INTER-BLOCK ANALYSIS OF INCOMPLETE BLOCK DESIGNS

by

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I. GENERAL REMARKS ABOUT DESIGNS

1.1 Introduction

In order for the reader to understand the work done on these designs, it will first be necessary to review briefly the characteristics of Balanced Incomplete Block and Partially Balanced Incomplete Block designs.

1.2 Balanced Incomplete Blocks (BIB)

1.21 **Definition.** A discussion of these designs may be found in Cochran and Cox [4, pp. 259]. Briefly, there are v treatments and b blocks, with each block containing k treatments and each treatment being found in r blocks.

Any two treatments occur together in the same block a certain fixed number of times; say λ, and λ is the same irrespective of which pair of treatments are being considered.

1.22 **Example.** As an illustration of the BIB designs, consider plan 11.1 from Table 11.3 of Cochran and Cox. The field plan is diagrammed in Table 1.1. This may then be expanded to the more easily studied form of Table 1.2 which we may call the incidence diagram. From it we can readily verify that $v = 4$, $k = 2$, $r = 3$, $b = 6$, and $\lambda = 1$.

Table 1.1

Blocks	Treatments
1	1,2
2	3,4
3	1,3
4	2,4
5	1,4
6	2,3

Table 1.2

	Treatments			
Blocks	1	2	3	4
1	x	x		
2			x	x
3	x		x	
4		x		x
5	x			x
6		x	x	

1.3 Partially Balanced Incomplete Blocks (PBIB)

1.31 Definition. A complete discussion of the properties of these designs may be found in Bose and Shimamoto [2, pp. 151]. The important characteristics are as follows:

(a) The parameters v, k, b, and r are as defined for BIB.

(b) Any two treatments are either 1st, 2nd, ..., or mth associates, where m is the number of associate classes in the design.

(c) Two treatments which are ith associates occur together in exactly λ_i blocks.

(d) Each treatment has exactly n_i ith associates.

(e) Given any two treatments which are ith associates, the number of treatments common to the jth associates of the first and the kth associates of the second is p^i_{jk}, and is the same no matter which two ith associates we are considering.

It is obvious that if m = 1, the design must be BIB. For the purposes of this paper we will confine our discussion to the cases m = 1 or 2.

1.32 Example. As an example of the PBIB designs, we will take Plan SR-1 from the B.C.S. Tables of Partially Balanced Designs with Two Associate Classes [1, pp. 139]. The incidence diagram is shown in Table 1.3. From this we may verify that $v = 6$, $k = 2$, $r = 3$, $b = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $n_1 = 1$, and $n_2 = 4$.

Table 1.3

	Treatments					
Blocks	1	2	3	4	5	6
1	x		x		x	
2	x			x		x
3		x	x			x
4		x		x	x	

Careful study will show that condition 1.31e of the properties of PBIB designs is satisfied for the above plan. Specifically:

$$p^1_{11} = 0 \qquad p^1_{12} = p^1_{21} = 0$$

$$p^1_{22} = 4 \qquad p^2_{11} = 0$$

$$p^2_{12} = p^2_{21} = 1 \qquad p^2_{22} = 2$$

for all cases.

1.4 Duality

1.41 Definition and Properties. There is one more characteristic of all incomplete block designs which is sometimes studied. It is called duality and is the focal point of this paper.

If we let D be our design, we can form a design D* as follows:

Let the treatments of D become the blocks of D*, and the blocks of D become the treatments of D*. D* will then be the dual of D.

The parameters of D* are called v*, k*, r*, and b*, and are analogous to the parameters of D.

1.42 Illustration of Duality. If we desire to study the dual of Plan 11.1, we merely interchange the treatments and blocks. This, in effect, is the same as tilting the incidence diagram on its side. Then we can find the method of association between each pair of the v* treatments in the same manner as we did for the BIB or PBIB designs.

This gives us v*, k*, r*, and b*, as well as the λ_i and n_i parameters. With these we can look up the apparent dual in a catalogue of BIB or PBIB designs.

Note that the designs which we used as examples of BIB and PBIB are mutually dual; i.e., Plan 11.1 (BIB) is the dual of Plan SR-1 (PBIB), and vice versa. A dual may turn out to be BIB, PBIB, or perhaps not balanced at all.

1.43 Requirements of Proof. If the dual has a constant λ throughout, then it is BIB, and no further proof is necessary. If, however, from consideration of the λ parameters it seems to be PBIB, the reader should note that it has not yet been proven to be so but to merely resemble some PBIB design.

Proof that a dual is actually PBIB, as suspected, involves condition 1.31e of the rules for PBIB designs; i.e., the constancy of p^i_{jk}. For more complicated designs, this is exceedingly difficult to demonstrate directly and is seldom used as a proof.

1.44 Nomenclature. Youden [2, pp. 159] applies the name "linked block" to designs whose duals are BIB. In keeping with this terminology, we call those designs whose duals are PBIB "partially linked block."

1.5 Model and Assumptions

1.51 General Model. Regardless of what kind of incomplete block design we are using, we assume that if the ith treatment occurs in the jth block, the model is

$$y_{ij} = \alpha + \tau_i + \beta_j + \epsilon_{ij} , \tag{1.51.1}$$

where y_{ij} is the measurement or score resulting from the ith treatment in the jth block, α is the mean effect, τ_i is the effect of the ith treatment, and β_j is the effect of the jth block.

The ϵ_{ij}'s are the error terms and are assumed to be normally and independently distributed with mean zero and variance σ^2. The model and assumptions are treated at length in Bose and Shimamoto [2, pp. 167].

1.52 Two Kinds of Assumptions. Briefly, the assumptions concerning the treatment and block effects are of two types. In one type, Model I, both the τ_i's and the β_j's are considered fixed, and there is no variance except that belonging to the error term.

In the second type, called the Mixed Model, the block effects are not considered fixed, but to be random variables which are normal and independent with zero mean and variance σ_1^2. In addition, the β_j's are independent of the ϵ_{ij}'s.

II. TWICE BALANCED DESIGNS

2.1 Purpose of this Paper

The purpose of this paper is to catalogue known BIB and PBIB designs having duals which are also balanced or partially balanced. Such designs we will call "twice balanced." Remember that throughout this paper, we consider PBIB designs with two associate classes only.

2.2 Reasons for Cataloguing Twice Balanced Designs

2.21 General Benefits. In the original Model I agricultural application of incomplete block designs, the experimenter is usually not interested in estimating the block effects. In some experiments, however, the block effects might be just as interesting as the treatment effects.

For instance, if, in a sawmill, we wish to study the output of different sawyers using different types of saw blades, we might consider the men to be the blocks and the blades the treatments. Here, we would be interested not only in the blades but also in the sawyers. In such an experiment, choosing a design which is twice balanced would:

(a) facilitate the estimation of the man effects and permit us to make this estimation without first having to evaluate the blade effects,

(b) insure an equality of precision in our estimates of the man effects.

In addition, if our twice balanced design is linked block PBIB, we may avoid having to analyse the PBIB design by first estimating the block effects and then finding the treatment estimates by using the equation (4.24.1).

2.22 Usefulness under Mixed Model. If the Mixed Model seems appropriate to the experimenter, he may be interested in the relative variability of the inter- and intra-block error. Thus, in our example, we may want to assume our men to be randomly chosen from all possible sawyers and see how variable their output is from one to another in relation to the variability in production of a typical man using a given saw blade.

A method has been devised by Thompson [5] for studying the ratio

$$\mu = \frac{\sigma_1^2}{\sigma^2} \tag{2.22.1}$$

which is a measure of the above mentioned variability. This method becomes simple for twice balanced designs as we will demonstrate in Chapter IV.

2.3 Verifying Designs to be Twice Balanced

2.31 Introduction. In 2.2, we have seen reasons for cataloguing twice balanced designs. In this and the next chapter, we attempt to study and catalogue these duality relationships for certain BIB and PBIB designs and to give methods by which we can determine the type of dual that a design possesses.

Such duality has been investigated by Bose, Clatworthy, Youden, Shrikhande, Shimamoto, and others, but no systematic study has been made of specific designs and their duals.

The principal stumbling block in the path of cataloguing these designs and their duals, is the necessity for proof that an apparently PBIB dual is really partially balanced. We have already said that such a proof depended on condition 1.31e which was difficult to demonstrate directly.

Despite this obstacle, there are several methods by which we can prove a dual to be PBIB. Some are mathematical and some merely diagrammatic schemes, but combined they take care of the bulk of known designs. The rest of this chapter will be devoted to these methods of proof.

2.32 Methods of Proof.

(a) Shrikhande [2, pp. 159] has shown that in a BIB design, if

(1) $\lambda = 1$, or

(2) $\lambda = 2$ and $k = r - 2$,

then the dual will be BIB or PBIB and formulae are available for computing the remaining parameters of the dual.

(b) Clatworthy [3, pp. 94] has presented a proof which shows all PBIB designs with parameters

$$v = k\left[(r-1)(k-1) + 1\right] ,$$

$$\lambda_1 = 1 , \qquad \lambda_2 = 0$$

$$n_1 = r(k-1) , \quad b = r\left[(r-1)(k-1) + 1\right]$$

$$r = r , \qquad k = k , \text{ and}$$

$$n_2 = (r-1)(k-1)^2$$

to have duals which are PBIB.

(c) The duals of some designs have the property that all the v* treatments can be arranged into m groups of n treatments each, such that any two treatments in the same group are 1st associates, while any two treatments not in the same group are 2nd associates. Such a dual will be a member of the group divisible type of PBIB designs (see Bose and Shimamoto [2, pp. 155]), and the original design is thus proven to be twice balanced.

Since this grouping property is sometimes quite easily spotted, it serves as a useful tool in verifying designs to be twice balanced. Its reliability may be readily demonstrated by finding all the p^{i}_{jk} for a general case of such a grouping. They will be constant, as is required by condition 1.31e of the rules for PBIB designs.

(d) Many designs have v = b and r = k, and hence have incidence diagrams which are square in shape. This diagram is often symmetric about one of the main diagonals. We may then rotate the design about its axis of symmetry and still retain the same diagram.

However, rotating a design about one of its main diagonals is equivalent to making all the rows into columns, or dualizing the design. Hence, if a design is symmetric about either of its main diagonals, it is self-dual.

Consider, for example, plan S-2 from the B.C.S. Tables [1, pp. 90], noting that its incidence diagram in Table 2.1 is symmetric about a diagonal. The reader can easily see that rotating the design about this diagonal does indeed change the rows into columns, and that the resulting diagram is the same as the dual of S-2, except that the blocks are in reverse order.

Table 2.1

	1	2	3	4	5	6
1	x	x		x	x	
2		x	x		x	x
3	x		x	x		x
4	x	x		x	x	
5		x	x		x	x
6	x		x	x		x

(e) A large number of PBIB designs have BIB duals. These can be immediately spotted because of mathematical relationships among their parameters. Utilizing the classification system for PBIB designs, as set forth in Bose and Shimamoto [2, pp. 154], we may prove the following results.

A PBIB design with two associate classes has a BIB design for its dual if, and only if:

(1) for singular group divisible designs

$$b = m \quad .$$

(2) for semi-regular group divisible designs

$$b = v - m + 1$$

and $$\lambda_1 = \frac{r(k/m - 1)}{n - 1} \quad .$$

(3) for triangular designs

$$n = b \quad \text{and} \quad 2\lambda_1 - \lambda_2 = r$$

or

$$v - b = n - 1 \qquad \text{and}$$

$$r = (4-n)\lambda_1 + (n-3)\lambda_2$$

Note that in all three cases v must be greater than b.

(f) In addition to the proofs involving PBIB designs, we have a useful relationship for BIB designs. It may be shown that a BIB design with $v = b$ and $r = k$ is self-dual.

III. TABLES

3.1 Introduction to Tables

In the tables of this chapter, we list the twice balanced designs which have been found. Designs shown are from either Cochran and Cox [4] or the B.C.S. Tables of Partially Balanced Designs with Two Associate Classes [1]. The plan number serves to identify the type of design as is shown in Key 3.1.

Key 3.1

Plan No.	Type of Design	Where Catalogued
10.__	Balanced Lattice	Cochran and Cox, Chapter 10
11.__	BIB	Cochran and Cox, Table 11.3
*	BIB	Cochran and Cox, Table 11.3
S-__	Singular Group Divisible PBIB	B.C.S. Tables IA and IB
SR-__	Semi-Regular Group Divisible PBIB	B.C.S. Tables IIA and IIB
R-__	Regular Group Divisible PBIB	B.C.S. Tables IIIA and IIIB
S-__.__	Simple PBIB	B.C.S. Tables IVA and IVB
T-__	Triangular PBIB	B.C.S. Tables VA and VB

In the tables we show the following:

(a) The plan number of the design considered. (Designs marked by an asterisk are constructed by forming all possible combinations of the v treatments in blocks of size k.)

(b) The v, k, r, and b parameters of this design and of the dual. Remember that the star identifies a parameter of the dual.

(c) The plan number of the dual.

(d) In the column labeled P, we indicate which of the proofs of 2.32 we invoke in proving that the design is twice balanced.

When the plan number of a dual is given as SD, we mean that the design is self-dual.

In some cases the dual of a design is BIB but is not listed in Cochran and Cox. We show this by giving the value of λ for the dual in place of the dual number.

The experimenter should note that if he cannot find in the primarily listed designs one which is suitable for his problem, he may then use the values of the dual parameters and work backwards in the tables.

For example, if an experimenter had a problem which required that he use 15 treatments in 9 blocks with approximately 10 treatments in each block, he would be unable to find a suitable design listed in the plan number column.

However, if he were to look under the v*, k*, r*, and b* columns, he would find parameters fitting his needs in number S-50, which is the dual of plan number S-16.

3.2 Tables of Twice Balanced Designs.

Plan No.	v b*	k r*	r k*	b v*	Dual No.	P
10.1	9	3	4	12	SR-20	a
10.2	16	4	5	20	SR-51	a
10.3	25	5	6	30	SR-70	a
10.4	49	7	8	56	SR-85	a
10.5	64	8	9	72	SR-89	a
11.1	4	2	3	6	SR-1	a
*	4	3	3	4	SD	f
11.2	5	2	4	10	T-1	a
*	5	3	6	10	T-15	e
*	5	4	4	5	SD	f
11.3	6	2	5	15	T-20	a
11.4	6	3	5	10	T-9	a
11.6	6	4	10	15	T-25	e
*	6	5	5	6	SD	f
*	7	2	6	21	T-31	a
11.7	7	3	3	7	SD	a
11.8	7	4	4	7	SD	f
*	7	6	6	7	SD	f

Plan No.	v b*	k r*	r k*	b v*	Dual No.	P
11.9	8	2	7	28	T-32	a
11.10	8	4	7	14	SR-32	c
*	8	7	7	8	SD	f
*	9	2	8	36	T-33	a
11.13	9	6	8	12	SR-26	c
*	9	8	8	9	SD	f
11.14	10	2	9	45	T-35	a
11.16	10	4	6	15	T-22	a
11.18	10	6	9	15	T-27	e
*	10	9	9	10	SD	f
*	11	2	10	55	T-36	a
11.19	11	5	5	11	SD	f
11.20	11	6	6	11	SD	f
*	11	10	10	11	SD	f
11.21	13	3	6	26	S-1.4	a
11.22	13	4	4	13	SD	a
11.23	13	9	9	13	SD	f
11.24	15	3	7	35	S-1.9	a
11.25	15	7	7	15	SD	f
11.26	15	8	8	15	SD	f
11.27	16	6	6	16	SD	f
11.29	16	10	10	16	SD	f

Plan No.	v b*	k r*	r k*	b v*	Dual No.	P
11.30	19	3	9	57	S-1.18	a
11.31	19	9	9	19	SD	f
11.32	19	10	10	19	SD	f
11.33	21	3	10	70	S-1.22	a
11.34	21	5	5	21	SD	a
11.36	25	4	8	50	S-1.17	a
11.37	25	9	9	25	SD	f
11.38	28	4	9	63	S-1.21	a
11.39	28	7	9	36	T-34	a
11.40	31	6	6	31	SD	a
11.41	31	10	10	31	SD	f
11.42	37	9	9	37	SD	f
11.43	41	5	10	82	S-1.25	a
11.44	57	8	8	57	SD	a
11.45	73	9	9	73	SD	a
11.46	91	10	10	91	SD	a
S-1	6	4	2	3	(1)	e
S-2	6	4	4	6	SD	d
S-3	6	4	6	9	S-13	c
S-4	6	4	8	12	S-25	c
S-5	6	4	10	15	S-48	c
S-6	8	4	3	6	SR-2	c

Plan No.	v b*	k r*	r k*	b v*	Dual No.	P
S-7	8	6	3	4	(4)	e
S-9	8	6	6	8	SD	c
S-11	8	6	9	12	S-31	c
S-12	9	6	2	3	(3)	e
S-14	9	6	6	9	SD	d
S-15	9	6	8	12	S-30	c
S-16	9	6	10	15	S-50	c
S-17	10	4	4	10	T-2	c
S-18	10	8	4	5	(6)	e
S-19	10	6	6	10	T-16	c
S-21	10	8	8	10	SD	d
S-22	12	8	2	3	(4)	e
S-23	12	6	3	6	SR-3	c
S-24	12	9	3	4	(6)	e
S-28	12	10	5	6	(8)	e
S-37	12	8	10	15	S-52	c
S-40	14	6	3	7	SD	d
S-41	14	8	4	7	SD	d
S-43	14	6	6	14	SD	d
S-44	14	8	8	14	SD	d
S-1.7	27	3	5	45	S-1.14	b
S-1.12	40	4	4	40	SD	b

Plan No.	v b*	k r*	r k*	b v*	Dual No.	P
S-1.26	85	5	5	85	SD	b
R-1	6	3	3	6	SD	d
R-2	6	4	4	6	SD	c

IV. ILLUSTRATIVE EXAMPLE

4.1 Introduction

As an illustration of the use of the tables we have introduced in this paper, we will consider a problem similar to the one mentioned in section 2.21, which involved the output of various sawyers using different saw blades. The example will also demonstrate:

(a) the analysis of linked block designs by first finding the estimates of the block effects,

(b) the interblock analysis of PBIB designs using a revised computing procedure,

(c) finding a confidence interval for the ratio of the two variance components in a mixed model experiment.

4.2 Fixed Effects Model

4.21 The Problem. Suppose that a saw manufacturer is planning to produce a band saw for use by sawmills in cutting a certain kind of lumber. He has six possible saw types to choose from and four sawyers to try them. He wants us to pick out the best blade to be manufactured and, as a by-product of the experiment, he would like to find out which of his sawyers are the most efficient. In addition, he feels that it is necessary for us to limit our observations to about twelve.

In sawing lumber from logs, both the type of saw blade and the skill of the sawyer are important contributors to the volume of lumber which the mill can produce. The "best" type of blade may vary with the species of wood and the kind of board being sawed, but under specified conditions we may find a most desirable one.

The sawyer is the most important man in the mill, and his speed and accuracy of judgment have a great deal to do with the output from the mill. We must insure that all sawyers will cut the same size boards and use the same method of sawing a log.

Since we desire estimates of both the blades and the sawyer effects, we may profitably use a twice balanced design in setting up our experiment. Looking in the tables of Chapter III, we find that the BIB design 11.1 fits our data very nicely. Its dual is the partially balanced design SR-1. Note that this is the same twice balanced design we discussed in 1.42.

Though it makes no difference how we label them, we will call our blades the treatments and consider our men to be blocks. We thus use in our analysis the parameters of the PBIB design, which are $v = 6$, $b = 4$, $r = 2$, and $k = 3$. This means that we will get our estimates of the man effects from a balanced design while the blade estimates must come from the PBIB design. Both effects will be considered fixed.

4.22 Finding Estimates of the Man Effects. The experiment is run as planned, and the resulting data, the number of board feet of lumber sawed out per unit of time by the jth sawyer using the ith blade, is tabled (see Table 4.1). This value is the y_{ij} specified in the model (1.51.1). From the data, we compute the treatment and block totals, T_i and B_j, the grand total, G, and the treatment and block means.

Table 4.1

Blocks (Men)	Treatments (Blades) 1	2	3	4	5	6
1	1101.26		1060.05		1103.16	
2	1007.44			965.56		961.88
3		1042.46	1015.33			1025.36
4		919.19		875.83	945.53	
T_i	2108.70	1961.65	2075.38	1841.39	2048.69	1987.24
T_i/r	1054.350	980.825	1037.690	920.695	1024.345	993.620

G = 12023.05

Table 4.1 (Con.)

Blocks (Men)	Treatments (Blades) B_j	B_j/k
1	3264.47	1088.16
2	2934.88	978.29
3	3083.15	1027.72
4	2740.55	913.52
T_i	12023.05	
T_i/r		

We will first find our estimates of the man effects, $\hat{b}_j$, from the balanced design 11.1. To do this we need the block totals, B_j, and the adjusted block totals, Q_{B_j}, which are found from

(4.22.1) $$Q_{B_j} = B_j - T_{B_j} \quad ,$$

where T_{B_j} is the sum of the treatment means for all treatments appearing in block j.

We remind the reader that here, while estimating the block effects, we are working with design 11.1, which is actually the dual of the design appearing in Table 4.1.

We may now calculate our $\hat{b}_j$'s from

(4.22.2) $$\hat{b}_j = \frac{1}{E} \; \frac{Q_{B_j}}{k}$$

The parameter E is the efficiency factor of the BIB design and is found in Cochran and Cox [4, pp. 327]. In this design E = .67.

The $\hat{b}_j$'s may be most easily calculated by use of Table 4.2.

Table 4.2

Man	B_j	T_{B_j}	Q_{B_j}	$\hat{b}_j$
1	3264.47	3116.385	148.085	74.04
2	2934.88	2968.665	- 33.785	-16.89
3	3083.15	3012.135	71.015	35.51
4	2740.55	2925.865	-185.315	-92.66
	12023.05	12023.05	0	0

Note as a check that

$\Sigma B = G$, $\Sigma T_B = G$, $\Sigma Q_B = 0$, and $\Sigma \hat{b} = 0$.

4.23 Example of Computation. As an example of this computing, to find $\hat{t}_1$ we first find

$$T_{B_1} = \frac{T_1}{r} + \frac{T_3}{r} + \frac{T_5}{r} ,$$

since block 1 contains treatments 1, 3, and 5. Therefore,

$$T_{B_1} = 1054.350 + 1037.690 + 1024.345$$
$$= 3116.385$$

Next,

$$Q_{B_1} = 3264.47 - 3116.385$$
$$= 148.085$$

Then

$$\hat{b}_1 = \frac{148.085}{(.67)(3)} = 74.04$$

4.24 <u>Finding Estimates of the Blade Effects</u>. Once we have found our $\hat{b}_j$'s we may compute the estimates of the blade effects directly by the relationship

(4.24.1) $\hat{t}_i = \frac{T_i}{r} - \frac{G}{N} - \frac{1}{r}$ (sum of the $\hat{b}_j$ for blocks containing treatment i).

Here, N = vr = 12, which is the number of observations.

The block estimates are shown in Table 4.3.

Table 4.3

Blade	$\hat{t}_i$
1	23.854
2	7.474
3	-19.006
4	-26.456
5	31.729
6	-17.611
Total	- .016

For a check, note that $\Sigma\hat{t} = 0$.

As an example,

$$\hat{t}_1 = \frac{T_1}{r} - \frac{G}{N} - \frac{1}{r}(\hat{b}_1 + \hat{b}_2)$$

since treatment 1 occurs in blocks 1 and 2 only. Therefore

$$\hat{t}_1 = 1054.350 - 1001.921 - \tfrac{1}{2}(74.04 - 16.89)$$
$$= 23.854 \quad .$$

4.25 <u>Variances of the Difference Between Two Estimates</u>. We may wish to find the variance of the difference between any two $\hat{b}_j$'s or between some two $\hat{t}_i$'s. In our example, the variance between two man effect estimates is

(4.25.1) $$\mathrm{Var}(\hat{b}_i - \hat{b}_j) = \frac{2\sigma^2}{kE} = \frac{2\sigma^2}{3(.67)}$$
$$= \sigma^2 \quad .$$

Since we might consider the blades to be the treatments of the PBIB design, there will be two different variances, depending on whether the two blades considered are 1st or 2nd associates. These variances are found from

(4.25.2) $$\mathrm{Var}(\hat{t}_i - \hat{t}_u) = \frac{2\sigma^2}{r} \; \frac{(k-c_j)}{k-1} \quad ,$$

where $j = 1$ if blades i and u are 1st associates, and $j = 2$ if they are 2nd associates. c_1 and c_2 are found in the B.C.S. Tables [1, pp. 140]. For design SR-1, $c_1 = 0$, and $c_2 = \frac{1}{2}$.

Therefore, the variance of the difference between two 1st associates is

$$\sigma^2 \, \frac{(3-0)}{2} = 1.5\,\sigma^2 \quad .$$

Similarly, the variance of the difference between two 2nd associates is

$$\frac{\sigma^2(3-.5)}{2} = 1.25\,\sigma^2 .$$

The reader should note that we have been able to estimate the man effects with equal precision and the blade effects with only two degrees of precision, because our design was twice balanced. If it were not so chosen, we might have many different degrees of precision.

4.3 Mixed Model

4.31 Introduction. Let us now look at the problem from a different point of view. Suppose we wish to assume that our four sawyers are randomly chosen from a population of all possible sawyers. Such a supposition means that we will have two variances entering into our model, and also, that we will get estimates of our blade effects different from those found under a fixed model.

4.32 Using an Analysis of Variance Table to Estimate μ. In the analysis of a mixed model experiment, we record our data in the same manner as Table 4.1. The first step in the "interblock analysis" will be to set up Table 4.4, which is the analysis of variance table for a BIB design.

Table 4.4

Source	D/F	Sum of squares	Mean Square
Men (adjusted)	3	31227.68	10409.23
Blades (unadjusted)	5	23285.34	
Error	3	235.30	78.43
Total	11	54748.32	

Table 4.4 (Cont.)

Source	Expectation of S/S
Men (adjusted)	$(b-1)\sigma^2 + (N-v)\sigma_1^2$
Blades (unadjusted)	
Error	$(N-b-v+1)\sigma^2$
Total	

The adjusted man sum of squares is found from

(4.32.1) $$\frac{r(b-1)\,\Sigma Q_B^2}{bk\,(r-1)} \quad ,$$

the unadjusted blade sum of squares is

(4.32.2) $$\frac{\Sigma\, T^2}{r} - \frac{G^2}{N} \quad ,$$

and the total sum of squares is

(4.32.3) $$\Sigma\, y^2 - \frac{G^2}{N}$$

The error sum of squares is found by subtraction.

We now find estimates of the error variance, σ^2, and the block variance, σ_1^2, by equating the error and block sums of squares to their expectations. Hence,

$$(N-b-v+1)\hat{\sigma}^2 = 235.30$$
$$3\hat{\sigma}^2 = 235.30$$
$$\hat{\sigma}^2 = 78.43$$

and

$$(b-1)\hat{\sigma}^2 + (N-v)\hat{\sigma}_1^2 = 31227.68$$
$$3\hat{\sigma}^2 + 6\hat{\sigma}_1^2 = 31227.68 \ .$$

Substituting for $\hat{\sigma}^2$, we get

$$\hat{\sigma}_1^2 = 5165.39 \ .$$

An estimate of $\mu = \dfrac{\sigma_1^2}{\sigma^2}$ is then

$$\hat{\mu} = \frac{\hat{\sigma}_1^2}{\hat{\sigma}^2} = \frac{5165.39}{78.43}$$
$$= 65.86 \ .$$

4.33 <u>Finding Interblock Estimates of the Blade Effects</u>. To find estimates of the blade effects under a mixed model, we will use a method slightly different from that prescribed in Bose and Shimamoto [2, pp. 170], but one which we feel may speed up the computing procedure, especially for large experiments.

First we compute A_i from

(4.33.1) $\quad A_i$ = sum of means for all blocks containing treatment i.

Next, we find L_i by

(4.33.2) $$L_i = (k\hat{\mu}+1)T_i - k\hat{\mu}A_i - \frac{rG}{N} \quad .$$

$S_1(L_i)$ is then computed by adding together the L's for all first associates of treatment i.

From this, we may go directly to the interblock estimates of the blade effects, $\hat{t}_i'$, using

(4.33.3) $$\hat{t}_i' = \frac{(k-d_2)L_i + (d_1-d_2)S_1(L_i)}{r(k+k^2\hat{\mu}-k\hat{\mu})} \quad .$$

Here,

(4.33.4) $$d_i = \frac{c_i\Delta(k\hat{\mu})^2 + r\lambda_i k\hat{\mu}}{(k\hat{\mu})^2 + rHk\hat{\mu} + r^2} \quad ,$$

where i = 1 or 2. The parameters c_1, c_2, Δ, λ_1, λ_2, and H may be found in the B.C.S. Tables [1, pp. 140]. For this example, $c_1 = 0$, $c_2 = \frac{1}{2}$, $\Delta = 8/3$, $\lambda_1 = 0$, $\lambda_2 = 1$, and $H = 10/3$.

The computing may be most easily done by setting up the table 4.5.

For a check, note that

$$\sum A_i = G, \sum L_i = 0, \sum S_1(L_i) = 0, \text{ and } \sum \hat{t}_i' = 0 \quad .$$

Table 4.5

Treatment	T_i	A_i	L_i	$S_1(L_i)$	$\hat{t}_i'$
1	2108.70	2066.450	8452.613	3991.720	24.07
2	1961.65	1941.233	3991.720	8452.613	7.26
3	2075.38	2115.873	- 7929.148	-10124.436	-18.59
4	1841.39	1891.810	-10124.436	- 7929.148	-26.86
5	2048.69	2001.673	9333.388	- 3725.179	31.66
6	1987.24	2006.010	- 3725.179	9333.388	-17.54
	12023.05	12023.07	- .042	- .042	0

4.34 <u>Example of Computation</u>. In our example, we computed $\hat{t}_i'$ in the following manner.

First,

$$A_1 = \frac{1}{k}\sum B_j$$

where the summation is over the r blocks containing treatment 1. Therefore,

$$A_1 = \frac{1}{3}(B_1+B_2)$$

$$= 1088.16 + 978.29$$

$$= 2066.450$$

because treatment 1 occurs only in blocks 1 and 2.

Then,

$$L_1 = \left[3(65.86) + 1\right] T_1 - 3(65.86)A_1 - \frac{2(12023.05)}{12}$$

$$= 8452.613$$

$$S_1(L_1) = L_2 = 3991.720$$

since treatment 2 is the only 1st associate of treatment 1.

Since $c_1 = 0$ and $\lambda_1 = 0$, $d_1 = 0$, also.

$$d_2 = \frac{(\frac{1}{2})(\frac{8}{3})(197.58)^2+(2)(197.58)}{(\frac{8}{3})(197.58)^2+2(\frac{10}{3})(197.58)+4}$$

$$= .9975$$

Lastly,

$$\hat{t}_1' = \frac{(3-.4975)L_1-(.4975)\ S_1(L_1)}{2\left[3+9(65.86)-197.58\right]}$$

$$= 24.07$$

4.35 A Confidence Interval for . If we wish to place confidence limits on our estimate of μ, we must follow the procedure given by Thompson [5, pp. 19].

First, we find

(4.35.1) $$e = \frac{1}{r}\left[k(r-1) - \lambda^*\right].$$

Then, we compute μ_{α_1} and μ_{α_2} from

(4.35.2) $$\mu_\alpha = \left[\frac{1}{e^2F_\alpha}\ \frac{(N-v-b+1)}{(b-1)}\ \frac{\Sigma Q_B^2}{S.S.E.}\right] - \frac{1}{e}$$

where F_α has $b - 1$ and $N - v - b + 1$ degrees of freedom. Then, if $\alpha_1 < \alpha_2$,

$$\mu_{\alpha_2} < \mu < \mu_{\alpha_1}$$

is a confidence interval for μ with confidence coefficient $\alpha_2 - \alpha_1$.

For example, let us find a 95% confidence interval for our μ, and choose $\alpha_1 = .025$ and $\alpha_2 = .975$. First,

$$e = \tfrac{1}{2}\left[(3)(2) - 1\right] = 2.5$$

since, for design 11.1, $\lambda = 1$.

Then,

$$\mu_{\alpha_1} = \left[\frac{1}{(6.25)\ F_{3,3}(.025)} \ \frac{3}{3} \ \frac{62455.373}{235.29}\right] - \frac{1}{2.5}$$

$$= \frac{62455.373}{(6.25)(.0648)(235.29)} - \frac{1}{2.5}$$

$$= 655.01 \ .$$

Similarly,

$$\mu_{\alpha_2} = \left[\frac{1}{(6.25)(15.439)} \ \frac{3}{3} \ \frac{62455.373}{235.29}\right] - \frac{1}{2.5}$$

$$= 2.351 \ .$$

Therefore, the

$$\text{Prob.} \left[2.351 < \mu < 655.01\right] = .95 \ .$$

This interval is exceptionally large because our small experiment gave us few degrees of freedom.

V. SUMMARY AND CONCLUSIONS

By a study of the duality relationships of a large number of balanced and partially balanced incomplete block designs, certain ones have been found which lend themselves nicely to interblock analysis. Besides facilitating this analysis, these designs make possible the use of a new method for studying the relative variability of the inter- and intra-block error.

These "nice" designs, which are called twice balanced, have the property that their duals are also balanced or partially balanced. In the partially balanced designs, the investigation has been confined to those with two associate classes.

Some methods are shown which may be used to prove that a dual is twice balanced.

The twice balanced designs which have been found are catalogued, showing the plan numbers of the design and the dual, and the necessary identifying parameters of both. The proofs used in verifying the designs to be twice balanced are also indicated.

Finally, there is an illustrative example making use of the methods and tables introduced in this paper. It includes a new computing method to be used for finding estimates of the treatment effects in a mixed model experiment.

VI. BIBLIOGRAPHY

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VIII. VITA

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APPENDIX

Designs Having Duals which are not BIB or PBIB with Two Associate Classes

11.5	S-8
11.11	S-10
11.12	S-20
11.15	S-33
11.17	
11.28	
11.35	