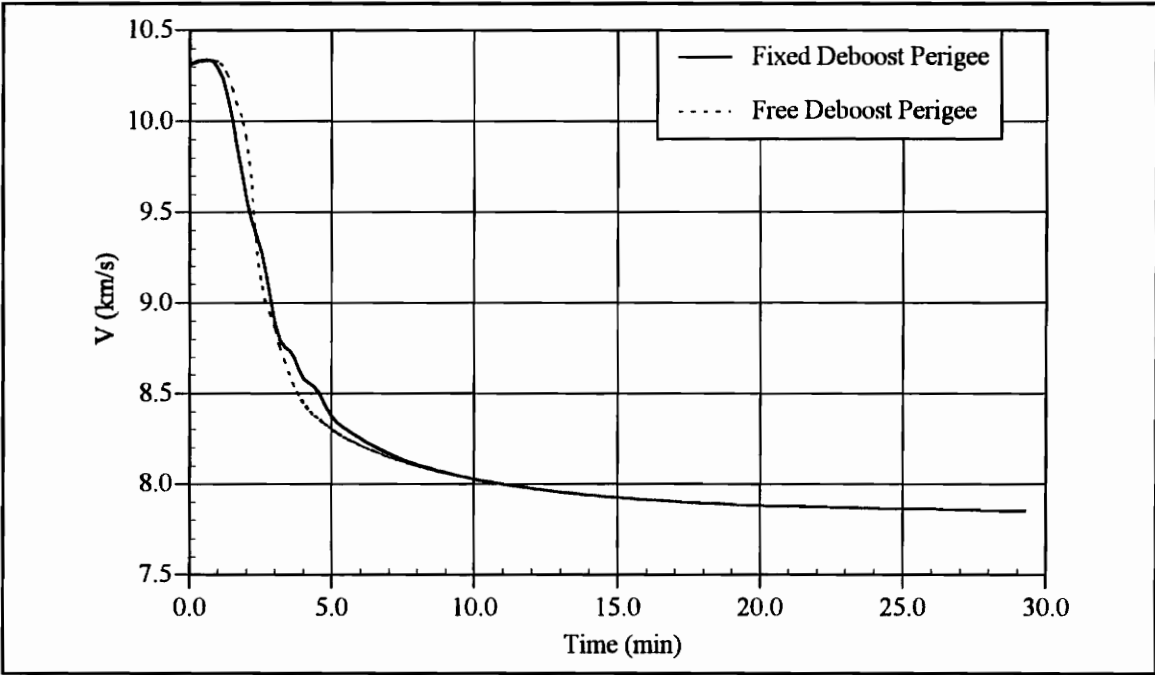


Figure 21. Velocity vs. Time for Approximate Optimal Control Solution

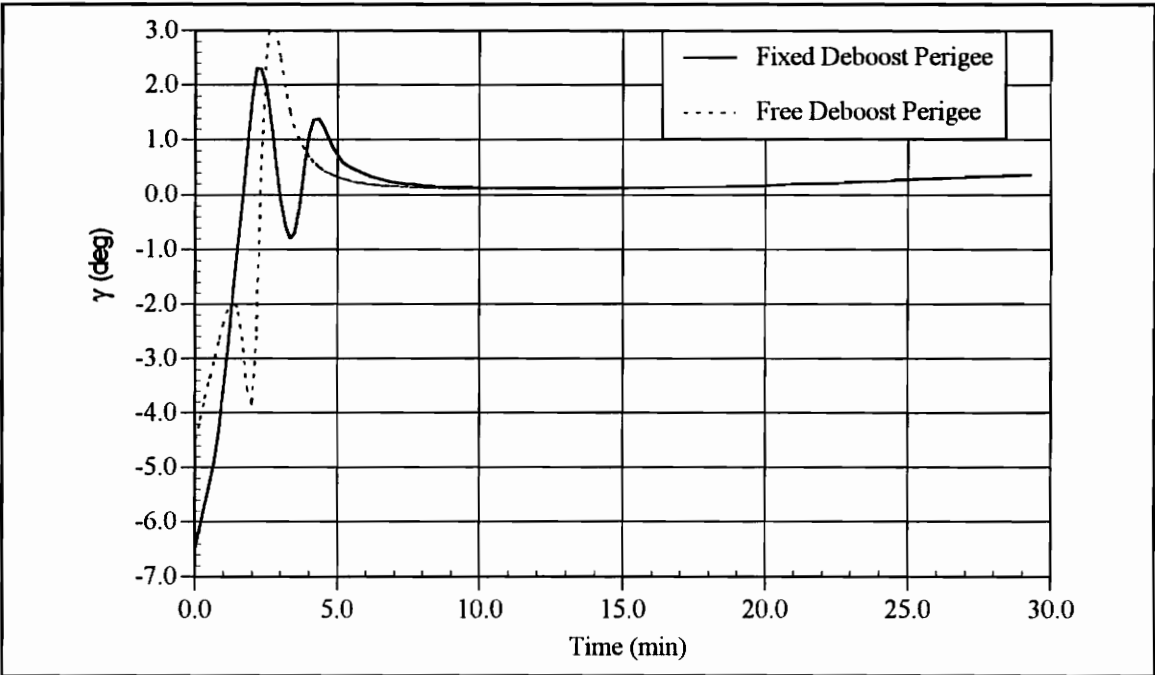


Figure 22. Flight Path Angle vs. Time for Approximate Optimal Control Solution

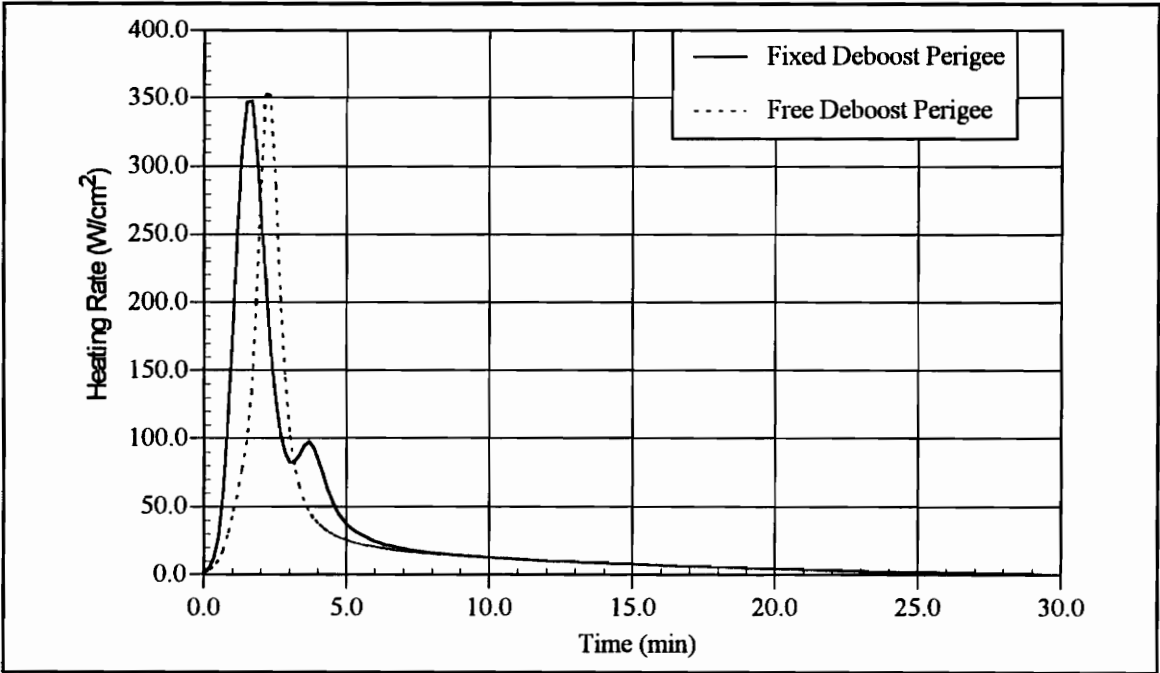


Figure 23. Heating Rate vs. Time for Approximate Optimal Control Solution

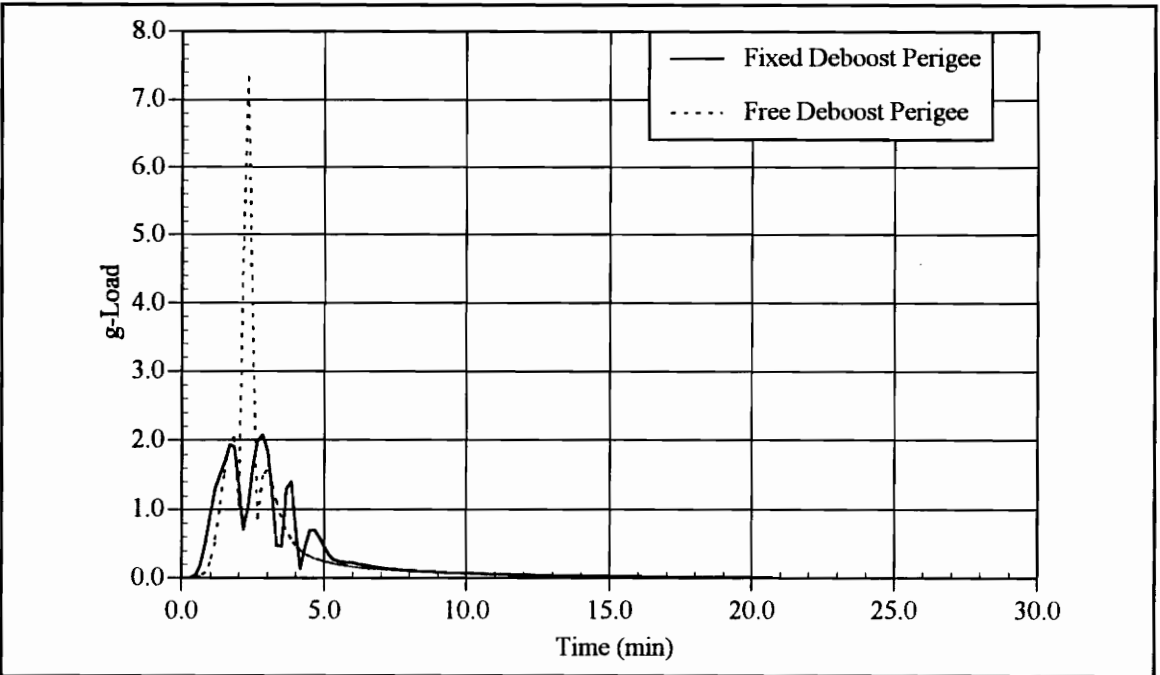


Figure 24. G-Load vs. Time for Approximate Optimal Control Solution

7.2 Piecewise-Constant Lift Coefficient Control

The two types of piecewise-constant lift coefficient control are distinguished by the switching time. The first case allows switching the lift coefficient value only when the flight path angle is zero. This case is referred to as the fixed switching time case and is based on the Martell [13] scheme of switching the control when the vehicle is at the minimum altitude. The second case allows the optimizer to choose the switching time so as to minimize the cost function (free switching time case). Both cases employ two segment piecewise-constant lift coefficient control and use a forty node skip trajectory for calculation with OTIS.

7.2.1 Fixed Switching Time

With the switching time and deboost perigee fixed, only two control parameters are needed to describe the skip trajectory (C_{L1} and C_{L2}). Table 5 gives a representative summary of converged solutions for this case. The solutions are arranged in order of decreasing cost (ΔV_2). While it is likely that some of the converged solutions are not actual local minima, but simply arise from the numerical integration of the trajectories, others may be local minima. In particular, the first entry, which is notably dissimilar to the rest of the entries, exhibits a constant maximum lift coefficient which coincides with the results of Martell and probably represents a local minimum. Nonetheless, a trend is apparent in the remaining entries as the solutions tend toward decreasing ΔV_2 . The initial value of the lift coefficient rises steadily from 0.08 to its maximum value of 0.90. The second value of the lift coefficient decreases from -0.0085 to -0.11. The time needed to complete the skip increases from 684 seconds to 2295 seconds. The cost decreases from 68 m/s to 34 m/s.

In reaching the solution, the value of C_{L1} increases to pull the vehicle out of the downward plunge. The value of C_{L2} then decreases to keep the vehicle in the atmosphere longer so as to bleed off enough energy to lower apogee to the target radius allowing a tangential ΔV_2 . With C_{L1} against the bound, this family of solutions is at its overall minimum. The minimum found with this control formulation is approximately 3.5 m/s greater than that found with the approximate continuous control of Section 7.1.

Table 5. Converged Solutions for Piecewise-Constant Lift Coefficient Control with Fixed Switching Time and Fixed Deboost Perigee

C_{L1}	t_{sw} (sec)	C_{L2}	t_{final} (sec)	ΔV_2 (m/s)
0.90000000	72.52	0.90000000	161.26	1413.27
0.08331963	97.73	-0.00850096	684.50	68.17
0.22336942	86.16	-0.04067982	1306.16	40.22
0.29206689	83.24	-0.05192941	1503.24	38.18
0.55813100	76.77	-0.08599651	1996.77	35.41
0.90000000	72.52	-0.11503531	2294.99	34.33

Figures 25 through 30 give the time histories of the lift coefficient, altitude, velocity, flight path angle, heating and g-load for the fixed switching time fixed deboost case. With the initial lift coefficient remaining constant at its maximum value, the minimum altitude reached by the vehicle will be higher than that produced by any other control law based on the same atmospheric entrance conditions. The minimum altitude for the piecewise-constant fixed switching time controller is 10 km greater than that for the approximate continuous controller. The higher minimum altitude leads to a maximum heating rate that is half that of the continuous case. The g-load for this piecewise-constant case is larger than that of the fixed deboost perigee continuous control case due to the large lift coefficient value being used at the minimum altitude. The velocity and flight path angle histories show the vehicle reaching nearly the same exit conditions as the continuous case. The similar exit conditions produce a rendezvous ΔV that is only 3.5 m/s larger than the continuous controller case.

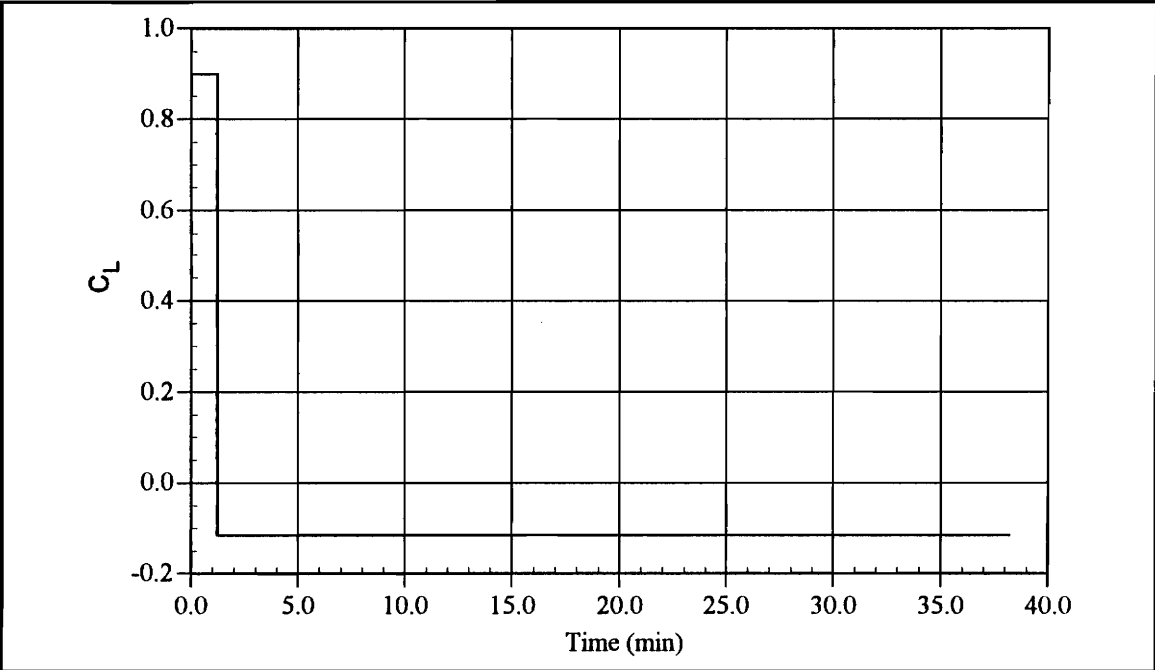


Figure 25. *Lift Coefficient vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Fixed Deboost Perigee*

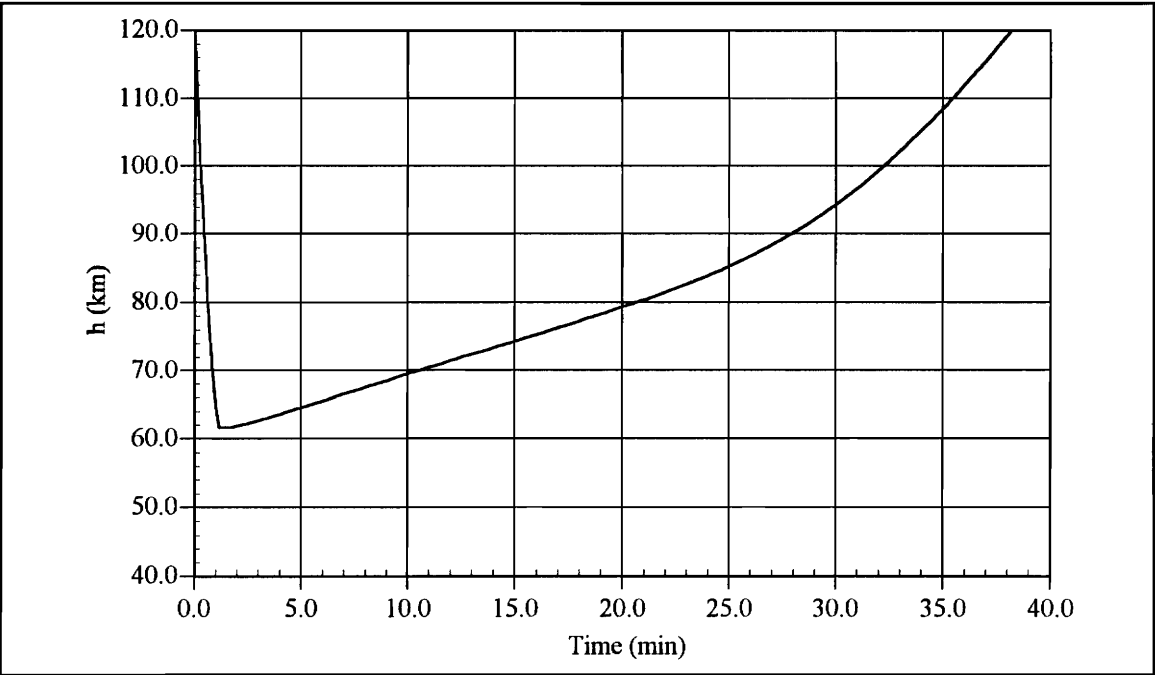


Figure 26. *Altitude vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Fixed Deboost Perigee*

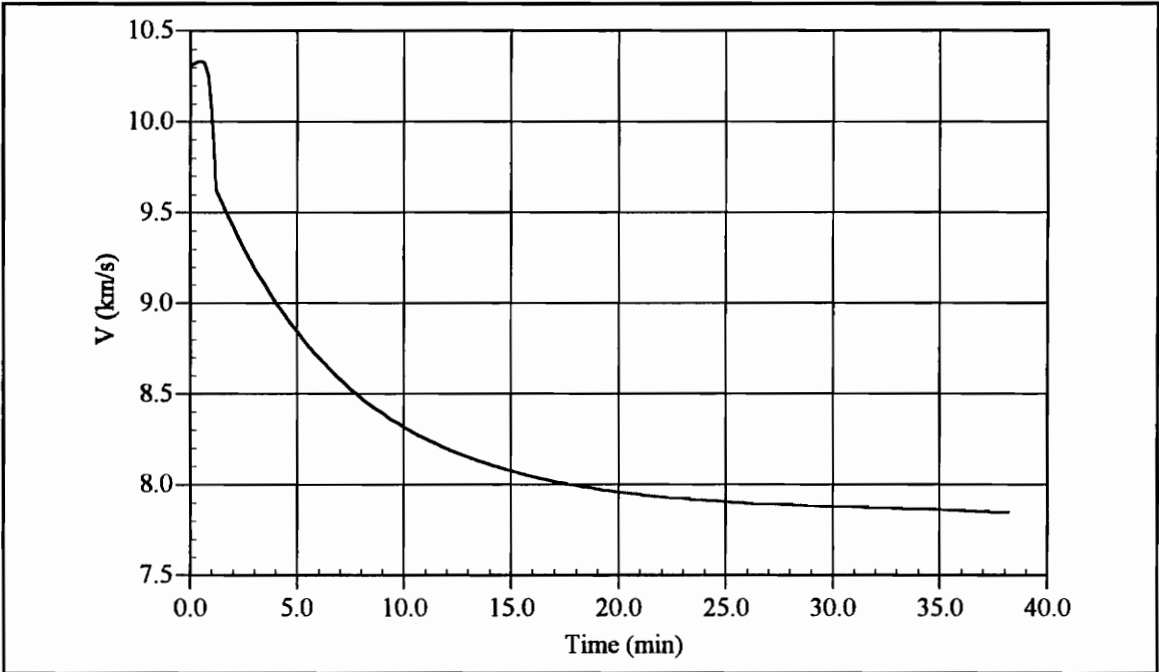


Figure 27. Velocity vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Fixed Deboost Perigee

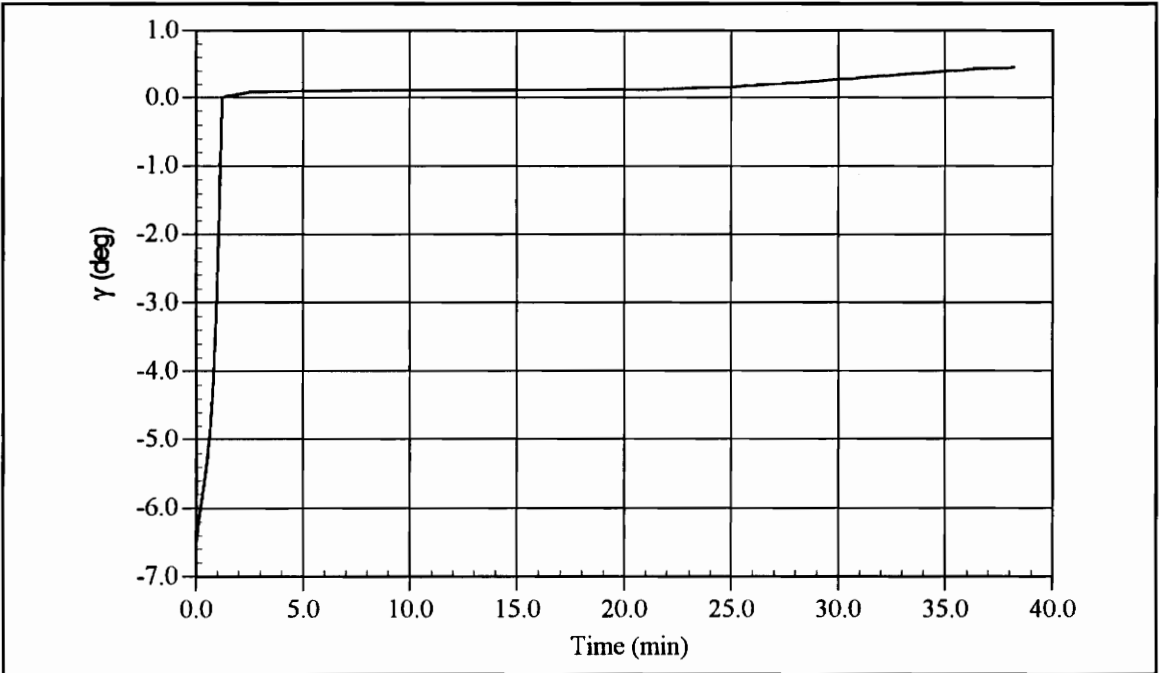


Figure 28. Flight Path Angle vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Fixed Deboost Perigee

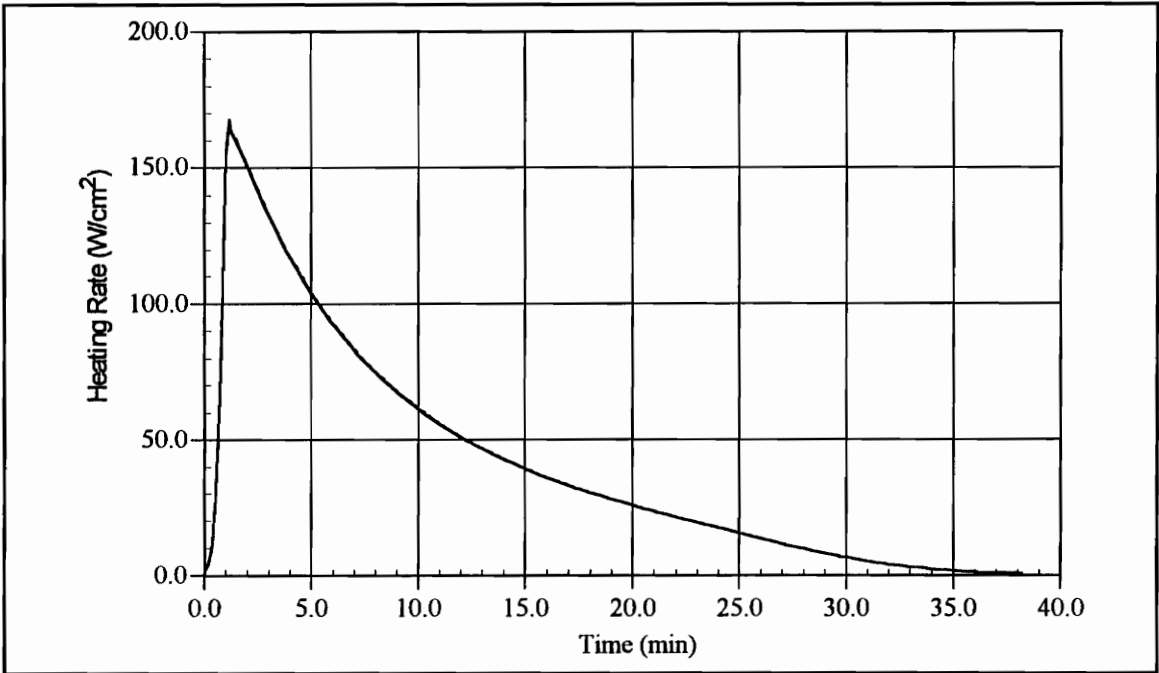


Figure 29. Heating Rate vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Fixed Deboost Perigee

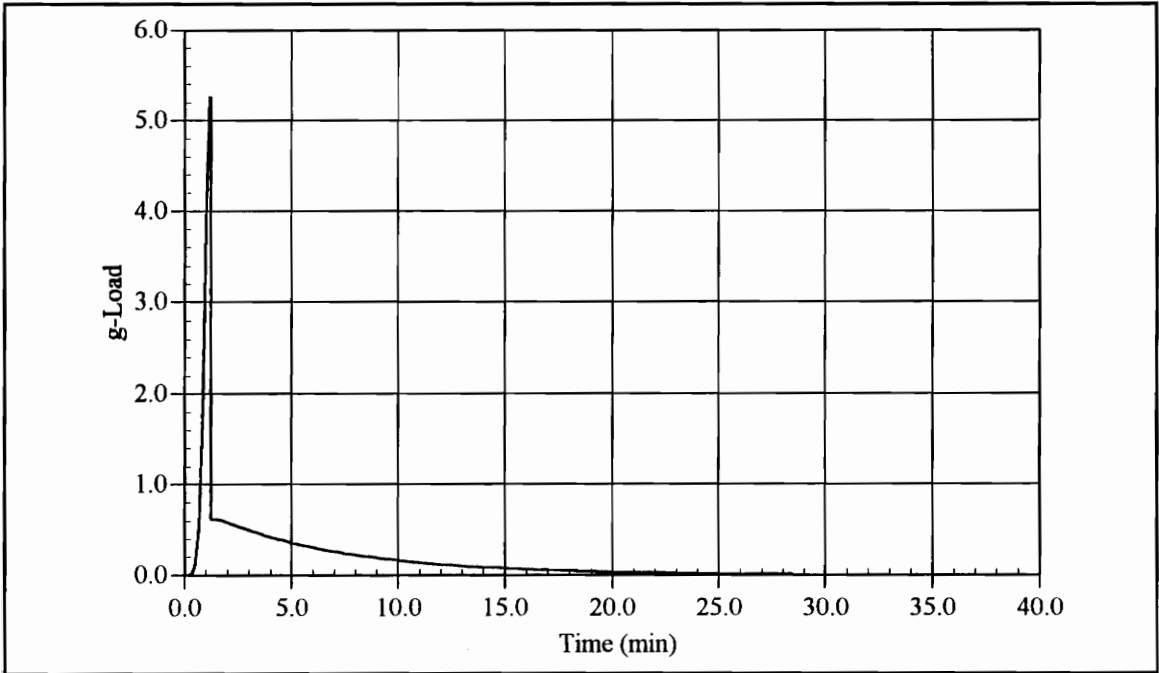


Figure 30. G-Load vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Fixed Deboost Perigee

7.2.2 Free Switching Time

The switching time is taken as the third control parameter for the fixed deboost perigee case. When the deboost perigee is not fixed, the velocity and flight path angle at entrance to the atmosphere are also taken as control parameters bringing the total number of parameters to five. Table 6 gives a summary of converged solutions for the fixed deboost perigee case. The solutions are arranged in order of decreasing cost (ΔV_2). Again, it is unlikely that all the solutions represent local minima. The solutions are included to illustrate the trends in the controls that lead to the minimizing solution.

Again the constant maximum lift coefficient solution is obtained as a likely local minima but proves a poor candidate for the global minimum. For the case with fixed deboost perigee, a similar trend to that found for the case with fixed switching time is discovered. The value of C_{L1} increases to pull the vehicle out of the downward plunge. The value of C_{L2} decreases to keep the vehicle in the atmosphere longer so as to bleed off enough energy to lower apogee to the target radius allowing a tangential ΔV_2 . However, it is when C_{L2} reaches the bound that the overall minimum is obtained. In contrast, the fixed switching time result is limited by C_{L1} . The minimum ΔV_2 of 30.91 m/s is approximately equal to the 30.78 m/s value obtained from the approximate continuous control solution and represents a savings of 3.4 m/s over the fixed switching time case. Also, the total time of the skip is only 1552 seconds which is 743 seconds shorter than the time needed for the fixed switching time case.

Table 7 shows two converged solutions for the free deboost perigee case. The first entry is taken from the minimum produced from the above fixed deboost perigee case. The solutions indicate a lesser ΔV_1 leading to a higher deboost perigee. To compensate, C_{L1} becomes negative to pull the vehicle into the atmosphere to prevent premature atmospheric exit. The result is a savings in the initial ΔV of 5 m/s. Also, the corresponding ΔV_2 is found to be 30.87 m/s which is within 0.1 m/s of the value obtained from the approximate optimal solution for either the fixed or free deboost perigee cases.

Figures 31 through 36 show the time histories for the lift coefficient, altitude, velocity, flight path angle, heating rate, and g-load for the minimum fuel solutions with free switching time for both

Table 6. Converged Solutions for Piecewise-Constant Lift Coefficient Control with Free Switching Time and Fixed Deboost Perigee

C_{L1}	t_{sw} (sec)	C_{L2}	t_{final} (sec)	ΔV_2 (m/s)
0.90000000	71.57	0.90000000	161.26	1413.27
0.04307151	160.57	-0.55882978	880.00	36.43
0.04697586	156.49	-0.59873754	1348.45	31.60
0.06328520	147.21	-0.84449102	1553.34	30.94
0.06938178	144.46	-0.90000000	1552.41	30.91

Table 7. Converged Solutions for Piecewise-Constant Lift Coefficient Control with Free Switching Time and Free Deboost Perigee

V_e (km/s)	γ_e (deg)	ΔV_1 (m/s)	C_{L1}	t_{sw} (sec)	C_{L2}	t_{final} (sec)	ΔV_2 (m/s)	ΔV_{total} (m/s)
*10.310	-6.5	1496.23	0.06938	144.46	-0.9	1552.41	30.91	1527.14
10.3102	-5.5	1493.14	-0.00038	289.35	-0.9	1314.18	31.60	1524.74
10.3106	-4.7	1491.20	-0.13282	392.32	-0.9	2203.64	30.87	1522.07

* Fixed deboost perigee case from Table 6

the fixed and free deboost perigee cases. Due to the differing entrance conditions, the free deboost perigee case shows a more gradual penetration into the atmosphere and leads to a minimum altitude that is much higher than that for the fixed deboost perigee case. The velocity and flight path angle histories indicate that both cases reach the same atmospheric exit conditions though the free deboost perigee case takes over ten minutes longer to do so. The peak heating rate again corresponds to the minimum altitude and is large, near 450 W/cm^2 , for the fixed deboost case and small, near 180 W/cm^2 , for the free deboost perigee case. The maximum g-load for the fixed deboost perigee case is lower than that for the case with fixed switching time, but the max g-load for the free deboost perigee free switching time case is an extremely low 1.3 times the acceleration due to gravity.

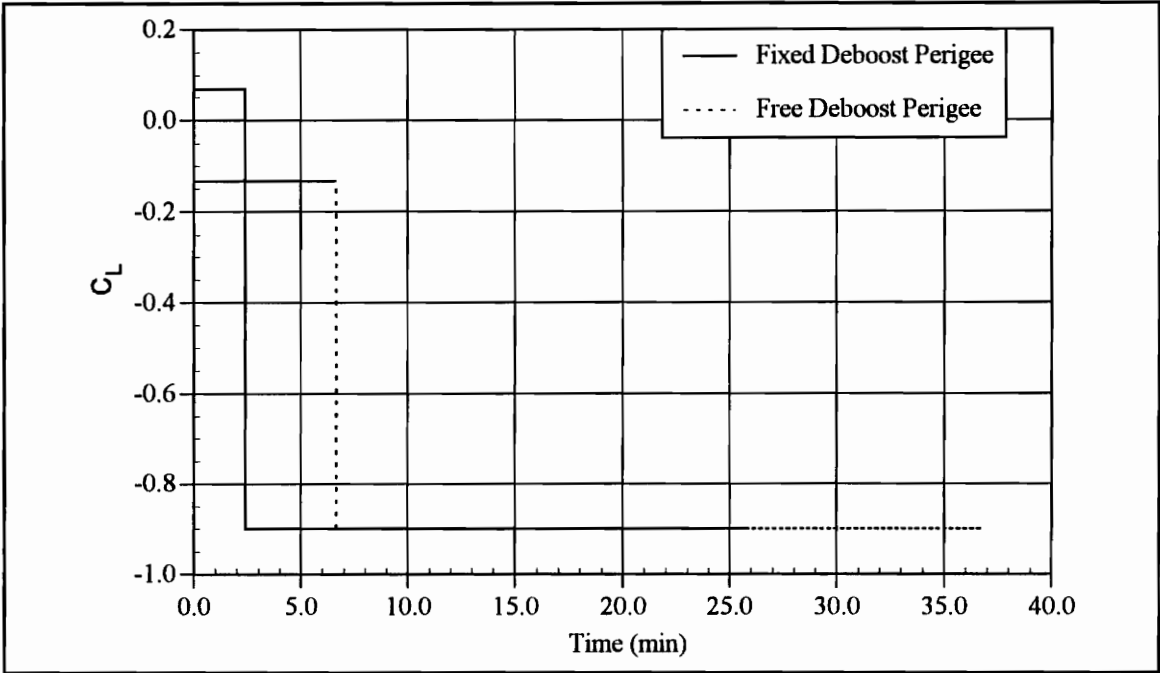


Figure 31. Lift Coefficient vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Free Switching Time

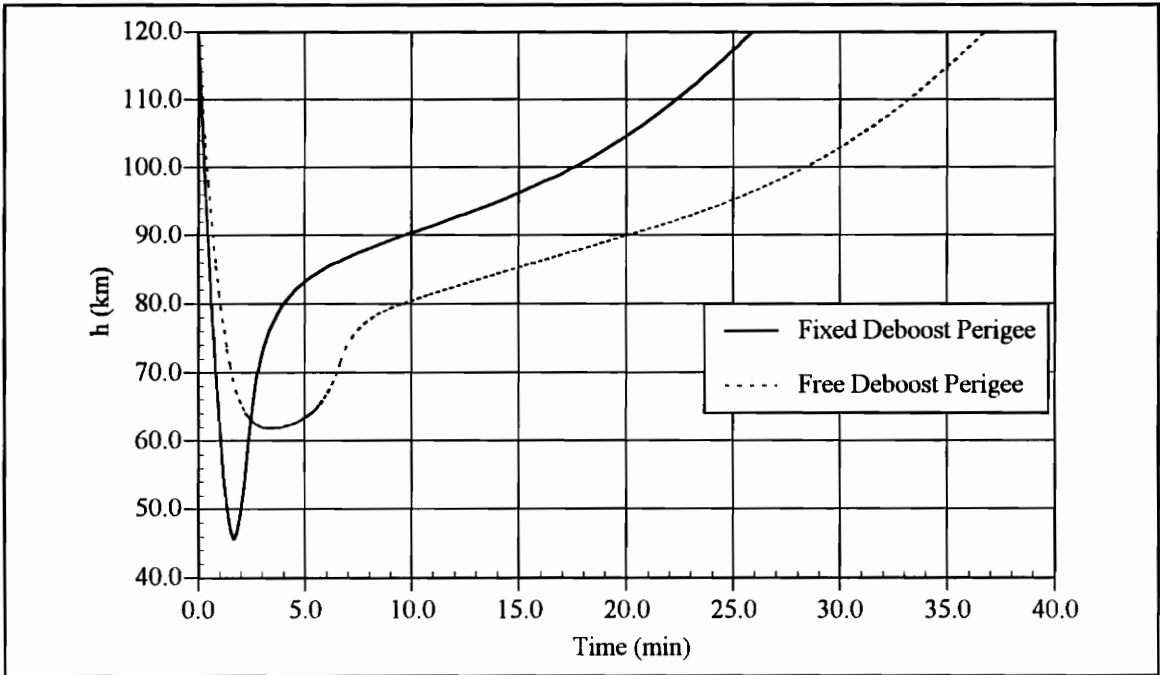


Figure 32. Altitude vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Free Switching Time

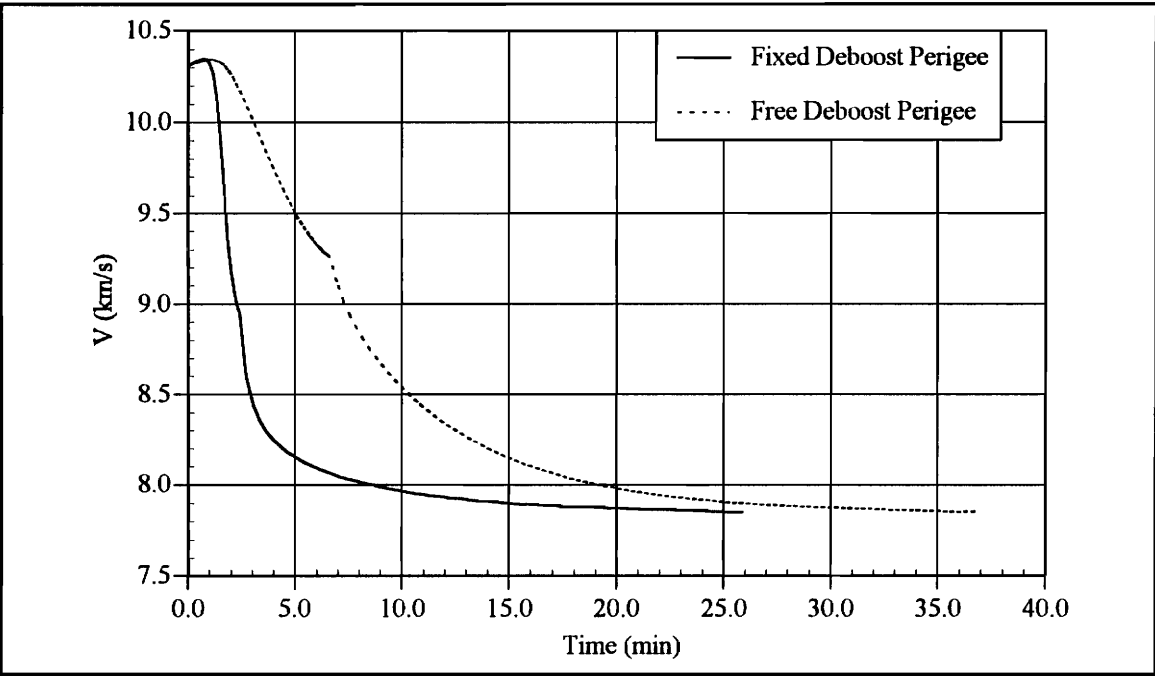


Figure 33. Velocity vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Free Switching Time

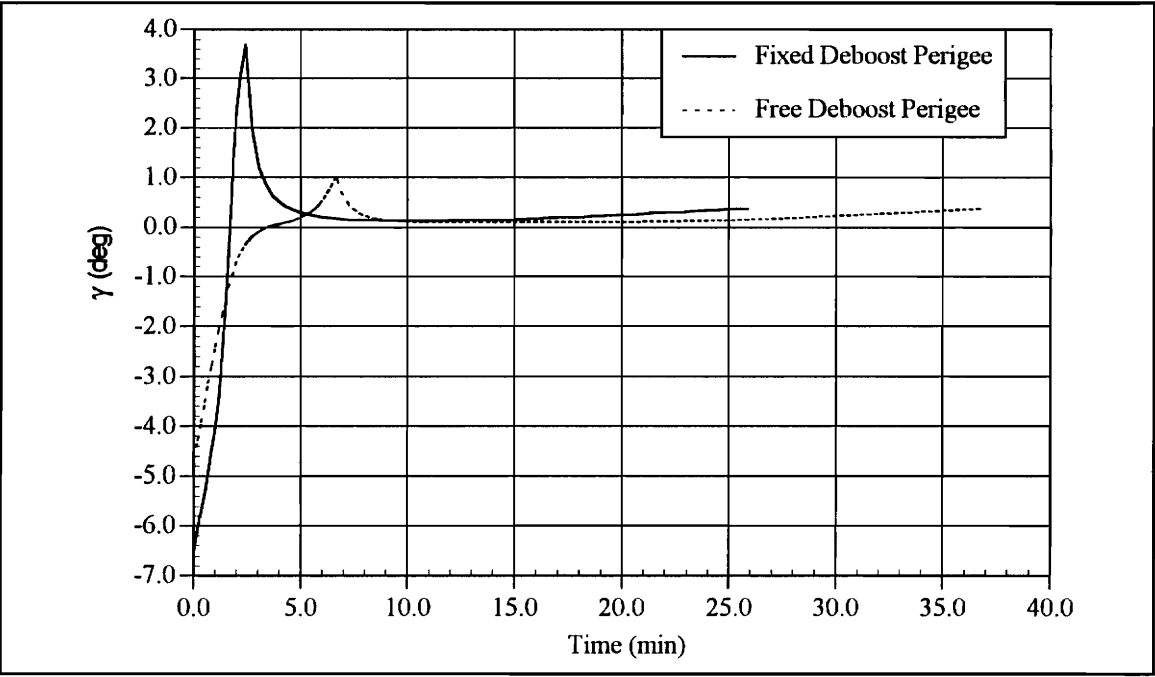


Figure 34. Flight Path Angle vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Free Switching Time

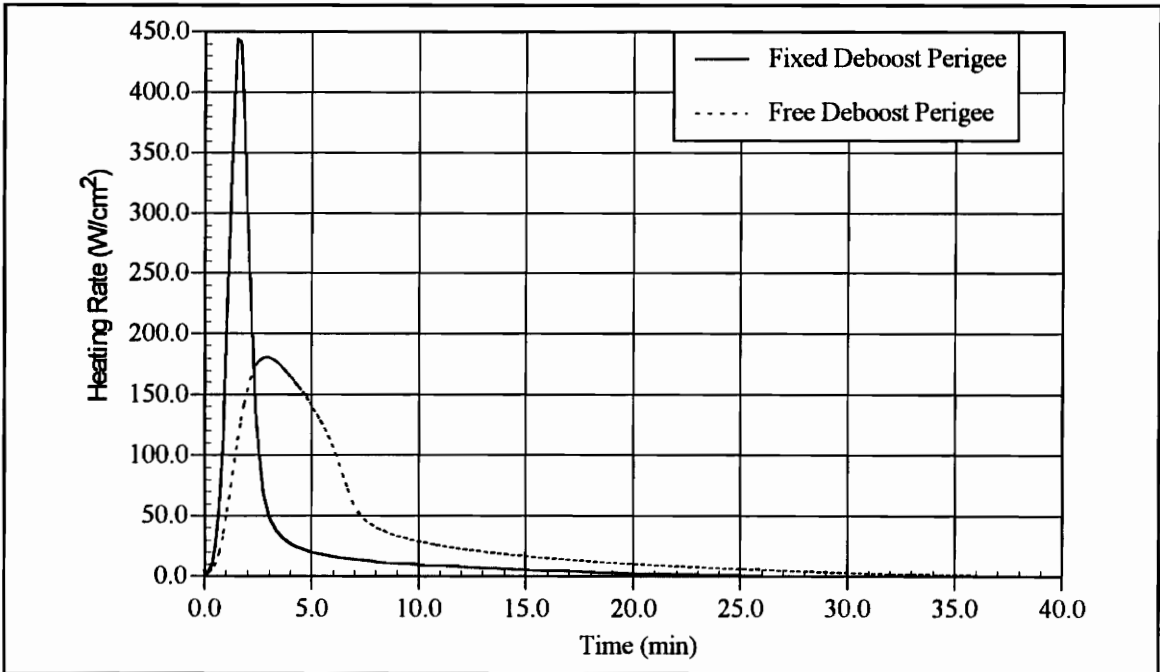


Figure 35. *Heating Rate vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Free Switching Time*

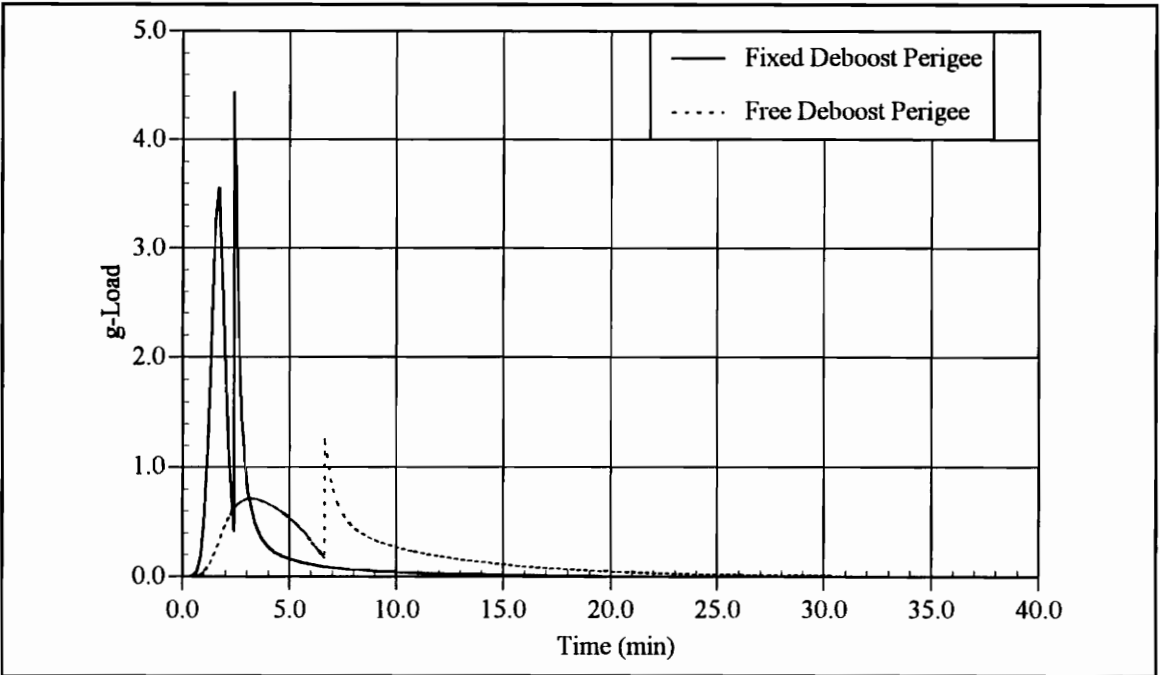


Figure 36. *G-Load vs. Time for Piecewise-Constant Lift Coefficient Control Solution with Free Switching Time*

7.3 Piecewise-Constant Pitch Angle Control

Kelley et al [10] indicate that use of a constant pitch angle control produces a better damped response than constant angle of attack control for certain problems. In hopes of improving the performance of the optimization, piecewise-constant pitch angle control is investigated for the case of fixed deboost perigee only. Again, a two segment controller with a free switching time is used. A forty node skip trajectory is employed with OTIS.

The minimum fuel solution is found to be: $\Theta_1=0.6917$, $t_w=90.48$ sec, $\Theta_2=-0.8216$. Again, the minimum rendezvous ΔV , which is found to be 30.90 m/s, is within 0.12 m/s of that obtained from the approximate continuous control formulation. Figures 37 through 42 show the time histories for the pitch angle, lift coefficient, altitude, velocity, flight path angle, heating rate, and g-load. Note the constant pitch angle case produces nearly constant lift coefficient. The reason the lift coefficient is nearly constant is two fold. First, since the relationship between pitch angle, angle of attack and flight path angle is given by

$$\Theta = \alpha + \gamma \quad (87)$$

and most of the flight takes place at small nearly constant flight path angles, the angle of attack is also constant. Secondly, since the current formulation considers angle of attack and lift coefficient to be equal, yielding direct lift coefficient control instead of angle of attack control, the lift coefficient will be constant as well.

In order to fully explore constant pitch angle solutions, a lift model would need to be implemented. For the current research it will suffice to state that constant pitch angle control yields solutions similar to constant lift coefficient control with no notable improvement in performance of the optimization routines.

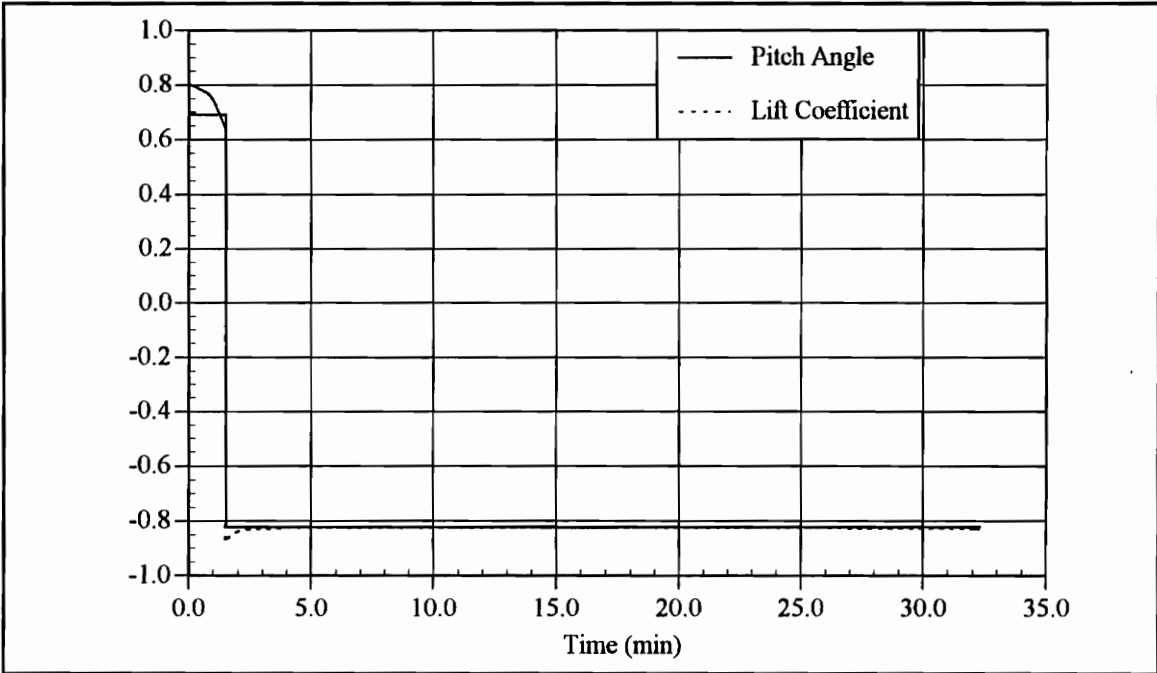


Figure 37. Pitch Angle and Lift Coefficient vs. Time for Piecewise-Constant Pitch Angle Control Solution with Fixed Deboost Perigee

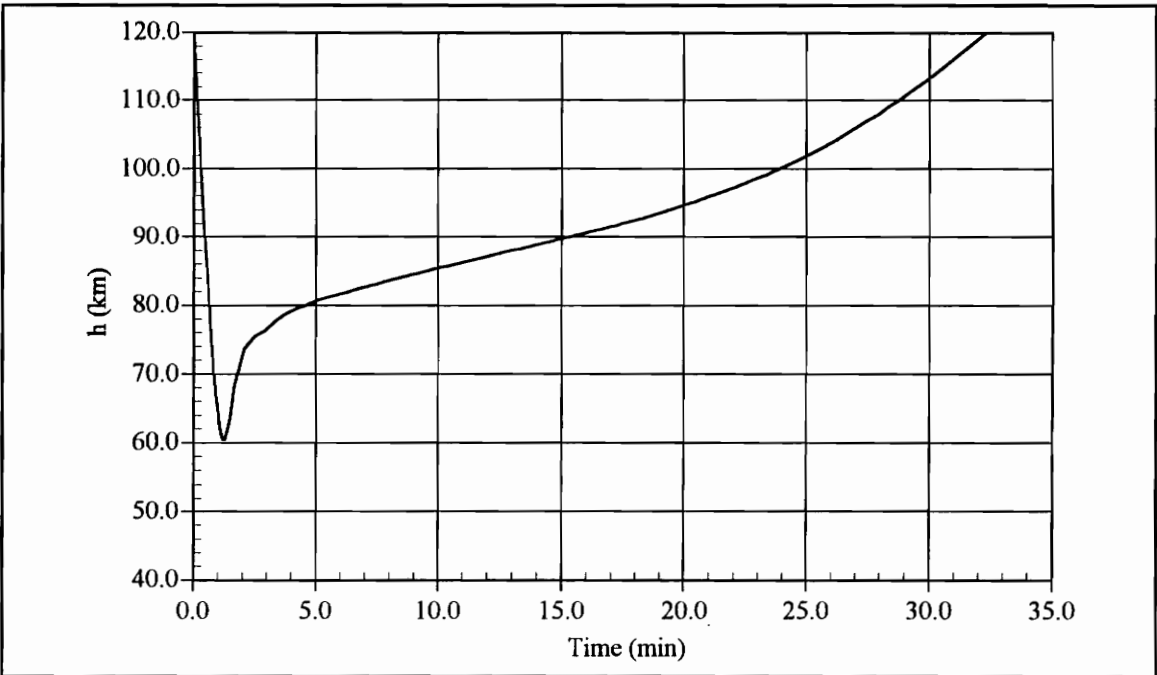


Figure 38. Altitude vs. Time for Piecewise-Constant Pitch Angle Control Solution with Fixed Deboost Perigee

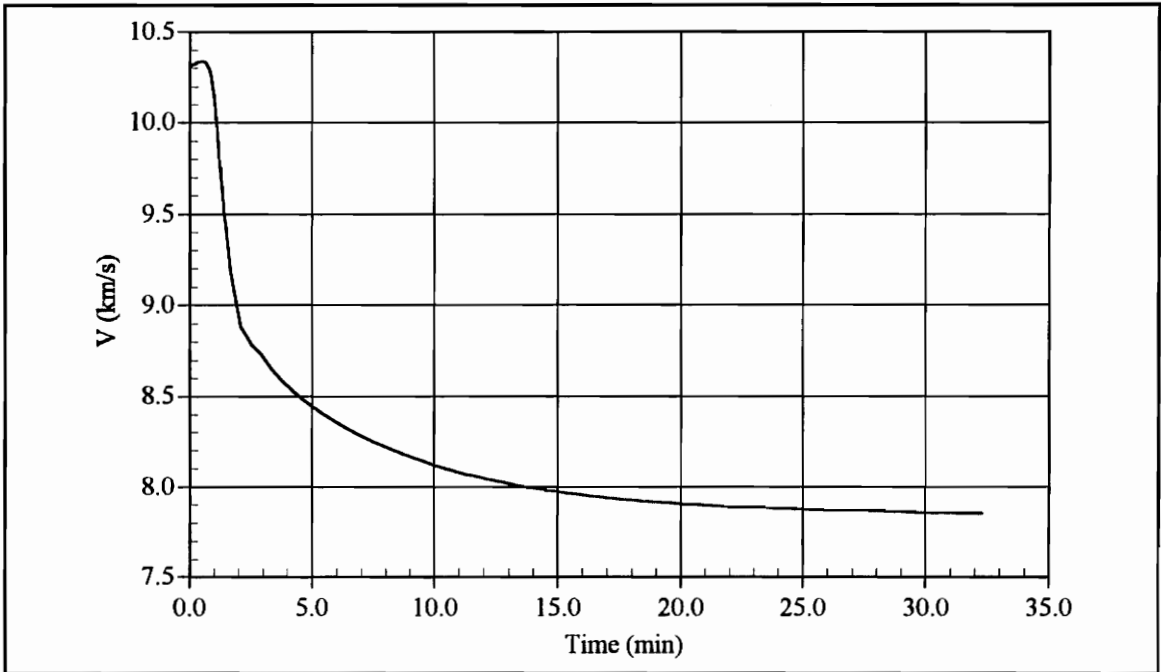


Figure 39. Velocity vs. Time for Piecewise-Constant Pitch Angle Control Solution with Fixed Deboost Perigee

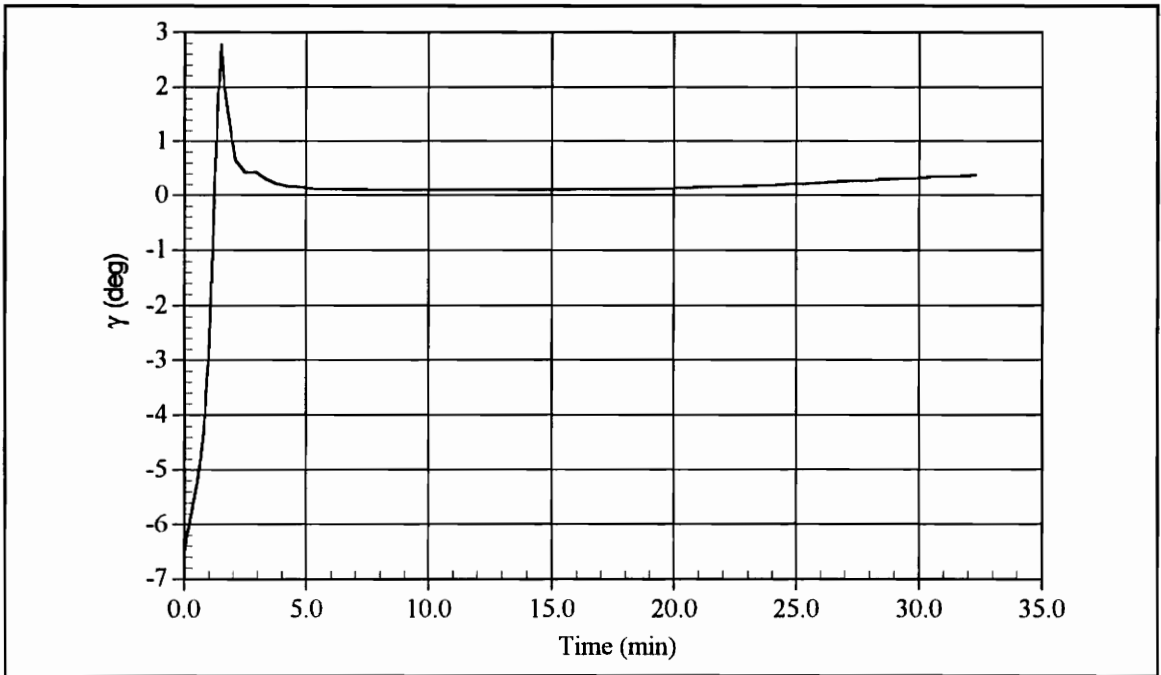


Figure 40. Flight Path Angle vs. Time for Piecewise-Constant Pitch Angle Control Solution with Fixed Deboost Perigee

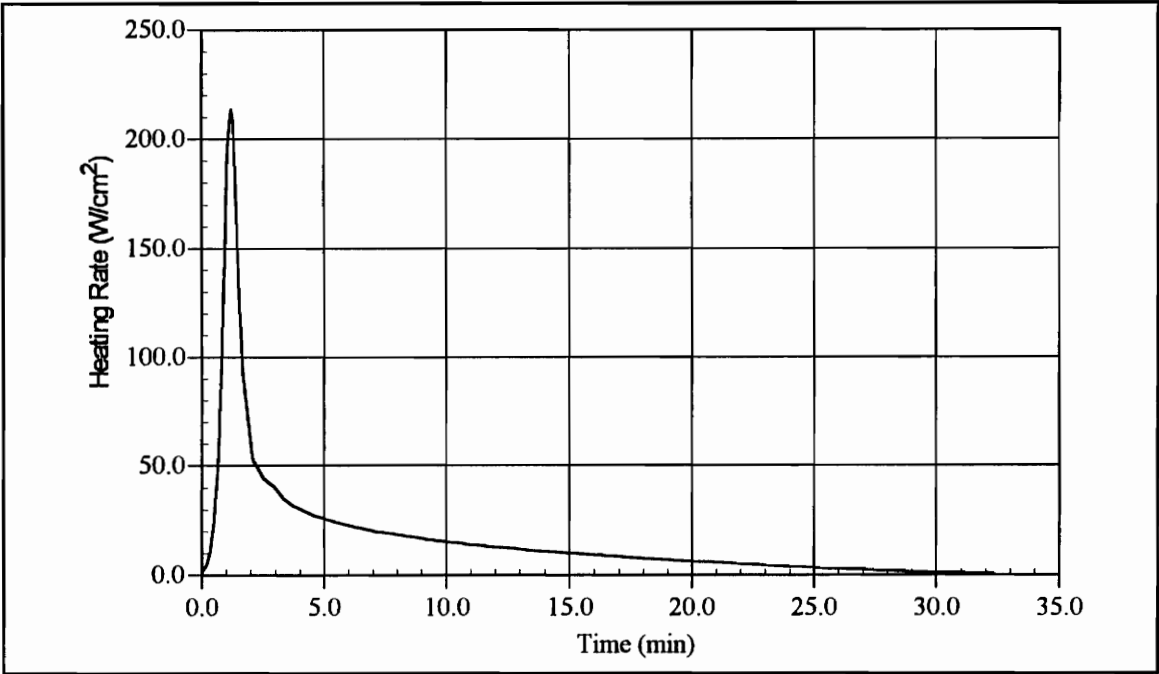


Figure 41. *Heating Rate vs. Time for Piecewise-Constant Pitch Angle Control Solution with Fixed Deboost Perigee*

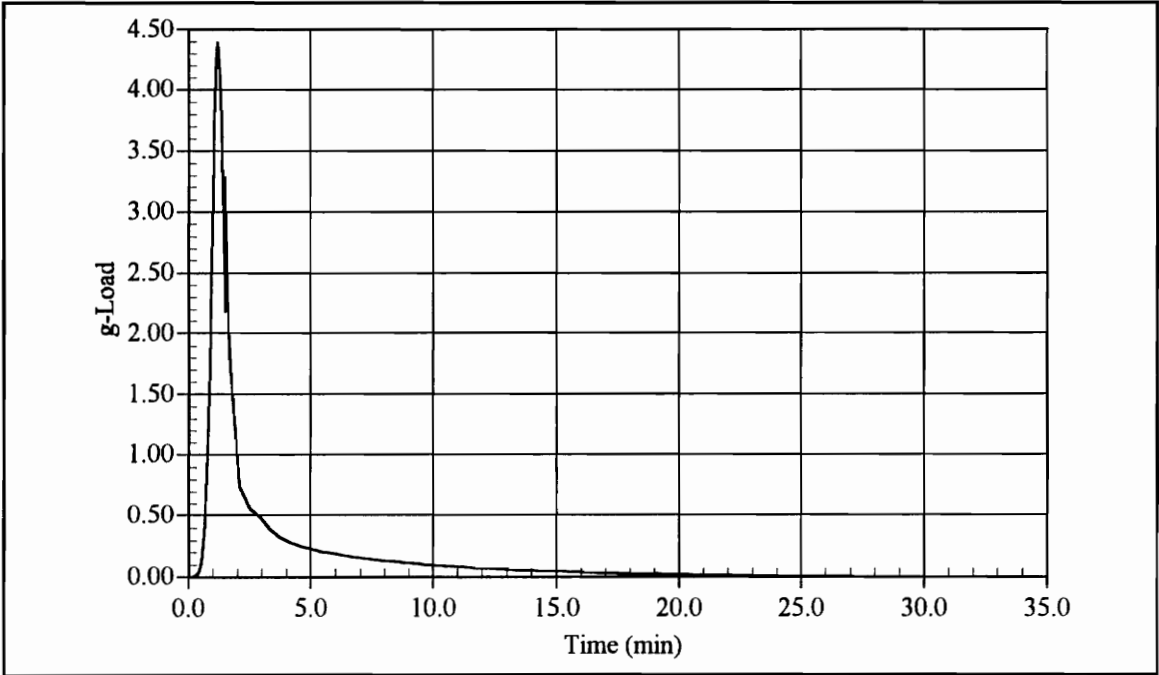


Figure 42. *G-Load vs. Time for Piecewise-Constant Pitch Angle Control Solution with Fixed Deboost Perigee*

7.4 Summary

The major features of the minimizing solutions presented in this chapter are given in Table 8. Also included in the table are relevant values obtained from the idealized aeroassisted transfer described in Section 2.3.1.2. The initial impulses obtained from the minimum fuel solutions vary from 5.3 m/s to 10.6 m/s greater than the idealized case, representing at most an increase of only seven tenths of one percent over the idealized value of 1485.6 m/s. The rendezvous impulses obtained from the minimum fuel solutions vary from 10.3 m/s to nearly 6.8 m/s greater than the idealized case. The rendezvous ΔV s represent increases of between 28 percent and 43 percent over the idealized value of 24 m/s. While the initial ΔV remains nearly the same for all solutions obtained, the rendezvous ΔV can increase by a factor of forty-five, as in the constant maximum lift coefficient solution given at the top of Table 6 which exhibits a rendezvous ΔV of 1413 m/s. The large discrepancies in the rendezvous ΔV s that arise from different control laws as compared with the nearly constant values for the initial ΔV s, validates the use of the fixed deboost perigee case, in which only the rendezvous ΔV is considered in the cost, in the prediction of overall fuel usage.

The total characteristic velocities for the solutions presented in Table 8 are within 1.4 percent of that obtained from the idealized aeroassisted transfer. Furthermore, the characteristic velocities obtained from the piecewise-constant control solutions are nearly identical to those obtained from the approximate optimal control solutions which are in turn nearly identical to the optimal control theory solutions of Mease and Vinh [16]. Therefore, the piecewise-constant control formulations presented in this research, are sufficient to obtain the same fuel usage as the continuous control solutions found from optimal control theory.

Also, a strong correlation can be seen between minimum altitude and maximum heating rate. The lower the minimum altitude, the larger the maximum heating rate. A similar correlation exists between the minimum altitude and the maximum g-load. The lower the minimum altitude, the greater the maximum g-load. However with the g-load, the value of the lift coefficient near the minimum altitude also plays an important role in determining the maximum value. Both the lowest maximum heating rate and the lowest maximum g-load are obtained from piecewise-constant lift solutions. Similarly low values could be obtained from the approximate optimal

case since it could always degenerate to the piecewise-constant case as long as no stringent constraints are placed on the derivatives of the control variables. Nonetheless, piecewise-constant control laws are seen to be able to easily affect the quantities presented in Table 8 while preserving the minimum fuel solutions.

As mentioned throughout this chapter, several converged solutions were obtained with each control formulation. The time to produce a single converged solution for a given formulation is difficult to measure in a quantitative manner that can be compared with the time to produce converged solutions for other formulations. The reason being that time to convergence is heavily dependent on the initial guess of the solution. However, it is impossible to avoid drawing a qualitative conclusion of the ease of solution during the countless hours spent investigating the various control formulations. The conclusions are themselves based on some quantitative observations, such as average time per iteration and average number of iterations per converged solution. For forty node trajectories, the piecewise-constant control formulations yield roughly three iterations per cpu minute (on an IBM RISC 6000) while the approximate optimal control formulation gives one iteration ever three minutes. Both the approximate optimal and piecewise-constant formulations produce converged solutions for most reasonable initial guesses. The problem with the approximate continuous solutions is the oscillation evident in the control law. When constraints are added on the control derivatives to attempt to eliminate the oscillations, converged solutions become harder to obtain. The draw back to the piecewise-constant formulation is that some of the solutions must be forced. That is, when a trend is seen that an increasing initial control value produces lower values of the cost function, the initial control value might need to be constrained to be a larger value to force a lower cost solution. In all, solutions are obtained with less effort with the piecewise-constant control formulations.

Table 8. Characteristics of Minimum Fuel Trajectories

	Approximate Optimal Control		Piecewise-Constant Control				Idealized Transfer
Control Variable	C_L	C_L	C_L	C_L	C_L	Θ	NA
Switching Time (sec)	NA	NA	Fixed	Free	Free	Free	NA
Deboost Perigee	Fixed	Free	Fixed	Fixed	Free	Fixed	R_{atm}
ΔV_1 (m/s)	1496.2	1490.9	1496.2	1496.2	1491.2	1496.2	1485.6
Minimum Altitude (km)	50.3	47.6	61.6	45.6	61.9	60.5	R_{atm}
Maximum Heating Rate (W/cm ²)	348	354	167	444	181	214	NA
Maximum G-Load (g's)	2.1	7.4	5.3	4.4	1.3	4.4	NA
Time of Skip (sec)	1755	1758	2295	1552	2204	1937	NA
ΔV_2 (m/s)	30.8	30.8	34.3	30.9	30.9	30.9	24.0
ΔV_{total} (m/s)	1527.0	1521.7	1530.5	1527.1	1522.1	1527.1	1509.6

8 Conclusions and Recommendations

Piecewise-Constant control strategies for the atmospheric portion of an aeroassisted orbital transfer are sufficient to calculate minimum fuel trajectories. Furthermore, a two segment piecewise-constant lift coefficient control solution with a free switching time and a free deboost perigee can not only reach the same minimum fuel result as that obtained by optimal control theory, but also proves to have enough flexibility to affect other important parameters (e.g. heating rate). Such piecewise-constant control strategies not only simplify the solution process but should also be easier to implement than continuously varying control strategies. The implicit integration routine establishes itself as a valuable asset in the solution of parameter optimization problems.

While much research has been done in the area of optimal aeroassisted orbital transfers, some topics have yet to be properly explored. An intriguing problem for future research is that of an active interceptor that must rendezvous with a passive target, and must do so under given time constraints. Also, a feedback control scheme based on the piecewise-constant control strategy should be devised and tested against various atmospheric models.

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