Discontinuities, Balance Laws, and Material Momentum

Harmeet Singh

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Engineering Mechanics

James Hanna, Chair
Mark Cramer
Sunghwan Jung
Robert Parker
Mark Stremler

December 3, 2018
Blacksburg, Virginia

Keywords: Discontinuities, Partial Contact, Material Forces, Conservation Laws

Discontinuities, Balance Laws, and Material Momentum

Harmeet Singh

Abstract

This dissertation presents an analytical study of a class of problems involving discontinuities, also referred to as shocks, propagating through one dimensional flexible objects such as strings and rods. The study entails interrogation of the classical balance laws of momentum, angular momentum, and energy across propagating discontinuities. A major part of this dissertation also concerns itself with a non-classical entity called the “material momentum”. The balance of material momentum is studied in a variational context, where both the local and singular forms of it are derived from an action principle.

A distinguishing aspect of discontinuities propagating in continua is that, unlike in the bulk, the balance of momentum and angular momentum are not sufficient to describe their mechanics, even when the discontinuities are energy conserving. In this work, it is shown that the additional information required to close the system of equations at propagating discontinuities can be obtained from the singular form of energy balance across them. This entails splitting of the energy balance by its invariance properties, and identifying the non-invariant and invariant part of the source term with the power input and energy dissipation respectively at the shock. This approach is in contrast with other treatments of such problems in the literature, where additional non-classical concepts such as “material momentum” and “configurational force” have been invoked.

To further our understanding of the connections between the classical and non-classical approaches to problems involving discontinuities, a detailed exposition of the concept of material momentum is presented. The balance and conservation laws associated with material momentum are derived from an action principle. It is shown that the conservation of material momentum is associated with the material symmetry of the continuum, and that the conditions for the conservation of physical and material momentum are independent of each other. A new classification of the deformed configurations of the planar Euler elastica based on conserved quantities associated with the spatial and material symmetry of the rod is proposed. The manifestation of the balance of material momentum in seemingly unrelated fields of research, such as fracture mechanics, ideal fluids, and the mechanics of rods with discontinuities, is also discussed.

Discontinuities, Balance Laws, and Material Momentum

Harmeet Singh

General Audience Abstract

One dimensional flexible bodies such as strings and rods can exhibit fascinating and counter-intuitive behavior when they interact with rigid obstacles. For instance, a chain falling on a rigid surface falls faster than it would have if it were falling freely. When one end of a long chain piled up in a container placed at an elevation is pulled across the rim and let go, the chain flows out of the container like a water fountain. Discontinuities in the cross-sectional properties of an elastic rod contained in a curved frictionless channel can result in the generation of forces that propel the rod along the channel. Such counter-intuitive phenomena are a consequence of the physics taking place at the point of partial contact where the flexible body comes in contact with a rigid surface. The purpose of this dissertation is to study the mechanics of such points of discontinuity.

Several such phenomena where effectively one dimensional bodies interact with rigid surfaces are all around us. A familiar example is the peeling of an adhesive tape, where the peeling front qualifies as a point of discontinuity propagating through the tape as the peeling progresses. A good understanding of the mechanics of the peeling front is crucial in estimating the strength of the adhesive. Another such example of practical importance is a mooring line being placed on the seabed. In such situations, the existence of a reaction force acting at the touchdown point depends on whether or not the cable develops a kink at that point. Similar questions of importance can be asked in the context of deployment and unspooling of space tethers.

In this dissertation, an analytical study of the general physics of the phenomena described above is presented. Standard theoretical tools of classical physics are employed to understand the mechanics of points of partial contact between flexible and rigid bodies. The conditions under which a flexible body could experience sharp changes in its geometry (e.g. a kink) at such points are investigated. In addition to that, we explore the implications of a non-classical law of physics called the balance of "material momentum" in the context of such problems.

This work was funded by the National Science Foundation under grant CMMI-1462501

Contents

List of Figures

Chapter 1

Introduction

The overarching theme of this dissertation is to study the balance laws of continuum mechanics across discontinuities propagating in continuous media. This includes the classical balance laws of physical and angular momentum, and energy, as well the balance of a non-classical entity called the “material momentum”. A major part of the dissertation focuses on partial contact problems associated with one dimensional flexible bodies such as strings and rods. The rest of this work pertains to a study of the balance of material momentum in the general setting of 3D compatible elasticity from a variational perspective, and its implications for problems involving ideal fluids and flexible rods.

Discontinuities arising in flexible objects in partial contact with rigid bodies are ubiquitous in nature. When a moving flexible body comes in partial contact with a rigid surface, the point of contact propagates with respect to the material coordinates of the body, resulting in a non-material discontinuity. The flexible body may suffer a jump in its geometry at the point of discontinuity due to such interactions. For instance, inextensible flexible chains with no bending resistance may develop a kink (i.e. a jump in the tangents), whereas elastic rods can develop a jump in curvature at the point of partial contact. Such discontinuities often lead to counter-intuitive dynamics of the medium that they propagate in. For instance, a chain falling on a rigid surface falls faster than it would if it were falling freely [1, 2, 3]. When one end of a long chain piled up in a container is pulled across the rim of the container and allowed to fall, the chain flows out in form of an inverted arch [4, 5]. Other common examples where contact discontinuities arise are processes like picking up or laying down of a string, peeling of adhesive tapes [6, 7, 8, 9], the vibration of certain Indian musical instruments such as the *sitar* [10, 11] and *veena*, elastic rods confined to move in frictionless channels [12, 13, 14, 15], deployment, unspooling and lay-down of space tethers [16, 17] and mooring lines [18].

The earliest studies of some practical engineering problems involving a moving point contact associated with a flexible body can be found in the works of Rivlin [6] and Kendall [7], where they analyze the peeling of an adhesive tape using energy considerations. In later years, Burridge and Keller [8] examined problems such as peeling and slipping of one dimensional media, and lengthwise splitting of a beam. They referred to the propagating discontinuities in their problems as free boundaries, since they cannot be prescribed *a priori*, and must be obtained as part of the solution. A particularly notable aspect of the mechanics of such free boundaries is that, unlike in the bulk, the mechanics of these points cannot be described by the balance of momentum and angular momentum alone. This is true even for energy conserving discontinuities. A treatment of propagating discontinuities in elastic bars can be found in the work of Ericksen [9]. Abeyaratne and Knowles [19, 20] have studied the propagation phase transition fronts in 3D bulk solids and bars. This work was extended to the mechanics of strings by Purohit and Bhattacharya [21] who incorporated discontinuities in the tangents in their framework.

A general mathematical formalism suited for analyzing such problems through local singular balances of momentum and energy at the discontinuity was initiated by Green and Naghdi [22]. Building upon this framework, O'Reilly [23, 24] presented a general theory for the treatment of discontinuities in one dimensional elastic rods [25, 26]. O'Reilly's framework allowed for the inclusion of source terms in the singular balance laws that accounted for the presence of point forces, moments, and energy injection or dissipation at the point of discontinuity. A distinguishing feature of O'Reilly's work was that he included an additional balance law, which he called the balance of "material momentum", at the points of singularity [25]. In his framework, the singular form of this balance law, which was assumed to be satisfied identically in the bulk, served the purpose of providing an extra piece of information without which the dynamics of the discontinuity could not be obtained.

In recent years, several interesting experiments done by Bigoni et al. [27, 14] involving elastic rods partly or fully constrained by frictionless channels have uncovered an unexpected axial reaction force that exists at points of discontinuity in contact. Their theoretical analysis, which is based on an approach that minimizes the global energy of such a system, claims that this force, by analogy, is similar to the force on a defect proposed by Eshelby [28]. They refer to this force as "Eshelby like" or a "configurational force". However, O'Reilly has shown using his theory of material momentum for rods that such reaction forces are in fact not configurational or material in nature [29], and that the material/configurational force at the point of discontinuity is actually zero.

This dissertation takes an approach which is different than the ones taken by other authors. It is demonstrated that the additional piece of information needed to close the system of equations at points of discontinuities can be obtained from the energy balance, without invoking non-classical concepts such as the balance of material momentum. The balances of momentum and energy across points of propagating discontinuities in one dimensional flexible objects such as strings and rods are studied. The singular form of these balance laws, also known as the jump conditions, are derived from an action principle with a time dependent spatial boundary. The jump conditions thus obtained are permitted to admit source terms which represent the presence of point forces or moments in the balances of linear and angular momentum respectively. For the balance of energy, the source term represents the net working of singular reaction forces and moments during the motion of the discontinuity through the body. This energy balance is then split based on its invariance properties, and the invariant part of the energy source is interpreted as the energy dissipation associated with the discontinuity.

The other half of this dissertation concerns itself with the study of the balance of material momentum from a variational perspective. The notion of the concept of material momentum in continuum mechanics dates back to the work of Eshelby [28] who proposed the notion of a "force" acting on a defect. This force is identified as the dynamic quantity, which when multiplied by the parameter characterizing the motion of the defect, would deliver the energy changes associated with its motion. A conceptual foundation of the idea of "material momentum" was laid down by some notable authors like Rogula [30, 31] and A. G. Herrmann [32, 33] who derived the associated balance and conservation laws from an action principle. They showed that the balance of material momentum arises out of variations in the material labels of the bodies. Kienzler and G. Herrmann [34, 35], and Maugin [36, 37] contributed to the literature by formulating laws of material momentum as projections of the physical momentum onto the "material manifold".

The work presented in this dissertation is organized in four standalone manuscripts that appear as four chapters. Starting with chapter 2, a simple problem of an inextensible string in partial contact with a rigid surface is considered. The balance of energy and momentum across the point of partial contact are examined with allowance for singular sources, and a compatibility relation between sources of momentum and energy at the point of discontinuity is derived. For a moving point of contact between a string and a smooth rigid plane in the presence of gravity, an asymmetry is observed between the processes of picking up and laying down, such that a kink in the geometry of the string is inadmissible during pickup. This asymmetry is shown to be a consequence of the compatibility relation and the second law of thermodynamics.

Chapter 3 consists of a unified classical treatment of partially constrained elastic rods. The general theory developed is applied to two cases, one where a rod is partially constrained by a straight frictionless sleeve, and another where it is partially attached to a rigid surface by an adhesive. The relationship between singular forces and moments developed at the point of partial contact is derived from the local energy balance during quasi-static motion. The classical approach presented is compared with other approaches based on the balance of material momentum [25], and the driving traction [20].

In chapter 4, the classical problem of the planar Euler elastica with forces and moments applied at its terminal ends is revisited. A classification of the deformed configurations based on conserved quantities associated with the spatial and material symmetries of the action is presented. The commonly used director, variational, and dynamical systems representations of the elastica are compared. It is shown that the widely used approach based on the shape equation for the tangential angle obscures physical information about the tension in the body.

Finally, chapter 5 presents a comprehensive treatment of the concept of material momentum using a variational framework. Balance laws for physical momentum, material momentum, and energy are derived for a first gradient field theory for a material body. It is shown using the calculus of variations that the balance of material momentum is nothing but the projection of the physical momentum on the material manifold. However, the conditions that must exist for the conservation of these two quantities are shown to be mutually independent. Several well known conservation laws of ideal fluid motion are derived using the balance of material momentum. The motion of an elastic rod moving in a curved frictionless channel is also understood within the framework of material momentum.

Bibliography

[1] E. Hamm and J-C. Géminard. The weight of a falling chain, revisited. *American Journal of Physics*, 78:828–833, 2010.

[2] A. Grewal, P. Johnson, and A. Ruina. A chain that speeds up, rather than slows, due to collisions: How compression can cause tension. *American Journal of Physics*, 79:723–729, 2011.

[3] E. G. Virga. Chain paradoxes. *Proceedings of the Royal Society A*, 471(20140657), 2015.

[4] J. S. Biggins. Growth and shape of a chain fountain. *Europhysics Letters*, 106:44001, 2014.

[5] E. G. Virga. Dissipative shocks in a chain fountain. *Physical Review E*, 89(053201), 2014.

[6] R. S. Rivlin. The effective work of adhesion. *Paint Technology*, IX(106):1–4, 1944.

[7] K. Kendall. Thin-film peeling–the elastic term. *Journal of Physics D: Applied Physics*, 8:1449–1452, 1975.

[8] R. Burridge and J. B. Keller. Peeling, slipping and cracking–some one-dimensional free-boundary problems in mechanics. *SIAM Review*, 20:31–61, 1978.

[9] J. L. Ericksen. *Introduction to the Thermodynamics of Solids*. Springer, New York, 1998.

[10] C. V. Raman. On some Indian stringed instruments. *Proc. Indian Assoc. Cultiv. Sci.*, 7:29–33, 1921.

[11] R. Burridge, J. Kappraff, and C. Morshedi. The sitar string, a vibrating string with a one-sided inelastic constraint. *SIAM Journal on Applied Mathematics*, 42:1231–1251, 1982.

[12] D. Bigoni, F. Dal Corso, D. Misseroni, and F. Bosi. Torsional locomotion. *Proceedings of the Royal Society A*, 470:20140599, 2014.

[13] G. Cicconofri and A. DeSimone. A study of snake-like locomotion through the analysis of a flexible robot model. *Proceedings of the Royal Society A*, 471:20150054, 2015.

[14] D. Bigoni, F. Dal Corso, F. Bosi, and D. Misseroni. Eshelby-like forces acting on elastic structures: Theoretical and experimental proof. *Mechanics of Materials*, 80:368–374, 2015.

[15] F. Dal Corso, D. Misseroni, N. M. Pugno, A. B. Movchan, N. V. Movchan, and D. Bigoni. Serpentine locomotion through elastic energy release. *Journal of the Royal Society Interface*, 14:20170055, 2017.

[16] S. K. Batra and W. B. Fraser. *Engineering Fundamentals of Ring Spinning/Twisting, Over-End Unwinding and Two-For-One Twisting in Textile Processes*. DEStech, Lancaster, 2015.

[17] V. V. Beletsky and E. M. Levin. *Dynamics of Space Tether Systems*. American Astronautical Society, San Diego, 1993.

[18] J. I. Gobat and M. A. Grosenbaugh. Dynamics in the touchdown region of catenary moorings. *International Journal of Offshore and Polar Engineering*, 11:273–281, 2001.

[19] R. Abeyaratne and J. K. Knowles. On the driving traction acting on a surface of strain discontinuity in a continuum. *Journal of the Mechanics and Physics of Solids*, 38:345 – 360, 1990.

[20] R. Abeyaratne and J. K. Knowles. Kinetic relations and the propagation of phase boundaries in solids. *Archive for Rational Mechanics and Analysis*, 114:119–154, 1991.

[21] P. K. Purohit and K. Bhattacharya. On beams made of a phase-transforming material. *International Journal of Solids and Structures*, 39:3907–3929, 2002.

[22] A. E. Green and P. M. Naghdi. A derivation of jump condition for entropy in thermomechanics. *Journal of Elasticity*, 8(2):179–182, 1978.

[23] O. M. O'Reilly and P. C. Varadi. A treatment of shocks in one-dimensional thermomechanical media. *Continuum Mechanics and Thermodynamics*, 11:339–352, 1999.

[24] O. M. O'Reilly and P. C. Varadi. On some peculiar aspects of axial motions of closed loops of string in the presence of a singular supply of momentum. *Journal of Applied Mechanics*, 71:541–545, 2004.

[25] O. M. O'Reilly. A material momentum balance law for rods. *Journal of Elasticity*, 86:155–172, 2007.

[26] O. M. O'Reilly. *Modeling Nonlinear Problems in the Mechanics of Strings and Rods*. Springer, New York, 2017.

[27] F. Bosi, D. Misseroni, F. Dal Corso, and D. Bigoni. An elastica arm scale. *Proceedings of the Royal Society A*, 470:20140232, 2014.

[28] J. D. Eshelby. The force on an elastic singularity. *Phil. Trans. R. Soc. Lond. A*, 244:87–112, 1951.

[29] O. M. O'Reilly. Some perspectives on Eshelby-like forces in the elastica arm scale. *Proceedings of the Royal Society A*, 471:20140785, 2015.

[30] D. Rogula. Noether's theorem for a continuous medium interacting with external fields. *Proceedings of Vibration Problems*, 7:337–344, 1966.

[31] D. Rogula. Forces in material space. *Archives of Mechanics*, 29:705–713, 1977.

[32] A. G. Herrmann. On conservation laws of continuum mechanics. *International Journal of Solids and Structures*, 17:1–9, 1981.

[33] A. G. Herrmann. On the lagrangian formulation of continuum mechanics. *Physica A*, 118:300–314, 1983.

[34] R. Kienzler and G. Herrmann. On material forces in elementary beam theory. *Journal of Applied Mechanics*, 53:561–564, 1986.

[35] R. Kienzler and G. Herrmann. *Mechanics in Material Space.* Springer, Berlin, 2000.

[36] G. Maugin. *Material Inhomogeneities in Elasticity.* Chapman & Hall, London, 1993.

[37] G. A. Maugin. *Configurational Forces.* CRC Press, Boca Raton, 2011.

Chapter 2

Pick-up and impact of flexible bodies

The contents of this chapter have appeared as is in the *Journal of the Mechanics and Physics of Solids*, volume 106, pages 46–59, 2017. The published article can be found at: `https://www.sciencedirect.com/science/article/pii/S0022509617301308`.

Attribution

The work presented in this chapter was done in collaboration with J. A. Hanna, who contributed to the inception and development of the main ideas in this work.

Abstract

Picking up, laying down, colliding, rolling, and peeling are partial-contact interactions involving moving discontinuities. We examine the balances of momentum and energy across a moving discontinuity in a string, with allowance for injection or dissipation by singular supplies. We split the energy dissipation according to its invariance properties, discuss analogies with systems of particles and connections with the literature on shocks and phase transition fronts in various bodies, and derive a compatibility relation between supplies of momentum and translation-invariant energy. For a moving contact discontinuity between a string and a smooth rigid plane in the presence of gravity, we find a surprising asymmetry between the processes of picking up and laying down, such that steady-state kinks in geometry and associated jumps in tension are not admissible during pick-up. This prediction is consistent with experimental observations. We briefly discuss related problems including the falling folded chain, peeling of an adhesive tape, and the "chain fountain". Our approach is applicable to the study of impact and locomotion, and to systems such as moored floating structures and some musical instruments that feature vibrating string and cable elements interacting with

a surface.

2.1 Introduction

Picking up or laying down a floppy object is not a particularly exotic activity, and one might assume the process to be well understood. However, dynamic interactions between flexible and rigid bodies are often counterintuitive. A chain falling onto a table experiences an additional acceleration beyond that of free-fall [1, 2], ropes and chains lift off of pulleys or other guiding surfaces [3, 4, 5, 6], and form arches [7, 8, 9] when pulled from piles at rest. In perfectly flexible objects such as idealized strings and some real chains, shape discontinuities— kinks— can form. Contact with obstacles or guides can help sustain such jumps in the geometry, and in other fields such as the axial tension, momentum, and energy.

In the present work, we consider a propagating discontinuity separating contacting and free regions of an inextensible string. We examine a commonplace and seemingly simple process, that of picking up or laying down a flexible extended body from or onto a rigid surface. We find a surprising asymmetry in this problem, namely that kinks and associated jumps in tension are admissible in lay-down impact onto, but generally inadmissible in pick-up from, a smooth surface. In lifting a string or chain from a table or the floor, one observes that its tangents are continuous. That this is a necessity can be determined from the jump conditions for linear momentum and energy across the contact discontinuity. No such requirement exists for impact. This prediction is consistent with our own limited experimental evidence, as well as experiments and theoretical results from other workers on linearized and related problems. The dynamics we describe arise in deployment, unspooling, and lay-down of space tethers, yarn, machine belts, and submarine cables [10, 11, 12, 13], as well as vibration of structures as disparate as the strings of the *veena* [14, 15], catenary moorings of offshore structures [16], and peeling adhesive tapes [17, 18, 19, 20, 21, 22, 23].

Our approach follows, and modifies, previous work on the mechanics of discontinuities in strings and rods. A general treatment of shocks in one dimensional media, starting from Green and Naghdi's framework [24], was undertaken by O'Reilly and Varadi [25, 26], further developed by O'Reilly [27, 28], and applied by Virga [29, 30] to several counterintuitive problems in chain dynamics. One important aspect of this approach is the inclusion of singular source terms in jump conditions for field quantities. Such terms allow for the injection of momentum and dissipation of energy by interaction with an obstacle, or other means. It is the relationships between these terms that will lead us to our results. A thermo-mechanical treatment of moving discontinuities in elastic bars, which are kink-free by definition, may be found in Ericksen [20]. The work of Abeyaratne and Knowles [31, 32] on propagating phase transition fronts in bulk solids and bars was extended to strings, which do admit kinks, by Purohit and Bhattacharya [33]. Source terms do not appear in these treatments. Podio-Guidugli and co-workers [21] treat the problem of peeling by employing a configurational approach, including a source-like term.

We begin with a derivation of the governing equations and the relevant jump conditions in Section 2.2. We also construct a natural orthogonal co-ordinate system at the discontinuity that proves useful in subsequent manipulations. In Section 2.3, we isolate the translation-invariant power dissipated at the discontinuity, and interpret this through two analogies with particle impact systems. The functional form of this invariant power is shown to be consistent with the notion of a driving traction proposed by Abeyaratne and Knowles [31, 32]. We also contrast our definition with those proposed by O'Reilly [27, 28] and Virga [29, 30]. In Section 2.4, we introduce a compatibility relation between linear momentum and energy source terms. In Section 2.5, we discuss conditions for the conservation of translation-invariant energy. In Section 2.6, we use our compatibility relation to determine the necessary conditions for a kink to exist in a particular physical situation of interest. We consider pick-up and impact involving an inextensible, perfectly flexible one-dimensional body and a smooth, flat, frictionless surface. We require that the jump in translation-invariant energy at the contact discontinuity be non-positive, and conclude that a steady-state kink may exist at the point of impact but not at pick-up. Some experimental evidence in support of this prediction is presented involving an axially moving chain perturbed into a "bubble" on a surface, as well as prior work by Gobat and Grosenbaugh [16] on the touchdown point of vibrated pendant chains. The appendices provide an alternative justification for the non-positivity of the power injection in pick-up and impact that employs a form of the Clausius-Duhem inequality, as well as brief discussions of related problems: a falling folded chain, peeling of an adhesive tape, and deployment from a pile, including the "chain fountain".

2.2 Geometry and governing equations

Consider a one-dimensional body described as a curve $\mathbf{X}(s,t)$ parameterized by its arc length s and the time t. We will be concerned with inextensible strings, the simplest example of such a body. The assumption of inextensibility means that the arc length s is also a material coordinate. Let there be a discontinuity in any or all field quantities at the point $s = s_0(t)$. The sum and difference (jump) of a field quantity across the discontinuity are denoted by $\{Q\} \equiv Q^+ + Q^-$ and $[\![Q]\!] \equiv Q^+ - Q^-$, respectively,[1] where $Q^\pm$ are the values of Q on either side of the point s_0. Nearly all of the equations in this paper will be jump conditions holding only at the non-material point s_0, which point we will refer to as a discontinuity, jump, or shock.

The following kinematic constraints on position and velocity hold at the shock,

$$[\![\mathbf{X}]\!] = 0\,, \tag{2.1}$$

$$[\![\partial_t\mathbf{X} + \partial_t s_0 \partial_s \mathbf{X}]\!] = 0\,, \tag{2.2}$$

[1] Our notation differs by a factor of two from that of O'Reilly [25, 26, 27, 28], who uses {} to denote the average.

where (2.1) is simply a statement that the curve is continuous, and the velocity compatibility condition (2.2) ensures continuity of the spatial velocity at the non-material point of discontinuity. Condition (2.2) is obtained by taking a total time derivative of $[\![\mathbf{X}(s_0(t),t)]\!]$. In our notation, ∂_t indicates a material time derivative (constant s) when applied to the position vector $\mathbf{X}(s,t)$, and a partial time derivative when applied to $s_0(t)$.[2] Condition (2.2) links spatial velocities of material points $\partial_t\mathbf{X}^{\pm}$, tangent vectors $\partial_s\mathbf{X}^{\pm}$, and the material velocity of the shock $\partial_t s_0$. Due to the continuity of the coordinate s, $\partial_t s_0$ commutes with the double bracket, and so can be moved in or out of it.

That s is also an arc length coordinate means that $\partial_s\mathbf{X}^{\pm}$ are unit vectors, and thus their sum and jump form an orthogonal pair: $[\![\partial_s\mathbf{X}]\!]\cdot\{\partial_s\mathbf{X}\}=0$. As long as the tangents are neither continuous ($[\![\partial_s\mathbf{X}]\!]=0$) nor perfectly folded-back ($\{\partial_s\mathbf{X}\}=0$), this pair of axes is quite useful at the point of interest, and we will often break vector quantities into their projections onto the sum and jump of tangents. Figure 2.1 shows the geometry of the relevant objects. Projections of the velocity compatibility condition (2.2) reveal relationships between the geometry and the material and spatial velocities,

$$[\![\partial_t\mathbf{X}]\!]\cdot\{\partial_s\mathbf{X}\}=0\,, \tag{2.3}$$

$$[\![\partial_t\mathbf{X}]\!]\cdot[\![\partial_s\mathbf{X}]\!]=-\partial_t s_0\;[\![\partial_s\mathbf{X}]\!]\cdot[\![\partial_s\mathbf{X}]\!]\;. \tag{2.4}$$

We will often make use of the identity

$$2\,[\![\mathbf{A}\cdot\mathbf{B}]\!]=[\![\mathbf{A}]\!]\cdot\{\mathbf{B}\}+\{\mathbf{A}\}\cdot[\![\mathbf{B}]\!] \tag{2.5}$$

for two vectors $\mathbf{A}$ and $\mathbf{B}$.

Our discussion will center on two additional jump conditions that may be derived in different ways, one way being the approach of O'Reilly and Varadi [25]. We briefly outline the different approach of a prior paper [34], which employs actions with time-dependent boundaries of the form

$$A=\int\!dt\int^{s_0(t)}\!ds\,\mathcal{L}\,, \tag{2.6}$$

$$2\mathcal{L}=\mu\partial_t\mathbf{X}\cdot\partial_t\mathbf{X}-\sigma\left(\partial_s\mathbf{X}\cdot\partial_s\mathbf{X}-1\right)\,, \tag{2.7}$$

one such action for each side of the jump. The quantity $\mathcal{L}$ is the constrained kinetic energy density of an inextensible string, with the constant μ a uniform mass density and the multiplier $\sigma(s,t)$ the tension. Variation of $\mathbf{X}$ leads to the string equations in the bulk,

$$\mu\partial_t^2\mathbf{X}-\partial_s(\sigma\partial_s\mathbf{X})=0\,, \tag{2.8}$$

while variations of $\mathbf{X}$ and t lead to jump conditions at the shared moving "boundary" $s=s_0(t)$,

$$\mathbf{P}+[\![\sigma\partial_s\mathbf{X}+\mu\partial_t s_0\partial_t\mathbf{X}]\!]=0\,, \tag{2.9}$$

$$\tilde{E}+\left[\!\!\left[\sigma\partial_s\mathbf{X}\cdot\partial_t\mathbf{X}+\tfrac{1}{2}\mu\partial_t s_0\partial_t\mathbf{X}\cdot\partial_t\mathbf{X}\right]\!\!\right]=0\,, \tag{2.10}$$

[2] Where we write $\partial_t s_0$, Virga [29, 30] writes $\dot{s}_0$ and O'Reilly [25, 26, 27, 28] writes $\dot{\gamma}$.

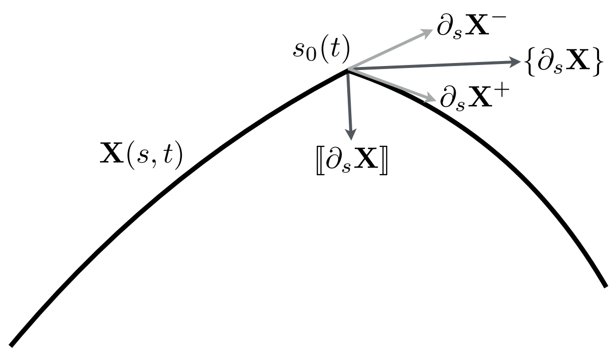

Figure 2.1: A one-dimensional body $\mathbf{X}(s,t)$ with a discontinuity at $s = s_0(t)$. Also shown are unit tangent vectors $\partial_s \mathbf{X}^{\pm}$ on either side of the discontinuity, and their sum $\{\partial_s \mathbf{X}\}$ and jump $[\![\partial_s \mathbf{X}]\!]$.

where, following Green and Naghdi [24] and O'Reilly and Varadi [25], we allow for singular sources of stress $\mathbf{P}$ and power per area $\tilde{E}$ that represent in general form the effect of obstacles, external forces, internal dissipative processes, and so on, that are not explicitly accounted for in the actions.[3] We will loosely refer to these jump conditions as those for (linear) momentum and energy, though the correct units are these quantities per time per area. A negative value of energy injection rate $\tilde{E}$ means dissipation of energy.[4] This quantity is not invariant under translations of the system.

We can express the momentum jump condition (2.9) in the sum-jump basis mentioned above. Using the velocity compatibility condition (2.2), we rewrite (2.9) as

$$\mathbf{P} = -\left[\!\left[(\sigma - \mu \partial_t s_0^2)\partial_s \mathbf{X}\right]\!\right] , \tag{2.11}$$

and then use the identity (2.5) to expand this as

$$\mathbf{P} = -\tfrac{1}{2} [\![\sigma]\!] \{\partial_s \mathbf{X}\} - \tfrac{1}{2}\{\sigma - \mu \partial_t s_0^2\} [\![\partial_s \mathbf{X}]\!] . \tag{2.12}$$

We will make use of this expression in subsequent discussion.

2.3 Translational invariance and energy dissipation

In this section, we isolate a translation-invariant measure of the dissipation of energy at the shock. This quantity is equivalent to what was introduced in an *ad hoc* manner in

[3]Without source terms, these jump conditions are often referred to as Rankine-Hugoniot conditions [35, 20].

[4]There are sign differences in our present definitions of singular sources and those in [34].

prior work [34]. The term $\tilde{E}$ in the energy jump condition (2.10) is the total areal power input or loss across the shock in some choice of inertial frame. Because some of this is associated with changes in kinetic energy, $\tilde{E}$ is not invariant under translations. We extract the translation-invariant part by using the identity (2.5) to rearrange (2.10) in terms of the (generally non-orthogonal) sum and jump of velocities, then rewrite using the momentum jump condition (2.9),

$$0 = \tilde{E} + \tfrac{1}{2} [\![\sigma\partial_s\mathbf{X} + \mu\partial_t s_0 \partial_t\mathbf{X}]\!] \cdot \{\partial_t\mathbf{X}\} + \tfrac{1}{2}\{\sigma\partial_s\mathbf{X}\} \cdot [\![\partial_t\mathbf{X}]\!] \,, \tag{2.13}$$

$$= \tilde{E} - \tfrac{1}{2}\mathbf{P} \cdot \{\partial_t\mathbf{X}\} + \tfrac{1}{2}\{\sigma\partial_s\mathbf{X}\} \cdot [\![\partial_t\mathbf{X}]\!] \,. \tag{2.14}$$

The total power is split into a non-invariant piece which consists of the external stress $\mathbf{P}$ acting on the average spatial velocity of material points, and an invariant piece which is the negative of the average contact force acting on the jump in this velocity. In the following subsections, we suggest interpretations for these quantities. We can loosely think of the non-invariant term as the external working on the material, and the invariant term as some kind of "internal" dissipation.

The splitting of the power in (2.14) differs from that of Podio-Guidugli [36], who defines the inertial power as that which changes the total kinetic energy. This is because the kinetic energy can also be split according to its invariance properties. While the kinetic energy of a system depends on the choice of frame, the changes in kinetic energy due to internal collisions are frame invariant (as shown, for example, in Section 96 of Whittaker [37]).

We define the translation-invariant energy injection rate as

$$E = \tilde{E} - \tfrac{1}{2}\mathbf{P} \cdot \{\partial_t\mathbf{X}\} \,, \tag{2.15}$$

the difference between the rate of change of the total energy of the shock and the working of external forces on the infinitesimal interval of the string that encloses the shock [31, 32]. This form of the power injection also arises when applying a form of the Clausius-Duhem inequality to isentropic, isothermal, purely mechanical systems, as shown in Appendix A.1. Note that for a rod with bending energy, a non-rotationally invariant term involving a source of areal torque and the rates of change of the tangent vectors would appear as well.

2.3.1 Two examples involving discrete particles

To illustrate some of the relevant concepts, we present two examples involving systems of rigid particles.

Consider first[5] the total kinetic energy of such a system, which may be written in a special

[5]We thank J. S. Biggins for suggesting this analogy.

way using the center of mass velocity $\boldsymbol{v}_{com}$:

$$\tilde{T} = \sum_{j=1}^{N} \tfrac{1}{2} m_j (\boldsymbol{v}_j - \boldsymbol{v}_{com})^2 + \tfrac{1}{2} m_j \boldsymbol{v}_{com} \cdot \boldsymbol{v}_{com} \,. \tag{2.16}$$

The first term on the right side is the kinetic energy of the particles as observed from the frame of the center of mass of the system, whereas the second term is the kinetic energy associated with the center of mass itself. The total kinetic energy $\tilde{T}$ in (2.16) is not invariant under translations, but the quantity $T \equiv \tilde{T} - \big(\sum_{j=1}^{N} \frac{1}{2} m_j\big) \boldsymbol{v}_{com} \cdot \boldsymbol{v}_{com}$ is. Taking a total time derivative of this quantity, we obtain

$$\frac{d}{dt}(T) = \frac{d}{dt}(\tilde{T}) - \mathbf{F}_{ext} \cdot \boldsymbol{v}_{com} \,, \tag{2.17}$$

where $\mathbf{F}_{ext}$ is any external force effectively acting on the center of mass. The invariant term on the left side is the rate of change of energy as observed from the center of mass of the system of particles. We may trivially identify terms in (2.17) and (2.15). The quantity E is analogous to the rate of change of energy as observed from the center of mass of the infinitesimal interval on the string that encloses the shock. This is the energy available for "internal" dissipation by *e.g.* particle-particle or chain link-link collisions, without the aid of external forces. The derivation above holds equally well if the center of mass frame is accelerating. A detailed discussion on the special properties of the center of mass frame and the validity of the work-energy theorem in inertial and non-inertial frames of reference can be found in [38].

A second example will help explain the form of the external working in (2.14) and (2.15). In these expressions, the external stress $\mathbf{P}$ is associated with the average material velocity $\frac{1}{2}\{\partial_t \mathbf{X}\}$ across the shock. This pairing also arises during the collision of two rigid bodies, where the working of the impact is the product of the impulse with the mean of the relative velocities describing approach and separation. The derivation we reproduce here is essentially that of Article 308 of Thomson and Tait [39], and is an elementary result in impact mechanics [40]. The change in kinetic energy due to a two-particle collision is

$$\Delta\tilde{T} = \tfrac{1}{2} m_1 (\boldsymbol{v}_{1f} \cdot \boldsymbol{v}_{1f} - \boldsymbol{v}_{1i} \cdot \boldsymbol{v}_{1i}) + \tfrac{1}{2} m_2 (\boldsymbol{v}_{2f} \cdot \boldsymbol{v}_{2f} - \boldsymbol{v}_{2i} \cdot \boldsymbol{v}_{2i}) \,, \tag{2.18}$$

$$= \tfrac{1}{2} m_1 (\boldsymbol{v}_{1f} - \boldsymbol{v}_{1i}) \cdot (\boldsymbol{v}_{1f} + \boldsymbol{v}_{1i}) + \tfrac{1}{2} m_2 (\boldsymbol{v}_{2f} - \boldsymbol{v}_{2i}) \cdot (\boldsymbol{v}_{2f} + \boldsymbol{v}_{2i}) \,, \tag{2.19}$$

$$= \mathbf{I} \cdot \tfrac{1}{2} (\boldsymbol{v}_{app} + \boldsymbol{v}_{sep}) \,, \tag{2.20}$$

where the symbols for initial and final velocities of particle 1 and particle 2 are straightforward, $\boldsymbol{v}_{app} \equiv \boldsymbol{v}_{2i} - \boldsymbol{v}_{1i}$ and $\boldsymbol{v}_{sep} \equiv \boldsymbol{v}_{2f} - \boldsymbol{v}_{1f}$ are the approach and separation velocities, and $\mathbf{I} = m_2(\boldsymbol{v}_{2f} - \boldsymbol{v}_{2i}) = -m_1(\boldsymbol{v}_{1f} - \boldsymbol{v}_{1i})$ is the impulse experienced by each particle due to the collision. It is the average of the approach and separation velocities that appears conjugate to the impulse. Our problem corresponds to the case where one "particle" is a stationary obstacle.

Our definition of the working of the source of momentum $\mathbf{P}$ differs from the approaches taken by O'Reilly [28] and Virga [30], who associate $\mathbf{P}$ with the spatial velocity of the non-material point $\mathbf{X}(s_0)$ corresponding to the shock. We provide further detail on their choice in the following subsection.

2.3.2 Rearrangements and relationships

Returning to the energy jump in the form (2.14), and the definition (2.15), we have

$$E = -\tfrac{1}{2}\{\sigma\partial_s\mathbf{X}\} \cdot [\![\partial_t\mathbf{X}]\!] \,. \tag{2.21}$$

Employing velocity compatibility (2.2), we may write

$$E = \partial_t s_0 \tfrac{1}{2}\{\sigma\partial_s\mathbf{X}\} \cdot [\![\partial_s\mathbf{X}]\!] \,. \tag{2.22}$$

Essentially this form for the dissipation at a moving shock can be found in the work of Abeyaratne and Knowles [31, 32]. They proposed the concept of a "driving traction" as a dynamic conjugate to the material velocity $\partial_t s_0$ of a strain discontinuity in a bulk solid or unkinked, extensible bar. If the driving traction vanishes everywhere on the surface of discontinuity, the motion is dissipation-free. Purohit and Bhattacharya [33] derived a thermomechanical expression for the driving force associated with a moving phase boundary, *sans* source terms, in a kinked string made of a phase-transforming material. This coincides with our expression (2.22) in the inextensible, purely mechanical case.[6]

Using (2.5), the expression (2.22) can be rewritten in an informative way in terms of the jumps in tension and tangents,

$$E = \tfrac{1}{4}\partial_t s_0 \, [\![\sigma]\!] \, [\![\partial_s\mathbf{X}]\!] \cdot [\![\partial_s\mathbf{X}]\!] \,. \tag{2.23}$$

This expression indicates that a propagating shock conserves translation-invariant energy if either the tensions or the tangents are continuous across the shock. A similar observation about the tensions was made by Calkin and March [41] for the case of a falling folded chain, where (2.23) reduces to $E = \partial_t s_0 \, [\![\sigma]\!]$, in agreement with equation (12) of [41].[7] We discuss that problem in Appendix A.2.

The energy supply at the shock defined by (2.23) differs from equation (2.25) of Virga [30] by the geometric factor $\frac{1}{4}\,[\![\partial_s\mathbf{X}]\!] \cdot [\![\partial_s\mathbf{X}]\!]$. This difference arises due to Virga's choice of the spatial velocity $\boldsymbol{v}_0$ of the non-material point of discontinuity,

$$\boldsymbol{v}_0 = \partial_t\mathbf{X}^{\pm} + \partial_t s_0 \partial_s\mathbf{X}^{\pm} = \tfrac{1}{2}\{\partial_t\mathbf{X}\} + \tfrac{1}{2}\partial_t s_0\{\partial_s\mathbf{X}\}\,, \tag{2.24}$$

[6] When the source $\mathbf{P} = 0$, (2.5) and (2.11) can be used to simplify the driving traction from $\frac{1}{2}\{\sigma\partial_s\mathbf{X}\}\cdot[\![\partial_s\mathbf{X}]\!]$ to just $[\![\sigma]\!]$.

[7] In that problem, $[\![\partial_s\mathbf{X}]\!]\cdot[\![\partial_s\mathbf{X}]\!] = 4$, and the $\dot{x}$ in equation (12) of [41] is twice the shock speed, $\dot{x} = 2\partial_t s_0$.

as conjugate to the external force $\mathbf{P}$, in contrast to our choice of the average spatial velocity of material points $\frac{1}{2}\{\partial_t \mathbf{X}\}$. From the above expression, we note that this average is simply the portion of $\boldsymbol{v}_0$ that is not translation-invariant. Although the energy supply derived in [30] does not display any dependence on the the jump in tangents, Virga assumes a constitutive law in equation (2.27) of [30] which does. In our derivation, the geometric dependence arises naturally from our splitting of the total energy supply. Equation (2.25) of [30], which Virga derives from a Rayleigh dissipation principle, can also be obtained from a different rearrangement of the energy jump condition (2.10). Substituting for $\frac{1}{2}\{\partial_t \mathbf{X}\}$ from (2.24) into equation (2.14), and using (2.21) and (2.23),

$$0 = \tilde{E} - \mathbf{P} \cdot \boldsymbol{v}_0 + \partial_t s_0 \tfrac{1}{2} \mathbf{P} \cdot \{\partial_s \mathbf{X}\} - \tfrac{1}{4} \partial_t s_0 [\![\sigma]\!] [\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!] \,, \tag{2.25}$$

then using (2.12) to insert for $\mathbf{P}$ in the third term on the right, and the identity $\{\partial_s \mathbf{X}\} \cdot \{\partial_s \mathbf{X}\} + [\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!] = 4$,

$$0 = \tilde{E} - \mathbf{P} \cdot \boldsymbol{v}_0 - \partial_t s_0 [\![\sigma]\!] \,. \tag{2.26}$$

By defining the dissipation $\mathscr{D} \equiv -\tilde{E} + \mathbf{P} \cdot \boldsymbol{v}_0$, one obtains Virga's equation (2.25) in [30]. Equation (2.26) also corresponds to equation (34) of O'Reilly [28] without director terms. The quantity B there is called the source of "material momentum", and in our example would correspond to $[\![\sigma]\!]$ in (2.26). In another paper [27], O'Reilly derives an additional configurational momentum balance law and shows that, given this additional balance, the energy jump condition reduces to an identity relating the power expended by the singular supplies of linear and material momentum. Similarly, if one uses the momentum and energy balances, O'Reilly's configurational balance reduces to an identity.

Finally, we write general expressions for the tensions in the vicinity of the shock in terms of the singular supplies and the geometry. Consider the momentum jump condition in the form (2.12). Projecting it onto the jump in tangents and rearranging, we obtain the tension sum

$$\{\sigma\} = 2\mu \partial_t s_0^2 - \frac{2\mathbf{P} \cdot [\![\partial_s \mathbf{X}]\!]}{[\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!]} \,. \tag{2.27}$$

The tension jump can be obtained from the energy jump in the form (2.23),

$$[\![\sigma]\!] = \frac{4E}{\partial_t s_0 [\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!]} \,. \tag{2.28}$$

Thus, the tensions on either side of the shock are

$$\sigma^{\pm} = \mu \partial_t s_0^2 - \frac{\mathbf{P} \cdot [\![\partial_s \mathbf{X}]\!]}{[\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!]} \pm \frac{2E}{\partial_t s_0 [\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!]} \,. \tag{2.29}$$

These expressions will be used in Appendix A.2.

2.4 A compatibility relation between the supplies

The momentum and energy supplies cannot be arbitrarily prescribed. Eliminating $[\![\sigma]\!]$ between (2.12) and (2.23), we obtain the following relation between the supplies,

$$E\{\partial_s \mathbf{X}\} \cdot \{\partial_s \mathbf{X}\} = -\tfrac{1}{2}\partial_t s_0 \mathbf{P} \cdot \{\partial_s \mathbf{X}\} \, [\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!] \ . \tag{2.30}$$

This relation places no assumptions on the kinematics of the system, and is applicable to steady or unsteady motions. We will use it in Section 2.6 to derive qualitative results about the admissibility of geometric discontinuities during pick-up and impact. Here we make a couple of general observations, valid as long as the body is not perfectly folded back, in which case $\{\partial_s \mathbf{X}\}$ and both sides of the relation would all vanish.

Given a non-zero jump in tangents $[\![\partial_s \mathbf{X}]\!]$, it is impossible to inject momentum into the shock without injecting or dissipating energy unless the momentum injection $\mathbf{P}$ is orthogonal to the sum of tangents $\{\partial_s \mathbf{X}\}$, that is, in the direction of the jump. O'Reilly and Varadi [42] have observed that this fact is inconsistent with some assumptions appearing in the literature on space-fixed loads acting on axially moving or rotating structures.

Given that the sign of the shock speed depends on a physically irrelevant parameterization, the sign of the energy supply E depends only on the orientation of the momentum supply $\mathbf{P}$ relative to the sum of tangents $\{\partial_s \mathbf{X}\}$. Letting $\partial_t s_0$ be negative, as it would be for an axially moving string parameterized in the direction of the axial flow, a positive projection $\mathbf{P} \cdot \{\partial_s \mathbf{X}\}$ would imply a supply of energy at the shock, a negative projection dissipation.

2.5 Shocks conserving translation-invariant energy

Before considering dissipative shocks, it is worthwhile to look at cases where $E = 0$. As mentioned earlier, equation (2.23) for the energy jump indicates that either the tension or the tangents must be continuous if $E = 0$, that is,

$$[\![\sigma]\!] = 0 \,, \quad \text{or} \tag{2.31}$$

$$[\![\partial_s \mathbf{X}]\!] = 0 \,. \tag{2.32}$$

Meanwhile, the compatibility relation (2.30) indicates that at least one of the following conditions must hold if $E = 0$. Either

$$\mathbf{P} = 0 \quad \text{if} \quad \{\partial_s \mathbf{X}\} \neq 0 \,, \quad \text{or} \tag{2.33}$$

$$\mathbf{P} \cdot \{\partial_s \mathbf{X}\} = 0 \quad \text{if} \quad \mathbf{P} \neq 0 \,, \quad \text{or} \tag{2.34}$$

$$[\![\partial_s \mathbf{X}]\!] = 0 \,. \tag{2.35}$$

Conditions (2.32) and (2.35) are the same. It turns out that conditions (2.31) and (2.34) are also equivalent, in that if $[\![\sigma]\!] = 0$, equation (2.11) reduces to $\mathbf{P} = -(\sigma - \mu\partial_t s_0^2) \, [\![\partial_s \mathbf{X}]\!]$, so that (2.34) holds by orthogonality of the sum and jump of tangents in our inextensible string. Although the conditions (2.31) and (2.34) are equivalent, the latter is more useful. The momentum injection $\mathbf{P}$ is often provided by an external obstacle or tool, so it may be possible to infer something about its orientation from the physical attributes of this object. Information about the tension on either side of the shock is usually harder to determine. There exists a third possibility, given by equation (2.33). Here there is no supply of momentum at the shock, and the sum of tangents across the shock is non-zero, meaning that the body is not perfectly folded back on itself. To understand this case better, let us look at the momentum jump condition written in the sum-jump basis (2.12). Assume a kink, such that the jump in tangents is also non-zero. In this case, the only solutions for which $\mathbf{P} = 0$ would require $\sigma^{\pm} = \mu \partial_t s_0^2$. This condition obtains for the lariats [43, 44], solutions that have fixed or rigidly translating shape and constant axial flow. Kinks are compatible with these solutions [42, 34].

It must be emphasized that the condition $\mathbf{P} = 0$ is not sufficient for conservation of translation-invariant energy. It can be inferred from (2.30) that a non-zero E can still exist if the sum $\{\partial_s \mathbf{X}\} = 0$. The falling folded chain discussed in Appendix A.2 is an example of such a case [45, 41, 46, 30].

2.6 Admissibility of a kink during pick-up and impact

In this section, we use the compatibility relation (2.30) to determine whether a discontinuity in tangents is admissible in certain physical circumstances. It is reasonable to assume that such a propagating kink will be dissipative, so that E is non-positive. Indeed, this assumption follows from a form of the Clausius-Duhem inequality for isentropic, isothermal, purely mechanical systems, as discussed in Appendix A.1. A more general system involving adhesion energy and a jump in entropy, which violates this assumption, is discussed in Appendix A.3. For $E \leq 0$ and a parameterization such that $\partial_t s_0 < 0$, we find

$$\mathbf{P} \cdot \{\partial_s \mathbf{X}\} \leq 0\,. \tag{2.36}$$

This inequality restricts the directions of $\mathbf{P}$ relative to the sum of tangents. Since the momentum injection comes from the interaction of the string with an external source, the physical settings of the system may dictate whether this source is capable of providing a $\mathbf{P}$ in a direction consistent with (2.36). For example, a smooth, frictionless, non-adhesive surface can only provide a $\mathbf{P}$ pointing outwards in the direction of the surface normal.

The assumption of the non-positivity of E, as defined in (2.15), can be examined in the light of the total energy balance (2.14). The implication is that $\tilde{E} \leq \frac{1}{2}\mathbf{P} \cdot \{\partial_t \mathbf{X}\}$, meaning that the rate of change of the total energy across the shock $\tilde{E}$ cannot be greater than the rate at which work is done on the material at the shock $\frac{1}{2}\mathbf{P} \cdot \{\partial_t \mathbf{X}\}$ by external forces. The difference between the two quantities tells us about the translation-invariant energy being dissipated in the shock. Note that given (2.12), (2.36) implies $[\![\sigma]\!] \geq 0$. In an inextensible string, the tension cannot decrease across the shock; the tension at a material point passing through the shock cannot decrease.

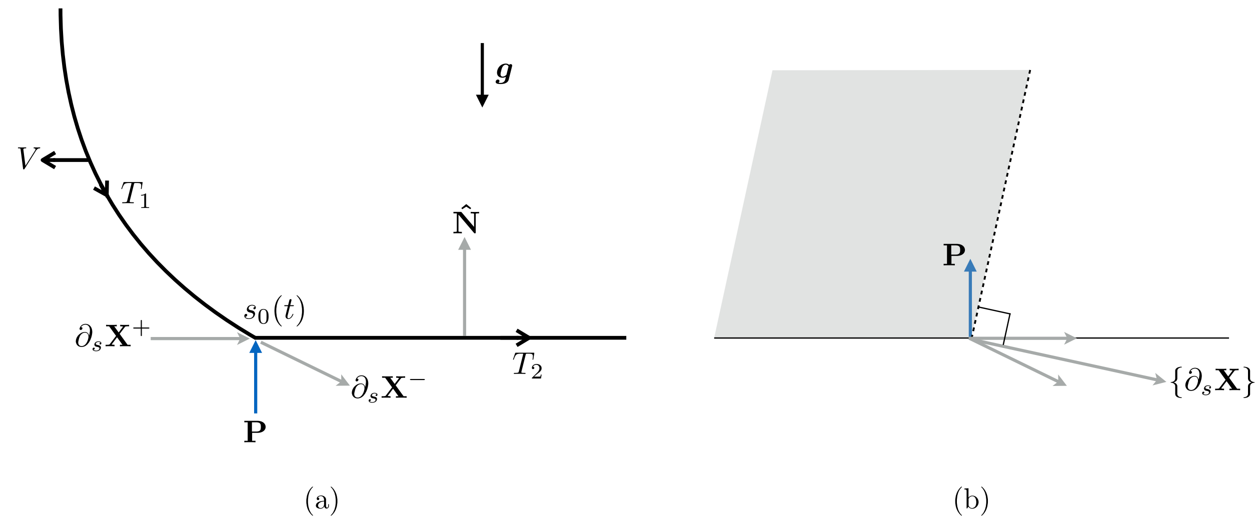

Figure 2.2: An axially moving string impacting a surface with unit normal $\hat{\mathbf{N}}$ antiparallel to gravity $\boldsymbol{g}$. (a) The free piece translates with velocity $-V\partial_s\mathbf{X}^+$ and flows with axial velocity $T_1\partial_s\mathbf{X}^-$, while the contacting piece flows with axial velocity $T_2\partial_s\mathbf{X}^+$. A singular supply $\mathbf{P}$ acts at the discontinuity at $s_0(t)$, which moves to the left as the free piece is laid down onto the surface. (b) The sum of tangents and the grey region of admissibility where $\mathbf{P} \cdot \{\partial_s\mathbf{X}\} \leq 0$ are shown.

We now apply condition (2.36) to determine the admissibility of kinks at the pick-up and impact points of a string interacting with a smooth, rigid plane normal to gravity. We include gravity simply to preclude a lariat-type solution in which the reaction stress $\mathbf{P}$ vanishes. Figures 2.2 and 2.3 show the kinematics of these closely related problems. Without loss of generality, we consider an axially flowing string with an additional rigid translation imposed on the free piece, and parameterize the string in the direction of the axial flow for each case. Defining the axial speeds T_1 and T_2 and the translational speed V as non-negative, the spatial velocity of the non-material kink is to the left in Figure 2.2a (impact) and to the right in Figure 2.3a (pick-up). The material velocities in the vicinity of the impact point are $\partial_t\mathbf{X}^- = T_1\partial_s\mathbf{X}^- - V\partial_s\mathbf{X}^+$ and $\partial_t\mathbf{X}^+ = T_2\partial_s\mathbf{X}^+$. Equation (2.3) requires $T_1 = T_2 + V$, and equation (2.4) then gives the shock velocity $\partial_t s_0 = -T_1$ in the material coordinates. The material velocities in the vicinity of the pick-up point are $\partial_t\mathbf{X}^- = T_1\partial_s\mathbf{X}^-$ and $\partial_t\mathbf{X}^+ = T_2\partial_s\mathbf{X}^+ - V\partial_s\mathbf{X}^-$. Equation (2.3) requires $T_2 = T_1 + V$, and equation (2.4) then gives the shock velocity $\partial_t s_0 = -T_2$ in the material coordinates.

The grey regions in Figures 2.2b and 2.3b are regions of admissibility, where $\mathbf{P} \cdot \{\partial_s\mathbf{X}\}$ is non-positive. The plane can provide only a normal reaction force $\mathbf{P}$. The tangents lie in the

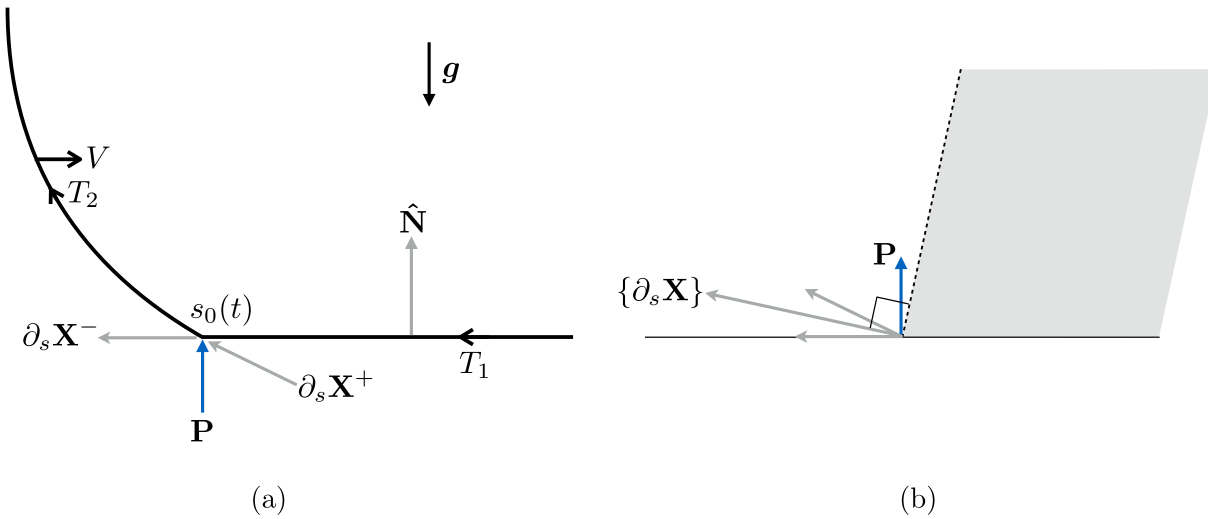

Figure 2.3: An axially moving string picked up from a surface with unit normal $\hat{\mathbf{N}}$ antiparallel to gravity $\boldsymbol{g}$. (a) The free piece translates with velocity $-V\partial_s\mathbf{X}^-$ and flows with axial velocity $T_2\partial_s\mathbf{X}^+$, while the contacting piece flows with axial velocity $T_1\partial_s\mathbf{X}^-$. A singular supply $\mathbf{P}$ acts at the discontinuity at $s_0(t)$, which moves to the right. (b) The sum of tangents and the grey region of admissibility where $\mathbf{P}\cdot\{\partial_s\mathbf{X}\} \leq 0$ are shown.

half-space above the plane, and for impact (Figure 2.2) it is clear that $\mathbf{P}$ is always admissible, and energy will be dissipated. However, during pick-up (Figure 2.3), the projection $\mathbf{P}\cdot\{\partial_s\mathbf{X}\}$ is always non-negative. A positive projection would imply an unphysical injection of energy at the kink. There are two ways to resolve this difficulty. One is if the surface has a bump at the pick-up point that could provide a $\mathbf{P}$ that points into the grey region. The other is if the reaction stress $\mathbf{P}$ vanishes. Excluding the perfectly folded case, this and the relation (2.30) would imply that $E = 0$. From the discussion in Section 2.5, we know that this implies that either the tension or tangents are continuous, $[\![\sigma]\!] = 0$ or $[\![\partial_s\mathbf{X}]\!] = 0$. Here we consider both possibilities, and show that in the presence of gravity and a vanishing $\mathbf{P}$, each implies the other in steady-state. Assuming first the continuity of tension, and no momentum injection, the momentum balance (2.12) during pick-up becomes $-(\sigma - \mu T_2^2)\,[\![\partial_s\mathbf{X}]\!] = 0$. In the presence of gravity, the steady-state bulk tension in an axially flowing string follows the catenary solution $\sigma = \mu T_2^2 - (\mathbf{g}\cdot\hat{\mathbf{n}}/\kappa)$ where both the curvature κ and the projection of gravity $\boldsymbol{g}$ onto the curve normal $\hat{\mathbf{n}}$ are always non-zero. Hence $\sigma - \mu T_2^2 \neq 0$, so it must follow that $[\![\partial_s\mathbf{X}]\!] = 0$. For an unsteady instantaneous configuration with a kink, gravity will be balanced by an acceleration that will smooth the kink. On the other hand, if we assume continuity of tangents, the momentum balance (2.12) becomes $-\,[\![\sigma]\!]\,\partial_s\mathbf{X} = 0$, and thus $[\![\sigma]\!] = 0$ for both steady and unsteady configurations. Note that we can say more. If the tangents are continuous, the momentum balance (2.12) says that any nonzero $\mathbf{P}$ must be parallel to the sum of tangents, an impossibility if $\mathbf{P}$ is normal to the surface. Hence, for any steady or unsteady smooth pick-up or impact from a smooth, frictionless surface, both $\mathbf{P}$ and $[\![\sigma]\!]$ vanish. From all of this, we see that the "chain fountain" and related problems are not so simple as one might hope (see Appendix A.4).

To summarize, a shock in the tangents of a string can only exist if the momentum supply satisfies the condition (2.36). For situations where a nonzero injection of momentum is required but cannot satisfy (2.36), including the pick-up of a string off of a smooth, frictionless surface, the tangents and the tension must be continuous at the contact discontinuity. For a kink to exist at the pick-up point, a properly oriented obstacle is necessary. An experimental illustration of this point will be shown shortly.

We have established a criterion for the admissibility of a kink. We now discuss another criterion that predicts the existence of a kink, associated with non-vanishing $\mathbf{P}$ and negative E. Consider again the impact problem shown in Figure 2.2. We know that the force $\mathbf{P}$, if it exists, can only point in the direction normal to the surface. Projecting the momentum balance (2.11) onto the unit surface normal $\hat{\mathbf{N}}$, we obtain

$$\mathbf{P} \cdot \hat{\mathbf{N}} = -(\sigma^- - \mu \partial_t s_0^2) \sin\theta \,, \tag{2.37}$$

where $\cos\theta = \partial_s \mathbf{X}^+ \cdot \partial_s \mathbf{X}^-$. For a non-adhesive surface, $\mathbf{P} \cdot \hat{\mathbf{N}}$ must be non-negative. A kink will have some angle $0 < \theta < \pi$, excluding the perfectly folded case. If $\sigma^- - \mu \partial_t s_0^2 \leq 0$, $\mathbf{P} \cdot \hat{\mathbf{N}} \geq 0$ and a kink is admissible. If $\sigma^- - \mu \partial_t s_0^2 > 0$, a kink is inadmissible, $\theta = 0$ and $\mathbf{P} \cdot \hat{\mathbf{N}} = 0$. The folded case $\theta = \pi$ is an exception, but not relevant to the impact geometry. An impact force and associated kink will exist if $|\partial_t s_0| > \sqrt{\sigma^-/\mu}$, that is, if the material velocity of the non-material impact point is greater than the local transverse wave speed in the string on the pre-impact side. Because $\partial_t s_0 = -T_2$ for impact, and $\sigma < \mu T_2^2$ for a concave-down catenary arch, this condition always holds for such a steady state. However, it need not hold for transient configurations. Burridge and co-workers [15] derived a similar criterion for the existence of a reaction force at the wrapping impact point for the geometrically linearized problem of a stretched string vibrating against a curved obstacle, as happens in certain Indian musical instruments. They simply state without justification that this criterion cannot be met for the corresponding unwrapping motion.

Gobat and Grosenbaugh's study of the touchdown region of catenary moorings [16] featured experiments on submerged link chains vertically vibrated against a platform. They present data on the shape of the chain during this process which clearly show the disappearance of a kink during the transition to upward motion of the chain, and smooth pick-up points during upward motion. Kinks appear during sufficiently rapid downward motion. Additional qualitative support for our predictions comes from the video stills shown in Figure 2.4. This laboratory demonstration of a pair of propagating contact discontinuities is essentially the same setup as in [47]. A ball-and-link chain is pulled at several m/s to the left over a smooth surface. A retractable blade near the right hand side creates a "bubble" in the chain. After the blade retracts, the disturbance moves freely to the left and eventually dies down. While this is not a steady-state configuration, the tangential speed of the chain is nearly constant, and is quite rapid compared to the shape dynamics. Many of the features we discuss above are apparent. The blade provides a sideways reaction force that can sustain a pick-up kink; immediately once it is retracted, the pick-up point becomes smooth. Later, the impact point undergoes a transition from smooth to kinked.

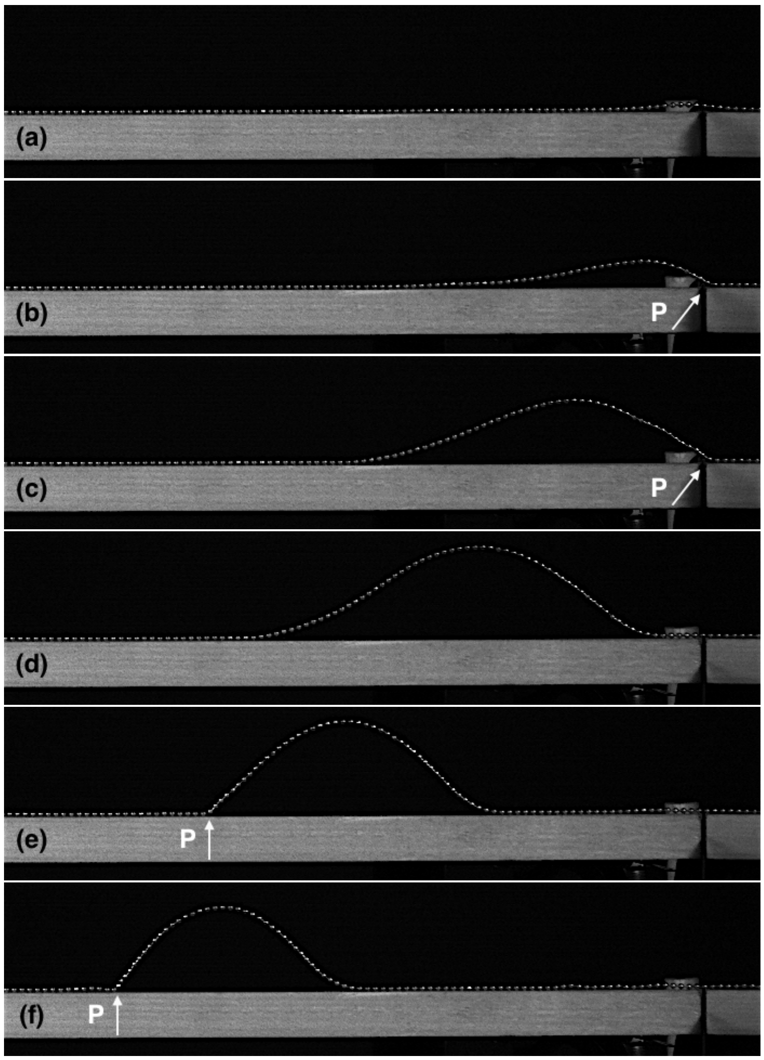

Figure 2.4: Still images from a video of pick-up and impact discontinuities. A ball-and-link chain is pulled at several m/s to the left over a smooth surface. In frames (a)-(c), a small blade protrudes above the surface on the right, causing a "bubble" to form. After the blade is retracted, the shape propagates freely. Kinks can be seen at the pick-up points in frames (b)-(c) and at impact in frames (e)-(f). Inferred reaction forces are labeled.

2.7 Further discussion and conclusions

We have examined a simple mechanical system, and made a surprising prediction that appears to have gone unnoticed in the literature, but is consonant with previous experimental observations. A kink in a flexible structure is generally inadmissible during pick-up. This statement follows from a compatibility relation (2.30) between stress and translation-invariant power supplies, along with basic assumptions about the reaction stress, at the contact discontinuity. While our conclusion is based on excluding exceptional solutions for which there is no reaction stress (including lariat-type solutions), which we have only been able to do explicitly for steady-state configurations, the limited experimental evidence suggests that the conclusions are more broadly applicable.

At the discontinuity, we supplemented momentum conservation with energy conservation, while allowing for source terms. These balances correspond to symmetries in spatial and temporal coordinates. An alternative is to use a balance of some configurational-like or "material" quantity [21, 28, 27], corresponding to a boundary shift or a symmetry in material coordinates.

The simple example of an inextensible string allowed us to use a very convenient orthogonal pair, the sum and jump of tangents, to transparently express vector quantities and perform calculations. A non-orthogonal pair, the sum and jump of velocities, was used to isolate the translation-invariant part of the energy. A natural consequence of our energy splitting is that the external stress supply is paired with the average velocity across the kink, in keeping with standard treatments of particle impact mechanics. This leads to the compatibility relation used in our analysis.

An immediate question is, how do these concepts extend to more complicated one-dimensional bodies? We expect that axial stretching will complicate the problem without providing much new physics, at least for a hyperelastic material with convex elastic energy. However, higher-order discontinuities in rods and other structures with bend and twist resistance are potentially interesting. Recent experiments on "torsional locomotion" show how such structures can be propelled away from partial constraints by jumps in curvature or twist [48]. Rotational invariance properties of the power dissipation terms will likely be important for such systems.

Other ways of looking at these problems might consider the dynamics of the geometry, or of the non-material point of contact. An example problem in which the latter type of formulation is possible, given certain assumptions about the geometry and dissipation, is discussed in Appendix A.2.

Acknowledgments

The chain bubble demo of Figure 2.4 was designed and filmed by R. B. Warner during his time as an undergraduate researcher in our group. We thank J. S. Biggins, E. Fried, A. Gupta, O. M. O'Reilly, and E. G. Virga for helpful and detailed discussions. This work was supported by U.S. National Science Foundation grant CMMI-1462501.

Bibliography

[1] E. Hamm and J-C. Géminard. The weight of a falling chain, revisited. *American Journal of Physics*, 78:828–833, 2010.

[2] A. Grewal, P. Johnson, and A. Ruina. A chain that speeds up, rather than slows, due to collisions: How compression can cause tension. *American Journal of Physics*, 79:723–729, 2011.

[3] B. C. Rennie. Chains and whips in the teaching of mathematics. *The Mathematical Gazette*, 56(398):271–274, 1972.

[4] M. G. Calkin. The dynamics of a falling chain: II. *American Journal of Physics*, 57(2):157–159, 1989.

[5] A. D. Cambou, B. D. Gamari, E. Hamm, J. A. Hanna, N. Menon, C. D. Santangelo, and L. Walsh. Unwrapping chains. [arXiv:1209.0481].

[6] P.-T. Brun, B. Audoly, A. Goriely, and D. Vella. The surprising dynamics of a chain on a pulley: lift off and snapping. *Proceedings of the Royal Society A*, 472:20160187, 2016.

[7] J. A. Hanna and H. King. An instability in a straightening chain. [arXiv:1110.2360].

[8] S. Mould. Amazing bead chain experiment in slow motion. `www.youtube.com/watch?v=6ukMId5fIi0` , 2013.

[9] J. S. Biggins. Growth and shape of a chain fountain. *Europhysics Letters*, 106:44001, 2014.

[10] V. V. Beletsky and E. M. Levin. *Dynamics of Space Tether Systems*. American Astronautical Society, San Diego, 1993.

[11] S. K. Batra and W. B. Fraser. *Engineering Fundamentals of Ring Spinning/Twisting, Over-End Unwinding and Two-For-One Twisting in Textile Processes*. DEStech, Lancaster, 2015.

[12] H. Irschik and A. K. Belyaev. *Dynamics of Mechanical Systems with Variable Mass*. Springer, 2014.

[13] E. E. Zajac. Dynamics and kinematics of the laying and recovery of submarine cable. *The Bell System Technical Journal*, 36:1129–1207, 1957.

[14] C. V. Raman. On some Indian stringed instruments. *Proc. Indian Assoc. Cultiv. Sci.*, 7:29–33, 1921.

[15] R. Burridge, J. Kappraff, and C. Morshedi. The sitar string, a vibrating string with a one-sided inelastic constraint. *SIAM Journal on Applied Mathematics*, 42:1231–1251, 1982.

[16] J. I. Gobat and M. A. Grosenbaugh. Dynamics in the touchdown region of catenary moorings. *International Journal of Offshore and Polar Engineering*, 11:273–281, 2001.

[17] R. S. Rivlin. The effective work of adhesion. *Paint Technology*, IX(106):1–4, 1944.

[18] K. Kendall. Thin-film peeling–the elastic term. *Journal of Physics D: Applied Physics*, 8:1449–1452, 1975.

[19] R. Burridge and J. B. Keller. Peeling, slipping and cracking–some one-dimensional free-boundary problems in mechanics. *SIAM Review*, 20:31–61, 1978.

[20] J. L. Ericksen. *Introduction to the Thermodynamics of Solids*. Springer, New York, 1998.

[21] N. Pede, P. Podio-Guidugli, and G. Tomassetti. Balancing the force that drives the peeling of an adhesive tape. *Il Nuovo Cimento*, 121 B(5):531, 2006.

[22] P.-P. Cortet, M.-J. Dalbe, C. Guerra, C. Cohen, M. Ciccotti, S. Santucci, and L. Vanel. Intermittent stick-slip dynamics during the peeling of an adhesive tape from a roller. *Physical Review E*, 87:022601, 2013.

[23] M.-J. Dalbe, S. Santucci, P.-P. Cortet, and L. Vanel. Strong dynamical effects during stick-slip adhesive peeling. *Soft Matter*, 10:132–138, 2014.

[24] A. E. Green and P. M. Naghdi. A derivation of jump condition for entropy in thermomechanics. *Journal of Elasticity*, 8(2):179–182, 1978.

[25] O. M. O'Reilly and P. C. Varadi. A treatment of shocks in one-dimensional thermomechanical media. *Continuum Mechanics and Thermodynamics*, 11:339–352, 1999.

[26] O. M. O'Reilly and P. C. Varadi. On energetics and conservations for strings in the presence of singular sources of momentum and energy. *Acta Mechanica*, 165:27–45, 2003.

[27] O. M. O'Reilly. The energy jump condition for thermomechanical media in the presence of configurational forces. *Continuum Mechanics and Thermodynamics*, 18:361–365, 2007.

[28] O. M. O'Reilly. A material momentum balance law for rods. *Journal of Elasticity*, 86:155–172, 2007.

[29] E. G. Virga. Dissipative shocks in a chain fountain. *Physical Review E*, 89(053201), 2014.

[30] E. G. Virga. Chain paradoxes. *Proceedings of the Royal Society A*, 471(20140657), 2015.

[31] R. Abeyaratne and J. K. Knowles. On the driving traction acting on a surface of strain discontinuity in a continuum. *Journal of the Mechanics and Physics of Solids*, 38:345 – 360, 1990.

[32] R. Abeyaratne and J. K. Knowles. Kinetic relations and the propagation of phase boundaries in solids. *Archive for Rational Mechanics and Analysis*, 114:119–154, 1991.

[33] P. K. Purohit and K. Bhattacharya. Dynamics of strings made of phase-transforming materials. *Journal of the Mechanics and Physics of Solids*, 51:393 – 424, 2003.

[34] J. A. Hanna. Jump conditions for strings and sheets from an action principle. *International Journal of Solids and Structures*, 62:239–247, 2015.

[35] S. S. Antman. *Nonlinear problems of elasticity.* Springer, New York, 2005.

[36] P. Podio-Guidugli. Inertia and invariance. *Annali di Matematica Pura ed Applicata*, CLXII:103–124, 1997.

[37] E. T. Whittaker. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies.* Cambridge University Press, Cambridge, 1988.

[38] R. A. Diaz, W. J. Herrera, and D. A. Manjarrés. Work and energy in inertial and noninertial reference frames. *American Journal of Physics*, 77:270–273, 2009.

[39] W. Thomson and P.G. Tait. *Treatise on Natural Philosophy, Vol I, Part I.* Dover, New York, 1962.

[40] W. J. Stronge. *Impact mechanics.* Cambridge University Press, Cambridge, 2004.

[41] M. G. Calkin and R. H. March. The dynamics of a falling chain: I. *American Journal of Physics*, 57(2):154–157, 1989.

[42] O. M. O'Reilly and P. C. Varadi. On some peculiar aspects of axial motions of closed loops of string in the presence of a singular supply of momentum. *Journal of Applied Mechanics*, 71:541–545, 2004.

[43] E. J. Routh. *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies.* Dover, New York, 1955.

[44] T. Healey and J. N. Papadopoulos. Steady axial motion of strings. *Journal of Applied Mechanics*, 57:785–787, 1990.

[45] W. A. Heywood, H. Hurwitz, and D. Z. Ryan. Whip effect in a falling chain. *American Journal of Physics*, 23(5):279–280, 1955.

[46] M. Schagerl, A. Steindl, W. Steiner, and H. Troger. On the paradox of the free falling folded chain. *Acta Mechanica*, 125:155–168, 1997.

[47] J. A. Hanna, W. R. Royston, and R. B. Warner. Chain separation bubbles. `gfm.aps.org/meetings/dfd-2014/54136d3469702d585c4c0000` .

[48] D. Bigoni, F. Dal Corso, D. Misseroni, and F. Bosi. Torsional locomotion. *Proceedings of the Royal Society A*, 470:20140599, 2014.

[49] M. E. Gurtin. *Configurational Forces as Basic Concepts of Continuum Physics*. Springer, New York, 2000.

[50] J. Satterly. Falling chains. *American Journal of Physics*, 19:383–384, 1951.

[51] W. Tomaszewski, P. Pieranski, and J-C. Géminard. The motion of a freely falling chain tip. *American Journal of Physics*, 74:776–783, 2006.

[52] C. W. Wong and K. Yasui. Falling chains. *American Journal of Physics*, 74(6):490–496, 2006.

[53] T. R. Kane and D. A. Levinson. Deployment of a cable-supported payload from an orbiting spacecraft. *Journal of Spacecraft and Rockets*, 14(7):409–413, 1977.

[54] A. Heck, P. Uylings, and E. Kędzierska. Understanding the physics of bungee jumping. *Physics Education*, 45(1):63–72, 2010.

[55] J.-C. Géminard and G. Palacio. A chain falling from a table. `http://perso.ens-lyon.fr/jean-christophe.geminard/a_chain_falling_from_a_table.htm`, 2012.

[56] R. Martins. The (not so simple!) chain fountain. [arXiv:1612.09319].

Chapter 3

Partial constraint singularities in elastic rods

This is a post-peer-review, pre-copyedit version of an article published in *Journal of Elasticity*. The final authenticated version is available online at: `http://dx.doi.org/10.1007/s10659-018-9673-6`.

Attribution

The work presented in this chapter was done in collaboration with J. A. Hanna and E. G. Virga. The main idea of this work was conceived by E. G. Virga, and J. A. Hanna contributed towards the development and implementation of this work.

Abstract

We present a unified classical treatment of partially constrained elastic rods. Partial constraints often entail singularities in both shapes and reactions. Our approach encompasses both sleeve and adhesion problems, and provides simple and unambiguous derivations of counterintuitive results in the literature. Relationships between reaction forces and moments, geometry, and adhesion energies follow from the balance of energy during quasistatic motion. We also relate our approach to the balance of material momentum and the concept of a driving traction. The theory is generalizable and can be applied to a wide array of contact, adhesion, gripping, and locomotion problems.

3.1 Introduction

Partially constrained flexible structures provide fascinating everyday examples of the consequences of balance laws at geometric discontinuities, and their associated mechanical singularities. Over the years, both theory and experiment have revealed challenging and counter-intuitive results linking reaction forces and moments, adhesion energy, and curvatures of rods and strips at points of discontinuous contact with sleeves and sticky surfaces [1, 2, 3, 4, 5]. These results pertain to structural stability questions [6, 7], paper handling and processing [8, 9], animal locomotion [10, 11, 12, 13], standard tests for adhesives [14, 15, 16], adhesion of lipid membranes and molecules [17, 18] or MEMS [19, 20], and elastocapillarity [21, 22, 23].

In this note, we present a simple classical derivation based on the local balance of force, moment, and energy that provides a unified framework for guide, contact, and adhesion problems for rods. We apply this framework to two recent examples from the literature, involving sliding in or out of a frictionless sleeve [1] and peeling from an adhesive surface [4]. We directly recover results on global force balances, and on boundary conditions relating reaction forces and moments, and bending and adhesion energies, avoiding assumptions and ambiguities present in prior derivations. For simplicity of exposition, we focus our comments on an inextensible, unshearable, planar Euler *elastica*, quasistatic dynamics, and discontinuities of second or higher order in position. For inextensible, unshearable rods, this means discontinuities in curvature, but not kinks in tangent vectors; the bulk equivalent is an “acceleration wave”. However, there is nothing to prevent generalization of the present approach.

Other authors have taken several different approaches to partial constraint problems. Bigoni and co-workers [1] derived and experimentally confirmed the relationship between end loads on an *elastica* partially contained, and free to slide within, a frictionless sleeve. In their derivation, they employed an angle-pendulum representation of the shape equation, and extremized the elastic energy with respect to virtual shifts in the position of the sleeve edge. One disadvantage of the commonly employed angle representation, in contrast to either a curvature representation or a rod mechanics approach featuring contact force and moment, is that it throws away local information about tangentially conserved quantities [24]. These quantities, the force and material force, both have prominent roles in rod problems and can be used to derive results more directly. Bigoni and co-workers also obtained a relation between the reaction force and moment at the sleeve edge using an argument that considers a limiting geometry of the sleeve. They interpret part of the reaction force as an “Eshelby-like” term analogous to a configurational force. However, O’Reilly [25] has shown that the results follow from an *absence* of a reaction “material force”, a configurational source term arising in his material momentum balance approach. His derivation is elegant and direct, but non-classical in that it requires prescription of a singular source of material force, or lack thereof, at the contact boundary. In contrast with the familiar concept of an energy source or sink, our understanding of sources of material force, and how to prescribe such a source for a given physical situation, is still being developed. For systems governed by an action principle, material forces in the bulk can be understood as conserved quantities associated with symmetries in material properties of the rod; for purely mechanical systems such as an *elastica* without adhesion, the material momentum balance is simply the projection of the linear momentum balance onto the tangents of the body [24]. The literature suggests that discontinuities in material properties or internal energy can serve as sources of material force [26, 27, 28, 29]. It seems clear that for the problem of a uniform rod in a frictionless sleeve, no material source term is present at the contact discontinuity, but we will see later that it is not obvious how to prescribe this term for problems involving adhesion. Majidi and co-workers examined adhesion of an *elastica* from a variety of perspectives, including that of O'Reilly's material momentum balance, for which they interpreted the material source term as the jump in adhesion energy across the peeling front [4, 10, 30]. This approach led them to a boundary condition with an extra term, which they then assumed to vanish in order to agree with their own alternate derivations and with established results relating the reaction moment and bending energy at the peeling front.

Here we will show that all of the above results can be derived from classical balance laws at a moving geometric discontinuity. These jump conditions are introduced, along with our notation and other preliminaries, in Section 3.2. They are also derived in Section 3.3 from invariance considerations applied to the internal energy balance at a discontinuity. We there also discuss the free energy imbalance at a discontinuity for adiabatic processes, derive an expression for dissipation at the singular point and relate it to the concept of a driving traction, and obtain constitutive restrictions on free energy from the second law of thermodynamics. In Section 3.4, we identify the conserved force and material force, and solve sleeve and adhesion problems for an *elastica*. In particular, we show that the natural boundary conditions of interest follow from the balance of energy during a quasistatic motion. We conclude with a comparison of our approach and the use of a material momentum balance, and some additional comments, in Sections 3.5 and 3.6. While for simple mechanical systems the two approaches provide essentially the same information, our energy balance approach unambiguously avoids the appearance of a phantom tangential reaction term in the adhesion problem.

3.2 Quasistatic balance laws

We first present the kinematics and bulk and singular balance laws for an inextensible rod in quasistatic equilibrium, neglecting inertial effects.

Consider a planar, twistless rod described as a curve $\boldsymbol{x}(s,t)$ parameterized by a material coordinate $s \in [0, L]$ and time t, with a single discontinuity in field quantities at the time-dependent non-material point $s = s_0(t)$. A prime $(\,)'$ will denote an s-derivative, and a dot $\dot{(\,)}$ a material time derivative when applied to quantities depending on s and a partial time derivative when applied to a non-material quantity such as $s_0(t)$. Thus, $\dot{s}_0$ denotes how fast the discontinuity moves through the material composing the rod. For an inextensible, unshearable rod, s is also the arc length and $\boldsymbol{x}'$ a unit tangent vector. For compactness of expressions, we will use the shorthand $\boldsymbol{\omega} \equiv \boldsymbol{x}' \times \dot{\boldsymbol{x}}'$ for angular velocity and $\boldsymbol{\Omega} \equiv \boldsymbol{x}' \times \boldsymbol{x}''$ for the Darboux vector. The jump and mean of a quantity $\mathbf{Q}$ across the discontinuity s_0 are denoted by $[\![\mathbf{Q}]\!] \equiv \mathbf{Q}^+ - \mathbf{Q}^-$ and $\{\mathbf{Q}\} \equiv \frac{1}{2}(\mathbf{Q}^+ + \mathbf{Q}^-)$, where $\mathbf{Q}^\pm$ are the values of $\mathbf{Q}$ immediately on either side of the discontinuity.

In the absence of body or applied forces or couples, the force and moment balances in the bulk of the rod are given in terms of a contact force $\boldsymbol{n}$ and contact moment $\boldsymbol{m}$ by [31]

$$\boldsymbol{n}' = \mathbf{0}\,, \tag{3.1}$$

$$\boldsymbol{m}' + \boldsymbol{x}' \times \boldsymbol{n} = \mathbf{0}\,. \tag{3.2}$$

We will refer to four jump conditions at the point of discontinuity $s = s_0(t)$,

$$\mathbf{R} + [\![\boldsymbol{n}]\!] = \mathbf{0}\,, \tag{3.3}$$

$$\mathbf{M} + [\![\boldsymbol{m}]\!] = \mathbf{0}\,, \tag{3.4}$$

$$\tilde{E} + [\![\boldsymbol{n} \cdot \dot{\boldsymbol{x}} + \boldsymbol{m} \cdot \boldsymbol{\omega} + \dot{s}_0 \varepsilon]\!] = 0\,, \tag{3.5}$$

$$Y + [\![\boldsymbol{n} \cdot \boldsymbol{x}' + \boldsymbol{m} \cdot \boldsymbol{\Omega} - \psi]\!] = 0\,, \tag{3.6}$$

where ε and ψ are the internal energy and the free energy, respectively, per unit length of the rod. The first three conditions (3.3)-(3.5) are classical force, moment, and energy jump conditions, with allowance for singular sources of force $\mathbf{R}$, moment $\mathbf{M}$, and power $\tilde{E}$ [32, 33, 34, 35]. We will use these conditions to solve the specific problems to which we apply our theory in Section 3.4. Note that $\tilde{E}$ is the power input at the singularity; it should not be confused with the *dissipation* $\mathcal{D}$ at the singularity, which will be introduced in Section 3.3 below. The fourth, non-classical, condition (3.6) is a material momentum jump condition with a singular source of material force Y [36, 29]. We will not use this condition, but will discuss its significance in Section 3.5. In fact, the bracketed term in (3.6) can be given a classical interpretation for many simple conservative systems, in terms of material symmetry of an action [24] or the quasistatic form of a conservation law for uniform hyperelastic rods [37], but we lack an unambiguous way to specify the source Y. Fully dynamic forms of the jump conditions (3.3)-(3.6) can be found elsewhere [34, 35]. Boundary conditions on force and moment simply involve the bracketed $\boldsymbol{n}$ and $\boldsymbol{m}$ terms in (3.3)-(3.4). For example, if an end load $\mathbf{P}$ is applied at $s = L$, with the body lying on the $^-$ side, we would write $\mathbf{P} - \boldsymbol{n}^-(L) = \mathbf{0}$. Similarly, with zero applied end moment, we would write $-\boldsymbol{m}^-(L) = \mathbf{0}$.

We assume continuity of the position vector, $[\![\boldsymbol{x}\,(s_0(t), t)]\!] = \mathbf{0}$, and the tangents, $[\![\boldsymbol{x}'\,(s_0(t), t)]\!] = \mathbf{0}$, across the point of discontinuity. Total time derivatives of these expressions, and subsequent application of $\boldsymbol{x}' \times$ to the second, give the following kinematic jump conditions,

$$[\![\dot{\boldsymbol{x}} + \dot{s}_0 \boldsymbol{x}']\!] = \mathbf{0}\,, \tag{3.7}$$

$$[\![\dot{\boldsymbol{x}}' + \dot{s}_0 \boldsymbol{x}'']\!] = \mathbf{0}\,, \tag{3.8}$$

$$[\![\boldsymbol{\omega} + \dot{s}_0 \boldsymbol{\Omega}]\!] = \mathbf{0}\,. \tag{3.9}$$

The first and third of these express continuity of linear and angular velocity of the non-material point of discontinuity,

$$\boldsymbol{v}_0 = \dot{\boldsymbol{x}}^{\pm} + \dot{s}_0 \dot{\boldsymbol{x}}'^{\pm} \ = \{\dot{\boldsymbol{x}}\} + \dot{s}_0 \{\boldsymbol{x}'\} = \dot{\boldsymbol{x}}(s_0) + \dot{s}_0 \boldsymbol{x}'(s_0)\,, \tag{3.10}$$

$$\boldsymbol{\omega}_0 = \boldsymbol{\omega}^{\pm} + \dot{s}_0 \boldsymbol{\Omega}^{\pm} = \{\boldsymbol{\omega}\} + \dot{s}_0 \{\boldsymbol{\Omega}\}\,. \tag{3.11}$$

3.3 Dissipation and invariance

Here, following the pattern set forth by Noll [38], we prove that the singular balance laws (3.3)-(3.5) can all be derived from an invariance property required of the balance of energy at the singularity. We also consider the free energy imbalance to arrive at restrictions on the constitutive assumptions made for the rod.

3.3.1 Energy balance

For a rod with an energy source only at the singularity, and no heat source, we may write the energy balance as

$$\frac{d}{dt} \int_{s_1}^{s_2} ds\, \varepsilon = E + W + \boldsymbol{n} \cdot \dot{\boldsymbol{x}}\,|_{s_1}^{s_2} + \boldsymbol{m} \cdot \boldsymbol{\omega}\,|_{s_1}^{s_2}\ , \tag{3.12}$$

for a material interval enclosing a moving discontinuity at $s_0(t)$, such that $s_0(t) \in [s_1, s_2]$. In the expression above, E represents a source of power at the singularity, and

$$W = \mathbf{R} \cdot \{\dot{\boldsymbol{x}}\} + \mathbf{M} \cdot \{\boldsymbol{\omega}\} \tag{3.13}$$

is the power expended there by the singular sources of force and moment.

In principle, W could instead be prescribed in a more general form,

$$W = \mathbf{R} \cdot \langle \dot{\boldsymbol{x}} \rangle_\lambda + \mathbf{M} \cdot \langle \boldsymbol{\omega} \rangle_\lambda, \tag{3.14}$$

where, for any $\lambda \in [0, 1]$, we define the convex combination $\langle\, \rangle_\lambda = \lambda(\,)^+ + (1-\lambda)(\,)^-$. Clearly, $\{\,\}$ corresponds to $\langle\, \rangle_\lambda$ for $\lambda = \frac{1}{2}$, and it is the only kinematic measure $\langle\, \rangle_\lambda$ symmetric under the exchange of end signs $\pm$. For the sake of generality, and only in this section, we shall write W as in (3.14).

Splitting the integral in (3.12) over two time-dependent intervals, applying the Leibniz rule, and letting s_1 and s_2 approach s_0 from below and above, respectively, we obtain

$$E + \mathbf{R} \cdot \langle \dot{\boldsymbol{x}} \rangle_\lambda + \mathbf{M} \cdot \langle \boldsymbol{\omega} \rangle_\lambda + [\![\boldsymbol{n} \cdot \dot{\boldsymbol{x}} + \boldsymbol{m} \cdot \boldsymbol{\omega} + \dot{s}_0 \varepsilon]\!] = 0\,, \tag{3.15}$$

which coincides with (3.5), provided that we set

$$\tilde{E} = E + \mathbf{R} \cdot \langle \dot{\boldsymbol{x}} \rangle_\lambda + \mathbf{M} \cdot \langle \boldsymbol{\omega} \rangle_\lambda \,. \tag{3.16}$$

Equation (3.16) expresses the balance of working at the singularity; it reveals $\tilde{E}$ as the *net* singular power, comprising a source E and the working of the singular sources $\mathbf{R}$ and $\mathbf{M}$.

We now require that the singular balance (3.15) be invariant under translations and rotations, assuming that both E and ε represent objective quantities. Transforming $\dot{\boldsymbol{x}} \mapsto \dot{\boldsymbol{x}} + \boldsymbol{v}^*$, $\boldsymbol{\omega} \mapsto \boldsymbol{\omega} + \boldsymbol{\omega}^*$, with arbitrary vectors $\boldsymbol{v}^*$ and $\boldsymbol{\omega}^*$, the former in the plane of the rod and the latter orthogonal to the plane, we easily see that (3.15) is also valid for the transformed motion, provided that

$$\big([\![\boldsymbol{n}]\!] + \mathbf{R} \big) \cdot \boldsymbol{v}^* + \big([\![\boldsymbol{m}]\!] + \mathbf{M} \big) \cdot \boldsymbol{\omega}^* = 0 \,, \tag{3.17}$$

from which, by the arbitrariness of $\boldsymbol{v}^*$ and $\boldsymbol{\omega}^*$, both (3.3) and (3.4) follow at once.[1] We now use these equations to arrive at a reduced form of the singular balance of energy (3.15). Using the algebraic identity

$$[\![\mathbf{A} \cdot \mathbf{B}]\!] = [\![\mathbf{A}]\!] \cdot \langle \mathbf{B} \rangle_\lambda + \langle \mathbf{A} \rangle_{1-\lambda} \cdot [\![\mathbf{B}]\!] \,, \tag{3.18}$$

equation (3.15) is given the form

$$E = -\langle \boldsymbol{n} \rangle_{1-\lambda} \cdot [\![\dot{\boldsymbol{x}}]\!] - \langle \boldsymbol{m} \rangle_{1-\lambda} \cdot [\![\boldsymbol{\omega}]\!] - \dot{s}_0 \, [\![\varepsilon]\!] \,. \tag{3.19}$$

Using the linear and angular kinematic jump conditions (3.7) and (3.9) to substitute for $[\![\dot{\boldsymbol{x}}]\!]$ and $[\![\boldsymbol{\omega}]\!]$, we rearrange the latter equation as

$$E = \dot{s}_0 \big(\langle \boldsymbol{n} \rangle_{1-\lambda} \cdot [\![\boldsymbol{x}']\!] + \langle \boldsymbol{m} \rangle_{1-\lambda} \cdot [\![\boldsymbol{\Omega}]\!] - [\![\varepsilon]\!] \big) \,, \tag{3.20}$$

which is our *reduced* singular energy balance.

Although these conclusions were achieved for the general definition of W in (3.14), in the following we shall take the familiar value $\lambda = \frac{1}{2}$, but only for simplicity. Thus, equations (3.20) and (3.16) become

$$E = \dot{s}_0 \big(\{\boldsymbol{n}\} \cdot [\![\boldsymbol{x}']\!] + \{\boldsymbol{m}\} \cdot [\![\boldsymbol{\Omega}]\!] - [\![\varepsilon]\!] \big) \tag{3.21}$$

and

$$\tilde{E} = E + \mathbf{R} \cdot \{\dot{\boldsymbol{x}}\} + \mathbf{M} \cdot \{\boldsymbol{\omega}\} \,, \tag{3.22}$$

respectively. Both of these equations are consequences of (3.12), which henceforth will be assumed to be valid.

Note that (3.21) could also be obtained by splitting the kinematic quantities in (3.5) into invariant and non-invariant parts. Recall that with our continuity assumptions, the linear

[1] This argument is presented for a planar problem, but can be fully generalized.

and angular velocities (3.10) and (3.11) of the non-material point have the forms $\{\dot{\boldsymbol{x}}\} + \dot{s}_0\{\boldsymbol{x}'\}$ and $\{\boldsymbol{\omega}\} + \dot{s}_0\{\boldsymbol{\Omega}\}$. Using the identity (3.18) with $\lambda = \frac{1}{2}$, (3.5) can be written as $\tilde{E} + [\![\boldsymbol{n}]\!] \cdot \{\dot{\boldsymbol{x}}\} + [\![\boldsymbol{m}]\!] \cdot \{\boldsymbol{\omega}\} = -\{\boldsymbol{n}\} \cdot [\![\dot{\boldsymbol{x}}]\!] - \{\boldsymbol{m}\} \cdot [\![\boldsymbol{\omega}]\!] - \dot{s}_0\, [\![\varepsilon]\!]$, which upon using the kinematic conditions (3.7) and (3.9) and the force and moment conditions (3.3) and (3.4) reduces to

$$\tilde{E} - \mathbf{R} \cdot \{\dot{\boldsymbol{x}}\} - \mathbf{M} \cdot \{\boldsymbol{\omega}\} = \dot{s}_0\big(\{\boldsymbol{n}\} \cdot [\![\boldsymbol{x}']\!] + \{\boldsymbol{m}\} \cdot [\![\boldsymbol{\Omega}]\!] - [\![\varepsilon]\!]\,\big)\,, \tag{3.23}$$

which corresponds to (3.21) and (3.22). The splitting of the energy balance in (3.23) has paired non-invariant terms with the reaction force $\mathbf{R}$ and moment $\mathbf{M}$, and invariant terms with the average contact force $\{\boldsymbol{n}\}$ and moment $\{\boldsymbol{m}\}$.

A further assumption will be made in solving the equilibrium problems detailed in the following section, namely that the net singular power input $\tilde{E}$ vanishes,

$$\tilde{E} = 0\,. \tag{3.24}$$

Assumptions (3.12) and (3.24) have a completely different status: the former is a general energy balance for rods with a (possibly dissipative) singularity, the latter, combined with (3.22), represents a *detailed* energy balance at the singularity, which may or may not be valid in general. The equilibrium problems we consider will also elucidate the different roles that the reduced singular energy balance (3.21) can play in our theory. It may either be an identity expressing compatibility between the singular balances of force, moment, and energy, or a prescription for the energy source E.

3.3.2 Free energy imbalance

We will use the free energy imbalance of a rod undergoing an adiabatic process to define the dissipation at the singularity. In section 3.4.2, the inferences drawn in the present section will help us gain insight into the physical meaning of the term E in a more general thermodynamic setting.

As we proceeded for the energy balance in 3.3.1, we may write the free energy imbalance for a material interval enclosing a moving discontinuity as

$$\frac{d}{dt}\int_{s_1}^{s_2} ds\,\psi + \int_{s_1}^{s_2} ds\,\dot{\theta}\eta = -\mathcal{D} + W + \boldsymbol{n} \cdot \dot{\boldsymbol{x}}|_{s_1}^{s_2} + \boldsymbol{m} \cdot \boldsymbol{\omega}|_{s_1}^{s_2}\,, \quad \mathcal{D} \geq 0\,, \tag{3.25}$$

where $\mathcal{D}$ is the dissipation at the singularity. The free energy ψ is related to the internal energy ε, entropy η, and temperature θ through the relation $\psi = \varepsilon - \theta\eta$. Localizing this imbalance by the same procedure as before, we obtain

$$\mathcal{D} = W + [\![\boldsymbol{n} \cdot \dot{\boldsymbol{x}} + \boldsymbol{m} \cdot \boldsymbol{\omega} + \psi]\!]\,, \quad \mathcal{D} \geq 0\,, \tag{3.26}$$

which upon using (3.13) and some algebraic manipulations can be written as

$$\mathcal{D} = -\dot{s}_0(\{\boldsymbol{n}\} \cdot [\![\boldsymbol{x}']\!] + \{\boldsymbol{m}\} \cdot [\![\boldsymbol{\Omega}]\!] - [\![\psi]\!])\,. \tag{3.27}$$

The quantity appearing in (3.27) conjugate to the motion $\dot{s}_0$ of the singularity through the body is the driving traction of Abeyaratne and Knowles [39, 40].

We prescribe the following expressions for the free energy and entropy,

$$\psi = \hat{\psi}(\boldsymbol{\Omega}, \theta) + \psi_c\,, \tag{3.28}$$
$$\eta = \hat{\eta}(\boldsymbol{\Omega}, \theta) + \eta_c\,. \tag{3.29}$$

The internal energy can be determined as $\varepsilon = \hat{\varepsilon}(\boldsymbol{\Omega}, \theta) + \varepsilon_c$, where $\varepsilon_c = \psi_c + \theta\eta_c$. The ψ_c and η_c are constants whose jumps will account for jumps in the *descriptions* of free energy and entropy across the singularity, as will be necessary when we discuss peeling in Section 3.4.2.

Without loss of generality, we assume a moving discontinuity and parameterize the rod such that $-\dot{s}_0 > 0$. Using the continuity of tangents $[\![\boldsymbol{x}']\!] = 0$ and the definition of contact moment for a hyperelastic rod $\boldsymbol{m} = \frac{\partial\psi}{\partial\boldsymbol{\Omega}}$, and substituting (3.28) in (3.27) we obtain for any admissible quasistatic motion,

$$\left\langle \frac{\partial\hat{\psi}}{\partial\boldsymbol{\Omega}} \right\rangle \cdot [\![\boldsymbol{\Omega}]\!] - \left[\!\!\left[\hat{\psi}\right]\!\!\right] - [\![\psi_c]\!] \geq 0\,. \tag{3.30}$$

Returning to the definition of E, we use $[\![\varepsilon]\!] = \left[\!\!\left[\hat{\psi}\right]\!\!\right] + [\![\varepsilon_c]\!] + [\![\theta\hat{\eta}]\!]$ and the definition of the contact moment $\boldsymbol{m}$ to re-write (3.21) as

$$E = \dot{s}_0\left(\{\boldsymbol{n}\} \cdot [\![\boldsymbol{x}']\!] + \left\langle \frac{\partial\hat{\psi}}{\partial\boldsymbol{\Omega}} \right\rangle \cdot [\![\boldsymbol{\Omega}]\!] - \left[\!\!\left[\hat{\psi}\right]\!\!\right] - [\![\varepsilon_c]\!] - [\![\theta\hat{\eta}]\!] \right)\,. \tag{3.31}$$

The first term inside the round brackets vanishes due to continuity of tangents, while (3.30) simplifies the other terms such that we obtain the inequality

$$E \leq -\dot{s}_0\,[\![\theta\eta]\!]\,. \tag{3.32}$$

3.4 Equilibrium problems

We now derive results pertaining to sleeve and adhesion problems in quasistatic equilibrium. We first identify two useful conserved quantities. While, for the sleeve example, these quantities could be derived from spatial and material symmetries of an action [24], we will obtain them directly from the balances (3.1)-(3.2).

Clearly, the contact force in (3.1) is conserved,

$$\boldsymbol{n} = \mathbf{P}\,, \tag{3.33}$$

in which expression we anticipate identifying the constant of integration with an end load $\mathbf{P}$ at $s = L$. Another conserved quantity arises by integrating the tangential projection of

(3.1). We write $\boldsymbol{n}' \cdot \boldsymbol{x}' = (\boldsymbol{n} \cdot \boldsymbol{x}')' - \boldsymbol{n} \cdot \boldsymbol{x}'' = 0$ and project the moment balance (3.2) onto the (planar, twistless) Darboux vector $\boldsymbol{\Omega} = \boldsymbol{x}' \times \boldsymbol{x}''$ to obtain $\boldsymbol{n} \cdot \boldsymbol{x}'' = -\boldsymbol{m}' \cdot \boldsymbol{\Omega}$, making use of the orthogonality of $\boldsymbol{x}'$ and $\boldsymbol{x}''$. Integrating, we obtain a constant c,

$$\boldsymbol{n} \cdot \boldsymbol{x}' + \boldsymbol{m} \cdot \boldsymbol{\Omega} - \int ds\, \boldsymbol{m} \cdot \boldsymbol{\Omega}' = c\,. \tag{3.34}$$

For a hyperelastic rod, the contact moment can be defined as $\boldsymbol{m} = \frac{\partial \psi}{\partial \boldsymbol{\Omega}}$, which reduces (3.34) to

$$\boldsymbol{n} \cdot \boldsymbol{x}' + \boldsymbol{m} \cdot \boldsymbol{\Omega} - \psi = c\,. \tag{3.35}$$

The conserved quantity c is the "material force"[2] [34, 25], which for a purely mechanical system corresponds to a symmetry with respect to constant shifts in the material coordinate s [41, 37, 24]. It is the same quantity that occurs in the material momentum jump condition (3.6). In general, a corresponding bulk balance law for this quantity can also be derived, but for purely mechanical systems it simply reduces to the tangential projection of the conservation law for linear momentum. For the present purposes, the interpretation of c is irrelevant.

With the conserved quantities $\mathbf{P}$ and c in hand, the solution of partial contact problems is simple and direct, as will be shown presently.

3.4.1 A sleeve problem

Consider the arrangement in Figure 3.1, a problem studied by Bigoni and co-workers in [1]. A rod is bilaterally constrained by a straight, frictionless sleeve for $s \in [0, s_0]$, while two forces $\mathbf{S}$ and $\mathbf{P}$ are respectively applied to the ends at $s = 0$ and $s = L$, with $\mathbf{S}$ being parallel to the sleeve and rod. We will treat the system as purely mechanical by ignoring entropic terms, and assuming no dissipation at the sleeve edge, $\mathcal{D} = 0$. With these assumptions, the free and internal energies are indistinguishable, and the results on this problem do not rely on the theory presented in Section 3.3.2.

The rod is an *elastica* with uniform bending modulus B and the constitutive prescription

$$\psi = \hat{\psi} = \tfrac{1}{2} B \boldsymbol{x}'' \cdot \boldsymbol{x}'' = \tfrac{\boldsymbol{m} \cdot \boldsymbol{m}}{2B}\,, \quad \boldsymbol{m} = B\boldsymbol{\Omega}\,, \tag{3.36}$$

so that condition (3.30) is identically satisfied, and expression (3.35) becomes

$$\boldsymbol{n} \cdot \boldsymbol{x}' + \tfrac{\boldsymbol{m} \cdot \boldsymbol{m}}{2B} = c\,. \tag{3.37}$$

[2] O'Reilly [34, 25] uses terms which can be easily translated into ours as $\mathsf{C} = -c$ and $\mathsf{B} = -Y$.

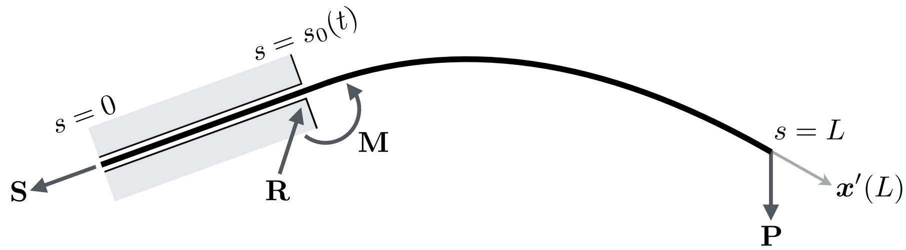

Figure 3.1: A rod $\boldsymbol{x}(s,t)$ partially constrained by a frictionless sleeve and subject to end loads $\mathbf{S}$ and $\mathbf{P}$. A reaction force $\mathbf{R}$ and moment $\mathbf{M}$ are present at the sleeve edge $s = s_0(t)$.

As shown in [1], the equilibrium relationship between the loads turns out to be $-\mathbf{S}\cdot\boldsymbol{x}'(0) = \mathbf{P}\cdot\boldsymbol{x}'(L)$. From our perspective, we can immediately see that this looks like a statement about conservation of the material force or the tangential projection of linear momentum, but it's not apparent that this should hold in the presence of a reaction force $\mathbf{R}$ and a reaction moment $\mathbf{M}$ at the sleeve edge. Indeed, one might be tempted to assume that a frictionless sleeve could only exert a reaction force $\mathbf{R}$ without any tangential component, but it turns out that this is not the case, and the magnitude of $\mathbf{M}$ and the tangential component of $\mathbf{R}$ are intimately related.

The rather idealized assumptions we are using are that the constrained part of the rod is perfectly straight and the sleeve perfectly frictionless, with no projection of external forces on the rod tangent. Thus, $\boldsymbol{m}^- = \mathbf{0}$, and (3.37) tells us that $\boldsymbol{n}^-\cdot\boldsymbol{x}'(s_0) = -\mathbf{S}\cdot\boldsymbol{x}'(0)$. We also note that any forces on the rod in the sleeve must be equal and opposite normal forces, and thus the force balance (3.1) tells us that $\boldsymbol{n}^- = -\mathbf{S}$ and $\boldsymbol{n}^+ = \mathbf{P}$. The force and moment jump conditions (3.3) and (3.4) at the sleeve edge can be written as

$$\mathbf{R} + \mathbf{P} + \mathbf{S} = \mathbf{0}\,, \tag{3.38}$$

$$\mathbf{M} + \boldsymbol{m}^+ = \mathbf{0}\,. \tag{3.39}$$

Equation (3.38) is also a global force balance. Using (3.37) on the free portion of the rod yields $\mathbf{P}\cdot\boldsymbol{x}'(L) + \frac{\boldsymbol{m}(L)\cdot\boldsymbol{m}(L)}{2B} = \mathbf{P}\cdot\boldsymbol{x}'(s_0) + \frac{\boldsymbol{m}^+\cdot\boldsymbol{m}^+}{2B}$, which in the current absence of an applied end moment is just

$$\mathbf{P}\cdot\boldsymbol{x}'(L) = \mathbf{P}\cdot\boldsymbol{x}'(s_0) + \tfrac{\boldsymbol{m}^+\cdot\boldsymbol{m}^+}{2B}\,. \tag{3.40}$$

Projecting (3.38) on the tangent $\boldsymbol{x}'(s_0)$ at the sleeve edge, and using (3.40) and (3.39), we obtain the relation

$$-\mathbf{S}\cdot\boldsymbol{x}'(0) - \mathbf{P}\cdot\boldsymbol{x}'(L) = \mathbf{R}\cdot\boldsymbol{x}'(s_0) - \tfrac{\mathbf{M}\cdot\mathbf{M}}{2B}\,. \tag{3.41}$$

This tangential force balance has been arranged so as to place the unknown reaction terms on the right hand side. We now obtain further information about these terms by employing the energy balance. By the continuity of tangents, and the constitutive prescriptions (3.36) for the *elastica*, the right hand side of (3.21) vanishes, as it should since no additional mechanism to inject or absorb power is envisioned here. If, in addition, we assume that the *net* power input across the sleeve edge vanishes, $\tilde{E} = 0$, then the expression (3.22) reduces to

$$W = \mathbf{R} \cdot \{\dot{\boldsymbol{x}}\} + \mathbf{M} \cdot \{\boldsymbol{\omega}\} = 0\,, \tag{3.42}$$

which has the transparent mechanical meaning of requiring the sleeve *edge* to exert a powerless system of reactions. The non-material point representing the edge neither translates nor rotates ($\boldsymbol{v}_0 = \mathbf{0}$ and $\boldsymbol{\omega}_0 = \mathbf{0}$), and the curvature inside the sleeve vanishes ($\boldsymbol{x}''^{-} = \mathbf{0}$), so the kinematics (3.10) and (3.11) tell us that $\{\dot{\boldsymbol{x}}\} = -\dot{s}_0 \boldsymbol{x}'(s_0)$ and $\{\boldsymbol{\omega}\} = -\dot{s}_0 \frac{1}{2}\mathbf{\Omega}^{+}$. These, in conjunction with the constitutive prescription for the moment (3.36) and the moment jump (3.39), transform (3.42) into

$$\dot{s}_0\big(\mathbf{R} \cdot \boldsymbol{x}'(s_0) - \tfrac{\mathbf{M}\cdot\mathbf{M}}{2B}\big) = 0\,. \tag{3.43}$$

Thus, for any quasistatic motion,

$$\mathbf{R} \cdot \boldsymbol{x}'(s_0) = \tfrac{\mathbf{M}\cdot\mathbf{M}}{2B}\,, \tag{3.44}$$

and (3.41) reduces to

$$-\mathbf{S} \cdot \boldsymbol{x}'(0) = \mathbf{P} \cdot \boldsymbol{x}'(L)\,. \tag{3.45}$$

The relations (3.44) and (3.45) are the results stated in equations (2.13) and (1.1) of [1].

We have chosen the solution that links smoothly with the fully static case. Other solutions may exist if $\dot{s}_0$ vanishes. If the rod is clamped instead of free to slide in a sleeve, the problem becomes that of a cantilever beam with the boundary conditions $\mathbf{R} = -\mathbf{P}$ and $\mathbf{M} = -[\boldsymbol{x}(L) - \boldsymbol{x}(s_0)] \times \mathbf{P}$.

3.4.2 A peeling problem

Consider the arrangement in Figure 3.2, a much-studied problem [2, 3, 19, 15, 16, 18, 4]. A rod of uniform mass density is constrained by a flat, adhesive surface for $s \in [0, s_0]$. Our interest is in the natural boundary condition for quasistatic peeling that relates bending and adhesion energies at the discontinuity.

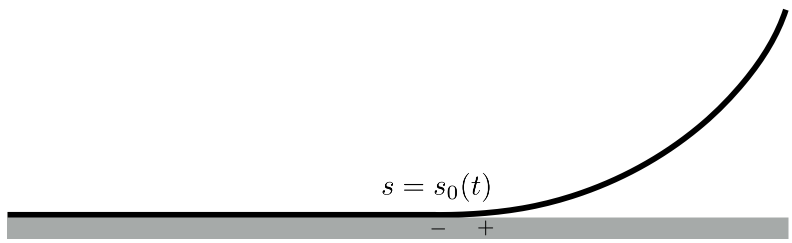

Figure 3.2: A rod $\boldsymbol{x}(s,t)$ partially constrained by an adhesive surface. Reactions at the contact discontinuity $s = s_0(t)$ may be present, but are not shown.

Unlike the sleeve problem, the peeling problem we study here entails irreversible processes, and entropy and dissipation can no longer be neglected. We first need an elementary model to justify a simple constitutive expression for η. One way of envisioning entropy in this problem is by imagining the adhesive as consisting of strong filamentous bonds with the substrate which can live in only two states: tight and loose. Tight bonds have no entropy, they cannot explore the space. Loose bonds have entropy, as they can take different shapes. Thus, the adhered rod has $\eta = 0$, whereas the detached rod has a nonzero positive entropy $\eta > 0$. Since no resistive forces hamper the detached portion of rod, the jump in entropy may only result from an entropy injection at the peeling singularity. Therefore, we make the following prescription for entropy,

$$\eta = \eta_c \,, \tag{3.46}$$

which, when combined with the constitutive prescription

$$\psi = \hat{\psi} + \psi_c = \tfrac{\boldsymbol{m}\cdot\boldsymbol{m}}{2B} + \psi_c \,, \quad \boldsymbol{m} = B\boldsymbol{\Omega} \,, \tag{3.47}$$

reduces equation (3.31) to

$$E = -\dot{s}_0 \left[\!\left[\varepsilon_c \right]\!\right] \,. \tag{3.48}$$

This relation describes E as the rate of change of internal energy at the singularity. The quantity $[\![\varepsilon_c]\!]$ will soon be related to the adhesive energy. If we still enforce $\tilde{E} = 0$, the balance of singular working (3.22) becomes

$$W = \mathbf{R} \cdot \{\dot{\boldsymbol{x}}\} + \mathbf{M} \cdot \{\boldsymbol{\omega}\} = \dot{s}_0 \left[\!\left[\varepsilon_c \right]\!\right] \,. \tag{3.49}$$

The adhered region is stationary ($\dot{\boldsymbol{x}}(s_0) = \mathbf{0}$), with vanishing curvature ($\boldsymbol{\Omega}^- = \mathbf{0}$), so the kinematics (3.10) and (3.11) tell us that the velocity of the non-material peeling point $\boldsymbol{v}_0 = \dot{s}_0 \boldsymbol{x}'(s_0)$ and the average material angular velocity $\{\boldsymbol{\omega}\} = -\dot{s}_0 \frac{1}{2} \boldsymbol{\Omega}^+$. We now identify the term $[\![\varepsilon_c]\!] = 0 - (-\Gamma)$, where Γ is a positive number representing the per length energy of adhesion between rod and surface. Thus, from (3.48), we see that E is a positive quantity, indicating just sufficient energy injection at the peeling front to break the adhesive bond. Finally, using the singular moment balance (3.4) with $\boldsymbol{m}^- = 0$, we transform (3.49) into

$$\dot{s}_0\left(\Gamma - \tfrac{\mathbf{M}\cdot\mathbf{M}}{2B}\right) = 0\,. \tag{3.50}$$

Thus, for any quasi-static motion,

$$\Gamma = \tfrac{\mathbf{M}\cdot\mathbf{M}}{2B}\,. \tag{3.51}$$

The reaction moment vanishes in the absence of adhesion energy. Furthermore, the adhesion energy Γ is subject to the constraint (3.30), which may be written as

$$\Gamma \leq [\![\theta\eta_c]\!]\ . \tag{3.52}$$

The relation (3.51) is well known in the literature, where it is derived in various ways. Similar results can be derived for plates. Majidi and Adams [5] showed that the effect of adhesion in partial contact of an elastic plate is equivalent to a discontinuity in the internal moment at the contact boundary. Hure and Audoly [22] derive the boundary condition at a two-dimensional elastocapillary peeling front; their equation (49g) is equivalent to our (3.51). An informal approach equivalent to the present one applied to the peeling of a flexible adhesive tape may be found in Appendix C of a prior paper by two of the present authors [42].

3.5 Comparison with material force balance

In the preceding section, we derived results for two partial constraint problems for an *elastica* using singular balances of force, moment, and energy (3.3)-(3.5), and some additional kinematic information. In the present section, we discuss an alternate framework in which a singular balance of material force (3.6) is used in lieu of the energy balance (3.5). O'Reilly proposed such a balance law for rods [34], compared it with the energy balance law [36], and employed it in a number of settings including that of Bigoni's sleeve problems [25, 29]. What this approach must necessarily entail is a prescription for the source term Y instead of the power $\tilde{E}$.

For the sleeve problem, $Y = 0$, entropic terms are neglected, and the two conditions (3.5) and (3.6) provide equivalent information. Using the constitutive prescription (3.36), the material jump (3.6) can be written as

$$Y + \left[\!\!\left[\boldsymbol{n}\cdot\boldsymbol{x}' + \tfrac{\boldsymbol{m}\cdot\boldsymbol{m}}{2B}\right]\!\!\right] = 0\,. \tag{3.53}$$

The quantity inside the brackets is the conserved material force c. The conservation of c implies $c^- = -\mathbf{S}\cdot\boldsymbol{x}'(0)$ and $c^+ = \mathbf{P}\cdot\boldsymbol{x}(L)$, so that setting $Y = 0$ recovers the result (3.45).

This appears to be the shortest possible derivation of this result. Additionally, using the force and moment jumps (3.3) and (3.4), we can rewrite the material jump as

$$Y - \mathbf{R} \cdot \{\boldsymbol{x}'\} - \mathbf{M} \cdot \{\boldsymbol{\Omega}\} = 0\,, \tag{3.54}$$

and use constitutive relations and the kinematic information $\{\boldsymbol{x}'\} = \boldsymbol{x}'(s_0)$ and $\{\boldsymbol{\Omega}\} = \frac{1}{2}\boldsymbol{\Omega}^+$ to recover the other result (3.44) when $Y = 0$. Another manipulation is possible when $\boldsymbol{v}_0 = \mathbf{0}$ and $\boldsymbol{\omega}_0 = \mathbf{0}$, as in the sleeve problem. Using (3.10) and (3.11), the energy jump in the form (3.21) can be rewritten $\tilde{E} + \dot{s}_0\mathbf{R} \cdot \{\boldsymbol{x}'\} + \dot{s}_0\mathbf{M} \cdot \{\boldsymbol{\Omega}\} = 0$, and the identification $\tilde{E} = -Y\dot{s}_0$ made by comparison with (3.54).

However, our course of action is no longer clear when we consider the peeling problem in the presence of an adhesive energy. Majidi and co-workers [10, 30, 4] treated the peeling problem as purely mechanical, so that a distinction between free and internal energy did not arise, and accounted for the jump in adhesion energy in the source term Y. This assumption makes (3.53) and (3.54) valid for their treatment of the peeling problem as well. They prescribe $Y = -\Gamma$, which when inserted into (3.54) gives $\Gamma = \frac{\mathbf{M}\cdot\mathbf{M}}{2B} - \mathbf{R} \cdot \boldsymbol{x}'(s_0)$. This differs from the correct result (3.51) by an additional term $-\mathbf{R} \cdot \boldsymbol{x}'(s_0)$. Recognizing this, these authors assume the additional term to vanish. Note that in order to obtain the reactions in this problem, one needs to know the contact force on the adhered side of the singularity, which requires some knowledge or assumptions about the behavior of the adhered portion of the rod. There is an alternate prescription for the material source term Y that would deliver the right boundary condition, namely $Y = -\Gamma + \mathbf{R} \cdot \boldsymbol{x}'(s_0)$, but we see no physical argument that would lead us to this prescription other than our foreknowledge of what the answer should be.[3] Using our energy balance approach, the phantom term simply does not arise.

3.6 Concluding comments

The problems we have been discussing involve either the real or virtual motion of an internal boundary through a material. The energy and material force balances are connected with symmetries in time and material coordinates, so there is a close relationship between them in problems of this type. The role of kinematic jump conditions in an energy balance approach is taken by compatibility conditions on boundary variations in a "configurational" approach.

These problems belong to the general class of free boundary problems [43]. Unlike in the bulk region, the energy balance across such free boundaries is not redundant, but provides additional information about the system once some constitutive assumption regarding the net energy input at singularities has been made. In the present cases, this assumption was the simple choice $\tilde{E} = 0$, indicating in one case that the sleeve edge exerts a powerless system of reactions, and in the other that the peeling front injects exactly that energy which breaks

[3] Equation (34) of [34] provides a relation between singular sources such that, given our prescription for $\tilde{E}$, the correct prescription for Y can be obtained. We thank O. M. O'Reilly for suggesting this approach.

the adhesive bond between rod and surface. Alternatively, instead of the energy balance, the balance of material momentum and a constitutive assumption about material momentum injection can be used [29].

Both problems treated here illuminate our quasistatic theory for singular rods. Extension of the theory to the fully dynamic case is called for. We anticipate that the dynamical counterpart of the reduced singular energy balance (3.21) will become an evolution law for the singularity. Another interesting extension to the present topics involves smooth inhomogeneities in material properties in the bulk, as for example in recent work on variable-property rods in variable-curvature channels [12, 13].

Acknowledgments

JAH and HS were supported by U.S. National Science Foundation grant CMMI-1462501. EGV acknowledges the kind hospitality of the Oxford Centre for Nonlinear PDE, where part of this work was done while he was visiting the Mathematical Institute at the University of Oxford.

Bibliography

[1] D. Bigoni, F. Dal Corso, F. Bosi, and D. Misseroni. Eshelby-like forces acting on elastic structures: Theoretical and experimental proof. *Mechanics of Materials*, 80:368–374, 2015.

[2] W. J. Bottega. Peeling and bond-point propagation in a self-adhered elastica. *Quarterly Journal of Mechanics and Applied Mathematics*, 44:17–33, 1990.

[3] A. J. Kinloch, C. C. Lau, and J. G. Williams. The peeling of flexible laminates. *International Journal of Fracture*, 66:45–70, 1994.

[4] C. Majidi. Remarks on formulating an adhesion problem using Euler's elastica (draft). *Mechanics Research Communications*, 34:85–90, 2007.

[5] C. Majidi and G. G. Adams. A simplified formulation of adhesion problems with elastic plates. *Proceedings of the Royal Society A*, 465:2217–2230, 2009.

[6] A. Humer and H. Irschik. Large deformation and stability of an extensible elastica with an unknown length. *International Journal of Solids and Structures*, 48:1301–1310, 2011.

[7] B. Phungpaingam, C. Athisakul, and S. Chucheepsakul. Instability of variable-arc-length elastica subjected to end moment. *The IES Journal Part A: Civil & Structural Engineering*, 5:85–89, 2012.

[8] L. Mansfield and J. G. Simmonds. The reverse spaghetti problem: Drooping motion of an elastica issuing from a horizontal guide. *Journal of Applied Mechanics*, 54:147–150, 1987.

[9] J. Stolte and R. C. Benson. An extending dynamic elastica: Impact with a surface. *Journal of Vibration and Acoustics*, 115:308–313, 1993.

[10] C. Majidi, O. M. O'Reilly, and J. A. Williams. On the stability of a rod adhering to a rigid surface: Shear-induced stable adhesion and the instability of peeling. *Journal of the Mechanics and Physics of Solids*, 60:827–843, 2012.

[11] A. DeSimone, P. Gidoni, and G. Noselli. Liquid crystal elastomer strips as soft crawlers. *Journal of the Mechanics and Physics of Solids*, 84:254–272, 2015.

[12] G. Cicconofri and A. DeSimone. A study of snake-like locomotion through the analysis of a flexible robot model. *Proceedings of the Royal Society A*, 471:20150054, 2015.

[13] F. Dal Corso, D. Misseroni, N. M. Pugno, A. B. Movchan, N. V. Movchan, and D. Bigoni. Serpentine locomotion through elastic energy release. *Journal of the Royal Society Interface*, 14:20170055, 2017.

[14] ASTM D6195: Standard test methods for loop tack. ASTM International, West Conshohocken, PA.

[15] R. Plaut, A. J. Dalrymple, and D. A. Dillard. Effect of work of adhesion on contact of an elastica with a flat surface. *Journal of Adhesion Science and Technology*, 15:565–581, 2001.

[16] R. Plaut, N. L. Williams, and D. A. Dillard. Elastic analysis of the loop tack test for pressure sensitive adhesives. *Journal of Adhesion*, 76:37–53, 2001.

[17] M. Deserno, M. M. Müller, and J. Guven. Contact lines for fluid surface adhesion. *Physical Review E*, 76:011605, 2007.

[18] N. J. Glassmaker and C. Y. Hui. Elastica solution for a nanotube formed by self-adhesion of a folded thin film. *Journal of Applied Physics*, 96:3429–3434, 2004.

[19] M. P. deBoer and T. A. Michalske. Accurate method for determining adhesion of cantilever beams. *Journal of Applied Physics*, 86:817–827, 1999.

[20] A. Pamp and G. G. Adams. Deformation of bowed silicon chips due to adhesion and applied pressure. *Journal of Adhesion Science and Technology*, 21:1021–1043, 2007.

[21] C. Py, P. Reverdy, L. Doppler, J. Bico, B. Roman, and C. N. Baroud. Capillarity induced folding of elastic sheets. *European Physical Journal Special Topics*, 166:67–71, 2009.

[22] J. Hure and B. Audoly. Capillary buckling of a thin film adhering to a sphere. *Journal of the Mechanics and Physics of Solids*, 61:450–471, 2013.

[23] H. Elettro, P. Grandgeorge, and S. Neukirch. Elastocapillary coiling of an elastic rod inside a drop. *Journal of Elasticity*, 127:235–247, 2017.

[24] H. Singh and J. A. Hanna. On the planar *elastica*, stress, and material stress. [arXiv:1706.03047].

[25] O. M. O'Reilly. Some perspectives on Eshelby-like forces in the elastica arm scale. *Proceedings of the Royal Society A*, 471:20140785, 2015.

[26] R. Kienzler and G. Herrmann. On material forces in elementary beam theory. *Journal of Applied Mechanics*, 53:561–564, 1986.

[27] R. Kienzler and G. Herrmann. *Mechanics in Material Space.* Springer, Berlin, 2000.

[28] G. A. Maugin. *Configurational Forces.* CRC Press, Boca Raton, 2011.

[29] O. M. O'Reilly. *Modeling Nonlinear Problems in the Mechanics of Strings and Rods.* Springer, New York, 2017.

[30] C. Majidi, O. M. O'Reilly, and J. A. Williams. Bifurcations and instability in the adhesion of intrinsically curved rods. *Mechanics Research Communications*, 49:13–16, 2013.

[31] S. S. Antman. *Nonlinear problems of elasticity.* Springer, New York, 1995;.

[32] A. E. Green and P. M. Naghdi. A derivation of jump condition for entropy in thermomechanics. *Journal of Elasticity*, 8(2):179–182, 1978.

[33] O. M. O'Reilly and P. C. Varadi. A treatment of shocks in one-dimensional thermomechanical media. *Continuum Mechanics and Thermodynamics*, 11:339–352, 1999.

[34] O. M. O'Reilly. A material momentum balance law for rods. *Journal of Elasticity*, 86:155–172, 2007.

[35] J. A. Hanna. Jump conditions for strings and sheets from an action principle. *International Journal of Solids and Structures*, 62:239–247, 2015.

[36] O. M. O'Reilly. The energy jump condition for thermomechanical media in the presence of configurational forces. *Continuum Mechanics and Thermodynamics*, 18:361–365, 2007.

[37] J. H. Maddocks and D. J. Dichmann. Conservation laws in the dynamics of rods. *Journal of Elasticity*, 34:83–96, 1994.

[38] W. Noll. La mécanique classique, basée sur un axiome d'objectivité. In *La Méthode Axiomatique dans les Mécaniques Classiques et Nouvelles*, Colloque International à Paris, 1959, pages 47–56. Gauthier-Villars, Paris, 1963. Reprinted in [44].

[39] R. Abeyaratne and J. K. Knowles. On the driving traction acting on a surface of strain discontinuity in a continuum. *Journal of the Mechanics and Physics of Solids*, 38:345–360, 1990.

[40] R. Abeyaratne and J. K. Knowles. Kinetic relations and the propagation of phase boundaries in solids. *Archive for Rational Mechanics and Analysis*, 114:119–154, 1991.

[41] L. J. F. Broer. On the dynamics of strings. *Journal of Engineering Mathematics*, 4:195–202, 1970.

[42] H. Singh and J.A. Hanna. Pick-up and impact of flexible bodies. *Journal of the Mechanics and Physics of Solids*, 106:46–59, 2017.

[43] R. Burridge and J. B. Keller. Peeling, slipping and cracking–some one-dimensional free-boundary problems in mechanics. *SIAM Review*, 20:31–61, 1978.

[44] W. Noll. *The Foundations of Mechanics and Thermodynamics.* Springer-Verlag, New York, 1974.

Chapter 4

On the planar *elastica*, stress, and material stress

This is a post-peer-review, pre-copyedit version of an article published in *Journal of Elasticity*. The final authenticated version is available online at: `http://dx.doi.org/10.1007/s10659-018-9690-5`.

Attribution

The work presented in this chapter was done in collaboration with J. A. Hanna, who conceived the main idea and contributed towards the development and implementation of this work.

Abstract

We revisit the classical problem of the planar Euler *elastica* with applied forces and moments, and present a classification of the shapes in terms of tangentially conserved quantities associated with spatial and material symmetries. We compare commonly used director, variational, and dynamical systems representations, and present several illustrative physical examples. We remark that an approach that employs only the shape equation for the tangential angle obscures physical information about the tension in the body.

4.1 Introduction

The planar Euler *elastica* [1] has the simplest curvature-dependent energy for a one-dimensional continuum. As a result, it appears in idealized models of a variety of physical systems, from structural cables and thin beams to biological macromolecules [2]. This breadth of application has given rise to numerous derivations of the governing equations across the scientific and engineering literature, often with specific boundary conditions imposed *a priori*. Some of these derivations are quite lengthy, and may give the student of elastic structures the impression that the system is far more complicated than it really is. The possibility of end moments is often ignored in expository treatments, and we know of no place where any two of the three most common approaches to the *elastica* are presented together, or the connections between them laid out. We suspect that much of this knowledge is well known among practitioners of mechanics, but feel that it is worth collecting in one place, alongside new results and perspectives.

In this note, we attempt to clarify some of these folkloric issues, as well as present a reclassification of the *elastica* curves in terms of quantities conserved under translations in material and physical space. The curves represent thin elastic bodies with material symmetry along their long axes, embedded in a symmetric Euclidean background. The associated material momentum and conventional (spatial) linear momentum balances express the conservation of material and conventional stresses. We will refer to the latter simply as the stress in what follows.[1] The relative magnitudes and sign of these quantities identify a mother curve from which a particular *elastica* may be cut in order to satisfy end moment conditions. The simplicity of the system is apparent when described in terms of conserved quantities, with angles or curvatures as dependent variables. Connections between forces, moments, and slopes are also relatively clean in this general viewpoint. Cumbersome expressions arise only when conditions on total length and boundary positions are imposed on particular solutions. In practice, these conditions are satisfied numerically. We do not dwell on these issues here, nor do we retread well known but sometimes impractical expressions for the shapes involving elliptic functions.

There are three widely used approaches to the planar *elastica*. Classical continuum mechanics employs Cosserat directors and begins with balances of linear and angular momentum [13], which can be combined into a single vector equation. The other two are variational treatments leading either to a single scalar pendulum equation for the tangential angle, or a

[1] Material forces and closely related quantities, which in our one-dimensional system have just one component, appear under many names in the literature, including Eshelbian force, quasimomentum, pseudomomentum, (Kelvin) impulse, and configurational force [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In the present case, one can derive everything from consideration of conventional force balance and its projection onto the tangents of the body, but the concept of material force is useful as a descriptive term, and also corresponds to an important symmetry of the Lagrangian. We generally prefer the term “spatial” to “conventional”, but try to avoid it in this note because of its alternative meaning as “non-planar” in the context of rods embedded in three dimensions rather than two.

single vector equation from which one can obtain a scalar equation for the curvature as the position of a particle in either a single- or double-well potential. These scalar shape equations represent the normal force balance on the body. In this note, we will relate the natural quantities in the vector and shape equations. In variational treatments, the magnitude of the stress and the only component of the material stress appear as first integrals. Though not typically expressed in these terms, in the pendulum approach it is customary to fix the stress and sweep the material stress to obtain a phase portrait. However, the other approach clearly indicates that changes in material stress lead to a pitchfork bifurcation between single- and double-well potentials. We show that this has physical meaning. Overall, it can be said that the pendulum description fails to capture information about tangential balance that can at times provide a more direct solution of physical problems involving elastic curves. Curiously, angular momentum balance and its associated conserved torque do not play any necessary role in this simple system, although the pendulum equation can be interpreted as a moment balance rather than a normal force balance.

We begin with a variational derivation of the vector equation and boundary conditions in Section 4.2, including a discussion of conserved quantities and classical rod theories in Section 4.2.1 and of the shape equation for the curvature in Section 4.2.2. This shape equation can be derived from the conserved stress and material stress, or by manipulation of the Euler-Lagrange equation. The variational scalar pendulum description is briefly reviewed in Section 4.3. In Section 4.4, we propose a classification scheme based on conserved quantities, and discuss phase portrait representations of both scalar shape equations. In Section 4.5, we present several examples involving physical boundary conditions on forces, moments, slopes, and the length of the elastic curves.

4.2 Vector approaches

The planar Euler *elastica* is a one-dimensional idealized model of a thin, uniform, inextensible, unshearable rod or tape without rest curvature. This body is represented by the position (embedding) vector of a planar curve $\boldsymbol{x}(s)$, where $s \in [0, l]$ is both a material coordinate and the arc length. An example is shown in Figure 4.1, decorated with various quantities to be introduced shortly. The elastic energy, given by the square of the curvature of $\boldsymbol{x}$, is expressible in several different ways. Perhaps the simplest form for the static Lagrangian is entirely in terms of derivatives of position,

$$L = \int_0^l ds\, \mathcal{L}(\boldsymbol{x}) = \int_0^l ds \left[\tfrac{1}{2} B \partial_s^2 \boldsymbol{x} \cdot \partial_s^2 \boldsymbol{x} + \tfrac{1}{2}\sigma \left(\partial_s \boldsymbol{x} \cdot \partial_s \boldsymbol{x} - 1\right)\right] , \tag{4.1}$$

where the first term is the bending energy with uniform modulus B (here with units of force), and the second is a quadratic constraint on the local length, enforced by a multiplier $\sigma(s)$. While σ contributes to the tension in the body, the full tension contains contributions from the curvature as well.[2]

Under a small shift in the position vector $\boldsymbol{x} \to \boldsymbol{x} + \delta\boldsymbol{x}$, the first order variation in the Lagrangian is

$$\delta L^{(1)} = \left[B\partial_s^2\boldsymbol{x} \cdot \partial_s\delta\boldsymbol{x} + (\sigma\partial_s\boldsymbol{x} - B\partial_s^3\boldsymbol{x}) \cdot \delta\boldsymbol{x}\right]\Big|_0^l + \int_0^l ds \left[B\partial_s^4\boldsymbol{x} - \partial_s(\sigma\partial_s\boldsymbol{x})\right] \cdot \delta\boldsymbol{x}\,. \tag{4.2}$$

Setting $\delta L^{(1)} = 0$ for arbitrary $\delta\boldsymbol{x}$ and $\partial_s\delta\boldsymbol{x}$ yields the bulk field equation

$$B\partial_s^4\boldsymbol{x} - \partial_s(\sigma\partial_s\boldsymbol{x}) = 0\,, \tag{4.3}$$

and boundary conditions of the form

$$\sigma\partial_s\boldsymbol{x} - B\partial_s^3\boldsymbol{x} = \mathbf{P} \quad \text{at } s = l\,, \tag{4.4}$$

$$B\partial_s^2\boldsymbol{x} = \mathbf{Q} \quad \text{at } s = l\,, \tag{4.5}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are per-area sources of force and torque at one boundary. Boundary conditions at the other end can be written analogously. While an equal but opposite force must balance $\mathbf{P}$, the torques at the two boundaries can differ. The force $\mathbf{P}$ could also be included in the Lagrangian using a term $-\mathbf{P}\cdot\boldsymbol{x}\,|_0^l = -\mathbf{P}\cdot\mathbf{R}$, as is done by several authors [18, 19, 20, 17]. The quantities in the boundary condition (4.5) live in the plane, but it is customary to write things in terms of the out-of-plane moment $\mathbf{M} = \partial_s\boldsymbol{x} \times \mathbf{Q}$ applied to the boundary. Applying the cross product with the tangent to (4.5), we may write an equivalent boundary condition

$$\partial_s\boldsymbol{x} \times B\partial_s^2\boldsymbol{x} = \mathbf{M} \quad \text{at } s = l\,. \tag{4.6}$$

Should we wish to obtain this boundary condition directly, one option is to write the bending energy in (4.1) as $\frac{1}{2}B\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}$, where $\boldsymbol{\Omega} = \partial_s\boldsymbol{x} \times \partial_s^2\boldsymbol{x}$ is the Darboux vector for a planar curve. Varying the position vector leads to the same variation of the Lagrangian (4.2), except that the first boundary term becomes $(\partial_s\boldsymbol{x} \times B\partial_s^2\boldsymbol{x}) \cdot (\partial_s\boldsymbol{x} \times \partial_s\delta\boldsymbol{x})$. We may then treat $\delta\boldsymbol{x}$ and $\partial_s\boldsymbol{x}\times\partial_s\delta\boldsymbol{x}$ as our two arbitrary variations yielding two boundary conditions. There are many ways to express the square of the curvature in this simple system. Another is to square the wryness, defined as $\boldsymbol{d} \times \nabla\boldsymbol{d}$, where $\boldsymbol{d}$ is the director, here identified with the unit normal.

We may connect our equations and boundary conditions with classical rod theory [13] by identifying the (per-area) contact force $\boldsymbol{n}(s)$ and contact moment $\boldsymbol{m}(s)$ as

$$\boldsymbol{n} \equiv \sigma\partial_s\boldsymbol{x} - B\partial_s^3\boldsymbol{x}\,, \tag{4.7}$$

$$\boldsymbol{m} \equiv \partial_s\boldsymbol{x} \times B\partial_s^2\boldsymbol{x}\,. \tag{4.8}$$

[2] Similar multipliers can be found in Burchard and Thomas [14], Singer [15], Tornberg and Shelley [16], who do not employ a variational derivation, and Guven and Vázquez-Montejo [17], who use multiple redundant multipliers. The multipliers in [14] and [16] are misidentified as the tension. For seemingly similar but qualitatively different multipliers, see the following footnote.

The contact force[3] can be identified as the term conjugate to the variation $\delta\boldsymbol{x}$ in the boundary term in (4.2), while the contact moment is conjugate to $\partial_s\boldsymbol{x}\times\partial_s\delta\boldsymbol{x}$ in the Darboux variation described above.

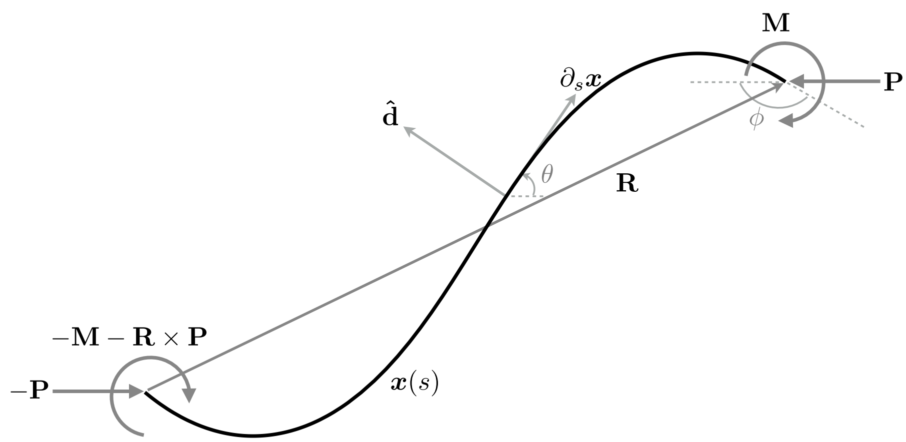

Figure 4.1: An elastic curve $\boldsymbol{x}(s)$ with end-to-end vector $\mathbf{R}$ and local frame of unit tangent $\partial_s\boldsymbol{x}$ and unit normal $\hat{\mathbf{d}}$. Also shown are a terminal force $\mathbf{P}$ and moment $\mathbf{M}$, the corresponding force and moment at the other end, the local tangential angle θ between $\partial_s\boldsymbol{x}$ and $\mathbf{P}$, and the angle ϕ between a terminal tangent and $\mathbf{P}$.

4.2.1 Symmetries

The Lagrangian (4.1) does not depend on position and orientation in space or on the choice of origin for the coordinate s. These translational and rotational symmetries of space and material symmetry of the body imply the existence of conserved quantities.

For a shift in the embedding vector $\boldsymbol{x}$ corresponding to a symmetry of the Euclidean plane, the current can be read off from the boundary term in (4.2). Translations correspond to a uniform $\delta\boldsymbol{x}$, so the contact force is conserved along the body,

$$\sigma\partial_s\boldsymbol{x}-B\partial_s^3\boldsymbol{x}=\boldsymbol{n}=\mathbf{P}\,. \tag{4.9}$$

[3]The tension $\boldsymbol{n}\cdot\partial_s\boldsymbol{x}$ can be associated with the multiplier T appearing in Nordgren [21] and Shelley and Ueda [22], who do not employ variational derivations, and Audoly [23], whose multiplier is different than our σ despite appearing similarly in a Lagrangian. The difference with Audoly arises due to his splitting of the variation in terms of an angular variation $\delta\theta$ on the boundary, rather than $\partial_s\delta\boldsymbol{x}$ as done here. A similar issue arises in plate and shell theories [24].

We have identified the conserved stress with $\mathbf{P}$ using the force boundary condition (4.4). That $\boldsymbol{n}$ is conserved may also be seen by integrating the bulk equation (4.3), which in terms of the contact force is simply

$$-\partial_s \boldsymbol{n} = 0\,. \tag{4.10}$$

Rotations correspond to a shift in the position vector of the form $\delta\boldsymbol{x} = \mathbf{C} \times \boldsymbol{x}$, where $\mathbf{C}$ is a constant vector. The boundary term yields the conserved torque,

$$\partial_s \boldsymbol{x} \times (B\partial_s^2 \boldsymbol{x}) + \boldsymbol{x} \times (\sigma \partial_s \boldsymbol{x} - B\partial_s^3 \boldsymbol{x}) = \boldsymbol{m} + \boldsymbol{x} \times \boldsymbol{n} = \mathbf{J}\,. \tag{4.11}$$

Unlike the contact force $\boldsymbol{n}$, the contact moment $\boldsymbol{m}$ is not conserved. Differentiating (4.11) with respect to s, and using (4.10), we obtain the bulk moment balance

$$\partial_s \boldsymbol{m} + \partial_s \boldsymbol{x} \times \boldsymbol{n} = 0\,. \tag{4.12}$$

For our *elastica*, it can be seen from (4.7) and (4.8) that $\partial_s \boldsymbol{x} \times \boldsymbol{n} = -\partial_s \boldsymbol{x} \times B\partial_s^3 \boldsymbol{x}$ and $\partial_s \boldsymbol{m} = \partial_s \boldsymbol{x} \times B\partial_s^3 \boldsymbol{x}$, so that the moment balance (4.12) is identically satisfied, and does not provide any additional information beyond the force balance. The torque $\mathbf{J}$ can always be eliminated by a shift in the origin $\boldsymbol{x} \to \boldsymbol{x} + \frac{\boldsymbol{n}}{\boldsymbol{n}\cdot\boldsymbol{n}} \times \mathbf{J}$ by noting that $\boldsymbol{m} \cdot \boldsymbol{n} = \mathbf{J} \cdot \boldsymbol{n} = \mathbf{0}$ in our system [20].

The third symmetry corresponds to a shift in the independent variable s by a uniform δs. This transformation is associated with the uniformity of the body, and has nothing to do with the embedding space. We denote the transformed material label as $s^* \equiv s + \delta s$. The vector $\boldsymbol{x}^*(s^*)$ corresponds to the same material point as $\boldsymbol{x}(s)$ did in the original coordinates, while $\bar{\delta}\boldsymbol{x} \equiv \boldsymbol{x}^*(s) - \boldsymbol{x}(s)$ is the shift in the position vector at the same material label due to the shift δs (*i.e.* $\boldsymbol{x}^*(s) \equiv \boldsymbol{x}^*(s^*)|_{s^*=s}$ is at the same value of s, not the same material point). Note that $\bar{\delta}$ and ∂_s commute. For a detailed presentation of action principles, including variations in the independent coordinates, the reader is referred to [25, 26]. Under a small uniform shift in the material coordinate $s \to s^*$, the variation in the Lagrangian is

$$L + \delta L = \int_{s_1+\delta s}^{s_2+\delta s} ds\, \mathcal{L}(\boldsymbol{x} + \bar{\delta}\boldsymbol{x}) \tag{4.13}$$

$$= L + \mathcal{L}(\boldsymbol{x})\delta s|_{s_1}^{s_2} + \int_{s_1}^{s_2} ds \left[B\partial_s^2 \boldsymbol{x} \cdot \partial_s^2 \bar{\delta}\boldsymbol{x} + \sigma \partial_s \boldsymbol{x} \cdot \partial_s \bar{\delta}\boldsymbol{x}\right] + \text{higher order terms.} \tag{4.14}$$

Integrating by parts and extremizing by setting the first order terms to zero,

$$0 = \left[\mathcal{L}\delta s + B\partial_s^2 \boldsymbol{x} \cdot \partial_s \bar{\delta}\boldsymbol{x} + (\sigma\partial_s \boldsymbol{x} - B\partial_s^3 \boldsymbol{x}) \cdot \bar{\delta}\boldsymbol{x}\right]\Big|_{s_1}^{s_2} + \int_{s_1}^{s_2} ds \left[B\partial_s^4 \boldsymbol{x} - \partial_s(\sigma \partial_s \boldsymbol{x})\right] \cdot \bar{\delta}\boldsymbol{x}\,. \tag{4.15}$$

For a uniform material translation δs, the current may be associated with the boundary term after using the condition $\boldsymbol{x}^*(s^*) = \boldsymbol{x}(s)$ to express the variations in the position and the tangent as $\bar{\delta}\boldsymbol{x} = -\partial_s\boldsymbol{x}\,\delta s$ and $\partial_s\bar{\delta}\boldsymbol{x} = -\partial_s^2\boldsymbol{x}\,\delta s$. Using the constitutive relations (4.7-4.8) for the contact force and moment, we obtain the only component of the conserved material force (per area),

$$-\tfrac{1}{2}B\partial_s^2\boldsymbol{x}\cdot\partial_s^2\boldsymbol{x} + \boldsymbol{m}\cdot\boldsymbol{\Omega} + \boldsymbol{n}\cdot\partial_s\boldsymbol{x} = \tfrac{\boldsymbol{m}\cdot\boldsymbol{m}}{2B} + \boldsymbol{n}\cdot\partial_s\boldsymbol{x} = c\,. \tag{4.16}$$

We have chosen to reverse the sign of the boundary term to align with the definition of stress in a more general setting that includes inertia, such that the time derivative of the momentum equals the divergence of stress, and the time derivative of the projection of the momentum onto the tangents equals the divergence of material stress— compare with equation (8) of Broer [28] and equation (3.2) of Maddocks and Dichmann [29]. O'Reilly [27, 12] defines the equivalent quantity with the opposite sign, in keeping with the conventional form for the Eshelby tensor. More general forms of c may be found in the context of static [30, 31, 32, 33] and dynamic [29] spatial hyperelastic rods. Another way to obtain this quantity is to integrate the tangential component of the force balance (4.10). Integrating $\partial_s\boldsymbol{n}\cdot\partial_s\boldsymbol{x}$ by parts and anticipating the identity of the constant, we can write $c = \boldsymbol{n}\cdot\partial_s\boldsymbol{x} - \int ds\,\boldsymbol{n}\cdot\partial_s^2\boldsymbol{x}$, which, upon inserting (4.7-4.8), simplifies to $c = \boldsymbol{n}\cdot\partial_s\boldsymbol{x} + \frac{\boldsymbol{m}\cdot\boldsymbol{m}}{2B}$.

Curiously, the conserved material stress can appear directly as a multiplier in an approach [34, 35] that employs a fully covariant description of a *geometric* energy, $\int\sqrt{g}\,d\alpha\,\left(\frac{1}{2}\nabla^2\boldsymbol{x}\cdot\nabla^2\boldsymbol{x} + c\right)$. In this formalism, common in some areas of physics, the volume form $\sqrt{g}\,d\alpha$ varies, and the energy density is to be interpreted as per present volume rather than per mass, a distinction that only matters when the rod is extensible.

4.2.2 Shape equation

The shape of a planar curve may be reconstructed from the tangential angle $\theta(s)$ or its derivative, the curvature $\kappa = \partial_s\theta$. For simplicity, we make use of an adapted orthonormal frame of tangent vector $\partial_s\boldsymbol{x}$ and normal $\hat{\mathbf{d}}$, such that $\partial_s^2\boldsymbol{x} = \kappa\hat{\mathbf{d}}$ and $\partial_s\hat{\mathbf{d}} = -\kappa\partial_s\boldsymbol{x}$. Using the definitions of the conserved quantities (4.9) and (4.16) along with the constitutive relations (4.7-4.8), we may write

$$(\sigma + B\kappa^2)\,\partial_s\boldsymbol{x} - B\partial_s\kappa\,\hat{\mathbf{d}} = \mathbf{P}\,, \tag{4.17}$$

$$\sigma + \tfrac{3}{2}B\kappa^2 = c\,. \tag{4.18}$$

Eliminating the multiplier σ and squaring (4.17) leads to the shape equation

$$(B\partial_s\kappa)^2 + \left(\tfrac{1}{2}B\kappa^2 - c\right)^2 = P^2\,, \tag{4.19}$$

where $P^2 = \mathbf{P}\cdot\mathbf{P}$. This is a restricted form of a more general equation derived by Lu and Perkins [36] for symmetric spatial Kirchhoff rods.

Alternatively, we can derive the shape equation (4.19) by projecting the bulk Euler-Lagrange equation (4.3) onto the frame to obtain two equations,

$$B\left(\partial_s^2\kappa - \kappa^3\right) - \sigma\kappa = 0\,, \tag{4.20}$$
$$-3B\kappa\partial_s\kappa - \partial_s\sigma = 0\,. \tag{4.21}$$

Integrating (4.21) and switching signs gives equation (4.18), and then inserting for σ in (4.20) gives

$$B\left(\partial_s^2\kappa + \tfrac{1}{2}\kappa^3\right) - c\kappa = 0\,. \tag{4.22}$$

Multiplying (4.22) by $2B\partial_s\kappa$, integrating, and calling the constant of integration $P^2 - c^2$ recovers the shape equation (4.19). The associated scalar boundary conditions derived from (4.4) and (4.6) are

$$\sigma + B\kappa^2 = \mathbf{P}\cdot\partial_s\boldsymbol{x} \quad \text{at } s = l\,, \tag{4.23}$$
$$-B\partial_s\kappa = \mathbf{P}\cdot\hat{\mathbf{d}} \quad \text{at } s = l\,, \tag{4.24}$$
$$B\kappa = M \quad \text{at } s = l\,, \tag{4.25}$$

where M is the only component of the moment at one boundary. Note that either (4.16), or (4.18) along with the boundary conditions (4.23) and (4.25), lead to the relation

$$c = \tfrac{M^2}{2B} + P\cos\phi\,, \tag{4.26}$$

where ϕ is the angle between $\mathbf{P}$ and the boundary tangent $\partial_s\boldsymbol{x}(l)$. Hence, the material stress can be determined easily from the magnitude and direction of the end force, and the single component of the end moment.

The curvature κ uniquely defines a planar curve up to rigid translations and rotations. There are the two parameters P and c and one additional degree of freedom corresponding to the final integration of the shape equation. However, the physical conditions one might impose on an elastic curve are related in a complicated way to these degrees of freedom. One might, for example, specify the ratio of the body's length to its end to end distance, along with the two angles made by the end tangents $\partial_s\boldsymbol{x}$ and the end to end vector $\mathbf{R}$. Or one might specify another quantity such as the moment M, or the angle ϕ between the force and the end tangent. Note that one can specify either the end position or $\mathbf{P}$, and one can specify either the end tangent or $\mathbf{M}$. The curve may be reconstructed using $\partial_s\theta = \kappa$ and $\partial_s\boldsymbol{x} = (\cos\theta, \sin\theta)$. We also have $\mathbf{R} = \boldsymbol{x}(l) - \boldsymbol{x}(0)$ and $l = s(l) - s(0) = \int_0^l ds\sqrt{\partial_s\boldsymbol{x}\cdot\partial_s\boldsymbol{x}} = \int_{\kappa(0)}^{\kappa(l)} d\tilde{\kappa}/\partial_s\tilde{\kappa}$. The significant complications that arise in applications of an *elastica* model are often due to specifications of length and position.

In Section 4.4, we will detail how the two parameters in the shape equation define shapes of infinite length, and how the moment boundary condition chooses the end point of any shape cut from such a mother curve.

4.3 Scalar approach

Before presenting our classification and examples, we briefly revisit the most well known description of the planar *elastica*, which gives rise to a pendulum equation. We may write the Lagrangian in terms of the tangential angle, rather than the position vector,

$$L(\theta) = \int_0^l ds \left[\tfrac{1}{2}B(\partial_s\theta)^2\right] + P\left[\int_0^l ds\,(\cos\theta) - R_P\right], \tag{4.27}$$

where P now appears as a global constraint on the distance R_P between the ends in the direction of the force. There is no constraint term corresponding to the direction orthogonal to the force. We have measured θ with respect to the direction of force, a quantity not always experimentally known, because it directly leads to the simplest form of the Euler-Lagrange equations.

Extremizing the Lagrangian with respect to shifts in the angle $\theta \to \theta + \delta\theta$ leads to the bulk field equation

$$B\partial_s^2\theta + P\sin\theta = 0\,, \tag{4.28}$$

and boundary conditions of the form

$$B\partial_s\theta = M \quad \text{at } s = l\,, \tag{4.29}$$

the latter equivalent to (4.25). The moment term could be included in the Lagrangian using a term $-M\theta(l)$. The pendulum equation (4.28) is equivalent to the projection of the conserved stress (4.9) onto the unit normal $\hat{\mathbf{d}}$, as well as the only component of the moment balance (4.12). Upon integration, we recover the conserved material stress,

$$\tfrac{1}{2}B(\partial_s\theta)^2 - P\cos\theta = c\,. \tag{4.30}$$

This shape equation is equivalent to the definition (4.16). With the identification of the curvature with the end moment (4.29), we recover (4.26). The pendulum equation (4.30) is a restricted form of a more general oscillator equation derived by van der Heijden and Thompson [33] for symmetric spatial Kirchhoff rods.

It is not immediately obvious that the shape equations (4.19) and (4.30) represent the same curves. However, using (4.28) and (4.30) to square and add the trigonometric terms, and recalling that $\kappa = \partial_s\theta$, we recover (4.19). We will return to the comparison between curvature and angle shape equations in Section 4.4, in the context of phase portrait representations of these equations.

4.4 Classification in terms of stress and material stress

The shapes of *elastica* curves are governed by the two parameters in the shape equation for the curvature (4.19), which under the rescalings $s \to s/\lambda$, $\kappa \to \kappa\lambda$, $c \to c\lambda^2/B$, $P \to P\lambda^2/B$, and $M \to M\lambda/B$ (λ an arbitrary length scale) takes the nondimensional form

$$(\partial_s \kappa)^2 + \left(\tfrac{1}{2}\kappa^2 - c\right)^2 = P^2 \,. \tag{4.31}$$

This equation describes the motion of a particle with position κ and energy $\frac{1}{2}P^2$ in a potential $\frac{1}{2}\left(\frac{1}{2}\kappa^2 - c\right)^2$. This potential has a pitchfork bifurcation at $c = 0$, such that for negative c there is a single minimum at $\kappa = 0$ and we require $P^2 > c^2$, and for positive c there are two wells with minima at $\kappa = \pm\sqrt{2c}$, the particle being confined to one of these wells if $P^2 < c^2$. Shapes are often classified as inflectional or non-inflectional [37]; if we define P as a positive quantity, the magnitude of the conserved stress, we can express the above relations as

$$\begin{aligned} &\frac{c}{P} \in [-\infty, -1) && \text{no solution}\,, \\ &\frac{c}{P} \in [-1, 0] && \text{inflectional single-well}\,, \\ &\frac{c}{P} \in (0, 1) && \text{inflectional double-well}\,, \\ &\frac{c}{P} \in [1, \infty] && \text{non-inflectional}\,. \end{aligned} \tag{4.32}$$

The division of the inflectional solutions into single-well and double-well type, depending on the sign of c, has a physical meaning. Recall the definition (4.16) of the material stress c. The squared moment term is strictly non-negative, so the tension $\boldsymbol{n} \cdot \partial_s \boldsymbol{x} \leq c$. When c is negative, the rod must be in compression everywhere: the tangents $\partial_s \boldsymbol{x}$ are always in opposition to $\mathbf{P}$, and the corresponding shapes can be represented by a single-valued height function above any line parallel to the direction of the force. Shapes and phase portraits corresponding to double-well and single-well potentials are shown in Figure (4.2). Other interesting landmarks include the straight line $\frac{c}{P} = -1$, the homoclinic solitary loop $\frac{c}{P} = 1$, the circle $P = 0$, and the figure-eight solution. Clearly our formulation says nothing about the global property of self-intersection, and we refrain from commenting on this matter here although it is clearly of interest in some physical problems. For example, in keeping with common experience, weaving somehow through the intersecting solutions must be a family of racket-like loops meeting at any angle.

The ratio $\frac{c}{P}$ can be simply related to parameters in elliptic integrals relating Cartesian coordinates on the curves [38, 39]. The connection with the planar restriction of a classification of Kirchhoff rods by Nizette and Goriely [32], who use the roots of a polynomial of a cosine of an Euler angle that generates the solutions through another elliptic integral, appears to be more complicated.

The physical significance of inflection points is that they are points where the moment vanishes, and so can correspond to boundaries loaded by a pure force. Hubbard [40] argues that if one end of an *elastica* is loaded like this, and the other end loaded with a force and a moment, then the curve can be extended to a fictitious curve loaded only with end forces. This is simply a consequence of the fact that an *elastica* with pure force loading

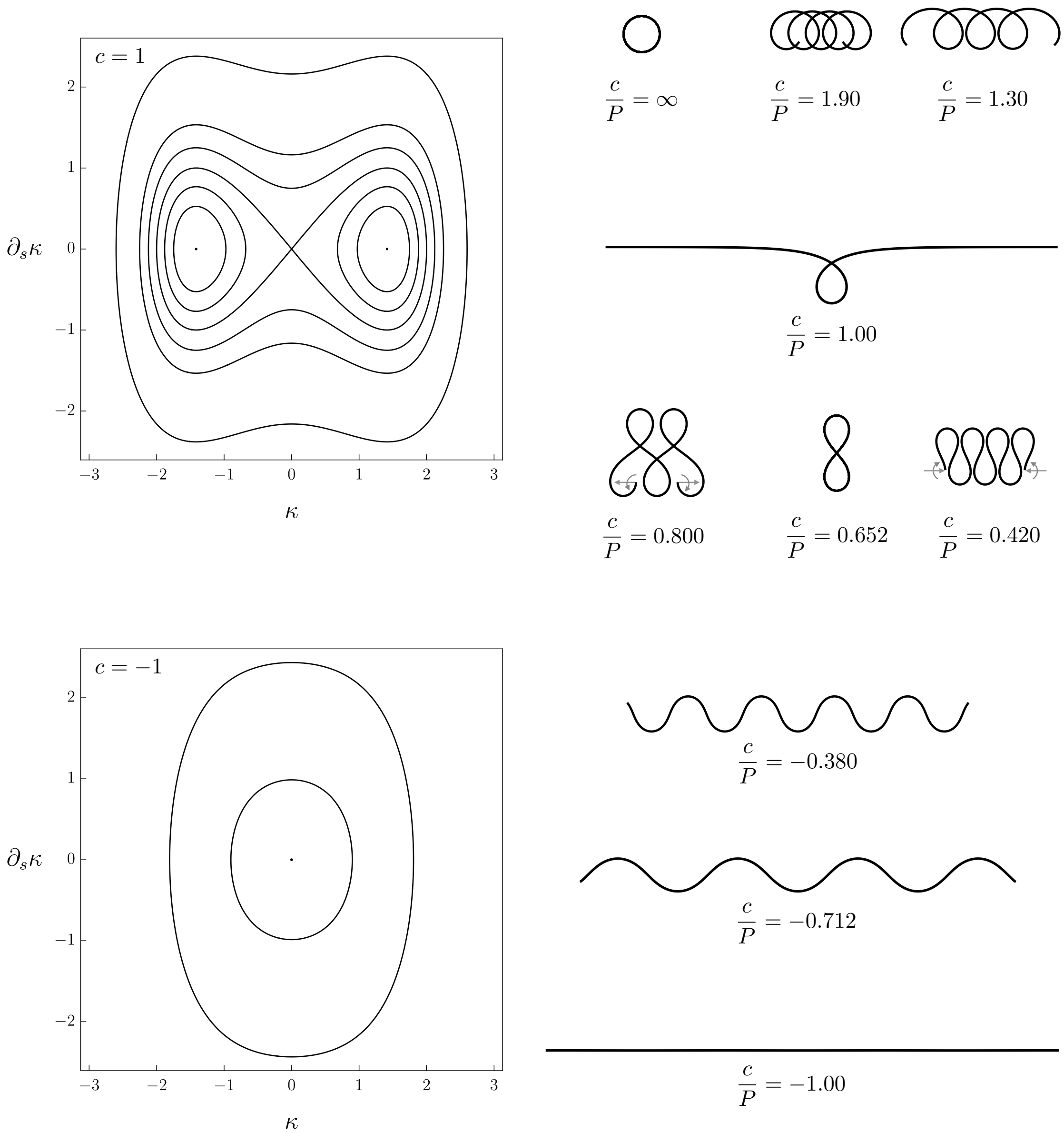

Figure 4.2: Phase portraits in κ-$\partial_s\kappa$ space and configurations of *elastica* curves, with larger contours corresponding to larger values of stress P. Arrows indicate the direction of forces for configurations in the sequence on either side of the figure-eight solution. Moments can be inferred from the curvature of the end points. At top, the material stress $c = 1$ and the potential has two wells at $(\pm\sqrt{2}, 0)$. Inside the $\frac{c}{P} = 1$ separatrix, the curves are non-inflectional. At bottom, $c = -1$ and the potential has one well at $(0, 0)$. These solutions are purely compressive. The length of each non-periodic configuration is 20.

on one end must be an inflectional *elastica*, and can be extended up to the next inflection point. Note that the absence of inflection points on a finite length of curve does not imply a non-inflectional *elastica*. Compare the two qualitatively similar curves in Figure 4.3, cut from distinctly different mother curves, one inflectional and one not.

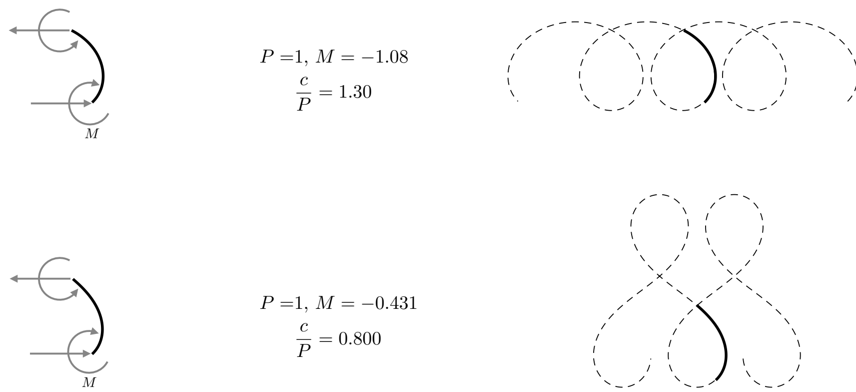

Figure 4.3: Two *elastica* curves of length $l = 1.5$ loaded with end forces and moments. From the ratio $\frac{c}{P}$, the top curve is identified as part of a non-inflectional shape, and the bottom curve part of an inflectional shape.

We can also interpret a rescaled form of the pendulum equation (4.30) in terms of motion with energy c in a sinusoidal potential with amplitude P, which again requires $\frac{c}{P} \geq -1$. A phase portrait is shown in Figure 4.4. This is something of an inversion of the curvature description, with the roles of stress and material stress reversed, and the inner librational contours of the phase portrait now representing inflectional solutions, and the outer rotational contours non-inflectional solutions. We may reproduce all the qualitative types of configurations shown in Figure 4.2 by varying the material stress c at some fixed stress P. Inside the $\frac{c}{P} = 0$ contour, which has the property that the tangential angle θ subtends exactly π radians, the curves are purely compressive. However, this fact is not apparent from the pendulum description, which does not contain any information about tangential forces and does not undergo a bifurcation. Although the curvature description is also in the form of a scalar equation representing normal force balance, it is derived using the tangential force balance and inherits important information from it.

4.5 Examples

Some examples will help relate our classification, which makes use of the somewhat esoteric conserved material force c, to physical quantities such as the magnitude P and direction of the end force, the end moment M, and the length of the body. All but the length are related

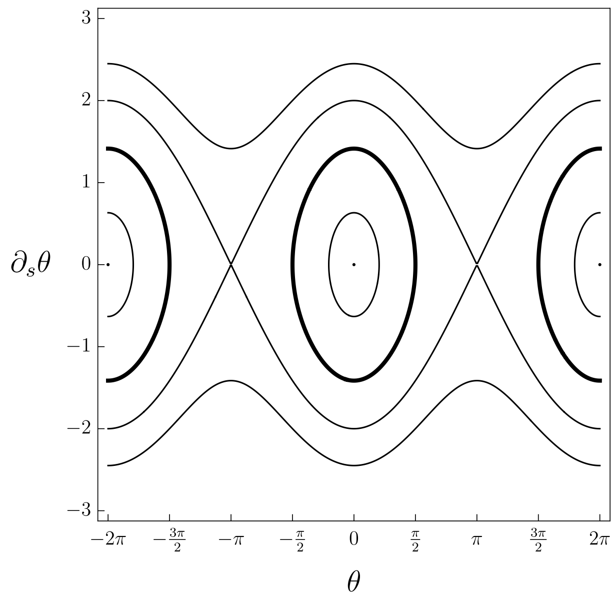

Figure 4.4: Phase portrait in θ-$\partial_s\theta$ space with stress $P = 1$ and increasing contours $\frac{c}{P} = (-1, -0.8, 0, 1, 2)$. The fixed points are straight lines, and the separatrices are solitary loop solutions separating inner inflectional and outer non-inflectional solutions. Not apparent in this pendulum description is the fact that solutions inside the bold $\frac{c}{P} = 0$ contour are purely compressive.

by the nondimensional form of (4.26),

$$c = \tfrac{1}{2}M^2 + P\cos\phi\,, \tag{4.33}$$

where P is defined as positive and ϕ is the angle between $\mathbf{P}$ and the boundary tangent $\partial_s\boldsymbol{x}(l)$. The end moment is simply the end curvature at this boundary, according to the nondimensional form of the boundary condition (4.25).

We first observe that many related finite length curves may be cut from a single mother curve of given ratio $\frac{c}{P}$, by applying appropriate moments at the ends. Figure 4.5 displays three such configurations of an *elastica* with both c and P fixed. We may slide along a contour and configuration by appropriate changes in the end moments. In Figure 4.6, we increase the load on a segment of elastica without moments, changing the potential and energy simultaneously. These shapes correspond to orbits subtending half a contour, beginning and ending on the $\partial_s\kappa$ axis (zero curvature). In the absence of moments, if the stress $\mathbf{P}$ is orthogonal to the end tangents, the material stress c vanishes and the curves transition from purely compressive to tensile near the ends and compressive in the middle. In Figure 4.7, we keep the load fixed and change one end moment, while keeping the other end slope fixed. This changes the other end moment, and the potential, while keeping the energy fixed. The *elastica* is transformed from a compressive single-well inflectional curve to a double-well inflectional curve to a non-inflectional curve living in one well of a double-well potential. An inflection

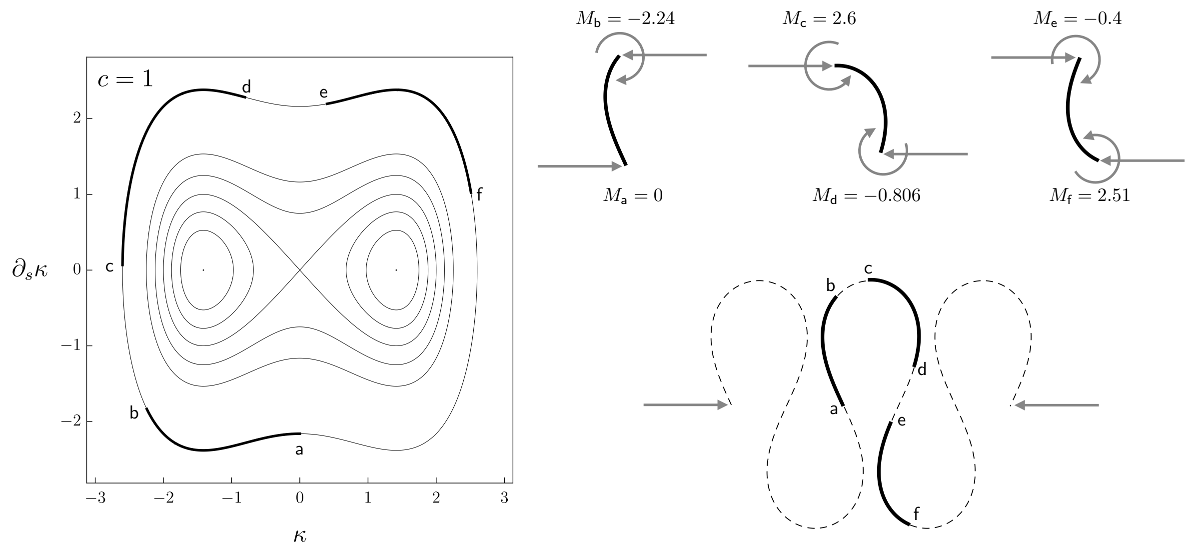

Figure 4.5: Orbits and configurations of a unit length *elastica* corresponding to different sections of a $\frac{c}{P} = 0.420$ mother curve.

point is expelled from the body into a virtual position on a mother curve, then ceases to exist entirely. A related sequence without the second end moment may be found in Griner's discussion of pole-vaulting [41].

Finally, we consider a system with fixed material force c and variable length. This example is inspired by O'Reilly's interpretation of a problem studied by Bigoni and co-workers [42, 43], featuring an elastic rod partially constrained by a straight, frictionless sleeve through which it is free to slide. There is a jump in curvature, along with a reaction force and moment, at the sleeve edge. A force of magnitude S acts on the rod in the sleeve to hold it in equilibrium with any end force and moment. One way to derive results on these configurations is to assume continuity of material stress at the sleeve edge [42]. If this is done, the quantity S required to maintain the rod in equilibrium can be identified with the material force c in equation (4.33). Note that our end moment M should not be confused with the reaction moment appearing in [43]. Figure 4.8 shows three equilibrium configurations with the same force in the sleeve and the same magnitude of end loading, so that they correspond to orbits on the same inflectional double-well contour. The three orbits correspond to three different freely hanging lengths l_h. The top and middle configurations correspond to loading by a follower force (same ϕ), while the bottom configuration has an additional moment and a change in ϕ. The top configuration shares the same shape (rotated) as the portion of the middle configuration far from the sleeve, while the bottom configuration shares the same shape and orientation as the portion of the middle configuration near the sleeve. It is not obvious whether one can easily direct the body along paths between these configurations, that is, slide the rod in and out of the sleeve by simple changes in the end loading.

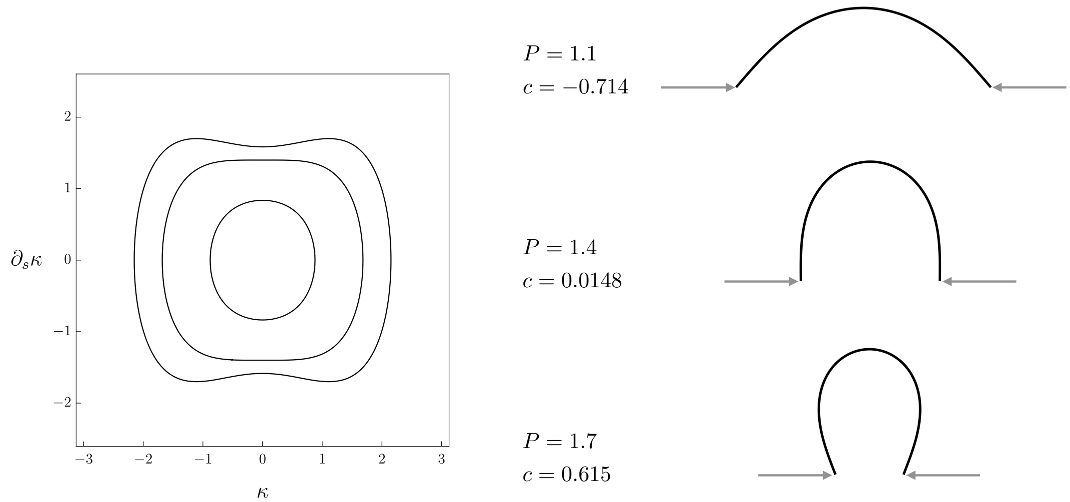

Figure 4.6: Configurations of an *elastica* of length $l = \pi$ under increasing end force and no end moments. The corresponding contours, enlarging with increasing stress and material stress, are shown; the orbits begin and end on the $\partial_s\kappa$ axis and subtend half a contour. At $c = 0$, the force is orthogonal to the end tangents, and the curves transition from purely compressive to tensile near the ends and compressive in the middle.

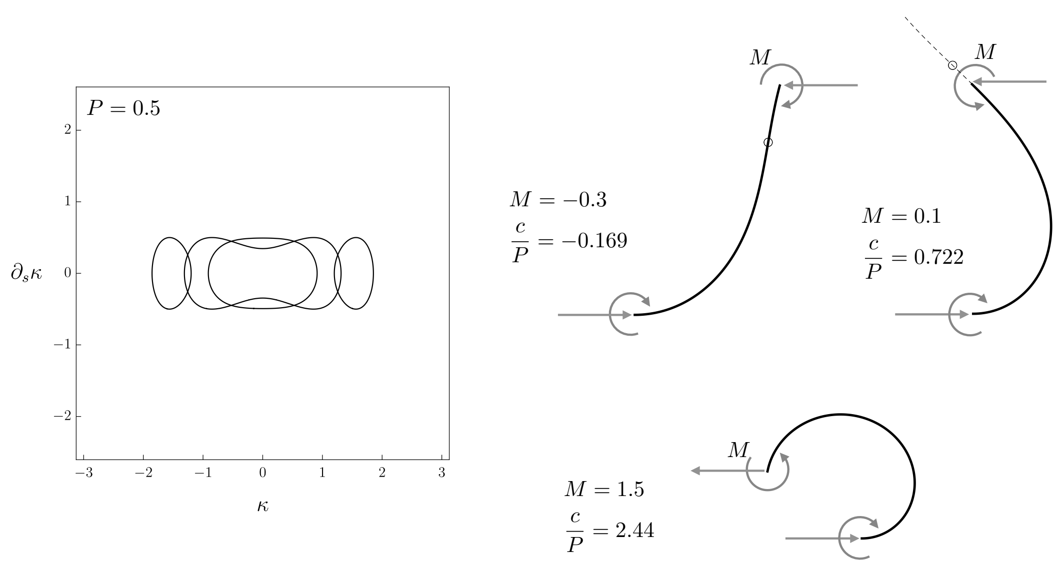

Figure 4.7: Configurations of an *elastica* of length $l = \pi$ under increasing end moment at one end, fixed slope at the other end, and fixed end forces. The corresponding contours transform from a single-well to a double-well potential. Inflection points, real and virtual, are denoted by open circles.

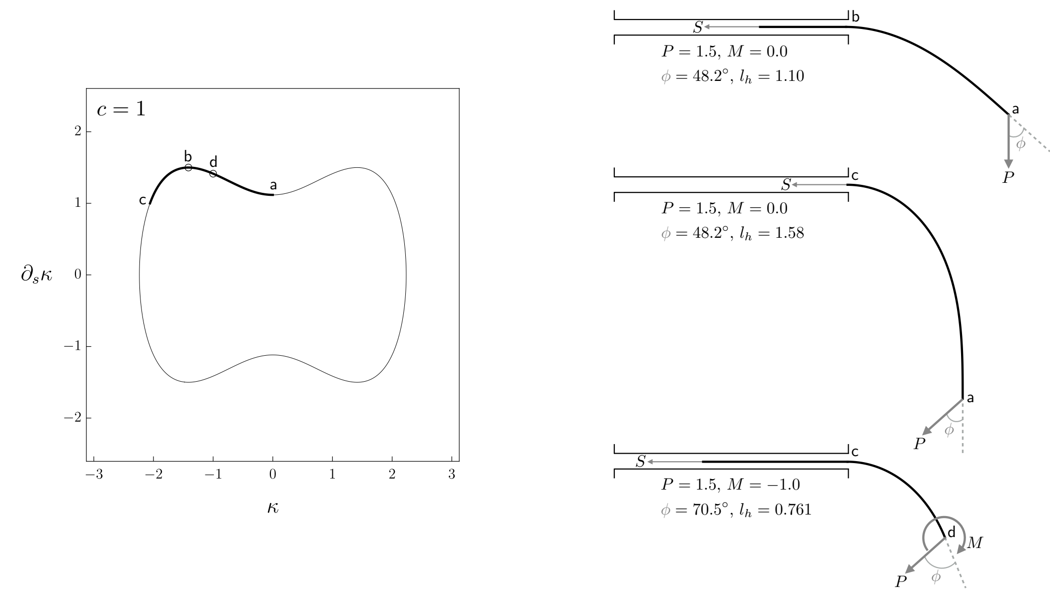

Figure 4.8: Configurations of a partially constrained *elastica* of total length $l = 1.6$, along with orbits corresponding to the freely hanging portions of lengths l_h on the same $\frac{c}{P} = \frac{2}{3}$ mother contour. The sleeve loading S, and thus the material stress, is the same for all configurations despite different free end loadings. A reaction force and moment (not shown) exist at the sleeve edge.

4.6 Summary

We have revisited the classical problem of the planar Euler *elastica*, compared commonly used representations, presented a classification in terms of conserved stress and material stress, and provided several physically inspired examples. A scalar shape equation for the curvature derived from a vector representation contains physical information not present in a scalar shape equation for the tangential angle.

Acknowledgments

We thank E. G. Virga for alerting us to the interesting features of the sleeve example, and J. H. Maddocks and O. M. O'Reilly for helpful discussions. This work was supported by U.S. National Science Foundation grant CMMI-1462501. This work has been available free of peer review on the arXiv since 6/2017.

Bibliography

[1] W. A. Oldfather, C. A. Ellis, and D. M. Brown. Leonhard Euler's elastic curves. *Isis*, 20(1):72–160, 1933.

[2] I. M. Mladenov and M. Hadzhilazova. *The Many Faces of Elastica.* Springer, Cham, 2017.

[3] D. Rogula. Forces in material space. *Archives of Mechanics*, 29:705–713, 1977.

[4] A. G. Herrmann. On conservation laws of continuum mechanics. *International Journal of Solids and Structures*, 17:1–9, 1981.

[5] T. B. Benjamin. Impulse, flow force and variational principles. *IMA Journal of Applied Mathematics*, 32:3–68, 1984.

[6] V. L. Gurevich and A. Thellung. Quasimomentum in the theory of elasticity and its conservation. *Physical Review B*, 42:7345–7349, 1990.

[7] D. F. Nelson. Momentum, pseudomomentum, and wave momentum: Toward resolving the Minkowski-Abraham controversy. *Physical Review A*, 44:3985–3996, 1991.

[8] G. A. Maugin. Material forces: Concepts and applications. *Applied Mechanics Reviews*, 48:213–245, 1991.

[9] R. Kienzler and G. Herrmann. *Mechanics in Material Space.* Springer, Berlin, 2000.

[10] M. E. Gurtin. *Configurational Forces as Basic Concepts of Continuum Physics*. Springer, New York, 2000.

[11] A. Yavari, J. E. Marsden, and M. Ortiz. On spatial and material covariant balance laws in elasticity. *Journal of Mathematical Physics*, 47:042903, 2006.

[12] O. M. O'Reilly. *Modeling Nonlinear Problems in the Mechanics of Strings and Rods*. Springer, New York, 2017.

[13] S. S. Antman. *Nonlinear Problems of Elasticity.* Springer, New York, 2005.

[14] A. Burchard and L. E. Thomas. On the Cauchy problem for a dynamical Euler's elastica. *Communications in Partial Differential Equations*, 28:271–300, 2003.

[15] D. A. Singer. Lectures on elastic curves and rods. *AIP Conference Proceedings*, 1002:3–32, 2008.

[16] A.-K. Tornberg and M. J. Shelley. Simulating the dynamics and interactions of flexible fibers in Stokes flows. *Journal of Computational Physics*, 196:8–40, 2004.

[17] J. Guven and P. Vázquez-Montejo. Confinement of semiflexible polymers. *Physical Review E*, 85:026603, 2012.

[18] H. Tsuru. Nonlinear dynamics for thin elastic rod. *Journal of the Physical Society of Japan*, 55:2177–2182, 1986.

[19] D. J. Steigmann and M. G. Faulkner. Variational theory for spatial rods. *Journal of Elasticity*, 33:1–26, 1993.

[20] S. Kehrbaum and J. H. Maddocks. Elastic rods, rigid bodies, quaternions and the last quadrature. *Philosophical Transactions of the Royal Society of London A*, 355:2117–2136, 1997.

[21] R. P. Nordgren. On computation of the motion of elastic rods. *Journal of Applied Mechanics*, 41:777–780, 1974.

[22] M. J. Shelley and T. Ueda. The Stokesian hydrodynamics of flexing, stretching filaments. *Physica D*, 146:221–245, 2000.

[23] B. Audoly. Introduction to the elasticity of rods. In C. Duprat and H. A. Stone, editors, *Fluid-Structure Interactions in Low-Reynolds-Number Flows*, pages 1–24. The Royal Society of Chemistry, Cambridge, 2016.

[24] D. J. Steigmann. Extension of Koiter's linear shell theory to materials exhibiting arbitrary symmetry. *International Journal of Engineering Science*, 51:216–232, 2012.

[25] E. Noether and M. A. Tavel. Invariant variation problems and Noether's theorem. *Transport Theory and Statistical Physics*, 3:183–207, 1971.

[26] E. L. Hill. Hamilton's principle and the conservation theorems of mathematical physics. *Reviews of Modern Physics*, 23:253–260, 1951.

[27] O. M. O'Reilly. A material momentum balance law for rods. *Journal of Elasticity*, 86:155–172, 2007.

[28] L. J. F. Broer. On the dynamics of strings. *Journal of Engineering Mathematics*, 4:195–202, 1970.

[29] J. H. Maddocks and D. J. Dichmann. Conservation laws in the dynamics of rods. *Journal of Elasticity*, 34:83–96, 1994.

[30] J. L. Ericksen. Simpler static problems in nonlinear theories of rods. *International Journal of Solids and Structures*, 6:371–377, 1970.

[31] S. S. Antman and K. B. Jordan. Qualitative aspects of the spatial deformation of non-linearly elastic rods. *Proceedings of the Royal Society of Edinburgh*, 73A(5):85–105, 1974/75.

[32] M. Nizette and A. Goriely. Towards a classification of Euler-Kirchhoff filaments. *Journal of Mathematical Physics*, 40(6):2830–2866, 1999.

[33] G. H. M Van Der Heijden and J. M. T. Thompson. Helical and localised buckling in twisted rods: A unified analysis of the symmetric case. *Nonlinear Dynamics*, 21:71–99, 2000.

[34] R. E. Goldstein and S. A. Langer. Nonlinear dynamics of stiff polymers. *Physical Review Letters*, 75(6):1094–1097, 1995.

[35] R. Capovilla, C. Chryssomalakos, and J. Guven. Hamiltonians for curves. *Journal of Physics A*, 35:6571–6587, 2002.

[36] C.-L. Lu and N. C. Perkins. Nonlinear spatial equilibria and stability of cables under uni-axial torque and thrust. *Journal of Applied Mechanics*, 61:879–886, 1994.

[37] A. E. H. Love. *A treatise on the mathematical theory of elasticity*. Dover, 1944.

[38] Craig G. Fraser. Mathematical technique and physical conception in Euler's investigation of the elastica. *Centaurus*, 34:211–246, 1991.

[39] R. Levien. The elastica: a mathematical history. *University of California, Berkeley, Technical Report No. UCB/EECS-2008-103*, 2008.

[40] M. Hubbard. An iterative numerical solution for the elastica with causally mixed inputs. *Journal of Applied Mechanics*, 47:200–202, 1980.

[41] G.M. Griner. A parametric solution to the elastic pole-vaulting pole problem. *Journal of Applied Mechanics*, 51:409–414, 1984.

[42] O. M. O'Reilly. Some perspectives on Eshelby-like forces in the elastica arm scale. *Proceedings of the Royal Society of London A*, 471:20140785, 2015.

[43] D. Bigoni, F. Dal Corso, F. Bosi, and D. Misseroni. Eshelby-like forces acting on elastic structures: Theoretical and experimental proof. *Mechanics of Materials*, 80:368–374, 2015.

Chapter 5

Material momentum in continuum mechanics

The work presented in this chapter is part of a larger ongoing project on the concept of material momentum. The contents of this chapter have not been published yet.

Attribution

The work presented in this chapter was done in collaboration with J. A. Hanna, who contributed to the inception and development of the main ideas in this work.

Abstract

In this paper we discuss the concept of material momentum in continuum mechanics.

5.1 Introduction

In continuum mechanics, the concept of material momentum traces its origins to Eshelby's pioneering work on the "force" on an elastic singularity [1]. In this work, Eshelby laid down the notion of a "force" on a defect in elastic materials as the dynamic quantity conjugate to the kinematic parameter characterizing the defect. This "force", which is commonly referred to as a material or configurational force, is related to the material momentum just as a Newtonian force is related to the physical momentum. Material momentum, which is also often referred to as pseudomomentum [5, 6], has long been deemed as a distinguishing feature of field theories associated with deformable matter [9, 10, 11].

One of the earliest identifications of this distinguishing feature of field theories describing material media was made by Rogula [9]. He remarked that the introduction of a material body imparts a special feature to the associated field theories, and that it requires introducing additional coordinates associated with the internal state of the material. These additional coordinates naturally lead to the concept of material momentum emanating from the variations associated with them. Unlike the standard physical momentum, which a material body acquires by virtue of its motion in the physical space, material momentum is a property associated with the motion of a defect in the material space. As commented by Peierls [11], without the presence of a material medium, the distinction between the momentum and the material momentum of a field (such as an electromagnetic field in a vacuum) disappears. In what seems to be one of the first derivations of the balance of material momentum in continuum mechanics, Rogula [9] defines a Lagrangian density per unit mass of the material body and constructs conservation laws associated with shifts in the material particle labels. He simply refers to this conservation law, without prescribing any name to it, as being associated with the homogeneity of the material body. Similar concepts are discussed by Gilbert and Mollow [10] in the context of elastic vibrations where they refer to material momentum as "tensor momentum".

Building on his prior work [1], Eshelby introduced the concept of a 4 dimensional "energy-momentum tensor" [12, 13] drawing inspiration from the energy-momentum tensors that already existed in the classical field theories [14]. Although his treatment in [12] constructs a 4×4 tensor which includes a time component, future workers [2, 8] have referred to "Eshelby's tensor" as only the material components of it. This tensor proved to be a useful tool in obtaining material forces acting on defects, and the path independent integrals of Rice [15] and Cherepanov [16] in fracture mechanics [3]. But unlike the energy-momentum tensor of classical fields associated with space and time, the non-temporal components of the tensor of Eshelby correspond to the material momentum instead of the physical momentum of the body [8].

Other treatments of the balance of material momentum as arising from variations in the material coordinates can be found in the literature concerned with conservation laws in continuum mechanics. The mathematical machinery of the calculus of variations required to obtain these conservation laws can be found in Hill [17], Rosen [18], and Barbashov [19]. Knowles and Sternberg [20] obtained the material conservation laws for finite elastostatics by Noether's theorem with invariance considerations under translations and "rotation" in material coordinates. Fletcher [21] extended their elastostatic results to linear elastodynamics. A detailed mathematical commentary on "impulse" from a variational principle was presented by Benjamin in [22]. In this work, the word "impulse" is used for Noether's charge appearing in a conservation law. Benjamin's treatment, being primarily mathematical in nature, concerns itself with variations in the independent coordinates without distinguishing material coordinates from spatial ones. However, he realizes that the "impulse" in many cases is not the same as the physical momentum of certain systems such as an ideal fluid. Material forces and the corresponding conservation laws in the presence of localized inhomogeneities have been presented by Rogula [23]. A comprehensive treatment of variational elasticity and the conservation laws associated with it can be found in Edelen [24].

In a series of papers in the 1980s [2, 3, 4], A.G. Herrmann presented a unified Lagrangian treatment of continuum mechanics for systems that admit a variational formulation. In [2] she derived a complete set of dynamic balance laws from an action principle and demonstrated that the balance of material momentum follows from varying the material or Lagrangian coordinates in an action. She emphasized the distinction between the physical and material space by showing that while the conservation of physical momentum corresponds to the homogeneity of the ambient space, the conservation of material momentum implies a homogeneous material space. She then constructs additional balance laws corresponding to the isotropy of the material space in [3] and relates them to path independent integrals, such as J and L integrals, that appear in fracture mechanics.

Some other notable works on the concept of material momentum include a non-variational derivation of Eshelby's tensor for elastostatics using the principle of virtual work presented by Hill [25]. A fairly mathematical treatment of conservation laws in elastodynamics using exterior calculus can be found in Şuhubi [26]. The balance of material momentum derived from a variational principle in the context of the Abraham-Minkowski controversy is given by Nelson [27] and Thellung [28]. A derivation of Eshelby's tensor from the entropy inequality has been done by Buratti et al. [29], where they liken it to the free enthalpy or Gibbs free energy.

The development of the concept of material/configurational forces has not been without its share of controversies [30]. Some researchers like Gurtin [31], Podio-Guidugli [32], and Fried [33] have adopted a viewpoint that the balance of material momentum is a basic law of continuum mechanics, at par with the balance of physical momentum, and is independent of any constitutive considerations. Their work primarily focuses on the role of configurational forces in dynamical phase transition problems, where they consider the balance of configurational forces to be a consequence of the invariance of the second law of thermodynamics at the interface. O'Reilly, who has produced extensive literature on the balance of material momentum for elastic rods and its applications [34, 35], also agrees with this interpretation. This view stands opposed to the view of others like Rogula [23], Maugin [5, 36, 6], Kienzler and Herrmann [7, 37], Rajagopal and Srinivasa [38], and Yavari [39] who consider this balance law to be simply the projection of the physical momentum on the configuration. Rajagopal and Srinivasa [38] in particular attribute the existence of configurational forces to evolving reference configurations of the material body, and emphasize that the balance of configurational forces is not an additional balance law of continuum mechanics.

The invariance of an action with respect to infinitesimal shifts in the material coordinates is an intrinsic feature of field theories of continuum mechanics and is applicable to solids and fluids alike. In solids, the idea of material symmetry has been a subject of considerable research [23, 36, 6, 7], which has in particular led to an improved understanding of the mechanics of defects in solids [1, 13] and propagating interfaces [31, 34]. However, the idea of material symmetry has not been exploited in fluid mechanics to the same extent. A broad commentary on the subject of variational principles and conservation laws in fluids was made by Benjamin [22], where he discussed conservation laws for continuous media as arising from spatial symmetries. He avoids physical interpretations (as either momentum or material momentum [40]) of the conserved quantities arising out of these spatial symmetries and deems the physical interpretations of these conserved quantities as irrelevant to the mathematically established physical conclusions. Material symmetry, often referred to as relabelling symmetry in fluids, has been explored by Eckart [41], Newcomb [42], Bretherton [43] and later by Padhye and Morrison [44], to show that well known conservation laws such as Kelvin's circulation theorem, and the conservation of vorticity and helicity, are a consequence of the relabelling symmetry of the material points of an ideal fluid. Attempts have been made by workers like Cherepanov [16], Atilgan [45], A. G. Herrmann [4], and Maugin [36] to construct analogies between the functional forms of certain path independent integrals in solids and fluids, but no direct relation with material symmetry has been made in case of fluids by them.

Recently, some very interesting experiments done by Bigoni and coworkers [46, 47, 48] on elastic rods moving through frictionless constraints have initiated interest in material/configurational forces in the context of elastic rods. In one particular experiment, Bigoni et al. [47] have shown that a terminally loaded elastic rod partially constrained by a straight frictioness sleeve experiences a tangential reaction force at the edge of the sleeve. They described this force as "Eshelby like" or a "configurational" force by analogy with Eshelby's work on force on a defect [1]. However, it was shown by O'Reilly [49] using the balance of material forces that the material/configurational force at the sleeve edge is actually zero, and that the force observed by Bigoni et al. was a standard Newtonian force.

In view of the long history of the concept of material momentum, and the persisting confusion between the nature of Newtonian and material forces, this paper is intended towards discussing this concept in a variational setting. We present a derivation of the balance of physical and material momentum, and the balance of energy from an action principle. In addition to the balance laws, we also derive from the same action principle the jump conditions associated with these physical quantities at non-material singular interfaces propagating through the medium. The derivation is constructed in a systematic way such that a clear conjugation between the variations and the associated physical quantities can be explicitly seen. We present some examples from seemingly disparate systems such as an ideal fluid, elastic rods, membranes, and plates, and highlight the applicability and utility of the concept of material momentum across these systems.

The paper is structured as follows. In section 5.1, we begin by highlighting the procedure for varying an action representing a material body with respect to the reference configuration and time (the independent fields), and the current configuration (the dependent field) simultaneously. Without prescribing specific dependencies of the Lagrangian density on the gradients of the fields involved, we identify the changes in the Lagrangian density induced by the shifts in the independent and dependent fields.

Thereafter, the balance laws for a first gradient mechanical field theory are presented, where the Lagrangian density is assumed to be a function of material velocity and the deformation gradient along with explicit dependencies on the independent field variables. In addition to the balance laws and boundary conditions, we obtain the jump conditions at non-material singular interfaces propagating within the continuum from the action principle. As a direct consequence of the variational treatment, we show that the bulk equations for the balance of material momentum and energy are nothing but the dot product of the physical momentum with the deformation gradient and time respectively. It is shown in section 5.2.2, through a rearrangement of the balance of material momentum and energy, that the source terms in these balance laws are the explicit partial derivative of the Lagrangian density with respect to the material coordinate and time respectively. In section 5.3, we discuss Noether's theorem and the conservation laws arising due to invariance under translational and rotational shifts in the spatial configuration, and translational shifts in the reference configuration. Thereafter, the balance of material momentum obtained is related to the the path independent J integral in fracture mechanics of hyperelastic solids in section 5.4.

We then consider material symmetry in an ideal fluid in section 5.5 and derive the balance of physical and material momentum for it. We demonstrate that certain well known objects in fluid mechanics such as *Cauchy's invariants*, *Weber's integral* can be derived directly from the balance of material momentum. We also show *Kelvin's circulation theorem* to be a consequence of material symmetry. Thereafter, we discuss the concept of *Helicity* and identify the form of material symmetry associated with it.

In section 5.6 we present a variational derivation of the balance of physical and material force for static elastic rods with variable bending stiffness. We then apply the balance of material force to an elastic rod moving in a curved frictionless channel as studied by DeSimone [50] and Bigoni et al. [48]. We demonstrate that the propulsive force on the elastic rod with variable bending stiffness can be obtained directly by integrating the balance of material momentum. Also, reaction forces experienced by the rod at points of geometric and material discontinuities, which were derived by Bigoni et al. [48] using micromechanical arguments, are derived from the singular version of the balance of material momentum at those discontinuities.

We end with a discussion on the "energy-momentum tensor" of the classical field theories and the one derived by Eshelby in [12, 13]. We argue that even though the formal mathematical derivations of these two quantities are quite identical, physically they represent distinct entities.

5.2 Balance laws

We begin by presenting a derivation of balance laws for physical and material momentum and energy from an action principle. Several variational treatments, such as those by Rogula [9], Knowles and Sternberg [20], Fletcher [21], Edelen [24], A. G. Herrmann [2, 4], and Yavari et al. [39], exist in the literature. Mathematical presentations of the general variational principle in field theories, detached from the context of continuum mechanics, can be found in Hill [17], Rosen [18], Barbashov and Nesterenko [19]. We construct our derivation in a way that delineates the Noether charges and currents associated with the appropriate variations. Also, unlike most derivations that exist in the literature, we allow for a propagating non-material singular interface in the body and obtain the relevant jump conditions from the action principle itself.

Let us assume that there exists a stress free reference configuration $\bar{\mathcal{B}}$ of the material manifold represented by the embedding vector $\bar{\boldsymbol{x}}$. A current configuration $\mathcal{B}$ of the body at any time t will be represented by a mapping $\boldsymbol{x} \equiv \boldsymbol{x}(\bar{\boldsymbol{x}}, t)$. We also define the operators $\bar{\nabla} \equiv \frac{d}{d\bar{\boldsymbol{x}}}$ and $\nabla \equiv \frac{d}{d\boldsymbol{x}}$ as representing the gradients in the reference and current configuration. The material time derivative of any field quantity will be denoted by d_t. To describe a material body, we assume a Lagrangian density $\bar{\mathcal{L}}\,(\bar{\boldsymbol{x}}, t; \boldsymbol{x})$ where the dependence of $\bar{\mathcal{L}}$ on the velocity $d_t\boldsymbol{x}$ and the successive material gradients of $\boldsymbol{x}$ is implied but not explicitly shown for brevity of notation. The reference configuration $\bar{\boldsymbol{x}}$ and time t are to be treated as the independent variables over which the deformed configuration $\boldsymbol{x}$ is defined. In keeping with the distinction made by Rogula [9] between fields associated with empty space and material bodies, we assume $\bar{\mathcal{L}}(\bar{\boldsymbol{x}}, t; \boldsymbol{x})$ to be per unit *reference* volume (equivalent to per unit mass) of the body.

The following action is defined by integrating the material Lagrangian density over the reference volume of the body,

$$A = \int_{t_0}^{t_1} dt \int_{\bar{\mathcal{B}}} d\bar{V}\, \bar{\mathcal{L}}(\bar{\boldsymbol{x}}, t; \boldsymbol{x})\,, \tag{5.1}$$

where $d\bar{V} = \sqrt{\bar{g}}\, du^1 du^2 du^3$ is the volume form, and $\bar{g}$ is the determinant of the metric in the reference configuration. Here $\{u^1, u^2, u^3\}$ are an arbitrary set of coordinates that uniquely identify a material point throughout the motion. Similarly $dV = \sqrt{g}\, du^1 du^2 du^3$ is used to denote the volume form in the current configuration, where g represents the determinant of the metric in the current configuration. The arguments of the Lagrangian density in (5.1) have been written such that the fields appearing on the right of the semicolon are considered to be a function of the independent fields on the left of it. The volume form itself is an invariant geometric object under coordinate transformations (i.e. changes in the material coordinates u^i), but under physical deformations (i.e. changes in the position vector) it transforms as $dV = J d\bar{V}$ where $J = \sqrt{\frac{g}{\bar{g}}}$ is called the determinant of the Jacobian associated with the deformation.

One consequence in the upcoming calculations of choosing a Lagrangian density per unit reference volume is that, unlike theories based on geometric energies, a variation of the action (5.1) would not involve varying the volume form, i.e. $\delta d\bar{V} = 0$. When compared to geometric actions appended with an isometric constraint, this results in a modified definition of the Lagrange multiplier enforcing the constraint. However, the resulting deformation of the material body predicted by the two approaches would be the same for isometric deformations.

Following prior works of Rogula [9], Maugin [5], and Kienzler and G. Herrmann [7], we subject the action (5.1) to a set of transformations $t \rightarrow t'$, $\bar{\boldsymbol{x}} \rightarrow \bar{\boldsymbol{x}}'$, $\boldsymbol{x}(\bar{\boldsymbol{x}}, t) \rightarrow \boldsymbol{x}'(\bar{\boldsymbol{x}}', t')$. Note that the dependent and the independent fields have both been transformed. The resulting action under these set of transformations is written as,

$$A' = \int_{t_0'}^{t_1'} dt' \int_{\bar{\mathcal{B}}'} d\bar{V}' \, \bar{\mathcal{L}}(\bar{\boldsymbol{x}}', t'; \boldsymbol{x}') \,, \tag{5.2}$$

where it is understood that transformed field variable $\boldsymbol{x}'$ and its successive material (and time) gradients are functions of the transformed independent variables $\{\bar{\boldsymbol{x}}', t'\}$. Since the independent variables have been shifted, the domains of integration in time and space have transformed as well. One implicit assumption made in going from (5.1) to (5.2) is that the functional form of the Lagrangian is not affected by the transformation, i.e. the Lagrangian density [1] is assumed to be a scalar function.

We now assume that the transformed coordinates $\bar{\boldsymbol{x}}'$ and t', and the field $\boldsymbol{x}'$, differ infinitesimally from their original selves as follows,

$$\bar{\boldsymbol{x}}' = \bar{\boldsymbol{x}} + \delta\bar{\boldsymbol{x}} \,, \qquad t' = t + \delta t \,, \qquad \boldsymbol{x}'(\bar{\boldsymbol{x}}', t') = \boldsymbol{x}(\bar{\boldsymbol{x}}, t) + \delta\boldsymbol{x}(\bar{\boldsymbol{x}}, t) \,. \tag{5.3}$$

The time variable has a special status in classical physics such that only constant shifts in it are admissible. Note that the variational quantities introduced above, which are infinitesimal in nature, can be considered a function of either the transformed or the original independent fields since $\delta\boldsymbol{x}(\bar{\boldsymbol{x}}', t') = \delta\boldsymbol{x}(\bar{\boldsymbol{x}}, t) + \frac{\partial \delta\boldsymbol{x}}{\partial \bar{\boldsymbol{x}}} \cdot \delta\bar{\boldsymbol{x}} + \frac{\partial \delta\boldsymbol{x}}{\partial t}\delta t + ...$, and all terms on the right except the first one are higher order [17]. For keeping the calculations simple, we assume the variations to be a function of the original un-transformed fields $\{\bar{\boldsymbol{x}}, t\}$.

The δ operator in (5.3) is a measure of the change in $\boldsymbol{x}$ due to changes in the independent fields, as well as the changes in $\boldsymbol{x}$ itself (physical deformation) at a *fixed material point* (at a fixed $\bar{\boldsymbol{x}}$). Since the two sides of equation $(5.3)_3$ are functions of two different fields ($\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}}'$), the δ operator does not commute with the gradient $\bar{\nabla}$ when applied to $\boldsymbol{x}$. Due to this reason we define an operator $\tilde{\delta}$ which measures the shift in the field variable at *a fixed material label*, and therefore commutes with the covariant derivative,

[1] not to be confused with "density" in the tensorial sense.

$$\boldsymbol{x}'(\bar{\boldsymbol{x}},t) = \boldsymbol{x}(\bar{\boldsymbol{x}},t) + \tilde{\delta}\boldsymbol{x}(\bar{\boldsymbol{x}},t)\,. \tag{5.4}$$

Using $(5.3)_3$ and (5.4), the two variational operators can be related as,

$$\delta\boldsymbol{x} = \tilde{\delta}\boldsymbol{x} + \mathbf{F}\cdot\delta\bar{\boldsymbol{x}} + d_t\boldsymbol{x}\,\delta t\,, \tag{5.5}$$

where $\mathbf{F} \equiv \bar{\nabla}\boldsymbol{x}$ is defined as the deformation gradient. The second term on the right side is a measure of the shift in the current configuration $\boldsymbol{x}$ due to shifts in the reference configuration $\bar{\boldsymbol{x}}$ alone. On subtracting (5.1) from (5.2) the change in the action due to the transformation (5.3) can be written as,

$$\Delta A = \int_{t_0'}^{t_1'} dt' \int_{\bar{\mathcal{B}}'} d\bar{V}'\,\bar{\mathcal{L}}(\bar{\boldsymbol{x}}',t';\boldsymbol{x}') - \int_{t_0}^{t_1} dt \int_{\bar{\mathcal{B}}} d\bar{V}\,\bar{\mathcal{L}}(\bar{\boldsymbol{x}},t;\boldsymbol{x})\,. \tag{5.6}$$

To evaluate the difference in the action in a meaningful way, we manipulate the first integral (see equation (10) of [54] for details) so that it is integrated over the original domain. On doing that we obtain upto first order,

$$\delta A = \int_{t_0}^{t_1} dt \int_{\bar{\mathcal{B}}} d\bar{V}\,\left[d_t\big(\bar{\mathcal{L}}\,\delta t\big) + \bar{\nabla}\cdot\big(\bar{\mathcal{L}}\,\delta\bar{\boldsymbol{x}}\big)\right] + \int_{t_0}^{t_1} dt \int_{\bar{\mathcal{B}}} d\bar{V}\tilde{\delta}\bar{\mathcal{L}}\,. \tag{5.7}$$

The integrals above clearly delineate the changes in the action due to shifts in the independent and dependent fields. The two terms in the first integral in (5.7) account for the shift in the domain of integration in time t and reference configuration $\bar{\boldsymbol{x}}$, whereas the second integral represents the change due to shifts in the dependent variables $\boldsymbol{x}$ at a *fixed material label* $\bar{\boldsymbol{x}}$.

On integrating by parts, the variation $\tilde{\delta}$ of the Lagrangian in the second integral can in general be written as,

$$\tilde{\delta}\bar{\mathcal{L}} = \boldsymbol{\mathcal{E}}(\bar{\mathcal{L}})\cdot\tilde{\delta}\boldsymbol{x} + d_t\big(\mathcal{E}_T(\bar{\mathcal{L}})\big) + \bar{\nabla}\cdot\big(\boldsymbol{\mathcal{E}}_M(\bar{\mathcal{L}})\big)\,. \tag{5.8}$$

Here $\boldsymbol{\mathcal{E}}(\bar{\mathcal{L}})$ is the Euler-Lagrange operator, and $\mathcal{E}_T(\bar{\mathcal{L}})$ and $\boldsymbol{\mathcal{E}}_M(\bar{\mathcal{L}})$ are the temporal and material boundary terms that come from the shifts $\tilde{\delta}\boldsymbol{x}$. On using (5.8) the variation of the action (5.7) can then be written in the following compact notation,

$$\delta A = \int_{t_0}^{t_1} dt \int_{\bar{\mathcal{B}}} d\bar{V}\left[d_t\mathcal{Q} + \bar{\nabla}\cdot\boldsymbol{\mathcal{J}} + \boldsymbol{\mathcal{E}}(\bar{\mathcal{L}})\cdot\tilde{\delta}\boldsymbol{x}\right]\,, \tag{5.9}$$

where,

$$\mathcal{Q} = \bar{\mathcal{L}}\,\delta t + \mathcal{E}_T(\bar{\mathcal{L}})\,. \tag{5.10}$$
$$\boldsymbol{\mathcal{J}} = \bar{\mathcal{L}}\,\delta\bar{\boldsymbol{x}} + \boldsymbol{\mathcal{E}}_M(\bar{\mathcal{L}})\,, \tag{5.11}$$

Here $\mathcal{Q}$ and $\boldsymbol{\mathcal{J}}$ represent Noether charge and current, respectively, which in general are functions of the variations of all the fields. The third term in the integral (5.9) delivers the bulk equations corresponding to various shifts. When the variations in the fields are symmetries of the system, the first two terms inside the integral (5.9) give us the conservation laws associated with the symmetries on the solution trajectories, i.e. the path in the state space where $\boldsymbol{\mathcal{E}}(\bar{\mathcal{L}}) = 0$. In the next section, we identify the balance and conservation laws that emerge from variation in the time t, the reference configuration $\bar{\boldsymbol{x}}$, and the current configuration $\boldsymbol{x}$ for a Lagrangian density dependent on, at most, the first material gradient of the position vector.

5.2.1 First gradient theory

In this section we consider one particular case of the field theory where the Lagrangian density is assumed to be a function of, at most, the first material gradient of $\boldsymbol{x}$, i.e. $\bar{\mathcal{L}} \equiv \bar{\mathcal{L}}(\bar{\boldsymbol{x}}, t; \boldsymbol{x}, d_t\boldsymbol{x}, \mathbf{F})$. We denote the material gradient of $\boldsymbol{x}$ by $\mathbf{F} \equiv \bar{\nabla}\boldsymbol{x} \equiv \frac{d\boldsymbol{x}}{d\bar{\boldsymbol{x}}}$ which is commonly known in continuum mechanics as the deformation gradient [55].

The theory of elasticity for simple materials (i.e. materials with no microstructure) is a first order gradient theory [5]. For such a case, the temporal and material boundary terms can be computed as $\mathcal{E}_T = \frac{\partial\bar{\mathcal{L}}}{\partial d_t\boldsymbol{x}} \cdot \tilde{\delta}\boldsymbol{x}$ and $\boldsymbol{\mathcal{E}}_M = \frac{\partial\bar{\mathcal{L}}}{\partial\mathbf{F}} \cdot \tilde{\delta}\boldsymbol{x}$. Following are the explicit expressions of the Euler-Lagrange operator, and Noether charge and current for a first gradient theory,

$$\boldsymbol{\mathcal{E}}(\bar{\mathcal{L}}) = \frac{\partial\bar{\mathcal{L}}}{\partial\boldsymbol{x}} - d_t\left(\frac{\partial\bar{\mathcal{L}}}{\partial d_t\boldsymbol{x}}\right) - \bar{\nabla}\cdot\left(\frac{\partial\bar{\mathcal{L}}}{\partial\mathbf{F}}\right)\,, \tag{5.12}$$
$$\mathcal{Q}(\bar{\mathcal{L}}) = \bar{\mathcal{L}}\,\delta t + \frac{\partial\bar{\mathcal{L}}}{\partial d_t\boldsymbol{x}} \cdot \tilde{\delta}\boldsymbol{x}\,, \tag{5.13}$$
$$\boldsymbol{\mathcal{J}}(\bar{\mathcal{L}}) = \bar{\mathcal{L}}\,\delta\bar{\boldsymbol{x}} + \frac{\partial\bar{\mathcal{L}}}{\partial\mathbf{F}} \cdot \tilde{\delta}\boldsymbol{x}\,. \tag{5.14}$$

For the sake of clarity in identifying the correct power conjugates to all the variational quantities, we would like to express the Noether current and charge in terms of the total variation $\delta\boldsymbol{x}$ at a *fixed material point* rather than the variation $\tilde{\delta}\boldsymbol{x}$ at a *fixed material label.* We use the relation (5.5) to substitute for $\tilde{\delta}\boldsymbol{x}$ in (5.13) and (5.14) and rearrange equations (5.13) and (5.14) to obtain the following,

$$\mathcal{Q} = \frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}} \cdot \delta \boldsymbol{x} + \left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}} \cdot \mathbf{F} \right) \cdot (-\delta \bar{\boldsymbol{x}}) + \left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}} \cdot d_t \boldsymbol{x} - \bar{\mathcal{L}} \right) (-\delta t) \ , \tag{5.15}$$

$$\boldsymbol{\mathcal{J}} = \frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot \delta \boldsymbol{x} + \left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot \mathbf{F} - \bar{\mathcal{L}}\, \mathbf{I} \right) \cdot (-\delta \bar{\boldsymbol{x}}) + \left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot d_t \boldsymbol{x} \right) (-\delta t) \ . \tag{5.16}$$

We can already see familiar terms appearing in the expressions for Noether's charge and current. In (5.15) we identify the terms conjugate to $\delta \boldsymbol{x}$, $-\delta \bar{\boldsymbol{x}}$, and $-\delta t$ [2] as the physical momentum, material momentum, and the Hamiltonian density respectively. The material momentum appearing conjugate to $-\delta \bar{\boldsymbol{x}}$ in (5.15) differs from the general definition of pseudomomentum by Peierls in equation (2.10) of [11] by a minus sign. However, our definition of the material momentum here is consistent with the definition of A.G. Herrmann [3]. Similarly, the Noether currents in (5.16) conjugate to $\delta \boldsymbol{x}$, $-\delta \bar{\boldsymbol{x}}$, and $-\delta t$ are identified as the physical stress, the material stress, and the power expended by the physical stress. Note that the quantity $\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}}$ in classical continuum mechanics is well known as the first Piola-Kirchoff stress tensor that appears in the material description of the deformation. Similarly, the material stress tensor appearing conjugate to $-\delta \bar{\boldsymbol{x}}$ in (5.16) is the Eshelby tensor.

Of all these terms, the charge and the current conjugate to $-\delta \bar{\boldsymbol{x}}$ are non-classical entities. The Noether current corresponding to $-\delta \bar{\boldsymbol{x}}$ is identified as the material components of the 4-dimensional *energy-momentum tensor* of Eshelby [12], with $\mathbf{I}$ being the identity tensor in the reference configuration. A four dimensional construction of the energy-momentum tensor where time is treated at par with the material coordinates can be found in Kienzler and G. Herrmann [8] where they identify the material part of it as the "Eshelby tensor". Hill derives the Eshelby energy momentum tensor in equation (14) of [25] using the principle of virtual work.

We now proceed to integrate (5.9) and obtain bulk and boundary conditions, along with singular balance laws at a surface of discontinuity. The two terms in (5.9) involving a total time derivative of $\mathcal{Q}$ and a total divergence of $\boldsymbol{\mathcal{J}}$ yield the boundary conditions as well as the jump conditions at a propagating non-material interface in the body.

Imagine a non-material surface $\bar{\mathcal{S}}(t)$ propagating through the reference configuration of the body, across which all the field quantities (including the normal vectors) are assumed to be discontinuous. The surface in the current configuration is denoted by $\mathcal{S}(t)$. This non-material mathematical surface deforms as it moves along in the material body. The presence of a discontinuity in the fixed reference domain $\bar{\mathcal{B}}$ entails the use of the following forms of the divergence theorem and the transport theorem applicable to piecewise continuous fields for integrating out the boundary terms,

[2] We choose to identify quantities appearing conjugate to the negatives of the independent variables to be consistent with the definition of the *Hamiltonian* [56], and the *elastic energy-momentum tensor* [12].

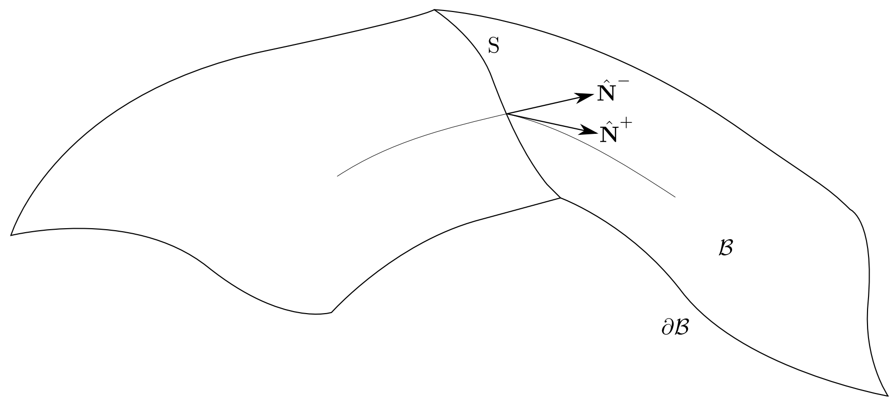

Figure 5.1: A two dimensional surface $\mathcal{B}$ with a line of discontinuity $S(t)$ propagating through it. The unit normal vectors $\hat{\mathbf{N}}^{\pm}$ denote the normal vectors on either side of S.

$$\int_{\bar{\mathcal{B}}} d\bar{V}\, \bar{\nabla} \cdot \mathbf{P} = \int_{\partial\bar{\mathcal{B}}} d\bar{A}\, \mathbf{P} \cdot \mathbf{n} - \int_{\bar{\mathcal{S}}(t)} d\bar{A}\, [\![\mathbf{P} \cdot \mathbf{N}]\!]\ , \tag{5.17}$$

$$d_t \int_{\bar{\mathcal{B}}} d\bar{V}\, \mathbf{P} = \int_{\bar{\mathcal{B}}} d\bar{V}\, d_t\mathbf{P} - \int_{\bar{\mathcal{S}}(t)} d\bar{A}\, U_N\, [\![\mathbf{P}]\!]\ , \tag{5.18}$$

where $\mathbf{n}$ and $\mathbf{N}$ are the unit vectors normal to the boundary $\partial\bar{\mathcal{B}}$ and the surface of discontinuity $\bar{\mathcal{S}}(t)$ respectively, and U_N is the velocity of propagation of $\bar{\mathcal{S}}(t)$ in the material space in the direction $\mathbf{N}$ [55, 57]. Using these two theorems, and equation (5.5) in (5.9), we obtain the following,

$$\begin{aligned} 0 = \int_{t_0}^{t_1} dt \int_{\partial\bar{\mathcal{B}}} d\bar{A}\, \boldsymbol{\mathcal{J}} \cdot \boldsymbol{n} + \int_{t_0}^{t_1} dt \int_{\bar{\mathcal{S}}(t)} d\bar{A}\, [\![-\boldsymbol{\mathcal{J}} \cdot \boldsymbol{N} + U_N \mathcal{Q}]\!] \\ + \int_{t_0}^{t_1} dt \int_{\bar{\mathcal{B}}} d\bar{V}\, \left[\boldsymbol{\mathcal{E}}(\bar{\mathcal{L}}) \cdot (\delta\boldsymbol{x} - \mathbf{F} \cdot \delta\bar{\boldsymbol{x}} - d_t\boldsymbol{x}\, \delta t)\right]\ . \end{aligned} \tag{5.19}$$

We are now in a position to obtain balance laws, boundary conditions, and jump conditions at propagating interfaces conjugate to the variation in physical coordinates, material coordinates, and time. Assuming the variations to be purely in the current configuration $\boldsymbol{x}$ (i.e. $\delta\bar{\boldsymbol{x}} = \delta t = 0$) we obtain the bulk equation, the boundary condition, and the jump condition at the propagating interface for physical momentum,

$$\boldsymbol{\mathcal{E}}(\bar{\mathcal{L}}) \equiv \frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{x}} - d_t\left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}}\right) - \bar{\nabla} \cdot \left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}}\right) = \mathbf{0} \qquad \text{for} \qquad \boldsymbol{x} \in \mathcal{B}\,, \tag{5.20}$$

$$\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot \mathbf{n} = \mathbf{0} \qquad \text{for} \qquad \boldsymbol{x} \in \partial\mathcal{B}\,, \tag{5.21}$$

$$\left[\!\left[-\frac{\partial \mathcal{L}}{\partial \mathbf{F}} \cdot \mathbf{N} + U_N \frac{\partial \mathcal{L}}{\partial d_t \boldsymbol{x}} \right]\!\right] = \mathbf{0} \qquad \text{for} \qquad \boldsymbol{x} \in \mathcal{S}(t)\,. \tag{5.22}$$

Assuming the variations to be purely in the reference configuration $\bar{\boldsymbol{x}}$ (i.e. $\delta \boldsymbol{x} = \delta t = 0$), we obtain the bulk equation, the boundary condition, and the jump condition at the propagating interface for material momentum,

$$\boldsymbol{\mathcal{E}}(\bar{\mathcal{L}}) \cdot \mathbf{F} \equiv \left[\frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{x}} - d_t\left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}}\right) - \bar{\nabla} \cdot \left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}}\right)\right] \cdot \mathbf{F} = 0 \qquad \text{for} \qquad \boldsymbol{x} \in \mathcal{B}\,, \tag{5.23}$$

$$\left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot \mathbf{F} - \bar{\mathcal{L}}\,\mathbf{I}\right) \cdot \mathbf{n} = 0 \qquad \text{for} \qquad \boldsymbol{x} \in \partial\mathcal{B}\,, \tag{5.24}$$

$$\left[\!\left[-\left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot \mathbf{F} - \bar{\mathcal{L}}\,\mathbf{I}\right) \cdot \mathbf{N} + U_N \left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}} \cdot \mathbf{F}\right) \right]\!\right] = 0 \qquad \text{for} \qquad \boldsymbol{x} \in \mathcal{S}(t)\,. \tag{5.25}$$

Assuming purely temporal variations (i.e. $\delta \boldsymbol{x} = \delta \bar{\boldsymbol{x}} = 0$), we obtain the bulk equation, the boundary condition, and the jump condition at the propagating interface for energy,

$$\boldsymbol{\mathcal{E}}(\bar{\mathcal{L}}) \cdot d_t\boldsymbol{x} \equiv \left[\frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{x}} - d_t\left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}}\right) - \bar{\nabla} \cdot \left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}}\right)\right] \cdot d_t\boldsymbol{x} = 0 \qquad \text{for} \qquad \boldsymbol{x} \in \mathcal{B}\,, \tag{5.26}$$

$$\left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot d_t\boldsymbol{x}\right) \cdot \mathbf{n} = 0 \qquad \text{for} \qquad \boldsymbol{x} \in \partial\mathcal{B}\,, \tag{5.27}$$

$$\left[\!\left[-\left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot d_t\boldsymbol{x}\right) \cdot \mathbf{N} + U_N \left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}} \cdot d_t\boldsymbol{x} - \bar{\mathcal{L}}\right) \right]\!\right] = 0 \qquad \text{for} \qquad \boldsymbol{x} \in \mathcal{S}(t)\,. \tag{5.28}$$

The interface conditions for physical momentum (5.22), material momentum (5.25), and energy (5.28) appear here without any source terms, since they were not accounted for explicitly in the Lagrangian. For lower dimensional bodies, such as strings, rods, and membranes, source terms in the jump conditions have been incorporated by various authors [34, 35, 58, 49, 59, 60, 61, 62] to model partial contact phenomena.

Note that the bulk equations for material momentum (5.23) and energy (5.26) turn out to be the projection of the balance of physical momentum (5.20) onto the present configuration and the velocity, respectively. For the balance of material momentum, this fact, as pointed out by Yavari [39], tells us that the in the absence of discontinuities the balance of material momentum is identically satisfied if the balance of physical momentum holds. This observation however obscures a crucial point, which is that even though the balance laws (5.23) and (5.20) are equivalent, the conserved quantities that arise due to symmetry of physical and material space are not. This point becomes clear when the balance of material momentum (5.23) is rearranged to be written in the form of a balance law, as shown in the next section, and it is seen that the source term appearing in the balance of material momentum is independent of the one appearing in the balance of physical momentum (5.20).

5.2.2 Physical and material forces

In this section, following A. G. Herrmann [2] and Maugin [5], we present rearrangements of the energy (5.26) and material momentum balances (5.23) written in the standard form of a balance law, and identify the analogous source terms in both cases. The balance of physical momentum (5.20) can be written with a source term on the right side as follows,

$$d_t\left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}}\right) + \bar{\nabla}\cdot\left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}}\right) = \frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{x}}\,. \tag{5.29}$$

This source term emanates from the explicit dependence of the Lagrangian density on the spatial position vector $\boldsymbol{x}$, which breaks the symmetry of the ambient space. Such an explicit dependence on the spatial position vector can occur in situations where body forces such as gravity are present.

The balance of energy (5.26) can be written as,

$$\frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{x}}\cdot d_t\boldsymbol{x} - d_t\left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}}\right)\cdot d_t\boldsymbol{x} - \bar{\nabla}\cdot\left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}}\right)\cdot d_t\boldsymbol{x} = 0\,. \tag{5.30}$$

To rearrange this equation in the form of a balance law we make use of the following chain rule for the total time derivative of the Lagrangian density,

$$d_t\bar{\mathcal{L}} = \left(\frac{\partial \bar{\mathcal{L}}}{\partial t} + \frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{x}}\cdot d_t\boldsymbol{x} + \frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}}\cdot d_t d_t\boldsymbol{x} + \frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} : (\bar{\nabla} d_t\boldsymbol{x})\right)\,. \tag{5.31}$$

The explicit dependence of the Lagrangian density on $\bar{\boldsymbol{x}}$ did not show up in the chain rule above since both these arguments are independent of time. Using this chain rule, and integrating by parts the second and the third term, equation (5.30) can be written in the following balance law form,

$$d_t\left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}}\cdot d_t\boldsymbol{x} - \bar{\mathcal{L}}\right) + \bar{\nabla}\cdot\left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}}\cdot d_t\boldsymbol{x}\right) = -\frac{\partial \bar{\mathcal{L}}}{\partial t}\,. \tag{5.32}$$

The energy balance can now be seen as a standard balance law with the term on the right side serving as a source. The term on the right is familiar and expected, as we know that explicit dependence of the Lagrangian on time introduces a source term in the energy balance.

On similar lines, let us now consider the balance of material momentum (5.23) and rewrite it in the form of a standard balance law. From (5.23) we have,

$$\frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{x}} \cdot \mathbf{F} - d_t \left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}} \right) \cdot \mathbf{F} - \bar{\nabla} \cdot \left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \right) \cdot \mathbf{F} = 0 \,. \tag{5.33}$$

In order to rearrange this equation, we need the following chain rule for the total material gradient of the Lagrangian density,

$$\bar{\nabla} \bar{\mathcal{L}} = \frac{\partial \bar{\mathcal{L}}}{\partial \bar{\boldsymbol{x}}} + \frac{\partial \bar{\mathcal{L}}}{\partial \boldsymbol{x}} \cdot \mathbf{F} + \frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}} \cdot d_t \mathbf{F} + \frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} : (\bar{\nabla} \mathbf{F}) \,. \tag{5.34}$$

On using the above equation and integrating by parts we obtain,

$$d_t \left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}} \cdot \mathbf{F} \right) + \bar{\nabla} \cdot \left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot \mathbf{F} - \bar{\mathcal{L}} \, \mathbf{I} \right) = - \frac{\partial \bar{\mathcal{L}}}{\partial \bar{\boldsymbol{x}}} \,. \tag{5.35}$$

The term on the right hand side of the above equation serves as a source of material momentum. This balance law can be found in Rogula [9] without the source term. Maugin [63, 5, 36], Kienzler and Herrmann [7], Fletcher [21], and Knowles and Sternberg [64] present this law for static finite deformations. It can also be found in Thellung [28] as a conservation law without a source term. An alternative derivation of these laws by simply differentiating the Lagrangian function and using the Euler-Lagrange equations can be found in [2].

The balance of material momentum (5.35) is clearly a balance law with $-\frac{\partial \bar{\mathcal{L}}}{\partial \bar{\boldsymbol{x}}}$ acting as the source term. Even though (5.23) is derived from a projection of the physical momentum balance (5.20), the interpretation of this balance law (5.23) is not trivial. This law is concerned with the symmetry of the continuum in the material space, and has nothing to do with the symmetry of the ambient space. This fact can be immediately seen from the source terms in (5.20) and (5.23), which are independent of each other. In other words, there could be a source term present in one of the balance laws while not in the other. This makes physical sense, since the symmetry of either space can be broken without affecting the symmetry of the other.

Just as the source term in (5.20) is traditionally interpreted as a “physical body force”, we interpret the source term in (5.23) as the “material body force”. It is important to note that this force is not the same as the familiar “Newtonian” force in general, most importantly in situations where the material manifold is of a lower dimension (such as strings, rods, and membranes) than the ambient space. Material and Newtonian forces reside in different spaces, namely the material and the embedding space, and therefore, in general, operations such as adding a material and a Newtonian force cannot be defined in a meaningful way. Newtonian forces are always associated with the motion of material points in the physical space, whereas material forces govern the motion of non-material inhomogeneities in the material space.

All the results in this section have been derived without referring to the material body as solid or fluid. Therefore, these results, although mostly employed to study the behavior of solids in the literature, are equally applicable to fluids. We must also remark that since equations (5.23) to (5.25) exist due to the continuum description of material objects, there exists no analogue of them for a discrete set of particles [2, 4].

5.3 Symmetries and Conservation of Material Momentum

In this section we apply *Noether's theorem* to exploit physical and material symmetry of the action (5.1) to obtain conservation laws corresponding to the physical and the material momentum respectively. Since we are interested in the conservation laws that hold on the solution trajectory, we will assume throughout this section that the Euler-Lagrange equation (5.20) is satisfied.

We begin by considering (5.9) and requiring the variation in the action to vanish. Since the Euler-Lagrange equation is assumed to be satisfied, the corresponding term in the integral drops out and we are eventually left with the following on the solution trajectory,

$$\int_{t_0}^{t} dt \int_{\partial \bar{\mathcal{B}}} d\bar{V} \left[d_t \mathcal{Q} + \bar{\nabla} \cdot \boldsymbol{\mathcal{J}} \right] = 0 \,, \tag{5.36}$$

where $\mathcal{Q}$ and $\boldsymbol{\mathcal{J}}$ are given by (5.15) and (5.16) respectively. From the above integral we can conclude that on the solution trajectory the conservation law $d_t \mathcal{Q} + \bar{\nabla} \cdot \boldsymbol{\mathcal{J}} = 0$ must always hold. A static version of this general statement for the theory of elasticity was reported in Edelen [24] (equation (6.7)).

We consider the translational and rotational symmetry of the embedding space by assuming $\delta \bar{\boldsymbol{x}} = \delta t = 0$, $\delta \boldsymbol{x} = \mathbf{C}$ and $\delta \boldsymbol{x} = \mathbf{C} \times \boldsymbol{x}$ ($\mathbf{C}$ being an arbitrary constant vector) to obtain the balance of physical momentum and angular momentum.

$$d_t\left(\frac{\partial\bar{\mathcal{L}}}{\partial d_t\boldsymbol{x}}\right)+\bar{\nabla}\cdot\left(\frac{\partial\bar{\mathcal{L}}}{\partial\mathbf{F}}\right)=0\,, \tag{5.37}$$

$$d_t\left(\boldsymbol{x}\times\frac{\partial\bar{\mathcal{L}}}{\partial d_t\boldsymbol{x}}\right)+\bar{\nabla}\cdot\left(\boldsymbol{x}\times\frac{\partial\bar{\mathcal{L}}}{\partial\mathbf{F}}\right)=0\,. \tag{5.38}$$

On comparing the conservation law (5.37) to the balance (5.20) we can conclude that for the physical momentum to be conserved we must have $\frac{\partial\bar{\mathcal{L}}}{\partial\boldsymbol{x}}=0$, i.e. the Lagrangian must not be an explicit function of the position vector.

Similarly, to extract the balance laws corresponding to symmetries w.r.t. shifts in the material coordinates, we consider the following balance with $\delta\bar{\boldsymbol{x}}$ unspecified,

$$d_t\left(\frac{\partial\bar{\mathcal{L}}}{\partial d_t\boldsymbol{x}}\cdot\mathbf{F}\cdot\delta\bar{\boldsymbol{x}}\right)+\bar{\nabla}\cdot\left[\left(\frac{\partial\bar{\mathcal{L}}}{\partial\mathbf{F}}\cdot\mathbf{F}-\bar{\mathcal{L}}\,\mathbf{I}\right)\cdot\delta\bar{\boldsymbol{x}}\right]=0\,. \tag{5.39}$$

If the shift in the reference configuration $\delta\bar{\boldsymbol{x}}$ is a translational symmetry of the system, i.e. $\delta\bar{\boldsymbol{x}}=$ const., the above equation will deliver the following conservation law corresponding to it.

$$d_t\left(\frac{\partial\bar{\mathcal{L}}}{\partial d_t\boldsymbol{x}}\cdot\mathbf{F}\right)+\bar{\nabla}\cdot\left[\left(\frac{\partial\bar{\mathcal{L}}}{\partial\mathbf{F}}\cdot\mathbf{F}-\bar{\mathcal{L}}\,\mathbf{I}\right)\right]=0\,. \tag{5.40}$$

Conservation laws for translational and “rotational” symmetry in the material space have been reported in equations (4.4) of Rogula [9] and equation (7.4) of [28]. The quantity inside the time derivative in equation (5.40) is what we defined as the material momentum, and the quantity inside the divergence is the Eshelby tensor. Comparing it with the bulk equation (5.23) we conclude that for material symmetry to hold, $-\frac{\partial\bar{\mathcal{L}}}{\partial\bar{\boldsymbol{x}}}=0$, i.e. the Lagrangian density cannot be an explicit function of $\bar{\boldsymbol{x}}$.

5.4 Fracture in Hyperelastic solids

The balance laws pertaining to translational invariance in the material space presented in section 5.3 are intimately related to path independent integrals that exist in fracture mechanics [15, 16, 65]. In this section, we follow the lead of A. G. Herrmann [3, 4] to relate these balance laws to the J-integral in fracture mechanics.

Consider the law of conservation of material momentum as written in (5.40) for a hyperelastic solid. The Lagrangian for such a solid can be written as the difference between the kinetic and the potential energy. Integrating (5.23) over an arbitrary volume V on the material manifold we obtain,

$$d_t \int_{\bar{V}} d\bar{V} \left(\frac{\partial \bar{\mathcal{L}}}{\partial d_t \boldsymbol{x}} \cdot \mathbf{F} \right) + \int_{\bar{V}} d\bar{V}\, \bar{\nabla} \cdot \left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot \mathbf{F} - \bar{\mathcal{L}}\, \mathbf{I} \right) = 0\,. \tag{5.41}$$

If the volume $\bar{V}$ on the material manifold encapsulates a defect, a non-zero term on the right side of the equation could exist, and in the limit $\bar{V} \to 0$ it would represent the total "configurational/material force" acting on the defect. Equation (5.41) in the limit $\bar{V} \to 0$ is known as the dynamic generalization of the J-integral [1, 13, 15, 16, 66, 6, 67, 25].

For the static case, the dynamic term on the left of equation (5.41) drops out. Employing Gauss's divergence theorem on the second term we obtain,

$$\int_{\partial \bar{V}} d\bar{A} \left(\frac{\partial \bar{\mathcal{L}}}{\partial \mathbf{F}} \cdot \mathbf{F} - \bar{\mathcal{L}}\, \mathbf{I} \right) \cdot \bar{\boldsymbol{n}} = 0\,. \tag{5.42}$$

As in the dynamic case, the right side of the equation above gives the total configurational/material force on a defect encapsulated by the volume $\bar{V}$ in the limit $\bar{V} \to 0$. The integral on the left is called the *J-integral* which was proposed by Rice [15], and Cherepanov [16], building on the work of Eshelby [1]. Gupta and Markenscoff [68] have shown that the energy flux during an infinitesimal perturbation of the inhomogeneity's position is given by the dynamic J integral if and only if the linear momentum balance is preserved for a dynamic case. In the absence of any defects, the J-integral is path independent.

5.5 Ideal Fluid

In this section we present the balance of material momentum for an ideal fluid, along with the standard balances of physical momentum and energy following the line of thought presented in section 5.2. The concept of a reference configuration serves a different purpose in fluid mechanics than it does in solids. A reference configuration for a fluid can be chosen at will since a fluid has no memory of it. A reference configuration thus serves the sole purpose of labeling material particles that convect with the fluid as the flow progresses.

Variational treatments of ideal fluids in the Lagrangian framework can be found in Eckart [41] where both compressible and incompressible fluids are considered. Bretherton [43] and Salmon [69] have presented a derivation of both Lagrangian and Eulerian versions of equations of motion from variational principles. Since fluid mechanics is in general done in the Eulerian framework, the utility of material symmetry is less apparent in it than it is in solids. However, many of the classical results in fluid mechanics are of a material character, such as the conservation of the impulse evaluated over a material loop in an ideal fluid (Kelvin's circulation theorem) [41, 42, 43, 44], and the conservation of vorticity and helicity. These concepts are therefore best understood from a Lagrangian viewpoint.

Here we derive the balance of physical and material momentum for an ideal fluid from an action principle. We show that the conservation of vorticity or Cauchy's invariants [70], Kelvin's circulation theorem [43, 41, 44], and the conservation of helicity are a consequence of the material symmetry of an ideal fluid. All of the prior derivations of these conservation laws such as the ones by Bretherton [43], Salmon [69], and Newcomb [42], invoke material symmetry, but they fall short of deriving a local balance law of which the aforementioned conservation laws are a special case. Here we will derive this local balance law of material momentum and obtain the other conservation laws from it.

We imagine the current configuration of a fluid at time t to be described by a vector function $\boldsymbol{x} \equiv \boldsymbol{x}(u^i, t)$, where $u^i, i \in \{1, 2, 3\}$ are the material labels that convect with the material particles during the flow. These material labels can be chosen as the coordinates associated with an arbitrarily chosen reference configuration with a density $\bar{\rho}$. With the current configuration at time t, we associate a metric $g_{ij} \equiv \nabla_i \boldsymbol{x} \cdot \nabla_j \boldsymbol{x}$, where ∇_i represents the covariant derivative in the current configuration, and a density ρ. Consider the following action obtained by integrating a Lagrangian density defined per unit initial reference volume describing the motion of an ideal fluid,

$$A = \int dt \int d\bar{V} \bar{\mathcal{L}} = \int dt \int dV \left[\frac{1}{2} \rho d_t \boldsymbol{x} \cdot d_t \boldsymbol{x} + p \left(\sqrt{\frac{g}{\bar{g}}} - 1 \right) \right], \tag{5.43}$$

where p is a Lagrange multiplier enforcing the incompressibility constraint, and is identified with the pressure. The first term in the Lagrangian is just the kinetic energy density of the fluid per unit reference volume. Since the fluid is incompressible we have the Jacobian $J = 1$ and therefore $d\bar{V} = dV$ and $\bar{\rho} = \rho$. The full variation of the action involving shifts in the current configuration $\boldsymbol{x}$, the material labels u^i, and time t can be written as,

$$\delta A = \int_{t_0}^{t_1} dt \int_{\mathcal{B}} dV \Bigg[d_t \Big(\rho d_t \boldsymbol{x} \cdot \delta \boldsymbol{x} + \rho d_t \boldsymbol{x} \cdot \nabla_j \boldsymbol{x} \left(-\delta u^j \right) + \frac{1}{2} \rho d_t \boldsymbol{x} \cdot d_t \boldsymbol{x} (-\delta t) \Big) \tag{5.44}$$

$$+ \nabla_i \Big(p \nabla^i \boldsymbol{x} \cdot \delta \boldsymbol{x} + \left(p - \tfrac{1}{2} \bar{\rho} d_t \boldsymbol{x} \cdot d_t \boldsymbol{x} \right) \left(-\delta u^i \right) + p \nabla^i \boldsymbol{x} \cdot d_t \boldsymbol{x} (-\delta t) \Big) \tag{5.45}$$

$$+ \Big(- d_t (\rho d_t \boldsymbol{x}) - \nabla_i \left(p \nabla^i \boldsymbol{x} \right) \Big) \cdot \left(\delta \boldsymbol{x} - \nabla_j \boldsymbol{x} \, \delta u^j - d_t \boldsymbol{x} \, \delta t \right) \Bigg]. \tag{5.46}$$

Since the Lagrangian density is not explicitly dependent on the position vector or the material coordinates, the balance laws and conservation laws corresponding to $\delta \boldsymbol{x}$, δu^i, and δt coincide, and they can be written as,

$$\rho {d_t}^2 \boldsymbol{x} + \nabla_j \left(p \nabla^j \boldsymbol{x}\right) = 0\,, \tag{5.47}$$

$$d_t(\rho d_t \boldsymbol{x} \cdot \nabla_j \boldsymbol{x}) + \nabla_j \left(p - \frac{1}{2}\rho d_t \boldsymbol{x} \cdot d_t \boldsymbol{x}\right) = 0\,, \tag{5.48}$$

$$d_t\left(\frac{1}{2}\rho d_t \boldsymbol{x} \cdot d_t \boldsymbol{x}\right) + \nabla_i \left(p \nabla^i \boldsymbol{x} \cdot d_t \boldsymbol{x}\right) = 0\,. \tag{5.49}$$

Here (5.47), (5.48), and (5.49) are the balance of physical momentum, material momentum, and energy respectively. The quantity under the time derivative in (5.48) is the material momentum of the fluid. It is also known as the "impulse" associated with the fluid [22]. This term is also referred to as the "vortex momentum" by some authors [71].

Before moving on to the balance of material momentum (5.48), we would first like to highlight the correspondence between the material or Lagrangian description of the equations of physical momentum and energy with the more commonly used Eulerian form. The balance of physical momentum is relatively straightforward to check. The second term in (5.48) can be expanded as $\nabla^j \boldsymbol{x} \nabla_j p + p \nabla^j \nabla_j \boldsymbol{x}$ which reduces to just ∇p since $\nabla^j \boldsymbol{x} \nabla_j p = \nabla p$ and $\nabla_j \nabla^j \boldsymbol{x} = \nabla^2 \boldsymbol{x} = 0$ in a Euclidean space. This gives us the familiar Euler's equation of motion for an ideal fluid $\rho {d_t}^2 \boldsymbol{x} + \nabla p = 0$ in Eulerian coordinates.

Similarly, we consider the balance of energy and derive the unsteady Bernoulli equation from it. We will denote the material velocity of the particles by $\boldsymbol{v} = d_t \boldsymbol{x}$. Its contravariant components in the current configuration can be identified as $v^i = \nabla^i \boldsymbol{x} \cdot d_t \boldsymbol{x}$. We also recognize that since the flow is incompressible, we have $\nabla_i v^i = 0$. With these points in mind, and using $\frac{dp}{dt} = \frac{\partial p}{\partial t} + v^i \nabla_i p$, equation (5.49) can be rearranged as,

$$d_t\left(\frac{1}{2}\rho \boldsymbol{v} \cdot \boldsymbol{v} + p\right) = \frac{\partial p}{\partial t}\,. \tag{5.50}$$

This is the generalized Bernoulli equation in Lagrangian coordinates. This equation is the same as equation (3.14) of Eckart [41]. To see (5.50) in the context of Eulerian coordinates and streamlines, we consider our field variables to be a function of the current position $\boldsymbol{x}$ of the material particles. On doing that, the total time derivative d_t on the left of (5.50) is expanded out using the chain rule, and after some further manipulation we obtain,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho \boldsymbol{v} \cdot \boldsymbol{v} + p\right) + \boldsymbol{v} \cdot \nabla \left(\frac{1}{2}\rho \boldsymbol{v} \cdot \boldsymbol{v} + p\right) = \frac{\partial p}{\partial t}\,, \tag{5.51}$$

$$\implies \boldsymbol{v} \cdot \frac{\partial \boldsymbol{v}}{\partial t} + |\boldsymbol{v}| \frac{d}{ds}\left(\frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v} + \frac{p}{\rho}\right) = 0\,, \tag{5.52}$$

where the operator $\boldsymbol{v}\cdot\nabla = |\boldsymbol{v}|\frac{d}{ds}$, and s is a coordinate along a streamline. If the velocity can be represented as a gradient of a potential $\boldsymbol{v} = \nabla\phi$, the above equation can be rearranged as,

$$|\boldsymbol{v}|\frac{d}{ds}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} + \frac{p}{\rho}\right) = 0\,, \tag{5.53}$$

which is the familiar unsteady Bernoulli equation in Eulerian coordinates [41].

5.5.1 Vorticity

Here we consider the balance of material momentum for an ideal fluid (5.48) and show the conservation of vorticity to be a consequence of it. We apply the curl operator $\epsilon^{ijk}\nabla_k$ to equation (5.48), where ϵ^{ijk} is the contravariant permutation symbol in curvilinear coordinates, with its value being 0 for repeated indices, and $1/\sqrt{g}$ or $-1/\sqrt{g}$ for cyclic and acyclic permutations of its indices respectively [72]. It can be immediately seen that the curl operator acting on the second term would annihilate it since the curl of a gradient is identically zero. As for first term, we can show that for any vector v_j, the curl and the material time derivative commute for isochoric deformations, i.e. $\epsilon^{ijk}\nabla_k\left(d_t v_j\right) = d_t\big(\epsilon^{ijk}\nabla_k v_j\big)$, and therefore obtain the following result,

$$d_t\big[\epsilon^{ijk}\nabla_i\left(\rho d_t\boldsymbol{x}\cdot\nabla_j\boldsymbol{x}\right)\big] = \rho d_t\omega^k = 0\,, \tag{5.54}$$

where we have moved the constant ρ through the gradient and time derivative. The quantity $\omega^k = \epsilon^{ijk}\nabla_i(d_t\boldsymbol{x}\cdot\nabla_j\boldsymbol{x})$ is identified as the contravariant component of the vorticity field. Note that this messy looking expression is nothing but the curl of the velocity field in the current configuration written in convected coordinates. This correspondence may become clear once it is recognized that $d_t\boldsymbol{x}\cdot\nabla_j\boldsymbol{x}$ is nothing but the covariant component of the material velocity with respect to the convected basis. If the entire derivation was performed in a chosen reference configuration at a particular time, the contravariant components of the resulting conserved quantity could be recognized as Cauchy's invariants [73, 70].

Another apparent consequence of the balance of material momentum (5.48) can be seen by integrating it with respect to time to obtain the following,

$$\rho d_t\boldsymbol{x}\cdot\nabla_j\boldsymbol{x} + \nabla_j\int_0^t dt\left(p - \frac{1}{2}\rho d_t\boldsymbol{x}\cdot d_t\boldsymbol{x}\right) = 0\,. \tag{5.55}$$

This particular equation is known as the Weber-Cauchy integral relation, and the time integral inside the gradient operator is referred to as Weber's function. The Weber-Cauchy integral relation is a reformulation of the equations of motion of an ideal fluid in the Lagrangian description in terms of Weber's function [74, 75].

5.5.2 Kelvin's circulation theorem

To obtain *Kelvin's circulation theorem* from the balance of material momentum (5.48), we integrate the balance law along a closed curve in the material space. Imagine a closed curve in the material space $C \equiv C(s)$ parametrized by a parameter s (not necessarily the arc-length). We multiply (5.48) with $du^i = ds\frac{du^i}{ds}$ and integrate over the material curve $C(s)$ to obtain,

$$d_t \oint_C ds \left(\rho \dot{\boldsymbol{x}} \cdot \nabla_\alpha \boldsymbol{x} \frac{du^i}{ds} \right) + \oint_C ds \, \frac{du^i}{ds} \nabla_i \left(p - \frac{1}{2} \rho d_t \boldsymbol{x} \cdot d_t \boldsymbol{x} \right) = 0 \,, \tag{5.56}$$

$$\frac{d}{dt} \oint_C \rho d_t \boldsymbol{x} \cdot d\boldsymbol{x} + \oint_C ds \frac{d}{ds} \left(p - \frac{1}{2} \rho d_t \boldsymbol{x} \cdot d_t \boldsymbol{x} \right) = 0 \,. \tag{5.57}$$

We observe that the second integral in the above relation vanishes since it is an exact differential evaluated over a closed loop, and the remaining first term, when divided through by the density ρ, delivers *Kelvin's circulation theorem*,

$$d_t \oint_C d_t \boldsymbol{x} \cdot d\boldsymbol{x} = 0 \,. \tag{5.58}$$

The circulation theorem as obtained from material symmetry, also known as "relabelling symmetry" or "exchange invariance", can be found in [41, 43, 42, 69, 44].

5.5.3 Helicity

In this section we demonstrate that the conservation of *helicity* can be obtained by considering a specific functional form of the shifts in the material coordinates. Consider the following conservation law with respect to a general shift in the material coordinates,

$$d_t \big(\rho d_t \boldsymbol{x} \cdot \nabla_j \boldsymbol{x} \, \delta u^j \big) + \nabla_j \left[\big(p - \tfrac{1}{2} \rho d_t \boldsymbol{x} \cdot d_t \boldsymbol{x} \big) \, \delta u^j \right] = 0 \,, \tag{5.59}$$

We imagine the shift in the material coordinates to be of the form $\delta u^j = \lambda \omega^j$, where λ is an arbitrary infinitesimal constant parameter and ω^j are the contravariant components of the vorticity field in the material coordinates, defined as $\omega^j = \epsilon^{jkl} \nabla_k \left(\partial_t \boldsymbol{x} \cdot \nabla_l \boldsymbol{x} \right)$. This particular form of the shifts in the material coordinates is permissible since it preserves the mass of the system, i.e. $\nabla_j \left(\rho \sqrt{g} \lambda \omega^j \right) = 0$. With these shifts in place, equation (5.59) becomes,

$$d_t \big(\rho d_t \boldsymbol{x} \cdot \nabla_j \boldsymbol{x} \, \omega^j \big) + \nabla_j \left[\big(p - \tfrac{1}{2} \rho d_t \boldsymbol{x} \cdot d_t \boldsymbol{x} \big) \, \omega^j \right] = 0 \,. \tag{5.60}$$

Integrating the above equation over a material volume and applying the divergence theorem to the second term we obtain,

$$d_t \int dV \left(\rho d_t \boldsymbol{x} \cdot \nabla_j \boldsymbol{x}\, \omega^j\right) + \int dA \left[\left(p - \tfrac{1}{2}\rho d_t \boldsymbol{x} \cdot d_t \boldsymbol{x}\right) \omega^j n_j\right] = 0\,. \tag{5.61}$$

We now invoke the assumption that on a closed vortex surface (which is also a material surface by the Helmholtz second law [76]) we have $\omega^i n_i = 0$ [77], and divide through by the density ρ to obtain,

$$d_t \int dV \left(d_t \boldsymbol{x} \cdot \nabla_j \boldsymbol{x}\, \omega^j\right) = 0\,. \tag{5.62}$$

The term $d_t \boldsymbol{x} \cdot \nabla_j \boldsymbol{x}$ is the covariant component of the velocity field in the convected basis, which makes the above equation the statement of conservation of helicity [78].

5.6 Elastica with variable cross section

Elastic rods with variable cross sectional properties serve as useful examples in studying the concept of material forces. One of the earliest investigations of this idea in beam theory was conducted by Kienzler and G. Herrmann in [37], where they also considered singular jumps in the cross sectional properties of the beams. More recently, some very interesting experiments done by Bigoni et al. [47, 79] involving the motion of elastic rods fully or partially constrained by a sleeve have renewed interest in the application of the concepts of material symmetry to rods.

In this section, we study a recent example of an elastic rod with variable cross section, including singular jumps, moving through a curved frictionless channel. This problem was first considered by Cicconofri et al. [50] and then later examined by Dal Corso et al. [48]. The treatment of this problem proposed by Dal Corso et al. [48], where they also consider the singular jumps in the stiffness of the rod and the curvature of the channel, presents a variational framework to determine the total propulsive force on the rod. To determine the particular axial reaction forces at the ends and the points of discontinuity, they resort to using micromechanical arguments. They obtain the reaction forces by first computing the reactions for a case where there is a small clearance between the channel and the rod, and then taking the limit where this clearance goes to zero.

In the rest of this section, we present an alternative viewpoint for computing the propulsive force on an elastic rod moving through a frictionless channel. This approach is based on exploiting the balance of material momentum for an elastic rod. We will first present a variational formulation for an inextensible planar elastica assuming the bending stiffness

$B \equiv B(s)$ to be a function of the material coordinate s. The balance laws concerning the physical and material force will be obtained. It will then be demonstrated that, once certain ideas about material forces generated due to singularities in the material properties are understood, the total propulsive force of the rod can be obtained by directly integrating the balance of material forces. Using insights put forward by Kienzler and G. Herrmann [37] and O'Reilly [34, 58], we will then show that the reaction forces at the points of discontinuity in stiffness and curvature can be obtained from the singular balance of the material force in a straightforward manner.

5.6.1 Elastica: Balance laws

Consider a static inextensible planar elastica with stiffness $B(s)$ which has an explicit dependence on the material coordinate s. Due to the inextensibility constraint, it follows that s coincides with the arc-length coordinate of the rod. We begin by considering the following action describing such an elastic rod,

$$A = \int ds\, \mathcal{L} = \int_{s_1}^{s_2} ds \left[-\tfrac{1}{2}B(s)\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} - \tfrac{1}{2}\sigma\left(d_s\boldsymbol{x}\cdot d_s\boldsymbol{x} - 1\right)\right], \tag{5.63}$$

where $\boldsymbol{\Omega} = d_s\boldsymbol{x} \times d_s^2\boldsymbol{x}$ is the out of plane Darboux vector and therefore $\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} = \kappa^2$, κ being the curvature of the rod. Also, s_1 and s_2 denote the arc-length coordinates of the two ends of the rod. Since the rod is inextensible, the "volume form" does not vary due to shifts in the position vector $\boldsymbol{x} \equiv \boldsymbol{x}(t)$. The first variation of (5.63) can then be written as,

$$\delta A = \int_{s_1}^{s_2} ds\, \left[d_s(T^{ss}\delta s - \boldsymbol{n}\cdot\delta\boldsymbol{x} - \boldsymbol{m}\cdot\delta\boldsymbol{\Omega}) + d_s\boldsymbol{n}\cdot(\delta\boldsymbol{x} - d_s\boldsymbol{x}\,\delta s)\right], \tag{5.64}$$

where the contact force $\boldsymbol{n}$, the contact moment $\boldsymbol{m}$, and the only component of the material stress T^{ss} are given by the expressions,

$$\boldsymbol{n} = \sigma d_s\boldsymbol{x} - d_s\big(B(s)d_s^2\boldsymbol{x}\big), \quad \boldsymbol{m} = B(s)\boldsymbol{\Omega}, \quad T^{ss} = \boldsymbol{n}\cdot d_s\boldsymbol{x} + \tfrac{\boldsymbol{m}\cdot\boldsymbol{m}}{2B(s)}. \tag{5.65}$$

To obtain the balance laws at the points of discontinuity, we assume all the fields to be discontinuous at the arc-length value s_0. Using the one dimensional version of the divergence theorem over piecewise continuous fields to integrate the boundary term in (5.64) we obtain,

$$\begin{aligned}\delta A = (T^{ss}\delta s - \boldsymbol{n}\cdot\delta\boldsymbol{x} - \boldsymbol{m}\cdot\delta\boldsymbol{\Omega})\,\big|_{s_1}^{s_2} + [\![-(T^{ss}\delta s - \boldsymbol{n}\cdot\delta\boldsymbol{x} - \boldsymbol{m}\cdot\delta\boldsymbol{\Omega})]\!]\,\big|_{s=s_0} \\ + \int_{s_1}^{s_2} ds\, \left[d_s\boldsymbol{n}\cdot(\delta\boldsymbol{x} - d_s\boldsymbol{x}\,\delta s)\right].\end{aligned} \tag{5.66}$$

From (5.66) we can infer the balance of physical momentum (corresponding to $\delta s = 0$) and material momentum (corresponding to $\delta \boldsymbol{x} = 0$). Note that in this case, the balance law for the material momentum would contain a source term due to the explicit dependence of the action on the material coordinate s. To obtain the balance laws, we consider the variations $\delta \boldsymbol{x}$ and δs to be zero at the boundaries (i.e. s_1, s_2, and s_0) and obtain the balance laws corresponding to $\delta s = 0$ and $\delta \boldsymbol{x} = 0$ as,

$$d_s \boldsymbol{n} = \mathbf{0} \tag{5.67}$$

$$d_s T^{ss} = -\tfrac{1}{2}\partial_s B(s)\kappa^2, \tag{5.68}$$

where $d_s \boldsymbol{n} \cdot d_s \boldsymbol{x} = 0$ has been rearranged to be written as (5.68). The details of this calculation for a constant B can be found in Chapter 3. Note that even though the force balance (5.67) does not have a source term, the balance of material force (5.68) does, due to the explicit dependence of the Lagrangian density on s. The component T^{ss} of the material stress was referred to as the material force c in Chapter 4.

In addition to these balance laws, we can infer the singular versions of these balance laws at the point of discontinuity s_0 from (5.66). Assuming the variations δs, $\delta \boldsymbol{x}$ to vanish at s_1 and s_2, and the Euler-Lagrange equation (5.67) to hold, we can obtain the singular balance laws corresponding to $\delta s = 0$ and $\delta \boldsymbol{x} = 0$ as follows,

$$\mathbf{P} + [\![\boldsymbol{n}]\!] = 0\,, \tag{5.69}$$

$$Y + [\![T^{ss}]\!] = 0\,. \tag{5.70}$$

Here (5.69) and (5.70) are the singular balances of physical force and material force at a point of discontinuity with the source terms $\mathbf{P}$ and Y respectively. These source terms have been added to the jump conditions by hand, since they were not accounted for in the action explicitly. For a detailed treatment of the singular balance laws of rods, the reader is referred to [34, 58]. It is natural to interpret the source term in (5.69) as a point (physical) force, however the interpretation of the material force Y is not so trivial.

Given the lack of physical understanding of the concept of material forces, prescribing Y is not as obvious as prescribing a physical point force. The prescription of the source term at a point of discontinuity in an elastic beam has been deduced for a simple case of transversely loaded beam with no point loads by Kienzler and Herrmann [37]. O'Reilly [58] has argued in his work that the source term Y in general is a constitutive parameter which is related to the jump in energy $\tilde{E}$ across the discontinuity through the following relation,

$$Y \partial_t s_0(t) = \tilde{E} - \mathbf{P} \cdot \boldsymbol{v}_0 - \mathbf{M} \cdot \boldsymbol{\omega}_0\,. \tag{5.71}$$

Here $\partial_t s_0$ is the material velocity of the discontinuity, $\mathbf{P}$ and $\mathbf{M}$ are the point force and moment at the discontinuity, and $\boldsymbol{v}_0$ and $\boldsymbol{\omega}_0$ are the velocity and angular velocity of the spatial point associated with the discontinuity [34]. From this relation we can conclude that the material source Y can exist in the presence of a non-zero energy dissipation and/or point forces and moments at the singular point.

However, something general can be said about a particular class of problems where an elastic rod is moving through a frictionless constraint fixed in space in a given inertial frame of reference. For such systems, once can assume $\tilde{E} = 0$, which is equivalent to assuming that the singular reactions at the discontinuities must perform no net work on the moving rod. Furthermore, the discontinuities in such systems can be classified into two categories. One is associated with the discontinuity in the geometry of the fixed constraint (such as a jump in curvature of a frictionless channel), and another occurs due to a discontinuity in the material properties of the rod (such as a jump in the bending stiffness). For the first kind of discontinuity, the velocity and angular velocity of the spatial point of discontinuity would be zero, i.e. $\boldsymbol{v}_0 = 0$ and $\boldsymbol{\omega}_0 = 0$. This immediately tells us from (5.71) that in such cases the source term Y vanishes. For the second kind of discontinuity, we take a cue from the source term in the bulk law for material momentum in (5.70), and infer the following prescription for the material source,

$$Y = 0 \qquad \text{for discontinuities in the channel}\,, \tag{5.72}$$

$$Y = \frac{1}{2}\,[\![B(s)]\!]\,\boldsymbol{\Omega}\cdot\boldsymbol{\Omega} \qquad \text{for discontinuities in the material}\,. \tag{5.73}$$

With the knowledge of these prescriptions of a point material force, we now proceed to analyze the problem of the "serpentine locomotion" [48] of an elastic rod through a frictionless curved channel.

5.6.2 Serpentine locomotion

In this section, we consider the motion of an elastic rod with variable cross section moving in a constrained channel as studied by Cicconofri and DeSimone [50] and Dal Corso et al. [48]. They derive an expression for a propulsive force responsible for the motion of the rod along the channel. We will show that this propulsive force can be obtained directly by integrating the balance of material momentum in the bulk and by considering the prescriptions (5.72) and (5.73). The singular propulsive forces that arise due to the jumps in the axial component of the contact force at the points of discontinuity are obtained from the jump condition (5.70) and the prescriptions (5.72) and (5.73).

Consider a frictionless channel of total length L whose curvature is given by the function $\chi \equiv \chi(S)$ where S is the curvilinear coordinate along the channel length. The curvature $\kappa(s,t)$ of the rod at a material coordinate s can then be written as $\kappa(s) = \chi(\xi(t) + s)$ where

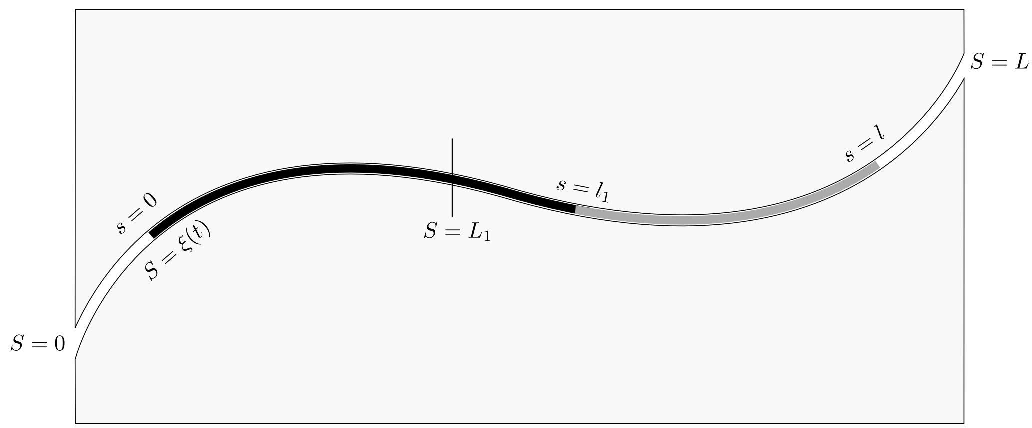

Figure 5.2: A planar elastica moving in a frictionless channel of prescribed shape. Here s and S denote the arc length values measured from the left end of the elastica and the channel respectively. The point $S = L_1$ denotes a discontinuity in the curvature of the channel, whereas $s = l_1$ depicts the point where the bending stiffness $B(s)$ suffers a jump (denoted by a color change).

$\xi(t)$ denotes the position of the left end of the rod from the left end of the channel. We consider an elastica of total length l with a discontinuity in the bending stiffness B at $s = l_1$. We also assume that the channel has a discontinuity in its curvature at the global coordinate $S = L_1$.

We consider the motion of the rod to be quasi-static and show that the propulsive force of Bigoni [48] can be directly obtained by integrating the balance of material momentum for the rod. To do that, we invoke the balance of material momentum (5.68) and define the propulsive force as its integral over the arc-length coordinate s,

$$P(t) = \int_0^l ds\ \left[d_s T^{ss} + \tfrac{1}{2}\partial_s B(s)\kappa^2\right] \tag{5.74}$$

$$= \int_0^{l_1^-} ds\ \left[d_s T^{ss} + \tfrac{1}{2}\partial_s B(s)\kappa^2\right] + \int_{l_1^+}^{l} ds\ \left[d_s T^{ss} + \tfrac{1}{2}\partial_s B(s)\kappa^2\right]\ , \tag{5.75}$$

$$= -T^{ss}(0,t) + T^{ss}(l,t) - [\![T^{ss}(l_1,t)]\!] + \int_0^{l_1^-} ds\left(\tfrac{1}{2}\partial_s B(s)\kappa^2\right) + \int_{l_1^+}^{l} ds\left(\tfrac{1}{2}\partial_s B(s)\kappa^2\right). \tag{5.76}$$

In the dynamic case, this propulsive force would be balanced by an inertial term. Combining the jump condition for the material momentum (5.70), and the prescription of the source term (5.73) delivers the following condition at any point where a jump in the bending stiffness of the rod occurs,

$$\tfrac{1}{2}\,[\![B(s)]\!]\,\kappa^2(s,t) + [\![T^{ss}(s,t)]\!] = 0\,, \tag{5.77}$$

It must be understood that the above relation, which is just the prescription (5.73) rewritten, is a constitutive prescription which has been inferred from the bulk balance law (5.68) only by way of analogy. From the above relation, one can easily compute all the terms in (5.76) as follows,

$$T^{ss}(0,t) = -\tfrac{1}{2}B(0)\kappa^2(0,t)\,, \quad T^{ss}(l,t) = -\tfrac{1}{2}B(l)\kappa^2(l,t)\,, \quad [\![T^{ss}(l_1,t)]\!] = -\tfrac{1}{2}\,[\![B(l_1)]\!]\,\kappa^2(l_1,t)\,, \tag{5.78}$$

Substituting the relations obtained above in the expression for the propulsive force (5.76) we obtain,

$$\begin{aligned} P(t) = \tfrac{1}{2}B(0)\kappa^2(0,t) - \tfrac{1}{2}B(l)\kappa^2(l,t) + \tfrac{1}{2}\,[\![B(l_1)]\!]\,\kappa^2(l_1,t) &+ \int_0^{l_1^-} ds\,\tfrac{1}{2}\partial_s B(s)\kappa^2(s,t) \\ &+ \int_{l_1^+}^{l} ds\,\tfrac{1}{2}\partial_s B(s)\kappa^2(s,t)\,, \end{aligned} \tag{5.79}$$

This is the propulsive force derived in [48] (equation 2.14) written as a function of the arc-length coordinate s and time t.

Next we discuss the tangential reactions present at the end points of the rod and at the point of discontinuity of bending stiffness and curvature. These reaction forces have been computed in [48] from micro-mechanical arguments. Here we compute these reactions directly from the jump conditions of the material momentum.

The expression for the material force T^{ss} reduces as follows from (5.65),

$$T^{ss} = \boldsymbol{n}\cdot d_s\boldsymbol{x} + \tfrac{1}{2}B(s)\kappa^2\,. \tag{5.80}$$

The first term above is the axial component of the contact force. The axial forces at the ends and the jump in the axial forces at the point of discontinuity of B and κ can be computed from (5.77) and the subsequent relations (5.78).

$$
\begin{aligned}
&\text{at } s = l_1 \qquad \llbracket \boldsymbol{n} \cdot d_s\boldsymbol{x} \rrbracket \,|_{l_1} = - \llbracket B(l_1) \rrbracket \, \kappa^2(l_1, t)\,, && (5.81)\\
&\text{at } s = 0^+ \qquad \boldsymbol{n} \cdot d_s\boldsymbol{x}|_{0^+} = -B(0)\kappa^2(0, t)\,, && (5.82)\\
&\text{at } s = l^- \qquad \boldsymbol{n} \cdot d_s\boldsymbol{x}|_{l^-} = -B(0)\kappa^2(l, t)\,. && (5.83)
\end{aligned}
$$

At a point where there is a jump in the curvature, i.e. $S = L_1$ along the channel, the prescription for the material force is $Y = 0$ from (5.72) and the axial force can be computed as,

$$\llbracket \boldsymbol{n} \cdot d_s\boldsymbol{x} \rrbracket = -\tfrac{1}{2}B(L_1 - \xi) \left\llbracket \chi^2(L_1) \right\rrbracket \tag{5.84}$$

All of these relations from (5.81) to (5.84) coincide with the equations (2.27) to (2.30) of [48].

5.7 Energy-momentum tensor

The *energy-momentum tensor* is a well known object in classical field theory. It is a second-order tensor that contains all the relevant information about the flux of the field momentum and energy of a system. In the context of classical continuum mechanics, Eshelby obtained a four dimensional tensor associated with the elastic field of a material object and called it the *(elastic) energy-momentum tensor*. Since then, this object has been extensively studied due to its utility in obtaining "forces" on inhomogeneities such as cracks in fracture mechanics [81, 3, 7, 8, 63, 5, 82, 36].

It must however be mentioned that despite having very similar formal mathematical origins, the *energy-momentum tensor* of classical field theories, such as electromagnetism and gravitation [14], is physically different from the ones that mechanicians have been referring to as the *energy-momentum tensor*, *elastic energy-momentum tensor* or simply the *Eshelby tensor*. In fact, this distinction is intimately related to the difference between the concept of physical momentum and pseudomomentum that this chapter has largely been focused on.

The key difference between the energy-momentum tensors in classical field theories and continuum mechanics is attributed to the presence of a material medium in the latter. Where classical fields are associated with the vacuum of space-time, the elastic field is attached to a set of material particles. This gives a special character to the field theories of continuum mechanics where the internal state of the material body is described using material coordinates [9], in addition to the spatial coordinates concerning with the embedding of the body in the ambient space. It is this internal structure of the material body that the *energy-momentum tensor* of Eshelby is associated with, unlike the classical fields where the *energy-momentum tensor* describes the flow of energy and momentum of the body in space-time.

Once this distinction between the role of material and spatial coordinates in describing a material body is understood, the difference between momentum and pseudomomentum renders itself clear. The state of a material body can be completely described when its embedding in the space-time, as well as its internal state associated with the material particles, has been determined. The *energy-momentum tensor* of Eshelby is obtained when the action describing the material body is varied with respect to the material coordinates, whereas varying the spatial coordinates and time delivers the four dimensional *energy-momentum tensor* commonly used to describe non-material fields. The material stress tensor of Eshelby has nothing to do with either the energy or momentum of the system. This tensor, for all practical purposes, can simply be called the "material stress" tensor by analogy with the balance of physical momentum, where the flux tensor corresponding to the physical momentum is simply referred to as the "stress". One could also just simply refer to it as the "material momentum flux" tensor, which could potentially prevent any confusion between the two concepts.

Bibliography

[1] J. D. Eshelby. The force on an elastic singularity. *Phil. Trans. R. Soc. Lond. A*, 244:87–112, 1951.

[2] A. G. Herrmann. On conservation laws of continuum mechanics. *International Journal of Solids and Structures*, 17:1–9, 1981.

[3] A. G. Herrmann. Material momentum tensor and path-independent integrals in fracture mechanics. *International Journal of Solids and Structures*, 18:319–326, 1982.

[4] A. G. Herrmann. On the lagrangian formulation of continuum mechanics. *Physica A*, 118:300–314, 1983.

[5] G. Maugin. *Material Inhomogeneities in Elasticity.* Chapman & Hall, London, 1993.

[6] G. A. Maugin. Material forces: Concepts and applications. *Applied Mechanics Reviews*, 48:213–245, 1991.

[7] R. Kienzler and G. Herrmann. *Mechanics in Material Space.* Springer, Berlin, 2000.

[8] R. Kienzler and G. Herrmann. On conservation laws in elastodynamics. *International Journal of Solids and Structures*, 41:3595–3606, 2004.

[9] D. Rogula. Noether's theorem for a continuous medium interacting with external fields. *Proceedings of Vibration Problems*, 7:337–344, 1966.

[10] I. H. Gilbert and B. R. Mollow. Momentum of longitudinal elastic vibrations. *American Journal of Physics*, 9:822–825, 1968.

[11] R. Peierls. Momentum and pseudomomentum of light and sound. In F. Bassani, F. Fumi, and M. P. Tosi, editors, *Highlights of Condensed-Matter Theory*, pages 237–255. Italian Physical Society, 1985.

[12] J. D. Eshelby. Energy relations and the energy-momentum tensor in continuum mechanics. In M. F. Kaninen, editor, *Inelastic behaviour of solids*, pages 77–114. McGraw-Hill, 1970.

[13] J. D. Eshelby. The elastic energy-momentum tensor. *Journal of Elasticity*, 5:321–335, 1975.

[14] L. D. Landau and E. M. Lifshitz. *The classical theory of fields.* Pergamon press, Oxford, 1971.

[15] J. R. Rice. A path independent integral and the approximate analysis of strain concentration by notches and cracks. *Journal of Applied Mechanics*, 35:379–386, 1968.

[16] G. P. Cherepanov. Invariant gamma-integrals and some of their applications in mechanics. *Prikladnaia Matematika i Mekhanika*, 41:399–412, 1977.

[17] E. L. Hill. Hamilton's principle and the conservation theorems of mathematical physics. *Reviews of Modern Physics*, 23:253–260, 1951.

[18] J. Rosen. Noether's theorem in classical field theory. *Annals of Physics*, 69:349–363, 1972.

[19] B. M. Barbashov and V. V. Nesterenko. Continuous symmetries in field theory. *Fortschritte der Physik*, 31:535–567, 1983.

[20] J. K. Knowles and E. Sternberg. On a class of conservation laws in linearized and finite elasticity. *Archive for Rational Mechanics & Analysis*, 44:187, 1972.

[21] D. C. Fletcher. Conservation laws in linear elastodynamics. *Archive of Rational Mechanics & Analysis*, 60:329–353, 1976.

[22] T. B. Benjamin. Impulse, flow force and variational principles. *IMA Journal of Applied Mathematics*, 32:3–68, 1984.

[23] D. Rogula. Forces in material space. *Archives of Mechanics*, 29:705–713, 1977.

[24] D. G. B. Edelen. Aspects of variational arguments in the theory of elasticity: Fact and folklore. *International Journal of Solids and Structures*, 17:729–740, 1981.

[25] R. Hill. Energy-momentum tensors in elastostatics: Some reflections on the general theory. *Journal of the Mechanics and Physics of Solids*, 34:305–317, 1986.

[26] E. S. Suhubi. Conservation laws in nonlinear elastodynamics. *International Journal of Engineering Science*, 27:441–453, 1989.

[27] D. F. Nelson. Momentum, pseudomomentum, and wave momentum: Toward resolving the Minkowski-Abraham controversy. *Physical Review A*, 44:3985–3996, 1991.

[28] A. Thellung. Momentum and quasimomentum in the physics of condensed matter. In T. Paszkiewicz and K. Rapcewicz, editors, *Die Kunst of Phonons*, pages 15–32. Springer, 1994.

[29] G. Buratti, Y. Huo, and I. Müller. Eshelby tensor as a tensor for free enthalpy. *Journal of Elasticity*, 72:31–42, 2003.

[30] G. A. Maugin. Recent advances in M^3 (mechanics on the material manifold). *Theoretical and Applied Mechanics*, 28-29:221–223, 2002.

[31] M. E. Gurtin. *Configurational Forces as Basic Concepts of Continuum Physics*. Springer, New York, 2000.

[32] P. Podio-Guidugli. Configurational balance via variational arguments. *Interfaces and Free Boundaries*, 3:223–232, 2001.

[33] E. Fried and M. E. Gurtin. The unifying nature of the configurational force balance. In P. Steinmann and G. A. Maugin, editors, *Mechanics of Material forces*, pages 25–32. Springer, Boston.

[34] O. M. O'Reilly. A material momentum balance law for rods. *Journal of Elasticity*, 86:155–172, 2007.

[35] O. M. O'Reilly. The energy jump condition for thermomechanical media in the presence of configurational forces. *Continuum Mechanics and Thermodynamics*, 18:361–365, 2007.

[36] G. A. Maugin. *Configurational Forces*. CRC Press, Boca Raton, 2011.

[37] R. Kienzler and G. Herrmann. On material forces in elementary beam theory. *Journal of Applied Mechanics*, 53:561–564, 1986.

[38] K. R. Rajagopal and A. R. Srinivasa. On the role of the Eshelby energy-momentum tensor in materials with multiple natural configurations. *Mathematics and Mechanics of Solids*, 10:3–24, 2005.

[39] A. Yavari, J. E. Marsden, and M. Ortiz. On spatial and material covariant balance laws in elasticity. *Journal of Mathematical Physics*, 47:042903, 2006.

[40] M. E. McIntyre. On the 'wave momentum' myth. *Journal of Fluid Mechanics*, 106:331–347, 1981.

[41] C. Eckart. Variational principles of hydrodynamics. *The Physics of Fluids*, 3:3, 1960.

[42] W. A. Newcomb. Exchange invariance in fluid systems. *Proceedings of Symposia in Applied Mathematics*, 18:152–161, 1967.

[43] F. P. Bretherton. A note on Hamilton's principle for perfect fluids. *Journal of fluid mechanics*, 44:19–31, 1970.

[44] N. Padhye and P. J. Morrison. Fluid element relabeling symmetry. *Physics letters A*, 219:287–292, 1996.

[45] A. R. Atilgan. Analogy between dislocation mechanics and aerodynamics. *Journal of Applied Mathematics and Mechanics*, 77:631–633, 1977.

[46] F. Bosi, D. Misseroni, F. Dal Corso, and D. Bigoni. An elastica arm scale. *Proceedings of the Royal Society A*, 470:20140232, 2014.

[47] D. Bigoni, F. Dal Corso, F. Bosi, and D. Misseroni. Eshelby-like forces acting on elastic structures: Theoretical and experimental proof. *Mechanics of Materials*, 80:368–374, 2015.

[48] F. Dal Corso, D. Misseroni, N. M. Pugno, A. B. Movchan, N. V. Movchan, and D. Bigoni. Serpentine locomotion through elastic energy release. *Journal of the Royal Society Interface*, 14:20170055, 2017.

[49] O. M. O'Reilly. Some perspectives on Eshelby-like forces in the elastica arm scale. *Proceedings of the Royal Society A*, 471:20140785, 2015.

[50] G. Cicconofri and A. DeSimone. A study of snake-like locomotion through the analysis of a flexible robot model. *Proceedings of the Royal Society A*, 471:20150054, 2015.

[51] R. Capovilla and J. Guven. Stresses in lipid membranes. *Journal of Physics A: Mathematical and General*, 35:6233–6247, 2002.

[52] R. Capovilla. Elastic bending energy: A variational approach. *Journal of Geometry and Symmetry in Physics*, 45:1–45, 2017.

[53] J. A. Hanna. Some observations on variational elasticity and its application to plates and membranes. [arXiv:1807.06426], 2018.

[54] E. Noether. Invariant variation problems. *Transport Theory and Statistical Physics*, 1:186–207, 1971. Translation by M. A. Tavel.

[55] M. E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua*. Cambridge University Press, Cambridge, 2010.

[56] H. Goldstein, J. L. Safko, and C. P. Poole. *Classical Mechanics*. Addison-Wesley, Boston, 2001.

[57] A. C. Eringen. *Mechanics of Continua*. Robert E. Krieger Publishing Company, New York, 1980.

[58] O. M. O'Reilly. *Modeling Nonlinear Problems in the Mechanics of Strings and Rods*. Springer, New York, 2017.

[59] H. Singh and J.A. Hanna. Pick-up and impact of flexible bodies. *Journal of the Mechanics and Physics of Solids*, 106:46–59, 2017.

[60] J. A. Hanna, H. Singh, and E. G. Virga. Partial constraint singularities in elastic rods. *Journal of Elasticity*, 2018.

[61] E. G. Virga. Dissipative shocks in a chain fountain. *Physical Review E*, 89(053201), 2014.

[62] E. G. Virga. Chain paradoxes. *Proceedings of the Royal Society A*, 471(20140657), 2015.

[63] G. A. Maugin. Applications of an energy-momentum tensor in nonlinear elastodynamics: Pseudomomentum and Eshelby stress in solitonic elastic systems. *Journal of the Mechanics and Physics of Solids*, 40:1543–1558, 1992.

[64] J. K. Knowles. On the dissipation associated with equilibrium shocks in finite elasticity. *Journal of Elasticity*, 9:131–158, 1979.

[65] L. B. Freund. *Dynamic fracture mechanics*. Cambridge University Press, Cambridge, 1990.

[66] X. Markenscoff. Eshelby generalization for the dynamic J, L, M integrals. *C. R. Mecanique*, 334:701–706, 2006.

[67] T. Nakamura, C. F. Shih, and L. B. Freund. Computational methods based on an energy integral in dynamic fracture. *International Journal of Fracture*, 27:1985, 229–243.

[68] A. Gupta and X. Markenscoff. A new interpretation of configurational forces. *Journal of Elasticity*, 108:225–228, 2012.

[69] R. Salmon. Hamiltonian fluid mechanics. *Annual Review of Fluid Mechanics*, 20:220–256, 1988.

[70] U. Frisch and B. Villone. Cauchy's almost forgotten Lagrangian formulation of the Euler equation for 3d incompressible flow. *The European Physical Journal H*, 39:325–351, 2014.

[71] G. A. Kuz'min. Ideal incompressible hydrodynamics in terms of the vortex momentum density. *Physics Letters*, 96A:88–90, 1983.

[72] W. Flügge. *Tensor analysis and continuum mechanics*. Springer, Berlin, 1972.

[73] J. Casey and P. M. Naghdi. On the Lagrangian description of vorticity. *Archive for Rational Mechanics and Analysis*, 115:1–14, 1991.

[74] H. Lamb. *Hydrodynamics*. Dover, Dover, New York.

[75] A. Bennett. *Lagrangian Fluid Dynamics*. Cambridge University Press, Cambridge, 2006.

[76] P. G. Saffman. *Vortex Dynamics*. Cambridge University Press, Cambridge, 1992.

[77] H. K. Moffatt. The degree of knottedness of tangled vortex lines. *Journal of Fluid Mechanics*, 35:117–129, 1969.

[78] A. Yahalom. Helicity conservation *via* the noether theorem. *Journal of Mathematical Physics*, 36:1324–1327, 1995.

[79] D. Bigoni, F. Dal Corso, D. Misseroni, and F. Bosi. Torsional locomotion. *Proceedings of the Royal Society A*, 470:20140599, 2014.

[80] H. Singh and J. A. Hanna. On the planar *elastica*, stress, and material stress. *Journal of Elasticity*, 2018. https://doi.org/10.1007/s10659-018-9690-5

[81] A. G. Herrmann and G. Herrmann. On energy-release rates for a plane crack. *Journal of Applied Mechanics*, 48:525–528, 1981.

[82] G. A. Maugin. On the J-integral and energy release rates. *Acta Mechanica*, 105:33–47, 1994.

Appendix A

Pick-up and impact of flexible bodies

A.1 The Clausius-Duhem inequality

The conclusions made in this work about the admissibility of a shock in a string rely on the assumed non-positivity of the invariant measure of energy dissipation E as defined in (2.15). In this appendix, we briefly note that this assumption can be inferred from the second law of thermodynamics[1]. For our simple string, we may write

$$\frac{d}{dt}\int_{s_1}^{s_2} ds\,\tfrac{1}{2}\mu\partial_t\mathbf{X}\cdot\partial_t\mathbf{X} \leq \sigma\partial_s\mathbf{X}\cdot\partial_t\mathbf{X}\,\big|_{s_1}^{s_2} + \tfrac{1}{2}\mathbf{P}\cdot\{\partial_t\mathbf{X}\}\,, \tag{A.1}$$

for a material interval enclosing a moving discontinuity at $s_0(t)$, such that $s_1 < s_0(t) < s_2$. This form of the Clausius-Duhem inequality, without the final term involving the stress source $\mathbf{P}$, can be arrived at from an energy balance and entropy imbalance following Chapters 6 & 7 of Gurtin [1] and noting that our string has no free energy— neither internal energy nor entropy terms exist. We split the integral over two time-dependent intervals, apply the Leibniz rule, and let s_1 and s_2 approach s_0 from below and above, respectively, to obtain

$$-\left[\!\left[\sigma\partial_s\mathbf{X}\cdot\partial_t\mathbf{X} + \tfrac{1}{2}\mu\partial_t s_0\partial_t\mathbf{X}\cdot\partial_t\mathbf{X}\right]\!\right] - \tfrac{1}{2}\mathbf{P}\cdot\{\partial_t\mathbf{X}\} \leq 0\,, \tag{A.2}$$

which from (2.10) and (2.15) is simply

$$E \leq 0\,. \tag{A.3}$$

For a rod, an internal bending energy term and the working of an areal torque source need to be considered as well.

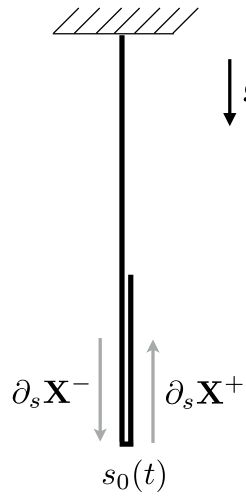

Figure A.1: A folded chain with one fixed and one falling segment.

A.2 Falling folded chain

Consider a chain fixed at one end $s = 0$, while the other end $s = l$ falls after being released from rest in a perfectly folded configuration (Figure A.1). This problem has a history as a challenging and counterintuitive example in dynamics [2, 3, 4, 5, 6, 7, 8], and also sees application in the context of cable deployment from orbiting spacecraft [9] and recreational bungee jumping ([10] and references therein). From the perspective of the current paper, the perfectly folded chain is a special configuration where the sum of tangents $\{\partial_s \mathbf{X}\} = 0$. Additionally, there is no external source $\mathbf{P}$, and our compatibility relation (2.30) is satisfied identically for any arbitrary choice of internal dissipation $E = \tilde{E}$.

We will combine the general expressions (2.29) with a constitutive prescription for E to recover Virga's [8] results for this problem. The prescription may be derived from O'Reilly and Varadi's general prescription for the jump in tension in equation (7.7) of [11],

$$[\![\sigma]\!] = -2 f \mu |\partial_t s_0| \partial_t s_0 \, , \tag{A.4}$$

where f is a constitutive parameter. Substituting (A.4) in (2.23) prescribes E as

$$E = -\tfrac{1}{2} f \mu |\partial_t s_0| (\partial_t s_0)^2 \, [\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!] \; , \tag{A.5}$$

or, using velocity compatibility (2.2), as

$$E = -\tfrac{1}{2} f \mu |\partial_t s_0| \, [\![\partial_t \mathbf{X}]\!] \cdot [\![\partial_t \mathbf{X}]\!] \; , \tag{A.6}$$

[1]We thank A. Gupta for suggesting this line of argument.

which agrees with equation (9) of [12] and equation (4.1) of [8].

With $\mathbf{P}$ set to zero and a constitutive prescription of E specified, the bulk equation $\mu\partial_t^2\mathbf{X} = \partial_s(\sigma\partial_s\mathbf{X}) + \mu\boldsymbol{g}$ can be projected onto the tangents on each side and integrated, with (2.29) providing the boundary conditions at the fold. Using (A.5), we obtain the boundary conditions

$$\sigma^{\pm} = \mu\partial_t s_0 \left(\partial_t s_0 \mp f|\partial_t s_0|\right) . \tag{A.7}$$

For these kinematics, velocity compatibility (2.2) indicates that on the falling side, $\partial_t\mathbf{X} = -2\partial_t s_0\partial_s\mathbf{X}$, and thus $\partial_t^2\mathbf{X} = -2\partial_t^2 s_0\partial_s\mathbf{X}$. Integrating the projected bulk equation $-2\mu\partial_t^2 s_0 = \partial_s\sigma - \mu|\boldsymbol{g}|$ and using $\sigma(l) = 0$ at the free end, we obtain

$$\sigma(s) = -\mu(|\boldsymbol{g}| - 2\partial_t^2 s_0)(l - s) \quad \text{for} \quad s_0 \leq s \leq l\,. \tag{A.8}$$

Evaluating this at $s = s_0$ and using (A.7), we may write a differential equation for the dynamics of the shock,

$$\partial_t s_0(\partial_t s_0 - f|\partial_t s_0|) + (|\boldsymbol{g}| - 2\partial_t^2 s_0)(l - s_0) = 0\,, \tag{A.9}$$

in agreement with the free-end case of equation (6.6) of [8]. Similarly, the reaction force at the support can be obtained by integrating the bulk equation on the stationary side and employing the boundary condition at the fold. We obtain

$$\sigma(0) = \mu\partial_t s_0(\partial_t s_0 + f|\partial_t s_0|) + \mu|\boldsymbol{g}|s_0\,, \tag{A.10}$$

in agreement with equation (6.7) of [8].

The freedom to prescribe E is a special feature of the perfectly folded chain that cannot be used in more realistic configurations such as those of the experiments in [6].

A.3 Peeling

Peeling [13, 14, 15, 16] is a moving-boundary problem related to standard adhesion tests, and is sometimes used as a toy model of fracture. Interesting inertial and material instabilities have been observed in peeling tapes [17, 18]. In this appendix, we briefly show how classical quasi-static results relate to our approach.

Consider a tape peeled from a plane rigid surface, as in Figure A.2. The stress supply $\mathbf{P}$ for an adhesive surface can point in any direction. To obtain the power balance, we modify the Lagrangian density (2.7) to be

$$2\mathcal{L} = \mu\partial_t\mathbf{X}\cdot\partial_t\mathbf{X} - \sigma(\partial_s\mathbf{X}\cdot\partial_s\mathbf{X} - 1) - 2\varepsilon\,, \tag{A.11}$$

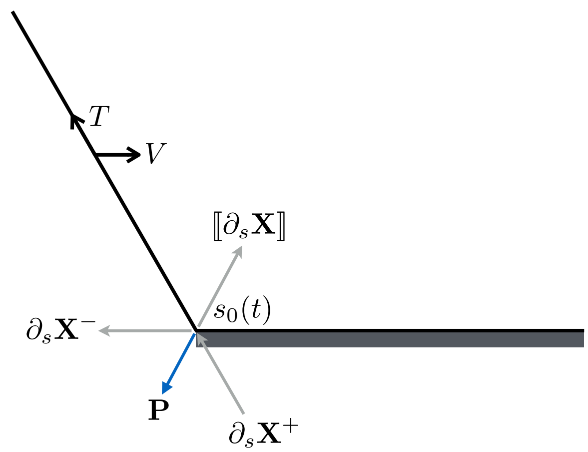

Figure A.2: A peeling tape, akin to the pick-up problem of Figure 2.3 without gravity and with a stationary contacting piece adhered to the surface. The reaction stress **P** points antiparallel to the jump in tangents.

where ε is the internal energy density of the system, a quantity we will relate to the adhesion energy below. We don't concern ourselves with constraint terms on the position of the adhered portion of the tape, as they would not end up in the jump condition. The jump condition (2.10) is modified accordingly,

$$\tilde{E} + [\![\sigma \partial_s \mathbf{X} \cdot \partial_t \mathbf{X} + \partial_t s_0 \left(\tfrac{1}{2} \mu \partial_t \mathbf{X} \cdot \partial_t \mathbf{X} + \varepsilon \right)]\!] = 0\,. \tag{A.12}$$

Note that in a typical analysis of this problem, $\partial_t \mathbf{X}^- = 0$. From the pick-up kinematics (Section 2.6), we know then that $\partial_t \mathbf{X}^+ = -\partial_t s_0 \, [\![\partial_s \mathbf{X}]\!]$. Let us write

$$[\![\varepsilon]\!] = 0 - (-\Gamma/d) = \Gamma/d\,, \tag{A.13}$$

where Γ, a positive number, is the surface adhesion energy and d is the thickness of the tape. In the quasi-static limit ($\mu = 0$), the assumption that $\tilde{E} = 0$ along with the association $\sigma^+ = F/bd$, where F is the magnitude of the axial pull force and b is the width of the tape, recovers the inextensible limit of the Rivlin-Kendall result [13, 14]

$$\Gamma = (F/b)(1 - \cos\theta)\,, \tag{A.14}$$

where $\cos\theta = \partial_s \mathbf{X}^+ \cdot \partial_s \mathbf{X}^-$. Note two special cases. If the peeling angle $\theta = 0$, peeling cannot occur, while if the adhesion energy $\Gamma = 0$, the peeling angle must be zero, a situation akin to our pick-up problem in Section 2.6.

If we rearrange (A.12) in the manner performed in the main text, we obtain

$$E + \partial_t s_0 \, [\![\varepsilon]\!] = \tfrac{1}{4} \partial_t s_0 \, [\![\sigma]\!] \, [\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!]\ . \tag{A.15}$$

We now recognize that $E + \partial_t s_0 [\![\varepsilon]\!] = 0$, meaning that the peeled tape has a higher translation-invariant energy density than the adhered tape, corresponding to an "injection" of energy $E > 0$. Thus, the tension σ is continuous for a kink, and from (2.12) we see that $\mathbf{P}$ points in the direction of the jump in tangents (antiparallel with the jump for a positive tension, as shown in Figure A.2). The compatibility relation of Section 2.4 will be modified to

$$\big(E + \partial_t s_0 [\![\varepsilon]\!]\big)\{\partial_s \mathbf{X}\} \cdot \{\partial_s \mathbf{X}\} = -\tfrac{1}{2}\partial_t s_0 \mathbf{P} \cdot \{\partial_s \mathbf{X}\} [\![\partial_s \mathbf{X}]\!] \cdot [\![\partial_s \mathbf{X}]\!] \,, \tag{A.16}$$

which is satisfied by having $E + \partial_t s_0 [\![\varepsilon]\!] = 0$ and $\mathbf{P} \cdot \{\partial_s \mathbf{X}\} = 0$.

The quasi-static Rivlin-Kendall result was also obtained by Pede and co-workers [19], who employed a configurational balance instead of an energy balance. Our energy approach, involving consideration of the invariant quantity E, is consistent with the main result and also gives us the direction of $\mathbf{P}$, but we only obtain the main result (as well as the magnitude of $\mathbf{P}$) by considering $\tilde{E}$. This is perhaps to be expected, as the adhesive surface breaks the translational symmetry of the problem. Application of an isothermal Clausius-Duhem inequality to the shock would tell us that our positive E needs to be compensated by a jump in entropy.

A.4 String pulled from a pile

Variations on another classic variable-mass problem [11] that have attracted much recent interest involve the deployment of a string-like body from a loose, stationary pile. As these variations are often improperly conflated, we briefly summarize known results here.

Two types of chain are typically used in the relevant experiments. A link chain offers essentially no bending resistance, though may offer some hard-wall twist resistance if sufficiently twisted. A ball-and-link chain offers a hard-wall bending resistance if sufficiently bent. Pulling one end of either type of chain with constant velocity leads to the formation of a small arch-like structure near the pile. The ball-and-link chain pushes back on the pile [21], while the link chain does not [22]. If the ball-and-link chain's deployment is an accelerated motion driven by gravity, a large arch known as a "chain fountain" forms [23, 24]. If observed for long enough, the structure resembles a very noisy catenary. The effect is small, if present at all, in a link chain, and so appears to require the existence of a push-back effect arising from bending resistance [24]. The smaller arch structure formed by either type of chain has been observed on short time scales, where gravity may be ignored. Though it does not require a bending push-back, we speculate that its formation may be connected to the twist resistance of the chain. We recall from Section 2.6 that the pick-up and impact effects discussed in the present paper have been observed in both types of chain.

Both Biggins [24] and Virga [12] have proposed models of the chain fountain which are not consistent with the analysis in the present paper. They make assumptions to simplify the geometry and kinematics of the system which violate velocity compatibility (and thus indirectly conservation of mass) and/or require energy injection. Both authors seem aware of these limitations, and our present treatment is also unable to capture the experimental observations of these deployment phenomena. Likewise, it was recognized by O'Reilly and Varadi [11] that textbook treatments of similar string problems lead to impossibilities when they beg the question of the geometry and kinematics. It is clear from the analysis in this paper that the question of matching a concave-down catenary with a surface-contacting piece is not trivial. We conclude that the geometry of these problems is likely to be a dynamic, three-dimensional one, as already suggested by experiments [22, 20].

Bibliography

[1] M. E. Gurtin. *Configurational Forces as Basic Concepts of Continuum Physics*. Springer, New York, 2000.

[2] J. Satterly. Falling chains. *American Journal of Physics*, 19:383–384, 1951.

[3] W. A. Heywood, H. Hurwitz, and D. Z. Ryan. Whip effect in a falling chain. *American Journal of Physics*, 23(5):279–280, 1955.

[4] M. G. Calkin and R. H. March. The dynamics of a falling chain: I. *American Journal of Physics*, 57(2):154–157, 1989.

[5] M. Schagerl, A. Steindl, W. Steiner, and H. Troger. On the paradox of the free falling folded chain. *Acta Mechanica*, 125:155–168, 1997.

[6] W. Tomaszewski, P. Pieranski, and J-C. Géminard. The motion of a freely falling chain tip. *American Journal of Physics*, 74:776–783, 2006.

[7] C. W. Wong and K. Yasui. Falling chains. *American Journal of Physics*, 74(6):490–496, 2006.

[8] E. G. Virga. Chain paradoxes. *Proceedings of the Royal Society A*, 471(20140657), 2015.

[9] T. R. Kane and D. A. Levinson. Deployment of a cable-supported payload from an orbiting spacecraft. *Journal of Spacecraft and Rockets*, 14(7):409–413, 1977.

[10] A. Heck, P. Uylings, and E. Kędzierska. Understanding the physics of bungee jumping. *Physics Education*, 45(1):63–72, 2010.

[11] O. M. O'Reilly and P. C. Varadi. A treatment of shocks in one-dimensional thermomechanical media. *Continuum Mechanics and Thermodynamics*, 11:339–352, 1999.

[12] E. G. Virga. Dissipative shocks in a chain fountain. *Physical Review E*, 89(053201), 2014.

[13] R. S. Rivlin. The effective work of adhesion. *Paint Technology*, IX(106):1–4, 1944.

[14] K. Kendall. Thin-film peeling–the elastic term. *Journal of Physics D: Applied Physics*, 8:1449–1452, 1975.

[15] J. L. Ericksen. *Introduction to the Thermodynamics of Solids*. Springer, New York, 1998.

[16] R. Burridge and J. B. Keller. Peeling, slipping and cracking–some one-dimensional free-boundary problems in mechanics. *SIAM Review*, 20:31–61, 1978.

[17] P.-P. Cortet, M.-J. Dalbe, C. Guerra, C. Cohen, M. Ciccotti, S. Santucci, and L. Vanel. Intermittent stick-slip dynamics during the peeling of an adhesive tape from a roller. *Physical Review E*, 87:022601, 2013.

[18] M.-J. Dalbe, S. Santucci, P.-P. Cortet, and L. Vanel. Strong dynamical effects during stick-slip adhesive peeling. *Soft Matter*, 10:132–138, 2014.

[19] N. Pede, P. Podio-Guidugli, and G. Tomassetti. Balancing the force that drives the peeling of an adhesive tape. *Il Nuovo Cimento*, 121 B(5):531, 2006.

[20] R. Martins. The (not so simple!) chain fountain. [arXiv:1612.09319].

[21] J.-C. Géminard and G. Palacio. A chain falling from a table. `http://perso.ens-lyon.fr/jean-christophe.geminard/a_chain_falling_from_a_table.htm` , 2012.

[22] J. A. Hanna and H. King. An instability in a straightening chain. [arXiv:1110.2360].

[23] S. Mould. Amazing bead chain experiment in slow motion. `www.youtube.com/watch?v=6ukMId5fIi0` , 2013.

[24] J. S. Biggins. Growth and shape of a chain fountain. *Europhysics Letters*, 106:44001, 2014.

Appendix B

Journal copyright permissions

Chapter 2 of this dissertation has appeared in the *Journal of the Mechanics and Physics of Solids*, which belongs to the publishing company Elsevier. According to the copyright policies of Elsevier, the authors can use their articles, in full or in part, for inclusion in a theses or dissertation (provided that this is not to be published commercially). Therefore, the permission to include this paper in this dissertation is granted. Detailed information on the copyright policies of Elsevier can be found at the link: `https://www.elsevier.com/about/policies/copyright`. The published article can be found at: `https://www.sciencedirect.com/science/article/pii/S0022509617301308`.

Chapter 3 of this dissertation has appeared in the *Journal of Elasticity* which belongs to the publishing company Springer. According to the copyright policies of Springer, the authors retain the right to reproduce the article in whole or in part in a thesis written by the authors provided acknowledgment is given to the original source of publication. An acknowledgment to the original published article has been prominently placed at the beginning of chapter 3. Detailed information on the copyright policy of Springer can be found on `https://www.springer.com/birkhauser?SGWID=0-40290-6-1420142-0`. The published article is available at: `http://dx.doi.org/10.1007/s10659-018-9673-6`.

Chapter 4 of this dissertation has appeared in the *Journal of Elasticity* which belongs to the publishing company Springer. According to the copyright policies of Springer, the authors retain the right to reproduce the article in whole or in part in a thesis written by the authors provided acknowledgment is given to the original source of publication. An acknowledgment to the original published article has been prominently placed at the beginning of chapter 4. Detailed information on the copyright policy of Springer can be found on `https://www.springer.com/birkhauser?SGWID=0-40290-6-1420142-0`. The published article is available at: `http://dx.doi.org/10.1007/s10659-018-9690-5`.

Chapter 5 of this dissertation has not yet appeared in any journal, and therefore, no permissions are required.